Flows on Conformal Manifolds with Machine Learning

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A thesis presented for the degree of
Bachelor of Science in Physics

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Greece
June 2023

Abstract

We explore an approach for solving the Conformal Bootstrap using reinforcement learning techniques. Specifically, we apply the SAC[5] reinforcement learning algorithm to a 2D compactified bosonic CFT and demonstrate its effectiveness in yielding reasonable results. Through extensive experimentation, we show that our reinforcement learning approach is capable of finding solutions that align with known results and exhibit desirable properties. Our findings highlight the potential of reinforcement learning as a valuable tool for solving the Conformal Bootstrap problem.

Acknowledgments

This work was performed as an undergraduate thesis under the supervision of professor Vasilis Niarchos (UoC) whom I want to deeply thank for his guidance, patience and permanent eagerness to be of help. I would also like to thank Dr. Paul Richmond (QMUL) for his crucial assistance on technical issues of using the BootSTOP algorithm.

As of people who assisted me indirectly, I would like to first and foremost thank my whole family and especially, my parents. My mother for always believing in me and my father for always making me feel someone has got my back. Many thanks should also go to my dear friends for having the mental fortitude to put up with my eccentricity. In particular, I want to thank Kostas Tazes, Nikos Kazantzakis, Irene Mantadaki and Nikos Triantafylledes. The prize on that domain however, goes to my best friend and roommate Giorgos Damigos who generously helped cover the computational needs for the project while constantly being insulted and yelled upon in the process. You guys are either heroes or imbeciles, I can't tell anymore.

Last but not least I want to thank my colleagues at the 3rd Disaster Management Special Unit of the Hellenic Fire Corps for always being supportive of my academic endeavors and actively helping me get the necessary time off to pursue them.

Contents

Introduction

Conformal field theories (CFTs) are quantum field theories that are invariant under the conformal group. Their applications in condensed matter physics (e.g. the study of the behavior of strongly correlated electron systems and the study of the critical behavior of phase transitions in condensed matter systems) and theoretical physics (e.g. String theory and black hole dynamics) have made their study crucial for research in these fields.

For years, theoretical physicists have turned their eye to analytic and conventional computational methods to probe the vast unanswered question on various important CFTs and their connection to the bigger picture of the cosmos.

One of the key challenges in studying CFTs is understanding their operator spectra and the constraints imposed by the crossing equations. The Conformal Bootstrap provides a powerful mathematical approach to investigate the properties of CFTs. By imposing consistency conditions on correlation functions, it allows us to derive constraints on the conformal dimensions and operator coefficients. Solving the Conformal Bootstrap problem entails finding solutions that satisfy these constraints, enabling us to gain insights into the symmetries and dynamics of the underlying CFTs.

Traditionally, solving the Conformal Bootstrap problem has relied on analytical techniques, perturbative expansions, or numerical methods such as semidefinite programming. However, due to the vast parameter space of CFTs and the complexity of the bootstrap equations, finding precise solutions remains a formidable task. This motivates the exploration of alternative computational approaches that can efficiently navigate the parameter space and uncover meaningful solutions.

In recent years, a new field of computer science has emerged, *machine learning*. This field focuses on the development of algorithms and models that allow machines to learn from data and improve their performance on a task. Machine learning is used to develop systems that can automatically recognize patterns in data, make predictions, and take actions based on those predictions. So far machine learning has had successful applications in a huge array of industries and science fields including Medicine, Astrophysics, Biology, Economics, Psychology and many others.

The numerous achievements of machine learning raise the following question; could machine learning also find applications in theoretical physics? Several attempts have been made, met with various degrees of success. In this work, we explore the possibility of implementing machine learning algorithms to solve conformal field theories via conformal bootstrap [9] [13].

A powerful for, of machine learning, reinforcement learning, has emerged as a promising tool for solving complex problems in various domains, ranging from robotics to natural language processing. The ability of reinforcement learning agents to learn from interactions with an environment, guided by rewards, makes it an intriguing candidate for tackling the Conformal Bootstrap problem. By formulating the problem as a reinforcement learning task, we can leverage the agent's exploration and optimization capabilities to search for conformal dimensions and operator coefficients that satisfy the bootstrap constraints.

The toy model theory we will make our discussion on will be a two dimensional compactified boson. We will explore ideas and limitations of the reinforcement learning approach to solve the theory at a point on the conformal manifold and also to flow away from that point and obtain a global solution on that manifold.

Of course to make that discussion, for completeness sake we must first go through the prerequisites of CFT, conformal bootstrap and the 2D compact boson. The first chapters aims to familiarize the reader with these topics while establishing our formalism and conventions. On chapter 1 we will make a brief review of CFT focusing on 2D CFTs and analytically solve the 2D compactified boson CFT. On chapter 2 we will make derive the crossing symmetry equations and see how they are further constrained by the existence of a global U(1) symmetry. On chapter 3 we discuss the need of truncations and spin partitions and further explore the 2D boson to get a feel of what would be an appropriate spin partition and truncation. On chapter 4 we setup the exact framework of our work and describe our methodology. On chapter 5 we present the results we obtained through our methods. Lastly, on chapter 6 we conclude the work and provide an outlook for future projects.

Note that throughout this work, it is assumed that the reader has at least introductory course level knowledge in quantum field theory and group theory. Basic knowledge on scalar fields, propagators, Wick's theorem, Noether's theorem, Ward's identities, symmetries, groups, representations, generators is essential to keep up with the discussion.

Chapter 1

A Brief Review of CFT

We begin our discussion by briefly reviewing the most important general aspects of CFTs and in particular two dimensional CFTs. After, we solve the S_1 theory of a compact boson as an example.

1.1 Introduction to Conformal Symmetry

Conformal field theories are quantum field theories that are invariant under the conformal group. We shall begin by studying this symmetry.

Definition 1 *Consider a D-dimensional flat space-time with a metric $\eta_{\mu\nu}$. A conformal transformation of the coordinates is defined as an invertible mapping $x \to x'$, which leaves the metric tensor invariant up to a scale factor:*

$$\eta_{\mu\nu} \to \Omega^2(x)\eta_{\mu\nu}$$

or equivalently:

$$\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}\eta'_{\rho\sigma} = \Omega^2(x)\eta_{\mu\nu} \tag{1.1}$$

The local *rescaling factor $\Omega^2(x)$ is called a Weil factor.*

The set of transformations of the form of (1.1) compose a group; the *conformal group.* Note that for $\Omega^2(x) = 1$, (1.1) becomes a Poincare transformation and thus the Poincare group is a sub-group of the conformal group. Let us now put some physical context on the conformal group transformations. Consider two vectors A, B. The angle θ between them is defined as

$$\cos\theta = \frac{A \cdot B}{|A||B|} = \frac{\eta_{\mu\nu}A^{\mu}B^{\nu}}{\sqrt{\eta_{\rho\sigma}A^{\rho}A^{\sigma}}\sqrt{\eta_{\tau\delta}B^{\tau}B^{\delta}}}$$

Suppose now we make a conformal transformation. The transformed angle (in the original frame of reference) is:

$$\cos\theta' = \frac{\Omega^2(x)\eta_{\mu\nu}A^\mu B^\nu}{\sqrt{\Omega^2(x)\eta_{\rho\sigma}A^\rho A^\sigma}\sqrt{\Omega^2(x)\eta_{\tau\delta}B^\tau B^\delta}} =$$

$$\frac{\eta_{\mu\nu}A^\mu B^\nu}{\sqrt{\eta_{\rho\sigma}A^\rho A^\sigma}\sqrt{\eta_{\tau\delta}B^\tau B^\delta}} = \cos\theta$$

This result leads to the following alternative, more intuitive, definition of a conformal transformation.

Definition 2 *A conformal transformation is a mathematical mapping that preserves angles, while allowing changes in sizes and orientation.*

Equation (1.1) holds for any *finite* conformal transformation. We can use that to see how an *infinitesimal* transformation acts and through that find the generators and Lie algebra of the conformal group.
Consider an infinitesimal conformal transformation:

$$x'^\mu = x^\mu + \xi^\mu(x) \qquad \Omega(x) = 1 + \kappa(x) \tag{1.2}$$

Substituting (1.2) in equation (1.1) and neglecting $\mathcal{O}(\xi^2), \mathcal{O}(\kappa^2), \mathcal{O}(\xi\kappa)$ (and higher) terms we obtain:

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2\kappa(x)\eta_{\mu\nu} \tag{1.3}$$

Equation (1.3) is known as the *conformal killing equation* for flat space-time[1]. If we contract (1.3) with $\eta^{\mu\nu}$ we obtain:

$$\kappa(x) = \frac{1}{D}\partial_\mu \xi^\mu \tag{1.4}$$

1.2 Conformal Symmetry in $D > 2$

Even though in this work we will focus on the two dimensional case it will be beneficial to briefly study how conformal symmetry works for $D > 2$. The reason for that is that elements of the $D > 2$ case can be trivially passed on the 2D case in the appropriate context.
For $D > 2$, the general solution of (1.3) is:

$$\xi^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \sigma x^\mu + b^\mu x^\rho x_\rho - 2b_\nu x^\nu x^\mu \tag{1.5}$$

$$\kappa(x) = \sigma - 2b_\nu x^\nu \tag{1.6}$$

where:

[1]For curved space-time: $\partial_\mu \to \nabla_\mu$

- a^μ is a constant
- ${\omega^\mu}_\nu$ is the infinitesimal Lorentz transformation (i.e. $\omega_{\mu\nu} = -\omega_{\nu\mu}$)
- σ is the *dilatation* factor
- b_μ is the *special conformal transformation* constant

1.2.1 Conformal Algebra

Writing (1.2) as $x'^\mu = (1 + \xi^\nu \partial_\nu) x^\mu$ we can see that we can choose a basis for our Lie algebra so that the conformal killing vector, $\xi^\mu \partial_\mu$ is a generator of said Lie algebra. With that established and by taking a look at equations (1.5) and (1.6) we can see there are four fundamental parts (or sub-groups) of the conformal symmetry:

i) The generator of translations:

$$P_\mu = \partial_\mu \tag{1.7}$$

which is our well known momentum.

ii) The generator of rotations:

$$L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \tag{1.8}$$

also known as Lorentz transformation.

iii) The generator of dilatation:

$$D = x^\mu \partial_\mu \tag{1.9}$$

that re-scales towards all directions and finnaly,

iv) The special conformal transformations' generator:

$$K_\mu = 2x_\mu x^\nu \partial_\nu - x^\rho x_\rho \partial_\mu \tag{1.10}$$

whose interpretation is to be discussed in 1.2.2

With these definitions, the most general generator, according to (1.5), is:

$$\xi^\mu \partial_\mu = a^\mu P_\mu + \frac{1}{2} {\omega^\mu}_\nu {L^\nu}_\mu + \sigma D + b^\mu K_\mu \tag{1.11}$$

As we can see by looking at equations (1.7), (1.8), (1.9) and (1.10), the number of independent generators is:

$$N = \frac{1}{2}(D+1)(D+2) \tag{1.12}$$

Since the conformal group, as mentioned before, is a Lie group, the elements of the corresponding Lie algebra (i.e. the generators) must be closed under the algebra's product (i.e. the commutator). The only non-vanishing commutators are:

Conformal generators' commutation relations ($D > 2$)

$$[P_\mu, L_{\nu\rho}] = \eta_{\mu\nu} P_\rho - \eta_{\mu\rho} P_\nu \tag{1.13}$$

$$[P_\mu, K_\nu] = -2\eta_{\mu\nu} D + 2L_{\mu\nu} \tag{1.14}$$

$$[K_\mu, L_{\nu\rho}] = \eta_{\mu\nu} K_\rho - \eta_{\mu\rho} K_\nu \tag{1.15}$$

$$[K_\mu, D] = -K_\mu \tag{1.16}$$

$$[P_\mu, D] = P_\mu \tag{1.17}$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = \eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho} \tag{1.18}$$

As these relations show, the generators form a closed algebra under the commutator, as is to be expected from the Lie algebra of a Lie group.

1.2.2 Finite Conformal Transformations

After our study of the conformal algebra we can now recover finite conformal transformations through the exponential map:

$$x'^\mu = e^{\xi^\nu \partial_\nu} x^\mu \tag{1.19}$$

Lets see how a finite transformation look like for the four fundamental parts of the conformal symmetry:

i) The most trivial case is just considering $\xi^\mu = a^\mu$:

$$x'^\mu = \left(1 + a^\nu \partial_\nu + \frac{1}{2}(a^\nu \partial_\nu)^2 + \mathcal{O}((a^\nu \partial_\nu)^3)\right) x^\mu$$

and since $\partial_\mu x^\nu = \delta_\mu{}^\nu$ we get:

$$x'^\mu = x^\mu + a^\mu \tag{1.20}$$

ii) Next, lets consider a rotation, i.e. $\xi^\mu = \omega^\mu{}_\nu x^\nu$. The result is:

$$x'^\mu = M^\mu{}_\nu x^\nu \tag{1.21}$$

as is well known from studying the Lorentz group

iii) Consider now a dilatation, $\xi^\mu = \sigma x^\mu$:

$$x'^\mu = \left(1 + \sigma x^\nu \partial_\nu + \frac{1}{2}(\sigma x^\nu \partial_\nu)^2 + \cdots\right) x^\mu = \left(1 + \sigma + \frac{\sigma^2}{2}\right) x^\mu \Rightarrow$$

$$x'^\mu = e^\sigma x^\mu \tag{1.22}$$

which, as mentioned before, is just a re-scaling towards all directions.

iv) Lets now study the non-trivial case of special conformal transformation, i.e. $\xi^\mu = b^\mu x^2 + 2b_\nu x^\nu x^\mu$. Through the exponential map, one can get:

$$x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2} \tag{1.23}$$

At first this result does not seem to have any special meaning. This is incorrect. Lets first evaluate the norm of the transformed four-vector. (1.23) implies that:

$$x'^2 = \frac{x^2}{1 + 2b \cdot x + b^2 x^2}$$

substituting back to (1.23) we get:

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} + b^\mu \tag{1.24}$$

Equation (1.24) implies that special conformal transformations are just translations in the inverted coordinates frame of reference ($x^\mu \to \frac{x^\mu}{x^2}$). This also means that translations (generated by P_μ) are special conformal transformations on the inverted coordinates frame.

1.3 Conformal Symmetry in $D = 2$

We will now continue our discussion for the special case of $D = 2$ which is of more interest in this work. What is special in the two dimensional case?

1.3.1 Conformal Group in Two Dimensions

The answer lies back to the way we studied the infinitesimal transformations and obtained the generators; the *conformal killing equation*.
Consider, without loss of generality, a euclidean two dimensional plain with coordinates (x^1, x^2). Now, consider a mapping $x^\mu \to y(x)^\mu$. By demanding that the mapping to be conformal, we know that the metric must transform as in (1.1). An explicit calculation, yields the following constraints on the transformation:

$$\partial_1 y^2 = \partial_2 y^1 \quad \text{and} \quad \partial_1 y^1 = -\partial_2 y^2 \tag{1.25}$$

or

$$\partial_1 y^2 = -\partial_2 y^1 \quad \text{and} \quad \partial_1 y^1 = \partial_2 y^2 \tag{1.26}$$

Equations (1.25) and (1.26) are the well known Cauchy-Riemann equations for holomorphic and anti-holomorphic complex functions respectively. This result, motivates the use of the following complex coordinates to study the problem:

Definition of complex coordinates

$$z = x^1 + ix^2 \qquad \bar{z} = x^1 - ix^2 \tag{1.27}$$

$$\partial = \frac{1}{2}(\partial_1 - i\partial_2) \qquad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2) \tag{1.28}$$

$$x^1 = \frac{1}{2}(z + \bar{z}) \qquad x^2 = \frac{1}{2i}(z - \bar{z}) \tag{1.29}$$

$$\partial_1 = \partial + \bar{\partial} \qquad \partial_2 = i(\partial - \bar{\partial}) \tag{1.30}$$

Note that for all intents and purposes, z and $\bar{z}$ are considered to be *independent* variables with the constraint $z^* = \bar{z}$.
In this coordinate system, the flat metric becomes $ds^2 = dz\, d\bar{z}$. An infinitesimal conformal transformation, $z \to w(z)$, is now:

$$w = z + \epsilon \qquad \bar{w} = \bar{z} + \bar{\epsilon} \tag{1.31}$$

The conformal Killing equation yields the following constraints:

$$\bar{\partial}\epsilon = 0 \qquad \partial\bar{\epsilon} = 0 \tag{1.32}$$

Which simply mean that ϵ is holomorphic and $\bar{\epsilon}$ is anti-holomorphic. This constraint leaves a huge freedom on possible conformal transformations. In two dimensions, the space of local solutions to the conformal Killing equation is infinite dimensional. This results in a well known complex analysis result:

Theorem 1 *Any analytic mapping, $z \to w(z)$, of the complex plain to itself is conformal.*

This results in the following definition of the conformal group in two dimensions:

Definition 3 *The conformal group in two dimensions is the set of all analytic mappings of the complex plain to itself with the group multiplication being the composition of these maps. The group is infinitely dimensional as there do exist locally analytic mappings in every point of the complex plain.*

1.3.2 Energy-Momentum Tensor, Charges and the Virasoro Algebra

As one would expect, the infinite dimensional conformal group in two dimensions, has generators obeying an infinite dimensional Lie algebra. This algebra is called *Virasoro algebra.*
To better understand this algebra we will start indirectly, from the energy momentum tensor. Since we are dealing with *local* theories, we can indeed define a conserved energy momentum tensor, $T_{\mu\nu}$. As we are dealing with conformal field theories, this tensor is *trace-less.* In the complex coordinates defined in (1.27-1.30) these constraints are:

$$T_{z\bar{z}} = T_{\bar{z}z} = 0 \qquad \partial T_{zz} = \partial T_{\bar{z}\bar{z}} = 0 \tag{1.33}$$

As (1.33) suggests, in two dimensional CFTs there are only two non-zero elements, one holomorphic and one anti-holomorphic. For later convenience, we re-define said non-zero elements as:

$$T(z) = -2\pi T_{zz} \qquad \bar{T}(\bar{z}) = -2\pi T_{\bar{z}\bar{z}} \tag{1.34}$$

The *conformal charge* associated with the holomorphic component of an infinitesimal transformation $z \to z + \epsilon(z)$ is defined via:

$$Q_\epsilon = \frac{1}{2\pi i} \oint dz\, \epsilon(z) T(z) \tag{1.35}$$

By mode expanding T(z) and $\epsilon(z)$ we obtain a relation between the energy-momentum tensor and the Virasoro generators L_n:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \quad \Leftrightarrow \quad L_n = \frac{1}{2\pi i} \oint dz\, z^{n+1} T(z) \tag{1.36}$$

The extension of (1.35) and (1.36) to their anti-holomorphic counterparts is trivial.
As discussed on 1.3.1 $\epsilon(z)$ is bound by conformal symmetry to be holomorphic in the neighborhood of z. This allows us to express it as a Laurent series as:

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1} \epsilon_n \qquad \bar{\epsilon}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{n+1} \bar{\epsilon}_n \tag{1.37}$$

Using this expansion, in combination with (1.36) we can write the conformal charge in a series of the form:

$$Q_\epsilon = \sum_{n\in\mathbb{Z}} \epsilon_n L_n \qquad Q_{\bar{\epsilon}} = \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \bar{L}_n \tag{1.38}$$

And thus, the conformal charge is just a linear combination of generators of local conformal transformations.

