Entanglement Entropy in the AdS/CFT Correspondence

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Abstract

In this thesis, I follow the calculations done in [1] to calculate the entanglement entropy of a system using the AdS/CFT correspondence in both AdS_3-thermal and not- and AdS_{d+2} in the case where the boundary between the entangled regions is a straight belt of d dimensions.

Dedication

Dedicated to Aphrodite Aretaki who has been with me through thick and thin.

Contents

Chapter 1

Introduction

A key feature of the AdS/CFT correspondence is the ability to reduce complex CFT calculations to simpler gravitational ones. We will illustrate this by calculating the entanglement entropy between two subsystems; first in 3 dimensions and then in d+2 dimensions and check the results with what is expected from the CFT as calculated in [1].

Chapter 2

The Correspondence

This section follows closely [2]

In a sentence: The AdS/CFT correspondence is a link between a d-dimensional Conformal Field Theory and a d+1-dimensional Classical Gravitational Theory.

This correspondence can be motivated from some observations such as:

- In large N limits of QFTs strings may appear, fact that gives way to gravitational theories at low energies

- The maximum energy that can fit into a sphere is equal to the Schwarzschild black hole mass for the same radius. This results in a maximum entropy given by:
$$S_{\max} = \frac{A}{4G_N}$$
  Meaning that the entropy of such system is proportional to the area of it's boundary.

It is important to note that the correspondence demands one theory be weakly coupled while the other be strongly coupled, making AdS/CFT a powerful tool for QFT computations. Instead of facing a strongly coupled system head on we may instead do the appropriate calculations in a classical gravitational theory.

Chapter 3

Entanglement Entropy

This section follows closely [1]

Suppose a system made up of two mutually entangled subsystems (A and B). The entanglement entropy of one (suppose A) can be found as:

$$S_A = \text{Tr}_A(\rho_A \log \rho_A) \tag{3.1}$$

$$\rho_A := \text{Tr}_B(\rho_{tot}) \tag{3.2}$$

Where ρ_{tot} is the density matrix of the complete system while ρ_A is the reduced density matrix pertaining to the subsystem A. The entropy "quantifies the entanglement of A with B". Do observe that $\rho_{tot}, \rho_A \propto t$ in general. Moreover, in thermal cases we simply have to define ρ as $\rho_T = e^{-\beta \mathcal{H}_{\text{tot}}}, \beta := \frac{1}{T}$.

Consider now a d-dimensional system in static frame made up of two sub-manifolds, A and B, with the boundary between them being $\partial A := \gamma_A$. The entanglement entropy ends up being proportional to the boundary that separate the two manifolds:

$$S_A \propto \frac{\text{Area}(\gamma_A)}{\alpha^{d-1}} \tag{3.3}$$

Where α is the UV cutoff of the theory and is related with the IR cut off through $e^{\rho_0} \sim \frac{L}{\alpha}$.

Importantly, this is reminiscent of the Bekenstein-Hawking entropy:

$$S_{BH} = \frac{\text{Area}(\gamma_A)}{4G_N} \tag{3.4}$$

Which approximates the entanglement entropy.

Chapter 4

CFT Results

4.1 In 2D CFTs

4.1.1 Non Thermal, Infinite CFTs

The entanglement entropy on the subsystem A was evaluated in [1] by taking the derivative of $\text{Tr}_A \rho_A^n$ with respect to n as:

$$S_A = \lim_{n \to 1} \frac{\text{Tr}_A \rho_A^n - 1}{1 - n} = -\frac{\partial}{\partial n} \log \text{Tr}_A \rho_A^n \tag{4.1}$$

By calculating $\text{Tr}_A \rho_A^n$ and combining with (4.1) we get the following result:

$$S_A = \frac{c}{3} \log \frac{l}{\alpha} \tag{4.2}$$

Later in sections 5.1 and 5.2 we will replicate (4.2) from the gravitational theory.

4.1.2 Thermal CFTs

Recalculating S_A for finite temperature but for an infinite spacial length, (4.2) now becomes:

$$S_A = \frac{c}{3} \log \left(\frac{\beta}{\pi \alpha} \sinh(\frac{\pi l}{\beta}) \right) \tag{4.3}$$

Observe that (4.3) reduces to (4.2) for $T \to 0$. This result will be replicated from the gravity side in section 5.3

4.2   In N>2D CFTs

The method of calculating S_A doesn't deviate from section 4.1.1, main difference being that now:

$$\text{Tr}_A \rho_A^n = \frac{Z_n}{Z_1^N} \tag{4.4}$$

Where Z is the partition function. Finding Z is highly non trivial and it depends a lot on the shape of A. A rough estimation in the case where the manifold is $\mathbb{R}^d$ and the shape of the boundary is that of a straight belt is:

$$S_A = \frac{c}{3(d-1) \cdot 2^{d-1} \pi^{\frac{d-1}{2}} \Gamma(\frac{d+1}{2})} \left[\frac{L^{d-1}}{\alpha^{d-1}} - \frac{L^{d-1}}{l^{d-1}} \right] \tag{4.5}$$

Where L is the length of the belt and l is it's width and the central charge c relates to the radius of AdS and the newton constant through:

$$c = \frac{3R}{2G_N} \tag{4.6}$$

.

