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Analysis and Numerical methods for Phase Transition type problems

PhD Thesis
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Chapter 1

Abstract - Brief Introduction

In this dissertation, we focus on phase transition type problems. Let us briefly discuss the physical motivation of such problems in the simple case of two phases. We are given some substance in a container, which may exhibit two phases, say a_1 and a_2, and we would like to describe it mathematically. One approach could be that the interface formulation is driven by a variational principle, that is the pattern in the outcome of the minimization of a certain energy. For this, we may consider a "double well" potential W such that $W(a_1) = W(a_2) = 0$ and $W > 0$ otherwise. Next, one introduces a gradient term that penalizes the formulation of interfaces and measures interface energy. This is the Van der Waals free energy functional. To be more precise,

$$J_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dx \quad , \quad u : \Omega \subset \mathbb{R}^n \to \mathbb{R}. \tag{1.1}$$

We often call J_ε as the Allen-Cahn functional. Such a gradient term reduces the number of interfaces of the minimizers of J_ε, which turn out to be smooth functions interpolating between the phases a_1 and a_2 with level sets approaching hypersurfaces of least possible area. Therefore, our problem is also closely related to the theory of minimal surfaces. Phase transition type problems arise in many experimental disciplines, such as material science.

For studying three or more phases, one naturally is lead to the vector case. The equations arising in phase transition type problems are called Allen-Cahn equations or systems of equations in the case of more than two phases. Particularly,

$$\Delta u = W_u(u) \quad , \quad \text{where } \ u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m. \tag{1.2}$$

This dissertation is consisted of five independent parts. In the first part contained in Chapter 2, which is a work together with professor N. Alikakos and professor A. Zarnescu that can be found in [1], we study entire solutions of the

Allen-Cahn systems that are also minimizers. In particular, the specific feature of our systems are potentials having a finite number of global minima (i.e. the phases), with sub-quadratic behavior locally near their minima. We focus on qualitative aspects and we show the existence of entire solutions in an equivariant setting connecting the minima of W at infinity, thus modeling many coexisting phases, possessing free boundaries and minimizing energy in the symmetry class. The existence of a free boundary can be related to the existence of a specific sub-quadratic feature, a dead core, whose size is also quantified.

In the second part which is in Chapter 3, motivated by the relationship of phase transition type problems with minimal surfaces, we determine a transformation that transforms equipartitioned solutions of the Allen-Cahn equations in dimension three to the minimal surface equation of one dimension less. This is an application of a more general transformation introduced in this work which relates the solutions of the Allen-Cahn equations that are equipartitioned to solutions of the incompressible Euler equations with constant pressure. Other applications are De Giorgi type results, that is, the level sets of entire solutions are hyperplanes. Also, we determine the structure of solutions of the Allen-Cahn system in two dimensions that satisfy the equipartition of the energy and we apply the Leray projection to provide explicit entire solutions to analyze this structure. In addition, we obtain some examples of smooth entire solutions of the Euler equations, some of which, can be extended to the Navier-Stokes equations for specific type of initial conditions. This work can be found in [2].

In the following part, which concerns the work in [3] and is contained in Chapter 4, we are dealing with the $\Gamma-$ convergence of the Allen-Cahn functional with Dirichlet boundary conditions in the vectorial case. Let us briefly describe the analog of the $\Gamma-$limit result in the scalar case. Assume F_ε is the $\varepsilon-$energy functional of the Allen-Cahn equation,

$$F_\varepsilon(u,\Omega) := \begin{cases} \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)dx & , \ u \in W^{1,2}(\Omega;\mathbb{R}) \\ +\infty & , \text{ elsewhere} \end{cases} \tag{1.3}$$

then it is a classical well known theorem that the $\Gamma-$limit of F_ε is the perimeter functional F_0 which measures the transitions between the two phases of the problem, i.e.

$$F_0(u,\Omega) := \begin{cases} \sigma\mathcal{H}^{n-1}(Su) & , \ u \in BV(\Omega;\{-1,1\}) \\ +\infty & , \text{ elsewhere} \end{cases} \tag{1.4}$$

$$\text{where } W:\mathbb{R}\to[0,+\infty)\ ,\ \{W=0\}=\{-1,1\}\ ,\ \sigma=\int_{-1}^{1}\sqrt{2W(u)}du \tag{1.5}$$

$$\textit{and } \ Su \text{ is the singular set of the function } u. \tag{1.6}$$

Thus, the interfaces of the limiting problem will be minimal surfaces. We provide all the appropriate references and previous fundamental contributions on the topic in this third part. So, in the vectorial case that we study, one expects that the $\Gamma-$limit turn out to be the perimeter functional that measures the transition between the $N-$ phases of the problem. We prove this fact with the constraint of boundary conditions. In this case, the minimizers of the limiting functional are closely related to minimizing partitions of the domain. Moreover, utilizing that the triod and the straight line are the only minimal cones in the plane together with regularity results for minimal curves, we determine the precise structure of the minimizers of the limiting functional, and thus the limit of minimizers of the $\varepsilon-$ energy functional as $\varepsilon \to 0$. We also prove that the minimizer of the limiting functional in the disc is unique.

Next, in the forth part which is in Chapter 5, we study fully nonlinear elliptic equations via the notion of $P-$ functions. $P-$ functions can be thought as quantities of the solution of a general fully nonlinear partial differential equation that satisfy the maximum principle. Perhaps the most well-known example is

$$P(u;x) = \frac{1}{2}|\nabla u|^2 - W(u) \tag{1.7}$$

that is related to the Allen-Cahn equation

$$\Delta u = W'(u) \ , \ u : \Omega \subset \mathbb{R}^n \to \mathbb{R} \tag{1.8}$$

and one important application is the Modica inequality

$$\frac{1}{2}|\nabla u|^2 \leq W(u) \tag{1.9}$$

for every bounded solution of (1.8). There are many generalizations to Quasi-linear elliptic equations among other types of equations with applications such as gradient bounds and Liouville theorems which we refer in detail in the respective chapter of this forth part. In our work, which can be found in [4], we introduce the notion of $P-$ functions for fully nonlinear equations and establish some general criterion for obtaining such quantities for this class of equations. Some applications are gradient bounds, De Giorgi-type properties of entire solutions and rigidity results. Furthermore, we prove Harnack-type inequalities and local pointwise estimates for the gradient of solutions to fully nonlinear elliptic equations. Additionally, we consider $P-$functions for higher order nonlinear equations and for equations of order greater than two we obtain Liouville-type theorems and pointwise estimates for the Laplacian.

Finally, in Chapter 6, the last part of our thesis includes a work with professor C. Makridakis and G. Gkanis that can be found in [5], in which we study applications of Physics Informed Neural Networks to partial differential equations. Physics Informed Neural Networks is a numerical method which uses neural networks to approximate solutions of partial differential equations. It has received a lot of attention and is currently used in numerous physical and engineering problems. The mathematical understanding of these methods is limited, and in particular, it seems that, a consistent notion of stability is missing. Towards addressing this issue we consider model problems of partial differential equations, namely linear elliptic and parabolic PDEs. We consider problems with different stability properties, and problems with time discrete training. Motivated by tools of nonlinear calculus of variations we systematically show that coercivity of the energies and associated compactness provide the right framework for stability. For time discrete training we show that if these properties fail to hold then methods may become unstable. Furthermore, using tools of $\Gamma-$convergence we provide new convergence results for weak solutions by only requiring that the neural network spaces are chosen to have suitable approximation properties. These techniques can be extended to various other, possibly nonlinear, problems.

Περίληψη - Συνοπτική Εισαγωγή

Σε αυτή την διατριβή, μελετάμε κυρίως προβλήματα αλλαγής φάσεων. Ας περιγράψουμε σύντομα την υποκείμενη φυσική φαινομενολογία στην απλή περίπτωση των δύο φάσεων. Δοθέντος μιας ουσίας σε ένα δοχείο, η οποία μπορεί να λαμβάνει δύο φάσεις, έστω a_1 και a_2, και θα θέλαμε να το περιγράψουμε με μαθηματικά. Μια προσέγγιση θα μπορούσε να είναι ότι η διαμόρφωση διεπιφανειών περιγράφεται από μια μεταβολική αρχή, το οποίο προκύπτει ως αποτέλεσμα της ελαχιστοποίησης μιας συγκεκριμένης ενέργειας. Για το λόγο αυτό, μπορούμε να θερήσουμε ένα "double well" δυναμικό W τέτοιο ώστε $W(a_1) = W(a_2) = 0$ και $W > 0$ αλλιώς. Έπειτα, εισάγει κανείς έναν όρο παραγώγου ο οποίος να επιβάλλει την δημιουργία διεπιφανειών και να μετράει την ενέργεια των διεπιφανειών αυτών. Αυτό είναι το Van der Waals συναρτησοειδές της ενέργειας. Πιο συγκεκριμένα,

$$J_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dx \quad , \quad u : \Omega \subset \mathbb{R}^n \to \mathbb{R}. \tag{1.10}$$

Συχνά ονομάζουμε το J_ε και ως το συναρτησοειδές Allen-Cahn. Αυτός ο όρος παραγώγου μειώνει τον αριθμό των διεπιφανειών στους ελαχιστοποιητές του J_ε, οι οποίοι προκύπτει ότι είναι ομαλές συναρτήσεις που παρεμβάλονται μεταξύ των φάσεων a_1 και a_2 και τα σύνολα στάθμης τους προσεγγίζουν υπερεπιφάνειες με το λιγότερο δυνατό εμβαδόν. Επομένως, το πρόβλημα μας είναι στενά συνδεδεμένο με την θεωρία ελαχιστικών επιφανειών. Τα προβλήματα αλλαγής φάσεων προκύπτουν σε πολλές πειραματικές επιστήμες, οπώς για παράδειγμα στην επιστήμη των υλικών.

Για την μελέτη τριών ή περισσότερων φάσεων, οδηγείται κανείς με φυσιολογικό τρόπο στη διανυσματική περίπτωση. Οι εξισώσεις που προκύπτουν σε προβλήματα αλλαγής φάσεων ονομάζονται εξισώσεις Allen-Cahn ή σύστημα εξισώσεων Allen-Cahn στην περίπτωση που υπάρχουν περισσότερες από δύο φάσεις. Συγκεκριμένα,

$$\Delta u = W_u(u) \quad , \quad \text{όπου} \;\; u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m. \tag{1.11}$$

Αυτή η διατριβή αποτελείται από πέντε ανεξάρτητα μέρη. Στο πρώτο μέρος που περιέχεται στο Κεφάλαιο 2, το οποίο είναι μια εργασία μαζί με τον καθηγητή Ν. Αλικάκο και τον καθηγητή A. Zarnescu και μπορεί να βρεθεί στο [1], μελετάμε ολικές λύσεις των συστημάτων Allen-Cahn οι οποίες είναι και ελαχιστοποιητές. Ειδικότερα, το χαρακτιριστικό γνώρισμα των συστημάτων που μελετάμε είναι δυναμικά που έχουν πεπερασμένο αριθμό ολικών ελαχίστων (δηλ. φάσεων), με τοπικά υποδευτεροβάθμια (sub-quadratic) συμπεριφορά κοντά στα ελάχιστα. Εστιάζουμε σε ποιοτικές πτυχές και δείχνουμε την ύπαρξη ολικών λύσεων σε μια 'ισαλλοίωτη κλάση' (equivariant class) και συνδένουν τα ελάχιστα του δυναμικού W στο άπειρο, επομένως μοντελοποιούν πολλές συνυπάρχουσες φάσεις, δημιουργώντας ελεύθερα σύνορα και ελαχιστοποιούν την ενέργεια στη συμμετρική κλάση. Η ύπαρξη του ελεύθερου συνόρου μπορεί να σχετιστεί με την ύπαρξη του ειδικού υποδευτεροβάθμιου (sub-quadratic) χαρακτήρα, ενός 'νεκρού πυρήνα' (dead core), του οποίου το μέγεθος επίσης ποσοτικοποιούμε.

Στο δεύτερο μέρος που βρίσκεται στο Κεφάλαιο 3, παρακινούμενοι από τη σχέση μεταξύ των προβλημάτων αλλαγής φάσεων με τις ελαχιστικές επιφάνειες, προσδιορίζουμε έναν μετασχηματισμό που μετασχηματίζει τις ισοκατανεμημένες λύσεις των εξισώσεων Allen-Cahn στη διάσταση τρία σε λύσεις της εξίσωσης της ελαχιστικής επιφάνειας μιας διάστασης λιγότερο. Αυτό είναι μια εφαρμογή ενός γενικότερου μετασχηματισμού που εισάγεται σε αυτή την εργασία ο οποίος συσχετίζει τις ισοκατανεμιμένες λύσεις την εξίσωσης Allen-Cahn με τις ασυμπίεστες εξισώσεις Euler με σταθερή πίεση. Άλλες εφαρμογές είναι De Giorgi τύπου αποτελέσματα, δηλαδή, ότι τα σύνολα στάθμης ολικών λύσεων είναι υπερεπιφάνειες. Επίσης, προσδιορίζουμε τη δομή των λύσεων του συστήματος Allen-Cahn στις δύο διαστάσεις οι οποίες ικανοποιούν την ισοκατανομή της ενέργειας και εφαρμόζουμε την προβολή Leray για να δώσουμε παραδείγματα λύσεων σε κλειστή μορφή και για να αναλύσουμε τη δομή αυτή. Επιπλέον, παρέχουμε παραδείγματα ομαλών ολικών λύσεων των εξισώσεων Euler, κάποια από τα οποία μπορούν να επεκταθούν και για τις εξισώσεις Navier-Stokes για ειδικού τύπου αρχικές συνθήκες. Η εργασία αυτή μπορεί να βρεθεί στο [2].

Το επόμενο μέρος, που αφορά την εργασία στο [3] και περιλαμβάνεται στο Κεφάλαιο 4, έχει να κάνει με την $\Gamma-$σύγκλιση του συναρτησοειδούς Allen-Cahn με Dirichlet συνοριακές συνθήκες στην διανυσματική περίπτωση. Ας περιγράψουμε συνοπτικά το ανάλογο αποτέλεσμα του $\Gamma-$ορίου στην βαθμωτή περίπτωση. Έστω F_ε το $\varepsilon-$συναρτησοειδές της ενέργειας των εξισώσεων Allen-Cahn,

$$F_\varepsilon(u,\Omega) := \begin{cases} \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)dx & , \ u \in W^{1,2}(\Omega;\mathbb{R}) \\ +\infty & , \ \text{αλλού} \end{cases} \tag{1.12}$$

τότε από κλασσικό γνωστό αποτέλεσμα προκύπτει ότι το $\Gamma-$όριο του F_ε είναι το συναρτησοειδές της περιμέτρου F_0 το οποίο μετράει τις μεταβιβάσεις μεταξύ των φάσεων του προβλήματος, δηλ.

$$F_0(u,\Omega) := \begin{cases} \sigma\mathcal{H}^{n-1}(Su) & , \ u \in BV(\Omega;\{-1,1\}) \\ +\infty & , \ \text{αλλού} \end{cases}$$

$$\text{όπου } W:\mathbb{R}\to[0,+\infty) \ , \ \{W=0\}=\{-1,1\} \ , \ \sigma = \int_{-1}^{1}\sqrt{2W(u)}du \tag{1.13}$$

και Su είναι το σύνολο ανωμαλιών της συνάρτησης u.

Επομένως, οι διεπιφάνειες του οριακού προβλήματος θα είναι ελαχιστικές επιφάνειες. Παρέχουμε όλες τις απαραίτητες αναφορές και τις προηγούμενες θεμελιώδεις συνεισφορές στο θέμα σε αυτό το τρίτο μέρος. Άρα, στη διανυσματική περίπτωση που μελετάμε, περιμένει κανείς ότι το $\Gamma-$όριο θα είναι το συναρτησοειδές της περιμέτρου που μετράει τις μεταβιβάσεις μεταξύ των $N-$φάσεων του προβλήματος. Το αποδεικνύουμε αυτό με την προσθήκη συνοριακών συνθηκών. Σε αυτή την περίπτωση, οι ελαχιστοποιητές του οριακού προβλήματος είναι στενά συνδεδεμένοι με τις ελαχιστικές διαμερίσεις του χωρίου. Επιπροσθέτως, χρησιμοποιώντας ότι το τρίποδο (triod) και η ευθεία είναι οι μοναδικοί ελαχιστικοί κώνοι στο επίπεδο, εφαρμόζοντας και αποτελέσματα ομαλότητας για ελαχιστικές καμπύλες, προσδορίζουμε την ακριβή δομή των ελαχιστοποιητών του οριακού προβληματος, και ως εκτούτου, του ορίου των ελαχιστοποιητών του $\varepsilon-$ συναρτησοειδούς της ενέργειας καθώς το $\varepsilon \to 0$. Αποδεικνύουμε επίσης ότι ο ελαχιστοποιητής του οριακού προβλήματος στον δίσκο είναι μοναδικός.

Στη συνέχεια, στο τέταρτο μέρος που βρίσκεται στο Κεφάλαιο 5, μελετάμε πλήρως μη γραμμικές ελλιπτικές (fully nonlinear elliptic) εξισώσεις μέσω της έννοιας των $P-$ συναρτήσεων. Τις $P-$ συναρτήσεις μπορούμε να τις σκεφτόμαστε σαν ποσότητες της λύσης μιας γενικής πλήρως μη γραμμικής μερικής διαφορικής εξίσωσης που ικανοποιούν την αρχή μεγίστου. Πιθανώς το πιο γνωστό παράδειγμα είναι το εξής

$$P(u;x) = \frac{1}{2}|\nabla u|^2 - W(u) \tag{1.14}$$

που σχετίζεται με την εξίσωση Allen-Cahn

$$\Delta u = W'(u) \ , \ u : \Omega \subset \mathbb{R}^n \to \mathbb{R} \tag{1.15}$$

και μια σημαντική εφαρμογή είναι η ανισότητα Modica

$$\frac{1}{2}|\nabla u|^2 \leq W(u) \tag{1.16}$$

για κάθε φραγμένη λύση της (1.15). Υπάρχουν πολλές γενικεύσεις σε Quasi-linear elliptic εξισώσεις μεταξύ άλλων με εφαρμογές όπως φράγματα στις παραγώγους και Liouville τύπου αποτελέσματα, για τις οποίες παρέχουμε τις αντίστοιχες αναφορές με λεπτομέρειες στο κεφάλαιο που αντιστοιχεί στο τέταρτο αυτό μέρος. Στη δουλειά μας αυτή, η οποία μπορεί να βρεθεί στο [4], εισάγουμε την έννοια των $P-$ συναρτήσεων για πλήρως μη γραμμικές εξισώσεις και παρέχουμε ένα γενικό κριτήριο για να βρίσκει κανείς τέτοιες ποσότητες για αυτή την κλάση των εξισώσεων. Μερικές εφαρμογές είναι φράγματα στις παραγώγους, αποτελέσματα τύπου De Giorgi για ολικές λύσεις και αποτελέσματα ακαμψίας (rigidity results). Επιπροσθέτως, αποδεικνύουμε Harnack τύπου ανισότητες και τοπικά σημειακές εκτιμήσεις για τις παραγώγους των λύσεων πλήρως μη γραμμικών ελλιπτικών εξισώσεων. Επιπλέον, θεωρούμε $P-$ συναρτήσεις για υψηλότερης τάξης μη γραμμικών εξισώσεων και για εξισώσεις βαθμού μεγαλύτερου του δύο παίρνουμε Liouville τύπου θεωρήματα και σημειακές εκτιμήσεις για τη Λαπλασιανή.

Τέλος, στο Κεφάλαιο 6, το τελευταίο μέρος της διατριβής μας περιλαμβάνει μια εργασία με τον καθηγητή Χ. Μακριδάκη και τον Γ. Γκανή η οποία μπορεί να βρεθεί στο [5], όπου μελετάμε εφαρμογές των Physics Informed Νευρωνικών Δικτύων στις μερικές διαφορικές εξισώσεις. Τα Physics Informed Νευρωνικά Δίκτυα είναι μια αριθμητική μέθοδος που χρησιμοποιεί νευρωνικά δίκτυα για να προσεγγίσει λύσεις μερικών διαφορικών εξισώσεων. Έχει αρχίσει να λαμβάνει ολοένα και περισσότερη προσοχή τα τελευταία χρόνια και χρησιμοποιείται για πολυάριθμα προβλήματα φυσικής καθώς και μηχανικής. Η μαθηματική κατανόηση των μεθόδων αυτών είναι περιορισμένη, και ειδικότερα, απ΄ ότι φαίνεται, μια συνεπής έννοια ευστάθειας λείπει. Για την αντιμετώπιση αυτού του προβλήματος θεωρούμε μοντέλα προβλημάτων μερικών διαφορικών εξισώσεων, συγκεκριμένα γραμμικές ελλειπτικές και παραβολικές ΜΔΕ. Θεωρούμε προβλήματα με διαφορετικές ιδιότητες ευστάθειας, και προβλήματα διακριτώς χρονικά προσεγγίσιμα. Παρακινούμενοι από εργαλεία μη γραμμικού λογισμού μεταβολών δείχνουμε με συστηματικό τρόπο ότι η coercivity των ενεργειών και η συσχετιζόμενη συμπάγεια παρέχουν το κατάλληλο πλαίσιο για ευστάθεια. Για τις προσεγγίσεις διακριτού χρόνου δείχνουμε ότι αν κάποια από αυτές τις ιδιότητες αποτυγχάνουν να ισχύουν τότε οι μέθοδοι μπορούν να γίνουν ασταθείς. Επιπρόσθετα, χρησιμοποιώντας εργαλεία της $\Gamma-$ σύγκλισης παρέχουμε νέα αποτελέσματα σύγκλισης για ασθενείς λύσεις απαιτώντας μόνο το ότι οι χώροι των νευρωνικών δικτύων επιλέγονται ώστε να έχουν τις κατάλληλες προσεγγιστικές ιδιότητες. Αυτές οι τεχνικές μπορούν να επεκταθούν σε πολλά άλλα, πιθανώς μη γραμμικά, προβλήματα.

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Chapter 2

Entire Minimizers of Allen-Cahn Systems with Sub-Quadratic Potentials

Abstract

We study entire minimizers of the Allen-Cahn systems. The specific feature of our systems are potentials having a finite number of global minima, with sub-quadratic behaviour locally near their minima. The corresponding formal Euler-Lagrange equations are supplemented with free boundaries.

We do not study regularity issues but focus on qualitative aspects. We show the existence of entire solutions in an equivariant setting connecting the minima of W at infinity, thus modeling many coexisting phases, possessing free boundaries and minimizing energy in the symmetry class. We also present a very modest result of existence of free boundaries under no symmetry hypotheses. The existence of a free boundary can be related to the existence of a specific sub-quadratic feature, a dead core, whose size is also quantified.

2.1 Introduction and Main Results

In this note we consider minimizers in the whole space $\mathbb{R}^n$ for the functional

$$J(u) = \int \frac{1}{2}|\nabla u|^2 + W(u)dx \tag{2.1}$$

with $u : \mathbb{R}^n \to \mathbb{R}^m$. We take $W \geq 0$ and $\{W = 0\} = \{a_1, ..., a_N\} := A$, for some distinct points $a_1, ..., a_N \in \mathbb{R}^m$ that can physically model the phases of a substance that can exist in $N \geq 2$ equally preferred states.

We assume that

$$\liminf_{|z|\to\infty} W(z) > 0 \tag{2.2}$$

If W is smooth then the first derivatives vanish at the minimum points and the generic local behaviour near such a minimum, say a_i, is locally of quadratic nature, of the type $|u - a_i|^2$. The minimizers satisfy the Euler-Lagrange system

$$\Delta u - W_u(u) = 0. \tag{2.3}$$

We are interested in the class of solutions that connect in some way the phases or a subset of them. The scalar case $m = 1$ has been extensively studied with $N = 2$ that is the natural choice. The reader may consult [20], [26], [41] where further references can be found. A well known conjecture of De Giorgi (1978) and its solution about thirty years later, played a significant role in the development of a large part of this work.

The vector case $m \geq 2$ by comparison has been studied very little. We note that for coexistence of three or more phases a vector order parameter is necessary and so there is physical interest for the system.

For $m \geq 2$, (2.3) has been mainly studied in the class of equivariant solutions with respect to reflection groups beginning with [13] and later [27] and significantly extended and generalized in various ways [6], [3], [24], [4], [7], [11]. We refer to [1] where existence under symmetry is covered and where more references can be found.

Degenerate, super-quadratic behavior at the minima has also been considered for (2.3), $m = 1$, in [12], [21].

The focus of our work will be on going beyond this classical setting and explore the phenomena that are associated having sub-quadratic behaviour at the minima. Specifically, our potentials are modelled near their minima $a \in A$ after $|u - a|^\alpha$, for $0 < \alpha < 2$. Furthermore we will consider also the limiting case $\alpha = 0$ (that appears in a Γ-limit setting as $\alpha \to 0$). Formally, the minimizers solve certain free boundary problems:

1. For $\alpha \in (0,2)$:
$$\begin{cases} \Delta u = W_u(u) \quad \text{for } \{u(x) \notin A\} \\ |\nabla u|^2 = 0 \quad \text{for } \partial\{u(x) \notin A\} \end{cases} \tag{2.4}$$

2. For $\alpha = 0$:
$$\begin{cases} \Delta u = 0 \quad \text{for } \{u(x) \notin A\} \\ |\nabla u|^2 = 2 \quad \text{for } \partial\{u(x) \notin A\} \end{cases} \tag{2.5}$$

   In Appendix 2.5 we give a formal justification of these, that can be made rigorous with suitable regularity results, [8]. We note that for $\alpha = 2$, Corollary 3.1 p.92 in [1] states that if both $W(u(x)) = 0$ and $|\nabla u(x)|^2 = O(W(u(x))$ then $u \equiv a_i$. This latter condition holds in the scalar case, $m = 1$, by the Modica inequality. Hence for $\alpha = 2, m = 1$ we have $\partial\{u(x) \notin A\} = \emptyset$. Thus a free boundary may be expected only in the non smooth case. The reason is rather simple and can be traced back to the non-uniqueness of the trivial solution of the ODE $u' = \frac{2}{2-\alpha} C^{\frac{\alpha}{2}} u^{\frac{\alpha}{2}}$ that describes the behavior of the one-dimensional solutions (connections) near the minimum of W of (2.4), (2.5).

Thus we focus on the range $0 \le \alpha < 2$. An important special case of the potentials we consider is given, for the set of minima $A = \{a_1, \ldots, a_N\}$ by

$$W^{\overline{\alpha}}(u) = \prod_{k=1}^{N} |u - a_k|^{\alpha_k} \quad , \ \overline{\alpha} = (\alpha_1, ..., \alpha_N); 0 < \alpha_k < 2, \ \forall\, k \in \{1, \ldots, N\} \tag{2.6}$$

More generally, motivated by the form of W in (2.6), we assume:

$$\textbf{(H1)} \begin{cases} \underline{0 < \alpha < 2} : W \in C(\mathbb{R}^m; [0, +\infty)) \text{ with } \ \{W = 0\} = \{a_1, ..., a_N\} \neq \emptyset \ (N \ge 2). \\ \text{For } \ a \in \ \{W = 0\} \text{ the function } W \text{ is differentiable in a deleted} \\ \text{neighborhood of } a \text{ and satisfies } \frac{d}{d\rho} W(a + \rho\xi) \ge \alpha C^* \rho^{\alpha-1} \ , \ \forall\, \rho \in (0, \rho_0] \ , \\ \forall\, \xi \in \mathbb{R}^m : |\xi| = 1, \text{ for some constants } \rho_0 > 0, C^* > 0 \text{ independent of } \alpha. \end{cases}$$

$$\begin{cases} \underline{\alpha = 0} : \{W = 0\} = \{a_1, ..., a_N\} := A \ , \ W(u) := W^0(u) := \chi_{\{u \in S_A\}} \\ S_A := \{\sum_{i=1}^N \lambda_i a_i \ , \ \lambda_i \in [0, 1) \ , \ \forall \ i = 1, ..., N \ , \sum_{i=1}^N \lambda_i = 1 \ , \ N = m + 1\} \\ \text{We assume that the simplex } S_A \text{ is nondegenerate, that is the vectors} \\ \{a_2 - a_1, ..., a_{m+1} - a_1\} \text{ are linearly independent and } m \ge 2. \end{cases}$$

Clearly $W^{\overline{\alpha}}$ in (2.6) satisfy **(H1)** $(0 < \alpha < 2)$.

We are primarily interested in bounded minimizers defined on $\mathbb{R}^n$. We note in passing that the only critical points of $J_{\mathbb{R}^n}$, $n \geq 2$, with bounded energy are trivial [2]. A minimizer u, by definition minimizes energy subject to its Dirichlet values on any open, bounded $\Omega \subset \mathbb{R}^n$. More precisely,

Definition 2.1.1. Let $\mathcal{O} \subset \mathbb{R}^n$ open. A map $u \in W^{1,2}_{loc}(\mathcal{O}, \mathbb{R}^m) \cap L^\infty(\mathcal{O}; \mathbb{R}^m)$ is called a *minimizer* of the energy functional J defined in (2.1) if

$$J_\Omega(u+v) \geq J_\Omega(u) \ \text{, for } \ v \in W^{1,2}_0(\Omega, \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m) \tag{2.7}$$

for every open bounded Lipschitz set $\Omega \subset \mathcal{O}$, with J_Ω denoting the value of the integral in (2.1) when integrating over the domain Ω.

The case, $\alpha = 0$ for $m = 1$ was introduced and extensively studied by Caffarelli and his collaborators, with particular attention to the optimal regularity of the solution and to the regularity of the free boundary. These are important classical results that can be found for example in the books [16] or [33]. There is recent interest in the vector case for free boundary problems. We mention below two papers which relate to our work and where additional references can be found.

In [17] the authors study minimizers of the functional

$$\int_\Omega (\frac{1}{2}|\nabla u|^2 + Q^2(x)\chi_{\{|u|>0\}})dx \tag{2.8}$$

with $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $u_i \geq 0$, Ω bounded and $u = g$ on $\partial\Omega$. This corresponds to a cooperative system, and is a one-phase Bernoulli-type problem. On the other hand, our nonlinearity is of the competitive kind and our problem is a two-phase Bernoulli-type problem.

In [29] the functional that is studied is

$$\sum_{i=1}^m \int_\Omega \frac{1}{2}|\nabla u_i|^2 + \Lambda\mathcal{L}^n(\cup_{i=1}^m \{u_i \neq 0\})dx \tag{2.9}$$

with $u_i = \phi_i$ on $\partial\Omega$. This is a two-phase type problem and it is quite close to our functional for $\alpha = 0$.

The emphasis in these works is on the regularity of the solution and of the free boundary, while the existence of the free boundary is forced by the Dirichlet condition on $\partial\Omega$, and is not an issue in that context.

For stating our main results we need some algebraic preliminaries.

A *reflection point group* G is a finite subgroup of the orthogonal group whose elements g fix the origin. We will be assuming for simplicity that $m = n$ (the general case is presented in [1], Chapter 7), and that G acts both on the domain space $\mathbb{R}^n$ and the target space $\mathbb{R}^m$. A map $u : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *equivariant* with respect to the action of G, simply equivariant, if

$$u(gx) = gu(x) \quad , \forall\, g \in G \;,\; x \in \mathbb{R}^n$$

A *reflection* $\gamma \in G$ is a map $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$\gamma x = x - 2(x \cdot n_\gamma)n_\gamma \;,\; for \;\; x \in \mathbb{R}^n$$

for some unit vector $n_\gamma \in \mathbb{S}^{n-1}$ which aside from its orientation is uniquely determined by γ. The hyperplane

$$\pi_\gamma = \{x \in \mathbb{R}^n : x \cdot n_\gamma = 0 \;\}$$

is the set of the points that are fixed by γ. The open half space $\mathcal{S}^+_\gamma = \{x \in \mathbb{R}^n : x \cdot n_\gamma > 0\}$ depends on the orientation of n_γ. We let $\Gamma \subset G$ denote the set of all reflections in G. Every finite subgroup of the orthogonal group $O(\mathbb{R}^n)$ has a *fundamental region*, that is a subset $F \subset \mathbb{R}^n$ with the following properties:

1. F is open and convex,
2. $F \cap gF = \emptyset$ for $I \neq g \in G$, where I is the identity,
3. $\mathbb{R}^n = \cup\{g\overline{F} : g \in G\}$.

The set $\cup_{\gamma\in\Gamma}\pi_\gamma$ divides $\mathbb{R}^n \setminus \cup_{\gamma\in\Gamma}\pi_\gamma$ in exactly $|G|$ congruent conical regions. Each one of these regions can be identified with the fundamental region F for the action of G on $\mathbb{R}^n$. We assume that the orientations of n_γ are such that $F \subset \mathcal{S}^+_\gamma$ and we have

$$F = \cap_{\gamma\in\Gamma}\mathcal{S}^+_\gamma$$

Given $a \in \mathbb{R}^n$, the *stabilizer* of a, denoted by $G_a \subset G$ is the subgroup of the elements $g \in G$ that fix a:

$$G_a = \{g \in G : ga = a\}.$$

We now introduce two more hypotheses:

(H2)(symmetry) The potential W is invariant under a reflection (point) group G acting on $\mathbb{R}^n$, that is

$$W(gu) = W(u) \quad \text{for all} \;\; g \in G \;\; \text{and} \;\; u \in \mathbb{R}^n.$$

Moreover we assume (2.2).

(H3)(Location and number of global minima) Let $F \subset \mathbb{R}^n$ be a fundamental region of G. We assume that $\overline{F}$ contains a single global minimum of W say $a_1 \neq 0$, and let G_{a_1} be the stabilizer of a_1. Setting $D := Int(\cup_{g \in G_{a_1}} g\overline{F})$, a_1 is also the unique global minimum of W in the region D.

Notice that, by the invariance of W, Hypothesis **(H3)** implies that the number of minima of W is

$$N = \frac{|G|}{|G_{a_1}|},$$

where $|\cdot|$ stands for the number of elements.

We can now state our first main result.

Theorem 2.1.1. $(0 < \alpha < 2)$ Under hypothesis **(H1)-(H3)**, there exists an equivariant minimizer u of J, $u : \mathbb{R}^n \to \mathbb{R}^n$, such that
1. $|u(x) - a_1| = 0$ for $x \in D$ and $d(x, \partial D) \geq d_0$, where d_0 a positive constant depending on $||u||_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}$, C^* and α ($d_0 \to +\infty$ as $\alpha \to 2$).
2. $u(\overline{F}) \subset \overline{F}$, $u(\overline{D}) \subset \overline{D}$ (positivity).

Hence by equivariance the statements above hold for all a_i , $i = 1, ..., N$, in the respective copy of D.

Remark 2.1.2. In [8] it is shown that $u \in C^{2,\alpha-1}_{loc}$ for $\alpha \in (1, 2)$, $u \in C^{1,\gamma}_{loc}$ for any $\gamma \in (0, 1)$ and $u \in C^{1,\frac{\alpha}{1-\alpha}}$ for $\alpha \in (0, 1)$. The regularity for $\alpha \in (0, 1)$ is optimal. In Lemma 2.2.10 we establish the (suboptimal) estimate $|u|_{C^\beta} < \infty$ (any $\beta \in (0, 1)$) that holds for all $\alpha \in [0, 2)$ which is sufficient for our purposes. We revisit this point also later.

The analog of Theorem 2.1.1 for $\alpha = 2$, $W \in C^2$ was established in a series of papers by the first author and G.Fusco. It can be found in [1] (Theorem 6.1) where detailed references are given. The main difference with Theorem 2.1.1 above is that the condition $|u(x) - a_1| = 0$ for $x \in D$, $d(x, D) \geq d_0$, is replaced by $|u(x) - a_1| \leq Ke^{-kd(x,\partial D)}$, $x \in D$, where k , K are positive constants. In that context the minimizer u is a classical solution of (2.3) while in the present context u is a weak $W^{1,2}_{loc}$ solution of (2.3) in the complement of the free boundary $\partial\{u(x) \notin A\}$. The theorem in the smooth case is utilized in our proof of Theorem 2.1.1 where we are constructing a minimizer with the positivity property via a C^2 regularization of the potential. We thus bypass the gradient flow argument used in the proof of the $\alpha = 2$ case in [1] that would be problematic in the present setting. The role of positivity can be seen in the following proposition, which does not presuppose symmetry.

Proposition 2.1.3. ($0 < \alpha < 2$) (i) Assume that W as in (**H1**) above, and u a bounded minimizer of J , $u : \mathbb{R}^n \to \mathbb{R}^m$, $||u||_{L^\infty(\mathbb{R}^n,\mathbb{R}^m)} < \infty$. Moreover, let $\mathcal{O} \subset \mathbb{R}^n$ open, assume that

$$d(u(\mathcal{O}), \{W = 0\} \setminus \{a\}) \geq k > 0 \tag{2.10}$$

d the Euclidean distance, k constant.

Then given $q \in (0, ||u||_{L^\infty(\mathbb{R}^n,\mathbb{R}^m)})$, $\exists \ r_q > 0$ such that

$$B_{r_q}(x_0) \subset \mathcal{O} \Rightarrow |u(x_0) - a| < q \tag{2.11}$$

(ii) Let further $0 < 2q \leq \rho_0$ (cfr (**H1**)). Then there exists an explicit constant $\hat{C} = \hat{C}(\alpha, n) > 0$ (see (2.88) , $\lim_{\alpha\to 2} \hat{C}(\alpha, n) = \infty$, $\lim_{\alpha\to 0} \hat{C}(\alpha, n) = \infty$), such that

$$B_{\hat{C}q^{-\alpha}(x_0)} \subset \mathcal{O} \Rightarrow u(x) \equiv a \ \ , \ in \ B_{\frac{\hat{C}}{2}q^{-\alpha}(x_0)} \tag{2.12}$$

Remark 2.1.4. Part (i) of Proposition 2.1.3 holds for $\alpha = 2$, and is a result obtained in [23]. It can be found also in [1] Theorem 5.3. Note that positivity allows the application of this with $\mathcal{O} = D$, since the solution in D stays away from all the minima except one. This reveals the nature of **(H3)**.