The set of infinite generators, L_n, comprising the Virasoro algebra obey the commutation relations:

Virasoro Algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{1.39}$$

$$[L_n, \bar{L}_m] = 0 \tag{1.40}$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{1.41}$$

where c is the theory dependent central charge and $n, m \in \mathbb{Z}$.

1.3.3 Möbius Subgroup and the Global Part of the Conformal Algebra

As stated in definition 3 of sector 1.3.1 *any* analytic mapping constitutes a conformal transformation in two dimensions. It was also mentioned that there are *infinite local* such mappings. What about *global* mappings? Are there conformal transformation in two dimensions that are well defined on the whole Riemann sphere?
The answer is indeed affirmative. Such mappings are called Möbius transformations and have the form:

$$f(z) = \frac{az+b}{cz+d}, \qquad ad - bc = 1, \qquad a, b, c, d \in \mathbb{C} \tag{1.42}$$

Transformations defined by (1.42) can be represented by 2×2 complex matrices with unit determinant and thus the subgroup of global transformations is $SL(2, \mathbb{C})$. Note that in this group the number of independent elements is 6. This is the same result we could have gotten from (1.12) with $D = 2$.
If we now take a look at the general commutation relations of (1.39 - 1.41) we can identify the global sub-algebra:

Global Sub-Algebra

$$[L_{-1}, L_1] = -2L_0, \qquad [L_0, L_1] = -L_1, \qquad [L_0, L_{-1}] = L_{-1} \tag{1.43}$$

and their anti-holomorphic counterparts:

$$[\bar{L}_{-1}, \bar{L}_1] = -2\bar{L}_0, \qquad [\bar{L}_0, \bar{L}_1] = -\bar{L}_1, \qquad [\bar{L}_0, \bar{L}_{-1}] = \bar{L}_{-1} \tag{1.44}$$

This is indeed the algebra $\mathfrak{sl}(2, \mathbb{C})$ and thus we associate the generators L_{-1}, L_0 and L_1 and their anti-holomorphic counterparts with global conformal transformations.
By comparing (1.43) and (1.44) with the "standard" $D > 2$ CFT generators' commutation relations (1.13 - 1.18), we identify them as follows:

- $L_{-1}, \bar{L}_{-1}$ generate translations
- $L_1, \bar{L}_1$ generate special conformal transformations
- $L_0, \bar{L}_0$ are associated with dilatations and rotations.

By writing the $D > 2$ generators as linear combinations of the $D = 2$ global generators according to the above schematic and forcing them to obey the commutation relations, we can get the following relations between the two:

2D generators relation to $D > 2$ generators

$$L_{-1} = \frac{1}{2}(P_1 - iP_2) \qquad \bar{L}_{-1} = \frac{1}{2}(P_1 + iP_2) \tag{1.45}$$

$$L_0 = \frac{1}{2}(D - iL_{12}) \qquad \bar{L}_0 = \frac{1}{2}(D + iL_{12}) \tag{1.46}$$

$$L_1 = \frac{1}{2}(K_1 + iK_2) \qquad \bar{L}_1 = \frac{1}{2}(K_1 - iK_2) \tag{1.47}$$

With all this discussion in mind, we arrive at the following conclusion that justifies our previous study of the $D > 3$ conformal symmetry:

Theorem 2 *The group theoretic findings on the conformal group in $D > 2$ apply in the two dimensional case* only *for the global subgroup.*

1.4 Primaries, Quasi-Primaries and the Operator Product Expansion

Now that we have grasped the conformal symmetry in two dimensions, we can talk about some general aspects of two dimensional conformal field theories that are of interest in this work.
Lets begin with some important definitions and theorems:

Definition 4 ***(Conformal Dimension)*** *Consider a quantum field $\mathcal{O}_{\Delta,\, s}(x)$. The dilatation weight or conformal dimension, Δ is defined to be equal to the mass dimension of said field. Equivalently, under a re-scaling, $x' = \lambda x$ the field transforms as:*

$$\mathcal{O}'_{\Delta,\, s}(\lambda x) = \lambda^{-\Delta}\mathcal{O}_{\Delta,\, s}(x)$$

Definition 5 ***(Spin)*** *Consider a quantum field $\mathcal{O}_{\Delta,\,s}$. The operator spin, s, is a quantum number that characterizes the transformation properties of said operator under the rotation group. It takes integer and half-integer values. For example a $s = 0$ operator transforms under the trivial representation of the rotation group, a $s = 1/2$ one under the spinor representation, a $s = 1$ one under the vector representation etc.*

These are general aspects of CFTs, regardless of dimension. From now on, we will focus on two dimensional CFTs.

1.4.1 Primaries and Quasi-Primaries

Consider a field $\varphi(z, \bar{z})$ with dilatation weight Δ and spin s. We define the *holomorphic conformal dimension* h and its antiholomorphic counterpart as:

$$h = \frac{\Delta + s}{2} \qquad \bar{h} = \frac{\Delta - s}{2} \tag{1.48}$$

which in turn implies that:

$$\Delta = h + \bar{h}, \qquad s = h - \bar{h} \tag{1.49}$$

Definition 6 ***(Primary Field)*** *A field φ of conformal dimensions $(h, \bar{h})$ that, under* any *conformal map $z \to \zeta(z)$, $\bar{z} \to \bar{\zeta}(\bar{z})$, transforms as:*

$$\varphi'\left(\zeta(z),\ \bar{\zeta}(\bar{z})\right) = \left(\frac{d\zeta}{dz}\right)^{-h} \left(\frac{d\bar{\zeta}}{d\bar{z}}\right)^{-\bar{h}} \varphi(z, \bar{z}) \tag{1.50}$$

is called a primary*.*

Note that for an infinitesimal conformal transformation, $\zeta = z + \epsilon$, $\bar{\zeta} = \bar{z} + \bar{\epsilon}$, the variation of a primary field is:

$$\delta_{\epsilon,\bar{\epsilon}}\varphi = \varphi'(z, \bar{z}) - \varphi(z, \bar{z}) = -(h\varphi\partial_z\epsilon + \epsilon\partial_z\varphi) - (\bar{h}\varphi\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\varphi) \tag{1.51}$$

Definition 7 ***(Quasi-Primary Field)*** *If said field transforms as in (1.50) and (1.51)* at least *under the Mobius global subgroup then it is called* quasi-primary*.*

It is a direct result of the above definitions that all primary fields are also quasi-primary but the reverse is not true. Note that if $D > 2$ primaries and quasi-primaries are the same thing; the distinction is a two dimensional CFT particularity.

1.4.2 Unitarity Bounds

Unitarity bounds on CFTs arise from the requirement that the theory possesses a positive definite inner product on its Hilbert space, which guarantees a positive norm for physical states.

Theorem 3 ***(Unitarity Bounds)*** *For a CFT in D dimensions, the unitarity bounds on the scaling dimensions Δ of primary operators $\mathcal{O}$ can be stated as:*

$$\begin{gathered}\Delta = 0 \qquad \mathcal{O} = \mathbb{1} \\ \Delta \geq \begin{cases} \frac{D-2}{2} & s = 0 \\ D + s - 2 & s > 0 \end{cases}\end{gathered} \tag{1.52}$$

Note that in the case of a two dimensional CFT, for a non-identity operator, (1.52) is simply:

$$\Delta \geq s \tag{1.53}$$

1.4.3 Ward Identities and Correlation Functions

Ward identities are the quantum counterpart to Noether's theorem in classical physics. In 2D CFTs they relate the behavior of correlation functions under infinitesimal conformal transformations.
Consider an infinitesimal transformation $z \to z + \epsilon(z)$, $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$, a string of fields X and their correlation function $\langle X \rangle$. The 2D conformal ward identity has the form:

$$\delta_{\epsilon,\bar{\epsilon}}\langle X \rangle = -\frac{1}{2\pi i}\oint_C dz\, \epsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C d\bar{z}\, \bar{\epsilon}(\bar{z})\langle \bar{T}(\bar{z})X\rangle \tag{1.54}$$

Where is counterclockwise and it needs to only contain the positions of operators in X.
According to the *conformal ward identity*, some complex analysis and the charge definition (1.35):

$$\delta_\epsilon \phi = -[Q_\epsilon, \phi] \tag{1.55}$$

This means that Q_ϵ is a generator of conformal transformations. A result we had previously found when we studied the Virasoro algebra.
Suppose now that ϵ is an infinitesimal Möbius transformation and that the fields X are quasi-primary. In that case, since $SL(2,\mathbb{C})$ is a true symmetry of 2D CFTs the variation of a correlation function must vanish; i.e. $\delta_\epsilon\langle X \rangle = 0$. (We only consider the holomorphic part for brevity as the extension to the anti-holomorphic part is trivial). In this case the integrals on (1.54) can be computed using the residue theorem yielding:

$$\delta_\epsilon \langle X \rangle = -\sum_i (\epsilon(z_i)\partial_i + \partial_i \epsilon(z_i) h_i \langle X \rangle = 0 \tag{1.56}$$

The constraint imposed by equation (1.56) fixes the form of correlation functions of Quasi-primary fields (and therefore also primaries):

- The two-point correlation function is:

$$\langle \varphi_i(z,\bar{z})\varphi_j(w,\bar{w})\rangle = \begin{cases} \frac{g_{ij}}{(z-w)^{2h}(z-w)^{2\bar{h}}} & \text{if } h_i = h_j = h \text{ and } \bar{h}_i = \bar{h}_j = \bar{h} \\ 0 & \text{else} \end{cases} \tag{1.57}$$

  Where g_{ij} is a constant. For scalar fields the operators spectrum can be diagonalized such that $g_{ij} = \delta_{ij}$.

- The three-point function is:

$$\langle \varphi_i(z_i,\bar{z}_i)\varphi_j(z_j,\bar{z}_j)\varphi_k(z_k,\bar{z}_k)\rangle =$$
$$f_{ijk} \cdot \frac{1}{z_{ij}^{h_i+h_j-h_k} z_{jk}^{h_j+h_k-h_i} z_{ik}^{h_i+h_k-h_j}} \frac{1}{\bar{z}_{ij}^{\bar{h}_i+\bar{h}_j-\bar{h}_k} \bar{z}_{jk}^{\bar{h}_j+\bar{h}_k-\bar{h}_i} \bar{z}_{ik}^{\bar{h}_i+\bar{h}_k-\bar{h}_j}} \tag{1.58}$$

  where $z_{ij} = |z_i - z_j|$.

- The four-point function has the form:

$$\langle \varphi_1(z_1,\bar{z_1})...\varphi_4(z_4,\bar{z_4})\rangle = g(\zeta,\bar{\zeta}) \prod_{i<j}^{4} z_{ij}^{h/3-h_i-h_j} \bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j} \tag{1.59}$$

  where $\zeta = \frac{z_{12}z_{34}}{z_{13}z_{24}}$ is a quantity called *cross ratio* and $h = \sum_{i=1}^4 h_i$, $\bar{h} = \sum_{i=1}^4 \bar{h}_i$. The function $g(\zeta,\bar{\zeta})$ is an arbitrary, theory dependent, function of the cross ratios.

1.4.4 Operator Product Expansion

Lets now turn our eye on a very important tool in dealing with CFTs.

Theorem 4 ***(Operator Product Expansion)***
Given two operators $\mathcal{O}_i(z), \mathcal{O}_j(w)$ there exists a converging expansion of the form:

$$\mathcal{O}_i^\alpha(z)\mathcal{O}_j^\beta(w) = \sum_{Quasi-primaries} C_{ij}^k \hat{f}_{ij\,\gamma}^{k\,\alpha\beta}(z-w,\partial_w)\mathcal{O}_k^\gamma(w) \tag{1.60}$$

where α, β, γ are spin indices (in the appropriate representation for each operator). The OPE coefficients C_{ij}^{k} are theory dependent c-numbers that are closely connected to the three-point function coefficients and $\hat{f}_{ij}^{k}$ are differential operators fixed by conformal symmetry. [8]

Equation (1.60) is to be understood in the context of a correlation function. A very crucial property of the operator product expansion is that it is associative.
A useful example of the OPE is between the energy-momentum tensor and a *primary* field, that can be obtained from the ward identities:

$$T(z)\varphi(w,\bar{w}) = h\frac{\varphi(w,\bar{w})}{(z-w)^2} + \frac{\partial_w \varphi(w,\bar{z})}{z-w} + \text{"}non-singular\ terms\text{"} \quad (1.61)$$

$$\bar{T}(\bar{z})\varphi(w,\bar{w}) = \bar{h}\frac{\varphi(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial_{\bar{w}} \varphi(w,\bar{w})}{\bar{z}-\bar{w}} + \text{"}non-singular\ terms\text{"} \quad (1.62)$$

equation (1.61) (or equivalently (1.62)) can be used (and are used by various authors) as a definition of a primary field.
Another interesting OPE is that of the energy momentum tensor with itself:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{"}non-singular\ terms\text{"} \quad (1.63)$$

where c is the theory dependent central charge that appears in the Virasoro algebra.
Comparing the OPE (1.63) with (1.61) we can come to the conclusion that the energy-momentum tensor is *not* a primary operator.
Let us now make here a terminology definition:

Definition 8 ***(CFT Data)*** *The complete set of quasi-primary operators $\{\mathcal{O}_i\}$ and their OPE coefficients are called CFT data. Given the CFT data, one can compute* any *correlation function and thus has solved the theory.*

1.4.5 Hilbert Space and Conformal Families

We define the vacuum by its property to be invariant under global transformations. This can be recovered and extended to a statement about local transformations by the natural argument that $T(z)|0\rangle$ and $\bar{T}(\bar{z})|0\rangle$ are well behaved at the origin. This requirement combined with (1.36) yields:

$$L_n|0\rangle = 0 \qquad \bar{L}_n|0\rangle = 0 \qquad (n \geq -1) \quad (1.64)$$

Now that we have defined the foundation of our Hilbert space, lets introduce a very useful tool in dealing with CFT's states:

Theorem 5 ***(State-Operator Correspondence)*** *Consider a* primary *field $\varphi(z,\bar{z})$ of conformal dimensions $(h,\bar{h})$. For each such operator we can define a state, dual to it, as:*

$$\varphi \longleftrightarrow \left|h,\bar{h}\right\rangle = \varphi(0,0)\left|0\right\rangle \tag{1.65}$$

This is called the state-operator correspondence.

We can see the action of L_n and $\bar{L}_n$ on $\left|h,\bar{h}\right\rangle$ by finding its commutator with φ. Using (1.36) and the OPE (1.61), (1.62) we calculate:

$$[L_n,\varphi(w,\bar{w})] = h(n+1)w^n\varphi(w,\bar{w}) + w^{n+1}\partial\varphi(w,\bar{w}), \qquad (n \geq -1) \tag{1.66}$$

$$[\bar{L}_n,\varphi(w,\bar{w})] = \bar{h}(n+1)\bar{w}^n\varphi(w,\bar{w}) + \bar{w}^{n+1}\bar{\partial}\varphi(w,\bar{w}), \qquad (n \geq -1) \tag{1.67}$$

This in turn yields:

$$L_0\left|h,\bar{h}\right\rangle = h\left|h,\bar{h}\right\rangle \qquad \bar{L_0}\left|h,\bar{h}\right\rangle = \bar{h}\left|h,\bar{h}\right\rangle \tag{1.68}$$

and

$$L_n\left|h,\bar{h}\right\rangle = 0, \qquad \bar{L}_n\left|h,\bar{h}\right\rangle, \qquad (n>0)$$

The above, motivate an alternative (and more useful in the context of our discussion) definition of *primaries* and *quasi-primaries*:

Definition 9 ***(Primary Field)***
A field $\varphi(z,\bar{z})$ with conformal weights $(h,\bar{h})$ is called a primary *if:*

$$L_n\left|h,\bar{h}\right\rangle = 0, \qquad \bar{L}_n\left|h,\bar{h}\right\rangle = 0, \qquad \forall\, n>0 \tag{1.69}$$

Definition 10 ***(Quasi-Primary Field)***
A field $\varphi(z,\bar{z})$ with conformal weights $(h,\bar{h})$ is called a quasi-primary *if* at least*:*

$$L_1\left|h,\bar{h}\right\rangle = 0, \qquad \bar{L}_1\left|h,\bar{h}\right\rangle = 0 \tag{1.70}$$

Equation (1.68) shows that $L_0, \bar{L}_0$ are diagonal. It is therefore convenient to use their eigenstates as a basis for our Hilbert space. The commutator:

$$[L_0,L_n] = -nL_n \qquad [\bar{L}_0,\bar{L}_n] = -n\bar{L}_n \tag{1.71}$$

and so $L_n, \bar{L}_n$ act as raising (for $n<0$) and lowering (for $n>0$) operators for h and $\bar{h}$. i.e.:

$$L_m\left|h,\bar{h}\right\rangle \propto \left|h-m,\bar{h}\right\rangle \qquad L_{-m}\left|h,\bar{h}\right\rangle \propto \left|h+m,\bar{h}\right\rangle \qquad (m>0) \tag{1.72}$$

with the trivial extension for $\bar{L}_m$.
To complete our discussion on the Hilbert space, we define two more useful definitions:

Definition 11 ***(Descendant States)*** *If φ is a* primary *field with corresponding state $\left|h, \bar{h}\right\rangle$, states of the form $L_{-k_N}...L_{-k_2}L_{-k_1}\left|h, \bar{h}\right\rangle$ are called* descendant *states of $\varphi(z, \bar{z})$. Such a state has holomorphic conformal weight $h + \sum_{i=1}^{N} k_i$.*

Definition 12 ***(Conformal Families)*** *The set consisting of the primary φ and all its descendants is called a* conformal family*. Members of a conformal family transform to linear combinations of themselves.*

In this work, we will explore the Hilbert space only with the $SL(2, \mathbb{C})$ subgroup representations since we want our methods to be easily generalizable on higher dimensions.

1.5 Example: Analytic solution of S^1 theory

1.5.1 Introduction

Now that we have familiarized ourselves with CFT, we can analyze a specific theory and see the concepts discussed before in action. It is this theory that will be our toy model throughout this work.
Consider a scalar field X with a two dimensional world-sheet and a circular target space of radius R. Such a system is described by the action and boundary condition:

$$S = \frac{1}{4\pi}\int d^2z \partial X(z, \bar{z})\bar{\partial}X(z, \bar{z}), \qquad X = X + 2\pi R \tag{1.73}$$

This action, solving the classic euler Lagrange equations, yields the field's equation of motion:

$$\partial\bar{\partial}X(z, \bar{z}) = 0 \tag{1.74}$$

Which is nothing more than the very familiar wave equation.
The two-point correlation function or *propagator* is by construction the Green's function of the differential operator $\partial\bar{\partial}$. Or explicitly:

$$\partial\bar{\partial}\left\langle X(z, \bar{z})X(w, \bar{w})\right\rangle = \pi\delta(z - w)\delta(\bar{z} - \bar{w}) \tag{1.75}$$

Solving equation

$$\left\langle X(z, \bar{z})X(w, \bar{w})\right\rangle = -[ln(z - w) + ln(\bar{z} - \bar{w})] + const \tag{1.76}$$

Comparing this to the two point-function of a primary field (1.57), we deduce that X *is not* a primary (or quasi-primary) field. In fact X is not even a well defined field (from the physical perspective) since the correlator (1.76) diverges at large distances.

