HOLOGRAPHY AND TRANSPORT AT FINITE DENSITY

A thesis presented by

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Abstract

This work is devoted to the study of holography as a tool to understand strongly correlated systems in condensed matter systems. After a review of the general idea of the AdS/CFT correspondence we proceed to topics that are relevant to condensed matter systems as the AdS/CFT at finite temperature and finite density. Finally, we are introducing the concept of Effective Holographic Theories for condensed matter systems and study the zero temperature infrared behaviour of an Einstein Maxwell Dilaton System with a Liouville potential.

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to Stauroula and to my family

1 Introduction

One approach for tackling physical problems is to identify a small parameter in the problem and first treat this parameter as zero - simplifying the problem and perhaps making it tractable. For example, one might take a weakly interacting system and model it as a solvable non-interacting system, then treat the interaction as a small perturbation. While this approach is one of the most frequently used tools of a physicist's toolbox, it is only reliable for systems sufficiently "near" to the non-interacting, or solvable, situation. Systems where the interaction is strong are often resistant to this method of attack, and indeed such nonperturbative problems are often found in nature.

Despite the difficulty of this problem, remarkable progress has been made in understanding strongly interacting quantum systems. A powerful technique to study strongly coupled system, that we will consider in this paper is Holography.

Motivated from calculations in string theory involving the dynamics of D-branes, Maldacena [1] in 1997 introduced the AdS/CFT correspondence. The correspondence states that certain non-Abelian gauge theories can be described in a wholly different way, as theories of quantum gravity living in a higher-dimensional spacetime, in particular a spacetime with the asymptotic behaviour of anti-de Sitter space (AdS). But it is not only the different dynamics of those two systems and the different number of dimensions they live in that make them look different. According to the correspondence, when the gauge theory is strongly coupled the gravity theory is weakly coupled and vice versa. The AdS/CFT correspondence is a one to one correspondence between a strongly coupled system and a weakly coupled system. Therefore, one ultimately could ask questions about the strongly coupled system, that could not be answered before, but now they could be answered in the language of perturbation theory.

Moreover, the AdS/CFT correspondence could be used to describe the thermodynamic properties of the gauge theory if the theory is defined at finite temperature[2]. This could be achieved if we consider black hole solutions for the dual gravitational theory, since black holes have an associate temperature and an entropy that is proportional to the one fourth of the area of the horizon. Hence, by exploring the thermodynamic properties of the gravitational system we could describe the phase structure of the dual gauge theory. It is found that the dual description of $\mathcal{N} = 4$ Super Yang Mills gauge theory at finite temperature indicates correctly the confined-deconfined phase transition.

In addition, the AdS/CFT correspondence can incorporate the case where there is a global conserved $U(1)$ current in the gauge theory by including a $U(1)$ gauge field in the dynamics of the dual gravitational system [3]. The gravitational solution to the system under consideration is charged black holes in Einstein Maxwell anti de Sitter space. The phase structure of the charged black hole systems reveal a rich phase structure with analogues to classic thermodynamic systems, which displays classic critical phenomena. Specifically, the structures are isomorphic to the Van der Waals-Maxwell liquid-gas system.

Even though the AdS/CFT was initially used as a tool to tackle high energy physics problems, lately [4] has been applied to understand the low-temperature dynamics of strongly-correlated electron systems with intention to develop a quantitative description of universality classes at low temperature. The gravitational laboratories this time are the Einstein-Maxwell-Dilaton systems with a scalar potential. The near-extremal (zero temperature) solutions of those systems provide IR quantum critical geometries [5] that seems to agree with the study of the dual condensed matter systems.

In this work, after a quick review of large N gauge theories that will indicate a geometrical description for gauge theories [6], we will try to introduce the concept of the AdS/CFT [7],[8]. In chapter 3 we will explore the decoupling principle and the near horizon limit of D_3 branes where we will first state the correspondence. Then, we will try to make a precise statement for the correspondence by stating how to map operators in the gauge theory to fields in the gravitational theory. In the sequence, we will describe the finite temperature regime and study the Hawking Page transition. Next, we will incorporate with additional degrees of freedom in the gravitational theory by including a vector field in the dynamics. Finally, we will study the Einstein-Maxwell-Dillaton system that is suitable for Effective Holographic Theories for Condensed matter systems and we will show that there is a quantum critical behaviour for those systems.

2 $\frac{1}{N}$

In this section we briefly review the $\frac{1}{N}$ expansion, in the limit where the number of colors N goes to infinity, for a four dimensional $SU(N)$ gauge theory coupled to fermions. This will be instructive for us because on the one hand this is the appropriate limit, where the low energy effective string description of the theory is weakly coupled and on the other hand, because it turns out that in this limit there is a topological description of the gauge theory.

The degrees of freedom are the gauge fields transforming in the adjoint

$$A^a_{\mu b} = A^{b\dagger}_{\mu a}, \quad A^a_{\mu a} = 0 \tag{2.1}$$

and the matter fields transforming in the fundamental

$$\psi^a. \tag{2.2}$$

The field strength tensor is

$$F^a_{\mu\nu b} = \partial_\mu A^a_{\nu b} + A^a_{\mu c}A^c_{\nu b} - (\mu \leftrightarrow \nu). \tag{2.3}$$

And the Lagrangian is written as

$$\mathcal{L} = \frac{N}{q^2_{YM}} \left(-\frac{1}{4} F^a_{\mu\nu b} F^{\mu\nu b}_a + \bar{\psi}_a (i\partial_\mu + A^a_{\mu b})\gamma^\mu \psi^b - m\bar{\psi}_a \psi^a \right) \tag{2.4}$$

In order to have a good perturbation expansion we should keep something constant and what we are keeping constant is the 't Hooft coupling constant $\lambda = g^2_{YM}N$. We notice here that there are two indices for the gauge field, one transforming in the fundamental, and the other in the anti-fundamental. This trick, first established by 't Hooft, enables us to adopt a double line notation for the Feynmann diagrams, which is useful for a geometrical interpretation of the diagrams. In this notation the propagators are written in the form

$$\begin{aligned} < A^a_{\mu b}(x) A^c_{\nu d}(y) > &= (\delta^a_d \delta^c_b - \frac{1}{N}\delta^a_b \delta^c_d) D_{\mu\nu}(x-y) \\ < \psi^a(x)\bar{\psi}_b(y) > &= \delta^a_b S(x-y). \end{aligned} \tag{2.5}$$

In the limit where $N \to \infty$ the $\frac{1}{N}$ in the gauge field propagator is suppressed. It is now easy to read from the Lagrangian which are the multiplication constants that appear in the Feynmann diagrams. Each vertex V gives a factor N, every internal propagator E carries a factor of $\frac{1}{N}$ and every closed loop counts a number of N (because we are summing over the colors).

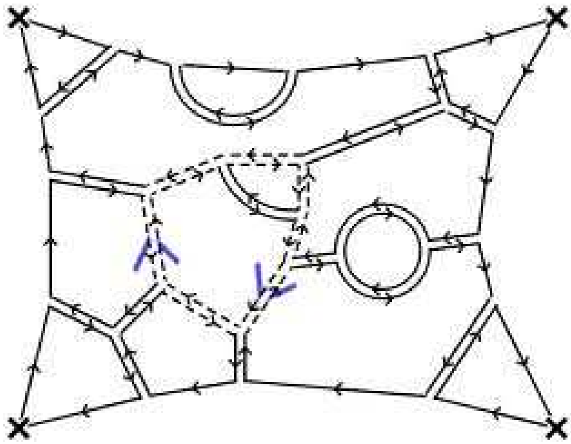

Figure 1: A planar diagram

For the sake of simplicity let us consider only vacuum to vacuum graphs. Due to the absence of external lines every index should close to a loop. Let us interpret each loop with a perimeter of a polygon. Furthermore, let us glue one edge of one polygon with one edge of another if they both lie on the same double line and give an orientation to each polygon by the right hand rule. In this manner we have constructed a two dimensional oriented surface which is topologically equivalent with a sphere with some number of holes cut out of it and some numbers of handles stuck on it. We now can see what are the leading diagrams. First notice that each graph is proportional to

$$N^{F-E+V} = N^{\chi}, \tag{2.6}$$

where F is the number of faces (closed loops), E are the number of edges (internal propagators), and V are the number of vertices and χ is the Euler characteristic which is equal to two minus twice the number of holes H minus the number of handles (boundaries) B

$$\chi = 2 - 2H - B. \tag{2.7}$$

Thus the leading connected vacuum to vacuum graphs are of order N^2. They are planar graphs consisted of gluons. The next order diagrams, vacuum graphs with quark loops, are of order N. Hence, we have shown that there is a geometrical interpretation for the Feynman diagrams of large N gauge theory that indicates a geometrical description. If we consider the parameter N of the colour gauge group SU(N) as a free parameter, then an expansion of the amplitudes at $N \to \infty$ arranges the Feynman diagrams into sets which have exactly the topology of the quantized dual string with quarks at its ends. What we will do in the rest of the text is to geometrically describe large N gauge theories.

3 AdS Space

The aim of this section is to review the geometrical structure of the Anti De Sitter space and to array the basic formulae such as the field equations in AdS space that will be useful in the sequence.