Part (ii) is utilizing a "Dead Core" estimate (Lemma 2.2.14 below) which shows that for a function $v \in W^{1,2}(B_R(x_0))$

$$\begin{cases} \Delta v \geq c^2 v^{\frac{\alpha}{2}} \ \ , \text{ weakly in } W^{1,2}(B_R(x_0)) \\ 0 \leq v \leq \delta \ , \ \delta > 0 \text{ sufficiently small depending on } c \end{cases} \tag{2.13}$$

Then if

$$\begin{cases} dist(y_0, \partial B_R(x_0)) > R_0 \Rightarrow \\ v(y_0) = 0 \text{ for } R > R_0 = \frac{\sqrt{n(n+2)}}{(1-\frac{\alpha}{2})c} \delta^{\frac{2-\alpha}{4}} \ , \ \alpha \in (0, 2) \end{cases} \tag{2.14}$$

"Dead Core" regions are sets where the solution is constant.

The first appearance of such a situation was in [16], [34], followed by more in depth study in [20].

Proposition 2.1.5. ($\alpha = 0$) Let

$$J(u) = \int (\frac{1}{2}|\nabla u|^2 + \chi_{A^c}(u))dx \tag{2.15}$$

where $A := \{W = 0\} = \{a_1, ..., a_N\} \subset \mathbb{R}^m$ $(N \geq 2)$, $A^c = \mathbb{R}^m \setminus A$. Let u be a nonconstant minimizer, $u : \mathbb{R}^n \to \mathbb{R}^m$, $||u||_{L^\infty(\mathbb{R}^n,\mathbb{R}^m)} < \infty$. Suppose that for some $a_i \in A$ we have

$$d(u(B_R(x_0)), \{W = 0\} \setminus a_i) > 0 \tag{2.16}$$

Then

$$\mathcal{L}^n(\{u = a_i\} \cap B_R(x_0)) \geq cR^n \ , \ R \geq R_0 \tag{2.17}$$

for some constant $c > 0$ independent of R.

What about existence of minimizer defined on $\mathbb{R}^n$ possessing a free boundary and without any symmetry assumptions? This is a difficult open problem for the coexistence of three or more phases. We have the following simple result in this direction.

Proposition 2.1.6. ($\alpha = 0$) Consider the functional

$$J(u) = \int (\frac{1}{2}|\nabla u|^2 + \chi_{A^c}(u))dx \tag{2.18}$$

where $A = \{a_1, ..., a_N\}$ distinct points in $\mathbb{R}^m$, $A^c = \mathbb{R}^m \setminus A$.
Let $u : \mathbb{R}^n \to \mathbb{R}^m$ be a nonconstant minimizer with $||u||_{L^\infty(\mathbb{R}^n,\mathbb{R}^m)} < \infty$ and $x_0 \in \mathbb{R}^n$, arbitrary and fixed. Then there exist an $R_0 > 0$ and at least two distinct points $a_i \neq a_j$ in A, such that the following estimates hold:

$$\mathcal{L}^n(\overline{B_R(x_0)} \cap \{u(x) = a_k\}) \geq c_k R^n \ , \ R \geq R_0 \ , \ k = i, j \tag{2.19}$$

$$||\partial\{u(x) = a_k\}||(B_R(x_0)) \ \geq \hat{c}_k R^{n-1} \ \ , \ R \geq R_0 \ , \ k = i, j \tag{2.20}$$

where c_k, $\hat{c}_k$ are positive constants, independent of x_0 and R (but depending on u). $||\partial E||$ stands for the perimeter measure of the set E and $||\partial E||(B_R(x_0))$ denotes the perimeter of E in $B_R(x_0)$ (see for instance [14]).

Remark 2.1.7. Proposition 2.1.6 holds for the whole range of potentials $0 < \alpha < 2$ defined in **(H1)** but with a significantly harder proof [8].

The natural way of constructing entire solutions u to (2.3) without symmetry requirements is by minimizing over balls B_R with appropriate boundary conditions forcing the phases on B_R:

$$\min J_{B_R}(v) \ \ , \ v = g_R \ \ , \text{ on } \ \partial B_R,$$

and taking the limit along subsequences of minimizers u_R

$$u = \lim_{R\to\infty} u_R$$

Remark 2.1.8. The result from Proposition 2.1.6 holds for the symmetric case as in Theorem 2.1.1 for $\alpha = 0$, and provides some quantitative information on the Dead Core. We have not been able to establish the exact analog of Theorem 2.1.1 for $\alpha = 0$.

Proposition 2.1.9. ($\alpha = 0$) Under the hypothesis **(H1)**-**(H3)** and $N = m + 1$, there exist a nontrivial equivariant minimizer of $J(u) = \int (\frac{1}{2}|\nabla u|^2 + \chi_{\{u \in S_A\}})dx$, $u : \mathbb{R}^n \to \mathbb{R}^n$, such that 1. $u(\overline{F}) \subset \overline{F}$, $u(\overline{D}) \subset \overline{D}$ (positivity).
2. $\mathcal{L}^n(D_R \cap \{u = a_1\}) \geq cR^n$, $R \geq R_0$, where $D_R = D \cap B_R(0)$ (D from **(H3)**).
3. $\mathcal{L}^n(D_R \cap \{u \neq a_1\}) \leq CR^{n-1}$, $R \geq R_0$.

A convenient hypothesis guaranteeing $||u||_{L^\infty} < \infty$ is [1]

$$\begin{cases} W_u(u) \cdot u \geq 0 & \text{, for } |u| \geq M \text{ , some } M \\ |g_R| \leq M \end{cases} \tag{2.21}$$

The existence of one-dimensional minimizers ($u : \mathbb{R} \to \mathbb{R}^n$, i.e. connections) for $\alpha \in (0, 2)$, can be obtained by Theorem 2.1, p.34 in [1]. For the $\alpha = 0$ case, where W is a characteristic function, one-dimensional minimizers are affine maps connecting the phases. More precisely,

$$u(x) = \begin{cases} a_1 & , \ x < -L \\ a_2 & , \ x > L \\ \frac{a_2 - a_1}{2L} x + \frac{a_1 + a_2}{2} & , \ x \in [-L, L] \end{cases} \tag{2.22}$$

and by minimality one can see that $L = \frac{|a_2 - a_1|}{2\sqrt{2}}$, which is formally what we expect from the free boundary condition $|\nabla u|^2 = 2$ (see (2.5)).

The basic question of course is whether a nontrivial minimizer u connecting the phases can be constructed. We know from the work on the De Giorgi referred above conjecture that for $m = 1$, and in low dimensions, any such minimizer will depend on a single variable, and so in a sense is trivial. For the system we expect otherwise, and indeed this was shown to be the case in the equivariant setting and for smooth potentials, in the book [1].

There are a few tools that we utilize in the sequel that because of their independent interest we mention explicitly.

[1]For $\Omega \subset \mathbb{R}^n$ open, by linear elliptic theory $u \in C^2(\Omega; \mathbb{R}^m)$. Set $v = |u|^2$, then $\Delta v = 2W_u(u) \cdot u + 2|\nabla u|^2 > 0$, for $u > M$. Hence $\max |u|^2 \leq M$ if v attains its max in the interior of Ω.

The Basic Estimate

For minimizers, $0 \leq \alpha < 2$ satisfying $|u(x)| \leq M$, $x \in \mathbb{R}^n$ we have that there exists $r_0 > 0$ such that for any $x_0 \in \mathbb{R}^n$

$$J_{B_r(x_0)}(u) \leq C_0 r^{n-1} \ , \ r \geq r_0 > 0 \ , \tag{2.23}$$

$C_0 > 0$ constant, independent of u, but depending on M.

For $\alpha \in [1, 2)$ elliptic theory applied to (2.3) implies $||\nabla u||_{L^\infty} < \infty$, and (2.23) follows easily (cfr. [1] Lemma 5.1). For $\alpha \in [0, 1)$, and $m = 1$, it is already mentioned in [3]. We prove it in Lemma 2.2.11. The estimate (2.23) is utilized in the proof of Proposition 2.1.6, and also in the proof of Proposition 2.1.3 on which Part 1 of Theorem 2.1.1 is based. Finally (2.23) is also utilized in the proof of the Density Estimate that we discuss below.

The Density Estimate

For minimizers u of the functional J in (2.1), $0 \leq \alpha < 2$ satisfying $|u(x)| \leq M$, we have

$$\begin{cases} \mathcal{L}^n(B_{r_0}(x_0) \cap \{|u - a| > \lambda\}) \geq \mu_0 > 0 \Rightarrow \\ \mathcal{L}^n(B_r(x_0) \cap \{|u - a| > \lambda\}) \geq C r^n \ , \ r \geq r_0 \end{cases} \tag{2.24}$$

$C = C(\mu_0, \lambda)$.

This is an important estimate of Caffarelli and Cordoba [3] established in the scalar case $m = 1$, and extended to the vector case by the first author and G.Fusco. We refer to [1] Theorem 5.2, where detailed references can be found. The proof in [1] has a gap for $0 \leq \alpha < 1$ since it is utilizing (2.23) that was taken for granted then but proved in the present paper.

The Hölder Estimate

For minimizers u of the functional J in (2.1), $0 \leq \alpha < 2$, satisfying $|u(x)| < M$, $x \in \mathbb{R}^n$, we have the estimate

$$|u(x) - u(y)| \leq C|x - y| \ln(|x - y|^{-1}) \quad , \ \forall \, x, y \ , \ |x - y| \leq \frac{1}{2} \tag{2.25}$$

which implies $u \in C^\beta(\mathbb{R}^n, \mathbb{R}^m)$, $\forall \beta \in (0, 1)$, $C = C(M)$, that has already be mentioned.

This is established in [10] for $m = 1$ and $\alpha = 0$. We give a detailed proof in Lemma 2.2.10. It is utilized in several places. For example in establishing Proposition 2.1.3 (i) we proceed by a contradiction argument that invokes the Density Estimate. Here uniform continuity is essential, and is provided by (2.25).

It is also instrumental for the derivation of the Basic Estimate (2.23).

The Hölder continuity is also needed in the proof of the Containment result presented in Appendix A, that we now describe.

The Containment
This states that for the special potentials

$$W(u) = \begin{cases} W^{\overline{\alpha}}(u) := \prod_{k=1}^{m+1} |u - a_k|^{\alpha_k} \ , \ \overline{\alpha} = (a_1, ..., a_{m+1}) \ , \ 0 < a_k < 2 \\ W^0(u) := \chi_{A^c}(u) \ , \ A = \{a_1, ..., a_{m+1}\} \end{cases} \tag{2.26}$$

where the vectors $\{a_2 - a_1, ..., a_{m+1} - a_1\}$ are linearly independent in $\mathbb{R}^m$, critical points of $J(u) = \int (\frac{1}{2}|\nabla u|^2 + W(u))dx \ , \ u : \mathbb{R}^n \to \mathbb{R}^n \ , \ |u(x)| < M$, map $\mathbb{R}^n$ inside the closure of the convex hull of A , $\overline{co}(A)$. This result was obtained jointly by the first author and P.Smyrnelis, in unpublished work. Its proof requires uniform continuity, and so for $\alpha \in (0, 1)$ we need to restrict ourselves to minimizers for which (2.25) holds.

This result shows that J^0 is in some natural way the limit of $J^{\overline{\alpha}}$, as $\overline{\alpha} \to 0$, and actually we establish a Γ-limit type relationship in Lemma 2.2.18.

This paper is structured as follows.
In section 2 we state and prove various Lemmas already mentioned in the introduction.

In section 3 we give the proofs of Theorem 2.1.1, Propositions 2.1.3, 2.1.5 and 2.1.6.
In Appendix 5.7 we state and prove the containment result, and in Appendix 2.5 we give a formal argument, taken essentially from [1], that explains the free boundary conditions in (2.4) and (2.5).

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2.2 Basic Lemmas

2.2.1 Regularity of u

We will prove a logarithmic estimate for bounded minimizers, following closely the proof of Theorem 2.1 in [10] (see also Lemma 2 in [17]). We have:

Lemma 2.2.10. ($0 \leq \alpha < 2$, Hölder Continuity)
Let $u : \mathbb{R}^n \to \mathbb{R}^m$ a minimizer of J , $|u(x)| < M$, W satisfying **(H1)** for $0 < \alpha < 2$ and $W = \chi_{A^c}(u)$ for $\alpha = 0$. Then there exists constant $C = C(M)$, such that

$$|u(x) - u(y)| \leq C|x-y|\ln(|x-y|^{-1}) \quad , \forall\, x, y \ , \ |x-y| \leq \frac{1}{2} \tag{2.27}$$

In paricular, $u \in C^{\beta}(\mathbb{R}^n; \mathbb{R}^m)$, $\forall\, \beta \in (0,1)$.

Proof. We restrict ourselves to $0 \leq \alpha < 1$, since the result follows immediately for $\alpha \in [1, 2]$ by linear elliptic theory. We begin with the case $0 < \alpha < 1$.

For an arbitrary $B_r(x_0)$ let v_r be the harmonic function equal to u on ∂B_r. Then by the maximum principle v_r is also bounded and taking into account the specific form of the potential (2.6) we have that there exists an M such that:

$$|u(x)|, |v_r(x)|, |W^{\overline{\alpha}}(u(x))|, |W^{\overline{\alpha}}(v_r(x))| \leq M \ , \ \forall x \in B_r(x_0), \ \alpha \in [0,1] \tag{2.28}$$

Then using the minimality of u and the non-negativity of the potentials $W^{\overline{\alpha}}$

together with (2.28) we have:

$$\int_{B_r} |\nabla u(x)|^2 \, dx \le \int_{B_r} |\nabla u(x)|^2 + W^{\overline{\alpha}}(u(x)) \, dx \le \int_{B_r} |\nabla v_r(x)|^2 + W^{\overline{\alpha}}(v_r(x)) \, dx$$
$$\le M|B_r| + \int_{B_r} |\nabla v_r(x)|^2 \, dx \tag{2.29}$$

hence

$$\int_{B_r} |\nabla u(x)|^2 - |\nabla v_r(x)|^2 \, dx \le Cr^n \tag{2.30}$$

On the other hand we have:

$$\begin{aligned}\int_{B_r} |\nabla u(x)|^2 - |\nabla v_r(x)|^2 \, dx &= \int_{B_r} (\nabla u(x) + \nabla v_r(x), \nabla u(x) - \nabla v_r(x)) \, dx \\ &= \int_{B_r} |\nabla(u(x) - v_r(x))|^2 \, dx + 2\int_{B_r} (\nabla u - \nabla v_r) \nabla v_r \, dx \\ &= \int_{B_r} |\nabla(u(x) - v_r(x))|^2 \, dx \end{aligned} \tag{2.31}$$

where for the last inequality we used that v_r is harmonic and equal to u on ∂B_r.

Thus we get:

$$\int_{B_r} |\nabla(u(x) - v_r(x))|^2 \, dx \le Cr^n \tag{2.32}$$

From the previous estimate, it suffices to show that

$$\int_{B_s} |\nabla u|^2 \le Cs^n[\ln^2(r/s) + 1] \tag{2.33}$$

This would imply (2.27).
To prove (2.33), we proceed as follows:

$$\int_{B_s} |\nabla u|^2 \le \int_{B_s} |\nabla v_{2s}|^2 + \int_{B_s} |\nabla(u - v_{2s})||\nabla(u + v_{2s})|$$

The first integral on the right side is estimated using the subharmonicity of $|\nabla v_{2s}|^2$, and then the minimality of v_{2s}. So,

$$\frac{1}{|B_s|} \int_{B_s} |\nabla v_{2s}|^2 \le \frac{1}{|B_{2s}|} \int_{B_{2s}} |\nabla v_{2s}|^2 \le \frac{1}{|B_{2s}|} \int_{B_{2s}} |\nabla u|^2$$

by (2.31).

The second integral is estimated by enlarging the domain to B_{2s}, then Cauchy-Schwartz, the established bound and the minimality of v_{2s}

$$\frac{1}{|B_s|}\int_{B_s} |\nabla(u-v_{2s})||\nabla(u+v_{2s})| \leq$$

$$\frac{|B_{2s}|}{|B_s|}\Big(\frac{1}{|B_{2s}|}\int_{B_{2s}} |\nabla(u-v_{2s})|^2\Big)^{\frac{1}{2}}\Big(\frac{2}{|B_{2s}|}\int_{B_{2s}} |\nabla u|^2 + |\nabla v_{2s}|^2\Big)^{\frac{1}{2}} \leq C\Big(\frac{1}{|B_{2s}|}\int_{B_{2s}} |\nabla u|^2\Big)^{\frac{1}{2}}$$

by (2.31), (2.32).

So if we set

$$x_k = \frac{1}{|B_{2^{-k}}|}\int_{B_{2^{-k}}} |\nabla u|^2$$

then

$$x_{k+1} \leq x_k + C x_k^{1/2}$$

Induction gives

$$x_{k+1} \leq C' k^2$$

from which you have (2.33).

Estimate (2.27) then follows from the proof of Morrey's embedding. Indeed, suppose x and y are given, of distance $2s$ apart. Let z be the midpoint. Then, by mean value theorem,

$$\frac{1}{|B_s|}\int_{B_s} |u(x)-u(p)|dp \leq Cs\frac{1}{|B_s|}\int_{B_s}\int_0^1 |\nabla u(p+t(x-p))|dtdp$$

Thus, interchanging the order of integration and using (2.33), we get

$$\frac{1}{|B_s|}\int_{B_s} |u(x)-u(p)|dp \leq Cs[\ln(1/s)+1]$$

The estimate for $|u(x)-u(y)|$ then follows from triangle inequality.

The proof for the case $\alpha = 0$ is similar, the only difference being that instead of the bound in (2.28) $|W^0(u(x))|, |W^0(v_r(x))| \leq 1$ is used. □

2.2.2 The Basic Estimate:

Lemma 2.2.11. Let $u : \mathbb{R}^n \to \mathbb{R}^m$ minimizer of J , $|u(x)| < M$, W satisfying **(H1)** for $0 < \alpha < 2$ and $W = W^0$ for $\alpha = 0$. Then there is a constant $C_0 = C_0(W, M)$ independent of x_0 and such that

$$J_{B_r(x_0)}(u) \leq C_0 r^{n-1} \quad , \ r > r_0$$

Proof.
1. For $\alpha \in [1, 2)$, utilizing elliptic estimates we obtain $|\nabla u(x)| < C(M)$, $x \in \mathbb{R}^n$. The estimate then follows by constructing a competitor $v(x)$ on a ball via

$$v(x) = \begin{cases} a & , \ |x - x_0| \leq r - 1 \\ (r - |x - x_0|)a + (|x - x_0| - r + 1)u(x) & , \ r - 1 < |x - x_0| \\ u(x) & , \ |x - x_0| > r \end{cases}$$

and utilizing the minimality of u (cfr Lemma 5.1 [1]). Here we can take $r_0 = 0$.

2. For $\alpha \in (0, 1)$, we aim to prove the estimate:

□

Lemma 2.2.12. Let $u : \mathbb{R}^n \to \mathbb{R}^m$ be a bounded local minimizer for the energy functional J in (2.1) with the potential W_α as in **(H1)**. Then there exists constant $C, R_0 > 0$ independent of u such that:

$$J(u; A(R)) \leq CR^{n-1}, \forall R \geq R_0 \tag{2.34}$$

where C is independent of $R \geq R_0$ and $A(R) := B_R(x_0) \setminus B_{R-1}(x_0)$.

Proof. We first claim that there exists a constant $\tilde{C} > 0$ such that for any $x_0 \in \mathbb{R}^n$ we have, for u a bounded local minimizer:

$$\int_{B_1(x_0)} |\nabla u(x)|^2 dx \leq \tilde{C} \tag{2.35}$$

To this end we consider the function $v \in W^{1,2}(B_1(x_0))$ with $v = u$ on $\partial B_1(x_0)$ and $\Delta v = 0$ in $B_1(x_0)$. Since u is bounded, by the maximum principle we have that v is also bounded and taking into account the hypothesis **(H1)** for the potential W_α we have that there exists $M > 0$ such that:

$$|u(x)|, |v(x)|, W_\alpha(u(x)), W_\alpha(v(x)) \leq M, \forall x \in \mathbb{R}^n, \alpha \in [0, 1] \tag{2.36}$$

We then have:

$$
\begin{aligned}
\int_{B_1(x_0)} |\nabla u(x)|^2\,dx &\le \int_{B_1(x_0)} |\nabla u(x)|^2 + W_\alpha(u(x))\,dx \le \int_{B_1(x_0)} |\nabla v(x)|^2 + W_\alpha(v(x))\,dx \\
&\le M|B_1| + \int_{B_1(x_0)} |\nabla v(x)|^2\,dx = M|B_1| + \int_{\partial B_1(x_0)} \frac{\partial v}{\partial \nu} v\,d\sigma \\
&\le M|B_1| + \|\frac{\partial v}{\partial \nu}\|_{H^{-\frac{1}{2}}(\partial B_1(x_0))} \|v\|_{H^{\frac{1}{2}}(\partial B_1(x_0))} \\
&\le M|B_1| + C\|\nabla v\|_{L^2(B_1(x_0))} \|v\|_{H^{\frac{1}{2}}(\partial B_1(x_0))} \\
&\le M|B_1| + \frac{1}{2}\|\nabla v\|^2_{L^2(B_1(x_0))} + C\|v\|^2_{H^{\frac{1}{2}}(\partial B_1(x_0))} \\
&= M|B_1| + \frac{1}{2}\|\nabla v\|^2_{L^2(B_1(x_0))} + C\|u\|^2_{H^{\frac{1}{2}}(\partial B_1(x_0))} \qquad (2.37)
\end{aligned}
$$

where for the first inequality we used the non-negativity of W_α, for the second the local minimality of u, and for the third the estimates (2.36). For the first equality we used the fact that v is a harmonic function and an integration by parts, while for the last equality we used that $u = v$ on $\partial B_1(x_0)$. For the penultimate inequality we used the continuity of the normal part of trace operator on the space $L^2_{div} = \{f \in L^2; \operatorname{div} f \in L^2\}$ (see for instance Prop. 3.47, (ii) in [18]).

We obtain thus:

$$
\int_{B_1(x_0)} |\nabla u(x)|^2\,dx \le M|B_1| + C\|u\|^2_{H^{\frac{1}{2}}(\partial B_1(x_0))} \qquad (2.38)
$$

On the other hand we have (see for instance [31]):

$$
\|u\|^2_{H^{\frac{1}{2}}(\partial B_1(x_0))} = \int_{\partial B_1(x_0)} \int_{\partial B_1(x_0)} \frac{|u(x) - u(y)|^2}{|x-y|^{n-1+1}}\,dx\,dy \le C \qquad (2.39)
$$

where for the last inequality we used the logarithmic estimate (2.2.10).

Combining the last two estimates we obtain the claimed uniform estimate (2.35). On the other hand, thanks to estimate (2.36) we have

$$
\int_{A(R)} W_\alpha(u(x))\,dx \le CR^{n-1} \qquad (2.40)
$$

which combined with the fact that one can cover $A(R)$ with CR^{n-1} balls of radius 1 and estimate (2.35) provides the desired estimate (2.34).

□

Note: Lemma 2.3 implies Lemma 2.2 ($\alpha \in (0,1)$) by considering the comparison function $v(x)$ as in $\alpha \in [1,2)$ case.

2.2.3 The "Dead Core" estimate:

Now, we proceed with a useful calculation. From the hypothesis **(H1)** for W we have that for $|u-a| << 1$, it holds that $W_u(u)\cdot(u-a) \geq \bar{c}^2|u-a|^\alpha$ with $\bar{c}^2 = \alpha C^*$, $\alpha \in (0,2)$. Set $v(x) = |u-a|^2$.

Then

$$\begin{gathered}\Delta v = \sum_{i=1}^{n} 2((u(x)-a)u_{x_i})_{x_i} = 2|\nabla u|^2 + 2(u(x)-a)\Delta u = \\ 2|\nabla u|^2 + 2W_u(u)\cdot(u(x)-a) \geq 2|\nabla u|^2 + 2\bar{c}^2|u-a|^\alpha\end{gathered} \tag{2.41}$$

Therefore,

$$\Delta v \geq c^2|u-a|^\alpha = c^2 v^{\frac{\alpha}{2}} \ \text{, where } c^2 = 2\alpha C^*. \tag{2.42}$$

Definition 2.2.2. Let $\Omega \subset \mathbb{R}^n$ open and $v \in W^{1,2}_{loc}(\Omega,\mathbb{R})$, a region $\Omega_0 \subset \Omega$ is called a *dead core* if $v \equiv 0$ in Ω_0.

For the convenience of the reader, let us now state some results from [20].

The article [20] is concerned with the problem

$$\begin{cases}\Delta u = c^2 u^p \ \text{ in } \Omega \subset \mathbb{R}^n \\ u = 1 \ \text{ on } \partial\Omega\end{cases} \tag{2.43}$$

with $p \in (0,1)$. We call that a "dead core" Ω_0 develops in Ω, i.e. a region where $u \equiv 0$.

Let $X(s)$ be a solution of

$$\begin{cases}X''(s) = c^2 X^p(s) \ \text{ in } (0,s_0) \\ X'(0) = 0 \ , X(s_0) = 1\end{cases} \tag{2.44}$$

As a first choice of a linear problem consider the "torsion problem" , i.e.

$$\begin{cases} \Delta\psi + 1 = 0 \text{ in } \Omega \\ \psi = 0 \text{ on } \partial\Omega \end{cases} \tag{2.45}$$

One then constructs a supersolution $\overline{u}(x)$ to (2.43) having the same level lines as the torsion function by setting

$$\overline{u}(x) = X(s(x)) \ , \ \ x \in \Omega \tag{2.46}$$

where

$$s(x) = \sqrt{2(\psi_m - \psi(x))} \ , \psi_m = \max_{\Omega} \psi \tag{2.47}$$

In problem (2.44) we choose $s_0 = \sqrt{2\psi_m}$.

Theorem 2.2.13. ([20]) Assume that the mean curvature of $\partial\Omega$ is nonnegative everywhere. Then

$$\begin{gathered} \overline{u}(x) = X(s(x)) \ \textit{is a supersolution}, \ \textit{i.e.} \\ \Delta\overline{u} \le c^2\overline{u}^p \text{ in } \Omega \\ \overline{u} = 1 \text{ on } \partial\Omega \end{gathered} \tag{2.48}$$

One of the corollaries of this Theorem is the information on the location and the size of the "dead core" Ω_0, which may be stated as

Corollary 2.2.1. ([20]) The dead core Ω_0 contains the set

$$\{x \in \Omega | \psi(x) \ge d(p,c)[\sqrt{2\psi_m} - \frac{1}{2}d(p,c)]\} \ ,$$

where $d(p,c) := \dfrac{\sqrt{2(p+1)}}{(1-p)c}$.

We will now utilize the above for the proof of the following Lemmas.

Lemma 2.2.14. Let $\Omega = B_R(x_0) \subset \mathbb{R}^n$ and $v \in C^2(\Omega; \mathbb{R}_+)$ satisfy the following assumptions:

$$\begin{aligned} \Delta v(x) \geq c^2 v^{\frac{\alpha}{2}}(x) \ , \ x \in \Omega \\ v(x) \leq \delta \ , \ x \in \partial\Omega \end{aligned} \tag{2.49}$$

$\alpha \in (0,2) \Leftrightarrow \frac{\alpha}{2} = p \in (0,1)$.
Then if $y_0 \in \Omega$ is such that $dist(y_0, \partial\Omega) > R_0 \Rightarrow v(y_0) = 0$.

where $R_0 := \begin{cases} \sqrt{n}d(p,\hat{c}) \ , R \geq \sqrt{n}d(p,\hat{c}) \\ 2R - \sqrt{n}d(p,\hat{c}) \ , \frac{1}{2}\sqrt{n}d(p,\hat{c}) < R < \sqrt{n}d(p,\hat{c}) \end{cases}$.
and $d(p,\hat{c}) := \dfrac{\sqrt{2(p+1)}}{(1-p)\hat{c}}$, $\hat{c} = \dfrac{c}{\delta^{\frac{1-p}{2}}}$.

Proof. From the maximum principle we have that $v(x) \leq \delta$ in Ω
Define $\hat{v} := \dfrac{v}{\delta}$ and $\hat{c} := \dfrac{c}{\delta^{\frac{1-p}{2}}}$, then we have:

$$\begin{cases} \Delta \hat{v}(x) \geq \hat{c}^2 \hat{v}^{\frac{\alpha}{2}}(x) \ , \ x \in \Omega. \\ \hat{v}(x) \leq 1 \ , \ x \in \partial\Omega \end{cases}$$

For $\Omega = B_R(x_0)$ we have that

$$\psi(x) = \frac{R^2}{2n} - \frac{1}{2n}|x - x_0|^2 \ , \ \psi_m = \frac{R^2}{2n} \tag{2.50}$$

is a solution to the problem:

$$\begin{cases} \Delta\psi(x) + 1 = 0, \ x \in \Omega \\ \psi(x) = 0 \ , \ x \in \partial\Omega \end{cases} \tag{2.51}$$

Also, we have that if:

$$\begin{cases} \Delta u \leq c^2 u^p, \ x \in \Omega \\ \Delta v \geq c^2 v^p, \ x \in \Omega \\ v \leq u \ , \ x \in \partial\Omega \end{cases} \tag{2.52}$$

then $v \leq u$, in Ω. So since $u, v \geq 0$, if $u(x_1) = 0 \Rightarrow v(x_1) = 0$.
Such u is defined in [20] via ψ in Theorem 2.2.13 (supersolution with $u = 1 \geq \hat{v}$ on the boundary). Then by Corollary 2.2.1 in [20], the dead core of $\overline{u}$ contains the set $\{x \in \Omega | \psi(x) \geq C_0 := d(p,\hat{c})[\frac{R}{\sqrt{n}} - \frac{1}{2}d(p,\hat{c})]\}$, that is if
$y_0 \in \{\psi(x) \geq C_0\} \Rightarrow \overline{u}(y_0) = 0$ and thus $\hat{v}(y_0) = v(y_0) = 0$.
Since ψ has the form (2.50) we can see that

$$\{x \in \Omega | \psi(x) \geq C_0\} = \{dist(x, \partial\Omega) \geq R_0\}$$

as follows:

$$\psi(x) \geq C_0 \Leftrightarrow \frac{R^2}{2n} - \frac{1}{2n}|x-x_0|^2 \geq C_0 \Leftrightarrow \sqrt{R^2 - 2nC_0} \geq |x-x_0|$$
$$\Leftrightarrow R - |x-x_0| \geq R - \sqrt{R^2-2nC_0} = R - \sqrt{R^2 - 2\sqrt{n}d(p,\hat{c})R + n(d(p,\hat{c}))^2} =$$
$$= R - |R - \sqrt{n}d(p,\hat{c})| = R_0$$

and notice that: $dist(x, \partial\Omega) = dist(x, \partial B_R(x_0)) = R - dist(x, x_0)$

□

Notes: (1) $\hat{c}$ depends on δ and tends to infinity as δ tends to zero.
(2) $d(p,\hat{c})$ tends to zero as δ tends to zero, and so does C_0.

Remark 2.2.15. If we take $\tilde{\Omega}$ open set, such that $B_R(x_0) \subset \tilde{\Omega}$ and

$$\begin{cases} \Delta\tilde{\psi}(x) + 1 = 0, \ x \in \tilde{\Omega} \\ \tilde{\psi}(x) = 0 \ \ , \ x \in \partial\tilde{\Omega} \end{cases}$$

then, we have: $\psi \leq \tilde{\psi} \Rightarrow \{\psi(x) \geq C_0\} \subset \{\tilde{\psi}(x) \geq C_0\} \Rightarrow \{x \in B_R(x_0) : dist(\partial B_R(x_0), x) \geq R_0\} \subset \{\tilde{\psi}(x) \geq C_0\}$.
Thus, the above theorem holds for more general open sets that contain a ball $B_R(x_0)$.

Lemma 2.2.16. Let D open, convex $\subset \mathbb{R}^n$ and for some $d_0 > 0$, $\Omega := \{x \in D : dist(x, \partial D) \geq d_0\}$ and let $v \in C^2(D; \mathbb{R}_+)$ satisfying:

$$\begin{gathered} \Delta v(x) \geq c^2 v^{\frac{\alpha}{2}}(x) \ , \ x \in \Omega \\ v(x) \leq \delta \ , \ x \in \Omega \end{gathered} \tag{2.53}$$

$\alpha \in (0,2) \Leftrightarrow \frac{\alpha}{2} = p \in (0,1)$.
Then if $x_0 \in D$ such that $dist(x_0, \partial D) \geq d_0 + 2\frac{\sqrt{2n(p+1)}}{(1-p)\hat{c}} \Rightarrow v(x_0) = 0$.

Proof. We have that:

$$\{x \in D : dist(x, \partial D) \geq d_0 + 2\frac{\sqrt{2n(p+1)}}{(1-p)\hat{c}}\} = \{x \in \Omega : dist(x, \partial\Omega) \geq 2\frac{\sqrt{2n(p+1)}}{(1-p)\hat{c}}\}$$

and Ω is convex (parallel sets have at the same side of supporting planes).
Let $x_0 \in D$ such that $dist(x_0, \partial D) \geq d_0 + 2\frac{\sqrt{2n(p+1)}}{(1-p)\hat{c}}$. Since $dist(\partial D, \partial\Omega) = d_0 \Rightarrow dist(x_0, \partial\Omega) \geq 2\frac{\sqrt{2n(p+1)}}{(1-p)\hat{c}}$ and since Ω is convex there exist a ball $B_R(x_0) \subset \Omega$ for $R = 2\frac{\sqrt{2n(p+1)}}{(1-p)\hat{c}} = 2\sqrt{n}d(p,\hat{c}) > R_0 = \sqrt{n}d(p,\hat{c})$, $d(p,\hat{c})$ as defined above.
Therefore we can apply Lemma 2.2.14 in the ball $B_R(x_0)$ and we have that $v(x) = 0\, , \forall x \in B_{R_0}(x_0) = \{x \in B_R(x_0) : dist(\partial B_R(x_0), x) \geq R_0\} \Rightarrow v(x_0) = 0$.

□

The results of Lemma 2.2.14 and Lemma 2.2.16 above were proved for the case $1 < \alpha < 2$, since $u \in C^{2,\alpha-1}$ by elliptic regularity. However, they also hold for the case where $0 < \alpha \leq 1$. The only difference in proving this, is that the differential inequality (2.42) holds weakly and we utilize it together with the weak maximum principle for the comparison argument as in the proof of lemma 2.2.14. So in order to extend the results of the lemmas above for the case where $0 < \alpha \leq 1$, it suffices to prove the following claim.

Lemma 2.2.17.

$$\Delta v \geq c^2 v^{\frac{\alpha}{2}} \quad \text{weakly in } \; W^{1,2}(B_R(x_0)).$$

Proof.

Let $v \in W^{1,2}(B_R(x_0))$, v continuous ($v = |u-a|^2$, by Lemma 2.2.10) and $v \geq 0$.
We define $v_\varepsilon := \max\{v, \varepsilon\}$, $0 < \varepsilon < \delta$ (where δ as in the above Lemmas). The set $\{v = \varepsilon\}$ is smooth by Sard's theorem, since v is smooth away from zero.
Let $\phi \in C_0^1(B_R(x_0))$, $B_R^\varepsilon(x_0) = \{v > \varepsilon\} \cap B_R(x_0)$, we have

$$\begin{aligned}
-\int_{B_R(x_0)} \nabla v \nabla \phi dx &= \lim_{\varepsilon\to 0}\int_{B_R^\varepsilon(x_0)} -\nabla v_\varepsilon \nabla\phi dx = \liminf_{\varepsilon\to 0}[-\int_{B_R^\varepsilon(x_0)} \nabla v\nabla\phi dx] \\
&\geq \liminf_{\varepsilon\to 0}[\int_{B_R^\varepsilon(x_0)} \Delta v\phi dx - \int_{\partial B_R^\varepsilon(x_0)} \frac{\partial v}{\partial \nu}\phi dS] \geq \liminf_{\varepsilon\to 0}[\int_{B_R^\varepsilon(x_0)} \Delta v \phi dx] \\
&\geq \liminf_{\varepsilon\to 0}[\int_{B_R^\varepsilon(x_0)} c^2 v^{\frac{\alpha}{2}}\phi dx] = \lim_{\varepsilon\to 0}[\int_{B_R^\varepsilon(x_0)} c^2 v^{\frac{\alpha}{2}}\phi dx] = \\
&\geq \lim_{\varepsilon\to 0}[\int_{B_R(x_0)} c^2 v_\varepsilon^{\frac{\alpha}{2}}\phi dx - c^2\varepsilon^{\frac{\alpha}{2}}\int_{B_R\setminus B_R^\varepsilon}\phi dx] = \int_{B_R(x_0)} c^2 v^{\frac{\alpha}{2}}\phi dx.
\end{aligned}$$

□

2.2.4 On the definition of W^0

In what follows we establish essentially that $\lim_{\alpha\to 0} J^\alpha = J^0$ in the $\Gamma-$ convergence sense. The containment result in Appendix 5.7 is essential here.

$$J^\alpha(\Omega, u) = \int_\Omega (\frac{1}{2}|\nabla u|^2 + W^\alpha(u))dx \tag{2.54}$$

with

$$W^\alpha(u) := \prod_{i=1}^{N} |u - a_i|^\alpha \ \ , \ i \in \{1, ..., N\} \ 0 < \alpha < 2. \tag{2.55}$$

We further denote:

$$W_0(u) := \chi_{\{u \in S_A\}} \tag{2.56}$$

where

$$A := \{a_1, \ldots, a_N\}$$

and

$$S_A := \left\{ \sum_{i=1}^{N} \lambda_i a_i, \text{ where } \sum_{i=1}^{N} \lambda_i = 1, \lambda_i \in [0, 1), i \in \{1, \ldots, N\} \right\} \tag{2.57}$$

(i.e. S_A is the convex hull of the points in A except the point themselves). Then

$$\bar{S}_A = S_A \cup A$$

We have the following:

Lemma 2.2.18. Let $(u^{\alpha_k})_{k\in\mathbb{N}}$ be a sequence of functions such that $\alpha_k \to 0$ as $k \to \infty$ and for any $k \in \mathbb{N}$ the function $u^{\alpha_k} : \mathbb{R}^n \to \mathbb{R}^m$ is an energy minimizer of J^{α_k} as defined in (2.54).