Using the Noether prescription on the theory's Lagrangian and (1.34), one can obtain the theory's energy momentum tensor. The energy momentum is:

$$T(z) = -\frac{1}{2} : \partial X \partial X : \tag{1.77}$$

where the $:\psi:$ notation denotes normal ordering, i.e. all annihilation operators to the right.

1.5.2 Primaries of the theory

By calculating the OPE with the energy momentum tensor using wick's theorem and comparing it to the conformal primaries defining OPE (1.61), we can see that in this theory, the basic primaries are the U(1) conserved currents:

$$j(z) = \frac{i}{2}\partial X(z), \qquad \bar{j}(\bar{z}) = \frac{i}{2}\bar{\partial} X(\bar{z}) \tag{1.78}$$

with conformal weights $(1,0)$ and $(0,1)$ respectively.
and the *vertex* operators:

$$V_{p,\bar{p}} =: e^{ipX(z)+i\bar{p}\bar{X}(\bar{z})} :, \qquad p = \frac{n}{R} + \frac{wR}{2},\ \bar{p} = \frac{n}{R} - \frac{wR}{2} \tag{1.79}$$

where $X(z)$ and $X(\bar{z})$ denote the holomorphic and anti-holomorphic parts of $X(z,\bar{z})$, n is the *integer momentum*, w is the *winding number*.
The charges p and $\bar{p}$ are called left (or holomorphic) and right (or anti-holomorphic) momentums.
The conformal scaling dimensions of a vertex operator are: $(h,\bar{h}) = (\frac{p^2}{2}, \frac{\bar{p}^2}{2})$.
By (1.48) the dilatation weight and spin of the vertex operators is:

$$\Delta = \frac{n^2}{R^2} + \frac{w^2R^2}{4} \qquad s = nw \tag{1.80}$$

As equation (1.80) suggests, a vertex operator without integer momentum *or* winding is spin-less.

1.5.3 Operator Product Expansions

To further explore the theory, we shall first write down some OPEs of the theory. All the results below are calculated using Wick's theorem and/or derivatives of (1.76).

- As stated before, ∂X is a primary and thus, it's 2 point correlation function is fixed by (1.57). The constant g_{ij} is fixed to be -1 to be consistent with the result we would obtain by differentiating (1.76) twice. That is:

$$\partial X(z)\partial X(w) = -\frac{1}{(z-w)^2} + \text{"}r.t.\text{"} \tag{1.81}$$

where "r.t." stands for "regular terms".

- By directly differentiating the above expression with respect to w, n times, we obtain the more general:

$$\partial X(z)\partial^n X(w) = -\frac{n!}{(z-w)^{n+1}} + \text{"}r.t.\text{"} \tag{1.82}$$

- The OPE of ∂X with a vertex operator e^{ipX}, given by Wick's theorem and (1.82) :

$$\partial X(z)e^{ipX(w)} = -ip\frac{e^{ipX}}{z-w} + \text{"}r.t.\text{"} \tag{1.83}$$

- The OPE with the energy momentum tensor with ∂X:

$$T(z)\partial X(w) = \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w} + \text{"}r.t.\text{"} \tag{1.84}$$

Note, again that the ∂X operator has the primary form OPE with the energy momentum tensor first introduced in (1.61)

- The OPE of the energy momentum tensor with the vertex operator e^{ipX}, too has that exact form:

$$T(z)e^{ipX(w)} = \frac{p^2}{2}\frac{e^{ipX(w)}}{(z-w)^2} + \frac{\partial e^{ipX(w)}}{z-w} + \text{"}r.t.\text{"} \tag{1.85}$$

1.5.4 Quasi-Primaries

We want to organise our spectrum of operators using the global, $\mathfrak{sl}(2,\mathbb{C})$, part of the 2D conformal algebra. To do this, we must be able to calculate the quasi-primary operators of the theory.
A (holomorphic) quasi-primary state is defined by its property to be annihilated by L_1. That is:

$$L_1 |\phi\rangle = 0 \Rightarrow \frac{1}{2\pi i}\oint dz\, z^2 T(z)\phi(0)|0\rangle = 0 \tag{1.86}$$

For (1.86) to hold true the OPE $T(z)\phi(0)$ must have no z^{-3} pole. That is because $res(\frac{1}{z^n}, 0) = \begin{cases} 0 \ (n>1) \\ 1 \ (n=1) \end{cases}$.
The general operator with holomorphic momentum p has the form:

$$\mathcal{O}_{m_1,\cdots,m_r;p} = \prod_{\alpha=1}^{r} (\partial^\alpha X)^{m_\alpha} e^{ipX} \tag{1.87}$$

A generic holomorphic quasi-primary is a linear combination of operators of the form (1.87) with the same conformal weights. The coefficients can then be fixed by demanding that the z^{-3} pole of its OPE, with the energy-momentum tensor, vanishes. For operators of the form (1.87), the z^{-3} pole is given by:

$$[T(z)\mathcal{O}_{m_1,\cdots,m_r;p}(0)]_3 =$$
$$\sum_{\alpha=1}^{r} [\alpha(\alpha-1)\mathcal{O}_{m_1,\cdots,m_{\alpha-1}+1,m_\alpha-1,m_{\alpha+1},\cdots,m_r;p}(0)] - ipm_1\mathcal{O}_{m_1-1,m_2,\cdots,m_r;p}(0) \tag{1.88}$$

Equations (1.87) and (1.88) can be used to calculate any right-moving (holomorphic) quasi-primary. For example the first 2 are:

$$\mathcal{Q}_2 = [(\partial X)^2 + ip\partial^2 X]e^{ipX},\ \mathcal{Q}_3 = [(\partial X)^3 + \frac{3ip}{2}\partial X\partial^2 X - \frac{p^2}{4}\partial^3 X]e^{ipX}$$

The full set of quasi-primaries of the theory can be obtained by combining any quasi-primary operator from the left-moving (holomorphic) sector with any quasi-primary operator from the right-moving (antiholomorphic) sector. There are no factors with mixed holomorphic-antiholomorphic derivatives as the equation of motion is $\partial\bar{\partial}X(z,\bar{z}) = 0$.
This means that a general quasi-primary is a linear combination of operators of the form:

$$\mathcal{O}_{\{m_\alpha\},\{\bar{m}_{\bar{\alpha}}\};p,\bar{p}} = \prod_{\alpha=1}^{r} (\partial^\alpha X)^{m_\alpha} \prod_{\bar{\alpha}=1}^{\bar{r}} (\bar{\partial}^{\bar{\alpha}}\bar{X})^{\bar{m}_{\bar{\alpha}}} e^{ipX+i\bar{p}\bar{X}} \tag{1.89}$$

The conformal dimensions of these operators are:

$$h = \ell + \frac{p^2}{2}, \qquad \bar{h} = \bar{\ell} + \frac{\bar{p}^2}{2} \tag{1.90}$$

where $\ell = \sum_{\alpha=1}^{r} \alpha m_\alpha$, $\bar{\ell} = \sum_{\bar{\alpha}=1}^{\bar{r}} \bar{\alpha}\bar{m}_{\bar{\alpha}}$ is the "level" of the operator i.e. the number of holomorphic or anti-holomorphic derivatives.

Chapter 2

Crossing Equations and U(1) Global Symmetry

In this work, we will calculate CFT data by exploiting the symmetries of four-point functions of the form $\left\langle \mathcal{O}\mathcal{O}\bar{\mathcal{O}}\bar{\mathcal{O}}\right\rangle$ with U(1) global symmetry where $\mathcal{O}$ is assumed to be a scalar primary operator of conformal dimensions $(h, \bar{h})$. These objects are by construction constrained by both OPE associativity and the U(1) symmetry. In this chapter we explore the details of these constrains.

2.1 Crossing Symmetry

Consider a four-point correlation function:

$$\langle \mathcal{O}_1(z_1, \bar{z}_1)\mathcal{O}_2(z_2, \bar{z}_2)\mathcal{O}_3(z_3, \bar{z}_3)\mathcal{O}_4(z_4, \bar{z}_4)\rangle$$

We can reduce this correlation function to a two-point function using the OPE (1.60) twice and then calculate it through (1.57). As stated in theorem 4, the operator product expansion is *associative*. This means that no matter what pair of operators we chose to do the first OPE with, the result has to be the same.
The OPE can be implemented with 3 possible combinations; the s-channel (12)(34), the t-channel (14)(23) and the s-channel (13)(24). These three channels must agree with each other and thus, yield two constraints on the CFT data. This symmetry is called *crossing symmetry* and the constraints it yields are called *crossing equations*.
For example, consider the s-channel OPE of said four point function:

$$\langle \mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\mathcal{O}_3(z_3,\bar{z}_3)\mathcal{O}_4(z_4,\bar{z}_4)\rangle = \frac{1}{z_{12}^{h_1+h_2} z_{34}^{h_3+h_4}} \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2} \bar{z}_{34}^{\bar{h}_3+\bar{h}_4}}$$
$$\times \left(\frac{z_{24}}{z_{14}}\right)^{h_{12}} \left(\frac{\bar{z}_{24}}{\bar{z}_{14}}\right)^{\bar{h}_{12}} \left(\frac{z_{14}}{z_{13}}\right)^{h_{34}} \left(\frac{\bar{z}_{14}}{\bar{z}_{13}}\right)^{\bar{h}_{34}} \sum_{\mathcal{O}_i\mathcal{O}_j} C_{12}^{O_i} g_{\mathcal{O}_i\mathcal{O}_j} C_{34}^{\mathcal{O}_j} g^{1234}_{h_i,\bar{h}_i}(z,\bar{z}) \tag{2.1}$$

where:

- $z_{ij} = z_i - z_j$
- $h_{ij} = h_i - h_j$
- the ratios

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad 1-z = \frac{z_{14}z_{23}}{z_{13}z_{24}}, \quad \frac{z}{1-z} = \frac{z_{12}z_{34}}{z_{14}z_{23}} \tag{2.2}$$

$$\bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}, \quad 1-\bar{z} = \frac{\bar{z}_{14}\bar{z}_{23}}{\bar{z}_{13}\bar{z}_{24}}, \quad \frac{\bar{z}}{1-\bar{z}} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{14}\bar{z}_{23}} \tag{2.3}$$

  are quantities called the conformal cross-ratios.

- C_{ij}^k are the OPE coefficients.
- $g_{\mathcal{O}_i,\mathcal{O}_j}$ is the two point function coefficient.
- $g^{1234}_{h,\bar{h}}(z,\bar{z})$ are functions called conformal blocks.

The conformal blocks are functions of the cross ratios that are completely fixed by conformal symmetry, i.e. they are theory independent.
In two dimensions their analytic expression is:

$$g^{1234}_{h,\bar{h}}(z,\bar{z}) = z^h \bar{z}^{\bar{h}} \cdot {}_2F_1(h-h_{12}, h+h_{34}; 2h; z) \cdot {}_2F_1(\bar{h}-\bar{h}_{12}, \bar{h}+\bar{h}_{34}; 2\bar{h}; \bar{z}) \tag{2.4}$$

where ${}_2F_1(a,b;c;z)$ is the ordinary hyper-geometric function.
The t-channel can be obtained from the s-channel by exchanging the insertions $1 \leftrightarrow 3$ and the cross-ratios $z \leftrightarrow 1-z$ and $\bar{z} \leftrightarrow 1-\bar{z}$.
To further suppress notation we set:

$${}_s\mathfrak{C}_{h,\bar{h}} = \sum_{\mathcal{O},\mathcal{O}'|\Delta_{\mathcal{O}}=\Delta_{\mathcal{O}'}=h+\bar{h}} C_{12}^{\mathcal{O}} g_{\mathcal{O}\mathcal{O}'} C_{34}^{\mathcal{O}} \tag{2.5}$$

By equating the s-channel and t-channel expansions, we arrive at a constraint called the 1st crossing equation:

$$
\begin{aligned}
&\sum_{h,\bar{h}} {}_s\mathfrak{C}_{h,\bar{h}} g^{(1234)}_{h,\bar{h}}(z,\bar{z}) = \\
&(-1)^{(h_{41}+h\bar{_4}1)} \frac{z^{h_1+h_2}}{(z-1)^{h_2+h_3}} \frac{\bar{z}^{\bar{h}_1+\bar{h}_2}}{(\bar{z}-1)^{\bar{h}_2+\bar{h}_3}} \sum_{h',\bar{h}'} {}_t\mathfrak{C}_{h',\bar{h}'} g^{(3214)}_{h',\bar{h}'}(1-z,1-\bar{z})
\end{aligned}
\tag{2.6}
$$

It is appropriate to note here that the sums, range across all quasi-primaries $(h,\bar{h})$ that appear on each channel. For each spinning operator that appears in the expansion the "exchanged weights" version of it appears, i.e. $(\bar{h},h)$. In general, $\mathfrak{C}_{h,\bar{h}} \neq \mathfrak{C}_{\bar{h},h}$.
However, when the external operators (i.e. the ones in the four-point function) are spin-less the coefficients When the external operators are spin-less, the OPE-squared coefficients are equal, ${}_s\mathfrak{C}_{h,\bar{h}} = {}_s\mathfrak{C}_{\bar{h},h}$, and we can collect together the $(h,\bar{h})$ and $(\bar{h},h)$ contributions to form a single conformal block of the form:

$$
\tilde{g}^{(1234)}_{h,\bar{h}}(z,\bar{z}) = \frac{1}{1+\delta_{h,\bar{h}}} \left(g^{(1234)}_{h,\bar{h}}(z,\bar{z}) + g^{(1234)}_{h,\bar{h}}(\bar{z},z) \right) \tag{2.7}
$$

In this manner, we can restrict the sums in (2.1) to only run over operators with $h \geq \bar{h}$, hence reducing by half the number of intermediate quasi-primary operators that we need to consider.
In our application we will, in fact, use spin-less external operators and thus use the above simplification.

2.2 Crossing Equations with U(1) Global Symmetry.

Consider now a four-point function of the form:

$$
\langle \mathcal{O}\mathcal{O}\bar{\mathcal{O}}\bar{\mathcal{O}} \rangle \tag{2.8}
$$

where $\mathcal{O}$ is a scalar primary operator.
Along with the crossing symmetry, the CFT data can be further bound by the existance of a global U(1) symmetry of such a four point function. The non-vanishing correlators must have zero total U(1) charge, p.
The two basic OPEs are:

$$
\mathcal{O} \times \bar{\mathcal{O}} = \mathbb{1} + \textit{spins } 0,1,2,\cdots \text{ (Charge 0 sector)}
$$

$$
\mathcal{O} \times \mathcal{O} = \textit{spins } 0,2,4,\cdots \text{ (Charge 2 sector)}
$$

[12]
where we have normalized the U(1) charge of $\mathcal{O}$,p , to 1.

With the above, we are ready to write down the stronger crossing equations that emerge from combining the two symmetries.
We consider a primary field $\mathcal{O}$ with conformal dimensions $(h, \bar{h})$ and the correlation function $\langle \mathcal{O}(z_1, \bar{z}_1)\mathcal{O}(z_2, \bar{z}_2)\bar{\mathcal{O}}(z_3, \bar{z}_3)\bar{\mathcal{O}}(z_4, \bar{z}_4)\rangle$. For this setup, the first crossing equation can be easily obtained from (2.6) for $h_i = h_j, \ \ \forall i, j$ and reducing the operators in the sums to the ones allowed by U(1) symmetry:

1st Crossing Equation with U(1) symmetry

$$\begin{aligned} |z-1|^{2\Delta_{\mathcal{O}}} \sum_{\substack{h \geq \bar{h} \\ h-\bar{h}=even}} {}_s\mathfrak{C}_{h,\bar{h}} \tilde{g}^{\mathcal{O}\mathcal{O}\bar{\mathcal{O}}\bar{\mathcal{O}}}_{h,\bar{h}}(z,\bar{z}) = \\ |z|^{2\Delta_{\mathcal{O}}}[1 + \sum_{\substack{h' \geq \bar{h}' \\ h,\bar{h} \in \mathbb{N}}} {}_t\mathfrak{C}_{h',\bar{h}'} \tilde{g}^{\bar{\mathcal{O}}\mathcal{O}\mathcal{O}\bar{\mathcal{O}}}_{h',\bar{h}'}(1-z, 1-\bar{z})] \end{aligned} \tag{2.9}$$

To get the second equation, the most intuitive to do would be to interchange $2 \longleftrightarrow 3$ in (2.1). However, this results in a $1/z$ argument in the confromal blocks that makes the expressions ill defined at the crossing symmetric point ($z = \bar{z} = 1/2$) due to the hypergeometric function's branch cut from 1 to infinity on the real axis.
The simplest way to obtain a well behaved equation is to consider the four-point function:

$$\langle \mathcal{O}\bar{\mathcal{O}}\mathcal{O}\bar{\mathcal{O}} \rangle$$

and take again the s and t channels equation. [10]

2nd Crossing Equation with U(1) symmetry

$$\begin{aligned} |z-1|^{2\Delta_{\mathcal{O}}}[1 + \sum_{\substack{h \geq \bar{h} \\ h,\bar{h} \in \mathbb{N}}} (-1)^{h-\bar{h}} {}_t\mathfrak{C}_{h,\bar{h}} g^{\mathcal{O}\bar{\mathcal{O}}\mathcal{O}\bar{\mathcal{O}}}_{h,\bar{h}}(z,\bar{z}) = \\ |z|^{2\Delta_{\mathcal{O}}}[1 + \sum_{\substack{h \geq \bar{h} \\ h,\bar{h} \in \mathbb{N}}} (-1)^{h-\bar{h}} {}_t\mathfrak{C}_{h,\bar{h}} g^{\mathcal{O}\bar{\mathcal{O}}\mathcal{O}\bar{\mathcal{O}}}_{h,\bar{h}}(1-z, 1-\bar{z})] \end{aligned} \tag{2.10}$$

On both cases we have separated the contribution of the identity, whenever it exists and assumed that $\langle \mathcal{O}\bar{\mathcal{O}} \rangle = 1$.
Theoretically, if one solved these equations, one has solved the theory.

Chapter 3

Spin-Partitions and Truncations

Since our goal is to solve the crossing equations, we may now star wondering how. As is obvious, an analytic approach is out of the question and thus we want to implement computational methods. This brings into the discussion the machine learning algorithms discussed on the introduction. Before talking about the specifics of these, we must first reformulate the problem in a computer-friendly manner.