We will compare (4.5) with it's equivalent in the gravitational theory in chapter 6

Chapter 5

Theory in Action: AdS$_\mathbf{d+2=3}$
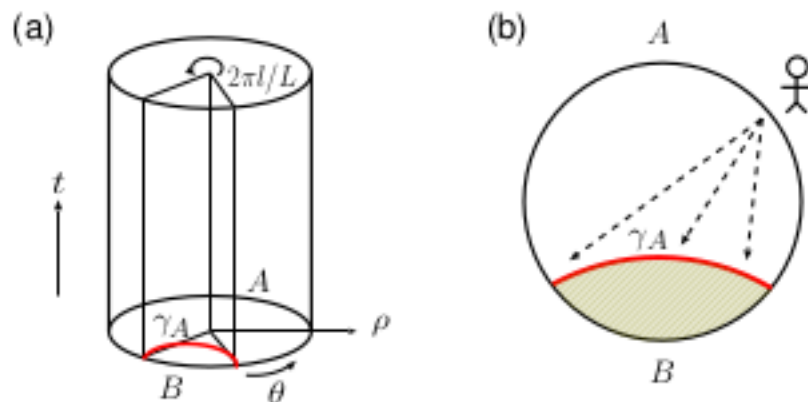
5.1 Global Coordinates

Figure 5.1: The boundary-geodesic, provided by [1]

Starting out, we have the AdS_3 global coordinates with line element:

$$ds^2 = R^2(-\cosh^2(\rho)dt^2 + d\rho^2 + \sinh^2(\rho)d\theta^2) \qquad (5.1)$$

We can find the Lagrangian as:

$$\mathcal{L} = \sqrt{\frac{d^2 s}{d\lambda^2}} = R\sqrt{-\cosh^2(\rho)\dot{t}^2 + \dot{\rho}^2 + \sinh^2(\rho)\dot{\theta}^2} \tag{5.2}$$

Since we are in d+2=3 dimensions, the horizon between two subspaces of the bulk will be a single line since time would be static. Thus the Lagrangian would now take the form:

$$\mathcal{L} = R\sqrt{\dot{\rho}^2 + \sinh^2(\rho)\dot{\theta}^2} \tag{5.3}$$

Taking as affine parameter λ s.t. $\frac{ds}{d\lambda} = 1$ we have the following set of Euler-Lagrange equations:

$$\text{for } \rho : \ddot{\rho} = \sinh(\rho)\cosh(\rho)(\dot{\theta}^2 - \dot{t}^2) \tag{5.4}$$

$$\text{for } \theta : \frac{d}{d\lambda}\left[\sinh^2(\rho)\dot{\theta}\right] = 0 \tag{5.5}$$

$$\text{for the affine parameter we have} : \dot{\rho}^2 + \sinh^2(\rho)\dot{\theta}^2 = \frac{1}{R^2} \tag{5.6}$$

However, (5.4) can be obtained by combining the (5.5) and (5.6).

From (5.5) we get a constant of motion:

$$\sinh^2(\rho)\dot{\theta} := C_\theta \tag{5.7}$$

Combining (5.7) with (5.6)we get the following formula for $\dot{\rho}$:

$$\dot{\rho} = \sqrt{\frac{1}{R^2} - \frac{C_\theta^2}{\sinh^2(\rho)}} \tag{5.8}$$

We can find a formula for C_θ from (5.6) by setting $\rho = \bar{\rho}$ where $\bar{\rho}$ s.t. $\dot{\rho}|_{\bar{\rho}} = 0$:

$$\dot{\rho}|_{\bar{\rho}} + \frac{C_\theta^2}{\sinh^2(\bar{\rho})} = \frac{1}{R^2} \Rightarrow C_\theta^2 = \frac{\sinh^2(\bar{\rho})}{R^2} \tag{5.9}$$

Rewriting (5.8) and preparing for integration we have:

$$\begin{aligned}
\int_{\lambda_0}^{\bar{\lambda}} \frac{d\lambda}{R} &= \int_{\rho_0}^{\bar{\rho}} \frac{d\rho}{\sqrt{1 - \frac{\sinh^2(\bar{\rho})}{\sinh^2(\rho)}}} \\
&= \int_{\rho_0}^{\bar{\rho}} \frac{\sinh(\rho)d\rho}{\sqrt{\sinh^2(\rho) - \sinh^2(\bar{\rho})}} = \int_{\rho_0}^{\bar{\rho}} \frac{sinh(\rho)d\rho}{\cosh(\bar{\rho})\sqrt{\frac{\cosh^2(\rho)}{\cosh^2(\bar{\rho})-1}}}
\end{aligned} \tag{5.10}$$