We will begin our discussion by giving the definition of the AdS space. The AdS space is a maximally symmetric Lorentzian manifold with constant negative scalar curvature and it is a vacuum solution of Einstein's field equation with a negative cosmological constant Λ. Specifically, AdS_{p+2} of dimension $p+2$ may be embedded in a $p+3$ dimensional flat space associated with a metric

$$ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2, \tag{3.1}$$

satisfying the constraint

$$X_0^2 + X_{p+2}^2 - \sum_{i=1}^{p+1} X_i^2 = L^2. \tag{3.2}$$

L is refereed in the literature as the AdS scale. We can parametrized this constraint by using the Poincaré coordinates

$$\begin{aligned} X_0 &= \frac{u}{2}\left(1 + \frac{1}{u^2}(L^2 + \vec{x}^2 - t^2)\right), \quad X_i = \frac{Lx^i}{u} \\ X_{p+1} &\frac{u}{2}\left(1 - \frac{1}{u^2}(L^2 - \vec{x}^2 + t^2)\right), \quad X_{p+2} = \frac{Lt}{u}. \end{aligned} \tag{3.3}$$

Then we obtain the what so called Poincaré metric

$$ds^2 = \frac{L^2}{u^2}\left(du^2 - dt^2 + d\vec{x}^2\right) \tag{3.4}$$

It will be useful to compute the field equation for a massive scalar field minimally coupled to gravity in AdS. The field equation is then

$$(\Box - m^2)\Phi = 0. \tag{3.5}$$

Using the Poincaré coordinates this is written as

$$\frac{u^2}{L^2}\left(\partial_u^2 - \frac{p}{u}\partial_u - \partial_t^2 + \partial\partial\right)\Phi = m^2\Phi \tag{3.6}$$

If we do a Fourier transform

$$\Phi(u;x) = \int \frac{d^{p+1}q}{(2\pi)^{p+1}}\Phi(u;q)e^{iqx} \tag{3.7}$$

the field equation is written in the form

$$\left(\partial_u^2 - \frac{p}{u}\partial_u - q^2 - \frac{m^2L^2}{u^2}\right)\Phi(u;q) = 0 \tag{3.8}$$

In the limit where $q = 0$ the solution scale as

$$\Phi\pm \sim u^{\Delta_\pm}, \quad \Delta_\pm = \frac{p+1}{2} \pm \frac{1}{2}\sqrt{(p+1)^2 + 4m^2L^2} \tag{3.9}$$

and the general solution it is given in terms of Bessel function

$$\Phi(u;q) \sim u^{\frac{p+1}{2}} Z_\nu(\sqrt{q^2}u). \tag{3.10}$$

It will be convenient for the sequence to define the bulk to boundary propagator satisfying the Laplace equation,

$$\left(\Box - m^2\right) K_\Delta(u;x;x') = 0, \tag{3.11}$$

and the boundary condition

$$K(u;x;x')|_{u=0} = \delta^{p+1}(x - x'). \tag{3.12}$$

So that the solution for the field Φ can be written as

$$\Phi(u,x) = \int d^4x' K(u;x;x')\Phi_0(x') \tag{3.13}$$

One solution for the propagator is

$$K_\Delta(u;x;x') = \frac{\Gamma[\Delta]}{\pi^{\frac{p+1}{2}}\Gamma[\Delta - \frac{p+1}{1}]}\frac{u^\Delta}{(u^2 + |x-x'|^2)^\Delta}, \tag{3.14}$$

where

$$\Delta(\Delta - p - 1) = m^2L^2. \tag{3.15}$$

4 AdS/CFT Correspondence

The AdS/CFT correspondence is a conjecture that relates conformal field theories living in $p+1$ dimensions and relativistic gravitational theories living in AdS_{p+2}. In the following section we will try to make the correspondence clear.

4.1 The decoupling principle and the near horizon limit

Consider N parallel coincident D_3-branes living in the ten dimensional spacetime. In this background the theory contains two types of excitations, closed strings and open strings, corresponding to the excitations of the ten dimensional bulk space and the excitations of the D-branes respectively. If we are interested in the energy regime bellow the string scale M_s then the action of the theory is

$$S = S_{bulk} + S_{branes} + S_{interactions}. \tag{4.1}$$

The first term is referring in the Type-II supergravity which could be described at low energy by the classical gravity. If we expand the bulk action around the free point with $g_{\mu\nu} = \eta_{\mu\nu} + kh_{\mu\nu}$, then

$$S_{bulk} \sim \frac{1}{2k^2} \int d^{10}\sqrt{-g}R + ... \sim \int d^{10}x(\partial h)^2 + \mathcal{O}(k). \tag{4.2}$$

The low energy description of the open degrees of freedom S_{branes} are described by $\mathcal{N} = 4$, U(N) super Yang-Mills theory. Furthermore, the interaction terms turns out to be proportional to k. A useful way to take the low energy limit of the theory is to keep the energy fixed but to take the characteristic scale of the theory l_s to be zero and to keep any other dimensionless parameters (N, g_s) finite. In this limit $k \sim g_s l_s^4 \to 0$ and the theory decouples to Yang-Mills gauge theory and to low dimensional supergravity. This is known as the decoupling limit.

Another point of view of the same system is to consider D_3-Branes as gravitational sources. The background field generated by the N D_3 branes is

$$ds^2 = -H^{-\frac{1}{2}}(r)(-dt^2 + d\vec{x}d\vec{x}) + H^{\frac{1}{2}}(r)(dr^2 + r^2 d\Omega_5^2) \tag{4.3}$$

where

$$H = 1 + \frac{L^4}{r^4}, \quad L^4 = 4\pi g_s l_s^4 N. \tag{4.4}$$

As we could see the tt component is r dependent and therefore the energy measured in a distance r, E_r would be due to the redshift

$$E_\infty = H^{-\frac{1}{4}} E_r \tag{4.5}$$

where E_∞ is the energy measured at infinity. Therefore, an observer at infinity sees two types of excitations that are decoupled at low energy. The first type is massless large wavelength excitations that propagate in the bulk where the space is essentially flat. The second type is any type of low energy (redshifted) excitations near $r = 0$ with geometry,

$$\begin{aligned} ds^2 =& \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2} \left(-dt^2 + d\vec{x} d\vec{x}\right) + L^2 d\Omega_5^2 \\ =& \frac{L^2}{u^2} \left(du^2 - dt^2 + d\vec{x} d\vec{x}\right) + L^2 d\Omega_5^2. \end{aligned} \tag{4.6}$$

Here the space is $AdS_5 \times S^5$. We conclude that in both description that were discussed above we end up with a theory that consist of two non interacting parts. One of the two parts was free ten dimensional supergravity. Hence, we believe that the remaining parts are the same and that leads to the conjecture that $\mathcal{N} = 4 \quad U(N) \quad SYM$ are the same with ten dimensional SUGRA that lives on $AdS_5 \times S^5$ known as the AdS/CFT correspondence.

We must make some remarks that will help us to understand the correspondence. First, in order to take the low energy limit $l_s \to 0$ in the near horizon limit $r \to 0$ we must keep the energy measured at infinity $E_\infty \sim (E_r l_s) \frac{r}{l_s^2}$ fixed since this is the energy measured in field theory. Furthermore, it is found that both the bosonic and fermionic symmetries are the same in those two theories.

Another remark is that when the gauge theory is defined over $\mathbb{R} \times S^3$ there is a discrete spectrum and a gap and the appropriate coordinates are the global coordinates. On the other hand, when the theory is defined on $\mathbb{R}^{1,3}$ the spectrum is continuous without a gap and the appropriate coordinates are the Poincare coordinates.

Moreover, we must match the three dimensionfull scales in string theory, the string length l_s, the AdS radius L and the ten dimensional Newton's constant G_N with the gauge theory dimensionless coupling constants via

$$\frac{L^4}{l_s^4} = 4\pi g_s N = g_{YM}^2 N = \lambda, \quad 16\pi \frac{G_N}{l_s^8} = (2\pi)^7 g_s^2 = 2^5 \pi^5 \frac{\lambda^2}{N^2}. \tag{4.7}$$

With this matching we infer an important remark. The Newton's constant is invert proportional to the square of the number of colors, $G \sim \frac{1}{N^2}$. That means that quantum effects in string theory are suppressed when $N >> 1$. Furthermore, higher stringy corrections are also small if the curvature of the background is much smaller that the string scale, $L >> l_s$. In other worlds, in the regime of strong 't Hooft coupling λ in the limit of large N, the string theory is described well by the two derivative action of the classical IIB SUGRA on $AdS_5 \times S^5$. But, large N gauge theory is weakly coupled in perturbation theory when $\lambda << 1$. Thus, the strongly coupled behaviour of the gauge theory, which is very difficult to be handled, is described by the weakly coupled string theory and vice versa.

Finally, we should mention that the correspondence could be generalized and valid for a CFT living in d dimensions with a dual bulk theory defined over $AdS_{d+1} \times K$ where K is a compact manifold. Moreover, it could be extended to non conformal field theories in d dimensions, but the price would be that the bulk theory will not have the conformal Killing symmetry any more.