The crossing equations (2.9)(2.10) are non-linear equations for the unknown positive (to obey unitarity bounds) conformal scaling dimensions $\Delta = h + \bar{h}$ and the corresponding OPE squared coefficients $\mathfrak{C}_{h,\bar{h}}$ in all channels.

For numerical methods the appearance of a typically infinite number of contributions to the conformal-block expansion makes the use of these equations impossible as they are. We address this problem by truncating the spectrum of intermediate quasi-primary operators, by setting some upper limit Δ_{max} on the scaling dimensions. The exponential convergence properties of the OPE expansion implies that good numerical results can be obtained without considering a very large Δ_{max}. Our approach on truncation is a bit different than the standard school of thought on the subject as will be presented in detail in a while. The difference is that instead of completely ignoring the contribution of high Δ operators we have a way of approximating their effect.

Unitarity bounds in 2D suggest that $\Delta_{max} \geq \Delta \geq s$ and thus the truncation method puts a constraint on the allowed spins on the crossing equations. Having in mind numerical methods, even with that constraint the problem is not solvable. The problem is that there is still a vast space of possibilities that an algorithm can explore associated with the freedom to choose any number of quasi-primaries at each spin. This final issue can be fixed by introducing a spin-partition. The spin-partition is a sequence of positive integers that specifies the number of quasi-primaries per spin contributing

to the conformal-block expansions of the truncated crossing equations.
Since spin partitions and truncation are very theory and setup specific, we will make the discussion using the S_1 theory we solved at section 1.5. To be able to decide for a spin partition for the theory, we must first calculate analytically its CFT Data. Obtaining the data, except from inspiring the algorithm's setup, will serve as a measure of accuracy for our numerical approach.

3.1 Classification of Quasi-Primaries in S_1 theory

To get a feeling of the above discussion we will classify, in terms of Δ and s, some quasi-primary operators of our "example at hand" theory; the $c = 1$ compactified boson. For reasons discussed on chapter 2 we need only consider cases where $h \geq \bar{h}$ and make a distinction between charge 0 and charge 2 sectors.
To be specific , we will specify the external operators we will consider in the four-point function of (2.8). The operators are the vertex operators:

$$\mathcal{O}_p = V_{p,p}(z, \bar{z}) = e^{ip[X(z)+\bar{X}(\bar{z})]}, \quad p = \bar{p} = \frac{n}{R},\ w = 0,\ \Delta_p = p^2 \tag{3.1}$$

3.1.1 Charge 0 Sector

Let us begin for simplicity with the U(1) charge, $p = 0$ sector. In the context of the previous chapter, these are t-channel operators. Using equations (1.87) and (1.88) we calculate the purely holomorphic quasi-primaries, i.e. the ones with weights $(h, 0)$.

U(1) charge 0, Holomorphic Operators		
Operator	Scaling Dimension (Δ)	Spin (s)
$\mathbb{1}$	0	0
$\partial X(z)$	1	1
$\partial X(z)^2$	2	2
$\vdots$	$\vdots$	$\vdots$
$\partial X(z)^n$	n	n
$\partial X \partial^3 X - \frac{3}{2}(\partial^2 X)^2$	4	4
$(\partial X)^2 \partial^3 X - \frac{3}{2}\partial X(\partial^2 X)^2$	5	5
$(\partial X)^3 \partial^3 X - \frac{3}{2}(\partial X)^2(\partial^2 X)^2$	6	6
$\vdots$	$\vdots$	$\vdots$
$(\partial X)^n \partial^3 X - \frac{3}{2}(\partial X)^{n-1}(\partial^2 X)^2$	$n+3$	$n+3$
$\partial X \partial^5 X - 10\partial^2 X \partial^4 X + 10(\partial^3 X)^2 \bar{\partial} X$	6	6
$\vdots$	$\vdots$	$\vdots$
$(\partial X)^n \partial^5 X - 10(\partial X)^{n-1}\partial^2 X \partial^4 X +$ $10(\partial X)^{n-1}(\partial^3 X)^2$	$n+7$	$n+7$
$\vdots$	$\vdots$	$\vdots$

Table 3.1: Holomorphic Quasi-Primaries with $p = 0$

Note that the above operators all have $s = \Delta$ since $\bar{h} = 0$. To obtain operators with any (Δ, s), one has to just combine the operators of Table 3.1 with their anti-holomorphic counterparts. For example all the operators with scaling dimension $\Delta \leq 6$ are:

U(1) charge 0 and $\Delta \leq 6$ Operators		
Operator	Scaling Dimension (Δ)	Spin (s)
$\mathbb{1}$	0	0
$\partial X \bar{\partial} X$	2	0
$(\partial X)^2 (\bar{\partial} X)^2$	4	0
$(\partial X)^3 (\bar{\partial} X)^3$	6	0
∂X	1	1
$(\partial X)^2 \bar{\partial} X$	3	1
$(\partial X)^2 \bar{\partial} X$	5	1
$(\partial X)^2$	2	2
$(\partial X)^3 \bar{\partial} X$	4	2
$(\partial X)^4 (\bar{\partial} X)^2$	6	2
$[\partial X \partial^3 X - \frac{3}{2}(\partial^2 X)^2](\bar{\partial} X)^2$	6	2
$(\partial X)^3$	3	3
$(\partial X)^4 \bar{\partial} X$	5	3
$[\partial X \partial^3 X - \frac{3}{2}(\partial^2 X)^2]\bar{\partial} X$	5	3
$(\partial X)^4$	4	4
$\partial X \partial^3 X - \frac{3}{2}(\partial^2 X)^2$	4	4
$(\partial X)^5 \bar{\partial} X$	6	4
$[(\partial X)^2 \partial^3 X - \frac{3}{2}\partial X(\partial^2 X)^2]\bar{\partial} X$	6	4
$(\partial X)^5$	5	5
$(\partial X)^2 \partial^3 X - \frac{3}{2}\partial X(\partial^2 X)^2$	5	5
$(\partial X)^6$	6	6
$(\partial X)^3 \partial^3 X - \frac{3}{2}(\partial X)^2(\partial^2 X)^2$	6	6
$\partial X \partial^5 X - 10\partial^2 X \partial^4 X + 10(\partial^3 X)^2$	6	6

Table 3.2: Quasi-Primaries with $p = 0$ and $\Delta \leq 6$

An important thing to note on Table 3.2 is the appearance of operator degeneracies; i.e. different operators with the same spin and scaling dimension. We know of their existence because we analytically solved the theory through its Lagrangian description. Our algorithms (that *do not* solve the theory in that way) cannot predict their existence and thus, for them, degeneracies are treated as one single effective operator.

3.1.2 Charge 2 Sector

We will now proceed to make a similar analysis for the case where the OPEs of the four point function is taken between same charge operators and thus

yield an expansion of charge $= 2p$ operators, where p is the U(1) charge of the external operators. In the context of the previous chapter, these are the s-channel operators. As expected, this makes operators' expressions more complicated.
To get a general feel, we start again with the holomorphic operators. It is important to note here that all operators have attached the full exponential $e^{2ip[X(z)+\bar{X}(\bar{z})]}$ (and *not* $e^{2ipX(z)}$). As stated in chapter 2 the charge 2 sector expansion does *not* include the identity and thus the minimum anti-holomorphic part an operator can have is $e^{2ip\bar{X}(\bar{z})}$

U(1) charge $= 2p$, minimum anti-holomorphic part Operators		
Operator	Scaling Dimension (Δ)	Spin (s)
$e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$4\Delta_p$	0
$\left((\partial X)^2 + 2ip\partial^2 X\right) e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$2{+}4\Delta_p$	2
$\left((\partial X)^3 + 3ip\partial X\partial^2 X - p^2\partial^3 X\right) e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$3{+}4\Delta_p$	3
$\left((\partial X)^4 + 4ip(\partial X)^2\partial^2 X - 4p^2(\partial^2 X)^2\right) e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$4{+}4\Delta_p$	4
$\left(\partial X\partial^3 X + \frac{ip}{6}\partial^4 X - \frac{3}{2}(\partial^2 X)^2\right) e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$4{+}4\Delta_p$	4
$\vdots$	$\vdots$	$\vdots$

Table 3.3: Quasi-Primaries with charge $= 2p$ and minimum anti-holomorphic part

As stated before and witnessed on Table 3.3 the expressions get very complex on, even as low as, level 4 operators. We also lose the ability to create and rely on general expressions as the one presented on Table 3.1. Just as on the t-channel operators, here $s = \ell$ (on the 0 charge sector $\ell = \Delta$)
In order to get operators of $s < \ell$, just as before, one has to just combine different holomorphic and anti-holomorphic parts of the above operators. For example all the operators with scaling dimension $\Delta \leq 6 + 4\Delta_p$ are:

U(1) charge $2p$ and $\Delta \leq 6+4\Delta_p$ Operators		
Operator	Scaling Dimension (Δ)	Spin (s)
$e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$4\Delta_p$	0
$[(\partial X)^2+2ip\partial^2 X][(\bar{\partial} X)^2+2ip\bar{\partial}^2 X]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$4+4\Delta_p$	0
$[(\partial X)^3+3ip\partial X\partial^2 X-p^2\partial^3 X][(\bar{\partial} X)^3+3ip\bar{\partial} X\bar{\partial}^2 X-p^2\bar{\partial}^3 X]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$6+4\Delta_p$	0
$[(\partial X)^3+3ip\partial X\partial^2 X-p^2\partial^3 X][(\bar{\partial} X)^2+2ip\bar{\partial}^2 X]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$5+4\Delta_p$	1
$[(\partial X)^2+2ip\partial^2 X]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$2+4\Delta_p$	2
$[(\partial X)^4+4ip(\partial X)^2\partial^2 X-4p^2(\partial^2 X)^2][(\bar{\partial} X)^2+2ip\bar{\partial}^2 X]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$6+4\Delta_p$	2
$[\partial X\partial^3 X+\frac{ip}{6}\partial^4 X-\frac{3}{2}(\partial^2 X)^2][(\bar{\partial} X)^2+2ip\bar{\partial}^2 X]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$6+4\Delta_p$	2
$[(\partial X)^3+3ip\partial X\partial^2 X-p^2\partial^3 X]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$3+4\Delta_p$	3
$[(\partial X)^4+4ip(\partial X)^2\partial^2 X-4p^2(\partial^2 X)^2]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$4+4\Delta_p$	4
$[\partial X\partial^3 X+\frac{ip}{6}\partial^4 X-\frac{3}{2}(\partial^2 X)^2]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$4+4\Delta_p$	4
$[(\partial X)^5+5ip(\partial X)^3\partial^2 X-p^2((\partial X)^2\partial^3 X+2ip\partial^2 X\partial^3 X+6\partial X(\partial^2 X)^2)]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$5+4\Delta_p$	5
$[(\partial X)^2\partial^3 X-\frac{3}{2}\partial X(\partial^2 X)^2+\frac{p}{2}(\frac{5i}{6}\partial X\partial^4 X-\frac{1}{24}\partial^5 X-i\partial^2 X\partial^3 X)]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$5+4\Delta_p$	5
$[(\partial X)^6+6ip(\partial X)^4\partial^2 X-12p^2\left((\partial X)^2\left(\partial^2 X\right)^2+\frac{2ip}{3}\left(\partial^2 X\right)^3\right)]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$6+4\Delta_p$	6
$[(\partial X)^3\partial^3 X-\frac{3}{2}\left((\partial X)^2\left(\partial^2 X\right)^2+\frac{2ip}{3}\left(\partial^2 X\right)^3\right)+3ip\left(\partial X\partial^2 X\partial^3 X-\left(\partial^2 X\right)^3+\frac{ip}{6}\left(\partial^3 X\right)^2\right)]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$6+4\Delta_p$	6
$[\partial X\partial^5 X+\frac{ip}{15}\partial^6 X-10\partial^2 X\partial^4 X+10\left(\partial^3 X\right)^2]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$6+4\Delta_p$	6
$[(\partial X)^3\partial^3 X-\frac{3}{2}(\partial X)^2\left(\partial^2 X\right)^2+\frac{3p^2}{2}\left(\partial^3 X\right)^2-\frac{4p^2}{3}\partial^2 X\partial^4 X+\frac{2ip}{3}(\partial X)^2\partial^4 X-ip\partial X\partial^2 X\partial^3 X]e^{2ip[X(z)+\bar{X}(\bar{z})]}$	$6+4\Delta_p$	6

Table 3.4: Quasi-Primaries with charge $= 2p$ and $\Delta \leq 6+4\Delta_p$

Same as before we observe degenerate operators on Table 3.4. As stated in the previous chapter, the $\mathcal{O} \times \mathcal{O}$ OPE *cannot* contain odd spin operators. That means that the OPE squared coefficients of them are expected to be zero.

3.2 OPE squared coefficients of S_1 theory

In the previous sector we developed a systematic way to find the expressions of the operators appearing on each conformal block expansion. That is half the CFT data. The other half is the OPE-squared coefficients. In this sector we analytically calculate them.
As stated before, in order to decide our spin partition and cut-off Δ we need a systematic way to calculate the analytic OPE squared coefficients $\mathfrak{C}_{h,\bar{h}}$. To do this we will use the analytic expression for the 4-point function we are interested in and compare it to its conformal block expansion. From there we will try to read off the coefficients. By calculating them in this way we do not need to worry about degeneracies of operators as all degenerate contributions will appear to be summed into a single term contribution.

3.2.1 Analytic Expressions

The four-point function of vertex operators is:

$$\langle V_{p_1}(z_1, \bar{z}_1) \cdots V_{p_n}(z_n, \bar{z}_n) \rangle = \prod_{i<j} |z_i - z_j|^{2p_i p_j} \tag{3.2}$$

In our case ($n = 4, p_1 = p_2 = -p_3 = -p_4$) this becomes:

$$\left\langle V_p(z_1, \bar{z}_1) V_p(z_2, \bar{z}_2) \bar{V}_p(z_3, \bar{z}_3) \bar{V}_p(z_4, \bar{z}_4) \right\rangle = \left(\frac{|z_{12}|^2 |z_{34}|^2}{|z_{13}|^2 |z_{14}|^2 |z_{23}|^2 |z_{24}|^2} \right)^{\Delta_p} \tag{3.3}$$

where Δ_p is the dilatation weight of the external operators. We now take the s and t conformal block expansions of (3.3).
The s channel expansion given by (2.1) is:

$$\left\langle V_p(z_1, \bar{z}_1) V_p(z_2, \bar{z}_2) \bar{V}_p(z_3, \bar{z}_3) \bar{V}_p(z_4, \bar{z}_4) \right\rangle = \left(\frac{1}{|z_{12}|^2 |z_{34}|^2} \right)^{\Delta_p} \sum_{h,\bar{h}} {}_s\mathfrak{C}_{h,\bar{h}} g_{h,\bar{h}}^{(VV\bar{V}\bar{V})}(z, \bar{z}) \tag{3.4}$$

The t channel expansion is obtained by exchanging $1 \leftrightarrow 3$ and $z \leftrightarrow 1 - z$

(and $\bar{z} \leftrightarrow 1-\bar{z}$):

$$\left\langle V_p\left(z_1, \bar{z}_1\right) V_p\left(z_2, \bar{z}_2\right) \bar{V}_p\left(z_3, \bar{z}_3\right) \bar{V}_p\left(z_4, \bar{z}_4\right)\right\rangle = \left(\frac{1}{|z_{32}|^2|z_{14}|^2}\right)^{\Delta_p} \sum_{h,\bar{h}} {}_t\mathfrak{C}_{h,\bar{h}} g_{h,\bar{h}}^{(\bar{V}VV\bar{V})}(1-z, 1-\bar{z}) \tag{3.5}$$

Equating (3.3) with the s and t expansions we get:

$$\left(\frac{|z|^4}{|1-z|^2}\right)^{\Delta_p} = \sum_{h,\bar{h}} {}_s\mathfrak{C}_{h,\bar{h}} g_{h,\bar{h}}^{(VV\bar{V}\bar{V})}(z, \bar{z}) \tag{3.6}$$

and:

$$|z|^{2\Delta_p} = \sum_{h,\bar{h}} {}_t\mathfrak{C}_{h,\bar{h}} g_{h,\bar{h}}^{(\bar{V}VV\bar{V})}(1-z, 1-\bar{z}) \tag{3.7}$$

s-channel

Let's begin with the first equation. By substituting the expression for the conformal blocks, and dividing with $|z|^{4\Delta_p}$ we obtain:

$$\left(\frac{1}{1-z}\right)^{\Delta_p}\left(\frac{1}{1-\bar{z}}\right)^{\Delta_p} = \sum_{h,\bar{h}=2\Delta_p} {}_s\mathfrak{C}_{h,\bar{h}}\ z^{h-2\Delta_p}\bar{z}^{\bar{h}-2\Delta_p} \times {}_2F_1(h,h,;2h;z)\,{}_2F_1(\bar{h},\bar{h},;2\bar{h};\bar{z}) \tag{3.8}$$

we now series expand both sides and get:

$$\sum_{n,m=0}^{\infty} \frac{(\Delta_p)_n(\Delta_p)_m}{n!m!} z^n\bar{z}^m = \sum_{h,\bar{h}=2\Delta_p}\sum_{n,m=0}^{\infty} {}_s\mathfrak{C}_{h,\bar{h}} \frac{(h)_n^2(\bar{h})_m^2}{(2h)_n(2\bar{h})_m n!m!} z^{h-2\Delta_p+n}\bar{z}^{\bar{h}-2\Delta_p+m} \tag{3.9}$$

Where $(A)_n = \begin{cases} 1 & \text{if } n=0 \\ A(A+1)\cdots(A+n-1) & \text{if } n>0 \end{cases}$ are the rising Pochhammer symbols.