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Now if we change variables:

$$u := \frac{\cosh(\rho)}{\cosh(\bar{\rho})}$$

$$du = \frac{\sinh(\rho)}{\cosh(\bar{\rho})} d\rho \tag{5.11}$$

$$\rho = \bar{\rho} \to u \equiv 1$$

$$\rho = \rho_0 \to u := u_0$$

The integration becomes much cleaner:

$$\frac{\lambda_\star}{2R} = \int_{u_0}^{1} \frac{du}{\sqrt{u^2 - 1}} \tag{5.12}$$

Where $\frac{\lambda_\star}{2} := \bar{\lambda} - \lambda_0$. The above integral is a known one and it comes out to be:

$$\frac{\lambda_\star}{2R} = \cosh^{-1}(u)|_{u_0}^{1}$$

$$\frac{\lambda_\star}{2R} = \cosh^{-1}(1) - acosh(u_0) \tag{5.13}$$

$$\frac{\lambda_\star}{R} = -\cosh^{-1}(2u_0^2 - 1)\cosh(-\frac{\lambda_\star}{R}) = 2u_0^2 - 1$$

Substituting back the formula for u_0 and taking into account the fact that cosh is an even function we get:

$$\cosh(\frac{\lambda_\star}{R}) = 2\frac{\cosh^2(\rho_0)}{\cosh^2(\bar{\rho})} - 1 \tag{5.14}$$

However, we are interested in expressing $\lambda_\star$ in terms of the cut off (ρ_0) and a characteristic length from the angular interval, not the turning point $\bar{\rho}$. To find a suitable expression between the angle and the turning point we

can turn back to (5.5) and using (5.8).

$$\dot{\theta} = \frac{C_\theta}{\sinh^2(\rho)} \Rightarrow$$

$$\frac{d\theta}{d\rho}\frac{d\rho}{d\lambda} = \frac{\sinh(\bar{\rho})}{R \cdot \sinh^2(\rho)} \Rightarrow$$

$$\frac{d\theta}{d\rho} = \frac{\sinh(\bar{\rho})}{\cdot \sinh^2(\rho)}\frac{1}{\sqrt{1 - \frac{\sinh^2(\bar{\rho})}{\sinh^2(\rho)}}} \Rightarrow \qquad (5.15)$$

$$= \frac{1}{\sinh(\rho)\sqrt{\frac{\sinh^2(\rho)}{\sinh^2(\bar{\rho})} - 1}}$$

Thus:

$$\int_0^{\frac{\pi l}{L}} d\theta = \int_{\rho_0}^{\bar{\rho}} \frac{d\rho}{\sinh(\rho)\sqrt{\frac{\sinh^2(\rho)}{\sinh^2(\bar{\rho})} - 1}} \Rightarrow$$

$$\frac{\pi l}{L} = \sin^{-1}(\tanh(\bar{\rho})\coth(\rho))|_{\rho_0}^{\bar{\rho}} \Rightarrow$$

$$\frac{\pi l}{L} = \frac{\pi}{2} - \sin^{-1}(\tanh(\bar{\rho})\coth(\rho_0)) \Rightarrow \qquad (5.16)$$

$$\frac{\pi l}{L} = \cos^{-1}(\tanh(\bar{\rho})\coth(\rho_0)) \Rightarrow$$

$$\cos(\frac{\pi l}{L}) = \tanh(\bar{\rho})\coth\rho_0$$

$$\bar{\rho} = \tanh^{-1}\left[\tanh(\rho_0)\cos(\frac{\pi l}{L})\right] \qquad (5.17)$$

Now, we can substitute (5.17) back into (5.14):

$$\cosh(\frac{\lambda_\star}{R}) = 2\frac{\cosh^2(\rho_0)}{\cosh^2(\tanh^{-1}\left[\tanh(\rho_0)\cos(\frac{\pi l}{L})\right])} - 1 \qquad (5.18)$$

We know that:

$$\cosh(\tanh^{-1}x) = \frac{1}{\sqrt{1 - x^2}}, \text{ and here } x = \tanh(\rho)\cos(\frac{\pi l}{L}) \qquad (5.19)$$

$$\cosh(\frac{\lambda_\star}{R}) = 2\cosh^2(\rho_0)\left[1 - (\tanh(\rho_0)\cos(\frac{\pi l}{L}))^2\right] - 1 \Rightarrow$$

$$\cosh(\frac{\lambda_\star}{R}) = 2\cosh^2(\rho_0) - 2\sinh^2(\rho_0) + 2\sinh^2(\rho)\sin^2(\frac{\pi l}{L}) - 1 \Rightarrow \qquad (5.20)$$

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$$\cosh(\frac{\lambda_\star}{R}) = 1 + 2\sinh^2(\rho_0)\sin^2(\frac{\pi l}{L})) \tag{5.21}$$