4.2 Fields and operators

Consider the natural objects living in a CFT_4 which are local operators $\mathcal{O}(x)$ to be added in the Lagrangian by the term

$$\int d^4x \Phi_0(x)\mathcal{O}(x) \tag{4.8}$$

This external source Φ_0 could be seen as the boundary value of a field propagating in AdS, at the boundary $u = 0$.

$$\Phi(u;x)|_{u=0} = u^{4-\Delta}\Phi_0(x) \tag{4.9}$$

A quantitative form of the AdS/CFT correspondence is

$$< e^{\int d^4x \Phi_0(x)\mathcal{O}(x)} >_{CFT} = \mathcal{Z}_{string}[\Phi(x;u)|_{u=0} = u^{4-\Delta}\Phi_0(x)]. \tag{4.10}$$

We need to define what we have written above. First, Z_{string} is the generational functional of the on shell string amplitudes in $AdS_5 \times S^5$ restricted to specific boundary condition for the bulk field. On the left hand side of the above equation the $< ... >$ is the expectation value for the gauge theory computed off-shell and Φ_0 is the associate source for the local operator $\mathcal{O}(x)$. At large N and λ the string theory could be approximated by supergravity and we can write

$$\log(\sum e^{I_{SUGRA}(\Phi_0)}) = -W_{gauge}, \tag{4.11}$$

where W_{gauge} is the generating functional for connected correlation functions and the sum is taken over semi classical extrema with the same asymptotics. In the case of AdS_5 there is only one extrema but in general there are more.

We should mention that there are UV divergences in the field theory side that correspond to IR (in the boundary) divergences in the string theory. In order to handle this divergences in the gravity side we impose the boundary conditions in a small distance ϵ from the boundary and then renormilize the action by subtracting the terms that diverge as ϵ goes to zero. Let us see how this works for the two point function for a massive scalar field.

The renormalized Euclidean action after the compactification of S^5 is

$$\begin{aligned} S =& \frac{1}{2}\int dud^4x\sqrt{g}\left((\partial\Phi)^2 + m^2\Phi^2\right) \\ =& -\frac{1}{2}\int d^4x\sqrt{g}g^{uu}\Phi(u;x)\partial_u\Phi(u;x)|_{u=0} \\ =& -\frac{1}{2}\int d^4x_1d^4x_2\Phi_0(x_1)\Phi_0(x_2)\int d^4x\frac{K(u;x_1;x)\partial_uK(u;x_2,x)}{u^3}|_{u=0}. \end{aligned} \tag{4.12}$$

Where we have used the bulk to boundary propagator (3.11). Expanding the propagator around $u = 0$ we find that

$$\int d^4x\frac{K(u;x;x_1)\partial_uK(u;x;x_2)}{u^3} = \\ \int d^4x\frac{\left(u^{4-\Delta_+}\delta^4(x-x_1) + u^{\Delta_+}\frac{c_3^{-1}}{|x-x_1|^{2\Delta_+}}\right)\left((4-\Delta_+)u^{3-\Delta_+}\delta^4(x-x_2) + \Delta u^{\Delta_+-1}\frac{c_3^{-1}}{|x-x_2|^{2\Delta_+}}\right)}{u^3} \tag{4.13}$$

where $c_3 = \pi^2\frac{\Gamma[\Delta_+-2]}{\Gamma[\Delta_+]}$ and $4-\Delta_+ = \Delta_-$. Therefore

$$\begin{aligned} \int d^4x\frac{K(u;x;x_1)\partial_uK(u;x;x_2)}{u^3} =& \Delta_-u^{2\Delta_--4}\delta^4(x_1-x_2) + \frac{4}{c_3}\frac{1}{|x_1-x_2|^{2\Delta_+}} \\ &+ \frac{\Delta_+}{c_3}u^{2\Delta_+-4}\int d^4x\frac{1}{|x-x_1|^{2\Delta_+}|x-x_2|^{2\Delta_+}} \end{aligned} \tag{4.14}$$

The first term of the RHS is diverging. In order to take a finite action we must add a counter term that will cancel the divergence. Therefore, we will introduce a cut off distance at $u^2 = \epsilon$ add the counter-term

$$S_{counter} = -\frac{\Delta_-}{2}\epsilon^{\Delta_--2}\int d^4x\Phi_0^2(x) \tag{4.15}$$

and finaly take the limit of $\epsilon \to 0$. Then the renormalized euclidean on shell action is written after a trivial rescaling of the field as

$$S_{ren}^{on-shell} = -\frac{1}{2}\int d^4x_1 d^4x_2 \frac{\Phi_0(x_1)\Phi_0(x_2)}{|x_1 - x_2|^{2\Delta_+}} \tag{4.16}$$

Therefore, in the CFT side for an operator with scaling dimension $\Delta = \Delta_+$ the two point function will be according with the correspondence

$$< \mathcal{O}(x_2)\mathcal{O}(x_2) >= \frac{1}{|x_1 - x_2|^{2\Delta}}. \tag{4.17}$$

4.3 AdS/CFT at finite temperature

In this section we will study the AdS/CFT in the regime where the field theory is defined at finite temperature. That will break the conformal symmetry, but the breaking is soft, i.e it happens at low energy and therefore we can still consider an AdS boundary. In other words, we can choose a metric space that in some limit reduces to the AdS space. Moreover the SUSY is broken and thus we are going through more realistic models as QCD at finite temperature. As we have argued before the correspondence may hold in that situation too. However, we will have to pay a price and the price is to replace the AdS background with something else.

The natural background to choose in the gravity theory is a black hole. In this background we can define thermodynamic quantities like the entropy, known as the Bekenstein-Hawking entropy or its conjugate variable, the temperature, known as the Hawking temperature. Moreover, for every other global conserved charge there is an associate thermodynamic potential. It is found the the BH entropy S is proportional to the one fourth of the area of the horizon of the black hole A

$$S[M] \equiv \frac{A}{4G}, \tag{4.18}$$

where M is the mass of the black hole. Furthermore, the Hawking temperature T is defined as the surface gravity κ divided by 2π

$$T = \frac{\kappa}{2\pi}. \tag{4.19}$$

The temperature and the mass are related by the condition

$$T \equiv \frac{\partial M}{\partial S}|_Q \tag{4.20}$$

where Q represents any other conserved charge. We can find the temperature in a general black hole background if we use the notion of the Euclidean compact time τ with a period $\beta = \frac{1}{T}$

$$ds^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2 \tag{4.21}$$

If we make a circle of around the outer horizon r_H and identify the time as the angle of this rotation then we can find the associate temperature

$$\beta = \frac{4\pi}{f'(r_H)}. \tag{4.22}$$

On the field theory side we are considering an Euclidean finite temperature gauge theory that lives on $S^1 \times S^3$ with associate radius β and R. Conformal invariance at zero temperature indicates to define the dimensionless temperature z as the ratio of the two radius

$$z = \frac{R}{\beta}. \tag{4.23}$$

This theory may have a phase transition at a finite value of z. It is known that at finite volume there are no discontinuities at the partition function and therefore no phase transition occur. On the other hand if we take the limit of infinite volume $R \to \infty$, we automatically take the large temperature limit by the definition of z and therefore no phase transition can occur at this limit. Fortunately, we are working in the large N regime where new phase transitions can occur at finite volume. Specifically, there are two phases in the theory. The deconfined phase that corresponds to large value of z above a critical value $z_\star$ and the confined phase that is found for temperature below this critical value. The criterion to distinguish a confined from a deconfined phase is the dependence of the free energy F on N. If the free energy is of order N^2 then the phase is deconfined and if the free energy is of order one then the phase is confined.

Let us now pause for the moment our discussion about the phase transition and investigate the thermodynamic relations in the near horizon limit of black D_3 branes. This is the gravitational background dual to the thermal $\mathcal{N} = 4$ SYM on $S^1 \times \mathcal{R}$. This theory is defined in infinite volume and we will see that is in the deconfined phase. The associate metric is

$$ds^2 = \frac{-f(r)dt^2 + d\vec{x}d\vec{x}}{\sqrt{H(r)}} + \sqrt{H(r)}\left(\frac{dr^2}{f(r)} + r^2 d\Omega_5^2\right), \tag{4.24}$$

where

$$H(r) = 1 + \frac{\tilde{L}^4}{r^4}, \quad f(r) = 1 - \frac{r_0^4}{r^4} \tag{4.25}$$

and we have put a tilde over L in order to distinguish it from the AdS scale. From this expression it is easy to read the temperature

$$T_H = \frac{\left(\frac{f(r)}{\sqrt{H(r)}}\right)' |_{r=r_0}}{4\pi} = \frac{r_0}{\pi\sqrt{r_0^4 + \tilde{L}^4}} \tag{4.26}$$

and the entropy is computed if we take into account that $G_{10} = 2^3\pi^6 l_s^8 g_s^2$ and that the area of the black hole is

$$A = \int \sqrt{h} d^3x d\Omega_5 = \sqrt{H(r_0)} r_0^5 V_3 \pi^3 \tag{4.27}$$

where h represents the determinant of the induced metric and V_3 the volume of the three dimensional space. The entropy then is

$$S = \frac{v_3 r_0^3 \sqrt{r_0^4 + \tilde{L}^4}}{2^5 \pi^3 g_s^2 l_s^8} \tag{4.28}$$