By order expanding (3.9) we obtain the following expression for the coefficients ${}_s\mathfrak{C}_{h,\bar{h}}$.

$$\sum_{n,\bar{n}=0}^{\ell,\bar{\ell}} \frac{(n+2\Delta_p)_{\ell-n}^2(\bar{n}+2\Delta_p)_{\bar{\ell}-\bar{n}}^2\,{}_s\mathfrak{C}_{n+2\Delta_p,\bar{n}+2\Delta_p}}{(2n+4\Delta_p)_{\ell-n}(2\bar{n}+4\Delta_p)_{\bar{\ell}-\bar{n}}(\ell-n)!(\bar{\ell}-\bar{n})!} = \frac{(\Delta_p)_\ell(\Delta_p)_{\bar{\ell}}}{\ell!\ \bar{\ell}!} \tag{3.10}$$

Where $\ell, \bar{\ell}$ are the holomorphic and anti-holomorphic levels of the highest order coefficient on the sum.
By replacing the right hand side two sums obtained by the same relation, one for $\ell = 0$ and one for $\bar{\ell} = 0$ we obtain the following constraint for the s-channel OPE squared coefficients:

$$ {}_s\mathfrak{C}_{n+2\Delta_p,\bar{n}+2\Delta_p} = {}_s\mathfrak{C}_{n+2\Delta_p,2\Delta_p}\ {}_s\mathfrak{C}_{2\Delta_p,\bar{n}+2\Delta_p} \tag{3.11} $$

This relation tells us that if we have the s-channel OPE-squared coefficients for $(\ell + 2\Delta_p, 2\Delta_p)$ we can calculate any coefficient.
The OPE squared coefficients, calculated from (3.10) are:

s-channel OPE Squared Coefficients		
Scaling Dimension (Δ)	Spin (s)	Analytic ${}_s\mathfrak{C}_{h,\bar{h}}$
$4\Delta_p$	0	1
$4 + 4\Delta_p$	0	$\frac{\Delta_p^4}{4(4\Delta_p+1)^2}$
$5 + 4\Delta_p$	1	0
$2 + 4\Delta_p$	2	$\frac{\Delta_p^2}{2(4\Delta_p+1)}$
$3 + 4\Delta_p$	3	0
$4 + 4\Delta_p$	4	$\frac{\Delta_p^2(\Delta_p+1)^2}{8(4\Delta_p+3)(4\Delta_p+5)}$
$5 + 4\Delta_p$	5	0
$\vdots$	$\vdots$	$\vdots$
$2n + 2\bar{n} + 4\Delta_p$	$2n - 2\bar{n}$	$\frac{(\Delta_p)_n^2}{(2n)!!\ (2n-1+4\Delta_p)_{n,2}} \frac{(\Delta_p)_{\bar{n}}^2}{(2\bar{n})!!\ (2\bar{n}-1+4\Delta_p)_{\bar{n},2}}$
$\vdots$	$\vdots$	$\vdots$

Table 3.5: s-channel OPE squared coefficients for $\Delta \leq 5 + 4\Delta_p$

Where $(A)_{n,2} = A(A+2)\cdots(A+n-2)$ is the $k = 2$ k-Pochhammer symbol. It is important to remember that since out external operators are spinless and thus $\mathfrak{C}_{h,\bar{h}} = \mathfrak{C}_{\bar{h},h}$.

t-channel

We now turn our eye on equation (3.7) and the t-channel. To avoid further expanding the expansion of the hyper-geometric functions of the RHS we change variables as $1 - z = \zeta$. This yields:

$$ |1-\zeta|^{2\Delta_p} = \sum_{h,\bar{h}} {}_t\mathfrak{C}_{h,\bar{h}} g_{h,\bar{h}}^{(\bar{V}VV\bar{V})}(\zeta,\bar{\zeta}) $$

By expanding the hyper-geometric functions of the RHS and the LHS with the binomial theorem we finally obtain:

$$\sum_{n,m=0}^{\infty} \binom{\Delta_p}{n}\binom{\Delta_p}{m}(-\zeta)^n(-\bar{\zeta})^m = \sum_{h,\bar{h}=0}\sum_{n,m=0}^{\infty} {}_t\mathfrak{C}_{h,\bar{h}} \frac{(h)_n^2(\bar{h})_m^2}{(2h)_n(2\bar{h})_m n!m!}\zeta^{h+n}\bar{\zeta}^{\bar{h}+m} \tag{3.12}$$

where $\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$ are the generalized binomial coefficient.
By order expanding (3.12) we can calculate the constants ${}_t\mathfrak{C}_{h,\bar{h}}$:

t-channel OPE Squared Coefficients		
Scaling Dimension (Δ)	Spin (s)	Analytic ${}_t\mathfrak{C}_{h,\bar{h}}$
0	0	1
2	0	Δ_p^2
4	0	$\frac{\Delta_p^4}{4}$
1	1	$-\Delta_p$
2	2	$\frac{\Delta_p^2}{2}$
3	1	$-\frac{\Delta_p^3}{2}$
3	3	$-\frac{\Delta_p^3}{6}$
$\vdots$	$\vdots$	$\vdots$
Δ	s	$\frac{(-1)^s\Delta_p^\Delta}{h!\bar{h}!} = \frac{(-1)^s\Delta_p^\Delta}{(\frac{\Delta+s}{2})!(\frac{\Delta-s}{2})!}$
$\vdots$	$\vdots$	$\vdots$

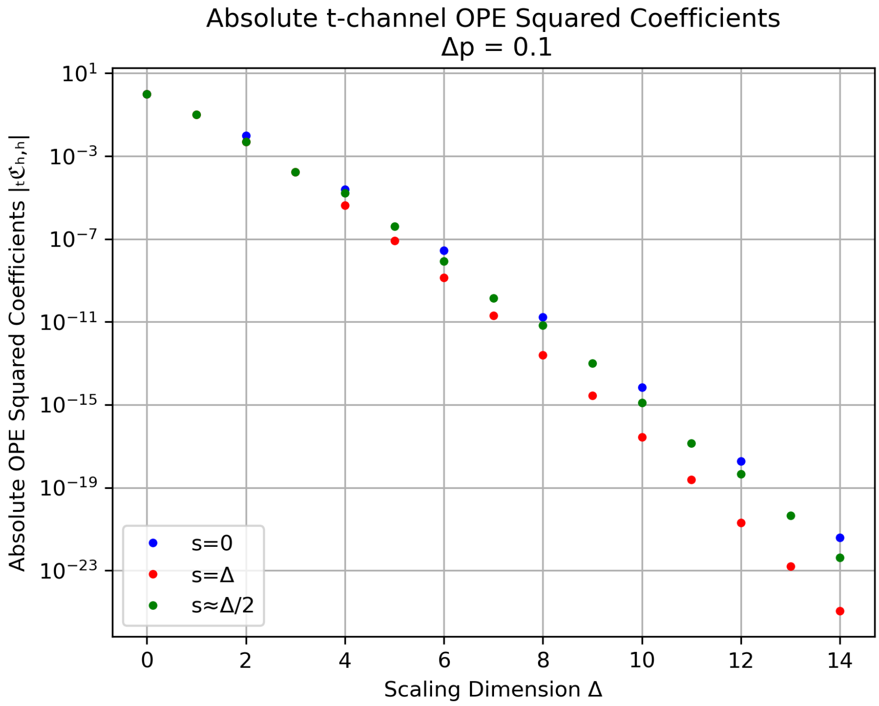
Table 3.6: t-channel OPE squared coefficients

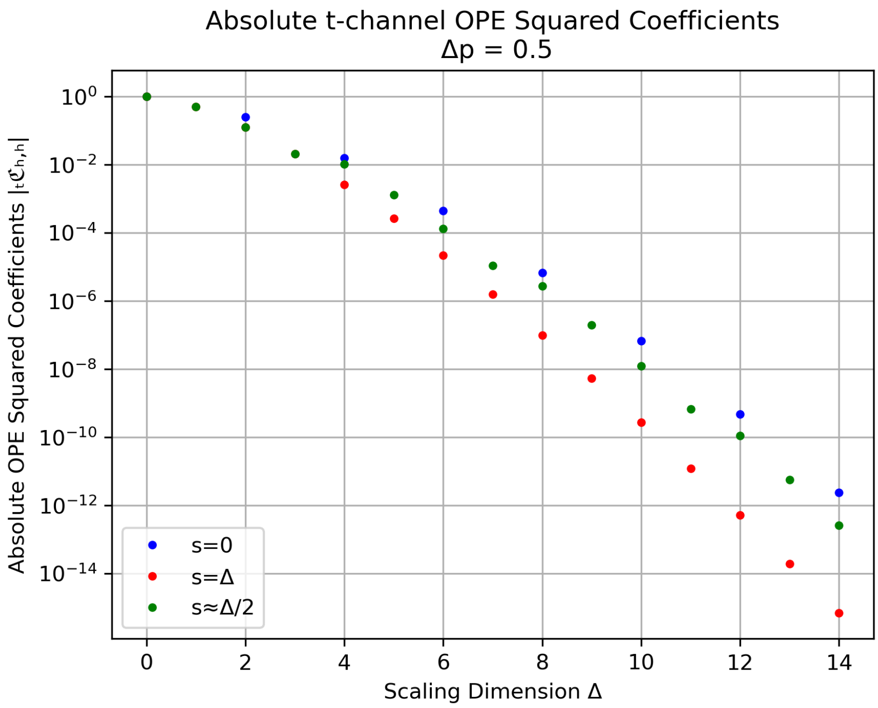
3.2.2 Convergence Rate

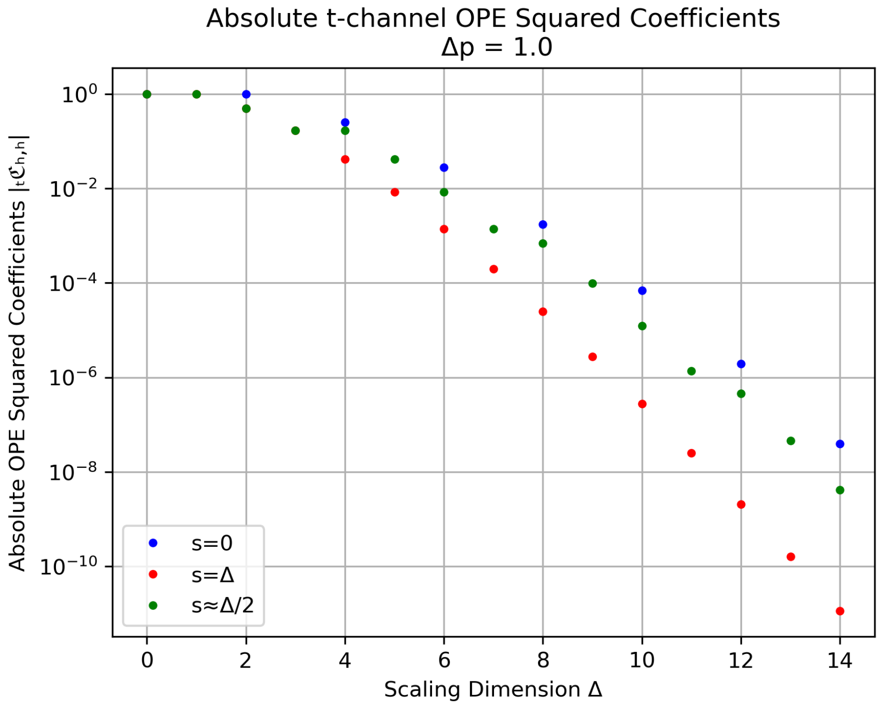
With the analytic expressions derived above, we can calculate the coefficients $\mathfrak{C}_{h,\bar{h}}$ up to any order we like. In order for us to decide out cut-off Δ and spin partition we need to know how fast are the sums converging, i.e. how fast are the coefficients approach zero as Δ rises.

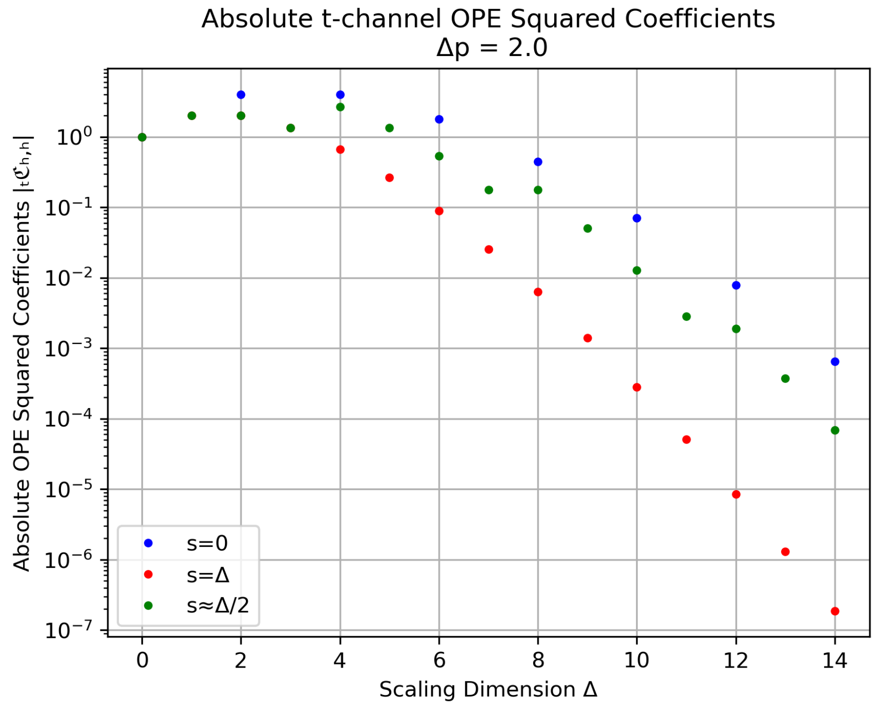
t-channel

With that in mind the following diagrams were plotted for the t-channel coefficients. As a logarithmic axis was necessary to visualize the data, we cannot plot negative values of ${}_t\mathfrak{C}_{h,\bar{h}}$. To see the order of these we plot the absolute $|{}_t\mathfrak{C}_{h,\bar{h}}|$ against Δ.


Absolute t-channel OPE Squared Coefficients
Δp = 0.1
Absolute OPE Squared Coefficients $|{}_t\mathfrak{C}_{h,h}|$
Scaling Dimension Δ
s=0
s=Δ
s≈Δ/2


Absolute t-channel OPE Squared Coefficients
Δp = 0.5
Absolute OPE Squared Coefficients $|{}_t\mathfrak{C}_{h,h}|$
Scaling Dimension Δ
s=0
s=Δ
s≈Δ/2





As we can see, OPE-squared coefficients convergence rate is getting slower as the external scaling dimension rises. This implies that truncating the the sums at a certain point will hold up to some Δ_p before starting to lose significant amounts of information.

s-channel

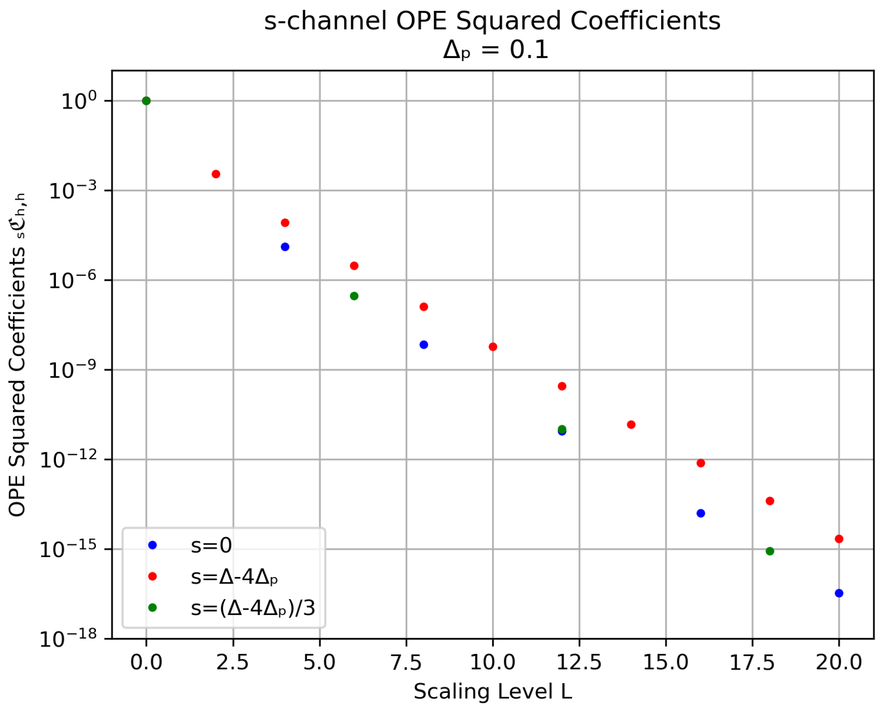
Now we do the same for the s-channel. Since we only have positive values on the s channel, we will not need to plot the absolute values. Here, another issue arises. The conformal block expansion operators' scaling dimensions are a function of the external operators' scaling dimension Δ_p due to the $e^{2ipX(z,\bar{z})}$ (see: 3.1.2). Explicitly, the dependence is:

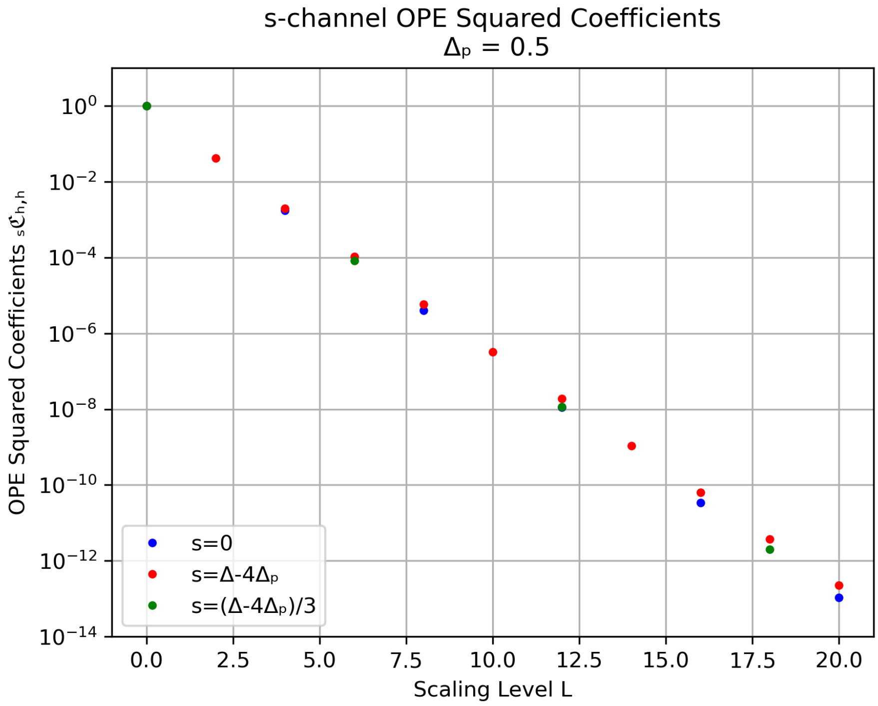
$$\Delta = \ell + \bar{\ell} + 4\Delta_p$$

To keep track of the operators in this channel, we must define a quantity that is invariant as we change Δ_p. The most appropriate such quantity seems to be the *level* of the operator:

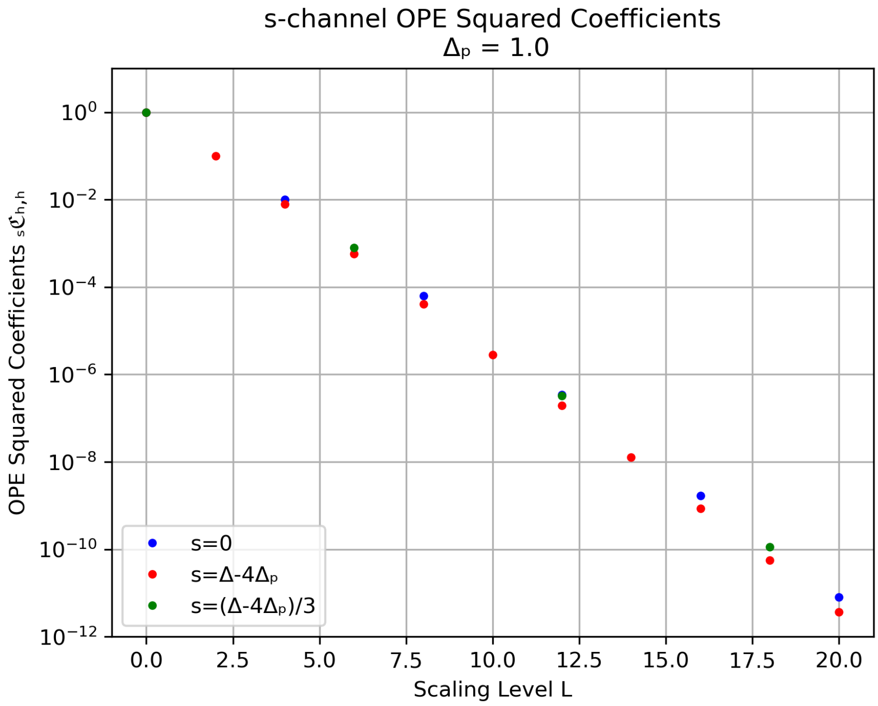
$$L = \ell + \bar{\ell} \tag{3.13}$$

Lets see how the OPE-squared coefficients behave on this channel:

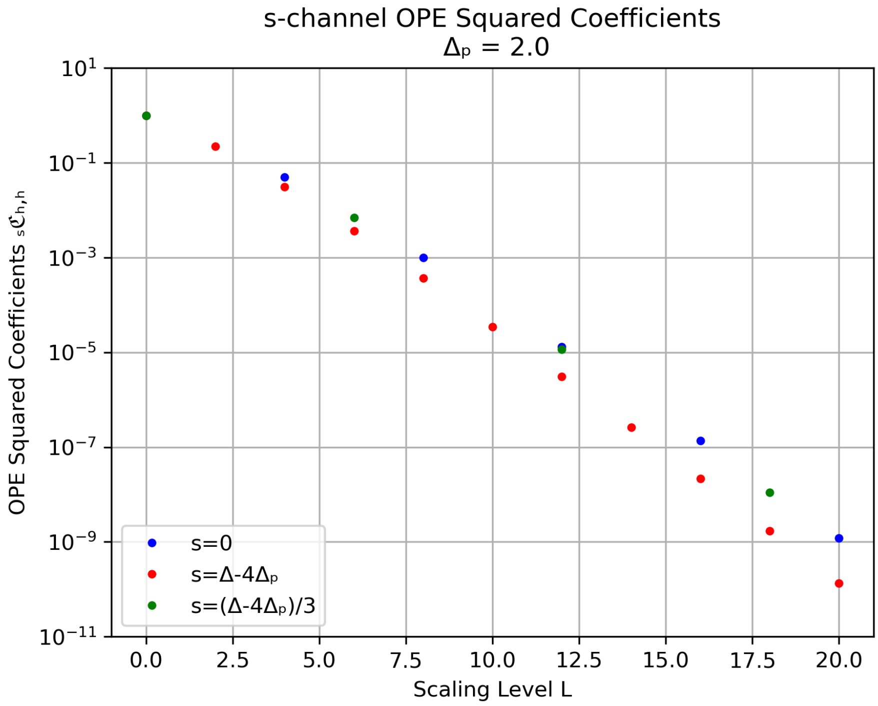




s-channel OPE Squared Coefficients
Δ_p = 0.5
OPE Squared Coefficients ${}_s\mathfrak{C}_{h,h}$
Scaling Level L
s=0
s=Δ-4Δ_p
s=(Δ-4Δ_p)/3



s-channel OPE Squared Coefficients
Δ_p = 1.0
OPE Squared Coefficients ${}_s\mathfrak{C}_{h,h}$
Scaling Level L
s=0
s=Δ-4Δ_p
s=(Δ-4Δ_p)/3



As we can see, the operators of the s-channel expansion are less sensitive to changing the external dimensions than their t-channel counterparts. Even thought hey get larger as we increase Δ_p, they do not get as large as the t-channel ones.
To truncate this channel, we must choose an L_{max} and not a Δ_{max} as in the latter case, we would lose operators as we changed Δ_p.

3.3 Tails of Truncated Sums

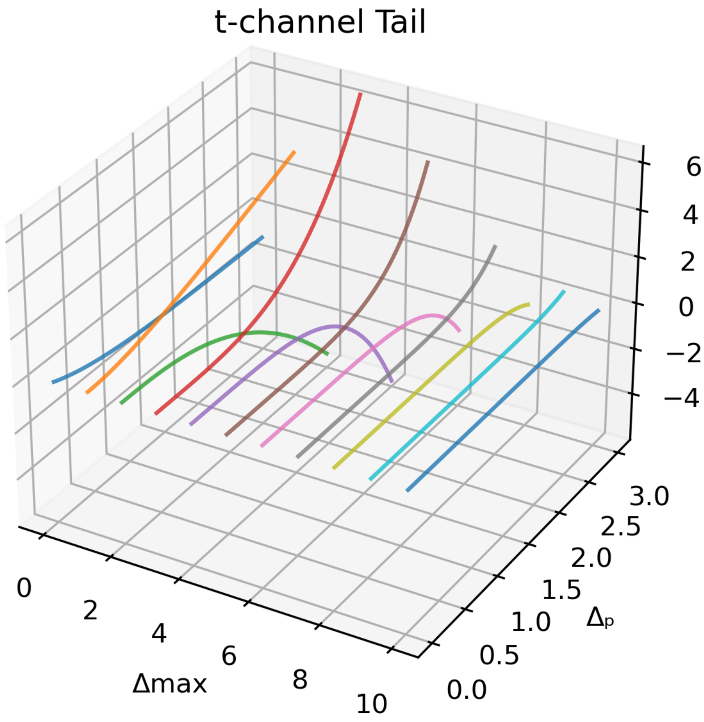
Since we will chose some Δ_{max} to limit the sums appearing in the crossing equations, we want to know the magnitude of the leftover sums or *tails*. To do this we can use equations (3.6) and (3.7). Note that here, we calculate the tails on the crossing symmetric point $z = \bar{z} = 1/2$.

3.3.1 t-channel

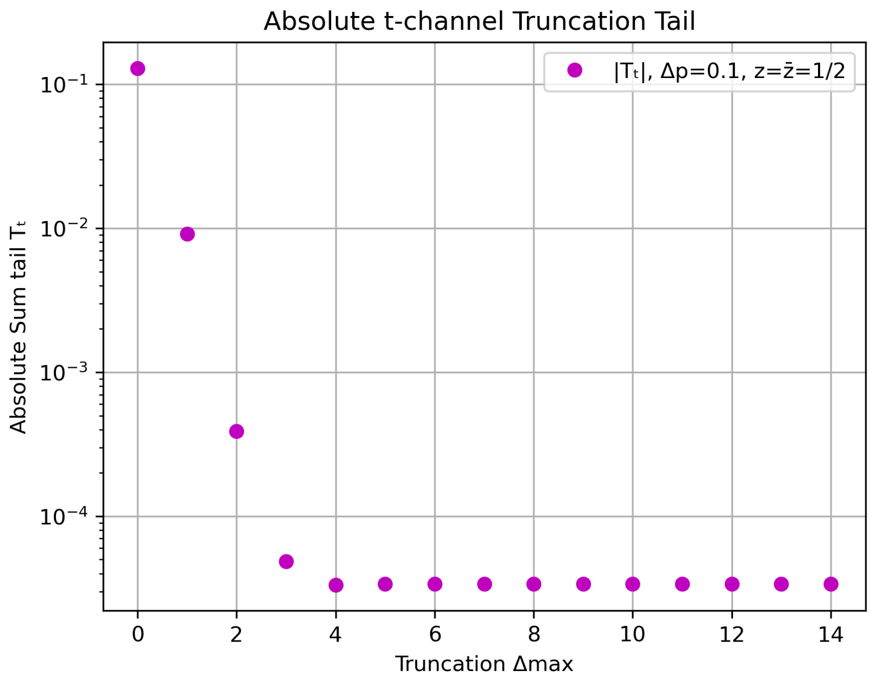
The t-channel tail T_t is:

$$T_t(\Delta_{max}, \Delta_p; z, \bar{z}) = |z|^{2\Delta_p} - \sum_{h,\bar{h}}^{\Delta_{max}} {}_t\mathfrak{C}_{h,\bar{h}} g_{h,\bar{h}}^{(\bar{V}V V\bar{V})}(1-z, 1-\bar{z}) \qquad (3.14)$$

The sum on the right hand side can be computed analytically for a given Δ_{max}. Below, we plot the t-tail for different values of Δ_{max} as Δ_p changes.



By taking a slice at $\Delta_p = 0.1$ we get:

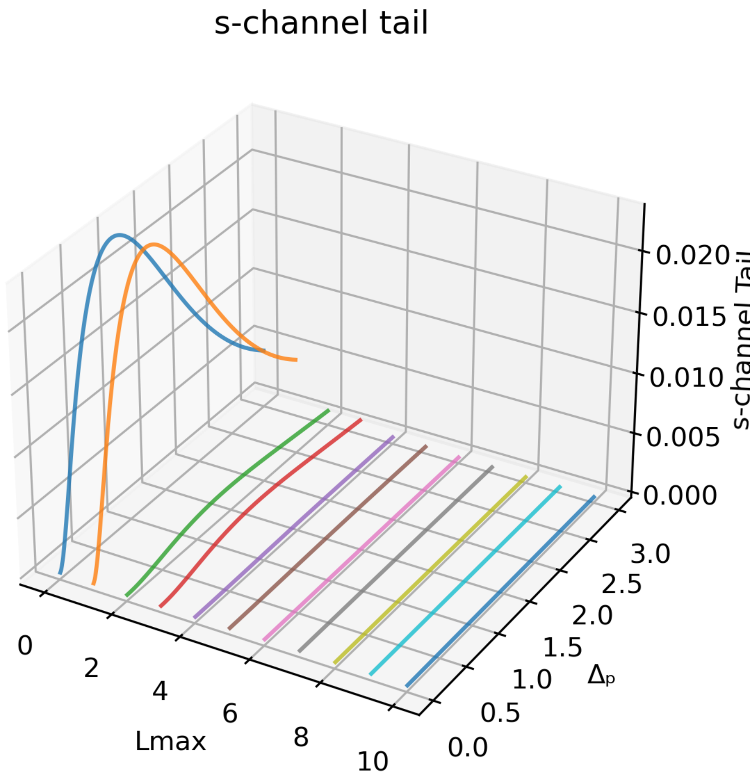


3.3.2 s-channel

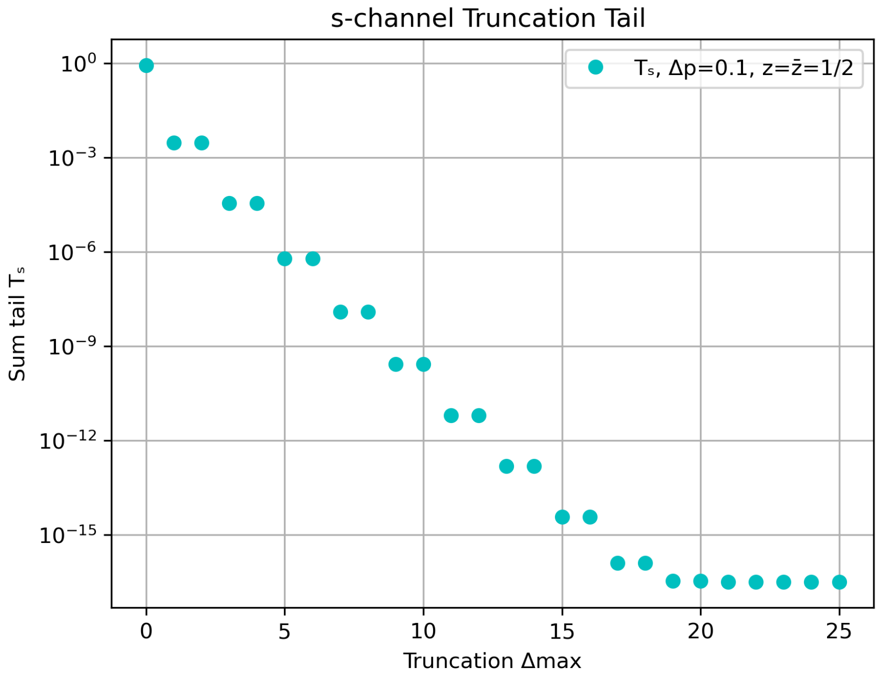
The s-channel tail T_s is:

$$T_s(\Delta_{max}, \Delta_p; z, \bar{z}) = \left(\frac{|z|^4}{|1-z|^2}\right)^{\Delta_p} - \sum_{h,\bar{h}}^{\Delta_{max}} {}_s\mathfrak{C}_{h,\bar{h}} g_{h,\bar{h}}^{(VV\bar{V}\bar{V})}(z, \bar{z}) \tag{3.15}$$

The sum on the right hand side can be computed analytically for a given L_{max}. Below, we plot the s-tail for different values of L_{max}, changing Δ_p.



Again, by taking a slice at $\Delta_p = 0.1$ we obtain:



3.3.3 Crossing Tail

We will now calculate the linear combination of these tails that will appear in the crossing equations. (2.9)(2.10).

1st Crossing Equation Tail

The 1st equation is:

$$\sum_{h,\bar{h}} {}_s\mathfrak{C}_{h,\bar{h}}\, g_{h,\bar{h}}^{VV\bar{V}\bar{V}}(z,\bar{z}) - \frac{|z|^{2\Delta_p}}{|1-z|^{2\Delta_p}} \left(1 + \sum_{h,\bar{h}\neq 0} {}_t\mathfrak{C}_{h,\bar{h}}\, g_{h,\bar{h}}^{(\bar{V}VV\bar{V})}(1-z,1-\bar{z})\right) = 0$$

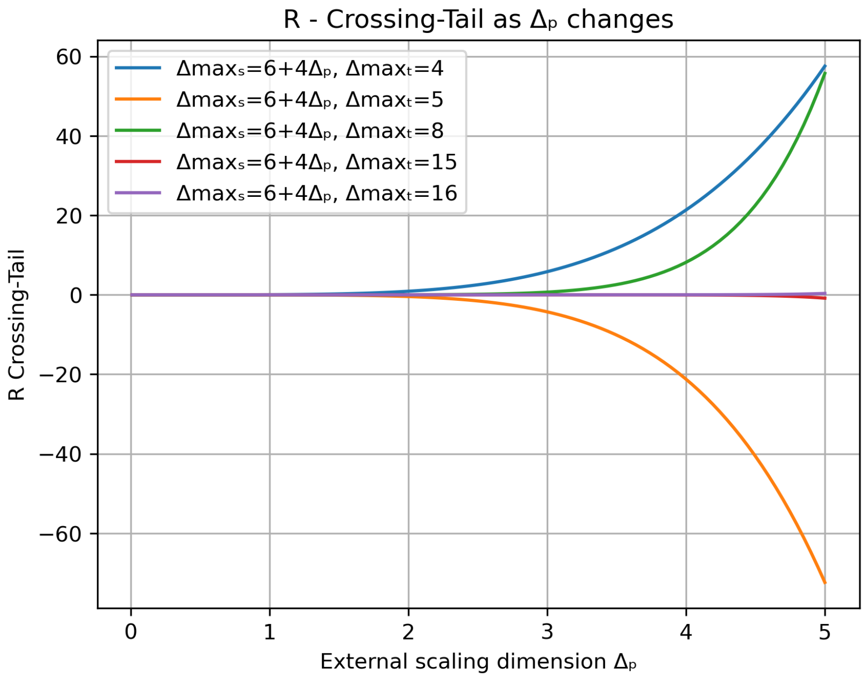
By taking out the tails for both sums, we get:

$$\sum_{h,\bar{h}=2\Delta_p}^{\Delta_{max}^s} {}_s\mathfrak{C}_{h,\bar{h}}\, g_{h,\bar{h}}^{VV\bar{V}\bar{V}}(z,\bar{z}) - \frac{|z|^{2\Delta_p}}{|1-z|^{2\Delta_p}} \left(1 + \sum_{h,\bar{h}\neq 0}^{\Delta_{max}^t} {}_t\mathfrak{C}_{h,\bar{h}}\, g_{h,\bar{h}}^{(\bar{V}VV\bar{V})}(1-z,1-\bar{z})\right)$$
$$+ \left[T_s(\Delta_{max}^s,\Delta_p;z,\bar{z}) - \frac{|z|^{2\Delta_p}}{|1-z|^{2\Delta_p}} T_t(\Delta_{max}^t,\Delta_p;z,\bar{z})\right] = 0 \tag{3.16}$$

We define the quantity

$$R(\Delta_{max}^s,\Delta_{max}^t,\Delta_p;z,\bar{z}) = T_s(\Delta_{max}^s,\Delta_p;z,\bar{z}) - \frac{|z|^{2\Delta_p}}{|1-z|^{2\Delta_p}} T_t(\Delta_{max}^t,\Delta_p;z,\bar{z}) \tag{3.17}$$

as the *first crossing tail*. From the previous calculations we can see that the t channel tail dominates as it tis the only term that diverges as we get to higher Δ_p. We now plot some tails against Δ_p for our usual example $z = \bar{z} = 1/2$.



2nd Crossing Equation Tail

The second crossing equation is:

$$[1 + \sum_{h,\bar{h} \neq 0} (-1)^{h-\bar{h}} {}_t\mathfrak{C}_{h,\bar{h}} g_{h,\bar{h}}^{V\bar{V}V\bar{V}}(z, \bar{z})] = \frac{|z|^{2\Delta_p}}{|z-1|^{2\Delta_p}} [1 + \sum_{h,\bar{h} \neq 0} (-1)^{h-\bar{h}} {}_t\mathfrak{C}_{h,\bar{h}} g_{h,\bar{h}}^{V\bar{V}V\bar{V}}(1-z, 1-\bar{z})] \tag{3.18}$$

In the same manner as before, we define the *second crossing tail* as:

$$U(\Delta_{max}^u, \Delta_p; z, \bar{z}) = T_u(\Delta_{max}^u, \Delta_p; z, \bar{z}) - \frac{|z|^{2\Delta_p}}{|1-z|^{2\Delta_p}} T_u(\Delta_{max}^u, \Delta_p; 1-z, 1-\bar{z}) \tag{3.19}$$

where:

$$T_u(\Delta_{max}^u, \Delta_p; z, \bar{z}) = \left| \frac{1}{1-z} \right|^{2\Delta_p} - \sum_{h,\bar{h}}^{\Delta_{max}^u} (-1)^{h-\bar{h}} {}_t\mathfrak{C}_{h,\bar{h}} g_{h,\bar{h}}^{(V\bar{V}V\bar{V})}(z, \bar{z})$$

Δ_u^{max} is the truncation scaling dimension for the u-channel sums, i.e. the sums with that have the coefficients $(-1)^s{}_t\mathfrak{C}_{h,\bar{h}}$. For simplicity, in this work we set $\Delta_u^{max} = \Delta_t^{max}$.
After this discussion, we can choose a well informed truncation and spin partition an proceed with the setup of the reinforcement learning environment and set our strategic approach on using the soft actor critic.