We assume that

$$u^{\alpha_k}(x) \in \bar{S}_A, \forall x \in \mathbb{R}^n, k \in \mathbb{N} \tag{2.58}$$

Then there exists a subsequence relabelled for simplicity as the initial sequence such that:

$$u^{\alpha_k} \rightharpoonup \tilde{u}, \text{ in } W^{1,2}(\mathbb{R}^n;\mathbb{R}^m), \text{ as } k \to \infty \tag{2.59}$$

with $\tilde{u}$ a local energy minimizer of the functional J^0 defined as:

$$J^0(\Omega, u) := \int_\Omega \frac{1}{2}|\nabla u|^2 + W^0(u(x))\,dx \tag{2.60}$$

(with W^0 from (2.56)).

Proof. We have

$$(P)\begin{cases} W^{\alpha_k}(u) \to W^0(u) & \text{in } \bar{S}_A \text{ as } k \to \infty \\ W^{\alpha_k} \geq 0,\, \forall \alpha_k > 0 \end{cases}$$

Arguing along the lines of Lemma 2.2.12, (while taking into account the properties (P) and the definition (2.55) of W^{α_k}s) we get:

$$J^{\alpha_k}(B_r, u^{\alpha_k}) \leq Cr^{n-1} \tag{2.61}$$

for all $r \geq 1$, where C depends only on the points $a_1, \ldots, a_N$ through the assumed inclusion (2.58) (and is independent of $\alpha_k, k \in \mathbb{N}$).

Out of this uniform bound we claim that there exists $\tilde{u} \in W^{1,2}(\mathbb{R}^n;\mathbb{R}^m)$ such that:
(1) $u^{\alpha_k} \rightharpoonup \tilde{u}$ in $W^{1,2}(\mathbb{R}^n;\mathbb{R}^m)$ as $k \to \infty$ on a subsequence
(2) $\tilde{u}$ is a local minimizer of J^0.

By the bound (2.61) , $W^\varepsilon \geq 0$ and by the Rellich- Kondrachov theorem, we can obtain, along a subsequence

$$u^{\alpha_k} \rightharpoonup \tilde{u} \quad on \;\; W^{1,2}(\mathbb{R}^n;\mathbb{R}^m)$$

and

$$u^{\alpha_k} \to \tilde{u} \quad on \;\; L^p_{loc}(\mathbb{R}^n;\mathbb{R}^m)$$

These provide claim (1).

In order to show claim (2) we note first we have:

$$J^0(\tilde{u},\Omega) \leq \liminf_{\alpha_k \to 0} J^{\alpha_k}(u^{\alpha_k},\Omega) \tag{2.62}$$

Indeed, we have by lower semicontinuity

$$\int_\Omega |\nabla \tilde{u}|^2\, dx \leq \liminf_{k\to\infty} \int_\Omega |\nabla u^{\alpha_k}|^2\, dx \tag{2.63}$$

We have that $\tilde{u} \in \bar{S}_A$ and we denote $A_{\tilde{u}} := \{x \in \mathbb{R}^n : \tilde{u}(x) \in S_A\}$. Taking into account the specific form (2.55) of the potential W^α we have, for $\alpha_k \to 0$ as $k \to \infty$:

$$\int_{A_{\tilde{u}}\cap\Omega} \chi_{\{\tilde{u}\in S_A\}}\, dx = \int_{A_{\tilde{u}}\cap\Omega} dx = \lim_{k\to\infty} \int_{A_{\tilde{u}}\cap\Omega} W_{\alpha_k}(u^{\alpha_k}(x))\, dx \tag{2.64}$$

Furthermore, since $W^\alpha \geq 0$ we have:

$$\int_{\Omega\setminus A_{\tilde{u}}} \chi_{\{\tilde{u}\in S_A\}}\, dx = 0 \leq \lim_{k\to\infty} \int_{\Omega\setminus A_{\tilde{u}}} W^{\alpha_k}(u^{\alpha_k}(x))\, dx \tag{2.65}$$

The last three estimates provide the claimed relation (2.62). One can then trivially see that:

$$\inf J^0(\cdot,\Omega) \leq J^0(\tilde{u},\Omega) \leq \liminf_{\alpha_k\to 0} \inf J^{\alpha_k}(\cdot,\Omega) \tag{2.66}$$

We claim now that for an arbitrary $u \in W^{1,2}_{loc}(\mathbb{R}^n;\mathbb{R}^m)$ with $u(x) \in \bar{S}_A$ for almost all $x \in \mathbb{R}^n$ we have:

$$\lim_{\alpha_k\to 0} J^{\alpha_k}(u,\Omega) = J^0(u,\Omega) \tag{2.67}$$

Indeed we have:

$$\int_{A_u\cap\Omega} \chi_{\{u\in S_A\}}\, dx = \int_{A_u\cap\Omega} dx = \lim_{k\to\infty} \int_{A_u\cap\Omega} W^{\alpha_k}(u(x))\, dx \tag{2.68}$$

$$\int_{\Omega\setminus A_u} \chi_{\{u\in S_A\}}\, dx = 0 = \lim_{k\to\infty} \int_{\Omega\setminus A_u} W^{\alpha_k}(u(x))\, dx \tag{2.69}$$

so

$$\int_\Omega |\nabla u|^2 + \chi_{\{u \in S_A\}}\,dx = \lim_{k\to\infty} \int_\Omega |\nabla u|^2 + W^{\alpha_k}(u(x))\,dx,$$

as claimed.

We note now that (2.67) implies:

$$J^0(u,\Omega) = \lim_{\alpha_k \to 0} J^{\alpha_k}(u,\Omega) = \limsup_{\alpha_k\to 0} J^{\alpha_k}(u,\Omega) \geq \limsup_{\alpha_k\to 0} \inf J^{\alpha_k}(\cdot,\Omega)$$

and since this holds for u arbitrary we get:

$$\inf J^0(\cdot,\Omega) \geq \limsup_{\alpha_k\to 0} \inf J^{\alpha_k}(\cdot,\Omega) \tag{2.70}$$

The last inequality, together with (2.66) provide the claimed local minimality of $\tilde{u}$.

□

Note: The above Lemma also holds for the class of local minimizers of the energy.

2.3 Proofs

2.3.1 Proof of Proposition 2.1.3

Proof. (i) (cfr [1] p.161). Let

$$|u(x) - a| < M \ , \ \ ||u||_{C^\beta} < \hat{C} = \hat{C}(M) \ , \ x \in \mathcal{O} \tag{2.71}$$

where for the Hölder bound we utilized Lemma 2.2.10. Given $q \in (0, M)$, assume that

$$|u(x_0) - a| \geq q \tag{2.72}$$

Then the Hölder continuity of u implies that the hypothesis of the Density Estimate (2.24) is satisfied for

$$\lambda = \frac{q}{2} \ , \ r_0 = (\frac{q/2}{\hat{C}})^{\frac{1}{\beta}} \ , \ \mu_0 = \mathcal{L}^n(B_{r_0}(x_0)) \tag{2.73}$$

Therefore

$$\mathcal{L}^n(B_r(x_0) \cap \{|u - a| > \frac{q}{2}\}) \geq Cr^n \ , \ B_r(x_0) \subset \mathcal{O} \ , \ r \geq r_0 \tag{2.74}$$

Let

$$0 < w_{\frac{q}{2}} := \min_{\Sigma} W(z) \ , \ \Sigma = \{|z - a| > \frac{q}{2}\} \cap \{d(z, \{W = 0\} \setminus a) \geq k\} \tag{2.75}$$

From this and the Basic Estimate Lemma 2.2.11 we obtain

$$w_{\frac{q}{2}} C_1 r^n \leq J_{B_r(x_0)}(u) \leq C_0 r^{n-1} \tag{2.76}$$

which is impossible for

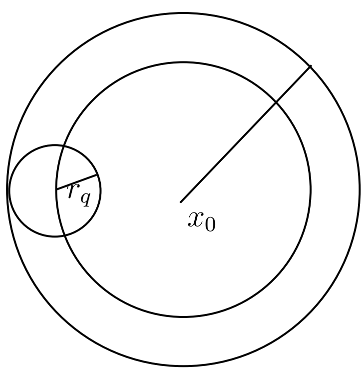
$$r > \frac{C_0}{w_{\frac{q}{2}} C_1} \tag{2.77}$$

Therefore if we set

$$r_q = \frac{2C_0}{w_{\frac{q}{2}} C_1} \tag{2.78}$$

then $B_{r_q}(x_0) \subset \mathcal{O}$ is incompatible with (2.72).
The proof of (i) is complete.

(ii) Consider the ball $B_R(x_0)$, R to be selected.
Let $\xi \in B_R(x_0)$



$B_R(x_0)$

$$d(\xi, \partial B_R(x_0)) = r_q \ , \ 0 < 2q < \rho_0 \tag{2.79}$$

where r_q as in (i) above. Note that by **(H1)**

$$w_{\frac{q}{2}} \geq C^*(\frac{q}{2})^{\alpha} \ , \ r_q = \frac{2C_0}{w_{\frac{q}{2}} C_1} \leq \frac{2C_0}{C_1 C^*}(\frac{q}{2})^{-\alpha} \tag{2.80}$$

and by (i) above

$$|u(\xi) - a| < q \tag{2.81}$$

Therefore by [1], Theorem 4.1 originally derived in [5]

$$|u(x) - a| < q \ , \ x \in B_{R-r_q}(x_0) \tag{2.82}$$

By (2.42) $v(x) := |u(x) - a|^2$ satisfies

$$\begin{cases} \Delta v \geq c^2 v^{\frac{\alpha}{2}} & \text{weakly in} W^{1,2}(B_{R-r_q}(x_0)) \\ v \leq \delta & \text{on } \partial B_{R-r_q}(x_0) \end{cases} \tag{2.83}$$

and therefore by Lemma 2.2.14

$$d(y_0, \partial B_{R-r_q}(x_0)) > R_0 \Rightarrow v(y_0) = 0 \tag{2.84}$$

where

$$R_0 = \frac{\sqrt{n(\alpha+2)}}{(1-\frac{\alpha}{2})c} q^{1-\frac{\alpha}{2}} \ , \ 0 < \alpha < 2 \ , \ c^2 = 2\alpha C^* \tag{2.85}$$

Therefore

$$u(x) = a \quad \text{in} \ B_{R-r_q-R_0}(x_0) \tag{2.86}$$

To conclude set $R = Cq^{-\alpha}$ and impose the requirement that

$$\frac{C}{2} q^{-\alpha} \leq Cq^{-\alpha} - r_q - R_0 \tag{2.87}$$

which is satisfied if

$$C \geq \frac{2^{\alpha+2} C_0}{C_1 C^*} + 2 \frac{\sqrt{n(\alpha+2)}}{(1-\frac{\alpha}{2})\sqrt{2\alpha C^*}} (\frac{\rho_0}{2})^{1+\frac{\alpha}{2}} =: \hat{C}(\alpha, n) \tag{2.88}$$

The proof of Proposition 2.1.3 is complete.

□

2.3.2 Proof of Theorem 2.1.1

Proof. Step 1 (Existence of a positive minimizer)

We will be establishing the existence of a map $u_R \in W^{1,2}(B_R, \mathbb{R}^n)$ that is equivariant, positive and also a minimizer in the equivariant class of

$$J_{B_R}(u) = \int_{B_R} (\frac{1}{2}|\nabla u|^2 + W(u))dx \ \ , \ B_R = \{|x| < R\} \subset \mathbb{R}^n, \tag{2.89}$$

that satisfies the Basic Estimate

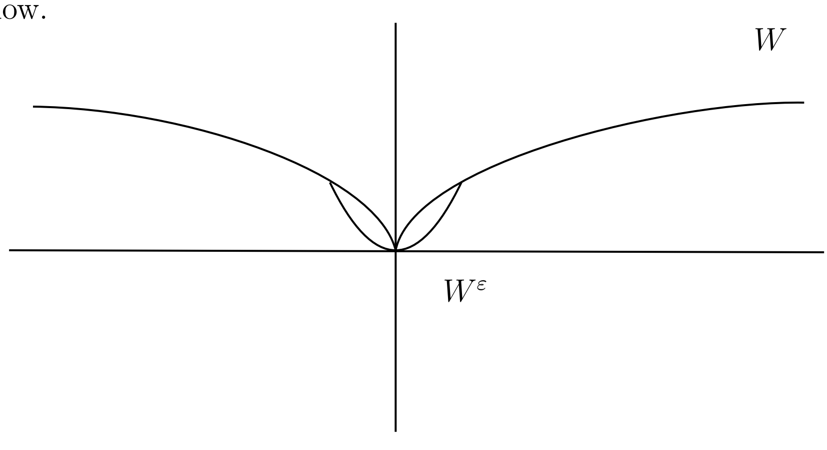
$$J_{B_r}(u_R) \leq Cr^{n-1} \ \ , \ r_0 < r < R \ , \ R \geq R_0 \tag{2.90}$$

C independent of $R\,, r$.

We introduce the regularized energy functional

$$J^{\varepsilon}_{B_R}(u) = \int_{B_R} (\frac{1}{2}|\nabla u|^2 + W^{\varepsilon}(u))dx \tag{2.91}$$

where W^{ε} is obtained from W by regularizing only at the minima as in Figure below.



with

$$(\star) \begin{cases} W^{\varepsilon} \to W(u) \ , \text{ uniformly on compacts} \\ W^{\varepsilon} \in C^2 \ , \ ||W^{\varepsilon}||_{C^{\alpha}} < C \ , \text{ for } W \text{ satisfying } (\mathbf{H1}) \ , \\ W^{\varepsilon} \geq 0 \ , \ \{W^{\varepsilon} = 0\} = \{W = 0\} \\ W^{\varepsilon}(gu) = W^{\varepsilon}(u) \ \ , \text{for all } g \in G \text{ and } u \in \mathbb{R}^n. \end{cases} \tag{2.92}$$

We can assume that

$$W^\varepsilon(u) = W(u) \quad \text{for } |u| \geq M > 0 \tag{2.93}$$

some $M > 0$, and that the minimizer of $J^\varepsilon_{B_R}$ in the equivariant class satisfies the bound

$$|u^\varepsilon_R| \leq M \quad , \ x \in B_R \tag{2.94}$$

with M independent of ε and R and that moreover u^ε_R is positive. Here we are utilizing [1] Lemma 6.1.

We begin by establishing the Hölder Estimate (2.25), for u^ε_R, with constant C independent of ε , R. Recall that u^ε_R is a minimizer in the equivariant class, while (2.25) was derived under the stronger hypothesis of being a minimizer under arbitrary perturbations. We point out only the necessary modifications of the proof of the Lemma 2.2.10.

We will derive

$$|u^\varepsilon_R(x) - u^\varepsilon_R(y)| \leq C|x-y| \ln|x-y|^{-1} \quad , \forall\, x, y \in B_R(0) \setminus B_1(0) \tag{2.95}$$

with $|x-y| \leq \frac{1}{2}$, $R \geq 2$.

Notice that we can cover $F_R \cap (B_R(0) \setminus B_1(0)) =: F_{R,D}$ where $F_R = F \cap B_R(0)$ by two types of balls $B_{\frac{1}{4}}(x_0)$:
(a) Balls entirely contained in $F_{R,D}$, $B_{\frac{1}{4}} \subset F_{R,D}$,
(b) balls $B_{\frac{1}{4}}(x_0)$ having their center in the wall of F_R which is made up of reflection planes in G_a.
Notice that both types can be equivariantly extended over $B_R(0) \setminus B_1(0)$ as sets.

Fix now $B_r(x_0)$, $r < \frac{1}{4}$ as in the proof of (2.27). Due to the equivariant extension of v_r there, and the minimality of u^ε_R in the equivariant class, we see that u^ε_R has the minimizing property on $B_r(x_0)$ and so (2.29) applies as before. The rest of the argument is unchanged.

Thus (2.95) is established.

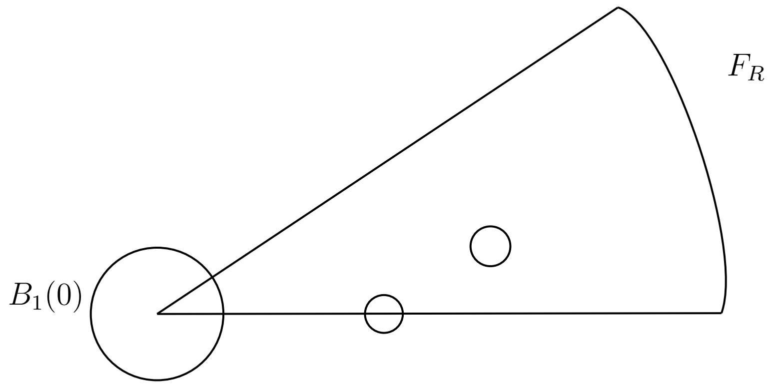


Fig :Typical $B_{\frac{1}{4}}(x_0)$ $'s$ covering the fundamental region and extensible equivariantly on $B_R(0) \setminus B_1(0)$.

Now we will proceed to establish (2.90),

$$J_{B_r(0)}(u_R^\varepsilon) \leq Cr^{n-1} \quad , \forall\, r \in (2, R-1) \tag{2.96}$$

with C constant independent of ε and R , $C = C(M)$.
We follow [1] Proposition 6.1, and for $2 < r < R-1$ we define

$$u_{aff}(x) = \begin{cases} d(x, \partial D)a_1 & , \text{for } x \in D_R \text{ and } d(x, \partial D) \leq 1 \\ a_1 & , \text{for } x \in D_R \text{ and } d(x, \partial D) \geq 1 \end{cases} \tag{2.97}$$

where $D_R = D \cap B_R$ and extend equivariantly in B_R. Since u_{aff} vanishes on ∂D, the extended map is also continuous. As it is well known, the distance is 1-Lipschitz and therefore in $W^{1,\infty}(B_R)$. Fix now a number $h \in (0, 1)$ and for $r \in (2, R-1)$ define

$$\hat{u}_R^\varepsilon(x) = \varphi(1 - \frac{|x| - (r-h)}{h})u_{aff}(x) + \phi(\frac{|x| - (r-h)}{h})u_R(x) \tag{2.98}$$

where $\phi : \mathbb{R} \to [0, 1]$ is a fixed C^1 function such that $\phi(s) = 0$, for $s \leq 0$ and $\phi(s) = 1$, for $s \geq 1$. Note that $\hat{u}_R^\varepsilon \in W_E^{1,2}(B_R(0); \mathbb{R}^n)$ (equivariant), and most importantly $\hat{u}_R^\varepsilon = u_R^\varepsilon$ on $\partial B_r(0)$. Moreover $\hat{u}_R = u_{aff}$ in $B_{r-h}(0)$ and $\hat{u}_R^\varepsilon = u_R^\varepsilon$ on $B_R(0) \setminus B_1(0)$ and $u_{aff} = a_1$ if $d(x, \partial D) \geq 1$. By the minimality of u_R^ε we have

$$\begin{gathered} J_{B_r(0)}(u_R^\varepsilon) \leq J_{B_r(0)}(\hat{u}_R^\varepsilon) \\ = \int_{B_{r-h} \cap \{d(x,\partial D) \leq 1\}} (\frac{1}{2}|\nabla \hat{u}_R^\varepsilon|^2 + W(\hat{u}_R^\varepsilon))dx + \int_{B_r \setminus B_{r-h}} (\frac{1}{2}|\nabla \hat{u}_R^\varepsilon|^2 + W(\hat{u}_R^\varepsilon))dx \\ \leq C_1(r-h)^{n-1} + C_2 r^{n-1} \end{gathered} \tag{2.99}$$

where for the estimate of the 2^{nd} term we used the Hölder estimate above and the analogous (2.38), (2.39).

Hence (2.96) is established.

Thus for any $R > 0$ there exists $C_R > 0$, independent of $\varepsilon > 0$, such that

$$\int_{B_R} (\frac{1}{2}|\nabla u_R^\varepsilon|^2 + W^\varepsilon(u_R^\varepsilon))dx < C_R \tag{2.100}$$

Out of the above uniform bounds we claim that there exists $u_R \in W^{1,2}(B_R;\mathbb{R}^m)$ such that

(1) $u_R^\varepsilon \rightharpoonup u_R$ weakly in $W^{1,2}(B_R;\mathbb{R}^m)$ as $\varepsilon \to 0$ on a subsequence,

(2) u_R is a minimizer of

$$J_{B_R}(u) = \int_{B_R} (\frac{1}{2}|\nabla u|^2 + W(u))dx \ ,$$

(3) $J_{B_r}(u_R) \leq Cr^{n-1}$ with C independent of ε and R ,

(4) u_R is equivariant and positive.

By (2.100) and $W^\varepsilon \geq 0$ and the Rellich-Kondrachov theorem, we can obtain, for a subsequence

$$u_R^\varepsilon \rightharpoonup u_R \quad \text{on} \quad W^{1,2}(B_R;\mathbb{R}^m)$$

and

$$u_R^\varepsilon \to u_R \quad on \quad L^p(B_R;\mathbb{R}^m)$$

These establish claims (1) and (4).

In order to show claim (2) we take $\phi \in C_c^\infty(\mathbb{R}^n)$, $supp\phi \subset K \subset B_R$. Then by minimality we have:

$$J_{B_R}^\varepsilon(u_R^\varepsilon + \phi) - J_{B_R}^\varepsilon(u_R^\varepsilon) \geq 0$$
$$\Leftrightarrow \int_{B_R} (\nabla u_R^\varepsilon \nabla\phi + \frac{1}{2}|\nabla\phi|^2 + W^\varepsilon(u_R^\varepsilon + \phi) - W^\varepsilon(u_R^\varepsilon))dx \geq 0$$

Let $I_1^\varepsilon := \int_{B_R} \nabla u_R^\varepsilon \nabla\phi dx$ and $I_2^\varepsilon := \int_{B_R}(W^\varepsilon(u_R^\varepsilon + \phi) - W^\varepsilon(u_R^\varepsilon))dx$.

Thanks to (1) before we have $I_1^\varepsilon \to I_1 = \int_{B_R} \nabla u_R \nabla \phi dx$
we split:

$$I_2 = \int_{B_R} (W^\varepsilon(u_R^\varepsilon + \phi) - W(u_R^\varepsilon + \phi))dx + \int_{B_R} (W(u_R^\varepsilon + \phi) - W^\varepsilon(u_R^\varepsilon))dx$$

Let $I_{21}^\varepsilon := \int_{B_R}(W^\varepsilon(u_R^\varepsilon + \phi) - W(u_R^\varepsilon + \phi))dx$ and $I_{22}^\varepsilon := \int_{B_R}(W(u_R^\varepsilon + \phi) - W^\varepsilon(u_R^\varepsilon))dx$, $I_{21}^\varepsilon \to 0$ as $\varepsilon \to 0$ because of the uniform bound $|u_R^\varepsilon(x)| \leq M$ the uniform convergence on compacts of W^ε to W and the dominated convergence theorem.
Also $I_{22}^\varepsilon \to I_{22} = \int_{B_R}(W(u_R + \phi) - W(u_R))dx$ because of the L^p convergence of u_R^ε to u_R, dominated convergence and continuity of W.

Thus we establish the claimed relation (2).
In order to get the claimed relation (3) we recall

$$J_{B_r}^\varepsilon(u_R^\varepsilon) = \int_{B_r} (\frac{1}{2}|\nabla u_R^\varepsilon|^2 + W^\varepsilon(u_R^\varepsilon))dx \leq Cr^{n-1}$$

with C depending only on M, but not on R nor on ε.
As $u_R^\varepsilon \rightharpoonup u_R$ in $W^{1,2} \Rightarrow \int_{B_R} |\nabla u_R|^2 dx \leq \liminf \int_{B_R} \frac{1}{2}|\nabla u_R^\varepsilon|^2 dx$ and we have

$$\int_{B_R} W^\varepsilon(u_R^\varepsilon)dx \to \int_{B_R} W(u_R)dx$$

arguing as in the treatment of the I_2 before.

Claim: There exists $\overline{u} \in W_{loc}^{1,2}(\mathbb{R}^n; \mathbb{R}^m)$ nontrivial equivariant, positive and minimizer of

$$J_\Omega(u) = \int_\Omega (\frac{1}{2}|\nabla u|^2 + W(u))dx \tag{2.101}$$

In addition, $\overline{u}$ satisfies the estimate

$$J_{B_r}(\overline{u}) \leq cr^{n-1} \tag{2.102}$$

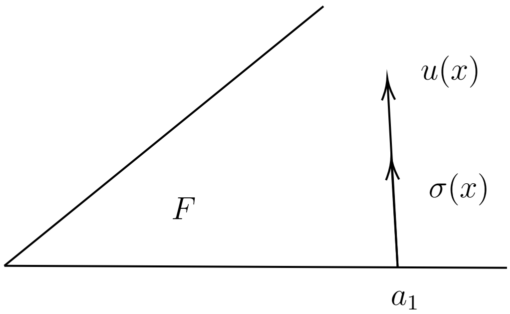
Proof.
We have that out of the uniform bound $J_{B_r}(u_R) \leq cr^{n-1}$, we get as before, in the proof of the claims (1)-(4) that $u_R \rightharpoonup \overline{u}$ in $W_{loc}^{1,2}(\mathbb{R}^n; \mathbb{R}^m)$ and that $\overline{u}$ is equivariant and positive. We can argue similarly as in the proof of (2) above to get that $\overline{u}$ is a minimizer of J_Ω defined in (2.101), (2.102) follows from (3).

Step 2. (Existence of a free boundary)

We utilize that D contains a unique zero a_1 of W and that by equivariance we can restrict u in D and note that

$$d(u(D), \{W = 0\} \setminus a_1) \geq k > 0$$

For implementing Proposition 2.1.3 we need a couple of observarions. Firstly u is minimizing in the class of equivariant positive maps. We recall that in the proof of Proposition 2.1.3 the density estimate (2.24) is utilized. We note that in the proof of the density estimate the energy comparison maps are obtained by reducing the modulus of the map $q^u(x) = |u(x) - a_1|$ and leaving the angular part $\nu^u(x)$ unchanged, $u(x) = a_1 + q^u(x)\nu^u(x)$, $\sigma(x) = a_1 + q^\sigma(x)\nu^u(x)$, $0 \leq q^\sigma(x) \leq q^u(x)$.



Therefore by the convexity of F the comparison map $\sigma(x)$ is also positive, $\sigma(\overline{F}) \subset \overline{F}$, and it can be extended equivariantly from F to $\mathbb{R}^n$ since $B_R(x_0) \subset F$ or $B_R(x_0) \subset D$ with $x_0 \in \partial F$, in the boundary of F, which consists of reflection planes in G_{a_1}.

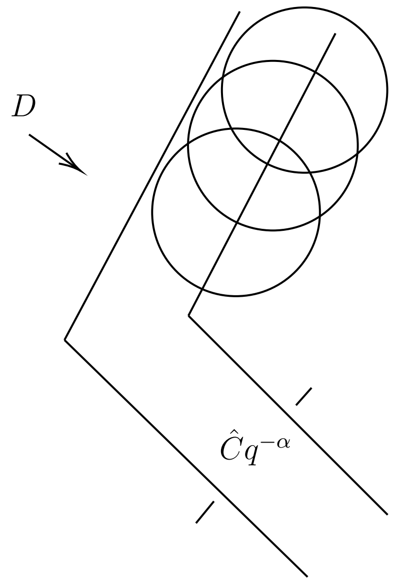
Thus Proposition 2.1.3 (ii) can be applied for a fixed q, with $2q \leq \rho$, to produce the estimate

$$B_{Cq^{-\alpha}}(x_0) \subset D \Rightarrow u(x) \equiv a_1 \text{ in } B_{\frac{C}{2}q^{-\alpha}}(x_0) \tag{2.103}$$

for $C \geq \hat{C}(\alpha, n)$.

By taking a sequence of $C's$ tending to infinity via a covering argument we see that

$$u(x) \equiv a_1 \text{ if } d(x, \partial D) \geq \hat{C}(\alpha, n)q^{-\alpha} \tag{2.104}$$



The proof of Theorem 2.1.1 is complete.

□

2.3.3 Proof of Proposition 2.1.5

Proof. From the assumption (2.16) and the Basic Estimate we have

$$\int_{B_R(x_0)} \chi_{\{u\neq a_i\}} dx = \int_{B_R(x_0)} \chi_{A^c}(u) dx \leq CR^{n-1}$$

But

$$\int_{B_R(x_0)} \chi_{\{u\neq a_i\}} dx = \mathcal{L}^n(\{|u-a_i|>0\} \cap B_R(x_0))$$

Hence

$$\mathcal{L}^n(\{u=a_i\} \cap B_R(x_0)) \geq |B_R(x_0)| - cR^{n-1} \geq CR^n \ \ , \ R > R_0.$$

□

2.3.4 Proof of Proposition 2.1.6

Proof. Let

$$0 < \theta < d_0 := \min\{|a_i - a_j| : i \neq j \ , \ i,j \in \{1,...,N\}\}$$

θ arbitrary otherwise.

1. We claim that there exist at least two distinct points $a_i \neq a_j$ in A such that

$$\mathcal{L}^n(B_R(x_0) \cap \{|u - a_k| \leq \theta\}) \geq C_k R^n \quad , \ R \geq R_0 \ , \ k = i,j$$

$C_k = C_k(\theta)$.

Proof of the Claim. Since u is a nonconstant minimizer, there is x_1 such that $u(x_1) \neq a_1$

$\Rightarrow \mathcal{L}^n(B_{\tilde{R}_0}(x_1) \cap \{|u - a_1| > \lambda\}) \geq \mu_0$ (by continuity, for some $\tilde{R}_0 \, , \mu_0 > 0$ and $\lambda > 0$ small)

and therefore by the Density Estimate (2.24) we have:

$$\mathcal{L}^n(B_R(x_1) \cap \{|u - a_1| > \lambda\}) \geq cR^n \ , \ R \geq \tilde{R}_0. \tag{2.105}$$

Notice that by (2.105), there is $R_1(x_0) > 0$ such that

$$\mathcal{L}^n(B_R(x_0) \cap \{|u - a_1| > \lambda\}) \geq c_1 R^n \ , \ R \geq R_1(x_0). \tag{2.106}$$

Similarly, since $u \neq a_k$ there is x_k such that $u(x_k) \neq a_k$ and we can repeat the arguments above with x_k in the place of x_1 to obtain

$$\mathcal{L}^n(B_R(x_0) \cap \{|u - a_k| > \lambda\}) \geq c_k R^n \ , \ R \geq R_k \ , \ k = 2,...,N \tag{2.107}$$

for some small $\lambda > 0$.
By Remark 5.4 in [1], $\forall \ \lambda_1, ..., \lambda_N \in (0, d_0)$ we have

$$\mathcal{L}^n(B_R(x_0) \cap \{|u - a_k| > \lambda_k\}) \geq c_k R^n \ , \ R \geq R_0 \ , \ (R_0 = \max_{k \in \{1,...,N\}} R_k). \tag{2.108}$$

So, if $\lambda < d_0 - \theta$ and $|u - a_1| \leq \theta < d_0 \leq |a_1 - a_2|$

$$\Rightarrow |u - a_2| \geq |a_1 - a_2| - \theta > \lambda > 0 \Rightarrow \{|u - a_1| \leq \theta\} \subset \{|u - a_2| > \lambda\}.$$

Thus

$$A_2 := \bigcup_{k=1\ ,\ k\neq 2}^{N} \{|u-a_k| \leq \theta\} \subset \{|u-a_2| > \lambda\} \tag{2.109}$$

$$\begin{aligned} &\Rightarrow A_2 \cup [\{|u-a_2| > \lambda\} \cap A_2^c] = \{|u-a_2| > \lambda\} \\ &\Leftrightarrow A_2 \cup [\{|u-a_2| > \lambda\} \cap (\bigcap_{k=1\ ,\ k\neq 2}^{N} \{|u-a_k| > \theta\})] = \{|u-a_2| > \lambda\} \end{aligned} \tag{2.110}$$

and from the Basic Estimate (2.23) and the hypothesis **(H1)** on W we have

$$\mathcal{L}^n(B_R(x_0) \cap \{|u-a_2| > \lambda\} \cap (\bigcap_{k=1\ ,\ k\neq 2}^{N} \{|u-a_k| > \theta\})) \leq \overline{c}R^{n-1}$$

Hence, by (2.108) and (2.110) it holds

$$\mathcal{L}^n(B_R(x_0) \cap A_2) \geq \overline{c}_2 R^n \Leftrightarrow \mathcal{L}^n(B_R(x_0) \cap (\bigcup_{k=1\ ,\ k\neq 2}^{N} \{|u-a_k| \leq \theta\})) \geq \overline{c}_2 R^n$$

and similarly, if $A_l := \bigcup_{k=1\ ,\ k\neq l}^{N} \{|u-a_k| \leq \theta\}$, $l = 1, 2, ..., N$, we have

$$\mathcal{L}^n(B_R(x_0) \cap (\bigcup_{k\neq l} \{|u-a_k| \leq \theta\})) \geq \overline{c}_l R^n \ \ ,\ R \geq R_0$$

for all $l = 1, 2, ..., N$.
Therefore there exist at least two $i, j \in \{1, ..., N\}$ such that

$$\mathcal{L}^n(B_R(x_0) \cap \{|u-a_k| \leq \theta\}) \geq \overline{c}_k R^n \ \ ,\ R \geq R_0 \ \ ,\ k = i, j,$$

and the claim is proved.

$\square$

2. We now proceed to conclude the proof of Proposition 2.1.6.
Let $\mathcal{A}_k^R := \overline{B_R(x_0)} \cap \{|u-a_k| \leq \theta\}$, $k = i, j$

$$\begin{aligned} \int_{\mathcal{A}_i^R} \chi_{\{u\neq a_i\}}(x)dx &= \mathcal{L}^n(\{|u-a_i| > 0\} \cap \mathcal{A}_i^R) \\ &= \mathcal{L}^n(\bigcap_{k=1}^{N} \{|u-a_k| > 0\} \cap \mathcal{A}_i^R) \quad \text{(by (2.109))} \\ &= \int_{\mathcal{A}_i^R} W^0(u)dx \leq cR^{n-1} \quad \text{(by the Basic Estimate (2.23))} \end{aligned} \tag{2.111}$$

$$
\begin{aligned}
\mathcal{L}^n(\{u = a_i\} \cap \mathcal{A}_i^R) &= \mathcal{L}^n(\mathcal{A}_i^R) - \mathcal{L}^n(\{u \neq a_i\} \cap \mathcal{A}_i^R) \\
&\geq c_i R^n - \mathcal{L}^n(\{u \neq a_i\} \cap \mathcal{A}_i^R) \quad \text{(by Step 1.)} \\
&\geq c_i R^n - cR^{n-1} \geq C_i R^n \ \ , \ R \geq R_0 \ \ \text{(by (2.111))}
\end{aligned} \tag{2.112}
$$

Similarly for $\{u = a_j\}$.

Now, for obtaining (2.20), we utilize the isoperimetric inequality (see for example [14])

$$
\min\{\mathcal{L}^n(\overline{B_R(x_0)} \cap E_i)\ ,\ \mathcal{L}^n(\overline{B_R(x_0)} \setminus E_i)\}^{1-\frac{1}{n}} \leq 2\hat{c}\, ||\partial E_i||(B_R(x_0)) \tag{2.113}
$$

with $E_i = \{u(x) = a_i\}$ $(E_j = \{u(x) = a_j\})$. Utilizing (2.19), we have

$$
\mathcal{L}^n(\overline{B_R(x_0)} \cap E_i) \geq c_i R^n
$$

On the other hand

$$
\overline{B_R(x_0)} \setminus E_i \supset \overline{B_R(x_0)} \cap E_j
$$

and once more by (2.19)

$$
\mathcal{L}^n(\overline{B_R(x_0)} \cap E_j) \geq c_j R^n
$$

Thus the lower bound (2.20) follows.

The proof of Proposition 2.1.6 is complete. □

2.3.5 Proof of Proposition 2.1.9

Proof. 1. Here we require $N = m + 1$ and invoke Lemma 2.2.18, and thus produce an equivariant, positive minimizer for $\alpha = 0$ satisfying the Basic Estimate (2.102). We note that from equivariance and (2.102) it follows that $u \neq$ constant (if $u \equiv$ constant, from equivariance we would have that $u \equiv (0, ..., 0)$ which contradicts the Basic Estimate (2.102) since $(0, ..., 0) \notin \{W = 0\}$).