Other thermodynamic quantities like the ADM mass M or the number of branes N are computed to be

$$M = \frac{V_3[5r_0^4 + 4\tilde{L}^4]}{2^7\pi^4 g_s^2 l_s^8}, \quad N = \frac{\tilde{L}^2\sqrt{r_0^4 + \tilde{L}^4}}{4\pi g_s l_s^2} \tag{4.29}$$

In the near horizon limit $r << \tilde{L}$ and if we want to stay outside the horizon we must conclude that $r_0 << \tilde{L}$ and therefore $T_H l_s << 1$. The AdS Scale is defined as $L^4 = l_s^4 \lambda = 4\pi g_s l_s^4 N$ and in the near horizon limit take the form

$$L^4 = \tilde{L}^4 \left((\frac{r_0}{\tilde{L}})^4 + 1 \right)^{\frac{1}{4}} \tag{4.30}$$

and if we expand the temperature in powers of $\frac{r_0}{\tilde{L}}$ we obtain that $r_0^4 = \tilde{L}^8(\pi T_H)^4$ and therefore

$$\tilde{L}^4 = L^4(1 - \frac{L^4}{2}(\pi T_H)^4) \tag{4.31}$$

If we change the coordinate by setting

$$r = \frac{L^2}{u} \tag{4.32}$$

then we will take the near horizon metric

$$ds^2 = \frac{L^2}{u^2}\left(\tilde{f}(u)d\tau^2 + d\vec{x}d\vec{x} + \frac{du^2}{\tilde{f}(u)} + L^2 d\Omega_5^2\right) \tag{4.33}$$

where

$$\tilde{f}(u) = 1 - (\pi T_H)^4 u^4. \tag{4.34}$$

In this limit the mass and the entropy can be written in the form

$$M = V_3 T_3 N + \frac{3}{8}\pi^2 V_3 N^2 T^4 = N T_3 V_3 + \frac{3}{4} E_{YM} \tag{4.35}$$

and the entropy as

$$S = \frac{1}{2}\pi^2 N^2 V_3 T^3 = \frac{3}{4} S_{YM} \tag{4.36}$$

As we have argued the energy scales as N^2 and therefore the theory is in the deconfined phase. The above exercise could be seen as an illustration of the fact that there is no phase transition at infinite volume.

We would now proceed with the exploration of phase transition in the context of the AdS/CFT by the investigation of a model at finite volume. What is important to be understood is that when we want to study a QFT living on a manifold M_4 its string dual involves a higher dimensional manifold X whose boundary is M_4. The precise correspondence is

$$\mathcal{Z}_{FT}(M_4) = \mathcal{Z}_{STRING}(X) \simeq e^{-I(X)}, \tag{4.37}$$

where $I(X)$ is the SUGRA action. However, there may be more than one competing manifolds X_i whose boundaries are M_4 then the correspondence is generalized in the form

$$\mathcal{Z}_{FT}(M_4) = \sum_i e^{-I(X_i)}. \tag{4.38}$$

Suppose now that the SUGRA action scales as $I(X_i) = N^a F(X_i)$ then in the large N limit there is a manifold $X_\star$ for which $F(X_\star)$ is the least and therefore

$$\lim_{N\to\infty} -\frac{\log \mathcal{Z}_{FT}(M_4)}{N^a} = F(X_\star) \tag{4.39}$$

However, there might be more cases in which two different manifolds can have the same energy $F(X_i) = F(X_j)$. At that point the field theory free energy is expected to have a singularity that indicates a first order phase transition.

4.4 The Hawking Page transition

In this section we will study the Hawking Page transition which is the dual bulk description of the phase transition that occur at the thermal $N = 4$ SYM. The low energy description of the Type II B SUGRA is given by the Einstein Hilbert action with a negative cosmological constant and the time to be compactified in a S^1.

$$I = -\frac{1}{16\pi} \int_{\mathcal{M}} d^5x\sqrt{g}[R + \frac{12}{L^2}] - \frac{1}{8\pi G_5} \int_{\partial\mathcal{M}} d^4x\sqrt{h}K, \tag{4.40}$$

where h is the determinant of the induced metric at the boundary and K is the trace of the second fundamental form $K_{\mu\nu}$ where

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} = \frac{1}{2} n_\rho g^{\rho\sigma}\partial_\sigma g^{\mu\nu}. \tag{4.41}$$

where n is a unit vector normal to the surface, $\mathcal{L}_n$ is the Lie derivative and $P_{\mu\nu}$ is the projection operator. The second term is called the Gibbons Hawking York term that is needed when the underlying manifold has a boundary.

As we have argued before, in order to reveal the thermodynamic relations of the dual field theory we must compute the former action on shell. The associate Einstein field equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{6}{L^2}g_{\mu\nu}, \tag{4.42}$$

hence $R = -\frac{20}{L^2}$ and the on-shell action is simplified to

$$I_{on-shell} = \frac{1}{2\pi G_5 L^2} \int_{\mathcal{M}} d^5x\sqrt{g} - \frac{1}{8\pi G_5} \int_{\partial\mathcal{M}} d^4x\sqrt{h}K. \tag{4.43}$$

Let us now discuss what are the solutions of the Einstein field equations. One solution that is spherically symmetric is the so called AdS black hole with the associate metric

$$ds^2 = (1 + \frac{r^2}{L^2} - \frac{wM}{r^2})dt^2 + (1 + \frac{r^2}{L^2} - \frac{wM}{r^2})^{-1}dr^2 + r^2 d\Omega_3^2, \tag{4.44}$$

where $w = \frac{16G_5}{3\pi^2}$, M is the ADM mass and t is the compact time with radius β. There are two distinguished manifolds here. The first manifold is that in which $M \neq 0$ and we will call it the manifold X. The black hole radius is the greatest root of the 00 component of the metric

$$1+\frac{r^2}{L^2} - \frac{wM}{r^2} = 0,$$

$$\frac{r_+^2}{L^2} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4wM}{L^2}} \tag{4.45}$$

and the period of the time it is found by requiring that there is no conical singularity

$$\beta = \frac{4\pi}{\frac{d}{dr}((1 + \frac{r^2}{L^2} - \frac{wM}{r^2})|_{r=r_+}} = \frac{2\pi L^2 r_+}{2r_+^2 + L^2}. \tag{4.46}$$

The boundary of manifold X ∂X has a $S^1 \times S^3$ topology at infinity since

$$ds^2 \sim \frac{r^2}{L^2}dt^2 + r^2 d\Omega_3^2, \tag{4.47}$$

where the radius of S^1 is $\beta_\infty = \frac{r}{L}\beta$ and the radius of S^3 is $R_\infty = r$. Therefore, the dimensionless temperature at the boundary is

$$z = \frac{R_\infty}{\beta_\infty} = \frac{L}{\beta} = \frac{2r_+ + L^2}{2\pi L r_+} \tag{4.48}$$

which could be identified with the temperature of the field theory. We should make two comments here. First, β as a function of r_+ has a maximum corresponding to the value $r_+ = \frac{L}{\sqrt{2}}$ and therefore the dimensionless temperature has a minimum for this manifold. In other words this manifold exists only when

$$z \geq \frac{\sqrt{2}}{\pi}. \tag{4.49}$$

Moreover, for every value of β there are two values for r_+

$$r_+ = \frac{\pi L^2}{2\beta}[1 \pm \sqrt{1 - \frac{2\beta^2}{\pi^2 L^2}}]. \tag{4.50}$$

We will call the black holes of the minus brunch, small black holes and the black holes of the plus branch large black holes. It is found that only the large black holes have a positive specific heat and thus are thermodynamically stable.

The second solution of the Einstein field equations is the thermal AdS manifold X' corresponding to the metric (4.44) when $M = 0$.

$$ds^2 = (1 + \frac{r^2}{L^2} - \frac{wM}{r^2})dt^2 + (1 + \frac{r^2}{L^2} - \frac{wM}{r^2})^{-1}dr^2 + r^2 d\Omega_3^2 \tag{4.51}$$

The boundary of X' is again in $r = \infty$ and has a topology $S^1 \times S^3$. Now we could see that both X and X' asymptotes correctly to $S^1 \times S^3$ and can give a bulk dual description of the thermal Yang Mills. The identification of the temperature of the field theory with the dimensionless temperature z will fix the black hole radius r_+ and consequently M for X while it will fix the S^1 radius for X'. In order to compute the partition function of the field theory we must compute the on-shell gravitational action. The good news are that the two manifolds have the same boundary and the same Gibbons Hawking term that we can omit, since we are interested in the difference of the Euclidean actions. Therefore, the difference is proportional to the difference of the volumes,

$$\Delta I = I(X) - I(X') = \frac{1}{16\pi G_5 L^2}[V(X) - V(X')] \tag{4.52}$$

In order to regulate the volumes and take the difference we will use a spatial cut-off close to the boundary $r \leq \frac{1}{\epsilon}$ and finally take the limit $\epsilon \to 0$ as usual. Then the volume of X is

$$V_\epsilon[X] = \int_0^\beta dt \int_{r=r_+}^{\frac{1}{\epsilon}} r^3 dr \int d\omega_3 = \beta(\frac{1}{\epsilon^4} - r_+^4)\frac{\pi^3}{4} \tag{4.53}$$

and for X'

$$V_\epsilon[X'] = \int_0^\beta dt \int_{r=0}^{\frac{1}{\epsilon}} r^3 dr \int d\omega_3 = \frac{\pi^3}{4}\frac{\beta'}{\epsilon^4} \tag{4.54}$$