Chapter 4

Methodology and RL Environment Setup

On this chapter, considering all said in the previous chapters, we will mark down the details of the setup we will work on.
What we want to do is calculate the CFT data appearing on the OPE of the four-point function $\langle V V \bar{V} \bar{V} \rangle$, where $V(z, \bar{z}) = e^{ipX(z,\bar{z})}$. To do this we will use the crossing and U(1) symmetries in the form of the crossing equations (2.9)(2.10). The way we extract information from these equations is to consider them in various $(z, \bar{z})$ points and try to solve the algebraic system of equations that occurs. The optimization task that emerges is handled by an appropriately modified version of the *BootSTOP* [7] [6] reinforcement learning algorithm.
To explore the conformal manifold defined by the external parameter Δ_p, we will start from a point where we accept that we have the analytic solution ($\Delta_p = 0.1$), and adiabatically flow away from that point towards the wanted Δ_p. To improve results, we will consider a tail contribution that we calculate analytically at $\Delta_p = 0.1$.

4.1 Truncations and Spin Partition

As the previous chapter showed, in order to solve the equations we need to truncate the sums appearing in them. As we saw, the s-channel's contribution is concentrated on its first operators and its tail goes to zero as we move higher in Δ_p. On the other hand, the t-channel's tail, inevitably gets to a Δ_p point where exponential growth starts and thus the whole system breaks down. This point, can be futher pushed away by considering a higher Δ^t_{max}. With that in mind, to have a wide enough range of Δ_p that flowing can work, we chose the following truncation Δ_{max}:

$$\Delta^s_{max} = 6 + 4\Delta_p \qquad \Delta^t_{max} = 8 \tag{4.1}$$

These choices fix our spin partition as follows:

Spin Partition		
$\Delta^s_{max} = 6 + 4\Delta_p$		$\Delta^t_{max} = 16$
Spin	s-channel	t-channel
0	2	4
1	0	4
2	2	4
3	0	3
4	1	3
5	0	2
6	1	2
7	0	1
8	0	1
total	6	24

Table 4.1: The specific spin partition of the exercise.

With this spin partition,
There are totally 30 operators. Note that in the above spin partition we do not include the identity since its contribution has been separated in the crossing equations (see below).

4.2 Crossing Equations

Now, we explicitly write down the two crossing equations in the exact form we set BootSTOP to solve.:

1st Crossing Equation:

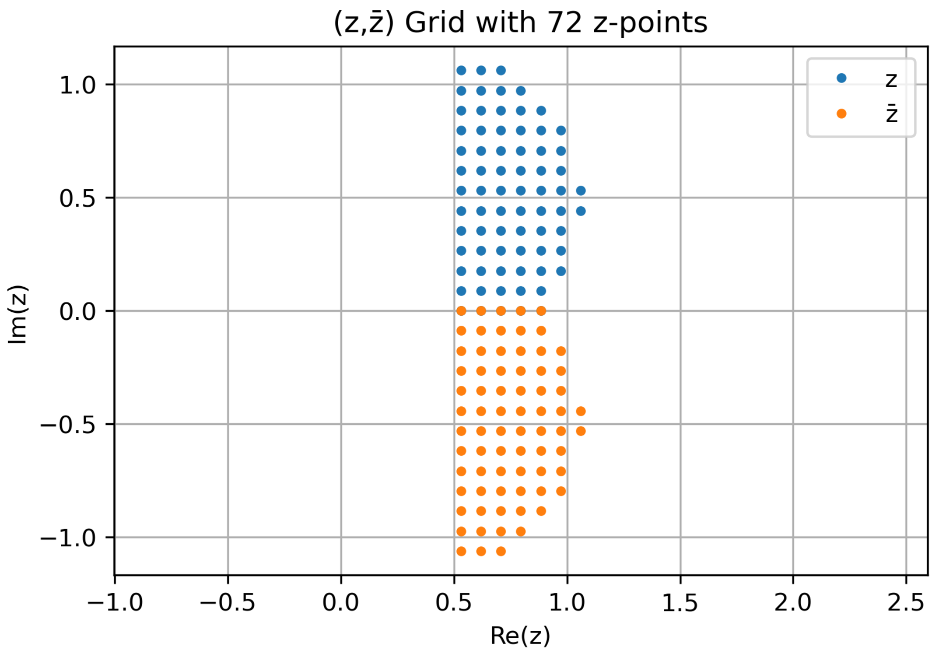
$$\sum_{\substack{h\geq\bar{h} \\ h-\bar{h}=even}} {}_s\mathfrak{C}_{h,\bar{h}}\, \tilde{g}^{VV\bar{V}\bar{V}}_{h,\bar{h}}(z,\bar{z}) - \frac{|z|^{2\Delta_p}}{|1-z|^{2\Delta_p}} \left(1 + \sum_{h>\bar{h}} {}_t\mathfrak{C}_{h,\bar{h}}\, \tilde{g}^{(\bar{V}VV\bar{V})}_{h,\bar{h}}(1-z,1-\bar{z}) \right) = 0 \tag{4.2}$$

2nd Crossing Equation:

$$1 - \frac{|z|^{2\Delta_p}}{|z-1|^{2\Delta_p}} + \sum_{h>\bar{h}} (-1)^{h-\bar{h}} {}_t\mathfrak{C}_{h,\bar{h}} \left(\tilde{g}^{V\bar{V}V\bar{V}}_{h,\bar{h}}(z,\bar{z}) - \frac{|z|^{2\Delta_p}}{|z-1|^{2\Delta_p}} \tilde{g}^{V\bar{V}V\bar{V}}_{h,\bar{h}}(1-z,1-\bar{z}) \right) = 0 \tag{4.3}$$

4.3 z-Points Grid

As said before, we need to consider the crossing equations on multiple $(z,\bar{z})$ points. The number of these points has to be such that the emerging system of algebraic equations is over-determined. Since we have 30 operators, we have 60 unknowns. One more thing one needs to consider, is that these points must be points where the crossing equations are well behaved.
The grid, we will use according to these specifications is an appropriately modified version of the one appearing on [2]



This grid avoids the hyper-geometric function's brunch cuts and is uniform on the area it covers. These points are expected to yield well behaved equations.

4.4 Strategy

Let us now, outline the algorithmic process we followed when running the algorithm.

Algorithm

- We start from $\Delta_p = 0.1$
- Add the tails at 0.1 as a constant contribution
- Repeat:
  - Start from best known solution Δ_p
  - Run SAC multiple times with $\Delta_p + = 0.05$
  - Calculate the weighted average of the results to get the solution. The weights are chosen to be the rewards of each run
  - The solution is now the new best solution

Following this strategy, we obtained the next chapter's results. Note that for the algorithm to work as intended, fine-tuning of its parameters is crucial. The details on that part will not be discussed as they are not important in the context of this work.

Chapter 5

Numerical Results

Let us now demonstrate some numerical results. For these results, we started on $\Delta_p = 0.1$. With step $+0.05$ we started flowing from that point making 10 runs per step.

5.1 BootSTOP's output

On each step, we obtain numerical results on the CFT data of the following form:

Table 5.1: s-channel, $\Delta_p = 0.15$

Spin	RL(Δ)	Analytic(Δ)	RL($\mathfrak{C}$)	Analytic($\mathfrak{C}$)
0	0.594008	0.6	1.0044724e+00	1.0000000e+00
0	4.396035	4.6	9.5831388e-05	4.9438477e-05
2	2.545494	2.6	7.8288222e-03	7.0312500e-03
2	6.574206	6.6	3.8405582e-06	1.2972695e-06
4	4.275060	4.6	2.0445386e-04	1.8450056e-04
6	6.445978	6.6	1.0794863e-05	7.0136006e-06

As is evident, the algorithm gets fairly good results, especially on the lower lying operators.

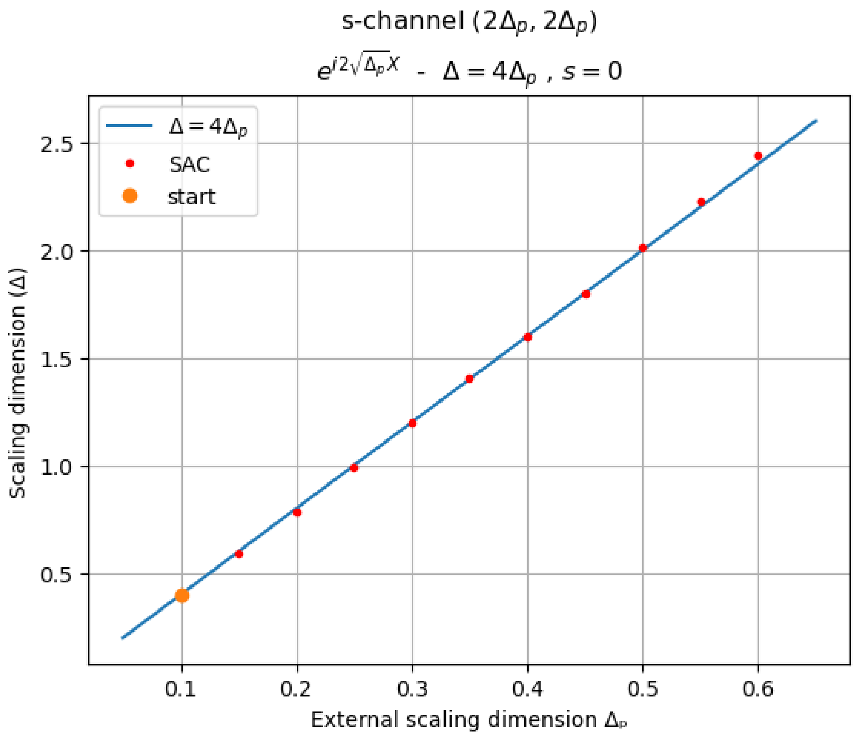
Table 5.2: t-channel, $\Delta_p = 0.15$

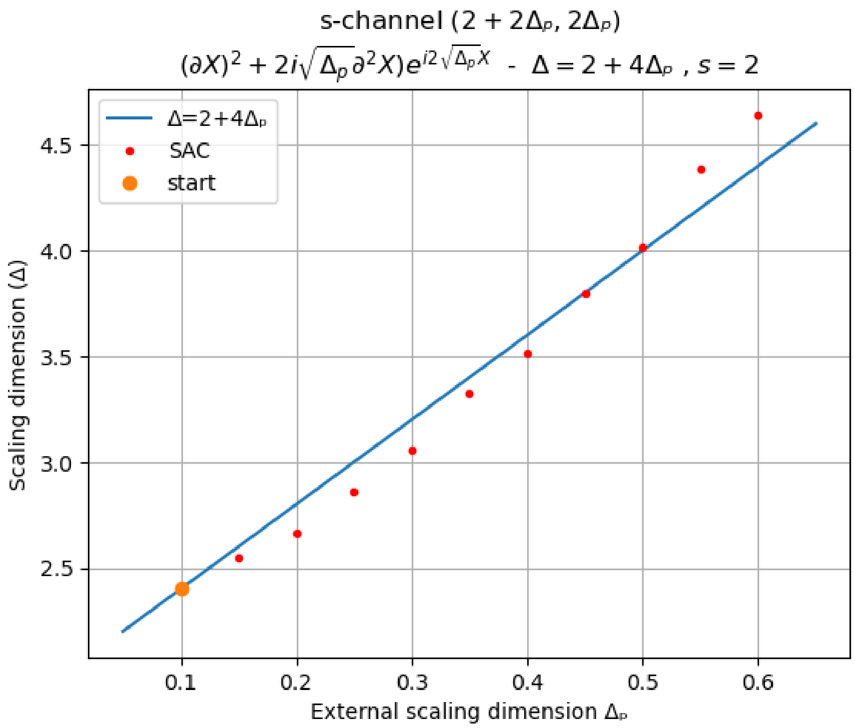
Spin	RL(Δ)	Analytic(Δ)	RL($\mathfrak{C}$)	Analytic($\mathfrak{C}$)
0	1.945099	2.0	2.2551485e-02	2.2500000e-02
0	3.969464	4.0	3.5728842e-04	1.2656250e-04
0	5.806528	6.0	4.3247064e-07	3.1640625e-07
0	7.994133	8.0	4.3942027e-10	4.4494629e-10
1	1.020799	1.0	1.4726147e-01	1.5000000e-01
1	3.061228	3.0	1.4483074e-03	1.6875000e-03
1	5.075957	5.0	9.6223790e-06	6.3281250e-06
1	6.828494	7.0	9.0792152e-09	1.1865234e-08
2	2.061307	2.0	1.0422554e-02	1.1250000e-02
2	4.055639	4.0	1.3342041e-04	8.4375000e-05
2	6.075670	6.0	9.0064162e-08	2.3730469e-07
2	7.925841	8.0	2.1960278e-10	3.5595703e-10
3	3.056391	3.0	8.0364710e-04	5.6250000e-04
3	5.139894	5.0	9.3805178e-06	3.1640625e-06
3	7.012824	7.0	6.6210649e-09	7.1191406e-09
4	4.182766	4.0	2.0779639e-05	2.1093750e-05
4	5.930093	6.0	9.4068200e-08	9.4921875e-08
4	7.984797	8.0	1.2140324e-10	1.7797852e-10
5	5.219733	5.0	8.5448097e-07	6.3281250e-07
5	6.889488	7.0	1.6304516e-09	2.3730469e-09
6	6.138786	6.0	8.7527580e-09	1.5820312e-08
6	7.928811	8.0	1.3156790e-11	5.0851004e-11
7	7.169603	7.0	3.7056103e-10	3.3900670e-10
8	8.035733	8.0	1.8530806e-12	6.3563756e-12

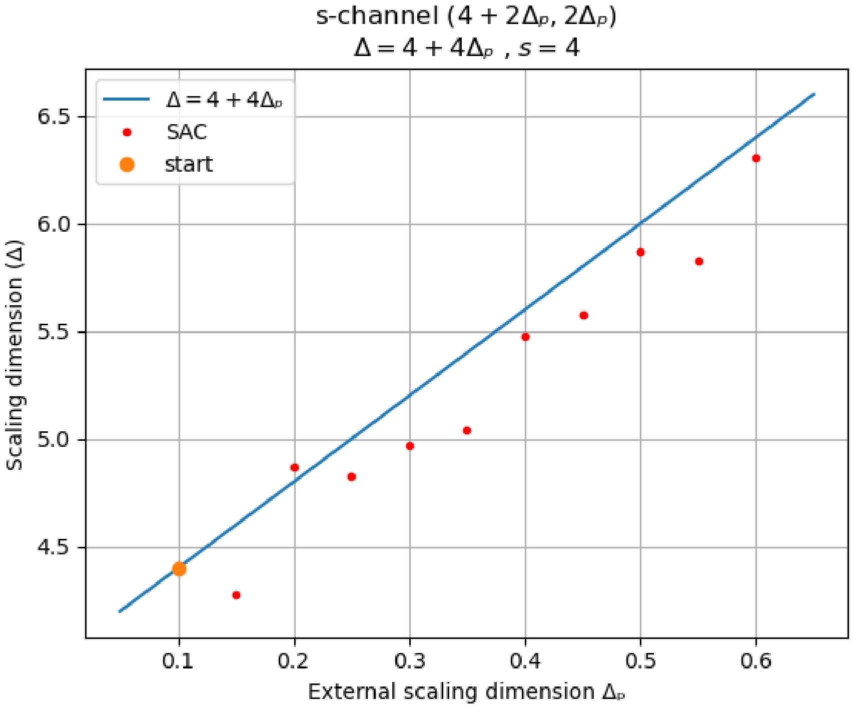
For each step, the algorithms' output is in the form of the above tables. For brevity, we shall not include the full output of all the steps and study some individual data themselves.

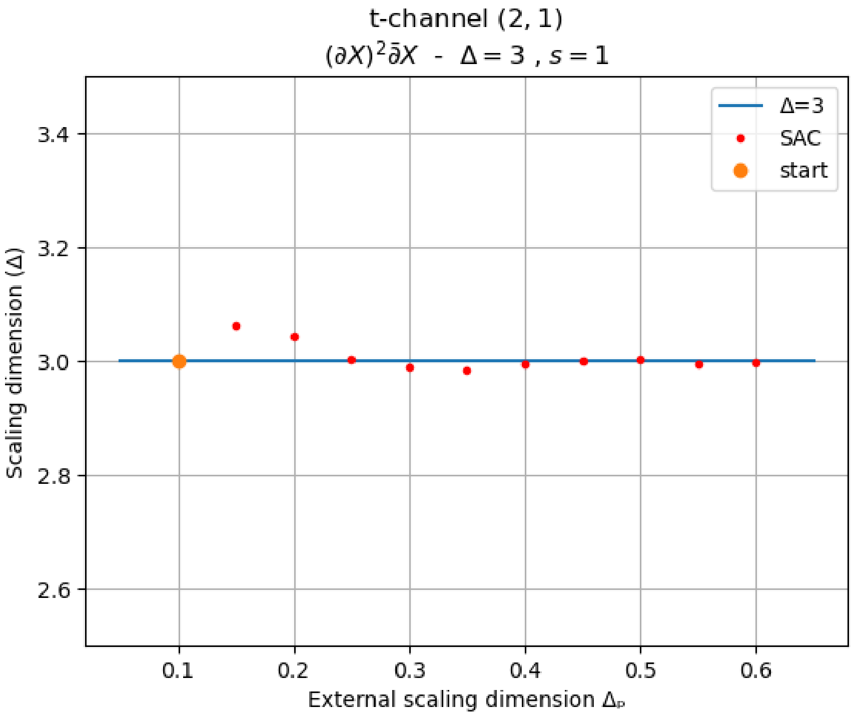
5.2 Scaling Dimensions

Lets take a look on how the algorithm keeps track of operators' scaling dimensions as we flow away from $\Delta_p = 0.1$. Below we plot some low lying operators and compare them to the analytic solution derived on chapter 3.


s-channel $(2\Delta_p, 2\Delta_p)$
$e^{i2\sqrt{\Delta_p}X}$ - $\Delta = 4\Delta_p$, $s = 0$
$\Delta = 4\Delta_p$
SAC
start
Scaling dimension (Δ)
External scaling dimension Δ_p
0.5
1.0
1.5
2.0
2.5
0.1
0.2
0.3
0.4
0.5
0.6


s-channel $(2 + 2\Delta_p, 2\Delta_p)$
$(\partial X)^2 + 2i\sqrt{\Delta_p}\partial^2 X)e^{i2\sqrt{\Delta_p}X}$ - $\Delta = 2 + 4\Delta_p$, $s = 2$
Δ=2+4Δ_p
SAC
start
Scaling dimension (Δ)
External scaling dimension Δ_p
2.5
3.0
3.5
4.0
4.5
0.1
0.2
0.3
0.4
0.5
0.6