2. By Proposition 2.1.6 we have that there exist $R_0 > 0$ and at least two distinct $a_i \neq a_j$ $(i, j \in \{1, ..., N+1\})$ such that

$$\mathcal{L}^n(B_R(0) \cap \{u = a_k\}) \geq c_k R^n \ \ , \ R \geq R_0 \ , \ k = i, j. \tag{2.114}$$

We partition $\mathbb{R}^n$ in $D^1, ..., D^{N+1}$ (see **(H3)**) where in each D^i there is a unique global minimum of W (i.e. a_i , and D^1 is denoted as D). Thus $u \neq a_j$ in the region D^i $(i \neq j)$, so from (2.114) we have

$$\mathcal{L}^n(B_R(0) \cap \{u = a_i\}) = \mathcal{L}^n(D^i_R \cap \{u = a_i\}) \geq c_i R^n \ \ , \ R \geq R_0 \ , \ D^i_R = D^i \cap B_R(0) \tag{2.115}$$

and from the equivariance of u we obtain

$$\mathcal{L}^n(D^k_R \cap \{u = a_k\}) \geq c_k R^n \ \ , \ R \geq R_0 \ , \ k = 1, ..., N+1. \tag{2.116}$$

3. Finally, from the Basic Estimate (2.102), we have

$$\mathcal{L}^n(B_R(0) \cap (\bigcap_{i=1}^{N+1} \{u \neq a_i\}) = \int_{B_R(0)} W^0(u) dx \leq C R^{n-1} \tag{2.117}$$

and therefore

$$\mathcal{L}^n(D^1_R \cap \{u \neq a_1\}) = \mathcal{L}^n(D^1_R \cap (\bigcap_{i=1}^{N+1} \{u \neq a_i\}) \leq C R^{n-1}. \tag{2.118}$$

The proof of Proposition 2.1.9 is complete. □

2.4 Appendix A: The Containment

The following result was established by the first author and P. Smyrnelis in unpublished work [9]. We reproduce it here for the convenience of the reader. For related applications of the method of proof we refer to [38].

Proposition 2.4.19. ([9])
Let $u : \mathbb{R}^n \to \mathbb{R}^m$ be a bounded ($|u(x)| < M$) critical point of the functional

$$J(u) = \int (\frac{1}{2}|\nabla u|^2 + W(u))dx$$

in the sense that $\forall \Omega \subset \mathbb{R}^n$, open, bounded,

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} J_\Omega(u + \varepsilon\phi) = 0 \quad , \forall \, \phi \in C_0^1(\Omega)$$

where

$$W(u) = \begin{cases} W^{\overline{\alpha}}(u) := \prod_{k=1}^{m+1} |u - a_k|^{\alpha_k} & , \, \overline{\alpha} = (\alpha_1, ..., \alpha_{m+1}) \ , \ 0 < \alpha_k \leq 2 \\ W^0(u) := \chi_{\{u \in S_A\}} \end{cases} \tag{2.119}$$

and S_A defined as the interior of the simplex with vertices $a_1, ..., a_m, a_{m+1}$,

$$S_A := \{\sum_{i=1}^{m+1} \lambda_i a_i \ ; \ \lambda_i \in [0,1) \ , \ \forall i = 1, ..., m+1 \ , \ \sum_{i=1}^{m+1} \lambda_i = 1\} \tag{2.120}$$

Then

$$u(x) \in \overline{S}_A \ , \ x \in \mathbb{R}^n \tag{2.121}$$

For $\alpha_k \in [0, 1)$ we require that u in addition is a minimizer in the sense of (2.3), so that (2.123) is available.

Proof. Following an idea from [4] we introduce the set
1. $\alpha_k \in (0,1)$, $k = 1, .., m$.

$$F_M := \{u : \mathbb{R}^n \to \mathbb{R}^m \ , \ u \ \text{ minimizer of } \ J \ , \ |u(x)| \leq M\} \tag{2.122}$$

By Lemma 2.2.10 we have the uniform Hölder estimate

$$|u|_{C^\beta(\mathbb{R}^n;\mathbb{R}^m)} \leq C(M) \quad , u \in F_M \tag{2.123}$$

Let Π be the face of the simplex $\overline{S}_A$ defined by $a_2, ..., a_{m+1}$, oppposite to a_1 and let $e \perp \Pi$.

Set

$$P(u; x) = \langle u(x) - a_2, e \rangle \tag{2.124}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^m$ and the orientation of e is such that $\langle a_2 - a_1, e \rangle > 0$. Set

$$P_M := \sup\{P(u;x) \ : \ u(\cdot) \in F_M \ , \ x \in \mathbb{R}^n\}$$

Claim: $P_M \leq 0$
Clearly the proposition follows from this claim. We proceed by contradiction. Suppose $P_M > 0$. Thus there is $\{u_k\} \in F_M$, $\{x_k\} \subset \mathbb{R}^n$, such that

$$P_M - \frac{1}{k} \leq P(u_k, x_k) \leq P_M. \tag{2.125}$$

Set

$$v_k(x) := u_k(x + x_k), \tag{2.126}$$

and note that $v_k \in F_M$ and

$$P_M - \frac{1}{k} \leq P(v_k, 0) \leq P_M \tag{2.127}$$

By (2.123),

$$|v_k|_{C^\beta(\mathbb{R}^n;\mathbb{R}^m)} \leq C(M)$$

hence by Arzela- Ascoli for a subsequence

$$v_k \xrightarrow{C^\beta} v \quad , \text{ on compacts} \tag{2.128}$$

We have

$$P(v;x) \leq P_M = P(v;0) > 0 \ , \ x \in \mathbb{R}^n \tag{2.129}$$

By the continuity of v there is $R > 0$ such that

$$\frac{P_M}{2} \leq P(v;x) \leq P_M \ , \ x \in B(0;R) \tag{2.130}$$

$$P(v_k;x) = \langle v_k(x) - a_2, e \rangle \geq \frac{P_M}{4} \ , \text{ on } \ B(0;R) \tag{2.131}$$

for k large.

Thus $v_k(x)$ uniformly away from $a_1, ..., a_m, a_{m+1}$, we have

$$\Delta v_k - W_u(v_k) = 0 \ , \text{ in } \ B(0;R) \tag{2.132}$$

classically, since $W_u(u) \in C^1$ away from $a_1, ..., a_m, a_{m+1}$ and $x \mapsto W_u(v_k(x))$ Holder by (A.10), thus $u \in C^{2+\beta}(B(0;R))$.

We now calculate:

$$\Delta P = \langle \Delta v, e \rangle = \langle W_u(u), e \rangle$$

$$\frac{\partial}{\partial v_j} W(v) = \frac{\partial}{\partial v_j} (\prod\nolimits_{\nu=1}^{m+1} |v - a_\nu|^{\alpha_\nu}) = \sum_{i=1}^{m+1} \frac{\partial}{\partial v_j} (|v - a_i|^{\alpha_i}) \prod\nolimits_{\nu \neq i} |v - a_\nu|^{\alpha_\nu}$$

Notice that

$$\frac{\partial}{\partial v_j} (|v - a_i|^2)^{\frac{\alpha_i}{2}} = \alpha_i |v - a_i|^{\alpha_i - 2} \cdot (v_j - a_i^j)$$

where $a_i = (a_i^1, ..., a_i^m)$

Hence

$$W_v(v) = \nabla_v W(v) = \sum_{i=1}^{m+1} a_i (|v - a_i|^{\alpha_i - 2})(v - a_i) \prod_{\nu \neq i} |v - a_\nu|^{\alpha_\nu} =$$
$$= \alpha_2 |v - a_2|^{\alpha_2 - 2} (v - a_2) \prod_{\nu \neq 2} |v - a_\nu|^{\alpha_\nu} + \sum_{i \neq 2} \alpha_i |v - a_i|^{\alpha_i - 2} (v - a_i) \prod_{\nu \neq i} |v - a_\nu|^{\alpha_\nu}.$$

Therefore

$$\Delta P = \alpha_2 |v - a_2|^{\alpha_2 - 2} \prod_{\nu \neq 2} |v - a_\nu|^{\alpha_\nu} \langle v - a_2, e \rangle$$
$$+ \sum_{i \neq 2} \alpha_i |v - a_i|^{\alpha_i - 2} \langle v - a_i, e \rangle \prod_{\nu \neq i} |v - a_\nu|^{\alpha_\nu}$$

Note that by the contradiction hypothesis, $\langle v(x) - a_i, e \rangle > 0$ (think of a_2 as the origin).

Hence $\Delta P > 0$ on $B(0;R)$ contradicting that $P(v;x)$ takes its maximum at $x = 0$.

2. $\overline{\alpha} = 0$
For $W(u) = W^0(u) := \chi_{\{u \in S_A\}}$, the proof proceeds similarly. The difference here is that $\Delta P = 0$, in $B(0;R)$ which also leads to a contradiction by the maximum principle since $P(v;x)$ takes its maximum at $x = 0$.

3. $\alpha_k \in [1,2],\ k = 1, ..., m$.
In this case we define

$$F_M := \{u : \mathbb{R}^n \to \mathbb{R}^m\ ,\ \Delta u - W_u(u) = 0\ ,\ |u(x)| \leq M\}$$

u a weak $W^{1,2}$ solution. By linear elliptic theory we have the estimate (2.123). The rest of the argument is as before.
The proof of the proposition is complete.

□

2.5 Appendix B: The free boundary

We follow closely the formal derivation from [1] p.140. We imbed the minimizer in a class of variations, $u(\tau) := u(\cdot,\tau)$, with $u(0)$ corresponding to the minimizer, $u(\tau) = u(0)$ outside a ball B centered at some x_0 and quite arbitrary otherwise.

Let

$$U(\tau) := \{|u(\cdot,\tau) - a| > 0\} \tag{2.133}$$

for

$$a \in \{W = 0\},\ u(\tau) = a \text{ on } \partial U(\tau)$$

Set

$$\lambda(\tau) := \frac{1}{2}\int_{U(\tau)} |\nabla u(\tau)|^2\, dx\ ,\ \mu(\tau) := \int_{U(\tau)} W(u(\tau))\, dx \tag{2.134}$$

We denote $V := \frac{\partial X}{\partial \tau}$ where $X(s,\tau)$ is a parametrisation of $\partial U(\tau)$, $s \in \Omega \subset \mathbb{R}^{n-1}$.

Then we have:

$$\begin{aligned}\dot{\lambda}(\tau) &= \int_{U(\tau)} \nabla u(\tau)\nabla u_\tau(\tau)\, dx + \frac{1}{2}\int_{\partial U(\tau)} |\nabla u(\tau)|^2 V\cdot\nu dS\\ &= \int_{U(\tau)} -\Delta u(\tau) u_\tau(\tau)\, dx + \int_{\partial U(\tau)} \frac{\partial u}{\partial \nu}\cdot u_\tau\, dS + \frac{1}{2}\int_{\partial U(\tau)} |\nabla u(\tau)|^2 V\cdot\nu\, dS\end{aligned} \tag{2.135}$$

where ν is the unit outward normal to $\partial U(\tau)$ (pointing outside $U(\tau)$).

Now from $u(X(s,\tau),\tau) = a$ we obtain:

$$\begin{aligned} 0 =& \frac{\partial}{\partial \tau}[u(X(s,\tau),\tau)] = \frac{\partial u}{\partial \tau} + \frac{\partial u}{\partial \nu}\frac{\partial X}{\partial \tau} \cdot \nu \\ =& u_\tau + \frac{\partial u}{\partial \nu} V \cdot \nu \end{aligned} \tag{2.136}$$

Hence

$$u_\tau \cdot \frac{\partial u}{\partial \nu} = -|\frac{\partial u}{\partial \nu}|^2 V \cdot \nu \tag{2.137}$$

Then from (2.135) and (2.137) and the equation $\Delta u = W_u(u)$ we get:

$$\dot{\lambda}(0) = \int_{U(0)} -W_u(u(0))u_\tau(0)\,dx - \frac{1}{2}\int_{\partial U(0)} |\nabla u(0)|^2 V \cdot \nu dS. \tag{2.138}$$

On the other hand

$$\dot{\mu}(\tau) = \int_{\partial U(\tau)} W(u(\tau)) V \cdot \nu dS + \int_{U(\tau)} W_u(u(\tau))u_\tau(\tau)\,dx \tag{2.139}$$

Here for $0 < \alpha < 2$ utilizing that $W(u(0)) = 0$ on $\partial U(0)$ we get:

$$\begin{aligned} 0 =& \dot{\mu}(0) + \dot{\lambda}(0) \\ =& -\frac{1}{2}\int_{\partial U(0)} |\nabla u(0)|^2 V \cdot \nu\, dS \end{aligned} \tag{2.140}$$

and since V is arbitrary

$$|\nabla u(0)| = 0 \text{ on } \partial U(0) \text{ for } \alpha \in (0,2). \tag{2.141}$$

(we note that $u \in C^{1,\beta-1}$,$\beta = \frac{2}{2-\alpha}$ by [8]).

Now, for $\alpha = 0$ we have $W(u(0)) = 1$ on $\partial U(0)$ and

$$\begin{aligned} 0 =& \dot{\mu}(0) + \dot{\lambda}(0) \\ =& \int_{\partial U(0)} V \cdot \nu\, dS - \frac{1}{2}\int_{\partial U(0)} |\nabla u(0)|^2 V \cdot \nu\, dS \end{aligned} \tag{2.142}$$

hence $\frac{1}{2}|\nabla_+ u(0)|^2 = 1$ (u is only Lipschitz, ∇_+ is the one-sided gradient).

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Chapter 3

A Relation of the Allen-Cahn equations and the Euler equations and applications of the equipartition

Abstract

We will prove that solutions of the Allen-Cahn equations that satisfy the equipartition of the energy can be transformed into solutions of the Euler equations with constant pressure. As a consequence, we obtain De Giorgi type results, that is, the level sets of entire solutions are hyperplanes. Also, we will determine the structure of solutions of the Allen-Cahn system in two dimensions that satisfy the equipartition. In addition, we apply the Leray projection on the Allen-Cahn system and provide some explicit entire solutions. Finally, we obtain some examples of smooth entire solutions of the Euler equations. For specific type of initial conditions, some of these solutions can be extended to the Navier-Stokes equations. The motivation of this paper is to find a transformation that relates the solutions of the Allen-Cahn equations to solutions of the minimal surface equation of one dimension less. We prove this result for equipartitioned solutions in dimension three.

3.1 Introduction

As it is well known, De Giorgi in 1978 [18] suggested a stricking analogy of the Allen Cahn equation $\Delta u = f(u)$ with minimal surface theory that led to significant developments in Partial Differential equations and the Calculus of Variations, by stating the following conjecture about bounded solutions on $\mathbb{R}^n$:
Conjecture:(De Giorgi) Let $u \in C^2(\mathbb{R}^n)$ be a solution to

$$\Delta u - u^3 + u = 0$$

such that: 1. $|u| < 1$, 2. $\dfrac{\partial u}{\partial x_n} > 0 \ \forall x \in \mathbb{R}^n$.
Is it true that all the level sets of u are hyperplanes, at least for $n \leq 8$?

De Giorgi's conjecture refers to an analogy between diffused interfaces and minimal surfaces. The relationship with the Bernstein problem for minimal graphs is the reason why $n \leq 8$ appears in the conjecture.

The first partial results on the De Giorgi conjecture was established by Modica and Mortola in [16] and [27].

In 1997 Ghoussoub and Gui in [12] proved the De Giorgi conjecture for $n = 2$. Building on [12], Ambrosio and Cabre in [3] proved the conjecture for $n = 3$. Also, Ghoussoub and Gui showed in [17] that the conjecture is true for $n = 4, 5$ for special class of solutions that satisfy an anti-symmetry condition.

In 2003 the conjecture was proved up to $n = 8$ by Savin in [27], under the additional hypothesis: $\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1$, using entirely different methods.

Finally, Del Pino, Kowalczyk and Wei in [11] gave a counterexample to the De Giorgi's conjecture for $n \geq 9$. This counterexample satisfies also the limiting assumption $\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1$. The construction is based on a careful perturbation argument building on the Bombieri, De Giorgi and Giusti [5] result for minimal surfaces.

The relation of the Allen-Cahn with minimal surfaces can be seen via the theory of Γ-convergence (see [22], [16] and [26] for further details). The family of functionals

$$J_\varepsilon(u) = \int_\Omega (\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon} W(u))dx \ \ , \ \varepsilon > 0$$

Γ-converges as $\varepsilon \to 0$ to the perimeter functional and the Euler-Lagrange equations are

$$\varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) = 0$$

therefore one expects that the level sets of the minimizers will minimize the perimeter.

So, one question could be, whether there exists a transformation that transforms the Allen-Cahn equation $\Delta u = f(u)$ $(u : \Omega \subset \mathbb{R}^n \to \mathbb{R})$ to the minimal surface equation of one dimension lower (i.e. $(n-1)$-dimensional minimal surface equation). The answer is positive for the class of solutions that satisfy the equipartition, at least in dimension 3, by Corollary 3.3.1 and then by applying a Bernstein-type theorem for the minimal surface equation (see [10], [13]) we obtain that the level sets of solutions are hyperlanes.

For bounded entire solutions of the Allen-Cahn equation that satisfy the equipartition holds a more general result (see Theorem 5.1 in [4]), that is, the level sets of entire solutions of the Allen Cahn equations that satisfy the equipartition are hyperplanes. This was already known by Modica and Mortola in 1980 (see final remark in [27]). In fact, any solution of the Allen-Cahn equation is smooth and satisfies the bound $|u| \leq 1$ (see Proposition 1.9 in [14]). The point in Corollary 3.3.1 is that, we can obtain that the level set of solutions are hyperplanes in any open, convex domain with the appropriate boundary conditions, utilizing the result in [13].

As we can see in Appendix B, we propose a De Giorgi type property for the 2D Euler equations. The relations between different classes of equations, allow us to obtain some explicit smooth entire solutions for the 2D and 3D Isobaric Euler equations. Those solutions can be extended when the pressure is linear function in the space variables. Some of these solutions have linear dependent components. Thus, if we impose linear dependency of the components of the initial conditions, we can obtain some explicit entire solutions and can be extended to other type of equations. In Appendix C we give some examples of smooth entire solutions of the Navier-Stokes equations with linear dependent components of the initial conditions.

One of the observations in this paper, is to view the equipartition as the Eikonal equation. As stated in Proposition 3.2.1, the Eikonal equation can be transformed to the Euler equations with constant pressure (without the divergence free condition). Thus, solutions of the Allen Cahn equations that satisfy the equipartition can be transformed into the Euler equations with constant pressure, and we obtain the divergence free condition from the Allen Cahn equations. This observation plays a crucial role in the proof of Corollary 3.3.1, which was the initial motivation of this work.

Furthermore, we state this result to the equation $a(u)\Delta u + b(u)|\nabla u|^2 = c(u)$, under the hypothesis that $u = \Phi(v)$ for some v that is also in this class of equation. This hypothesis is quite reasonable since the equation $a(u)\Delta u + b(u)|\nabla u|^2 = c(u)$ is invariant under such transformations, in the sence that if u is a solution then $v = F(u)$ is also in this class of equations.

In the last section, we propose an analogue of a De Giorgi type result for the vector Allen-Cahn equations and we will prove that entire solutions of the Allen-Cahn system in dimension 2 that satisfy the equipartition have such a specific structure. Finally, we apply the Helmholtz-Leray decomposition in the Allen-Cahn system and obtain an equation, independent from the potential W. Then we apply the Leray projection (i.e. only the divergence free term from the decomposition) and we can determine explicit entire solutions. In Appendix A, we give some examples of such solutions and compare them to the structure we have obtained from Theorem 3.4.6. One such example, for a particular potential $W \geq 0$ with finite number of global minima has the property that $\lim_{x\to\pm\infty} u(x,y) = a^{\pm}$, where $a^{\pm} \in \{W = 0\}$ and $\lim_{y\to\pm\infty} u(x,y) = U^{\pm}(x)$ where $U^{\pm}$ are heteroclinic connections of the system (i.e. $U^{\pm''} = W_u(U^{\pm})$). If fact, we can have infinitely many such solutions.

3.2 The Allen-Cahn equation and the equipartition

3.2.1 The equipartition of the energy and the Euler equations

We begin with a brief discussion on the equipartition of the energy. Let $u : \mathbb{R} \to \mathbb{R}^m$ be a minimizer of the energy functional

$$J(u) = \int_{\mathbb{R}} (\frac{1}{2}|u'|^2 + W(u))dx \tag{3.1}$$

where $W : \mathbb{R}^m \to \mathbb{R}$ is a C^2 function (the potential energy) such that $W > 0$ on $\mathbb{R}^m \setminus \{a^+, a^-\}$, $W_u(u) = (\frac{\partial W}{\partial u_1}, ..., \frac{\partial W}{\partial u_m})^T$.

Then u will satisfy

$$u'' - W_u(u) = 0 \tag{3.2}$$

We are interested in connecting the phases $u_1 = a^-$, $u_2 = a^+$, $W(a^{\pm}) = 0$. Consider also the length functional

$$L(u) = \sqrt{2} \int_{\mathbb{R}} \sqrt{W(u)}|u'|dx \tag{3.3}$$

which is invariant under the group of orientation preserving diffeomorphisms ψ : $\mathbb{R} \to \mathbb{R}$, $\psi' > 0$, that is, $L(u \circ \psi) = L(u)$.

So it holds that $L(u) \leq J(u)$ and equality holds when we have *equipartition of the energy* (or u is *equipartitioned*), that is,

$$\frac{1}{2}|u'|^2 = W(u) \tag{3.4}$$

In the case of heteroclinic connections (i.e. $u : \mathbb{R} \to \mathbb{R}^m$ minimizer of J such that $\lim_{x\to\pm\infty} u(x) = a^{\pm}$) the equipartition of the energy holds (see also Theorem 2.1 in [1]).

More generally, let u be a solution of

$$\Delta u - W_u(u) = 0 \ , \ u : \mathbb{R}^n \to \mathbb{R}^m \tag{3.5}$$

the equipartition of the energy takes the form

$$\frac{1}{2}|\nabla u|^2 = W(u) \tag{3.6}$$

However, when dealing with solutions $u : \mathbb{R}^n \to \mathbb{R}^m$ of (3.5), even in the scalar case $m = 1$, the equipartition of the energy does not hold for all solutions in general. In the scalar case, Modica in [29] proved a gradient bound $\frac{1}{2}|\nabla u|^2 \leq W(u)$ for entire solutions $u : \mathbb{R}^n \to \mathbb{R}$. If equality holds even at a single point, then it holds for every $x \in \mathbb{R}^n$ and the solutions will be one dimensional (see Theorem 5.1 in [4]). Therefore saddle-shaped solutions constructed in [6] or the counterexample for the De Giorgi's conjecture in [11] do not satisfy the equipartition of the energy. For solutions of the Allen-Cahn equation that satisfy the equipartition in an arbitrary domain $\Omega \subset \mathbb{R}^n$ there is no such a characterization in general.

In the vector case things are far more complicated, and there are examples that violate even the Modica inequality, see section 2 in [21]. Nevertheless, in contrast to the scalar case, we can have a wide variety of solutions that satisfy the equipartition. In the last section we analyze the structure of solutions to the Allen-Cahn system that satisfy the equipartition and in Appendix A we provide some examples of solutions related to that structure.

We now illustrate a transformation that relates the Eikonal equation and the Euler equation with constant pressure and without the incompressibility condition. Note that x_n plays the role of the "time parameter" and $x_n \in \mathbb{R}$ instead of $x_n > 0$. We could choose any of x_i , $i = 1, ..., n$, $n \geq 2$ as a "time parameter", supposing the monotonicity condition with respect to x_i.

Proposition 3.2.1. Let $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth solution of

$$|\nabla v|^2 = G(v) \tag{3.7}$$

where $G : \mathbb{R} \to \mathbb{R}$ is a smooth function and suppose that $v_{x_n} > 0$.

Then the vector field $F = (F_1, ..., F_{n-1})$ where $F_i = \dfrac{v_{x_i}}{v_{x_n}}$, $i = 1, ..., n-1$ satisfies the Euler equations

$$F_{x_n} + F\nabla_y F = 0 \ , \ y = (x_1, ..., x_{n-1}) \tag{3.8}$$

Proof. Differentiating (3.7) over x_i gives

$$2\sum_{j=1}^{n} v_{x_j} v_{x_j x_i} = G'(v) v_{x_i} \ \ , \ i = 1, ..., n \tag{3.9}$$

Now we have

$$F_{i\,x_j} = \frac{v_{x_i x_j} v_{x_n} - v_{x_i} v_{x_n x_j}}{v_{x_n}^2} \ \ , \ j = 1, ..., n \tag{3.10}$$

$$\Rightarrow F_j F_{i\,x_j} = \frac{v_{x_j} v_{x_i x_j} v_{x_n} - v_{x_j} v_{x_i} v_{x_n x_j}}{v_{x_n}^3} \tag{3.11}$$

Thus, by (3.10) and (3.11) (for $i = 1, ..., n-1$), we have

$$\begin{gathered} F_{i\,x_n} + \sum_{j=1}^{n-1} F_j F_{i\,x_j} = \\ \frac{v_{x_i x_n} v_{x_n}^2 - v_{x_n} v_{x_i} v_{x_n x_n} + \sum_{j=1}^{n-1} (v_{x_j} v_{x_i x_j} v_{x_n} - v_{x_j} v_{x_i} v_{x_n x_j})}{v_{x_n}^3} = \\ \frac{v_{x_n} \sum_{j=1}^{n} v_{x_j} v_{x_j x_i} - v_{x_i} \sum_{j=1}^{n} v_{x_j} v_{x_j x_n}}{v_{x_n}^3} \end{gathered} \tag{3.12}$$

finally, by (3.9), the last equation becomes

$$F_{i\,x_n} + \sum_{j=1}^{n-1} F_j F_{i\,x_j} = \frac{v_{x_n} \frac{G'(v)}{2} v_{x_i} - v_{x_i} \frac{G'(v)}{2} v_{x_n}}{v_{x_n}^3} = 0 \tag{3.13}$$

$$\Rightarrow F_{i\,x_n} + \sum_{j=1}^{n-1} F_j F_{i\,x_j} = 0 \ \ , \ i = 1, ..., n-1 \tag{3.14}$$

□

Remark 3.2.2. Note that since $v_{x_n} > 0$ it holds that $v(\Omega) \cap \{G = 0\} = \emptyset$. Indeed, if $v(x_0) \in \{G = 0\} \Rightarrow |\nabla v(x_0)|^2 = 0$ which contradicts $v_{x_n} > 0$. So, by setting $\tilde{v} = P(v)$, where $P'(v) = \frac{1}{\sqrt{G(v)}}$ we have $\nabla \tilde{v} = P'(v) \nabla v \Rightarrow |\nabla \tilde{v}|^2 = (P'(v))^2 |\nabla v|^2 \Rightarrow |\nabla \tilde{v}|^2 = 1$. Thus $\tilde{v}$ satisfies $|\nabla \tilde{v}|^2 = 1$ and $F_i = \dfrac{v_{x_i}}{v_{x_n}} = \dfrac{\tilde{v}_{x_i}}{\tilde{v}_{x_n}}$.

So, at first, it seems that this transformation can be inverted: $F_1^2 + ... + F_{n-1}^2 = \dfrac{\tilde{v}_{x_1}^2 + ... + \tilde{v}_{x_{n-1}}^2}{\tilde{v}_{x_n}^2} = \dfrac{1}{\tilde{v}_{x_n}^2} - 1 \Rightarrow \tilde{v}_{x_n} = \dfrac{1}{\sqrt{F_1^2 + ... + F_{n-1}^2 + 1}}$

$$\Rightarrow \tilde{v} = \int \frac{1}{\sqrt{F_1^2 + ... + F_{n-1}^2 + 1}} dx_n + a(x_1, ..., x_{n-1}) \tag{3.15}$$

That is, if F_i , $i = 1, ..., n-1$ satisfies the Euler equations $F_{x_n} + F\nabla_y F = 0$, then v defined by (3.15) will satisfy the Eikonal equation. This statement is true for $n = 2$ (see [2]). But to generalize for $n \geq 3$ it appears that further assumptions are needed. So, the class of solutions of the Euler equations with constant pressure seem to be "richer" in some sense than the class of solutions of the Eikonal equation, that is, for every smooth solution of the Eikonal equation, we can obtain a solution of the Euler equation, but not vice versa.

Theorem 3.2.3. Let $u, v : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that $u_{x_n} > 0$ satisfy the equations

$$\begin{aligned} a(u)\Delta u + b(u)|\nabla u|^2 &= f(u) \\ k(v)\Delta v + l(v)|\nabla v|^2 &= g(v) \end{aligned} \tag{3.16}$$

and suppose that $u = \Phi(v)$ for some $\Phi : \mathbb{R} \to \mathbb{R}$ ($\Phi' \neq 0$) and $p(t) \neq 0$, $a(t) \neq 0$, where
$p(t) := k(t)a(\Phi(t))\Phi''(t) + k(t)b(\Phi(t))(\Phi'(t))^2 - l(t)a(\Phi(t))\Phi'(t)$.

Then the vector field $F = (F_1, ..., F_{n-1})$ defined as $F_i = \dfrac{u_{x_i}}{u_{x_n}}$, $i = 1, .., n-1$, will satisfy the Euler equations

$$F_{x_n} + F\nabla_y F = 0 \quad , \; y = (x_1, ..., x_{n-1}) \tag{3.17}$$

Also, $div_y F = 0$ if and only if Φ is a solution of the ODE

$$a(\Phi(t))\Phi'(t)G'(t) + 2[b(\Phi(t))(\Phi'(t))^2 + a(\Phi(t))\Phi''(t)]G(t) = 2f(\Phi(t)) \tag{3.18}$$

where $G(t) := \dfrac{k(t)f(\Phi(t)) - g(t)a(\Phi(t))\Phi'(t)}{p(t)}$ (p as defined above)

Proof. We have $u = \Phi(v)$ and $\nabla u = \Phi'(v)\nabla v$, therefore

$$\begin{gathered}\Delta u = \Phi'(v)\Delta v + \Phi''(v)|\nabla v|^2 \\ \Rightarrow f(u) - b(u)|\nabla u|^2 = a(\Phi(v))(\Phi'(v)\Delta v + \Phi''(v)|\nabla v|^2) \\ \Rightarrow f(\Phi(v)) - b(\Phi(v))(\Phi'(v))^2|\nabla v|^2 = a(\Phi(v))(\Phi'(v)\Delta v + \Phi''(v)|\nabla v|^2)\end{gathered}$$

$$\Rightarrow \Delta v = \frac{f(\Phi(v)) - [b(\Phi(v))(\Phi'(v))^2 + a(\Phi(v))\Phi''(v)]\,|\nabla v|^2}{a(\Phi(v))\Phi'(v)} \quad (a, \Phi' \neq 0) \tag{3.19}$$

since u is a solution of $a(u)\Delta u + b(u)|\nabla u|^2 = f(u)$.

Now, since v is also solution of the second equation in (4.1.1), we have

$$\begin{gathered}k(v)(\frac{f(\Phi(v)) - [b(\Phi(v))(\Phi'(v))^2 + a(\Phi(v))\Phi''(v)]\,|\nabla v|^2}{a(\Phi(v))\Phi'(v)}) + l(v)|\nabla v|^2 = g(v) \\ \Leftrightarrow p(v)|\nabla v|^2 = k(v)f(\Phi(v)) - a(\Phi(v))\Phi'(v)g(v)\end{gathered}$$

where $p(v) = k(v)a(\Phi(v))\Phi''(v) + k(v)b(\Phi(v))(\Phi'(v))^2 - l(v)a(\Phi(v))\Phi'(v)$.

By hypothesis $p \neq 0$, thus

$$|\nabla v|^2 = G(v) \tag{3.20}$$

where

$$G(v) = \frac{k(v)f(\Phi(v)) - g(v)a(\Phi(v))\Phi'(v)}{p(v)} \tag{3.21}$$

Also note that $F_i = \dfrac{u_{x_i}}{u_{x_n}} = \dfrac{v_{x_i}}{v_{x_n}}$.

So we apply Proposition 3.2.1 and we obtain that

$$\begin{gathered}F_{i\,x_n} + \sum_{j=1}^{n-1} F_j F_{i\,x_j} = 0 \ \ , \ i = 1, ..., n-1 \\ \Leftrightarrow F_{x_n} + F\nabla_y F = 0\end{gathered} \tag{3.22}$$

Now, for the divergence of F:

$$\begin{gathered}F_{i\,x_i} = \frac{v_{x_i x_i} v_{x_n} - v_{x_i} v_{x_n x_i}}{v_{x_n}^2} \\ \Rightarrow div_y F = \sum_{i=1}^{n-1} F_{i\,x_i} = \frac{\sum_{i=1}^{n-1} v_{x_i x_i} v_{x_n} - \sum_{i=1}^{n-1} v_{x_i} v_{x_n x_i}}{v_{x_n}^2}\end{gathered}$$

$$\Rightarrow div_y F = \frac{v_{x_n}\Delta v - \frac{1}{2}(|\nabla v|^2)_{x_n}}{v_{x_n}^2} \tag{3.23}$$

Thus, from (3.19) and (3.20) the equation (3.23) becomes:

$$div_y F = \frac{v_{x_n} \frac{f(\Phi(v)) - [b(\Phi(v))(\Phi'(v))^2 + a(\Phi(v))\Phi''(v)]\, G(v)}{a(\Phi(v))\Phi'(v)} - \frac{G'(v)}{2} v_{x_n}}{v_{x_n}^2}$$

Therefore

$$div_y F = 0 \Leftrightarrow a(\Phi(v))\Phi'(v)G'(v) + 2[b(\Phi(v))(\Phi'(v))^2 + a(\Phi(v))\Phi''(v)]G(v) = 2f(\Phi(v))$$

$\square$

Notes: (1) It also holds that solutions of the Allen Cahn equations that satisfy the equipartition also satisfy $div(\frac{\nabla u}{|\nabla u|}) = 0$ has been proved for more general type of equations (see Proposition 4.11 in [6]).
(2) We could see the fact that $div_y F = 0$, can alternatively be obtained with calculations utilizing the stress-energy tensor (see [1] ,p.88), applied in the scalar case.
(3) If $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, then Theorem 3.2.3 implies that $F = \frac{u_x}{u_y}$ is solution of $F_y + FF_x = 0$ and in addition $F_x = 0$. This gives that the level sets of u are hyperplanes in any open and connected domain in $\mathbb{R}^2$. This property of solutions was known to hold for entire solutions in the case where $u = v$, $a(u) = 1 = l(u)$, $b(u) = 0 = k(u)$ and $g'(u) = 2f(u)$ to our setting (see [4], [27]).

3.3 Applications

As we will see now, in dimension 3 we can transform the Allen Cahn equation for the class of solutions that satisfy the equipartition of the energy, into the minimal surface equation of dimension 2 and then apply Bernstein's result to conclude that the level sets of the solution are hyperplanes. The one dimensionality of entire solutions that satisfy the equipartition is a special case of Theorem 5.1 in [4] and it was also known by Modica and Mortola (see the final remark in [27]). However, the result in Corollary 3.3.1 below holds for any open subset of $\mathbb{R}^n$, so by imposing the appropriate boundary conditions, utilizing the result in [13], we can obtain the result for any convex domain.

Corollary 3.3.1. Let $u \in C^2(\Omega;\mathbb{R})$ be a solution of $\Delta u = W'(u)$ such that $u_z > 0$, where $\Omega \subset \mathbb{R}^3$ is an open, convex set. If u satisfies

$$\frac{1}{2}|\nabla u|^2 = W(u) \tag{3.24}$$

then there exists a function ψ such that $\psi_y = -\dfrac{u_x}{u_z}$, $\psi_x = \dfrac{u_y}{u_z}$ that satisfies the minimal surface equation

$$\psi_{yy}(\psi_x^2+1) - 2\psi_x\psi_y\psi_{xy} + \psi_{xx}(\psi_y^2+1) = 0$$

In particular, if $\Omega = \mathbb{R}^3$ or if $\Omega \subset \mathbb{R}^3$ and $u_x = au_z$, $u_y = bu_z$ in $\mathbb{R}^3 \setminus \Omega$, then the level sets of u are hyperplanes.

Proof. From Theorem 3.2.3 we have that $div_{(x,y)}F = 0$, thus there exists some $\psi = \psi(x,y,z)$: $F_1 = -\psi_y$ and $F_2 = \psi_x$.

As we noted in Remark 2.1, $u(\Omega) \cap \{W = 0\} = \emptyset$ (by (3.24) and since $u_z > 0$).