We must insure that the two geometries are the same in the regulated boundary by relate β and β'. In other words

$$\beta'\sqrt{1 + \frac{1}{\epsilon^2 L^2}} = \beta\sqrt{1 + \frac{1}{\epsilon^2 L^2} - wM\epsilon^2}, \tag{4.55}$$

or

$$\beta' = \beta(1 - \frac{1}{2}wML^2\epsilon^4) + \mathcal{O}(\epsilon^6). \tag{4.56}$$

Finally, the difference of the volume is written as

$$\begin{aligned} V_\epsilon[X] - V_\epsilon[X'] &= \frac{\pi^3}{4}\beta[\frac{1}{\epsilon^4} - r_+^4 - \frac{1}{\epsilon^4} + \frac{1}{2}wML^2] + \mathcal{O}(\epsilon^2) \\ &= \frac{\pi^3}{4}\beta[\frac{wML^2}{2} - r_+^4] + \mathcal{O}(\epsilon^2) \\ &= \frac{\pi^3}{4}\beta[L^2\frac{r_+}{2} + \frac{r_+}{2}] + \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.57}$$

If we substitute β we find for the difference of the actions after taking the limit of $\epsilon \to 0$ that

$$\Delta I = \frac{\pi^3}{8G_5}\frac{r_+^3(L^2 - r_+^2)}{2r_+ + L^2}. \tag{4.58}$$

The energy is

$$\frac{\partial \Delta I}{\partial \beta} \tag{4.59}$$

and the entropy is computed to be

$$S = \beta E - \Delta I = \frac{A}{4G_5} \tag{4.60}$$

that scales correctly as $S \sim \beta^{-3}$.

4.5 Charged Black Holes in AdS Space

In this section we will explore the thermal properties of Einstein-Maxwell AdS charged black holes in $4+1$ dimensions. As we have mentioned earlier, this theory is dual to $SU(N)$, $\mathcal{N}=4$ Yang Mills theory coupled to a $U(1)$ current that lining in four dimensions. We will show that in the fixed charged ensemble the non zero charged density will modify the phase structure in comparison with the uncharged AdS black hole background that we studied earlier. Specifically, it turns out that there is a non-zero entropy at zero temperature. Furthermore, there is a similarity between the phase diagram of the system under consideration and phase diagram of the liquid-gas system.

The action for the model is the Einstein Hilbert action with a negative cosmological constant and a field strength tensor for the gauge field.

$$I = -\frac{1}{16\pi G_5}\int_{\mathcal{M}} d^5x\sqrt{g}[R - F^2 + \frac{12}{L^2}] - \frac{1}{8\pi G_5}\int_{\partial\mathcal{M}} d^4x\sqrt{h}K. \tag{4.61}$$

The static and spherical solution of the equation of motion of this action is given by the metric

$$ds^2 = V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_3^2, \tag{4.62}$$

where

$$V(r) = 1 - \frac{m}{r^2} + \frac{q^2}{r^4} + \frac{r^2}{L^2} \tag{4.63}$$

where $m = wM$ and w is what we have defined in (4.44), $q = 3\sqrt{3}wQ$ where Q is charge of the black hole. The horizon of the black hole r_+ is given by the root of $V(r)$

$$r^4 - mr^2 + q^2 + \frac{r^6}{L^2} = 0 \tag{4.64}$$

For the gauge field we find

$$A = (-\frac{\sqrt{3}q}{2}\frac{1}{r^2} + \Phi)dt \tag{4.65}$$

If we assume that the gauge field vanishes at the horizon, then the electrostatic potential difference between the horizon and the infinity is

$$\Phi = \frac{\sqrt{3}}{2}\frac{q}{r_+^2}. \tag{4.66}$$

The prescription to describe the phase structure of the dual field theory is to find the on-shell gravitational action, as we have done in the previous section, but now we will use the extremal black hole solution that corresponds to zero temperature as a background. If we fix the potential but not the charge then we must include a boundary term to the off shell action

$$I - \frac{1}{4\pi G_5}\int_{\partial\mathcal{M}} d^4x\sqrt{h}F^{\mu\nu}n_\mu A_\nu \tag{4.67}$$

where $n_\mu = (0, \frac{1}{\sqrt{g^{rr}}}, 0, 0, 0)$ is a unit vector normal to the boundary. Then the on-shell action then is

$$I = \frac{1}{16\pi G_5}\int_{\mathcal{M}} d^5\sqrt{g}[\frac{2F^2}{3} + \frac{8}{L^2}] - \frac{1}{4\pi G_5}\int_{\partial\mathcal{M}} d^4x\sqrt{h}F^{\mu\nu}n_\mu A_\nu - \frac{1}{8\pi G_5}\int_{\partial\mathcal{M}} d^4x\sqrt{h}K. \tag{4.68}$$

As earlier we must correctly match the asymptotic geometries for the extremal and the non extremal background by using a spacial cut-off $r \leq \frac{1}{\epsilon}$ and matching the radii of S^1 via

$$\beta'\sqrt{1 - m_e\epsilon^2 + \frac{1}{\epsilon^2 L^2} + q^2\epsilon^4} = \beta\sqrt{1 - m\epsilon^2 + \frac{1}{\epsilon^2 L^2} + q^2\epsilon^4} \tag{4.69}$$

or

$$\frac{\beta'}{\beta} = 1 + \frac{1}{2}(m_e - m)L^2\epsilon^4 + \mathcal{O}(\epsilon^6) \tag{4.70}$$

where

$$\beta = \frac{4\pi}{V'(r_+)} = \frac{4\pi L^2 r_+^5}{4r_+^6 + 2L^2 r_+^4 - 2q^2L^2} \tag{4.71}$$

is the radius of S^1 of the non extremal background and β' is the radius of S^1 of the extremal background. The extremal radius is given by the relation

$$4r_e^6 + 2L^2 r_e^4 = 2q^2L^2 \tag{4.72}$$

Let us split the action into four parts

$$I = I_1 + I_2 + I_b + I_{GHY} \tag{4.73}$$

where

$$I_1 = -\frac{1}{16\pi G_5}\int_{\mathcal{M}} d^5x\sqrt{g}\frac{2}{3}F^2, \tag{4.74}$$

$$I_2 = \frac{1}{2\pi G_5 L^2}\int_{\mathcal{M}} d^5x\sqrt{g} \tag{4.75}$$

$$I_b = -\frac{1}{4\pi G_5}\int_{\partial\mathcal{M}} d^4x\sqrt{h}F^{\mu\nu}n_\mu A_\nu \tag{4.76}$$

and

$$I_{GHY} = -\frac{1}{8\pi G_5}\int_{\partial\mathcal{M}} d^4x\sqrt{h}K \tag{4.77}$$

First we must first compute F^2 that is

$$F^2 = g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} = 2g^{00}g^{11}(F_{10})^2 = \frac{6q^2}{r^6} \tag{4.78}$$

Then I_1 is written in the form

$$\begin{aligned} I_1 &= \frac{1}{16\pi G_5 L^2}\beta\omega_3\int_{r=r_+}^{r=\frac{1}{\epsilon}} 4q^2L^2\frac{dr}{r^3} \\ &= -\frac{1}{16\pi G_5 L^2}\beta\omega_3 2q^2L^2(\frac{1}{r_+^2} + \mathcal{O}(\epsilon^2)). \end{aligned} \tag{4.79}$$

The second part of the action I_2 is

$$I_2 = \frac{1}{8\pi G_5 L^2}\beta\omega_3(\frac{1}{\epsilon^4} - r_+^4). \tag{4.80}$$

I_b is found to be

$$\begin{aligned} I_b &= -\frac{1}{4\pi G_5}\int_{\partial\mathcal{M}} d^4x\sqrt{h}F^{\mu\nu}n_\mu A_\nu \\ &= \frac{1}{4\pi G_5}\beta\omega_3 r^3\frac{\sqrt{3}q}{r^3}[-\frac{\sqrt{3}q}{2r^2} + \frac{\sqrt{3}q}{2r_+^2}])|_{r=\frac{1}{\epsilon}} \\ &= \frac{1}{16\pi G_5 L^2}\beta\omega_3\frac{6q^2L^2}{r_+^2} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{4.81}$$

For the Gibbons Hawking York term we must first compute the extrinsic curvature which is given gy (4.41) and is computed to be

$$K = \frac{rV'(r) + 6V(r)}{2r\sqrt{V(r)}}, \tag{4.82}$$

then the Gibbons Hawking York term is

$$
\begin{aligned}
I_{GHY} &= -\frac{1}{8\pi G_5}\int_{\partial\mathcal{M}} d^4x\sqrt{h}K \\
&= -\frac{1}{8\pi G_5}\beta\omega_3[\frac{r^2}{2}(rV'(r)+6V(r)]|_{r=\frac{1}{\epsilon}} \\
&= -\frac{1}{16\pi G_5 L^2}\beta\omega_3[\frac{8}{\epsilon^4}+6\frac{L^2}{\epsilon^2}-4mL^2]
\end{aligned}
\tag{4.83}
$$