Due to how we have chosen our affine parameter, the length of the geodesic (which in this case is the boundary) turns out to be $\lambda_\star$ and so:

$$\text{Length}(\gamma) = \int ds = 2\int_{\lambda_0}^{\bar{\lambda}} d\lambda = \lambda_\star \tag{5.22}$$

Since we have a large cut off $e^{\rho_0} >> 1$:

$$\cosh(\frac{\lambda_\star}{R}) \to \frac{e^{\frac{\lambda_\star}{R}}}{2}$$
$$\sinh(\rho) \to \frac{e^{\rho_0}}{2} \tag{5.23}$$

so:

$$\frac{e^{\frac{\lambda_\star}{R}}}{2} = 1 + \frac{1}{2}e^{2\rho_0}\sin(\frac{\pi l}{L}) \Rightarrow$$
$$\frac{e^{\frac{\lambda_\star}{R}}}{2} = e^{2\rho_0}(e^{-2\rho_0} + \frac{1}{2}\sin^2(\frac{\pi l}{L})) \Rightarrow \tag{5.24}$$
$$\lambda_\star = R\ln(e^{2\rho_0}\sin^2(\frac{\pi l}{L}))$$

Thus, finally:

$$\left\{S_A = \frac{\lambda_\star}{4G_N^{(3)}} = \frac{R}{4G_N^{(3)}}\ln(e^{2\rho_0}\sin^2(\frac{\pi l}{L})) \oplus (4.6)\right\} \Rightarrow S_A = \frac{c}{3}\ln(e^{\rho_0}\sin(\frac{\pi l}{L})) \tag{5.25}$$

5.2 Poincare Coordinates

In AdS_3's Poincare coordinates, the line element takes the for:

$$ds^2 = R^2\frac{-dt^2 + dx^2 + dz^2}{z^2} \tag{5.26}$$

Calculating the Christophel connections we get the following non zero contributions:

$$\Gamma_{tz}^t = -\frac{1}{z} \qquad \Gamma_{tt}^z = \frac{1}{z}$$
$$\Gamma_{xz}^x = -\frac{1}{z} \qquad \Gamma_{xx}^z = -\frac{1}{z} \tag{5.27}$$
$$\Gamma_{zz}^z = -\frac{1}{z}$$

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Then, the geodesic equations come out to (where $\dot{x}^i := \frac{\partial x^i}{\partial s}$):

$$\ddot{t} - \frac{\dot{t}\dot{z}}{z} = 0$$

$$\ddot{x} - \frac{\dot{x}\dot{z}}{z} = 0 \tag{5.28}$$

$$\ddot{z} + \frac{\dot{t}^2}{z} - \frac{\dot{x}^2}{z} = 0$$

However, for a given constant time, the equations simplify to:

$$\ddot{x} - \frac{\dot{x}\dot{z}}{z} = 0 \tag{5.29}$$

$$\ddot{z} - \frac{\dot{x}^2}{z} = 0 \tag{5.30}$$

Taking (5.29) we have:

$$\frac{\ddot{x}}{\dot{x}} = \frac{\dot{z}}{z} \Rightarrow$$

$$\int \frac{d\dot{x}}{\dot{x}} = \int \frac{dz}{z} \Rightarrow \tag{5.31}$$

$$ln(\dot{x}) = ln(z) \Rightarrow$$

$$\dot{x} = -z \tag{5.32}$$

Thus, now substituting into (5.30) we have:

$$\ddot{z} - z = 0 \Rightarrow$$

$$z(s) = A\sin(s) \tag{5.33}$$

Using (5.32) and the boundary condition $x(0) = \frac{l}{2}$

$$z(s) = \frac{l}{2}\sin(s) \tag{5.34}$$

$$x(s) = \frac{l}{2}\cos(s) \tag{5.35}$$

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Now, we can compute the length of the geodesic as:

$$\text{Length}(\gamma_A) = \int d\gamma \Rightarrow$$

$$\text{Length}(\gamma_A) = \int R\sqrt{\frac{dz^2 + dx^2}{z^2}} \Rightarrow$$

$$\text{Length}(\gamma_A) = R\int_\epsilon^{\pi/2} \sqrt{\frac{d(\cos(s))^2 + d(\sin(s)^2)}{\sin^2(s)}} = \tag{5.36}$$

$$\text{Length}(\gamma_A) = R\int_\epsilon^{\pi/2} \frac{ds}{\sin(s)} \Rightarrow$$

$$\text{Length}(\gamma_A) = -2R\ln|\frac{1}{\sin(s)} + \frac{\cos(s)}{\sin(s)}|\,|_\epsilon^{\pi/2} \Rightarrow$$

$$\text{Length}(\gamma_A) = -2R(\ln|1+0| - \ln|\frac{1}{\epsilon} + \frac{1}{\epsilon}|) \Rightarrow$$

$$\text{Length}(\gamma_A) = -2R\ln(\frac{\epsilon}{2}) = 2R\ln(\frac{l}{\alpha}) \tag{5.37}$$