So we set $v = G(u)$, with $G'(u) = \dfrac{1}{\sqrt{2W(u)}}$, thus

$$|\nabla v|^2 = 1 \quad \text{and} \quad F_1 = \frac{u_x}{u_z} = \frac{v_x}{v_z} \;,\; F_2 = \frac{u_y}{u_z} = \frac{v_y}{v_z}$$

$$\Rightarrow F_1^2 + F_2^2 = \frac{1}{v_z^2} - 1 \Rightarrow v_z = \frac{1}{\sqrt{F_1^2+F_2^2+1}}$$

$$v_{zx} = \frac{-F_1F_{1x} - F_2F_{2x}}{(F_1^2+F_2^2+1)^{\frac{3}{2}}}$$

and $v_x = F_1 v_z = \dfrac{F_1}{\sqrt{F_1^2+F_2^2+1}}$

$$\Rightarrow v_{xz} = \frac{F_{1z}(F_1^2+F_2^2+1) - F_1(F_1F_{1z}+F_2F_{2z})}{(F_1^2+F_2^2+1)^{\frac{3}{2}}}$$

Also, by Proposition 3.2.1, F satisfy

$$\begin{cases} F_{1z} + F_1F_{1x} + F_2F_{1y} = 0 \\ F_{2z} + F_1F_{2x} + F_2F_{2y} = 0 \end{cases}$$

and therefore, from the fact that $v_{zx} = v_{xz}$ since $v \in C^2(\Omega)$, we obtain

$$\begin{aligned}
F_{1z}(F_2^2+1) - F_1F_2F_{2z} + F_1F_{1x} + F_2F_{2x} = 0\\
\Rightarrow -F_1F_2F_{1x} - F_2^2F_{1y} - F_{1y} + F_1^2F_{2x} + F_1F_2F_{2y} + F_{2x} = 0\\
\Leftrightarrow \psi_{yy}(\psi_x^2+1) - 2\psi_x\psi_y\psi_{xy} + \psi_{xx}(\psi_y^2+1) = 0
\end{aligned}$$

Finally, if $\Omega = \mathbb{R}^n$, by Berstein's theorem (see Theorem 1.21 [10]) ψ must be a plane (in respect to the variables (x,y), since $\psi_{xx} = \psi_{xy} = \psi_{yy} = 0$): $\psi_x = b(z)$ and $\psi_y = -a(z)$ (for some functions $a, b : \mathbb{R} \to \mathbb{R}$) $\Rightarrow \psi(x,y,z) = b(z)x - a(z)y + c(z)$. This gives: $F_1 = -\psi_y = a(z)$, $F_2 = \psi_x = b(z)$

$$\begin{aligned}
\Rightarrow \frac{u_x}{u_z} = a(z) \quad \text{and} \quad \frac{u_y}{u_z} = b(z)\\
\Rightarrow u(x,y,z) = G(s,y) = H(t,x)\\
\text{where } s = x + \int \frac{1}{a(z)}dz \ , \ t = y + \int \frac{1}{b(z)}dz
\end{aligned}$$

Now we have

$$\begin{aligned}
u_x = a(z)u_z \Rightarrow H_x = \frac{a}{b}H_t \ \ (H_t \neq 0 \ \text{since} \ u_z > 0)\\
\text{and } \frac{1}{2}|\nabla u|^2 = W(u) \Rightarrow \frac{1}{2}[H_x^2 + H_t^2(1+\frac{1}{b^2})] = W(H)
\end{aligned}$$

Differentiating the last equation with respect to y, z respectively (and utilizing $H_x = \frac{a}{b}H_t$), we obtain

$$\begin{cases}
\frac{a}{b}H_tH_{xt} + H_tH_{tt}(1+\frac{1}{b^2}) = W'H_t\\
\frac{a}{b^2}H_tH_{xt} + H_tH_{tt}(\frac{1}{b}+\frac{1}{b^3}) - H_t^2\frac{b'}{b^3} = W'\frac{H_t}{b}
\end{cases}$$

$$\Rightarrow -H_t^2\frac{b'}{b^2} = 0 \Rightarrow b' = 0$$

thus, $b = b_0 = constant$. Arguing similarly for $G = G(s,y)$ we obtain $a = a_0 = constant$. Therefore,

$$u(x,y,z) = h(ax+by+z)$$

where h is a solution of the ODE

$$h''(t) = \frac{W'(h(t))}{a^2+b^2+1}$$

In the case where $\Omega \subset \mathbb{R}^3$, we utilize Theorem 1.1 in [13] to obtain that ψ is linear in Ω and similarly we conclude.

□

Now we will prove an analogue of Theorem 5.1 in [4] for subsolutions of the Allen Cahn equation and also, without excluding apriori some potential singularities of the solutions. The observation in Proposition 3.3.4 below, is to utilize the main result from [3].

Proposition 3.3.4. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a non constant, smooth subsolution of $\Delta u = W'(u)$, $W : \mathbb{R} \to [0, +\infty)$, except perhaps on a closed set S of potential singularities with $\mathcal{H}^1(S) = 0$ and $\mathbb{R}^n \setminus S$ is connected, such that

$$\frac{1}{2}|\nabla u|^2 = W(u) \tag{3.25}$$

where $\mathcal{H}^1$ is the Hausdorff 1-measure in $\mathbb{R}^n$.

Then

$$u(x) = g(a \cdot x + b) \quad , \quad \text{for some } \ a \in \mathbb{R}^n \ \ , \ |a| = 1 \ , \ b \in \mathbb{R}$$

and g is such that $g'' = W'(g)$.

Proof. First we see that W is strictly positive in $u(\mathbb{R}^n \setminus S)$. Indeed, if there exists $x_0 \in \mathbb{R}^n \setminus S$ such that $W(u(x_0)) = 0$, then u is a constant by Corollary 3.1 in [1] and since $\mathbb{R}^n \setminus S$ is connected.

So let $v = G(u)$, where $G'(u) = \dfrac{1}{\sqrt{2W(u)}}$, then

$$|\nabla v|^2 = (G'(u))^2 |\nabla u|^2 = 1 \quad , \quad \text{on } \ \mathbb{R}^n \setminus S$$

so v is a smooth solution of the Eikonal equation except perhaps of a closed set S of potential singularities with $\mathcal{H}^1(S) = 0$. Thus from the result of [3], we have that $v = a \cdot x + b$, $a \in \mathbb{R}^n$, $|a| = 1$, $b \in \mathbb{R}$ or $v = |x - x_0| + c$ for some $x_0 \in \mathbb{R}^n$, $c \in \mathbb{R}$.

Therefore,

$$u(x) = G^{-1}(a \cdot x + b) \quad \text{or} \quad u(x) = G^{-1}(|x - x_0| + c)$$

where $G : \mathbb{R} \to \mathbb{R}$, such that $G' = \dfrac{1}{\sqrt{2W}}$.

If $u = G^{-1}(d+c)$ where $d(x) = |x - x_0|$, then

$$\Delta u = (G^{-1})''(d+c) + \frac{n-1}{d}(G^{-1})'(d+c) \leq W'(u) = W'(G^{-1}(d+c))$$

and also,

$$|\nabla u|^2 = (G^{-1'}(d+c))^2 = 2W(u) \Rightarrow (G^{-1})'(d+c) = \sqrt{2W(G^{-1}(d+c))}$$

and thus, $(G^{-1})'' = W'(G^{-1})$, so we obtain

$$\begin{aligned}(G^{-1})''(d+c) + \frac{n-1}{d}(G^{-1})'(d+c) &\leq (G^{-1})''(d+c) \\ \Rightarrow (G^{-1})'(d+c) = 0 \Rightarrow \sqrt{2W(G^{-1}(d+c))} &= 0\end{aligned}$$

which contradicts the fact that W is strictly positive in $u(\mathbb{R}^n \setminus S)$.

Therefore $u(x) = g(a \cdot x + b)$ where $g = G^{-1}$.

□

Remark 3.3.5. (1) In Proposition 3.3.4 above, radially symmetric solutions are excluded as we see in the proof, but as it is well known (see [19]) if f is smooth and $u \in C^2(\overline{\Omega})$ is a positive solution of $-\Delta u = f(u)$ for $x \in B_1 \subset \mathbb{R}^n$ that vanishes on ∂B_1, it holds that then u is radially symmetric. So radially symmetric solutions of the Allen-Cahn equations are incompatible with the equipartition even if we do not exclude apriori singularities.

(2) Note that, in Theorem 3.2.3, if u, v are smooth entire solutions, by (3.20) in the proof and the monotonicity $u_{x_n} > 0$, arguing as in the proof of Proposition 3.3.4 above we can conclude that u, v are one dimensional and the radially symmetric solutions are also excluded in this case.

3.4 The Allen Cahn system

3.4.1 Applications of the Equipartition

We begin by proposing a De Giorgi like result for the Allen Cahn systems for solutions that satisfy the equipartition of the energy or as an analogy of [4] in the vector case. First, the property that the level sets of a solution are hyperplanes can be expressed equivalently as $\frac{u_{x_i}}{u_{x_n}} = c_i\ ,\ \ i = 1, ..., n-1\ \ (u : \mathbb{R}^n \to \mathbb{R}\ ,\ u_{x_n} > 0)$, that is, if we consider $v_i = \frac{u_{x_i}}{u_{x_n}}\ ,\ i = 1, ..., n,\ \ v_i : \mathbb{R}^n \to \mathbb{R}$, then

$$v_i = c_i\ ,\ i = 1, ..., n-1 \Leftrightarrow rank(\nabla v_i) < 1\ ,\ i = 1, ..., n-1$$

We can see the above statement as follows,
If $v_i = c_i$, $i = 1, ..., n-1$ then $\nabla v_i = 0 \Rightarrow rank(\nabla v_i) = 0 < 1$
Conversely, if $rank(\nabla v_i) < 1$, that is $rank(\nabla v_i) = 0$ since $v_i : \mathbb{R}^n \to \mathbb{R}$, we have by Sard's theorem that $\mathcal{L}^1(v_i(\mathbb{R}^n)) = 0$, $i = 1, ..., n-1$ (see for example [25]) where $\mathcal{L}^1$ is the lebesgue measure on $\mathbb{R}$. Thus, $\mathcal{L}^1(v_i(\mathbb{R}^n)) = 0 \Rightarrow v_i = c_i$ (constant) $i = 1, ..., n-1$.

Now, we can generalize the above to the vector case as follows:

Let $u : \mathbb{R}^n \to \mathbb{R}^m$, $u = (u_1, ..., u_m)$, $u_i = u_i(x_1, ..., x_n)$, we consider the functions

$$v_{ij} = \frac{u_{i\, x_j}}{u_{i\, x_n}} \ , \ i = 1, ..., m \ , \ j = 1, ..., n-1$$

and $\tilde{v}^k = (v_{1k}, ..., v_{mk})$, $\tilde{v}^k : \mathbb{R}^n \to \mathbb{R}^m$, $k = 1, ..., n-1$ and $\nabla \tilde{v}^k : \mathbb{R}^n \to \mathbb{R}^{m \times n}$. Thus, if u is a solution of the Allen Cahn system, we could ask (under appropriate assumptions) whether $rank(\nabla \tilde{v}^k) < \min\{n, m\} = \mu$ (and by Sard's Theorem we would have that $\mathcal{L}^\mu(\tilde{v}^k(\mathbb{R}^n)) = 0$, where $\mathcal{L}^\mu$ is the Lebesgue measure in $\mathbb{R}^\mu$).

Apart from u being a solution of the Allen Cahn system (and $u_{i\, x_n} > 0$)) we should need further assumptions, as in the scalar case. The geometric analog in the vector case is far more complicated than in the scalar case. In particular, there is a relationship with minimizing partitions. However, one possible assumption would be that u also satisfies the equipartition, i.e. $\frac{1}{2}|\nabla u|^2 = W(u)$. We will now prove that the above is true, at least for $n = m = 2$, that is, if $\tilde{v} = (v_1, v_2)$, $v_i = \dfrac{u_{i\, x}}{u_{i\, y}}$ and $u = (u_1, u_2)$ is a solution of the Allen-Cahn system that satisfy the equipartition, then $rank(\nabla v) < 2$. In fact, we can obtain a quite stronger result about the structure of solutions in two dimensions, as stated in Theorem 3.4.6 that follows.

Theorem 3.4.6. Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth solution of

$$\Delta u = W_u(u) \tag{3.26}$$

with $u_{iy} > 0$, $i = 1, 2$ and $W : \mathbb{R}^2 \to [0, +\infty)$ smooth.
If u satisfies

$$\frac{1}{2}|\nabla u|^2 = W(u) \tag{3.27}$$

Then

$$\begin{aligned} &\text{either} \quad u(x,y) = (U_1(c_1x+y), U_2(c_2x+y)) \quad , \\ &\text{where } U_i'' = \frac{W_{U_i}(U_1,U_2)}{c_i^2+1} \quad i=1,2 \end{aligned} \tag{3.28}$$

$$\text{or} \qquad \begin{cases} h(\dfrac{u_{1x}}{u_{1y}}, \dfrac{u_{2x}}{u_{2y}}) = 0 \quad , \\ \text{and} \quad u_{2y}^2 h_{v_1} - u_{1y}^2 h_{v_2} = 0 \end{cases} \tag{3.29}$$

for some $h : \mathbb{R}^2 \to \mathbb{R}$.

In particular, $\mathcal{L}^2(v(\mathbb{R}^2)) = 0$, where $v = (\dfrac{u_{1x}}{u_{1y}}, \dfrac{u_{2x}}{u_{2y}})$.

Proof. We differentiate (3.27) with respect to x , y

$$\begin{cases} u_{1x}u_{1xx} + u_{1y}u_{1yx} + u_{2x}u_{2xx} + u_{2y}u_{2yx} = W_{u_1}u_{1x} + W_{u_2}u_{2x} \\ u_{1x}u_{1xy} + u_{1y}u_{1yy} + u_{2x}u_{2xy} + u_{2y}u_{2yy} = W_{u_1}u_{1y} + W_{u_2}u_{2y} \end{cases} \tag{3.30}$$

and utilizing (3.26) we get

$$\begin{cases} u_{1x}u_{1xx} + u_{1y}u_{1yx} + u_{2x}u_{2xx} + u_{2y}u_{2yx} = u_{1x}\Delta u_1 + u_{2x}\Delta u_2 \\ u_{1x}u_{1xy} + u_{1y}u_{1yy} + u_{2x}u_{2xy} + u_{2y}u_{2yy} = u_{1y}\Delta u_1 + u_{2y}\Delta u_2 \end{cases} \tag{3.31}$$

$$\Leftrightarrow \begin{cases} u_{1y}u_{1yx} + u_{2y}u_{2yx} = u_{1x}u_{1yy} + u_{2x}u_{2yy} \\ u_{1x}u_{1xy} + u_{2x}u_{2xy} = u_{1y}u_{1xx} + u_{2y}u_{2xx} \end{cases} \tag{3.32}$$

Now we define $v_i := \dfrac{u_{ix}}{u_{iy}}$, $i = 1,2$ and by the second equation in (3.32) we have

$$\begin{gathered} v_{1x} = \frac{u_{1xx}u_{1y} - u_{1x}u_{1yx}}{u_{1y}^2} = \frac{u_{2x}u_{2xy} - u_{2y}u_{2xx}}{u_{1y}^2} = -\frac{u_{2y}^2}{u_{1y}^2}v_{2x} \\ \Leftrightarrow u_{1y}^2 v_{1x} + u_{2y}^2 v_{2x} = 0 \end{gathered} \tag{3.33}$$

similarly by the first equation in (3.32) we have

$$u_{1y}^2 v_{1y} + u_{2y}^2 v_{2y} = 0 \tag{3.34}$$

From (3.33), (3.34) and the assumption $u_{iy} > 0$, $i = 1,2$ we obtain that

$$v_{1x}v_{2y} - v_{1y}v_{2x} = 0 \Leftrightarrow det(\nabla v) = 0 \ \ , \ \forall \ (x,y) \in \mathbb{R}^2 \tag{3.35}$$

Since $det(\nabla v) = 0$, we have that $rank(\nabla v) < 2$ and by Sard's Theorem (see for example [25], p. 20) we have that $\mathcal{L}^2(v(\mathbb{R}^2)) = 0$. By Theorem 1.4.14 in [25], since

$rank(\nabla v) < 2$, we have that v_1, v_2 are functionally dependent, that is, there exists a smooth function $h : \mathbb{R}^2 \to \mathbb{R}$ such that

$$h(v_1, v_2) = 0 \Leftrightarrow h(\frac{u_{1x}}{u_{1y}}, \frac{u_{2x}}{u_{2y}}) = 0 \ \ , \ \forall \ (x, y) \in \mathbb{R}^2 \tag{3.36}$$

Thus we have

$$h_{v_1} v_{1x} + h_{v_2} v_{2x} = 0 \ \text{ and } \ h_{v_1} v_{1y} + h_{v_2} v_{2y} = 0 \tag{3.37}$$

so, together with (3.33), (3.34) we get

$$(u_{1y}^2 h_{v_2} - u_{2y}^2 h_{v_1}) v_{2x} = 0 \ \text{ and } \ (u_{1y}^2 h_{v_2} - u_{2y}^2 h_{v_1}) v_{2y} = 0 \tag{3.38}$$

which gives

$$\begin{gathered} v_{2x} = 0 \ \text{ and } \ v_{2y} = 0 \\ \text{or } \ u_{1y}^2 h_{v_2} - u_{2y}^2 h_{v_1} = 0 \end{gathered} \tag{3.39}$$

in the first case we also have

$$v_{1x} = 0 \ \text{ and } \ v_{1y} = 0 \tag{3.40}$$

and therefore

$$\begin{gathered} \frac{u_{1x}}{u_{1y}} = c_1 \ \text{ and } \ \frac{u_{2x}}{u_{2y}} = c_2 \\ \Rightarrow u_1(x, y) = U_1(c_1 x + y) \ \text{ and } \ u_2(x, y) = U_2(c_2 x + y) \end{gathered} \tag{3.41}$$

where

$$U_i'' = \frac{W_{U_i}(U_1, U_2)}{c_i^2 + 1} \quad i = 1, 2.$$

In the second case we see that both equations of (3.29) are satisfied. □

Note: If $W(u_1, u_2) = W_1(u_1) + W_2(u_2)$, then (3.26) becomes

$$\Delta u_i = W_i'(u_i) \ \ , \ i = 1, 2$$

so, by analogy with the scalar case we should suppose $u_{iy} > 0$ as we see in Theorem 3.4.6 above.

3.4.2 The Leray projection on the Allen-Cahn system

We begin with a calculation with which we will obtain an equation independent of the potential W.

Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth solution of the system

$$\Delta u = W_u(u) \Leftrightarrow \begin{cases} \Delta u_1 = W_{u_1}(u_1, u_2) \\ \Delta u_2 = W_{u_2}(u_1, u_2) \end{cases} \tag{3.42}$$

where $W : \mathbb{R}^2 \to \mathbb{R}$, a C^2 potential.

From (3.42), differentiating over x , y we obtain

$$\begin{cases} \Delta u_{1y} = W_{u_1u_1}u_{1y} + W_{u_1u_2}u_{2y} \\ \Delta u_{1x} = W_{u_1u_1}u_{1x} + W_{u_1u_2}u_{2x} \\ \Delta u_{2y} = W_{u_2u_1}u_{1y} + W_{u_2u_2}u_{2y} \\ \Delta u_{2x} = W_{u_2u_1}u_{1x} + W_{u_2u_2}u_{2x} \end{cases} \tag{3.43}$$

and therefore

$$\begin{aligned} u_{1x}\Delta u_{1y} + u_{2x}\Delta u_{2y} = W_{u_1u_1}u_{1y}u_{1x} + W_{u_2u_2}u_{2y}u_{2x} \\ +W_{u_1u_2}(u_{1x}u_{2y} + u_{1y}u_{2x}) \end{aligned} \tag{3.44}$$

thus we have

$$u_{1x}\Delta u_{1y} + u_{2x}\Delta u_{2y} = u_{1y}\Delta u_{1x} + u_{2y}\Delta u_{2x} \tag{3.45}$$

Now we will apply the Helmholtz-Leray decomposition, that resolves a vector field u in $\mathbb{R}^n$ $(n = 2, 3)$ into the sum of a gradient and a curl vector. Regardless of any boundary conditions, for a given vector field u can be decomposed in the form

$$u = \nabla\phi + \tilde{\sigma} = (\phi_x + \tilde{\sigma}_1, \phi_y + \tilde{\sigma}_2)$$

where $div\ \tilde{\sigma} = 0 \Leftrightarrow \tilde{\sigma}_{1x} + \tilde{\sigma}_{2y} = 0$ since we are in two dimensions, and thus $\tilde{\sigma}_1 = -\sigma_y$, $\tilde{\sigma}_2 = \sigma_x$. So, we have that

$$u = (\phi_x - \sigma_y, \phi_y + \sigma_x)$$

for some $\phi, \sigma : \mathbb{R}^2 \to \mathbb{R}$.

Utilizing now this decomposition of u, we obtain

$$\begin{aligned} &(\phi_{xx} - \sigma_{yx})\Delta(\phi_{xy} - \sigma_{yy}) + (\phi_{yx} + \sigma_{xx})\Delta(\phi_{yy} + \sigma_{xy}) \\ &= (\phi_{xy} - \sigma_{yy})\Delta(\phi_{xx} - \sigma_{yx}) + (\phi_{yy} + \sigma_{xy})\Delta(\phi_{yx} + \sigma_{xx}) \end{aligned} \tag{3.46}$$

Thus, if in particular we apply the Leray projection, $v = \mathbb{P}(u)$, we have that $v = \tilde{\sigma}$, that is, $v = (-\sigma_y, \sigma_x)$. So, from (3.46) we have

$$\begin{gathered}\sigma_{yx}\Delta\sigma_{yy} + \sigma_{xx}\Delta\sigma_{xy} = \sigma_{yy}\Delta\sigma_{yx} + \sigma_{xy}\Delta\sigma_{xx} \\ \Leftrightarrow (\sigma_{xx} - \sigma_{yy})\Delta\sigma_{xy} = \sigma_{xy}\Delta(\sigma_{xx} - \sigma_{yy})\end{gathered} \tag{3.47}$$

Note that a class of solutions to (3.47) is σ that satisfy

$$c_1\sigma_{xy} = c_2(\sigma_{xx} - \sigma_{yy}) \tag{3.48}$$

and we can solve explicitly in $\mathbb{R}^2$,

$$\begin{gathered}\sigma(x,y) = A(x) + B(y) \quad , \quad \text{if } \; c_2 = 0 \\ \sigma(x,y) = F(cx+y) + G(x-cy) \quad , \quad \text{where } \; c = \frac{c_1 + \sqrt{c_1^2 + 4c_2^2}}{2c_2} \; , \; \text{ if } \; c_2 \neq 0\end{gathered} \tag{3.49}$$

for arbitrary functions $A, B, F, G : \mathbb{R} \to \mathbb{R}$.

In the first case, the Leray projection of the solution is of the form

$$v = \mathbb{P}(u) = (b(y), a(x)) \tag{3.50}$$

and in the second case

$$v = \mathbb{P}(u) = (cg(x-cy) - f(cx+y), g(x-cy) + cf(cx+y)) \tag{3.51}$$

Similarly, if we take the projection to the space of gradients, we have $\tilde{v} = (\phi_x, \phi_y)$ that will also satisfy

$$(\phi_{xx} - \phi_{yy})\Delta\phi_{xy} = \phi_{xy}\Delta(\phi_{xx} - \phi_{yy}) \tag{3.52}$$

so again, the projection to the space of gradients of the solution will be of the form

$$\begin{gathered}\text{either } \; \tilde{u}(x,y) = (\tilde{a}(x), \tilde{b}(y)) \\ \text{or } \; \tilde{u}(x,y) = (c\tilde{f}(cx+y) + \tilde{g}(x-cy), \tilde{f}(cx+y) - c\tilde{g}(x-cy))\end{gathered} \tag{3.53}$$

Therefore, if we determine a class of potentials W, such that the solutions (or some solutions) are invariant under the Leray projection (or the projection to the space of gradients), we can obtain explicit solutions of the form (3.49) or (3.53). In the Appendix we give such examples.

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3.5 Appendix A: Some examples of entire solutions of the Allen-Cahn system

We note that solutions of the form (3.49) and (3.53) are equivalent in the special case that (3.48) is satisfied. So in the class of solutions of (3.48) the Leray projection is, in some sense equivalent with the projection to the space of gradients. Suppose now that $u = \nabla\phi$ for some $\phi : \mathbb{R}^2 \to \mathbb{R}$, that is, a solution of the Allen-Cahn system remains invariant under the projection to the space of gradients. Then, as (3.52) we have

$$\phi_{xy}\Delta(\phi_{xx} - \phi_{yy}) = (\phi_{xx} - \phi_{yy})\Delta\phi_{xy} \tag{3.54}$$

So a simple solution to (3.54) is

$$\phi_{xx} - \phi_{yy} = 0 \Rightarrow \phi(x, y) = F(x + y) + G(x - y) \tag{3.55}$$

and $u(x, y) = (\phi_x, \phi_y)$, so in this case u has the form

$$u(x, y) = (f(x + y) + g(x - y), f(x + y) - g(x - y)) \tag{3.56}$$

for some $f, g : \mathbb{R} \to \mathbb{R}$.

If u has the form (3.56), we can see that it also satisfies the equipartition. Indeed, (3.42) becomes

$$\begin{cases} 2f'' + 2g'' = W_{u_1} \\ 2f'' - 2g'' = W_{u_2} \end{cases} \Rightarrow \begin{cases} 2(f'' + g'')(f' + g') = W_{u_1}(f' + g') \\ 2(f'' - g'')(f' - g') = W_{u_2}(f' - g') \end{cases} \tag{3.57}$$

$$\begin{aligned} &\Rightarrow 4f''f' + 4g''g' = W_{u_1}(f' + g') + W_{u_2}(f' - g') \\ &\Rightarrow 2(f')^2 + 2(g')^2 = W(f + g, f - g) + c \end{aligned} \tag{3.58}$$

and the equipartition can be written as

$$\frac{1}{2}|\nabla u|^2 = W(u)$$
$$\Leftrightarrow 2(f'(x+y))^2 + 2(g'(x-y))^2 = W(f(x+y)+g(x-y), f(x+y)-g(x-y)) \tag{3.59}$$

(the system (A.1) remains equivalent if we add a constant to the potential)

First we note that solutions of the form (3.56) satisfy (3.29) in Theorem 3.4.6. Indeed, if u is of the form (3.56),

$$\begin{gathered} u_1 = f(x+y)+g(x-y) \text{ and } u_2 = f(x+y)-g(x-y) \\ \Rightarrow \frac{u_{1x}}{u_{1y}} = \frac{f'+g'}{f'-g'} = v_1 \quad \text{and} \quad \frac{u_{2x}}{u_{2y}} = \frac{f'-g'}{f'+g'} = v_2 \end{gathered} \tag{3.60}$$

so the function $h : \mathbb{R}^2 \to \mathbb{R}$ in (3.29) is $h(s,t) = st - 1$. Also,

$$\frac{u_{1y}^2}{u_{2y}^2} = \frac{(f'-g')^2}{(f'+g')^2} \quad \text{and} \quad \frac{h_{v_1}}{h_{v_2}} = \frac{v_2}{v_1} = \frac{(f'-g')^2}{(f'+g')^2} = \frac{u_{1y}^2}{u_{2y}^2} \tag{3.61}$$

Now we will see some examples of solutions to the Allen Cahn system that are not in the form (3.28) in Theorem 3.4.6 (which are more similar to the ones in the scalar case). Some of the examples of such solutions are in the form (3.56) and for all solutions in this form the function h in (3.29) is, as mentioned above, $h(s,t) = st - 1$.

Example 3.5.7 (1)**.** If $W(u_1, u_2) = u_1 u_2$, then

$$u(x,y) = (\cosh(\frac{x+y}{\sqrt{2}}) + \sin(\frac{x-y}{\sqrt{2}}), \cosh(\frac{x+y}{\sqrt{2}}) - \sin(\frac{x-y}{\sqrt{2}}))$$

where $\cosh(t) = \dfrac{e^t + e^{-t}}{2}$, is a solution of $\Delta u = W_u(u)$ that satisfies the equipartition and is of the form (3.29). A more general solution is

$$\begin{aligned} u(x,y) = (&c_1 e^{a_1 x + b_1 y} + c_2 e^{a_2 x + b_2 y} + c_3 \sin(a_3 x + b_3 y) + c_4 \cos(a_4 x + b_4 y), \\ &c_1 e^{a_1 x + b_1 y} + c_2 e^{a_2 x + b_2 y} - c_3 \sin(a_3 x + b_3 y) - c_4 \cos(a_4 x + b_4 y)) \\ &\text{where} \quad a_i^2 + b_i^2 = 1 \ , \ \ i = 1,2,3,4 \ \ , \ c_i \in \mathbb{R} \ \ i = 1,2,3,4. \end{aligned}$$

However, not all solutions in this form satisfy the equipartition. In this example the zero set of the potential is $\{W = 0\} = \{u_1 = 0\} \cup \{u_2 = 0\}$. Such potentials W belong in a class of potentials that have been thoroughly studied in [9].

Example 3.5.8 (2). If $W(u_1, u_2) = \dfrac{[(u_1+u_2)^2-4]^2 + [(u_1-u_2)^2-4]^2}{16}$, then

$$u(x,y) = (\tanh(\frac{x+y}{\sqrt{2}}) + \tanh(\frac{x-y}{\sqrt{2}}), \tanh(\frac{x+y}{\sqrt{2}}) - \tanh(\frac{x-y}{\sqrt{2}})) \qquad (3.62)$$

is a solution of $\Delta u = W_u(u)$ that satisfies the equipartition (and is of the form (3.29) and $h(s,t) = st - 1$). In addition, u above connects all four phases of the potential W at infinity, that is

$$\lim_{x\to\pm\infty} u(x,y) = (\pm 2, 0) \quad \text{and} \quad \lim_{y\to\pm\infty} u(x,y) = (0, \pm 2)$$

$\{W = 0\} = \{(2,0), (-2,0), (0,2), (0,-2)\}$.

This solution is a saddle solution (see [15]) and is invariant under rotations of $\frac{\pi}{2}$ angle (i.e. $u(\omega(x,y)) = \omega u(x,y)$, where ω is the $\frac{\pi}{2}$-rotation matrix.

Also, another solution of $\Delta u = W_u(u)$ for such potential is

$$u(x,y) = (\tanh x + \tanh(\frac{x+y}{\sqrt{2}}), \tanh x - \tanh(\frac{x+y}{\sqrt{2}})) \qquad (3.63)$$

for this solution the function h in (3.29) is $h(s,t) = s + t - 2$ but u in (3.63) does not satisfy the equipartition. Thus, the class of solutions of the Allen-Cahn system that are of the form (3.29) in Theorem 3.4.6, is more general than that of solutions to the Allen-Cahn system that satisfy the equipartition. Note that u in (3.63) has the property that

$$\lim_{x\to\pm\infty} u(x,y) = (\pm 2, 0) \quad \text{and} \quad \lim_{y\to\pm\infty} u(x,y) = (\tanh x \pm 1, \tanh x \mp 1) \qquad (3.64)$$

and $W(-u_1, u_2) = W(u_1, u_2)$. The general existence of solutions with property similar to (3.64) for potentials with such symmetry hypothesis can be found in [4].

More generally, if $a^2 + b^2 = 1 = c^2 + d^2$, then

$$u(x,y) = (\tanh(ax+by) + \tanh(cx+dy), \tanh(ax+by) - \tanh(cx+dy)) \qquad (3.65)$$

solves (3.42) and we obtain infinitely many solutions which connect the four minima of W in sectors of variable angle.

Example 3.5.9 (3)**.** If $W(u_1, u_2) = u_1^2 + u_2^2 - 1$, then

$$\begin{aligned} u(x,y) = (&c_1 e^{a_1 x + b_1 y} + c_2 e^{a_2 x + b_2 y} + c_3 e^{a_3 x + b_3 y} + c_4 e^{a_4 x + b_4 y}, \\ &c_1 e^{a_1 x + b_1 y} + c_2 e^{a_2 x + b_2 y} - c_3 e^{a_3 x + b_3 y} - c_4 e^{a_4 x + b_4 y}) \end{aligned}$$

is a solution of $\Delta u = W_u(u)$, where $a_i^2 + b_i^2 = 2$, $c_i \in \mathbb{R}$.

In this case, $\{W = 0\} = \{u_1^2 + u_2^2 = 1\}$.

Also, if $W(u_1, u_2) = W(u_1^2 + u_2^2)$ and $W' < 0$, we have that

$$u(x,y) = (\cos(ax + by + c), \sin(ax + by + c))$$

with $a^2 + b^2 = -2W'(1)$, is a solution to $\Delta u = W_u(u)$.

3.6 Appendix B: Entire solutions of the Euler equations

In this Appendix we will determine some smooth entire solutions of the 2D and 3D Euler equations and the pressure being a linear function with respect to the space variables.

We begin by illustrating an analogy for steady solutions of the incompressible Euler equations in two space dimensions and the De Giorgi conjecture.

Let $u = (u_1, u_2) : \mathbb{R}^2 \times (0, +\infty) \to \mathbb{R}^2$, $u_i = u_i(x,t)$, $x = (x_1, x_2)$ be a smooth solution of the Euler equations. The incompressibility condition $div\, u = 0$ gives that there exists a (unique up to an additive constant) stream function $\psi(x,t)$ such that

$$u = (-\psi_{x_2}, \psi_{x_1})$$

In addition, by Proposition 2.2 in [20], a stream function ψ on a domain $\Omega \subset \mathbb{R}^2$ defines a steady solution (i.e. time independent) of the 2D Euler equation on Ω if and only if

$$\Delta\psi = F(\psi) \ , \text{ for some function } F$$

So, if ψ is a bounded, entire solution such that $\psi_{x_2} \geq 0$, then by De Giorgi's conjecture (see Theorem 1.1 in [12]) it holds that

$$\psi(x_1, x_2) = g(ax_1 + bx_2)$$

Therefore we raise the following question.

Question: Let $u : \mathbb{R}^2 \times (0, +\infty) \to \mathbb{R}^2$, $(u = u(x, y, t) = (u_1, u_2))$ be a smooth, bounded entire solution of the Isobaric 2D Euler equations

$$\begin{cases} u_{1\,t} + u_1 u_{1\,x} + u_2 u_{1\,y} = 0 \\ u_{2\,t} + u_1 u_{2\,x} + u_2 u_{2\,y} = 0 \\ u_{1\,x} + u_{2\,y} = 0 \end{cases} \tag{3.66}$$

Is it true that then

$$u_1 = c_1\, g(\beta x + \gamma y - (\beta \tilde{c}_1 + \gamma \tilde{c}_2)t) + \tilde{c}_1 \ \ , \ \ u_2 = c_2 g(\beta x + \gamma y - (\beta \tilde{c}_1 + \gamma \tilde{c}_2)t) + \tilde{c}_2 \ \ ? \tag{3.67}$$

where $c_1\beta + c_2\gamma = 0$, $c_1, c_2, \tilde{c}_1, \tilde{c}_2, \beta, \gamma \in \mathbb{R}$.

From the form of solution (3.67) we can obtain a solution of the 2D Euler equation with pressure being a linear function in respect to the space variables.

Let $u : \mathbb{R}^3 \to \mathbb{R}^2$, $(u = u(x, y, t) = (u_1, u_2))$ is such that

$$\begin{gathered} u_1 = c_1\, g(\beta x + \gamma y - (\beta \tilde{c}_1 + \gamma \tilde{c}_2)t) + \lambda A(t) + \tilde{c}_1 \ \ , \\ u_2 = c_2 g(\beta x + \gamma y - (\beta \tilde{c}_1 + \gamma \tilde{c}_2)t) + \xi A(t) + \tilde{c}_2 \\ \text{and} \ \ p(x, y, t) = -a(t)(\lambda x + \xi y) + b(t) \end{gathered} \tag{3.68}$$

where $A'(t) = a(t)$, $a, b : \mathbb{R} \to \mathbb{R}$ and $c_1, \tilde{c}_1, c_2, \tilde{c}_2, \beta, \gamma, \lambda, \xi \in \mathbb{R}$ are such that $c_1\beta + c_2\gamma = 0$ and $\lambda\beta + \xi\gamma = 0$.

Then $u = (u_1, u_2)$ satisfies

$$\begin{cases} u_{1\,t} + u_1 u_{1\,x} + u_2 u_{1\,y} = -p_x \\ u_{2\,t} + u_1 u_{2\,x} + u_2 u_{2\,y} = -p_y \\ u_{1\,x} + u_{2\,y} = 0 \end{cases} \tag{3.69}$$

Now we give some examples of smooth entire solutions for the three dimensional

Euler equations. If $u = (u_1, u_2, u_3) : \mathbb{R}^4 \to \mathbb{R}^3$ where $u_i = u_i(x, y, z, t)$ is such that

$$\begin{gathered} u_1(x,y,z,t) = G(c_1 t - y + c_2 z) \quad , \quad u_2(x,y,z,t) = H(c_1 t - y + c_2 z) - A(t) \\ u_3(x,y,z,t) = \frac{1}{c_2} H(c_1 t - y + c_2 z) - \frac{1}{c_2} A(t) + C \\ \text{and } \; p(x,y,z,t) = a(t)(y + \frac{z}{c_2}) + b(t) \\ \text{where } \; A'(t) = a(t) \; , \; a, A, G, H : \mathbb{R} \to \mathbb{R} \quad , \quad c_2 \neq 0 \; , \; C = \frac{-c_1}{c_2} \end{gathered} \tag{3.70}$$

then $u = (u_1, u_2, u_3)$ is an entire solution of the Euler equations, that is u satisfies

$$\begin{cases} u_{1\,t} + u_1 u_{1\,x} + u_2 u_{1\,y} + u_3 u_{1\,z} = -p_x \\ u_{2\,t} + u_1 u_{2\,x} + u_2 u_{2\,y} + u_3 u_{2\,z} = -p_y \\ u_{3\,t} + u_1 u_{3\,x} + u_2 u_{3\,y} + u_3 u_{3\,z} = -p_z \\ u_{1\,x} + u_{2\,y} + u_{3\,z} = 0 \end{cases} \tag{3.71}$$

Note that from symmetry properties of the Euler equations and from (3.70) we can also have the following solution of (3.71):

$$\begin{gathered} u_1(x,y,z,t) = \frac{1}{c_2} H(c_1 t - z + c_2 x) - \frac{1}{c_2} A(t) + C \quad , \quad u_2(x,y,z,t) = G(c_1 t - z + c_2 x) \\ u_3(x,y,z,t) = H(c_1 t - z + c_2 x) - A(t) \\ \text{and } \; p(x,y,z,t) = a(t)(z + \frac{x}{c_2}) + b(t) \\ \text{where } \; A'(t) = a(t) \; , \; a, A, G, H : \mathbb{R} \to \mathbb{R} \quad , \quad c_2 \neq 0 \; , \; C = \frac{-c_1}{c_2} \end{gathered} \tag{3.72}$$

and also,

$$\begin{gathered} u_1(x,y,z,t) = H(c_1 t - x + c_2 y) - A(t) \; , \\ u_2(x,y,z,t) = \frac{1}{c_2} H(c_1 t - x + c_2 y) - \frac{1}{c_2} A(t) + C \\ u_3(x,y,z,t) = G(c_1 t - x + c_2 y) \; \text{ and } \; p(x,y,z,t) = a(t)(x + \frac{y}{c_2}) + b(t) \\ \text{where } \; A'(t) = a(t) \; , \; a, A, G, H : \mathbb{R} \to \mathbb{R} \quad , \quad c_2 \neq 0 \; , \; C = \frac{-c_1}{c_2} \end{gathered} \tag{3.73}$$

Finally, another example of smooth entire solution of (3.71) is the following

$$\begin{gathered} u_1(x,y,z,t) = G([k\tilde{c}_1 + l\tilde{c}_2]t + [kc_1 + lc_2]x - ky - lz) - A(t) \\ u_2(x,y,z,t) = c_1 u_1(x,y,z,t) + \tilde{c}_1 \quad , \quad u_3(x,y,z,t) = c_2 u_1(x,y,z,t) + \tilde{c}_2 \\ \text{and } \; p(x,y,z,t) = a(t)(x + c_1 y + c_2 z) \\ \text{where } \; A'(t) = a(t) \;\; , \; a, A, G : \mathbb{R} \to \mathbb{R} \; \text{ and } \; c_1, c_2, \tilde{c}_1, \tilde{c}_2, k, l \in \mathbb{R}. \end{gathered} \tag{3.74}$$

(we can choose A such that $A(0) = 0$)

Therefore we conclude to the following result

Theorem 3.6.10. Let $u = (u_1, u_2, u_3)$, u_i , $p : \mathbb{R}^3 \times (0, +\infty) \to \mathbb{R}$ and consider the initial value problem

$$\begin{cases} u_t + u\nabla u = -\nabla p \\ \operatorname{div} u = 0 \\ u(x, y, z, 0) = g(x, y, z) \end{cases} \tag{3.75}$$

where $g = (g_1, g_2, g_3)$ is either of the form

$$\begin{gathered} g = (g_1, c_1 g_1 + \tilde{c}_1, c_2 g_1 + \tilde{c}_2) \text{ and } g_1(x, y, z) = g_1([kc_1 + lc_2]x - ky - lz) \\ c_1, c_2, \tilde{c}_1, \tilde{c}_2, k, l \in \mathbb{R} \ , \ g_1 \text{ smooth} \end{gathered} \tag{3.76}$$

or

$$\begin{gathered} g = (g_1, g_2, \frac{1}{c_2} g_2 - \frac{c_1}{c_2}) \text{ and } g_1(x, y, z) = G(c_2 z - y) \ , \ g_2(x, y, z) = H(c_2 z - y) \\ c_1, c_2, \tilde{c}_1 \in \mathbb{R} \ , \ G, H \text{ smooth} \end{gathered} \tag{3.77}$$

Then there exists a smooth, globally defined in $t > 0$, solution of (3.75).
In particular, either u and p are given by (3.74) if the initial value g is of the form (3.76) or u and p are given by (3.70) if g is of the form (3.77).