Putting (4.79),(4.80),(4.81) and (4.83) together we obtain

$$
I_{ON-SHELL} = \frac{1}{16\pi G_5 L^2}\beta\omega_3[-\frac{6}{\epsilon^4}-\frac{6L^2}{\epsilon^2}-2r_+^4\frac{4q^2L^2}{r_+^2}+4mL^2]
\tag{4.84}
$$

Now what is left is to subtract the on-shell action for the extremal black hole and use (4.64),(4.72) and match the geometry at the boundary for the two actions via (4.70). After some algebra the difference of the on shell actions is then

$$
\begin{aligned}
\Delta I &= \frac{\beta\omega_3}{16\pi G_5 L^2}[-2r_+^4+\frac{4q^2L^2}{r_+^2}+mL^2+2r_e^4-\frac{4q^2L^2}{r_e^2}-m_eL^2] \\
&= \frac{\beta\omega_3}{16\pi G_5 L^2}[L^2r_+^2-r_+^4+\frac{5q^2L^2}{r_+^2}-\frac{3}{2}L^2r_e^2-\frac{9}{2}\frac{q^2L^2}{r_e^2}].
\end{aligned}
\tag{4.85}
$$

By keeping the charged fix we are working in the canonical ensemble and the difference of the actions represents the free energy multiplied by β

$$
\Delta I = \beta F.
\tag{4.86}
$$

The energy measured above the ground state can be computed as

$$
E = M - M_e = \left(\frac{\partial \Delta I}{\partial \beta}\right)_Q
\tag{4.87}
$$

The thermodynamic potential as

$$
\Phi = \frac{1}{\beta}\left(\frac{\partial \Delta I}{\partial Q}\right)_\beta = \frac{\sqrt{3}}{2}\left(\frac{q}{r_+^2}-\frac{q}{r_e^2}\right),
\tag{4.88}
$$

and the entropy

$$
S = \beta\left(\frac{\partial \Delta I}{\partial \beta}\right)_Q - \Delta I = \frac{A}{4G}.
\tag{4.89}
$$

The difference between that model and that of Hawking and Page is that there is a non zero extremal horizon r_e that corresponds to zero temperature and therefore there is a non zero entropy at zero temperature.

It is useful to examine the inverse temperature β as a function of r_+. The turning points of β determined by the condition

$$\left(\frac{\partial \beta}{\partial r_+}\right)_{r_+=r_t} = \left(\frac{\partial^2 \beta}{\partial r_+^2}\right)_{r_+=r_t} = 0, \tag{4.90}$$

It turns out that there are two turning points that they are different below a critical value for the charge, the same for that critical value and they disappear if the charge is greater than the critical value. The associate critical values are

$$r_{crit}^2 = \frac{L^2}{3}, \quad q_{crit}^2 = \frac{L^2}{3\sqrt{15}}. \tag{4.91}$$

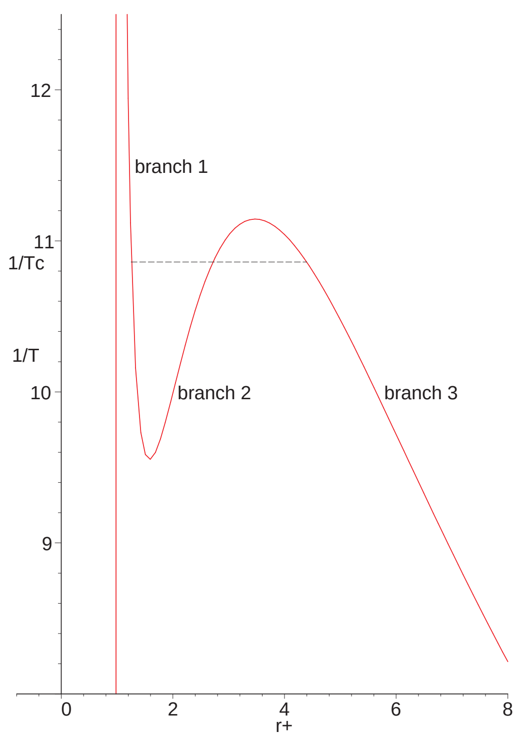

Figure 2: The $\beta(r_+)$ -r_+ plot

Therefore, for $q < q_{crit}$ there are three branches of solutions (different horizons) for a given temperature. Let us call the branch of solution with small radius branch 1 and the branch with big radius branch 3 while the middle branch with branch 2. The middle branch 2 has a negative specific heat since $\partial_\beta S \sim r^2 \partial_\beta r_+$ and therefore is unstable. For low temperatures the non extremal black hole with the smallest radius $r_+^{(2)}$ dominates the thermodynamic ensemble. There is a critical temperature T_c where the free energy of branch 3 is more negative and therefore branch 3 dominates the ensemble. Since the entropy is proportional to $r_+^{(3)}$ there is a jump and therefore a release of latent heat. At high temperature the branches 1 and 2 disappear and there is a single branch. The situation is similar for large charge. The phase structure of that system is isomorphic to the classic liquid-gas system of Van der Waals if we map

β to the pressure, r_+ to the volume and the charge to the temperature. There is one order parameter for the phase transitions that occurs at the critical point q_{crit}. This is the difference of the two radii $\rho_+ = r_+^{(3)} - r_+^{(1)}$, as it is zero above q_{crit} and non zero below. In the regime where $q \geq q_{crit}$,($\rho_+ = 0$), there is no phase transition by increasing the temperature.

5 Effective Holographic Theories for Condensed Matter Systems

We have seen in this text that a string theory can be approximated by a Supergravity action if the strings can be approximated as point-like objects. We can think about integrating-out massive string modes, in order to obtain an effective description of a small number of low energy modes. The key idea of Effective Holographic Theories is to select a set of operators in the strongly coupled field theory that are expected to dominate the dynamics and parametrize the dual field two derivative gravitational action.

In this section we will attempt to describe the strongly coupled low temperature dynamics for condensed matter systems via holography. In condensed matter systems an important ingredient is the presence of the source which is described in the dual gravitational theory by a vector field A_μ in addition to the metric $g_{\mu\nu}$ that controls the energy. However, we have seen in the previous section that if we include only the vector field in the dynamics we end up with a non zero entropy at zero temperature, but this would violate the third law of thermodynamics. A next step in modelling the dynamics of strongly coupled systems at finite density is to include the leading relevant operator, which is a scalar field ϕ controlling the coupling constant and its running in the IR. Moreover, for a condensed matter system the scalar filed would represent the strong interactions of the ion lattice, or the effect of spins on the charge carriers. The associate gravitational system is the Einstein-Maxwell-Dillaton system with a scalar potential.

$$S = M^{p-1} \int d^{p-1}x\sqrt{-g}\left(R - \frac{1}{2}(\partial\Phi)^2 + V(\Phi) - \frac{Z(\Phi)}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{5.1}$$

where p is the number of spatial dimensions and is left arbitrary here.

We have seen so far that there is one special dimension in the context of the AdS/CFT, the radial coordinate. In the gravity side it is a coordinate that tells us how far are we from the boundary while form the field theory side it could be seen as an energy scale that tells us how far are we from the UV theory. In an EHT we want to determine the low energy correlation functions by knowing the boundary conditions at the AdS boundary of the field equations, analogously with an Effective Field Theory in which we want to derive the low energy Lagrangian. In this section are mostly interested in infrared dynamics and we will parametrize the potential $V(\Phi)$ and $Z(\Phi)$ accordingly. In the absence of IR fixed point one parametrization for the scalar potential is

$$V(\Phi) = 2\Lambda e^{-\delta\Phi} \tag{5.2}$$

where $-\delta\Phi \to \infty$. Furthermore, gauge supergravity actions stemming from string theory indicate that we should take the coupling constant $Z(\Phi)$ to be also exponential

$$Z(\Phi) = e^{\gamma\Phi}. \tag{5.3}$$

It turns out that there is a criterion for this parametrization of the potential to be quite general. Indeed sub-leading changes in the potential do not change the qualitative IR physics if $C_p \neq 0$ where

$$C_p(\gamma, \delta) \equiv 2(p-1) + \gamma^2 + 2(p-2)\gamma\delta - (2p-3)\delta^2. \tag{5.4}$$

On the other hand if $C_p = 0$ then sub-leading changes in the potential are expected to change the IR behaviour. In the case of zero density the phenomenon occurs at $\delta = \delta_c = \sqrt{\frac{2}{p-1}}$ and leads to non conformal IR dynamics and in the following section we will try to show how this happens.