Thus, using (3.4) and (4.6) we get:

$$S_A = \frac{\text{Length}(\gamma_A)}{4G_N^{(3)}} = \frac{R}{2G_N^{(3)}}\ln(\frac{l}{\alpha}) = \frac{c}{3}\ln(\frac{l}{\alpha}) \tag{5.38}$$

5.3   Finite Temperature in d+2=3

Since we are in a thermal theory, we must use a metric with Euclidean signature. Take the 3D Euclidean BTZ Black Hole metric:

$$ds^2 = (r^2 - r_+^2)d\tau^2 + \frac{R^2}{(r^2 - r_+^2)}dr^2 + r^2 d\phi^2 \tag{5.39}$$

Note as well that:

$$\frac{\beta}{L} = \frac{R}{r_+} << 1 \text{ for r} \to \infty \tag{5.40}$$

We can turn the BTZ Black Hole metric into an equivalent AdS metric

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with the following coordinate transformations:

$$r = r_+ \cosh \rho \quad \Rightarrow \quad dr^2 = r_+^2 \sinh^2(\rho) d\rho^2 \tag{5.41}$$

$$\tau = \frac{R}{r_+} \theta \quad \Rightarrow \quad d\tau^2 = \frac{R^2}{r_+^2} d\theta^2 \tag{5.42}$$

$$\phi = \frac{R}{r_+} t \quad \Rightarrow \quad d\phi^2 = \frac{R^2}{r_+^2} dt^2 \tag{5.43}$$

Substituting into (5.39) we get:

$$ds^2 = R^2 \left[\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\theta^2 \right] \tag{5.44}$$

Calculating the entanglement entropy now is a mirror process of what was done in 5.1 but now the time coordinate corresponds to θ. Thus, for a static time frame:

$$ds^2 = R^2 \left[\cosh^2 \rho dt^2 + d\rho^2 \right] \tag{5.45}$$

Again, the Euler-Lagrange equations will be:

$$\mathcal{L} = R\sqrt{\cosh^2(\rho)\dot{t}^2 + \dot{\rho}^2} \tag{5.46}$$

$$\text{For } \rho : \quad \ddot{\rho} = \cosh(\rho)\sinh(\rho)\dot{t} \tag{5.47}$$

$$\text{For } t : \quad \cosh^2(\rho)\dot{t} = C_t \tag{5.48}$$

$$\text{Affine Parameter} : \quad R^2(\cosh^2(\rho)\dot{t}^2 + \dot{\rho}^2) = 1 \tag{5.49}$$

Combining (5.48) and (5.49) we get:

$$\frac{C_t^2}{\cosh^2(\rho)} + \dot{\rho}^2 = \frac{1}{R^2} \tag{5.50}$$

We can find an expression for C_t as:

$$\dot{\rho}|_{\bar{\rho}} = 0 \therefore C_t^2 = \frac{\cosh^2(\bar{\rho})}{R^2} \tag{5.51}$$

So (5.50) becomes:

$$1 = \frac{R\dot{\rho}}{\sqrt{1 - \frac{\cosh^2(\bar{\rho})}{\cosh^2(\rho)}}} \tag{5.52}$$

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We can integrate to get:

$$\int_{\bar\lambda}^{\lambda_0} \frac{d\lambda}{R} = \int_{\bar\rho}^{\rho_0} \frac{d\rho}{\sqrt{1 - \frac{\cosh^2(\bar\rho)}{\cosh^2(\rho)}}} \Rightarrow$$

$$\frac{\lambda_\star}{R} = 2\cosh^{-1}\left[\frac{\sinh(\rho)}{\sinh(\bar\rho)}\right]_{\rho_0}^{r\bar ho} \Rightarrow \tag{5.53}$$

$$\frac{\lambda_\star}{R} = -2\cosh^{-1}\left(\frac{\sinh\rho_0}{\sinh\bar\rho}\right) \Rightarrow$$

$$\frac{\lambda_\star}{R} = \cosh^{-1}\left(2\frac{\sinh^2(\rho_0)}{\cosh^2(\bar\rho)} - 1\right) \Rightarrow$$

$$\cosh\left(\frac{\lambda_\star}{R}\right) = 2\frac{\sinh^2(\rho_0)}{\sinh^2(\bar\rho)} - 1 \tag{5.54}$$