The condition (3.77) could be easily modified in order to obtain the solutions given by (3.72) and (3.73).

Remark 3.6.11. Such solutions can be extended to general dimensions, i.e. solutions of (3.8) and $n \geq 4$, together with the divergence free condition and a pressure being a linear function with respect to space variables.

3.7 Appendix C: Some examples of entire solutions of the Navier-Stokes equations

First we note that some solutions of the 3D Euler equations in Appendix B have the form $u = (u_1, c_1 u_1 + \tilde{c}_1, c_2 u_1 + \tilde{c}_2)$, that is, we have linear dependence of the components of the solution. So, now we will determine some specific examples of solutions of the Navier-Stokes equations with linear dependent components.

Let $u = (u_1, u_2)$, $u_i = u_i(x, y, t) : \mathbb{R}^2 \times (0, +\infty) \to \mathbb{R}$ defined as

$$\begin{gathered} u_1(x,y,t) = c_1 g(x - c_1 y, t) - c_1 A(t) + c_2 \ \ , \ \ u_2(x,y,t) = g(x - c_1 y, t) - A(t) \\ \text{and} \ \ p(x,y,t) = a(t)(c_1 x + y) + b(t) \ \ , t > 0 \ \ , \ c_1, c_2 \in \mathbb{R} \\ \text{where} \ \ g_t + c_2 g_s = \mu(c_1^2 + 1) g_{ss} \ \ , \ g = g(s,t) : \mathbb{R}^2 \to \mathbb{R} \\ \text{and } A'(t) = a(t) \ \ , \ a, b, A : \mathbb{R} \to \mathbb{R} \end{gathered} \tag{3.78}$$

then u is a solution of

$$\begin{cases} u_{1\,t} + u_1 u_{1\,x} + u_2 u_{1\,y} = -p_x + \mu \Delta u_1 \\ u_{2\,t} + u_1 u_{2\,x} + u_2 u_{2\,y} = -p_y + \mu \Delta u_2 \\ u_{1\,x} + u_{2\,y} = 0 \end{cases} \quad , \ \mu > 0 \tag{3.79}$$

Similarly in the three dimensional case, we give some examples of solutions of

$$\begin{cases} u_{1\,t} + u_1 u_{1\,x} + u_2 u_{1\,y} + u_3 u_{1\,z} = -p_x + \mu \Delta u_1 \\ u_{2\,t} + u_1 u_{2\,x} + u_2 u_{2\,y} + u_3 u_{2\,z} = -p_y + \mu \Delta u_2 \\ u_{3\,t} + u_1 u_{3\,x} + u_2 u_{3\,y} + u_3 u_{3\,z} = -p_z + \mu \Delta u_3 \\ u_{1\,x} + u_{2\,y} + u_{3\,z} = 0 \end{cases} \quad , \ \mu > 0 \tag{3.80}$$

Let $g = g(s, \eta, t)$, $g : \mathbb{R}^2 \times (0, +\infty) \to \mathbb{R}$ be a solution of

$$g_t - (\frac{\tilde{c}_1}{2c_1} + \frac{\tilde{c}_2}{2c_2}) g_s + (\frac{\tilde{c}_1}{2c_1} - \frac{\tilde{c}_2}{2c_2}) g_\eta = \mu(\frac{1}{4c_1^2} + \frac{1}{4c_2^2})(g_{ss} + g_{\eta\eta}) + \mu g_{ss} \tag{3.81}$$

where $\mu > 0$, $c_1, c_2, \tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$ and $t > 0$.

Then $u = (u_1, u_2, u_3)$, $u_i : \mathbb{R}^3 \times (0, +\infty) \to \mathbb{R}$, $i = 1, 2, 3$ defined as

$$\begin{gathered} u_1(x,y,z,t) = g(x - \frac{c_2 y + c_1 z}{2c_1 c_2}, \frac{c_2 y - c_1 z}{2c_1 c_2}, t) - A(t) \ \ , \ (x,y,z) \in \mathbb{R}^3 \ , \ t > 0 \\ u_2(x,y,z,t) = c_1 u_1(x,y,z,t) + \tilde{c}_1 \ \ , \ \ u_3(x,y,z,t) = c_2 u_1(x,y,z,t) + \tilde{c}_2 \\ \text{and} \ \ p(x,y,z,t) = a(t)(x + c_1 y + c_2 z) + b(t) \\ \text{where} \ \ A'(t) = a(t) \ , \ a, A : \mathbb{R} \to \mathbb{R} \end{gathered} \tag{3.82}$$

is a solution of (3.80).

Therefore we conclude to the following

Proposition 3.7.12. Let $u = (u_1, u_2, u_3)$, u_i, $p : \mathbb{R}^3 \times (0, +\infty) \to \mathbb{R}^3$ and consider the initial value problem

$$\begin{cases} u_t + u\nabla u = -\nabla p + \mu\Delta u \\ \operatorname{div} u = 0 \\ \lim_{t\to 0^+} u(x,y,z,t) = h(x,y,z) \end{cases} \quad , \ \mu > 0 \ , \ (x,y,z,t) \in \mathbb{R}^3 \times (0,+\infty) \quad (3.83)$$

where $h = (h_1, c_1h_1 + \tilde{c}_1, c_2h_1 + \tilde{c}_2)$ and $h_1(x,y,z) = H(2c_1c_2x - c_2y - c_1z)$, $c_1, c_2, \tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$ such that $\tilde{c}_1c_2 + c_1\tilde{c}_2 = 0$ and H smooth.

Then there exists a smooth, globally defined in $t > 0$, solution to (3.83).

In particular,

$$\begin{gathered} u(x,y,z,t) = (u_1, c_1u_1 + \tilde{c}_1, c_2u_1 + \tilde{c}_2) \ \text{ and } \ p(x,y,z,t) = a(t)(x + c_1y + c_2z) + b(t) \\ \text{where } \ u_1(x,y,z,t) = g(2c_1c_2x - c_2y - c_1z, t) - A(t) \\ \text{and } \ g = g(s,t) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} e^{-\frac{|s-w|^2}{4\tilde{\mu}t}} H(w)dw \ , \ \tilde{\mu} = \mu(4c_1^2c_2^2 + c_1^2 + c_2^2) \\ (A'(t) = a(t) \ , \ A(0) = 0) \end{gathered} \quad (3.84)$$

Remark 3.7.13. We can also have the same result for a bit more general initial values h in Proposition 3.7.12, as we can see from (3.81), (3.82). It suffices to have linear dependency of the components of h and h_1 above can also be for example of the form $h_1(x,y,z) = H(2c_1c_2x - c_2y - c_1z, c_2y - c_1z)$.

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Chapter 4

On the $\Gamma-$convergence of the Allen-Cahn functional with boundary conditions

Abstract

We study minimizers of the Allen-Cahn system. We consider the $\varepsilon-$energy functional with Dirichlet values and we establish the Γ-limit. The minimizers of the limiting functional are closely related to minimizing partitions of the domain. Finally, utilizing that the triod and the straight line are the only minimal cones in the plane together with regularity results for minimal curves, we determine the precise structure of the minimizers of the limiting functional, and thus the limit of minimizers of the ε-energy functional as $\varepsilon \to 0$.

4.1 Introduction

In this work we are concerned with the study of vector minimizers of the Allen-Cahn ε-functional,

$$J_\varepsilon(u,\Omega) := \int_\Omega \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right)dx, \quad u:\Omega\to\mathbb{R}^m, \tag{4.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open set and W is a N-well potential with N global minima.

Let

$$u_\varepsilon := \underset{v\in W^{1,2}(\Omega;\mathbb{R}^m)}{\operatorname{argmin}} \left\{J_\varepsilon(v,\Omega) : v|_{\partial\Omega} = g_\varepsilon|_{\partial\Omega}\right\}, \text{ where } g_\varepsilon \in W^{1,2}(\Omega;\mathbb{R}^m). \tag{4.2}$$

Thus $u_\varepsilon \in W^{1,2}(\Omega;\mathbb{R}^m)$ is a weak solution of the system

$$\begin{cases} \varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W_u(u_\varepsilon) = 0 \ , \text{ in } \Omega, \\ u_\varepsilon = g_\varepsilon \ , \text{ on } \partial\Omega, \end{cases} \tag{4.3}$$

We study the asymptotic behavior of u_ε within the framework of Γ-convergence. Moreover, we analyze the relationship between minimizers of the Allen-Cahn system and minimizing partitions subject to Dirichlet boundary conditions. For some particular assumptions on the limiting boundary conditions, we will prove uniqueness for the limiting geometric problem and we will determine the structure of the minimizers of the limiting functional.

4.1.1 Main Results

Hypothesis on W:

(H1) $W \in C^{1,\alpha}_{loc}(\mathbb{R}^m;[0,+\infty))$, $\{W=0\} = \{a_1,a_2,...,a_N\}$, $N\in\mathbb{N}$, a_i are the global minima of W. Assume also that

$$W_u(u)\cdot u > 0 \text{ and } W(u) \geq c_1|u|^2 \text{ , if } |u|>M.$$

Hypothesis on the Dirichlet Data:

(H2)(i) $|g_\varepsilon| \leq M$, $g_\varepsilon \overset{L^1(\Omega)}{\longrightarrow} g_0$ and $J_\varepsilon(g_\varepsilon,\Omega_{\rho_0}\setminus\Omega) \leq C$, where $\partial\Omega$ is Lipschitz and Ω_{ρ_0} is a small dilation of Ω , $\rho_0>1$, in which g_ε is extended (C, M indep. of ε). And either

(ii) $g_\varepsilon \in C^{1,\alpha}(\overline{\Omega})$, $|g_\varepsilon|_{1,\alpha} \leq \dfrac{M}{\varepsilon}$ and $\partial\Omega$ is C^2, where we denote with $|\cdot|_{1,\alpha}$ as the $C^{1,\alpha}$ norm.

Or **(ii')** $g_\varepsilon \in H^1(\Omega)$ and $J_\varepsilon(u_\varepsilon, \Omega) \leq C$.

For $i \neq j$, $i, j \in \{1, 2, ..., N\}$, let $U \in W^{1,2}(\mathbb{R}; \mathbb{R}^m)$ be the 1D minimizer of the action

$$\begin{gathered} \sigma_{ij} := \min \int_{-\infty}^{+\infty} \left(\frac{1}{2}|U'|^2 + W(U) \right) dt < +\infty \quad , \\ \lim_{t \to -\infty} U(t) = a_i \ , \ \lim_{t \to +\infty} U(t) = a_j \ , \ U(\mathbb{R}) \in \mathbb{R}^m \setminus \{W = 0\} \end{gathered} \tag{4.4}$$

where U is a connection that connects a_i to a_j , $i, j \in \{1, 2, ..., N\}$.

The existence of such geodesics has been proved under minimal assumptions on the potential W in [38].

Let J_ε defined in (4.1), we define

$$\tilde{J}_\varepsilon(u, \Omega) := \begin{cases} J_\varepsilon(u, \Omega) \ , \text{ if } \ u = g_\varepsilon \ \text{ on } \ \Omega_{\rho_0} \setminus \Omega \ , \ u \in H^1_{loc}(\mathbb{R}^n; \mathbb{R}^m) \\ +\infty \qquad , \ \text{ otherwise} \end{cases} \tag{4.5}$$

where $\Omega \subset \Omega_{\rho_0}$ as in **(H2)(i)** and let

$$J_0(u, \Omega) := \sum_{1 \leq i < j \leq N} \sigma_{ij} \mathcal{H}^{n-1}(\partial^* \Omega_i \cap \partial^* \Omega_j \cap \Omega) = \sum_{1 \leq i < j \leq N} \sigma_{ij} \mathcal{H}^{n-1}(S_{ij}(u) \cap \Omega), \tag{4.6}$$

where $S_{ij}(u) := \partial^*\{u = a_i\} \cap \partial^*\{u = a_j\}$, $u \in BV(\Omega; \{a_1, a_2, ..., a_N\})$ and we denote as $\partial^* \Omega_k$ the reduced boundary of Ω_k.

Finally we define the limiting functional subject to the limiting boundary conditions

$$\tilde{J}_0(u, \Omega) := \begin{cases} J_0(u, \Omega) \ , \text{ if } \ u \in BV(\Omega; \{a_1, a_2, ..., a_N\}) \ \text{ and } u = g_0 \ \text{ on } \ \Omega_{\rho_0} \setminus \Omega \\ +\infty \qquad , \ \text{ otherwise} \end{cases} \tag{4.7}$$

We can write $J_\varepsilon, J_0, \tilde{J}_\varepsilon, \tilde{J}_0 : L^1(\Omega; \mathbb{R}^n) \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and the Γ-convergence will be with respect to the L^1 topology.

Our first main result is the following

Theorem 4.1.1. Let J_ε be defined by (4.1) and $\tilde{J}_\varepsilon$, $\tilde{J}_0$ defined in (4.5) and (4.7) respectively.

Then

$$\Gamma - \lim_{\varepsilon \to 0} \tilde{J}_\varepsilon(u, \Omega) = \tilde{J}_0(u, \overline{\Omega}). \tag{4.8}$$

Remark 4.1.2. Note that the domain of $\tilde{J}_0$ is the closure of Ω, which means that there is a boundary term (see also (2.9) in [32] for the analog in the scalar case). More precisely, by Proposition 4.3.13 and Theorem 5.8 in [14] we can write

$$\tilde{J}_0(u,\overline{\Omega}) = \frac{1}{2}\sum_{i=1}^{N}\int_{\overline{\Omega}} |D(\phi_i \circ u)|$$
$$= \frac{1}{2}\sum_{i=1}^{N}\int_{\Omega} |D(\phi_i \circ u)| + \frac{1}{2}\sum_{i=1}^{N}\int_{\partial\Omega} |T(\phi_i \circ u) - T(\phi_i \circ g_0)|\, d\mathcal{H}^{n-1}$$
$$\text{where } \phi_i \text{ defined in (4.15) and } T \text{ is the trace operator for } BV \text{ functions.} \tag{4.9}$$

The overview of the strategy of the proof of Theorem 4.1.1 is as follows. First we observe that the $\Gamma-$limit established in [7], in particular Theorem 2.5, holds also without the mass constraint (see Theorem 4.2.5 in Preliminaries section). Next, we apply a similar strategy to that of [6, Theorem 3.7] in which there is a Γ-convergence result with boundary conditions in the scalar case which states that we can incorporate the constraint of Dirichlet values in the $\Gamma-$limit, provided that this $\Gamma-$limit is determined. Since by Theorem 4.2.5 we have that J_ε Γ-converges to J_0, we establish the Γ-limit of $\tilde{J}_\varepsilon$, that is, the $\Gamma-$limit of the functional J_ε with the constraint of Dirichlet values. For the proof of the $\Gamma-$limit we can assume either **(H2)(ii)** or **(H2)(ii')**.

Next, we study the solution of the geometric minimization problem that arise from the limiting functional.

In order to obtain precise information about the minimizer of the limiting functional $\tilde{J}_0(u,\overline{B}_1)$, $B_1 \subset \mathbb{R}^2$, we impose that the limiting boundary conditions g_0 have connected phases. So we assume,

(H2) (iii) Let $g_0 = \sum_{i=1}^{3} a_i\, \chi_{I_i}(\theta)$, $\theta \in [0,2\pi)$, $I_i \subset [0,2\pi)$, $\cup_{i=1}^{3} I_i = [0,2\pi)$ be the limit of g_ε. Assume that I_i are connected and that

$$\theta_0 < \frac{2\pi}{3}\ , \text{ where } \theta_0 \text{ is the largest angle of the points } p_i = \partial I_k \cap \partial I_l$$
$$k \neq l\ ,\ i \in \{1,2,3\} \setminus \{k,l\}.$$

The assumption $\theta_0 < \frac{2\pi}{3}$ arises from the Proposition 3.2 in [30] that we utilize for the proof (see Proposition 4.2.7 in Preliminaries section) and guarantees that the boundary of the partition defined by the minimizer will be line segments meeting at a point inside B_1.

Our second main result is the following

Theorem 4.1.3. Let $u_0 = a_1\chi_{\Omega_1} + a_2\chi_{\Omega_2} + a_3\chi_{\Omega_3}$ be a minimizer of $\tilde{J}_0(u, \overline{B}_1)$ subject to the limiting Dirichlet values **(H2)(iii)**.

Then the minimizer is unique and in addition,

$$\partial\Omega_i \cap \partial\Omega_j \text{ are line segments meeting at } 120^o \text{ in a point in } B_1 \ (i \neq j). \quad (4.10)$$

For proving Theorem 4.1.3, we first prove that the partition defined by u_0 is $(M, 0, \delta)-$minimal as in the Definition 2.1 in [30] (see Definition 4.2.3). This is proved by a comparison argument by defining a Lipschitz perturbation of the partition of the minimizer with strictly less energy. Then, by utilizing a uniqueness result for $(M, 0, \delta)-$minimal sets in [30] (see Proposition 4.2.7), we can conclude that the minimizer of the limiting energy is unique and the boundaries of the partition that the minimizer defines are are line segments meeting at 120^o degrees in an interior point of the unit disc.

In the last subsection, we note that the result in Theorem 4.1.3 can be extended also to the mass constraint case (see [7]). However, in this case the uniqueness will be up to rigid motions of the disc (see Theorem 3.6 and Theorem 4.1 in [11]).

4.1.2 Previous fundamental contributions

We will now briefly introduce some of the well known results in the scalar case. The notion of Γ-convergence was introduced by E. De Giorgi and T. Franzoni in [16] and in particular relates phase transition type problems with the theory of minimal surfaces. One additional application of Γ-convergence is the proof of existence of minimizers of a limiting functional, say F_0, by utilizing an appropriate sequence of functionals F_ε that we know they admit a minimizer and the Γ-limit of F_ε is F_0. And also vice versa ([25]), we can obtain information for the F_ε energy functional from the properties of minimizers of the limiting functional F_0. We can think of this notion as a generalization of the Direct Method in the Calculus of Variations i.e. if F_0 is lower semicontinuous and coercive we can take $F_\varepsilon = F_0$ and then $\Gamma-\lim\, F_\varepsilon = F_0$.

There are many other ways of thinking of this notion, such as a proper tool in finding the limiting functional among a sequence of functionals.

Let X be the space of the measurable functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ endowed with the L^1 norm and

$$F_\varepsilon(u,\Omega) := \begin{cases} \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)dx & , \ u \in W^{1,2}(\Omega;\mathbb{R}) \cap X \\ +\infty & , \text{ elsewhere in } X \end{cases}$$

$$F_0(u,\Omega) := \begin{cases} \sigma \mathcal{H}^{n-1}(Su) & , \ u \in SBV(\Omega;\{-1,1\}) \cap X \\ +\infty & , \text{ elsewhere in } X \end{cases}$$

$$\text{where } W : \mathbb{R} \to [0,+\infty) \ , \ \{W = 0\} = \{-1,1\} \ , \ \sigma = \int_{-1}^{1} \sqrt{2W(u)}du$$

$$\textit{and } Su \text{ is the singular set of the SBV function } u.$$

Let now u_ε be a minimizer of F_ε subject to a mass constraint, that is, $\int_\Omega u = V \in (0,|\Omega|)$. The asymptotic behavior of u_ε was first studied by Modica and Mortola in [27] and by Modica in [16, 29]. Also, later Sternberg [34] generalized these results for minimizers with volume constraint. Furthermore, Owen, Rubinstein and Sternberg in [32] and Ansini, Braides and Piat in [6], among others, studied the asymptotic behavior of the minimizers subject to Dirichlet values for the scalar case.

As mentioned previously, one of the most important outcomes of Γ-convergence in the scalar phase transition type problems is the relationship with minimal surfaces. More precisely, the well known theorem of Modica and Mortola states that the ε-energy functional of the Allen-Cahn equation Γ-converges to the perimeter functional that measures the perimeter of the interface between the phases (i.e. $\Gamma - \lim F_\varepsilon = F_0$). So the interfaces of the limiting problem will be minimal surfaces.

This relationship is deeper as indicated in the De Giorgi conjecture (see [15]) which states that the level sets of global entire solutions of the scalar Allen-Cahn equation that are bounded and strictly monotone with respect to x_n, are hyperplanes if $n \leq 8$. The relationship with the Bernstein problem for minimal graphs is the reason why $n \leq 8$ appears in the conjecture. The Γ-limit of the ε-energy functional of the Allen-Cahn equation is a possible motivation behind the conjecture.

In addition, Baldo in [7] and Fonseca and Tartar in [20] extended the Γ-convergence analysis for the phase transition type problems to the vector case subject to a mass constraint and the limiting functional measures the perimeter of the interfaces separating the phases, and thus there is a relationship with the problem of minimizing partitions. In section 5 we analyze this in the set up of Dirichlet boundary conditions. Furthermore, the general vector-valued coupled case has been thoroughly studied in the works of Borroso-Fonseca and Fonseca-Popovici in [8] and [21] respectively.

There are many other fundamental contributions on the subject, such as the works of Gurtin [17, 18], Gurtin and Matano [19] on the Modica-Mortola functional and its connection with materials science, the work of Hutchingson and Tonegawa on the convergence of critical points in [24], the work of Bouchitté [9] and of Cristoferi and Gravina [13] on space-dependent wells and extensions on general metric spaces in the work of Ambrosio in [5]. Several extensions to the non-local case and fractional setting have also been studied by Alberti-Bellettini in [3], by Alberti-Bouchitté-Seppecher in [4] and by Savin-Valdinoci in [33] among others.

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4.2 Preliminaries

4.2.1 Specialized definitions and theorems for the $\Gamma-$limit

First, we will define the supremum of measures that allow us to express the limiting functional in an alternative way. Let μ and ν be two regular Borel measures on Ω we denote by $\mu \bigvee \nu$ the smallest regular positive measure which is greater than or equal to μ and ν on all borel subsets of Ω, for μ , ν being two regular positive Borel measures on Ω. We have

$$(\mu \bigvee \nu)(\Omega) := \sup\{\mu(A) + \nu(B) : A \cap B = \emptyset,\ A \cup B \subset \Omega,\ A \text{ and } B \text{ are open sets in } \Omega\}.$$

Now let

$$\begin{aligned}\bigvee_{k=1}^{N} \int_{\Omega} |D(\phi_k \circ u_0)| := \sup\{\sum_{k=1}^{N} \int_{A_k} |D(\phi_k \circ u_0)| : \cup_{k=1}^{N} A_k \subset \Omega,\\ A_i \cap A_j = \emptyset\ ,\ i \neq j,\ A_i \text{ open sets in } \Omega\}.\end{aligned}$$

We will now provide a Lemma from [6] that is crucial in the description of the behavior of the $\Gamma-$limit with respect to the set variable. Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote by $\mathcal{A}_\Omega$ the family of all bounded open subsets of Ω.

Lemma 4.2.4. ([6]) Let J_ε defined in (4.1). Then for every $\varepsilon > 0$, for every bounded open sets U , U' , V, with $U \subset\subset U'$, and for every $u, v \in L^1_{loc}(\mathbb{R}^n)$, there exist a cut-off function ϕ related to U and U', which may depend on ε , U , U' , V , u , v such that

$$J_\varepsilon(\phi u + (1-\phi)v, U \cup V) \leq J_\varepsilon(u, U') + J_\varepsilon(v, V) + \delta_\varepsilon(u, v, U, U', V),$$

where $\delta_\varepsilon : L^1_{loc}(\mathbb{R}^n)^2 \times \mathcal{A}^3_\Omega \to [0, +\infty)$ are functions depending only on ε and J_ε such that

$$\lim_{\varepsilon \to 0} \delta_\varepsilon(u_\varepsilon, v_\varepsilon, U, U', V) = 0,$$

whenever U , U' , $V \in \mathcal{A}_\Omega$, $U \subset\subset U'$ and u_ε , $v_\varepsilon \in L^1_{loc}(\mathbb{R}^n)$ have the same limit as $\varepsilon \to 0$ in $L^1((U' \setminus \overline{U}) \cap V)$ and satisfy

$$\sup_{\varepsilon > 0}(J_\varepsilon(u_\varepsilon, U') + J_\varepsilon(v_\varepsilon, V)) < +\infty.$$

The above result is Lemma 3.2 in [6] and has been proved in the scalar case. The proof also works in the vector case with minor modifications. In [6], there is an assumption on W, namely $W \leq c(|u|^\gamma + 1)$ with $\gamma \geq 2$ (see (2.2) in [6]). This assumption however is only utilized in the proof of Lemma 2.1 above to apply the dominated convergence theorem in the last equation. In our case this assumption is not necessary since $W(u_\varepsilon)$ and $W(g_\varepsilon)$ are uniformly bounded (see **(H2)(i)** and Lemma 4.3.9). In fact, the only reason we assume in **(H1)** that $W(u) \geq c_1|u|^2$ for $|u| > M$ is to apply the above Lemma.

In [7] it has been proved that J_ε $\Gamma-$converges to J_0 with mass constraint, but it also holds without mass constraint (see Theorem 2.5). We will point out this more clearly in the proof of Theorem 4.1.1. In particular, it holds

Theorem 4.2.5. ([7]) Let J_ε defined in (4.1) and J_0 defined in (4.6). Then $\Gamma - \lim_{\varepsilon \to 0} J_\varepsilon(u, \Omega) = J_0(u, \Omega)$ in $L^1(\Omega; \mathbb{R}^m)$. That is, for every $u \in L^1(\Omega; \mathbb{R}^m)$, we have the following two conditions:
(i) If $\{v_\varepsilon\} \subset L^1(\Omega; \mathbb{R}^m)$ is any sequence converging to u in L^1, then

$$\liminf_{\varepsilon \to 0} J_\varepsilon(v_\varepsilon, \Omega) \geq J_0(u, \Omega), \tag{4.11}$$

and
(ii) There exist a sequence $\{w_\varepsilon\} \subset L^1(\Omega; \mathbb{R}^m)$ converging to u in L^1 such that

$$\lim_{\varepsilon \to 0} J_\varepsilon(w_\varepsilon, \Omega) = J_0(u, \Omega). \tag{4.12}$$

Remark 4.2.6. We note that in [7], there is also a technical assumption for the potential W (see (1.2) in p.70). However for the proof of the $\Gamma-$limit this assumption is only utilized for the proof of the liminf inequality in order to obtain the equiboundedness of the minimizers u_ε (see proof of (2.8) in [7]). However in our case we obtain equiboundedness from Lemma 4.3.9 in the following section. Therefore in our case this assumption is dismissed.

4.2.2 Specialized definitions and theorems for the Geometric problem

In addition, we introduce the notion of $(M, 0, \delta)$-minimality as defined in [30] together with a Proposition that certifies the shortest network connecting three given points in $\mathbb{R}^2$ as uniquely minimizing in the context of $(M, 0, \delta)-$ minimal sets. This characterization is one of the ingredients for the solution of the geometric minimization problem in the last section. In fact, in [30] the more general notion of (M, ε, δ)-minimality (or (M, cr^α, δ)-minimality) is introduced and regularity results for such sets are established. Particularly, $(M, 0, \delta)-$ minimality implies (M, cr^α, δ)-minimality (see [30]).

Definition 4.2.3. ([30]) Let $K \subset \mathbb{R}^n$ be a closed set and fix $\delta > 0$. Consider $S \subset \mathbb{R}^n \setminus K$ be a nonempty bounded set of finite m-dimensional Hausdorff measure. S is $(M, 0, \delta)$-minimal if $S = spt(\mathcal{H}^m \lfloor S) \setminus K$ and

$$\mathcal{H}^m(S \cap W) \leq \mathcal{H}^m(\phi(S \cap W)),$$

whenever

(a) $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is lipschitzian,
(b) $W = \mathbb{R}^n \cap \{z \ : \ \phi(z) \neq z\}$,
(c) $\mathrm{diam}(W \cup \phi(W)) < \delta$,
(d) $\mathrm{dist}(W \cup \phi(W), K) > 0$.

Proposition 4.2.7. ([30]) Let $K = \{p_1, p_2, p_3\}$ be the vertices of a triangle in the open δ-ball $B(0,\delta) \subset \mathbb{R}^2$, with largest angle θ for some fixed $\delta > 0$. Then there exist a unique smallest $(M, 0, \delta)-$minimal set in $B(0,\delta)$ with closure containing K, in particular:

(a) if $\theta \geq 120^o$, the two shortest sides of the triangle;
(b) if $\theta < 120^o$, segments from three vertices meeting at 120^o.

Here by the "unique smallest" we mean any other such $(M, 0, \delta)-$minimal set S has larger one-dimensional Hausdorff measure.

We now state a well known Bernstein-type theorem in $\mathbb{R}^2$.

Theorem 4.2.8. ([2]) Let A be a complete minimizing partition in $\mathbb{R}^2$ with $N = 3$ (three phases), with surface tension coefficients satisfying

$$\sigma_{ik} < \sigma_{ij} + \sigma_{jk} \quad , \text{ for } j \neq i, k \text{ with } i, j, k \in \{1, 2, 3\}. \tag{4.13}$$

Then ∂A is a triod.

For a proof and related material we refer to [37] and the expository [2].

4.3 Basic Lemmas

Lemma 4.3.9. For every critical point $u_\varepsilon \in W^{1,2}(\Omega; \mathbb{R}^m)$, satisfying (4.3) weakly together with the assumptions **(H1)** and **(H2)(i),(ii)**, it holds

$$||u_\varepsilon||_{L^\infty} < M \quad \text{and} \quad ||\nabla u_\varepsilon||_{L^\infty} < \frac{\tilde{C}}{\varepsilon}.$$

Proof. By linear elliptic theory, we have that $u_\varepsilon \in C^2(\Omega; \mathbb{R}^m)$ (see for example Theorem 6.13 in [14]). Set $v_\varepsilon(x) = |u_\varepsilon(x)|^2$, then

$$\Delta v_\varepsilon = 2W_u(u_\varepsilon) \cdot u_\varepsilon + 2|\nabla u_\varepsilon|^2 > 0 \quad for \;\; |u_\varepsilon| > M \,,$$

Hence $\max_\Omega |u_\varepsilon|^2 \leq M^2$.

On the other hand (from **(H2)**), $\max_{\partial\Omega} |u_\varepsilon| \leq M$. Thus $\max_{\overline{\Omega}} |u_\varepsilon| \leq M$. For the gradient bound, consider the rescaled problem $y = \frac{x}{\varepsilon}$, denote by $\tilde{u}$, $\tilde{g}$ the rescaled u_ε , g_ε, so by elliptic regularity (see for example Theorem 8.33 in [14]),

$$|\tilde{u}|_{1,\alpha} \le C(||\tilde{u}||_{L^\infty} + |\tilde{g}|_{1,\alpha}) \le 2CM$$
$$\Rightarrow ||\nabla\tilde{u}||_{L^\infty} \le 2CM \Rightarrow |\nabla u_\varepsilon| \le \frac{\tilde{C}}{\varepsilon}.$$

□

Lemma 4.3.10. Let u_ε defined in (4.2), then

$$J_\varepsilon(u_\varepsilon) = \int_\Omega \left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\right) dx \le C\ ,$$

C independent of $\varepsilon > 0$, if Ω is bounded.

Proof. Without loss of generality we will prove Lemma 4.3.10 for $\Omega = B_1$ (or else we can cover Ω with finite number of unit balls and the outside part is bounded by **(H2)**(i)).

Substituting $y = \dfrac{x}{\varepsilon}$,

$$J_\varepsilon(u_\varepsilon) = \int_{B_{\frac{1}{\varepsilon}}} \left(\frac{\varepsilon}{2}|\nabla_y \tilde{u}_\varepsilon|^2 \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}W(\tilde{u}_\varepsilon)\right) \varepsilon^n dy,$$

where $\tilde{u}_\varepsilon = u_\varepsilon(\varepsilon y)$ and for $\varepsilon = \dfrac{1}{R}$,

$$\Rightarrow J_\varepsilon(u_\varepsilon) = \varepsilon^{n-1}\int_{B_{\frac{1}{\varepsilon}}} \left(\frac{1}{2}|\nabla_y \tilde{u}_\varepsilon|^2 + W(\tilde{u}_\varepsilon)\right) dy = \frac{1}{R^{n-1}}\int_{B_R} \left(\frac{1}{2}|\nabla_y \tilde{u}_R|^2 + W(\tilde{u}_R)\right) dy$$
$$= \frac{1}{R^{n-1}}\tilde{J}_R(\tilde{u}_R).$$

So, $\tilde{u}_R$ is minimizer of $\tilde{J}_R(v) = \int_{B_R}(\frac{1}{2}|\nabla v|^2 + W(v))dx$.

By Lemma 4.3.9 applied in u_ε, it holds that $|\tilde{u}_R|, |\nabla\tilde{u}_R|$ are uniformly bounded independent of R and via the comparison function (see [1] p.135), for $R > 1$

$$v(x) := \begin{cases} a_1\ , & \text{for } |x| \le R-1 \\ (R-|x|)a_1 + (|x|-R+1)\tilde{u}_R(x)\ , & \text{for } |x| \in (R-1, R] \\ \tilde{u}_R(x)\ , & \text{for } |x| > R \end{cases}\ ,$$

we have

$$\tilde{J}_R(\tilde{u}_R) \le J(v) \le CR^{n-1}\ \ ,\ C \text{ independent of } R.$$

Thus

$$J_\varepsilon(u_\varepsilon) = \frac{1}{R^{n-1}}\tilde{J}_R(\tilde{u}_R) \le C\ \ (C \text{ independent of } \varepsilon > 0).$$

□

Lemma 4.3.11. Let u_ε defined in (4.2), then $u_\varepsilon \xrightarrow{L^1} u_0$, along subsequences and $u_0 \in BV(\Omega;\mathbb{R}^m)$. Moreover, $u_0 = \sum_{i=1}^N a_i \chi_{\Omega_i}$, $\mathcal{H}^{n-1}(\partial^*\Omega_i) < \infty$ and $|\Omega \backslash \cup_{i=1}^N \Omega_i| = 0$.

Proof. By Lemma 4.3.9 we have that u_ε is equibounded. Now arguing as in the proof of Proposition 4.1 in [7] (see also Remark 4.2.6), we obtain that $||u_\varepsilon||_{BV(\Omega;\mathbb{R}^m)}$ is uniformly bounded, $u_\varepsilon \to u_0$ in L^1 along subsequences and also $u_0 \in BV(\Omega;\mathbb{R}^m)$.

From Lemma 4.3.10, it holds

$$\frac{1}{\varepsilon}\int_\Omega W(u_\varepsilon(x))dx \leq C \quad (C \text{ independent of } \varepsilon > 0).$$

Since $|u_\varepsilon| \leq M$ and W is continuous in $\overline{B}_M \subset \mathbb{R}^m \Rightarrow W(u_\varepsilon) \leq \tilde{M}$, therefore by the dominated convergence theorem we obtain

$$\int_\Omega W(u_0(x))dx = 0 \Rightarrow u_0 \in \{W = 0\} \ \ a.e. \ \ \Rightarrow u_0 = \sum_{i=1}^N a_i \chi_{\Omega_i}$$

where χ_{Ω_i} have finite perimeter since $u_0 \in BV(\Omega;\mathbb{R}^m)$ (see [14]).

The proof of Lemma 4.3.11 is complete. □

Also, g_0 takes values on $\{W = 0\}$.