Let us now write the equation of motion for the Einstein Maxwell Dilaton model. By varying the action (5.1) with respect to A_μ and $g_{\mu\nu}$ we find

$$\partial_\mu \left(\sqrt{-g} Z F^{\mu\nu}\right) =0, \tag{5.5}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R =T_{\mu\nu}, \tag{5.6}$$

$$T_{\mu\nu} =\frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{4} g_{\mu\nu}(\partial\Phi)^2 + \frac{1}{8} g_{\mu\nu} Z F^2 - \frac{1}{2} F_\mu^\rho F_{\rho\nu} + \frac{1}{2} V g_{\mu\nu} \tag{5.7}$$

We will define the general coordinates as ,

$$ds^2 = -D(r)dt^2 + B(r)dr^2 + C(r)dx_i dx^i, \tag{5.8}$$

Then the vector field is written as

$$A_t' = \frac{q\sqrt{DB}}{ZC^{\frac{p-1}{2}}}. \tag{5.9}$$

and the associate Einstein equations are written in the form

$$\Phi'^2 + (p-1)\frac{C''}{C} - \frac{p-1}{2}\left(\frac{D'}{D} + \frac{B'}{B} + \frac{C'}{C}\right)\frac{C'}{C} = 0 \tag{5.10}$$

$$\frac{D''}{D} - \frac{C''}{C} + \frac{1}{2}\left(\frac{C'}{C} - \frac{D'}{D}\right)\left(\frac{B'}{B} + \frac{D'}{D} + (3-p)\frac{C'}{C}\right) - \frac{q^2 B}{ZC^{p-1}} = 0 \tag{5.11}$$

$$-\frac{1}{2}\Phi'^2 - BV + \frac{p-1}{2}\frac{C'}{C}\left(\frac{p-2}{2}\frac{C'}{C} + \frac{D'}{D}\right) + \frac{q^2 B}{2ZC^{p-1}} = 0. \tag{5.12}$$

5.1 Zero density and zero temperature

In this section we will solve the Einstein equation (5.10), (5.11) and (5.12) in the neutral case, i.e. when the global charge is zero. Furthermore, we will investigate the case where the temperature is zero. For convenience we will work in the conformally flat coordinate system, where $D(r) = B(r) = C(r) = e^{A(r)}$ and the metric is in the form

$$ds^2 = e^{A(r)}\left(-dt^2 + dr^2 + dx^i dx_i\right). \tag{5.13}$$

In that coordinate system you can identify the r coordinate with the energy scale via

$$\log E \leftrightarrow 2A(r) \tag{5.14}$$

As we will see below the scale factor is monotonically decreasing from the UV to the IR and the space time is either conformal with an AdS scale smaller than the UV scale (the gauge theory flows to an IR conformal fixed point) or the scale factor vanishes at finite or infinite r which indicates a curvature singularity.

What we will show below is that there is a solution of the Einstein equations-when the parameter δ is not taking the critical value $\delta_c^2 = \frac{2}{p-1}$-where the scale factor has a power law behaviour and the scalar field has a logarithmic behaviour. Furthermore, sub-leading changes in the potential do not change this behaviour and result to sub-leading corrections for the scale factor and the scalar field. On the other hand when the parameter δ is taking the critical value then the scale factor goes to zero exponentially.

The equations (5.10), (5.11) and (5.12) when the metric is in the conformal frame (5.13) are written in the form

$$\begin{aligned}(\Phi'(r))^2 + (p-1)A''(r) - \frac{p-1}{2}(A'(r))^2 = 0\\ \frac{p-1}{2}A''(r) + \frac{(p-1)^2}{4}(A'(r))^2 = e^{A(r)}V(\Phi(r)).\end{aligned} \tag{5.15}$$

First, we will examine the case where $V(\Phi) = 2\Lambda e^{-\delta\Phi}$. We impose that the asymptotic behaviour for the scale factor is of the form

$$e^{A_0(r)} = r^k, \quad k\epsilon(-\infty, -2). \tag{5.16}$$

Then the first equation is written in the form

$$\Phi_0' = \sqrt{\frac{p-1}{2}(k^2+2k)}\frac{1}{r}, \tag{5.17}$$

or

$$\Phi_0 = a\log(r) + c, \quad a = \sqrt{\frac{p-1}{2}(k^2+2k)} \tag{5.18}$$

By substitute the above solution to the second equation of (5.15) we take that

$$\left(\frac{p-1}{2}k(k-1) + \frac{(p-2)(p-1)}{4}k^2\right) r^{-2} = 2\Lambda r^{k-\delta a} e^{-\delta c} \tag{5.19}$$

and the constant c is found to be

$$c = -\frac{1}{\delta}\log((\frac{p-1}{2}k)(\frac{p-1}{2}k-1)) \tag{5.20}$$

By equalizing the exponents we find that

$$k - \delta a = -2 \tag{5.21}$$

and by substitution of a from (5.18) we take

$$\frac{k+2}{k} = \delta^2\frac{p-1}{2}. \tag{5.22}$$

The exponent k and the multiplication constant are written in the form

$$k = \frac{-2}{1-\delta^2\frac{p-1}{2}} \tag{5.23}$$

$$a = \frac{\delta^2\left(\frac{p-1}{2}\right)}{\delta\left(1-\delta^2\frac{p-1}{2}\right)} \tag{5.24}$$

Therefore, in the limit case where $\delta \to \delta_c$ we take $k \to -\infty$ and thus we must exclude this case. We now want to solve the case in which we are taking into account sub-leading corrections for the scalar potential, i.e. $V(\Phi) = 2\Lambda e^{-\delta\Phi}\Phi^m$. An approach would be to decompose the dilaton and the scale factor in two pieces. The first will satisfy the Einstein equations when $m = 0$ and the second will be small compared to the first, in the limit of large r.

$$\Phi(r) = \Phi_0(r) + \Delta\Phi(r), \quad A(r) = A_0 + \Delta A(r). \tag{5.25}$$

Then the E.E. (5.10), (5.11) and (5.12) will be after using the equation (5.18)

$$\begin{aligned} 2\Phi_0'\Delta\Phi' + (p-1)\Delta A'' - (p-1)A_0'\Delta A' = 0, \\ \frac{p-1}{2}\Delta A'' - \frac{(p-1)^2}{2}A_0'\Delta A' = 2\Lambda e^{A_0-\delta\Phi_0}\Phi_0^m, \end{aligned} \tag{5.26}$$

After some algebra we obtain

$$\begin{aligned} 2\frac{a}{r}\Delta\Phi' + (p-1)\Delta A'' - (p-1)\frac{k}{r}\Delta A' = 0, \\ \frac{p-1}{2}\Delta A'' - \frac{(p-1)^2}{2}\frac{k}{r}\Delta A' = 2\Lambda e^{-\delta c}r^{k-a\delta}a^m \log^m r, \end{aligned} \tag{5.27}$$

Since we are looking for the IR behaviour we are taking the limit of large r and the solution for $\Delta\Phi$ and ΔA is

$$\Delta A = \frac{\zeta}{m+1}\log^{m+1} r, \quad \Delta\Phi = \frac{\beta}{m+1}\log^{m+1} r \tag{5.28}$$

where the constants β and ζ are given by the relations

$$\beta = \frac{2\Lambda e^{-\delta c}}{a}, \quad \zeta = \frac{4\Lambda e^{-\delta c}}{(1-p)(k+1)}, \quad m < 0 \tag{5.29}$$

Therefore the fields now are written as

$$\Phi = (\log r)(a + \frac{\beta}{m+1}\log^m r) \tag{5.30}$$

$$A = (\log r)(k + \frac{\zeta}{m+1}\log^m r). \tag{5.31}$$

We could see now that this solution stands for $m < 0$. Another possible solution would be to decompose the fields as follow

$$\Phi = a\log r + \tilde{\beta}\log\log r \tag{5.32}$$

$$A = k\log r + \tilde{\zeta}\log\log r \tag{5.33}$$

Where again k, a are given from the $m = 0$ case as expected. Using the same logic as before we find the new perturbations $\Delta\Phi = \tilde{\beta}\log\log r$ and $\Delta A = \tilde{\zeta}\log\log r$ by substituting this ansatz in the equations of motion and keep in mind that we will keep the ΔA and $\Delta\Phi$ terms in the exponent factor.

$$2\Phi_0'\Delta\Phi' + (p-1)\Delta A'' - (p-1)A_0'\Delta A' = 0,$$
$$\frac{p-1}{2}\Delta A'' - \frac{(p-1)^2}{2}A_0'\Delta A' = 2\Lambda e^{A_0+\Delta A-\delta(\Phi_0+\Delta\Phi)}\Phi_0^m, \tag{5.34}$$

which are written in the form

$$\left(2a\tilde{\beta} - (p-1)(k+1)\tilde{\zeta}\right)\frac{1}{r^2\log r} = 0, \tag{5.35}$$

$$-\frac{p-1}{2}\tilde{\zeta}\left(1+(p-1)k\right)\frac{1}{r^2\log r} = 2\Lambda a^m r^{k-\delta a}(\log r)^{\tilde{\zeta}-\delta\tilde{\beta}+m}. \tag{5.36}$$