Again, we want the solution in terms of ρ_0 and the parametrization of t so:

$$\frac{dt}{d\lambda} = \frac{\cosh(\bar\rho)}{R\cosh^2(\rho)} \Rightarrow$$

$$\frac{dt}{d\rho}\dot\rho = \frac{\cosh(\bar\rho)}{R\cosh^2(\rho)} \Rightarrow$$

$$\frac{dt}{d\rho} = \frac{\cosh(\bar\rho)}{R\dot\rho\cosh^2(\rho)} \Rightarrow$$

$$\int_0^{\frac{\pi l r_+}{LR}} dt = \int_{\rho_0}^{\bar\rho} \frac{\cosh(\bar\rho)d\rho}{\sqrt{1 - \frac{\cosh^2(\bar\rho)}{\cosh^2(\rho)}}} \Rightarrow$$

$$\frac{\pi l r_+}{LR} = \sinh^{-1}(-i\coth(\bar\rho)\tanh(\rho))|_{\rho_0}^{\bar\rho} \Rightarrow$$

$$\frac{\pi l r_+}{LR} = \sinh^{-1}(-i) - \sinh^{-1}(-i\coth(\bar\rho)\tanh(\rho_0))$$

$$\frac{\pi l r_+}{LR} = \sinh^{-1}(-i\sqrt{1 - \coth^2(\bar\rho)\tanh^2(\rho)} + i\coth(\bar\rho)\tanh(\rho_0)\sqrt{1 + (-i)^2})$$

$$\frac{\pi l r_+}{LR} = \sinh^{-1}(\sqrt{\coth^2(\bar\rho)\tanh^2(\rho_0) - 1}) \Rightarrow$$

$$\sinh^2\left(\frac{\pi l r_+}{LR}\right) + 1 = \coth(\bar\rho)\tanh(\rho_0)$$

$$\tag{5.55}$$

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Giving us:

$$\bar{\rho} = \coth^{-1}\left[\coth(\rho_0)\cosh^2(\frac{\pi l r_+}{LR})\right] \tag{5.56}$$

Observe that $\frac{\beta}{L} = \frac{R}{r_+} \Rightarrow \frac{r_+}{LR} = \frac{1}{\beta}$ so, plugging in the (5.56) into (5.54) we get:

$$\cosh(\frac{\lambda_\star}{R}) = 2\frac{\sinh^2(\rho_0)}{\sinh^2\left[\coth^{-1}(\coth(\rho_0)\cosh^2(\frac{\pi l}{\beta}))\right]} - 1 \Rightarrow$$

$$\cosh(\frac{\lambda_\star}{R}) = 2\sinh^2(\rho_0)\left[\coth^2(\rho)\cosh^2(\frac{\pi l}{\beta}) - 1\right] - 1 \Rightarrow \tag{5.57}$$

$$\cosh(\frac{\lambda_\star}{R}) = 2\cosh^2(\rho)\sinh^2(\frac{\pi l}{\beta}) + 2\cosh^2(\rho_0) - 2\sinh^2(\rho_0) - 1 \Rightarrow$$

$$\cosh(\frac{\lambda_\star}{R}) = 1 + 2\cosh^2(\rho)\sinh^2(\frac{\pi l}{\beta}) \tag{5.58}$$

Again, the length of the geodesic -and thus the boundary area- is equal to $\lambda_\star$. Since, again, $\rho_0 >> 1$ and $\lambda_\star >> R$ the final expression for $\lambda_\star$ becomes:

$$\lambda_\star = R\ln\left[e^{2\rho_0}\sinh^2(\frac{\pi l}{\beta})\right] \tag{5.59}$$

Thus:

$$S_A = \frac{\text{Length}(\gamma_A)}{4G_N^{(3)}} = \frac{R\ln\left[e^{2\rho_0}\sinh^2(\frac{\pi l}{\beta})\right]}{4G_N^{(3)}} = \frac{c}{3}\ln(e^{\rho_0}\sinh(\frac{\pi l}{\beta})) \tag{5.60}$$

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Chapter 6

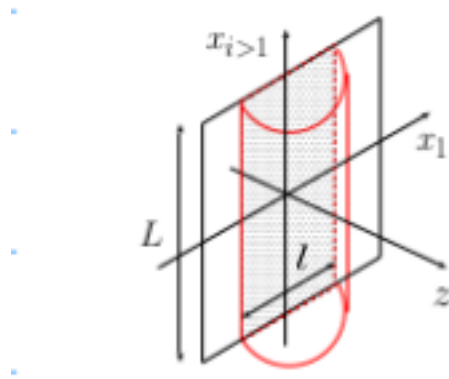# Higher Dimensions: Entanglement Entropy for a Straight Belt

Figure 6.1: A figure of the belt provided in [1]

Suppose now that we want to calculate the Entanglement Entropy in higher dimensional theories. In this case, the shape of the screen plays a crucial role. For simplicity, suppose a straight belt in d+2 AdS in Poincare

coordinates. The area can be found as (for a static frame):

$$\text{Area} = \int dx^\mu \sqrt{\gamma} \tag{6.1}$$

Where $x^\mu = \{x^1, x^3, ..., x^{d+2}\}$ are the coordinates tangent to the area, $\gamma_{\mu\nu}$ is the induced metric on the belt and γ it's determinant.