Lemma 4.3.12. Let g_0 be the limiting boundary condition of g_ε.
Then

$$g_0 = \sum_{i=1}^N a_i \chi_{I_i} \ \ , \text{ where } I_i \text{ have finite perimeter and } |\partial\Omega \setminus \cup_{i=1}^N I_i| = 0.$$

Proof. By **(H2)(i)**,

$$\begin{aligned} &J_\varepsilon(g_\varepsilon, \Omega_{\rho_0} \setminus \Omega) \leq C \\ \Rightarrow &\frac{1}{\varepsilon}\int_{\Omega_{\rho_0}\setminus\Omega} W(g_\varepsilon)dx \leq C \end{aligned}$$

So, arguing as in the proof of Lemma 4.3.11, we have that $g_0 \in \{W = 0\}$ and we conclude. □

Proposition 4.3.13. It holds that

$$\int_{\Omega'} |D(\phi_k \circ u_0)| = \sum_{i=1, i\neq k}^{N} \sigma_{ik} \mathcal{H}^{n-1}(\partial^*\Omega_k \cap \partial^*\Omega_i \cap \Omega') \qquad (4.14)$$
$$k = 1,2,..,N \ \text{ ,for every open } \Omega' \subset \Omega,$$

where $\phi_k(z) = d(z, a_k)$, $k = 1, 2, ..., N$, and a_k are the zeros of W and d is the Riemannian metric derived from $W^{1/2}$, that is

$$d(z_1, z_2) := \inf \left\{ \int_0^1 \sqrt{2} W^{1/2}(\gamma(t)) |\gamma'(t)| dt : \gamma \in C^1([0,1]; \mathbb{R}^2), \gamma(0) = z_1,\ \gamma(1) = z_2 \right\}. \qquad (4.15)$$

Proof. The proof can be found in Proposition 2.2 in [7].

□

Furthermore, reasoning as in the proof of Proposition 2.2 in [7] we have,

$$\bigvee_{k=1}^{N} \int_{\Omega} |D(\phi_k \circ u_0)| = \sum_{1\leq i<j\leq N} \sigma_{ij} \mathcal{H}^1(\partial^*\Omega_i \cap \partial^*\Omega_j \cap \Omega) = J_0(u_0, \Omega). \qquad (4.16)$$

The above equation is an alternative way to express the limiting functional.

4.4 Proof of the Γ-limit

Throughout the proof of the $\Gamma-$limit we will assume **(H1)** and **(H2)(i),(ii)**. The proof if we assume **(H2)(ii')** instead of **(H2)(ii)** is similar with minor modifications.

Proof of Theorem 4.1.1.

We begin by proving the $\Gamma - \liminf$ inequality.

Let $u_\varepsilon \in L^1(\Omega; \mathbb{R}^m)$ such that $u_\varepsilon \to u$ in $L^1(\Omega; \mathbb{R}^m)$. If $u_\varepsilon \notin H^1_{loc}$ or $u_\varepsilon \neq g_\varepsilon$ on $\Omega_{\rho_0} \setminus \Omega$, where $\Omega \subset \Omega_{\rho_0}$ as in **(H2)(i)**, then $\tilde{J}_\varepsilon(u_\varepsilon, \Omega) = +\infty$ and the liminf inequality holds trivially. So, let $u_\varepsilon \in H^1_{loc}(\Omega; \mathbb{R}^m)$ such that $u_\varepsilon \to u$ in L^1 and $u_\varepsilon = g_\varepsilon$ on $\Omega_{\rho_0} \setminus \Omega$.

Let $\rho > 1$ such that $\rho < \rho_0$ in **(H2)(i)**, we have

$$\tilde{J}_\varepsilon(u_\varepsilon, \Omega) = J_\varepsilon(u_\varepsilon, \Omega_\rho) - J_\varepsilon(g_\varepsilon, \Omega_\rho \setminus \Omega), \qquad (4.17)$$

where $\partial\Omega_\rho \in C^2$ since it is a small dilation of Ω and there is a unique normal vector $\nu \perp \partial\Omega_\rho$, such that each $x \in \partial\Omega$ can be written as $x = y + \nu(y)d$, $d = dist(x, \partial\Omega_\rho)$ (see the Appendix in [14]).

So,

$$J_\varepsilon(g_\varepsilon, \Omega_\rho \setminus \Omega) = \int_1^\rho \int_{\partial\Omega_r} \left(\frac{\varepsilon}{2}|\nabla g_\varepsilon|^2 + \frac{1}{\varepsilon}W(g_\varepsilon)\right) dSdr \leq C(\rho - 1), \tag{4.18}$$

by Fubini's Theorem and **(H2)(i)**.

Hence, by (4.17), for every u_ε converging to u in L^1 such that $u_\varepsilon = g_\varepsilon$ on $\Omega_{\rho_0} \setminus \Omega$ and $\liminf_{\varepsilon\to 0} \tilde{J}_\varepsilon(u_\varepsilon, \Omega) < +\infty$, we have that

$$\liminf_{\varepsilon\to 0} \tilde{J}_\varepsilon(u_\varepsilon, \Omega) \geq \liminf_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon, \Omega_\rho) - O(\rho - 1). \tag{4.19}$$

Also, by the liminf inequality for J_ε (see Theorem 4.2.5 and (4.16)), we can obtain

$$\liminf_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon, \Omega_\rho) \geq \sum_{1\leq i<j\leq N} \sigma_{ij}\mathcal{H}^1(\partial^*\Omega_i \cap \partial^*\Omega_j \cap \Omega_\rho) = J_0(u, \Omega_\rho). \tag{4.20}$$

Thus, by (4.19) and (4.20), passing the limit as ρ tends to 1 we have the liminf inequality

$$\liminf_{\varepsilon\to 0} \tilde{J}_\varepsilon(u_\varepsilon, \Omega) \geq J_0(u, \overline{\Omega}), \tag{4.21}$$

utilizing also the continuity of measures on decreasing sets.

We now prove the $\Gamma-$limsup inequality. Let $u \in BV(\Omega; \{a_1, a_2, ..., a_N\})$ be such that $u = g_0$ on $\Omega_{\rho_0} \setminus \Omega$.

a) We first assume that $u = g_0$ on $\Omega \setminus \Omega_{\rho_1}$ with $\rho_1 < 1$ and $|\rho_1 - 1|$ small.

As we observe in the proof of Theorem 2.5 in [7] the Γ-limsup inequality for J_ε also holds without the mass constraint, see in particular the proof of Lemma 3.1 in [7]. Since the Γ-liminf inequality holds, the Γ-limsup inequality is equivalent with

$$J_0(u, \Omega) = \lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon, \Omega), \tag{4.22}$$

for some sequence u_ε converging to u in $L^1(\Omega; \mathbb{R}^m)$. So let u_ε be a sequence converging to u in $L^1(\Omega_{\rho_1}; \mathbb{R}^m)$ such that (4.22) is satisfied. In particular u_ε converges to g_0 on $\Omega \setminus \Omega_{\rho_1}$, where Ω_{ρ_1} is a small contraction of Ω.

Now, utilizing the sequence u_ε obtained from (4.22), we will modify it by a cut-off function so that the boundary condition is satisfied. By Lemma 4.2.4, there exist a cut-off function ϕ between $U = \Omega_{\frac{1+\rho_1}{2}}$ and $U' = \Omega$ such that

$$J_\varepsilon(u_\varepsilon\phi + (1 - \phi)g_\varepsilon, \Omega) \leq J_\varepsilon(u_\varepsilon, \Omega) + J_\varepsilon(g_\varepsilon, V) + \delta_\varepsilon(u_\varepsilon, g_\varepsilon, U, U', V), \tag{4.23}$$

where $V = \Omega \setminus \overline{\Omega}_{\rho_1}$ and g_ε is extended in V trivially.

By the assumptions on u_ε and **(H2)** we also have

$$u_\varepsilon \to g_0 \,, \qquad g_\varepsilon \to g_0 \quad \text{in } L^1(V).$$

Hence, again by Lemma 4.2.4 we get

$$\lim_{\varepsilon \to 0} \delta_\varepsilon(u_\varepsilon, g_\varepsilon, U, U', V) = 0.$$

Note that the condition $\sup_{\varepsilon>0}(J_\varepsilon(u_\varepsilon, U') + J_\varepsilon(g_\varepsilon, V)) < +\infty$ in Lemma 4.2.4 is satisfied. To be more precise, from Lemma 4.3.10 it holds

$$\sup_{\varepsilon>0} J_\varepsilon(u_\varepsilon, U') < +\infty \ \ , \text{ where } U' = \Omega,$$

and by **(H2)(i)**,

$$\sup_{\varepsilon>0} J_\varepsilon(g_\varepsilon, V) < +\infty \ \ , \text{ where } V = \Omega \setminus \overline{\Omega}_{\rho_1}.$$

So, by (4.17), (4.18) and (4.23)

$$\Gamma - \limsup_{\varepsilon \to 0} \tilde{J}_\varepsilon(\tilde{u}_\varepsilon, \Omega) \leq \tilde{J}_0(u, \Omega),$$

where $\tilde{u}_\varepsilon = u_\varepsilon \phi + (1-\phi) g_\varepsilon$ and $\tilde{u}_\varepsilon = g_\varepsilon$ in $\Omega_{\rho_0} \setminus \Omega$.

b) In the general case we consider $\rho_1 < 1$ and we define $u_{\rho_1}(x) = u(\frac{1}{\rho_1}x)$ and without loss of generality we may asume that the origin of $\mathbb{R}^n$ belongs in Ω.

By the previous case (a) and (4.6),

$$\begin{aligned}
\Gamma - \limsup_{\varepsilon \to 0} \tilde{J}_\varepsilon(u_{\rho_1}, \Omega) \leq \tilde{J}_0(u_{\rho_1}, \Omega) &= \sum_{1 \leq i < j \leq N} \sigma_{ij} \mathcal{H}^{n-1}(S_{ij}(u_{\rho_1}) \cap \Omega) \\
&\leq \sum_{1 \leq i < j \leq N} \sigma_{ij} \mathcal{H}^{n-1}(S_{ij}(u) \cap \overline{\Omega}) + O(1 - \rho_1^{n-1}) \\
&= \tilde{J}_0(u, \overline{\Omega}) + O(1 - \rho_1^{n-1}).
\end{aligned} \tag{4.24}$$

Since u_{ρ_1} converges to u as ρ_1 tends to 1, if we denote

$$J'(u_{\rho_1}, \Omega) := \Gamma - \limsup_{\varepsilon \to 0} \tilde{J}_\varepsilon(u_{\rho_1}, \Omega),$$

then by the lower semicontinuity of the $\Gamma-$upper limit (see e.g. Proposition 1.28 in [10]) and (4.24),

$$\Gamma - \limsup_{\varepsilon \to 0} \tilde{J}_\varepsilon(u_{\rho_1}, \Omega) \leq \liminf_{\rho_1 \to 1} J'(u_{\rho_1}, \Omega) \leq \tilde{J}_0(u, \overline{\Omega}). \tag{4.25}$$

Hence by (4.21) and (4.25) we get the required equality (4.8). □

4.5 Minimizing partitions and the structure of the minimizer

In this section we begin with the basic definitions of minimizing partitions. Then we underline the relationship of minimizing partitions in $\mathbb{R}^2$ with the minimizers of the functional $\tilde{J}_0$ and we analyze the structure of the minimizer of $\tilde{J}_0$ that we obtain from the Γ-limit. Utilizing a Bernstein type theorem for minimizing partitions we can explicitly compute the energy of the minimizer in Proposition 4.5.16 and by regularity results in [30] we can determine the precise structure of a minimizer subject to the limiting boundary conditions in Theorem 4.1.3 and prove uniqueness. In subsection 4.5.2 we make some comments for the limiting minimizers in dimension three. Finally, in the last subsection we note that we can extend these results to the mass constraint case.

Let $\Omega \subset \mathbb{R}^n$ open, occupied by N phases. Associated to each pair of phases i and j there is a surface energy density σ_{ij}, with $\sigma_{ij} > 0$ for $i \neq j$ and $\sigma_{ij} = \sigma_{ji}$, with $\sigma_{ii} = 0$. Hence, if A_i denoted the subset of Ω occupied by phase i, then Ω is the disjoint union

$$\Omega = A_1 \cup A_2 \cup ... \cup A_N$$

and the energy of the partition $A = \{A_i\}_{i=1}^N$ is

$$E(A) = \sum_{1 \leq i < j \leq N} \sigma_{ij} \mathcal{H}^{n-1}(\partial^* A_i \cap \partial^* A_j), \tag{4.26}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-Hausdorff measure in $\mathbb{R}^n$ and A_i are sets of finite perimeter. If Ω is unbounded, for example $\Omega = \mathbb{R}^n$ (we say then that A is complete), the quantity above in general will be infinity. Thus, for each W open, with $W \subset\subset \Omega$, we consider the energy

$$E(A; W) = \sum_{0 < i < j \leq N} \sigma_{ij} \mathcal{H}^{n-1}(\partial^* A_i \cap \partial^* A_j \cap W). \tag{4.27}$$

Definition 4.5.4. The partition A is a *minimizing* N-partition if given any $W \subset\subset \Omega$ and any N-partition A' of Ω with

$$\bigcup_{i=1}^N (A_i \triangle A'_i) \subset\subset W, \tag{4.28}$$

we have

$$E(A; W) \leq E(A'; W).$$

The symmetric difference $A_i \triangle A_i'$ is defined as their union minus their intersection, that is, $A_i \triangle A_i' = (A_i \cup A_i') \setminus (A_i \cap A_i')$.

To formulate the Dirichlet problem, we assume that $\partial\Omega$ is C^1 and given a partition C of $\partial\Omega$ up to a set of $\mathcal{H}^{n-1}$-measure zero, we may prescribe the boundary data for A:

$$(\partial_\Omega A)_i = \partial A_i \cap \partial\Omega = C_i \ , \quad i = 1, ..., N.$$

Now the energy is minimized subject to such a prescribed boundary.

Remark 4.5.14. Note that the minimization of the functional $\tilde{J}_0(u, \Omega)$ is equivalent to minimizing the energy $E(A; \Omega)$ under the appropriate Dirichlet conditions.

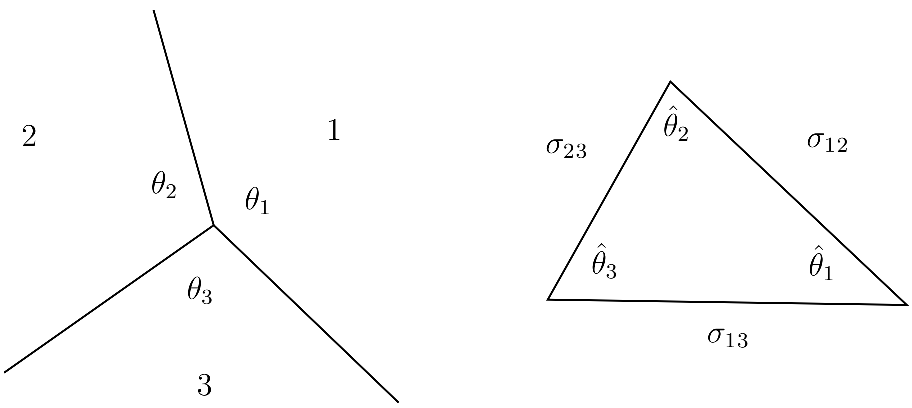

Figure 1.

In Figure 1 we show a triod with angles $\theta_1, \theta_2, \theta_3$, and the corresponding triangle with their supplementary angles $\hat{\theta}_i = \pi - \theta_i$. For these angles Young's law holds, that is,

$$\frac{\sin\hat{\theta}_1}{\sigma_{23}} = \frac{\sin\hat{\theta}_2}{\sigma_{13}} = \frac{\sin\hat{\theta}_3}{\sigma_{12}}. \tag{4.29}$$

Definition 4.5.5. Let $\mathcal{A}_{x_0} = \{A_1, A_2, A_3\}$ be a $3-$partition of $\mathbb{R}^2$ such that A_i is a single infinite sector emanating from the point $x_0 \in \mathbb{R}^2$ with three opening angles θ_i that satisfy (4.29). We call as a *triod* $C_{tr}(x_0)$ the boundary of the partition $\mathcal{A}_{x_0}$, that is, $C_{tr}(x_0) = \{\partial A_i \cap \partial A_j\}_{1 \le i < j \le 3}$.

So, in other words, the triod is consisted of three infinite lines meeting at a point x_0 and their angles between the lines satisfy the Young's law (4.29) (see Figure 1). As we see in Theorem 4.2.8, the triod is the unique locally 3−minimizing partition of $\mathbb{R}^2$. The point x_0, i.e. the center of the triod, is often called a *triple junction point*.

4.5.1 The structure of the minimizer in the disk

Throughout this section we will assume that $\sigma_{ij} = \sigma > 0$ for $i \neq j$, therefore we have by Young's law $\theta_i = \dfrac{2\pi}{3}$, $i = 1, 2, 3$. As a result of Theorem 4.2.8, we expect that, by imposing the appropriate boundary conditions, the minimizer u_0 of $\tilde{J}_0(u, \overline{B}_1)$, $B_1 \subset \mathbb{R}^2$ which we obtain from the Γ-limit will be a triod with angles $\dfrac{2\pi}{3}$ restricted in B_1 and centered at a point $x \in B_1$.

We now recall *Steiner's problem* that gives us some geometric intuition about this fact.

Let us take three points A, B and C, arranged in any way in the plane. The problem is to find a fourth point P such that the sum of distances from P to the other three points is a minimum; that is we require $AP + BP + CP$ to be a minimum length.

If the triangle ABC possesses internal angles which are all less than 120^o, then P is the point such that each side of the triangle, i.e. AB, BC and CA, subtends an angle of 120^o at P. However, if one angle, say $A\hat{C}B$, is greater than 120^o, then P must coincide with C.

The *Steiner's problem* is a special case of the Geometric median problem and has a unique solution whenever the points are not collinear. For more details and proofs see [22].

The problem of minimizing partitions subject to boundary conditions, in contrast to the mass constraint case, might not always admit a minimum, we provide an example in Figure 2 below.

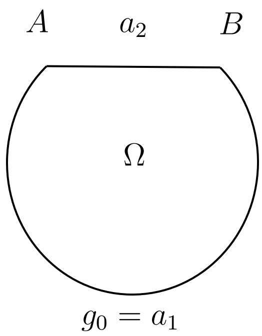

Figure 2.

However a minimizer will exist for the minimization problem $\min_{u \in BV(\Omega;\{W=0\})} \tilde{J}_0(u, \overline{\Omega})$, for instance the one we obtain from the Γ-limit, which will form a "boundary layer" in the boundary of the domain instead of internal layer (i.e. the interface separating the phases). Particularly, in Figure 2 above, $u_0 = a_1$, a.e. will be a minimizer of $\tilde{J}_0$ and

$$\tilde{J}_0(u_0, \overline{\Omega}) = \frac{1}{2} \sum_{i=1}^{3} \int_{\partial\Omega} |T(\phi_i \circ u_0) - T(\phi_i \circ g_0)| d\mathcal{H}^1 = \sigma \mathcal{H}^1(\partial\Omega_{AB}),$$

where $\partial\Omega_{AB}$ is the part of the boundary of Ω in which $g_0 = a_2$. When there are no line segments in the boundary of the domain or when g_0 does not admit jumps nearby such line segments, then we expect that there are no boundary layers and the boundary term in the energy of $\tilde{J}_0$ vanishes (see Remark 4.1.2), otherwise we could find a minimizer with strictly less energy. In the cases where the boundary term vanishes we can write $\tilde{J}_0(u_0, \overline{\Omega}) = \tilde{J}_0(u_0, \Omega)$. This can be proved rigorously in the case where $\Omega = B_1$ and assuming **(H2)(iii)**, utilizing also Proposition 4.2.7 as we will see in the proof of Theorem 4.1.3.

Remark 4.5.15. For the mass constraint case, by classical results of Almgren's improved and simplified by Leonardi in [26] for minimizing partitions with surface tension coefficients σ_{ij} satisfying the strict triangle inequality (see (4.13)), Ω_j can be taken open with $\partial\Omega_j$ real analytic except possibly for a singular part with Hausdorff dimension at most $n-2$. Therefore $\partial^*\Omega_i \cap \partial^*\Omega_j = \partial\Omega_i \cap \partial\Omega_j$, $\mathcal{H}^{n-1}$-a.e., where $u_0 = \sum_{i=1}^{N} a_i \chi_{\Omega_i}$ is the minimizer of J_0 with a mass constraint. These regularity results have been stated by White in [36] but without providing a proof. Also, Morgan in [31] has proved regularity of minimizing partitions in the plane subject to mass constraint. However, we deal with the problem with boundary conditions, so we cannot apply these regularity results.

Notation: We set as $x_0 \in B_1$ the point such that the line segments starting from $p_i = \partial I_k \cap \partial I_l$, $k \neq l$, $i \in \{1, 2, 3\} \setminus \{k, l\}$ and ending at x_0 meet all at angle $\frac{2\pi}{3}$ (see **(H2)(iii)** and Proposition 4.2.7). Also we denote by C_0 the sum of the lengths of these line segments. The following Proposition measures the energy of the limiting minimizer.

Proposition 4.5.16. Let (u_ε) be a minimizing sequence of $\tilde{J}_\varepsilon(u, B_1)$. Then $u_\varepsilon \to u_0$ in L^1 along subsequence with $u_0 \in BV(B_1; \{a_1, a_2, a_3\})$ and u_0 is a minimizer of $\tilde{J}_0(u, \overline{B}_1)$ subject to the limiting Dirichlet values **(H2)(iii)**, where we extend u by setting $u = g_0$ on $\mathbb{R}^2 \setminus B_1$.

In addition, we have

$$\sum_{1 \leq i < j \leq 3} \mathcal{H}^1(\partial^* \Omega_i \cap \partial^* \Omega_j \cap \overline{B}_1) = C_0 \ , \tag{4.30}$$

where $u_0 = a_1 \chi_{\Omega_1} + a_2 \chi_{\Omega_2} + a_3 \chi_{\Omega_3}$.

Proof. From Lemma 4.3.10, 4.3.11 it holds that if u_ε is a minimizing sequence for $\tilde{J}_\varepsilon(u, B_1)$, then $\tilde{J}_\varepsilon(u_\varepsilon, B_1) \leq C$ and thus $u_\varepsilon \to u_0$ in L^1 along subsequence. The fact that u_0 is a minimizer of $\tilde{J}_0$ is a standard fact from the theory of $\Gamma-$convergence. It can be seen as follows.

Let $w \in BV(\overline{B_1}, \{a_1, a_2, a_3\})$ such that $w = g_0$ on $\mathbb{R}^2 \setminus B_1$, then from the limsup inequality in Theorem 4.1.1, we have that there exists $w_\varepsilon \in H^1_{loc}(\mathbb{R}^2; \mathbb{R}^m)$, $w_\varepsilon = g_\varepsilon$ on $\mathbb{R}^2 \setminus B_1$ such that $w_\varepsilon \to w$ in L^1 and $\limsup_{\varepsilon \to 0} \tilde{J}_\varepsilon(w_\varepsilon, B_1) \leq \tilde{J}_0(w, \overline{B}_1)$. Now since u_ε is a minimizing sequence for $\tilde{J}_\varepsilon(u, B_1)$ and from the liminf inequality in Theorem 4.1.1, we have

$$\begin{aligned} \tilde{J}_0(u_0, \overline{B}_1) &\leq \liminf_{\varepsilon \to 0} \tilde{J}_\varepsilon(u_\varepsilon, B_1) \leq \liminf_{\varepsilon \to 0} \tilde{J}_\varepsilon(w_\varepsilon, B_1) \\ &\leq \limsup_{\varepsilon \to 0} \tilde{J}_\varepsilon(w_\varepsilon, B_1) \leq \tilde{J}_0(w, \overline{B}_1) \end{aligned} \tag{4.31}$$

For proving (4.30), we utilize Theorem 4.2.8 (i.e. Theorem 2 in [2]). Since the triod is a minimizing 3-partition in $\mathbb{R}^2$ we have that for any $W \subset\subset \mathbb{R}^2$ and any partition it holds that $E(A, W) \leq E(V, W)$, where suppose that $A = \{A_1, A_2, A_3\}$ is the partition of the triod and $V = \{V_1, V_2, V_3\}$ is a 3-partition in $\mathbb{R}^2$.

We have $u_0 = a_1\chi_{\Omega_1} + a_2\chi_{\Omega_2} + a_3\chi_{\Omega_3}$ such that $u_0 = g_0$ on ∂B_1 and extend u_0 in $\mathbb{R}^2$, being the triod with $\theta_i = \dfrac{2\pi}{3}$ in $\mathbb{R}^2 \setminus B_1$ centered at x_0. This defines a 3-partition in $\mathbb{R}^2$, noted as $\tilde{\Omega} = \{\tilde{\Omega}_i\}_{i=1}^3$. Since the triod is a minimizing 3-partition in the plane, we take any $W \subset\subset \mathbb{R}^2$ such that $B_2 \subset\subset W$ and $\bigcup_{i=1}^3 (A_i \triangle \tilde{\Omega}_i) \subset\subset W$, so we have

$$E(A, W) = E(A, \overline{B}_1) + E(A, W \setminus \overline{B}_1) \leq E(\tilde{\Omega}, W) = E(\tilde{\Omega}, \overline{B}_1) + E(\tilde{\Omega}, W \setminus \overline{B}_1) \tag{4.32}$$

where A is the partition of the triod.

Now since

$$E(A, W \setminus \overline{B}_1) = E(\tilde{\Omega}, W \setminus \overline{B}_1)$$

from the way we extended u_0 in $\mathbb{R}^2$ and

$$E(A, \overline{B}_1) = \sigma \sum_{1 \leq i < j \leq 3} \mathcal{H}^1(\partial A_i \cap \partial A_j \cap \overline{B}_1) = C_0 \sigma$$

since $\partial A_i \cap \partial A_j \cap \overline{B}_1$ are line segments inside B_1 with sum of their lengths equals C_0, we conclude

$$\begin{aligned} & C_0 \sigma \leq E(\tilde{\Omega}, \overline{B}_1) = \tilde{J}_0(u_0, \overline{B}_1) \\ \Leftrightarrow\ & C_0 \leq \sum_{1 \leq i < j \leq 3} \mathcal{H}^1(\partial^* \Omega_i \cap \partial^* \Omega_j \cap \overline{B}_1) \end{aligned} \tag{4.33}$$

For the upper bound inequality $\sum_{1 \leq i < j \leq 3} \mathcal{H}^1(\partial^* \Omega_i \cap \partial^* \Omega_j \cap \overline{B}_1) \leq C_0$, we consider as a comparison function $\tilde{u} = a_1\chi_{A_1} + a_2\chi_{A_2} + a_3\chi_{A_3}$, where $C_{tr}(x_0) = \{A_1, A_2, A_3\}$ is the partition of the triod centered at $x_0 \in B_1$ and angles $\theta_i = \frac{2\pi}{3}$ (see Definition 4.5.5).

Then $\tilde{u}$ satisfies the boundary condition $\tilde{u} = g_0$ on $\mathbb{R}^2 \setminus B_1$ and therefore by the minimality of u_0 we have

$$\begin{aligned} & \tilde{J}_0(u_0, \overline{B}_1) \leq \tilde{J}_0(\tilde{u}, \overline{B}_1) = C_0 \sigma \\ \Rightarrow & \sum_{1 \leq i < j \leq 3} \mathcal{H}^1(\partial^* \Omega_i \cap \partial^* \Omega_j \cap \overline{B}_1) \leq C_0. \end{aligned} \tag{4.34}$$

□

Corollary 4.5.1. Assume for simlicity that x_0 in Proposition 4.5.16 above is the origin of $\mathbb{R}^2$. Then for every $R > 0$ the energy of the limiting minimizer will satisfy

$$\tilde{J}_0(u_0, \overline{B}_R) = 3\sigma R. \tag{4.35}$$

In addition, there exists an entire minimizer in the plane and the partition that defines is a minimal cone.

Proof. Since x_0 is the origin of $\mathbb{R}^2$, it holds that C_0 in (4.30) equals 3. Arguing as in Proposition 4.5.16 above we can similarly obtain a minimizer of $\tilde{J}_0(u_0, \overline{B}_R)$ that satisfies (4.35). By a diagonal argument the minimizer can be extended in the entire plane and will also satisfy

$$\frac{\mathcal{H}^1(\partial\Omega_i \cap \partial\Omega_j \cap B_R)}{\omega_1 R} = C \quad , \forall\, R > 0.$$

Thus, the partition that it defines is a minimal cone (see [37] or [2]). □

Finally, we will prove that the minimizer of $\tilde{J}_0$ in $\overline{B}_1$ is unique, that is, the only minimizer is the triod restricted to B_1 centered at a point in B_1. In Figure 3 below we provide the structure of the minimizer u_0 obtained in Theorem 4.1.3.

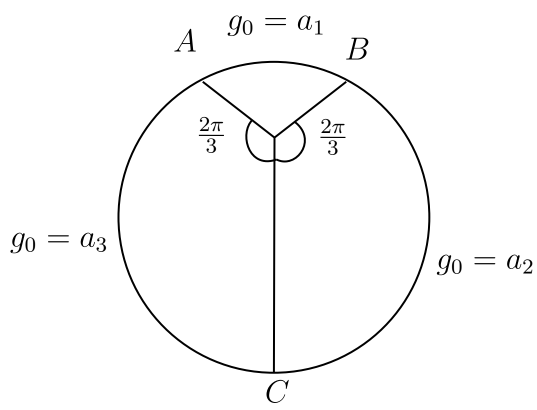

Figure 3.

Proof of Theorem 4.1.3. Firstly, we show that the minimizing partition of B_1 with respect to the boundary conditions defined from g_0, is a $(M, 0, \delta)$-minimal for $\delta > 0$ (see Definition 4.2.3). If not, let S be the partition defined from u_0, we can find a Lipschitz function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\mathcal{H}^1(S \cap W) > \mathcal{H}^1(\phi(S \cap W)),$$

with

$$W = \mathbb{R}^2 \cap \{x : \phi(x) \neq x\} \ , \ \operatorname{diam}(W \cup \phi(W)) < \delta$$
$$\text{and } \operatorname{dist}(W \cup \phi(W), \mathbb{R}^2 \setminus B_1) > 0.$$

So if we consider the partition

$$\tilde{S} := \begin{cases} S \ , \ S \cap W = \emptyset \\ \phi(S \cap W) \ , \ S \cap W \neq \emptyset \end{cases} ,$$

then the boundary of the partition defined by $\tilde{S}$ will satisfy the boundary conditions (since $\mathrm{dist}(W \cup \phi(W), \mathbb{R}^2 \setminus B_1) > 0$) and also $\mathcal{H}^1(\tilde{S}) < \mathcal{H}^1(S)$ which contradicts the minimality of S.

Thus, by **(H2)(iii)** we apply Proposition 4.2.7 and we have that the unique smallest $(M, 0, \delta)$-minimal set consists of three line segments from the three vertices defined from g_0 (i.e. the jump points in ∂B_1) meeting at $\frac{2\pi}{3}$. The meeting point is unique and belongs in the interior of B_1. Thus, $\partial\Omega_i \cap \partial\Omega_j = \partial^*\Omega_i \cap \partial^*\Omega_j$ are line segments meeting at $\frac{2\pi}{3}$ in an interior point of B_1. $\square$

Corollary 4.5.2. Let $u_0 = a_1\chi_{\Omega_1} + a_2\chi_{\Omega_2} + a_3\chi_{\Omega_3}$ be a minimizer of $\tilde{J}_0(u, \overline{B}_1)$ subject to the limiting Dirichlet values $g_0(\theta) = a_1\chi_{(0,\frac{2\pi}{3})} + a_2\chi_{(\frac{2\pi}{3},\frac{4\pi}{3})} + a_3\chi_{(\frac{4\pi}{3},2\pi)}$, $\theta \in (0, 2\pi)$. Then $\partial\Omega_i \cap \partial\Omega_j$ are radi of B_1 , $|\Omega_i| = \frac{1}{3}|B_1|$ and the minimizer is unique.

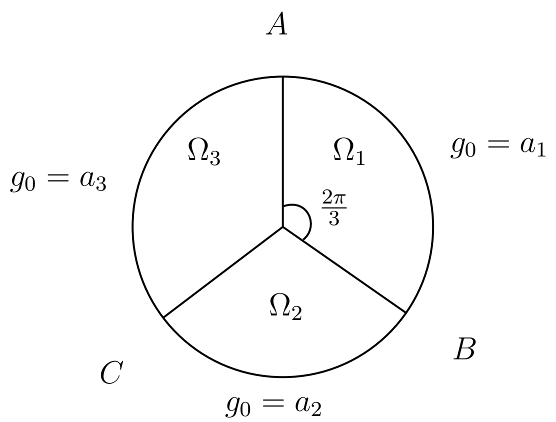

Figure 4.

In Figure 4 above we illustrate the structure of the minimizer u_0 obtained in Corollary 4.5.2.

4.5.2 Minimizers in dimension three

In this subsection we will briefly make some comments for the structure of minimizers in $\mathbb{R}^3$. If we impose the appropriate boundary conditions in $B_R \subset \mathbb{R}^3$ and $\{W = 0\} = \{a_1, a_2, a_3\}$, $g_\varepsilon \to g_0$ in $L^1(B_R; \mathbb{R}^3)$ such that the partition in ∂B_R defined by g_0 is equal to the partition of $(C_{tr} \times \mathbb{R}) \cap \partial B_R$, where C_{tr} is the triod as in Figure 1 (with equal angles), then by Theorem 3 in [2], arguing as in Proposition 4.5.16 (see also Corollary 4.5.1), we can obtain

$$\tilde{J}_0(u, B_R) = \frac{3}{2}\sigma\pi R^2 ,$$

which gives

$$\frac{\mathcal{H}^2(\partial\Omega_i \cap \partial\Omega_j \cap B_R)}{\omega_2 R^2} = \frac{3}{2} ,$$

where ω_2 is the volume of the 2-dimensional unit ball (see [37]). That is, the partition that the minimizer defines can be extended to a minimal cone in $\mathbb{R}^3$. Now since the only minimizing minimal cones are the triod and the tetrahedral cone (see [35]), then the minimizer of $\tilde{J}_0$ is such that $u_0 = \sum_{i=1}^3 a_i \chi_{\Omega_i}$, where $\Omega = \{\Omega_i\}_{i=1}^3$ is the partition of $(C_{tr} \times \mathbb{R}) \cap B_R$.

Similarly, if $\{W = 0\} = \{a_1, a_2, a_3, a_4\}$ and we impose the Dirichlet conditions such that g_0 defines the partition of the tetrahedral cone intersection with ∂B_R, then again $u_0 = \sum_{i=1}^4 a_i \chi_{\Omega_i}$, where $\Omega = \{\Omega_i\}_{i=1}^4$ is the partition of the tetrahedral cone restricted in B_R.

4.5.3 Minimizers in the disc for the mass constraint case

Throughout this subsection we will assume that a_i , $i = 1, 2, 3$, are affinely independent, that is, they are not contained in a single line. This can also be expressed as

$$\text{whenever } \sum_{i=1}^{3} a_i \lambda_i = 0 \text{ with } \sum_{i=1}^{3} \lambda_i = 0 \text{ , then } \lambda_i = 0 \text{ , } i = 1, 2, 3. \tag{4.36}$$

In addition, we consider that $m = (m_1, m_2) \in \mathbb{R}^2$ such that $m_1, m_2 > 0$ (as in [7]).

Let u_0 be a minimizer of $J_0(u, B_1)$, $B_1 \subset \mathbb{R}^2$ defined in (4.6) subject to the mass constraint

$$\int_{B_1} u(x)dx = m , \tag{4.37}$$

(i.e. the minimizer u_0 of Theorem p.70 in [7]) and $\{W=0\}=\{a_1,a_2,a_3\}$. Then $u_0=\sum_{i=1}^{3}a_i\chi_{\Omega_i}$, where $\Omega_1,\Omega_2,\Omega_3$ is a partition of B_1 which minimizes the quantity

$$\sum_{1\leq i<j\leq 3}\sigma\mathcal{H}^1(\partial^*\Omega_i\cap\partial^*\Omega_j)\ , \tag{4.38}$$

among all other partitions of B_1 such that $\sum_{i=1}^{3}|\Omega_i|a_i=m$.

Theorem 4.5.17. Let u_0 be a minimizer of $J_0(u,B_1)$ as above and assume that

$$m=\sum_{i=1}^{3}c_ia_i\ \ ,\text{ where }\ c_i>0\ ,\text{ with }\ \sum_{i=1}^{3}c_i=|B_1|. \tag{4.39}$$

Then

$$\begin{gathered}|\Omega_i|=c_i\ ,\ i=1,2,3\ ,\ \partial^*\Omega_i\cap\partial^*\Omega_j=\partial\Omega_i\cap\partial\Omega_j\ \text{ are piecewise smooth}\\ \text{and the minimizer is unique up to a rigid motion of the disc.}\end{gathered} \tag{4.40}$$

In particular, the boundary of the partition is consisted of three circular arcs or line segments meeting at an interior vertex at 120 degrees angles, reaching orthogonally ∂B_1 and so that the sum of geodesic curvature is zero.

Proof. We have that $u_0=\sum_{i=1}^{3}a_i\chi_{\Omega_i}$, where Ω_i are such that $\sum_{i=1}^{3}|\Omega_i|=|B_1|$ and u_0 minimizes the quantity (4.38).

By the assumption (4.39), since u_0 satisfies (4.37), we have

$$\begin{gathered}\sum_{i=1}^{3}a_i|\Omega_i|=\sum_{i=1}^{3}c_ia_i\ \text{ and }\ \sum_{i=1}^{3}(|\Omega_i|-c_i)=0\\ \Rightarrow|\Omega_i|=c_i\ \ ,\ i=1,2,3\ ,\ \text{ and }\ c_i\in(0,|B_1|),\end{gathered} \tag{4.41}$$

since a_i are affinely independent.

Now by Theorem 4.1 in [11] we conclude that the minimizer is a standard graph i.e. it is consisted of three circular arcs or line segments meeting at an interior vertex at 120 degrees angles, reaching orthogonally ∂B_1 and so that the sum of geodesic curvature is zero. So, $\partial^*\Omega_i\cap\partial^*\Omega_j=\partial\Omega_i\cap\partial\Omega_j$ are piecewise smooth.