After some algebra we are evaluate the coefficients $\tilde{\zeta}$ and $\tilde{\beta}$,

$$\tilde{\zeta} = m - 1 - \delta\frac{(m-1)(p-1)(1+k)}{2a+\delta(p-1)(1+k)}, \tag{5.37}$$

$$\tilde{\beta} = \frac{(m-1)(p-1)(1+k)}{2a+\delta(p-1)(1+k)} \tag{5.38}$$

We notice here that $\frac{\log^{m+1} r}{\log\log r} \to 0$ for $m \leq -1$. Therefore the $\log\log r$ subleading terms are dominant for $m \leq -1$ and the $\log^{m+1} r$ terms are dominant for $m > -1$. Summarizing, we have found that when $\delta \neq \delta_c$ then if we change slightly the scalar potential that will yield to subleading corrections for the scale factor and the scalar field. Specifically, if $V(\Phi) = 2\Lambda e^{-\delta\Phi}\Phi^m$ with $m < 0$ then the solution is written for $m < -1$

$$\Phi = (\log r)(a + \frac{\beta}{m+1}\log^m r), \tag{5.39}$$

$$A = (\log r)(k + \frac{\zeta}{m+1}\log^m r) \tag{5.40}$$

and for $m\epsilon(-1,0)1$ as

$$\Phi = a\log r + \tilde{\beta}\log\log r, \tag{5.41}$$

$$A = k\log r + \tilde{\zeta}\log\log r. \tag{5.42}$$

Now we will investigate the case when $\delta = \delta_c$. In that case we have found that the scale factor does not admit power law solution. On the other hand, if we take the scale factor to be

$$e^A = e^{-r^l}, \tag{5.43}$$

then the first equation of (5.15) will be

$$\Phi' = \sqrt{\frac{p-1}{2}} l r^{l-1}, \tag{5.44}$$

or

$$\Phi = \sqrt{\frac{p-1}{2}} r^{l} + c. \tag{5.45}$$

Then if we take the scalar potential to be $V(\Phi) = 2\Lambda e^{-\delta\Phi}\Phi^m$ then the second equation of (5.15) will be

$$\frac{(p-1)^2}{4} l^2 r^{2l-2} = 2\Lambda(\sqrt{\frac{p-1}{2}} r^l)^m \tag{5.46}$$

with

$$l = \frac{2}{2-m}. \tag{5.47}$$

Thus, when $\delta = \delta_c$ the scale factor has an exponential behaviour and subleading corrections to the potential do not yield to subleading changes for the scale factor and for the scalar field.

5.2 Finite density and zero temperature

In this section we will try to illustrate that the things are significantly more complicated when we include a conserved global $U(1)$ current in the theory. The problem is to solve the system of the differential equations (5.10), (5.11), (5.12) when C_p in (5.4) is zero in the limit where $\delta\Phi \to \infty$ and for the general class of potentials

$$\begin{aligned} V(\Phi) =& 2\Lambda e^{-\delta\Phi}\Phi^m, \\ Z(\Phi) =& e^{\gamma\Phi}\Phi^n. \end{aligned} \tag{5.48}$$

We are looking for asymptotic solution of the form $\Phi \sim \Phi_0 r^k$, $D \sim e^{-dr^{a_1}}, B \sim e^{-br^{a_2}}, C \sim e^{-cr^{a_3}}$ by using our experience for the zero density case. So the problem is to relate eight asymptotic parameters $\Phi_0, d, b, c, k, a_1, a_2, a_3$ and one integration parameter q with five Lagrangian parameters $\Lambda, \gamma, \delta, m, n$.

Then the Einstein equations take the form

$$
\begin{aligned}
&\frac{cd}{2}(1-p)a_1a_3r^{-2+a_1+a_3}+\frac{cb}{2}(1-p)a_2a_3r^{-2+a_2+a_3}\\
&-\frac{c^2}{2}(1-p)a_3^2r^{-2+2a_3}\\
&+\frac{c}{2}(1-p)(a_3-1)a_3r^{-2+a_3}+\Phi_0^2k^2r^{-2+2k}=0,
\end{aligned}
\tag{5.49}
$$

$$
\begin{aligned}
&\frac{d^2a_1^2}{2}r^{-2+2a_1}+\frac{c^2a_3^2}{2}(1-p)r^{-2+2a_3}\\
&-\frac{bd}{2}a_1a_2r^{-2+a_1+a_2}+\frac{cd}{2}(p-1)a_1a_3r^{-2+a_1+a_3}\\
&+\frac{bc}{2}a_2a_3r^{-2+a_2+a_3}-da_1(a_1-1)r^{-2+a_1}+ca_3(a_3-1)r^{-2+a_3}\\
&-q^2\Phi_0^{-n}r^{-nk}e^{-br^{a_2}-r^k\gamma\Phi_0-c(1-p)r^{a_3}}=0,
\end{aligned}
\tag{5.50}
$$

$$
\begin{aligned}
&\frac{cd}{2}(p-1)a_1a_3r^{-2+a_1+a_3}+\frac{c^2}{4}(p-2)(p-1)a_3^2r^{-2+2a_3}-\frac{k^2\Phi^2}{2}r^{-2+2k}\\
&-2\Lambda\Phi_0^mr^{km}e^{-br^{a_2}-\delta\Phi_0r^k}\\
&+\frac{1}{2}q^2\Phi_0^{-n}r^{-nk}e^{-br^{a_2}-r^k\gamma\Phi_0-c(1-p)r^{a_3}}=0.
\end{aligned}
\tag{5.51}
$$

One way to tackle this problem is to assume that some exponents in the former equations are equal and then take those terms as the leading terms. By looking at equation (5.49) we notice that there are $\sum_{k=2}^{5}\frac{5!}{k!(5-k)!}=26$ differing ways of matching the exponents. However it turns out that only eighteen matches give non trivial results. We tabulate them in Table I.

Table I			
(i)	$a_1+a_3=a_2+a_3$	(x)	$a_1+a_3=a_2+a_3=2a_3$
(ii)	$a_1+a_3=2a_3$	(xi)	$a_1+a_3=a_2+a_3=a_3$
(iii)	$a_1+a_3=a_3$	(xii)	$a_1+a_3=a_2+a_3=2k$
(iv)	$a_1+a_3=2k$	(xiii)	$a_1+a_3=2a_3=2k$
(v)	$a_2+a_3=2a_3$	(xiv)	$a_1+a_3=2a_3=k$
(vi)	$a_2+a_3=a_3$	(xv)	$a_2+a_3=2a_3=2k$
(vii)	$a_2+a_3=2k$	(xvi)	$a_2+a_3=a_3=2k$
(viii)	$2a_3=2k$	(xvii)	$a_1+a_3=a_2+a_3=2a_3=2k$
(ix)	$a_3=2k$	(xviii)	$a_1+a_3=a_2+a_3=a_3=2k$

Unfortunately, this approach is quite complicated since system of the algebraic equations is non linear and admits a solution only for specific values of m and n. However, it is possible that a better guess for the asymptotic ansantz could lead to a simpler system for the unknowns, something that we do not attempt in this project and we are postponing it for further investigation in the future.

6 Conclusions

In this paper we saw how holographic techniques could be applied to condensed matter systems.

In chapter 2 we indicated that there is a geometrical description of $SU(N)$ gauge theories when $N \to \infty$ that was an appetizer for the string theory description of those gauge theories. In chapter 3 we briefly reviewed some useful results from general relativity, like the geometrical description of the AdS space and the field equation in that space, that where used in chapter 3.

In the sequence, we described the AdS/CFT correspondence and showed that this is a correspondence between conformal gauge theories and gravitational theories when we augment the space time dimensions by one and assume that the gauge theory lives on the boundary of the bulk space which is the AdS space. That was done by taking the near horizon limit of extremal N D_3 branes. Specifically, we showed that the low energy description of this system could be seen either as a gravity theory living on ten dimensional space time plus excitation of the D_3 branes, i.e. $SU(N)$ gauge fields or as a gravity theory living on ten dimensional space time with a geometry near the branes of $AdS_5 \times S^5$. Even more, we found that when one theory is weakly coupled the other is strongly coupled and that gave us the opportunity to study strongly coupled field theories.

If we break the conformal symmetry softly and compactify the time direction then we can describe gauge theories at finite temperature. The associate gravitational background for the Yang Mills theory was black holes in AdS space. There we incorporated the situation when two different backgrounds had the same boundary. It was shown that a first order phase transition could occur at large N, known as the Hawking Page transition that is dual to the confined deconfined phase transition of the gauge theory. In the next subsection, we modelled the gravitational theory when there is $U(1)$ conserved current in the gauge theory. We computed the on shell action for the Einstein Maxwell charged black hole in the grand canonical ensemble, when the potential at infinity was varied but the charge was kept fixed. Finally, we concluded that the phase structure of that system is isomorphic to the classic liquid-gas system of Van der Waals if we map $\beta = \frac{1}{T}$ to the pressure, the horizon of the black hole r_+ to the volume and the charge to the temperature.

In chapter 5 we studied Effective Holographic Theories for Condensed Matter Systems.

Specifically, we identified the relative degrees of freedom for those systems and wrote the dual gravitational action. The model was the Einstein Maxwell Dilaton with a Liouvile potential. Moreover, we sought the infrared behaviour of the system first for the neutral case and then for the charged case. A set of parameters was used as an index of the infrared behaviour of the system. When the parameters satisfied a specific constraint then the conformal symmetry was broken in the infrared and the form of the fields was changed. When, the constrained was not satisfied the system was scale invariant in the infrared and subleading changes of the scalar potential led to subleading changes for the field.

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