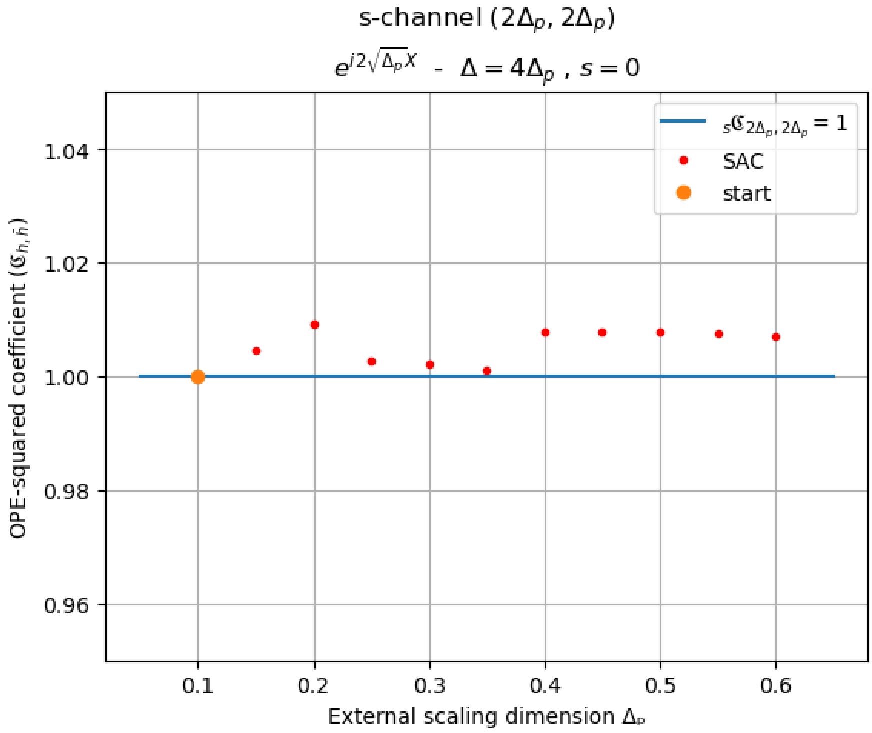

s-channel $(4+2\Delta_p, 2\Delta_p)$
$\Delta = 4 + 4\Delta_p$, $s = 4$
$\Delta = 4 + 4\Delta_p$
SAC
start
Scaling dimension (Δ)
External scaling dimension Δ_p

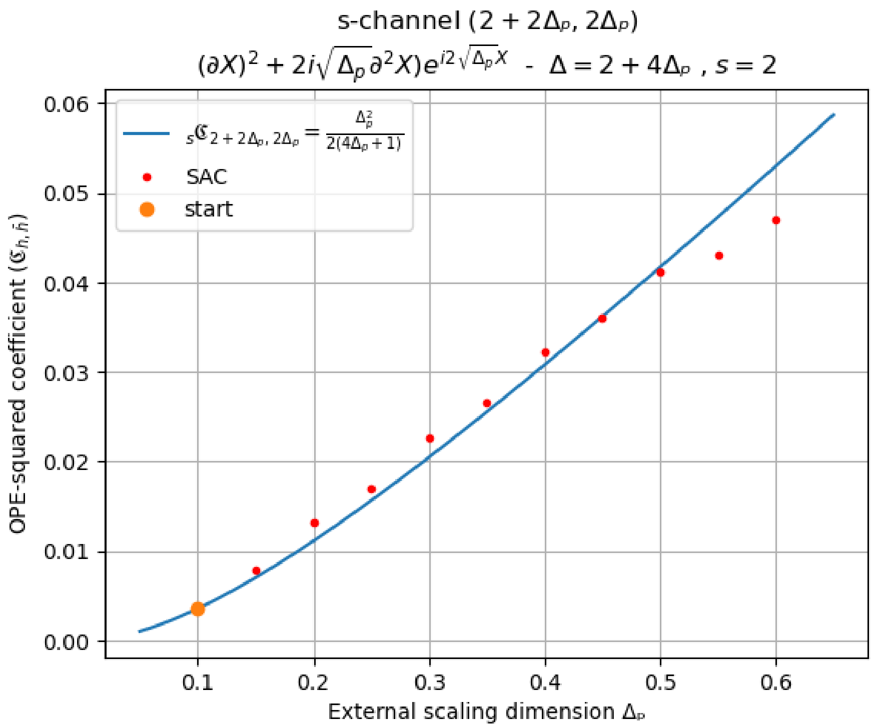

t-channel (2, 1)
$(\partial X)^2\bar{\partial}X$ - $\Delta = 3$, $s = 1$
Δ=3
SAC
start
Scaling dimension (Δ)
External scaling dimension Δ_p

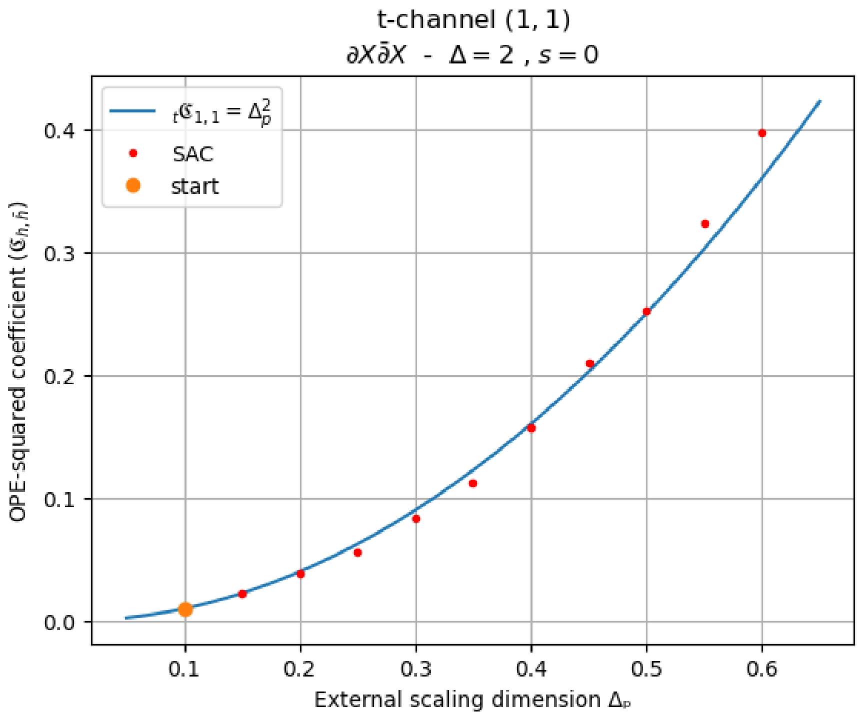
As we can see, results are better for coefficients with lower scaling dimension. That is to be expected as the lower the coefficient is in terms of scaling dimension and spin, the larger contribution is has on the crossing equations and thus larger contribution on the network's loss function.

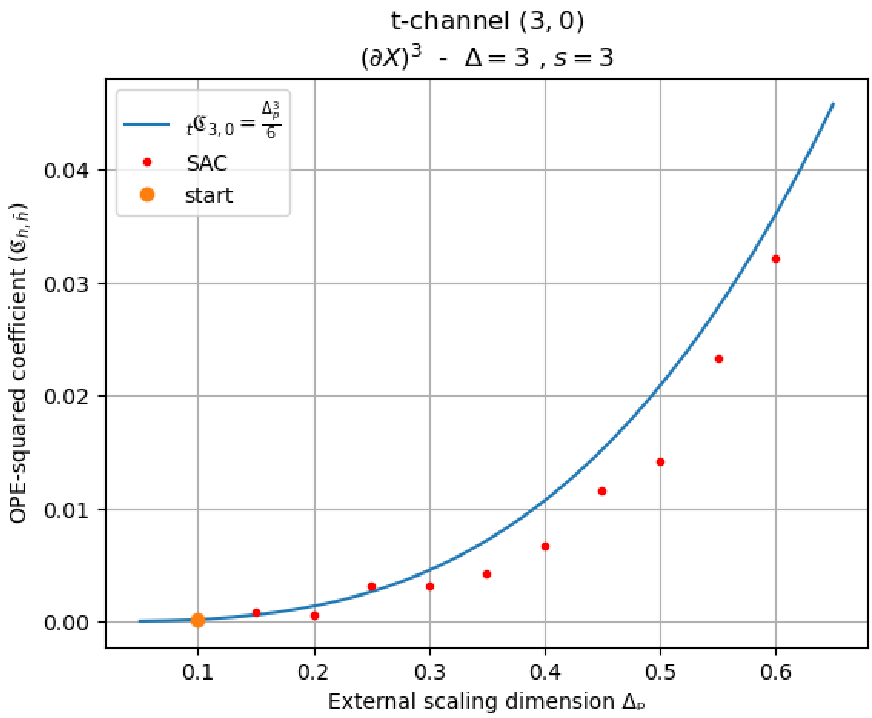
5.3 OPE-squared Coefficients

Now, we do the same for some of the the OPE squared coefficients. Again, the results are very good, but their quality declines as we move to higher scaling dimensions.




s-channel $(2+2\Delta_p, 2\Delta_p)$
$(\partial X)^2 + 2i\sqrt{\Delta_p}\partial^2 X)e^{i2\sqrt{\Delta_p}X}$ - $\Delta = 2 + 4\Delta_p$, $s = 2$
${}_s\mathfrak{C}_{2+2\Delta_p, 2\Delta_p} = \frac{\Delta_p^2}{2(4\Delta_p+1)}$
SAC
start
OPE-squared coefficient $(\mathfrak{C}_{h,\bar{h}})$
External scaling dimension Δ_p
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.1
0.2
0.3
0.4
0.5
0.6


t-channel $(1, 1)$
$\partial X\bar{\partial}X$ - $\Delta = 2$, $s = 0$
${}_t\mathfrak{C}_{1,1} = \Delta_p^2$
SAC
start
OPE-squared coefficient $(\mathfrak{C}_{h,\bar{h}})$
External scaling dimension Δ_p
0.0
0.1
0.2
0.3
0.4
0.1
0.2
0.3
0.4
0.5
0.6



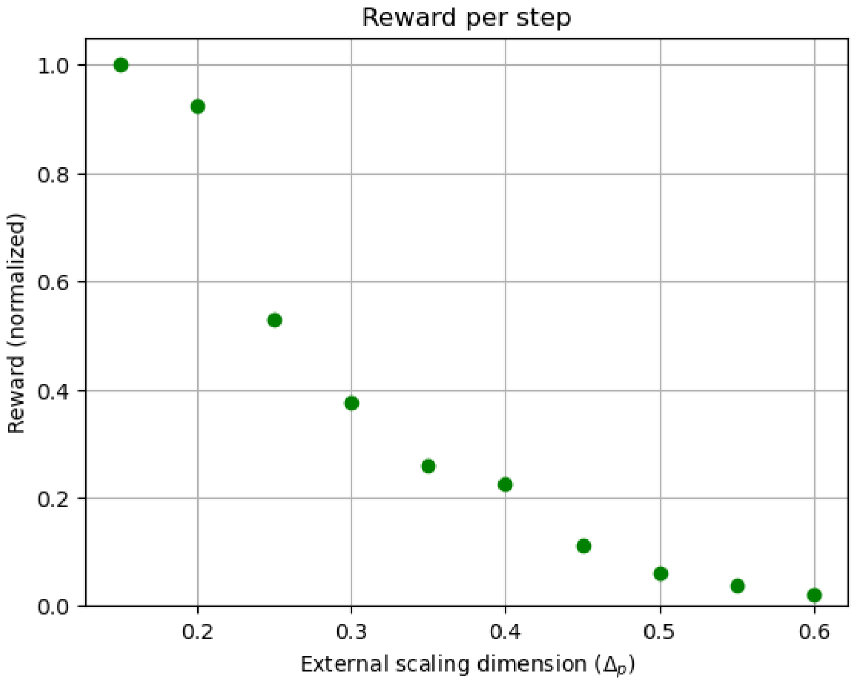
5.4 General Performance

Previously we saw how some of the data were tracked by the algorithm as we flew from $\Delta_p = 0.1$. Of course, these were just some examples of the 60 data we tried to follow. Although they individually seem to be reasonably accurate, the overall performance of the algorithm is something hard to measure.

In our example, where an analytic solution is available, checking the algorithms performance is an easy task. In general, for a theory with completely unknown features this would not be possible. As of now, it is not entirely clear what is the best way to evaluate the results.

The only measure that currently exists is the *reward* (1/loss) the agent achieves on each run. Even though, the correct solution is sure to have a high reward, nonsensical configurations also exist with relatively high reward. So how can we tell the difference? The answer is not clear yet but as of now, the best available answer is that one needs to do a great number of runs and treat the answers statistically.

Below, we plot the (normalized, weighted) average reward of the runs that contributed to the results.



As expected, the performance of the algorithm plummets as we flow away from the start. The reason lies back to the tail analysis of chapter 3. As Δ_p rises, the constant tail approximation becomes worse and at some point it breaks down completely.

Note that even on as low reward as in $\Delta_p = 0.5$ SAC is still able to get the low lying data with very decent accuracy.

In general, the results seem very good considering how minimal computational power was used to obtain them. Note however that for higher operators, the results are not that great.

Chapter 6

Conclusion and Outlook

In this thesis, we have presented an application of reinforcement learning techniques to solve the Conformal Bootstrap problem in Conformal Field Theories (CFTs). Our results demonstrate the effectiveness of using reinforcement learning as a tool for navigating the parameter space and finding solutions that satisfy the constraints imposed by the conformal bootstrap crossing equations.

Through extensive experimentation, we have shown that our approach yields reasonable results that align with known findings and exhibit desirable properties. The SAC agent effectively learns to explore the parameter space, guided by a reward function that encourages adherence to the conformal bootstrap constraints. This suggests that reinforcement learning can play a crucial role in assisting researchers in the field of theoretical physics to uncover new insights into CFTs.

It is worth noting that the results obtained here are more accurate than the ones obtained at [7]. The likeliest reasons are probably the inclusion of two (instead of one) crossing equations, the addition of the *tail* contribution and the superior z-grid used. The use of more operators (30 against 18) although it makes the best, theoretically, possible result better, is not bound to make the algorithm perform better as it might as well exponentially increase the number of high-reward low quality configurations the agent might get trapped in. Another determining factor seems to be the use of a better informed spin partition with the inclusion of the OPE selection rules imposed by the global U(1) symmetry. The latter one highlights how the inclusion of as much analytic information as possible can boost performance.

However, it is important to note that obtaining proper results using this approach requires a significant number of runs and subsequent statistical analysis. The Conformal Bootstrap problem is inherently complex, with a vast parameter space to explore, and the reinforcement learning agent needs time to converge to meaningful solutions. Therefore, to ensure the reliability and robustness of the results, we recommend conducting hundreds, if not thousands, of runs and applying statistical methods to analyze the outcomes.

Looking ahead, there are several promising directions for further research. Firstly, it would be valuable to extend our methodology to explore other CFT models. Different CFT setups, such as fermionic or supersymmetric theories, may present new challenges and insights. By adapting our reinforcement learning approach to these setups, we can gain a deeper understanding of their underlying properties and uncover novel conformal dimensions and operator coefficients.

Additionally, investigating the scalability of our approach to more complex CFTs is an intriguing avenue for future exploration. As the number of operators and dimensions increases, the parameter space expands exponentially, making the Conformal Bootstrap problem increasingly challenging. Utilizing reinforcement learning techniques may provide a scalable solution, allowing us to tackle more intricate CFTs and gain further insights into their behavior.

Appendix A

Reinforcement Learning Overview

A.1 Generalities

Reinforcement learning (RL) is a branch of machine learning that deals with the interaction of an agent with an environment to learn optimal decision-making policies. RL has gained significant attention due to its ability to solve complex problems by learning from trial-and-error experiences.

In RL, an agent (can be thought as the brain of the algorithm) interacts with an environment in a sequential manner. At each time step, the agent observes the current state of the environment, takes an action based on its policy, and receives a numerical reward signal as feedback. The goal of the agent is to learn a policy that maximizes its cumulative reward over time.

The RL framework can be formalized as a Markov Decision Process (MDP), defined by a tuple (S, A, P, R), where:

- S is the set of possible states in the environment.
- A is the set of possible actions the agent can take.
- P is the state transition probability function, which defines the probability of transitioning to a new state given the current state and action.
- R is the reward function, which assigns a numerical value to each state-action pair, indicating the desirability of taking that action in that state.

The agent's behavior is determined by its policy, which maps states to actions. Policies can be deterministic or stochastic. A deterministic policy selects a single action for each state, while a stochastic policy selects actions probabilistically.

To train the agent, RL algorithms often employ exploration-exploitation strategies. Exploration involves taking actions to gather new information and discover potentially better policies, while exploitation involves leveraging the current knowledge to select actions that yield high rewards.

Reinforcement learning has been successfully applied to various domains, including robotics, game playing, and natural language processing. It offers a powerful framework for solving problems that can be reduced to a search in a parameter space.

A.2 Soft Actor Critic (SAC)

The Soft Actor-Critic (SAC) is a state-of-the-art reinforcement learning algorithm that combines the principles of deep Q-learning and maximum entropy reinforcement learning. SAC aims to learn optimal policies for both discrete and continuous action spaces by maximizing the expected cumulative reward while also encouraging exploration and learning diverse behaviors.

The key idea behind SAC is to incorporate an entropy regularization term into the objective function. By maximizing the entropy of the policy, SAC promotes exploration and encourages the agent to discover a wide range of actions that can lead to high rewards. This helps address the exploration-exploitation trade-off in RL and enables the agent to discover novel solutions.

SAC leverages the actor-critic framework, which consists of two main components: an actor network and a critic network. The actor network is responsible for learning the policy and generating actions based on the observed states, while the critic network evaluates the quality of the actions by estimating the state-action value function.

The actor network is trained to maximize the expected cumulative reward, taking into account both the rewards obtained and the entropy of the policy. By maximizing entropy, the actor network explores the action space more extensively, leading to better policy exploration and improved generalization.

The critic network, on the other hand, is trained to estimate the state-action value function. It provides feedback to the actor network by estimating the expected cumulative reward. The critic network's role is crucial in guiding the learning process and ensuring that the actor network converges to an optimal policy.

SAC employs a soft update mechanism for the target networks to stabilize the learning process. Instead of performing a strict update using the current estimates, SAC performs a soft update by blending the current network parameters with the target network parameters. This soft update strategy helps prevent oscillations and improves the stability of the learning process.

Furthermore, SAC utilizes a replay buffer, which stores the agent's experiences, to break the correlation between consecutive samples and improve sample efficiency. By randomly sampling from the replay buffer during training, SAC can learn from a diverse set of experiences and reduce the variance in the learning process.

The combination of entropy regularization, actor-critic architecture, soft target updates, and experience replay makes Soft Actor-Critic a powerful algorithm for reinforcement learning in continuous action spaces. SAC has achieved state-of-the-art performance on various challenging tasks, including robotic control, locomotion, and manipulation.

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