6.1 The Induced Metric Determinant

$$\gamma_{\mu\nu} = \partial_\mu x^\alpha \partial_\nu x^\beta \bar{g}_{\alpha\beta}, \quad \bar{g}_{\alpha\beta} = \frac{R^2}{z^2} \delta_{\alpha\beta} \Rightarrow$$

$$\gamma_{\mu\nu} = \partial_\mu x^\alpha \partial_\nu x^\alpha \frac{R^2}{z^2} \tag{6.2}$$

$$\text{if } x^\alpha = x^{\mu/\nu} \Rightarrow \partial_{\mu/\nu} x^\alpha = 1$$

$$\text{if } x^\alpha = z \propto x := x^1 \Rightarrow \partial_{\mu=\nu=2} z = \dot{z}$$

In every other case the derivatives are zero, thus:

$$\gamma_{22} = \frac{R^2(\dot{z}^2 + 1)}{z^2}$$

$$\gamma_{\mu\nu} = \frac{R^2}{z^2} \ \forall \, \mu, \nu \neq 2 \tag{6.3}$$

Since γ is d dimensional and diagonal:

$$\gamma = \prod_{\mu=2}^{\mu=d+2} \gamma_{\mu\mu} = \frac{R^{2d}(\dot{z}^2 + 1)}{z^{2d}} \tag{6.4}$$

6.2 The Area Lagrangian

Using (6.1) and (6.4) we can now calculate the full formula for the area as:

$$\text{Area} = R^d \int_{\mathbb{R}^{d-1}} dx^i \int_{-l/2}^{l/2} dx \frac{\sqrt{1 + \dot{z}}}{z^d} = R^d L^{d-1} \int_{-l/2}^{l/2} dx \frac{\sqrt{1 + \dot{z}}}{z^d} \tag{6.5}$$

Where $i \in \{3, ..., d+2\}$, coordinates which all span from 0 to L on the belt and l is the characteristic width of the belt. We can now compute the Lagrangian of the area as:

$$\mathcal{L} = \frac{L^{d-1} R^d \sqrt{1 + \dot{z}^2}}{z^d} \tag{6.6}$$

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From the Hamiltonian we can find an expression for $z(x)$:

$$\mathcal{H} = \dot{z}\frac{\partial \mathcal{L}}{\partial \dot{z}} - \mathcal{L} \Rightarrow$$

$$= \frac{L^{d-1}R^d}{z^d}\left[\frac{\dot{z}^2}{\sqrt{1+\dot{z}^2}} - \sqrt{1+\dot{z}^2}\right] = \frac{L^{d-1}R^d}{z^d}\frac{-1}{\sqrt{1+\dot{z}^2}} = E \Rightarrow \qquad (6.7)$$

$$\left\{\sqrt{1+\dot{z}^2} = \frac{-L^{d-1}R^d}{z^d E}, \quad z_\star^d := \frac{-L^{d-1}R^d}{E}\right\} \Rightarrow 1+\dot{z}^2 = \frac{z_\star^{2d}}{z^{2d}} \Rightarrow$$

$$\dot{z} = \frac{\sqrt{z_\star^{2d} - z^{2d}}}{z^d} \qquad (6.8)$$

Observe that $z_\star$ is the turning point pf the belt as $\dot{z}|_{z_\star} = 0$. Noting that for $x = \pm\frac{l}{2} \rightarrow z = \varepsilon$ and $x = 0 \rightarrow z = z_\star$ we can compute the integral (6.5) by substituting x for z as the integration variable:

$$\text{Area} = R^d L^{d-1} \int_{-l/2}^{l/2} dx \frac{\sqrt{1+\dot{z}}}{z^d} \Rightarrow$$

$$= R^d L^{d-1}\left[\int_{-l/2}^{0} dx \frac{\sqrt{1+\dot{z}}}{z^d} + \int_{0}^{l/2} dx \frac{\sqrt{1+\dot{z}}}{z^d}\right] \Rightarrow$$

$$\qquad (6.9)$$

$$= 2R^d L^{d-1} \int_{-l/2}^{0} dx \frac{\sqrt{1+\dot{z}}}{z^d} = 2R^d L^{d-1} \int_{\alpha}^{z_\star} dz \frac{z_\star^d}{z^d} \frac{1}{\sqrt{z_\star^{2d} - z^{2d}}} \Rightarrow$$

$$= \frac{2R^d L^{d-1}}{z_\star^{d-1}} \int_{\varepsilon}^{1} \frac{dy}{y^d} \frac{1}{\sqrt{1-y^{2d}}}$$

Where in the last equality we substituted the integration variable with $y := \frac{z}{z_\star}$ and $\varepsilon := \frac{\alpha}{z_\star}$ We cannot calculate this integral as is since the lower bound is not zero. We can manipulate the expression to change the lower bound:

$$\text{Area} = \frac{2R^d L^{d-1}}{z_\star^{d-1}}\left[\int_{\varepsilon}^{1} \frac{dy}{y^d} \frac{1}{\sqrt{1-y^{2d}}} + \int_{0}^{\varepsilon} \frac{dy}{y^d} \frac{1}{\sqrt{1-y^{2d}}} - \int_{0}^{\varepsilon} \frac{dy}{y^d} \frac{1}{\sqrt{1-y^{2d}}}\right] \Rightarrow$$

$$= \frac{2R^d L^{d-1}}{z_\star^{d-1}}\left[\int_{0}^{1} \frac{dy}{y^d} \frac{1}{\sqrt{1-y^{2d}}} - \int_{0}^{\varepsilon} \frac{dy}{y^d} \frac{1}{\sqrt{1-y^{2d}}}\right]$$

$$\qquad (6.10)$$

Since in the interval $[0,\varepsilon]$ y<<1:

$$\frac{1}{y^d\sqrt{1-y^{2d}}} \approx \frac{1}{y^d} \qquad (6.11)$$

We have:

$$
\begin{aligned}
\text{Area} &= \frac{2R^d L^{d-1}}{z_\star^{d-1}} \left[\int_0^1 \frac{dy}{y^d} \frac{1}{\sqrt{1-y^{2d}}} - \int_0^\varepsilon \frac{dy}{y^d} \right] \Rightarrow \\
&= \frac{2R^d L^{d-1}}{z_\star^{d-1}} \left[\int_0^1 \frac{dy}{y^d} \frac{1}{\sqrt{1-y^{2d}}} - \int_0^1 \frac{dy}{y^d} + \int_\varepsilon^1 \frac{dy}{y^d} \right] \Rightarrow \\
&= \frac{2R^d L^{d-1}}{z_\star^{d-1}} \left[\int_0^1 \frac{dy}{y^d} \left(\frac{1}{\sqrt{1-y^{2d}}} - 1 \right) + \int_\varepsilon^1 \frac{dy}{y^d} \right] \Rightarrow \\
&= \frac{2R^d L^{d-1}}{z_\star^{d-1}} \left[\int_0^1 \frac{dy}{y^d} \left(\frac{1}{\sqrt{1-y^{2d}}} - 1 \right) + \frac{1}{d-1} \left(\frac{1}{\varepsilon^{d-1}} - 1 \right) \right]
\end{aligned}
\tag{6.12}
$$

By defining $I := \frac{1}{d-1} - \int_0^1 \frac{dy}{y^d} \left(\frac{1}{\sqrt{1-y^{2d}}} - 1 \right)$ and plugging in the expression for ε we can rewrite the above expression as:

$$
\text{Area} = \frac{2R^d}{d-1} \left(\frac{L}{\alpha} \right)^{d-1} - 2R^d I \left(\frac{L}{z_\star} \right)^{d-1}
\tag{6.13}
$$

A yet to be addressed issue is the fact that we don't have a concrete expression for $z_\star$. However, we can go back to (6.8) and integrate[1]:

$$
\begin{aligned}
\int_0^{z_\star} dz \frac{z^d}{\sqrt{z_\star^{2d} - z^{2d}}} &= \int_0^{\frac{l}{2}} dx \\
\frac{l}{2} &= z_\star \int_0^1 dy \frac{y^d}{\sqrt{1-y^{2d}}} \Rightarrow \\
\frac{l}{2} &= \frac{\sqrt{\pi}\Gamma(\frac{d+1}{2d})}{2d\Gamma(\frac{1}{2d})} z_\star \Rightarrow
\end{aligned}
\tag{6.14}
$$

$$
z_\star = \frac{l}{2} \frac{2d\Gamma(\frac{1}{2d})}{\sqrt{\pi}\Gamma(\frac{1+d}{2d})}
\tag{6.15}
$$

Now, we can calculate the Area from (6.13) as:

$$
\text{Area} = \frac{2R^d}{d-1} \left(\frac{L}{\alpha} \right)^{d-1} - \frac{2^d R^d \pi^{d/2}}{d-1} \left(\frac{\Gamma(\frac{d+1}{2d})}{\Gamma(\frac{1}{2d})} \right)^d \left(\frac{L}{l} \right)^{d-1}
\tag{6.16}
$$

---

[1] $\int_0^1 dx\, x^{\mu-1}(1-x^\lambda)^{\nu-1} = \frac{\Gamma(\mu/\lambda)\Gamma(\nu)}{\Gamma(\mu/\lambda+\nu)\lambda}$

While the entropy is just (6.16) divided by $4G_N^{d+2}$

$$S_A = \frac{1}{4G_N^{(d+2)}} \left(\frac{2R^d}{d-1} \left(\frac{L}{\alpha}\right)^{d-1} - \frac{2^d R^d \pi^{d/2}}{d-1} \left(\frac{\Gamma(\frac{d+1}{2d})}{\Gamma(\frac{1}{2d})}\right)^d \left(\frac{L}{l}\right)^{d-1} \right) \qquad (6.17)$$

Chapter 7

Conclusions

We have successfully managed to replicate the CFT entanglement entropy (4.2) both in the global coordinates (5.25) and Poincare coordinates (5.38). Moreover we replicated the thermal case results (4.3) in AdS (5.60) as well as the d+2 case for a straight belt shaped boundary (4.5) (6.17)

Bibliography

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