Finally, the minimizer is unique up to rigid motions of the disc by Theorem 3.6 in [11]. □

Note that in the case where $m=\frac{1}{3}|B_1|\sum_{i=1}^{3}c_ia_i$, it holds that $|\Omega_i|=\frac{1}{3}|B_1|\ ,\ i=1,2,3\ ,$ and $\partial\Omega_i\cap\partial\Omega_j$ are line segments meeting at the origin and the minimizer is unique up to rotations.

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Chapter 5

Applications of $P-$functions to Fully Nonlinear Elliptic Equations: Gradient Estimates and Rigidity Results

Abstract

We introduce the notion of $P-$functions for fully nonlinear equations and establish a general criterion for obtaining such quantities for this class of equations. Some applications are gradient bounds, De Giorgi-type properties of entire solutions and rigidity results. Particularly, we establish a gradient bound and a rigidity result for Pucci's equations. Furthermore, we prove Harnack-type inequalities and local pointwise estimates for the gradient of solutions to fully nonlinear elliptic equations. In addition, we consider such quantities for higher order nonlinear equations and for equations of order greater than two we obtain Liouville-type theorems and pointwise estimates for the Laplacian.

5.1 Introduction

In this work we introduce the notion of *$P-$ function* for fully nonlinear partial differential equations or differential inequalities and we incorporate the "*$P-$ function* technique" in a general setting. This abstract setting allow us to obtain many applications by only determining an example of a *$P-$function*. Some of these applications are gradient bounds for entire solutions, Harnack -type inequalities for the gradient of solutions and rigidity results.

The structure of this paper is as follows. In section 2 we define the notion of *$P-$ functions* and study such quantities for fully nonlinear elliptic equations. We provide two general criteria for obtaining *$P-$ functions* for this class of equations. Different examples of such quantities may give various types of gradient bounds for solutions of a particular equation. For instance, in section 3, we obtain a gradient bound for entire solutions of the Allen-Cahn equation that differs from the Modica inequality.

Moreover, in section 3, we prove an abstract pointwise estimate, i.e. Theorem 5.3.5, for a class of *$P-$ functions* that are associated to any given fully nonlinear equation. This pointwise estimate is in fact a gradient bound for entire solutions in a wide variety of fully nonlinear elliptic equations and some examples of such gradient bounds are given. As an application, we prove a gradient bound for entire solutions of Pucci's equations. Other consequences are also illustrated in section 4 where we establish a Liouville-type theorem and a De Giorgi-type property for entire solutions. One additional application is Theorem 5.4.10, that is an abstract rigidity result for entire solutions of fully nonlinear elliptic equations such as Pucci's equations. This result also recovers as particular cases the classical results of J. Serrin in [19].

In section 5 we establish a Harnack-type inequality and local pointwise estimates for the gradient of solutions to quasi-linear equations. These estimates can be extended to fully nonlinear elliptic equations, such as for the Monge-Ampère equation.

Additionally, in section 6, we study such quantities for nonlinear equations of order greater than two together with some applications. For example, we establish an a priori bound for the Laplacian and pointwise estimates through the mean value properties for higher order equations. Also, some Liouville-type properties can be established for nonlinear equations of order greater than two. In this setting, one can obtain many other types of bounds for any order of derivatives, assuming a $C^{k,\alpha}$ a priori estimate and provided that we have an appropriate *$P-$ function* related to the respective equation.

We will now briefly discuss some of the most important contributions on the "*$P-$ function* technique" and it's applications. Perhaps the most well-known example is $P(u,x) = \frac{1}{2}|\nabla u|^2 - W(u)$ that is related to the Allen-Cahn equation

$$\Delta u = W'(u) \ , \ u : \Omega \subset \mathbb{R}^n \to \mathbb{R} \tag{5.1}$$

and Modica in [16] proved the well-known gradient bound

$$\frac{1}{2}|\nabla u|^2 \leq W(u) \tag{5.2}$$

for every bounded entire solution of (5.1).

Later, Caffarelli et al in [4] generalized this gradient bound for a class of varational quasi-linear equations and proved Liouville-type and De Giorgi-type properties for a particular choise of $P-$*function* related to the equation $div(\Phi'(|\nabla u|^2)\nabla u) = F'(u)$. This bound was extended for anisotropic partial differential equations and other general types of equations in [5, 8, 9]. The gradient bound (5.2) also holds in unbounded domains with nonnegative mean curvature as proved in [10].

Furthermore, $P-$*functions* had been already studied by Sperb in [20], Payne and Philippin in [17, 18] who studied other types of quasilinear equations for the form $div\,(A(u,|\nabla u|^2)\nabla u) = B(u,|\nabla u|^2)$, which are not necessarily Euler-Lagrange equations of an elliptic integrand. They derived maximum principles for some appropriate $P-$*functions*. Due to the greater generality, however, the relevant P and the conditions under which satisfies an elliptic differential inequality are rather implicitly given while in [4, 6] are given explicitly.

There are many other applications of $P-$*functions* that can be found in [20], among others, such as lower bounds for eigenvalue problems. One additional important application is in [1], where they showed that the monotonicity assumption $u_{x_n} > 0$, that is also stated in the De Giorgi's conjecture, does in fact imply the local minimality of u. Such implication is by no means trivial and it is based on the construction of a so-called *calibration* associated to the energy functional. Such notion is intimately connected to the theory of null-Lagrangians, see [13], chapter 1 and chapter 4, section 2.4. In Theorem 4.4 in [1], they carry out the construction of the appropriate calibration for general integrands of the calculus of variations and such construction relies explicitly on the $P-$ *function*.

Last but not least, there are applications such as gradient bounds similar to (5.2) and Liouville-type properties for vector equations. To be more precise, in Theorem 3.5 in [21], is a gradient bound is proved for the Ginzburg-Landau system of equations.

5.2 $P-$functions for Fully Nonlinear Elliptic equations

We begin by defining the notion of P-function

Definition 5.2.6. Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^d$ be a smooth solution or subsolution of

$$F(x, u, \nabla u, ..., \nabla^m u) = 0 \tag{5.3}$$

where F is a continuous function.

We say that $P = P(x, u, \nabla u, ..., \nabla^{m-1} u)$ is a $P-$function of (5.3) if there exists an elliptic operator L and a non negative function $\mu = \mu(x) \geq 0$

$$\begin{gathered} L = -\sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} + \sum_{i=1}^{n} b_i \partial_{x_i} + c \ \ , \text{ with } \ c \geq 0 \\ \text{such that } \ \mu L\, P \leq 0 \ \ , \text{ in } \ \Omega. \end{gathered} \tag{5.4}$$

An immediate corollary is that any $P-$function related to an equation or to a differential inequality attains its maximum at the boundary $\partial\Omega$ or at a point $x \in \Omega$ such that $\mu(x) = 0$.

We initially state as a direct consequence a strong maximum principle that holds in general (see Theorem 2.2 in [4] or Theorem 4.7 in [6]).

Theorem 5.2.1. Let u be a smooth solution or subsolution of

$$\begin{gathered} F(x, u, \nabla u, ..., \nabla^m u) = 0 \quad , \quad u : \Omega \to \mathbb{R}^d \\ \text{where } \ \Omega \ \text{ is a connected, bounded subset of } \ \mathbb{R}^n \end{gathered} \tag{5.5}$$

such that $\inf_{\overline{\Omega}} g(\nabla^k u) > 0$ for some $g : \mathbb{R}^{n^k \times d} \to [0, +\infty)$, $k \in \{1, ..., m-1\}$ and suppose that $P = P(x, u, \nabla u, ..., \nabla^{m-1} u)$ is a $P-$function of (5.5) with $\mu = \mu(g(\nabla^k u))$, $\mu(t) > 0$, $\forall\, t > 0$.

If there exists $x_0 \in \Omega$ such that

$$P(x_0, u(x_0), ..., \nabla^{m-1} u(x_0)) = \sup_{\Omega} P(x, u, ..., \nabla^{m-1} u) \tag{5.6}$$

then $P(x, u, \nabla u, ..., \nabla^{m-1} u)$ is constant in Ω.

Proof. The proof is an immediate consequence of the strong maximum principle since
$\mu(g(\nabla^k u)) > 0$ in Ω. $\square$

The most common choice of g in Theorem 5.2.1 above is the Euclidean norm. For example, if $k=1$, $g(\nabla u)=|\nabla u|$. If $\mu>0$, $\forall\, t\geq 0$, then the assumption $\inf_{\overline{\Omega}} g(\nabla^k u)>0$ is dismissed.

Remark 5.2.2. The constancy of $P-$functions with a particular form hides geometric information on the level sets $\{x\in\mathbb{R}^n \mid u(x)=t\}$ of the solution u, such as the property of being surfaces of zero mean curvature (see Proposition 4.11 in [6]).

We now focus on fully nonlinear elliptic equations. Let $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}$ be a smooth solution of

$$F(x,u,\nabla u,\nabla^2 u)=0 \tag{5.7}$$

where $F:\Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n}\to\mathbb{R}$ is a continuous function and satisfies the ellipticity condition

$$\lambda|\xi|^2\leq\sum_{i,j}F_{a_{ij}}(x,u,\nabla u,\nabla^2 u)\xi_i\xi_j\leq\Lambda|\xi|^2 \quad ,\ \forall\, x\in\Omega\ ,\ \forall\,\xi\in\mathbb{R}^n \tag{5.8}$$

Here we use the notation $F=F(x,s,q,A)$, $s\in\mathbb{R}$, $q\in\mathbb{R}^n$, $A\in\mathbb{R}^{n\times n}$ and $F_{a_{ij}}=\dfrac{\partial F}{\partial a_{ij}}$. Some important examples of fully nonlinear elliptic equations are

(1) Pucci's equations.

We introduce the Pucci's extremal operators. Let $0<\lambda\leq\Lambda$ and $A\in\mathcal{S}$, where $\mathcal{S}$ is the class of symmetric $n\times n$ matrices, we define

$$\begin{aligned}\mathcal{M}^-(A,\lambda,\Lambda)=\mathcal{M}^-(A)=\lambda\sum_{e_i>0}e_i+\Lambda\sum_{e_i<0}e_i\\ \mathcal{M}^+(A,\lambda,\Lambda)=\mathcal{M}^+(A)=\lambda\sum_{e_i<0}e_i+\Lambda\sum_{e_i>0}e_i\end{aligned} \tag{5.9}$$

where $e_i=e_i(A)$ are the eigenvalues of A. It holds that $\mathcal{M}^-$ and $\mathcal{M}^+$ are uniformly elliptic with ellipticity constants λ and $n\Lambda$ (see [2]).

Pucci's equations are

$$\mathcal{M}^-(\nabla^2 u)=f(x,u) \tag{5.10}$$

and

$$\mathcal{M}^+(\nabla^2 u)=f(x,u) \tag{5.11}$$

respectively.

(2) Monge-Ampère's equation

$$det(\nabla^2 u) = f(x)$$

for strictly convex solutions u and $f > 0$.

(3) Equation of Prescribed Gauss curvature

$$det(\nabla^2 u) = K(x)(1 + |\nabla u|^2)^{\frac{n+2}{2}}$$

$K(x)$ is the Gauss curvature of the graph u at $(x, u(x))$. Again, this equation is elliptic for strictly convex solutions u.

(4) Quasi-Linear equations of the form

$$\sum_{i,j} a_{ij}(\nabla u) u_{x_i x_j} = F(x, u, \nabla u)$$

where a_{ij} satisfy the ellipticity condition $\lambda|\xi|^2 \leq \sum_{i,j} a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$. The $p-$Laplace equation, the Allen-Cahn equation and the minimal surface equation belong in this class of equations. Such equations are thoroughly studied in [4, 6, 9] among others.

There are many other examples of fully nonlinear elliptic equations, such as Bellman equation and Isaacs equation (see [2]).

We now provide two general criteria for obtaining $P-$functions.

Lemma 5.2.3. Let u be a smooth solution of (5.7) and F satisfies (5.8). Consider the quantity

$$I := (\lambda g''(u) - 2F_u)|\nabla u|^2 + (g'(u)\nabla_q F - 2\nabla_x F)\nabla u + g'(u) \sum_{i,j} F_{a_{ij}} u_{x_i x_j} \quad (5.12)$$

and assume that $I \geq 0$ for some function $g : \mathbb{R} \to \mathbb{R}$.

Then $P(u, \nabla u) = |\nabla u|^2 + g(u)$ is a $P-$function of (5.7).

Proof. Assume that $I \geq 0$ for some $g : \mathbb{R} \to \mathbb{R}$ and let $P(u, |\nabla u|^2) = |\nabla u|^2 + g(u)$ (i.e. $P(s,t) = t + g(s)$).

We have

$$P_{x_i} = 2\sum_k u_{x_k} u_{x_k x_i} + g'(u) u_{x_i}$$

$$\Rightarrow P_{x_i x_j} = 2\sum_k (u_{x_k x_j} u_{x_k x_i} + u_{x_k} u_{x_k x_i x_j}) + g''(u) u_{x_i} u_{x_j} + g'(u) u_{x_i x_j}$$

$$\begin{aligned} \Rightarrow \sum_{i,j} d_{ij} P_{x_i x_j} &= 2\sum_{i,j,k} (d_{ij} u_{x_k x_j} u_{x_k x_i} + d_{ij} u_{x_k} u_{x_k x_i x_j}) + g''(u) \sum_{i,j} d_{ij} u_{x_i} u_{x_j} \\ &\quad + g'(u) \sum_{i,j} d_{ij} u_{x_i x_j} \\ &\geq 2\lambda |Hes\ u|^2 + 2\sum_{i,j,k} d_{ij} u_{x_k} u_{x_k x_i x_j} + \lambda g''(u) |\nabla u|^2 + g'(u) \sum_{i,j} d_{ij} u_{x_i x_j} \end{aligned} \tag{5.13}$$

where $d_{ij} = F_{a_{ij}} = \dfrac{\partial F}{\partial a_{ij}}$.

Differentiating (5.7) over x_k, and then multiplying by u_k, we obtain

$$F_{x_k} + F_u u_{x_k} + \sum_m F_{q_m} u_{x_m x_k} + \sum_{m,l} d_{ml} u_{x_m x_l x_k} = 0$$

$$\Rightarrow \sum_{m,l,k} d_{ml} u_k u_{x_m x_l x_k} = -\nabla_x F \nabla u - F_u |\nabla u|^2 - \frac{1}{2} \nabla_q F \nabla_x P + \frac{1}{2} g'(u) \nabla_q F \nabla u \tag{5.14}$$

Therefore the last equation of (5.13) becomes

$$\begin{aligned} \sum_{i,j} d_{ij} P_{x_i x_j} + \nabla_q F \nabla_x P \geq 2\lambda |Hes\ u|^2 + g'(u) \nabla_q F \nabla u + (\lambda g''(u) - 2F_u) |\nabla u|^2 \\ -2\nabla u \nabla_x F + g'(u) \sum_{i,j} F_{a_{ij}} u_{x_i x_j} \geq 0 \end{aligned} \tag{5.15}$$

$\square$

Lemma 5.2.4. Let u be a smooth solution of (5.7) and F satisfies (5.8). Consider the quantity

$$\begin{aligned} J := (\lambda B''(u) - 2A'(|\nabla u|^2) F_u) |\nabla u|^2 + (B'(u) \nabla_q F - 2A'(|\nabla u|^2) \nabla_x F) \nabla u \\ + B'(u) \sum_{i,j} F_{a_{ij}} u_{x_i x_j} + \frac{\lambda (B'(u))^2}{2A'(|\nabla u|^2)} \end{aligned} \tag{5.16}$$

and assume that $J \geq 0$ for some functions $A, B : \mathbb{R} \to \mathbb{R}$ with $A' > 0$, $\forall t > 0$ and $A'' \geq 0$.

Then $P(u, \nabla u) = A(|\nabla u|^2) + B(u)$ is a $P-$function of (5.7).

Proof. We argue as in Lemma 5.2.3 and obtain

$$\begin{aligned}\sum_{i,j} d_{ij} P_{x_i x_j} &= A''(\sum_{i,j} d_{ij} [|\nabla u|^2]_{x_i} [|\nabla u|^2]_{x_j}) + 2A'(\sum_{i,j,k} d_{ij} [u_{x_k x_j} u_{x_k x_i} + u_{x_k} u_{x_k x_i x_j}]) \\ &\quad + \sum_{i,j} d_{ij} (B''(u) u_{x_i} u_{x_j} + B'(u) u_{x_i x_j}) \\ &\geq 2A'\lambda |Hes\, u|^2 + 2A' \sum_{i,j,k} d_{ij} u_{x_k} u_{x_k x_i x_j} + \lambda B'' |\nabla u|^2 + B' \sum_{i,j} d_{ij} u_{x_i x_j}\end{aligned} \tag{5.17}$$

by (5.8) and since $A'' \geq 0$, where $d_{ij} = \dfrac{\partial F}{\partial a_{ij}}$.

We also calculate from the first equation of (5.13) and the Cauchy-Schwarz inequality

$$\begin{aligned}\sum_i (P_{x_i} - B' u_{x_i})^2 &\leq 4(A')^2 |\nabla u|^2 |Hes\, u|^2 \\ \Rightarrow 2A' |Hes\, u|^2 &\geq \frac{1}{2|\nabla u|^2 A'} (|\nabla P|^2 - 2B' \nabla u \nabla P + (B')^2 |\nabla u|^2)\end{aligned} \tag{5.18}$$

In addition, similarly to (5.14) we have

$$\sum_{m,l,k} d_{ml} u_k u_{x_m x_l x_k} = -\nabla_x F \nabla u - F_u |\nabla u|^2 - \frac{1}{2A'} (\nabla_q F \nabla_x P - B' \nabla u \nabla_q F) \tag{5.19}$$

We plug (5.18) and (5.19) into (5.17) and thus

$$\begin{aligned}\sum_{i,j} d_{ij} P_{x_i x_j} &\geq \frac{\lambda}{2|\nabla u|^2 A'} (|\nabla P|^2 - 2B' \nabla u \nabla P + (B')^2 |\nabla u|^2) + \lambda B'' |\nabla u|^2 \\ &+ B' \sum_{i,j} d_{ij} u_{x_i x_j} + 2A'(-\nabla_x F \nabla u - F_u |\nabla u|^2 - \frac{1}{2A'} (\nabla_q F \nabla_x P - B' \nabla u \nabla_q F))\end{aligned} \tag{5.20}$$

which gives

$$\sum_{i,j} d_{ij} P_{x_i x_j} + (\frac{\lambda B'(u)}{A'(|\nabla u|^2)|\nabla u|^2} \nabla u + \nabla_q F) \nabla_x P \geq J \tag{5.21}$$

and we conclude. □

A direct consequence of Lemma 5.2.4 is the following.

Corollary 5.2.1. Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth solution of

$$\Delta u = f(u) \tag{5.22}$$

and let $P(s,t) = A(t) + B(s)$ such that $A' > 0$ for $t > 0$, $A'' \geq 0$ and assume that

$$t^2 B''(s) + B'(s) f(s) + \frac{(B'(s))^2}{2A'(t^2)} + 2A'(t^2) t^2 f'(s) \geq 0 \tag{5.23}$$

Then $P = P(u, |\nabla u|^2)$ is a $P-$function of (5.22).

5.2.1 Examples of $P-$functions

(1) The well known $P-$function of (5.22) is

$$\begin{gathered} P(u, |\nabla u|^2) = \frac{|\nabla u|^2}{2} - F(u) \\ \text{where } \; F'(u) = f(u) \end{gathered} \tag{5.24}$$

(see [16] or Chapter 5 in [20]).

It is easy to see that (5.24) satisfies (5.23) in Corollary 5.2.1.

(2) Another general example of $P-$function of (5.22) is

$$\begin{gathered} P(u, |\nabla u|^2) = \frac{|\nabla u|^4}{2} + 2\int_0^u (\int_0^y \sqrt{f(z)f'(z)}dz)^2 dy \quad , \text{ if } \; f(t)f'(t) \geq 0 \; , \; \forall \, t \in \mathbb{R} \\ P(u, |\nabla u|^2) = \frac{|\nabla u|^4}{2} - 2\int_0^u (\int_0^y \sqrt{-f(z)f'(z)}dz)^2 dy \quad , \text{ if } \; f(t)f'(t) \leq 0 \end{gathered} \tag{5.25}$$

and satisfies condition (5.23) of Corollary 5.2.1.

Note that the above example is not in the form $P = g(u)|\nabla u|^2 + h(u)$ that we see in [20] as general form for P related to equation (5.22).

(3) Let u be a solution of

$$F(|\nabla u|^2 - cu, \nabla^2 u) = 0 \tag{5.26}$$

where F satisfies the ellipticity condition (5.8) and assume $\sum_{i,j} F_{a_{ij}} u_{x_i x_j} \leq \dfrac{\lambda c}{2}$, $c > 0$.

Then $P = P(u, |\nabla u|^2) = |\nabla u|^2 - cu$ is a $P-$function of (5.26) since P satisfies condition (5.16) of Lemma 5.2.4.

(4) The following example is in [18] (see Theorem 1).

Let u be a solution of

$$div(\Phi'(|\nabla u|^2)\nabla u) = \rho(|\nabla u|^2)F'(u) \tag{5.27}$$

with $\Phi'(t), \rho(t) > 0$ and $\Phi'(t) + 2t\Phi''(t) > 0$, $\forall\, t \geq 0$.

Consider the function

$$P(s,t) = \int_0^t \frac{\Phi'(y) + 2y\Phi''(y)}{\rho(y)} dy - 2F(s) \tag{5.28}$$

Then $P = P(u, |\nabla u|^2)$ is a $P-$function of (5.27).

Note that for $\rho \equiv 1$, we have the one studied in [4].

5.3 Gradient Bounds for entire solutions of Fully Nonlinear equations

In this section we will see that utilizing the techniques of [4, 9], we can obtain gradient bounds for solutions of equations of the form (5.7). To be more precise, for any explicit example of $P-$function, we obtain a particular gradient bound.

Some of the regularity assumptions in this work can be relaxed for some classes of equations. In the study of Quasi-linear equations for example, we can only assume that $u \in W^{1,p}_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, as in assumption (i) in Theorem 1.6 in [4] and utilize regularity results in [22] afterwords. For fully nonlinear elliptic equations also, we can relax the regularity of solutions and then utilize regularity results from [2, 23]. However, our main goal is not the optimal regularity assumptions since we state the results in an abstract form. Therefore, we will assume that the solutions are smooth and satisfy an analog of assumption (ii) in Theorem 1.6 in [4]. One application of interest is a gradient bound for entire solutions of Pucci's equation that we establish in subsection 3.1.

Assumption.

$$\begin{gathered} u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\ ,\ \nabla u \in C^\alpha_{loc}(\mathbb{R}^n;\mathbb{R}^n)\ \text{ for some }\ \alpha \in (0,1) \\ \text{and there exists }\ C = C(||u||_{L^\infty(\mathbb{R}^n)}) > 0\ \text{ such that }\ |\nabla u(x)| \leq C\ ,\ \forall\ x \in \mathbb{R}^n \end{gathered} \tag{5.29}$$

The next theorem provides an a priori pointwise estimate for solutions of (5.7). In contrast to the gradient bounds in [4, 6], the theorem below holds for any $P-$function that satisfies $P(u,0) \leq 0$. When P is of the form $P = P(u,\nabla u)$ we use the notation $P(u,0)$ instead of $P(u,0,...,0)$ and also we sometimes write $P = P(u;x)$ for simplicity.

Theorem 5.3.5. Let u be an entire solution of

$$F(x,u,\nabla u,\nabla^2 u) = 0 \tag{5.30}$$

that satisfy assumption (5.29). If $P = P(u,\nabla u)$ is a $P-$function of (5.30), with $\mu = \mu(|\nabla u|)$, $\mu(t) > 0\ ,\ \forall\, t > 0$, such that $P(s,0) \leq 0$,

Then

$$P(u(x),\nabla u(x)) \leq 0\ \ ,\ \forall\, x \in \mathbb{R}^n. \tag{5.31}$$

Proof. Let u be a solution of (5.30) that satisfies assumption (5.29) and consider the family of all translations of u,

$$\mathscr{F} = \{v\ : \mathbb{R}^n \to \mathbb{R} \mid \exists\, z \in \mathbb{R}^n\ \text{ such that }\ v(x) = u(x+z)\ \ \forall\ x \in \mathbb{R}^n\} \tag{5.32}$$

$\mathscr{F}$ is non empty since $u \in \mathscr{F}$.

Let P be a $P-$function of (5.30), with $\mu = \mu(|\nabla u|)$, $\mu(t) > 0\ ,\forall t > 0$, such that $P(u,0) \leq 0$. For simplicity, we denote $P = P(u;x)$ instead of $P = P(u(x),\nabla u(x))$.

Consider now

$$P_0 = \sup\{P(v;x) \mid v \in \mathscr{F}\ ,\ \ x \in \mathbb{R}^n\} \tag{5.33}$$

We will prove that

$$P_0 \leq 0 \tag{5.34}$$

and from this we conclude.

We argue by contradiction, so we suppose that

$$P_0 > 0 \tag{5.35}$$

Then, by (5.33) there exist $(w_k)_{k\in\mathbb{N}}$ in $\mathscr{F}$ and $(x_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ such that

$$\lim_{k\to+\infty} P(w_k; x_k) = P_0 \tag{5.36}$$

Let $v_k(x) = w_k(x+x_k)$. Also, by definition we have that $v_k \in \mathscr{F}$ and $P(v_k; 0) = P(w_k; x_k)$, so that (5.36) can be rewritten as

$$\lim_{k\to+\infty} P(v_k; 0) = P_0 \tag{5.37}$$

Since $v_k \in C^{1,\alpha}_{loc}(\mathbb{R}^n)$, by the Ascoli-Arzela theorem together with a diagonal argument, we can extract from $(v_k)_{k\in\mathbb{N}}$ a subsequence, denoted by $(v_k^{(k)})_{k\in\mathbb{N}}$ that converges with its first-order derivatives, uniformly on compact subsets of $\mathbb{R}^n$. Denote by $\tilde{v}$ the limit function.

By the assumption (5.29) we have that

$$\mathscr{F} \text{ is relatively compact in } C^{1,\alpha}_{loc}(\mathbb{R}^n) \tag{5.38}$$

Thus $\tilde{v} \in \mathscr{F}$ and

$$P(\tilde{v}; 0) = \lim_{k\to+\infty} P(v_k^{(k)}; 0) = P_0 \tag{5.39}$$

by (5.37).

Consider now the set

$$U = \{x \in \mathbb{R}^n \mid P(\tilde{v}; x) = P_0\} \tag{5.40}$$

from the continuity of P on $\mathbb{R}^n$, U is closed and non empty since $0 \in U$. We will prove that U is also open. Let $x_0 \in U$, we observe that $|\nabla\tilde{v}(x_0)| \neq 0$, otherwise we would have

$$P_0 = P(\tilde{v}; x_0) = P(\tilde{v}(x_0), \nabla\tilde{v}(x_0)) = P(\tilde{v}(x_0), 0) \leq 0$$

against the assumption that $P_0 > 0$.

By continuity, there exists $\delta > 0$ such that

$$\inf_{\overline{B}_\delta(x_0)} |\nabla\tilde{v}| > 0 \text{ and thus } \inf_{\overline{B}_\delta(x_0)} \mu(|\nabla\tilde{v}|) > 0 \tag{5.41}$$

and by Theorem 5.2.1 we conclude that

$$P(\tilde{v}; x) \equiv P_0 \quad \text{in } B_\delta(x_0) \tag{5.42}$$

So U is open and it follows that $U = \mathbb{R}^n$ by connectedness.

On the other hand, since $\tilde{v}$ is bounded it holds that there exists a sequence $(y_l)_{l\in\mathbb{N}}$ in $\mathbb{R}^n$ such that

$$\lim_{l\to+\infty} \nabla\tilde{v}(y_l) = 0 \tag{5.43}$$

By the boundedness of $\tilde{v}$ we also have $\tilde{v}(y_l) = \tilde{v}_l \to v_0$ up to a subsequence that we still denote as y_l, and so we obtain

$$0 < P_0 = \lim_{l\to\infty} P(\tilde{v}(y_l), \nabla\tilde{v}(y_l)) = P(v_0, 0)$$

which contradicts the assumption $P(s, 0) \leq 0$. Therefore $P_0 \leq 0$ and we conclude. $\square$

A direct consequence is a general gradient bound for fully nonlinear equations.

Corollary 5.3.1. Let u be an entire solution of

$$F(x, u, \nabla u, \nabla^2 u) = 0 \tag{5.44}$$

that satisfy assumption (5.29) and F satisfies the ellipticity condition (5.8). Consider $P = P(u, \nabla u) = A(|\nabla u|^2) + B(u)$ is a $P-$function from Lemma 5.2.4 such that $B \leq 0$.

Then

$$\begin{gathered} |\nabla u|^2 \leq \Psi(u) \\ \text{where } \Psi(s) = A^{-1}(-B(s)) \end{gathered} \tag{5.45}$$

Similarly, if P is obtained from Lemma 5.2.3 and $g \leq 0$, we have $|\nabla u|^2 \leq G(u)$, where $G = -g \geq 0$.

Remark 5.3.6. Note that the condition $P(u, 0) \leq 0$ can be removed. Consider for example $P = P(u, \nabla u)$ be a $P-$function of (5.30) such that the condition $P(u, 0) \leq 0$ is not satisfied and set $\tilde{P}(u, \nabla u) = P(u, \nabla u) - \sup_{\mathbb{R}^n} P(u, 0)$. Then $\tilde{P}$ is also a $P-$function of (5.30) and satisfies $\tilde{P}(u, 0) \leq 0$. The only difference is that the gradient bound in this case takes the form $P(u(x), \nabla u(x)) \leq \sup_{\mathbb{R}^n} P(u(x), 0)$.

5.3.1 Gradient Bound for entire solutions of Pucci's equations

We denote as $\mathcal{M}^-$ and $\mathcal{M}^+$ as the Pucci's extremal operators defined in (5.9). The first application of Theorem 5.3.5 is the following.

Theorem 5.3.7. Let $u : \mathbb{R}^n \to \mathbb{R}$ be an entire solution of

$$\mathcal{M}^-(\nabla^2 u) = F'(u) \quad , \; F \geq 0 \tag{5.46}$$

that satisfy assumption (5.29).

Consider the quantity

$$R^- = \lambda \sum_{i<j \; , \; e_k>0} \frac{\partial e_k}{\partial u_{x_i x_j}} u_{x_i x_j} + \Lambda \sum_{i<j \; , \; e_k<0} \frac{\partial e_k}{\partial u_{x_i x_j}} u_{x_i x_j} \tag{5.47}$$

and assume $R^- \cdot F' \leq 0$.

Then

$$\frac{1}{2}|\nabla u|^2 \leq \frac{F(u)}{\lambda} \tag{5.48}$$

Proof. We set in Lemma 5.2.4, $A(t) = \frac{t}{2}$, $B(s) = -\frac{F(s)}{\lambda}$ and we calculate

$$J = (-\lambda \frac{F''(u)}{\lambda} + F'')|\nabla u|^2 - \frac{F'(u)}{\lambda} \sum_{i,j} \frac{\partial \mathcal{M}(\nabla^2 u)}{\partial u_{x_i x_j}} u_{x_i x_j} + \frac{(F'(u))^2}{\lambda}$$

$$\Rightarrow J = \frac{(F'(u))^2}{\lambda} - \frac{F'(u)}{\lambda} \sum_{i,j} \frac{\partial \mathcal{M}(\nabla^2 u)}{\partial u_{x_i x_j}} u_{x_i x_j}$$

Also we have the following,

$$\frac{\partial e_k}{\partial u_{x_i x_j}} = \begin{cases} (v_i^k)^2 & , \quad i = j \\ 2 v_i^k v_j^k & , \quad i \neq j \end{cases} \tag{5.49}$$

where v^k is the unit length eigenvector of e_k (this identity holds since e_k are distinct).

Identity (5.49) is an algebraic fact and can be proved as follows. Let A be an $n \times n$ symmetric matrix with eigenvalues e_k and respective unit length eigenvectors v^k,

$$A \cdot v^k = e_k v^k$$
$$\Rightarrow (dA) v^k + A(dv^k) = de_k v^k + e_k dv^k$$

where d is a differential operator, say for example the partial derivative with respect to the element a_{ij} of A. Since v^k is unit length, $v^k dv^k = \frac{1}{2}d(|v^k|^2) = 0$, so

$$(v^k)^T(dA)v^k = de_k$$

since $(v^k)^T A = e_k(v^k)^T$.

Thus,

$$\sum_{i,j}\frac{\partial\mathcal{M}^-(\nabla^2 u)}{\partial u_{x_ix_j}}u_{x_ix_j} = \lambda\sum_{e_k>0}\sum_{i,j}\frac{\partial e_k}{\partial u_{x_ix_j}}u_{x_ix_j} + \Lambda\sum_{e_k<0}\sum_{i,j}\frac{\partial e_k}{\partial u_{x_ix_j}}u_{x_ix_j} =$$
$$\lambda\sum_{e_k>0}\left(\sum_{i=1}^{n}(v_i^k)^2u_{x_ix_j} + \sum_{i\neq j}2v_i^kv_j^ku_{x_ix_j}\right) + \Lambda\sum_{e_k<0}\left(\sum_{i=1}^{n}(v_i^k)^2u_{x_ix_j} + \sum_{i\neq j}2v_i^kv_j^ku_{x_ix_j}\right) \tag{5.50}$$

by (5.49).

Also, $(v^k)^T(\nabla^2u)v^k = e_k$, since $(v^k)^Tv^k = 1$, so

$$\sum_{i,j}\frac{\partial\mathcal{M}^-(\nabla^2 u)}{\partial u_{x_ix_j}}u_{x_ix_j} = \lambda\sum_{e_k>0}\left(e_k + \sum_{i\neq j}v_i^kv_j^ku_{x_ix_j}\right) + \Lambda\sum_{e_k<0}\left(e_k + \sum_{i\neq j}v_i^kv_j^ku_{x_ix_j}\right)$$
$$\Rightarrow \sum_{i,j}\frac{\partial\mathcal{M}^-(\nabla^2 u)}{\partial u_{x_ix_j}}u_{x_ix_j} = \lambda\sum_{e_k>0}e_k + \Lambda\sum_{e_k<0}e_k$$
$$+\lambda\sum_{e_k>0}\sum_{i<j}2v_i^kv_j^ku_{x_ix_j} + \Lambda\sum_{e_k<0}\sum_{i<j}2v_i^kv_j^ku_{x_ix_j}$$
$$= \mathcal{M}^-(\nabla^2u) + \lambda\sum_{e_k>0}\sum_{i<j}\frac{\partial e_k}{\partial u_{x_ix_j}}u_{x_ix_j} + \Lambda\sum_{e_k<0}\sum_{i<j}\frac{\partial e_k}{\partial u_{x_ix_j}}u_{x_ix_j} = F'(u) + R^- \tag{5.51}$$

by (5.46) and (5.49).

Therefore,

$$J = \frac{(F'(u))^2}{\lambda} - \frac{F'(u)}{\lambda}\sum_{i,j}\frac{\partial\mathcal{M}(\nabla^2u)}{\partial u_{x_ix_j}}u_{x_ix_j} = -\frac{F'(u)}{\lambda}R^- \geq 0$$

thus, by Lemma 5.2.4, we have that $P(u,|\nabla u|^2) = \frac{1}{2}|\nabla u|^2 - \frac{F(u)}{\lambda}$ is a $P-$function of (5.46).

In addition, $P(u,0) = -\frac{F(u)}{\lambda} \leq 0$, therefore by Theorem 5.3.5 we conclude that

$$\frac{1}{2}|\nabla u|^2 \leq \frac{F(u)}{\lambda} \quad , \quad \forall x \in \mathbb{R}^n.$$

□

Remark 5.3.8. **(1)** The gradient bound (5.48) also holds for the operator $\mathcal{M}^+$ with similar calculations by considering

$$R^+ = \lambda \sum_{i<j\ ,\ e_k<0} \frac{\partial e_k}{\partial u_{x_i x_j}} u_{x_i x_j} + \Lambda \sum_{i<j\ ,\ e_k>0} \frac{\partial e_k}{\partial u_{x_i x_j}} u_{x_i x_j} \tag{5.52}$$

and assuming that $R^+ \cdot F' \leq 0$.
(2) Note that in the case where $\lambda = \Lambda = 1$, we have $\mathcal{M}^- = \mathcal{M}^+ = \Delta u$ and $R^- = R^+ = 0$, so the assumption $R^- F' \leq 0$ (or $R^+ \cdot F' \leq 0$) is dismissed. In this view, Theorem 5.3.7 can be seen as a generalization of the Modica inequality for the Pucci's operators.

We will now see that in two dimensions the quantity R^- is non positive, and therefore the assumption $R^- F' \leq 0$ can be relaxed.

Theorem 5.3.9. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be an entire solution of

$$\mathcal{M}^-(\nabla^2 u) = F'(u) \tag{5.53}$$

that satisfy (5.29) and assume that $F, F' \geq 0$.
Then

$$\frac{1}{2}|\nabla u|^2 \leq \frac{F(u)}{\lambda} \tag{5.54}$$

Proof. In the cases where $e_1 \cdot e_2 > 0$ we have that $\mathcal{M}^- = (\lambda + \Lambda)\Delta u = F'(u)$ and by setting $A(t) = \frac{t}{2}$ and $B(s) = -\frac{F(s)}{\lambda}$ in Lemma 5.2.4 we calculate J,

$$J = -\frac{F'(u)}{\lambda} \sum_{i,j} (\lambda + \Lambda)\delta_{ij} u_{x_i x_j} + \frac{(F'(u))^2}{\lambda} = 0$$

and thus the elliptic inequality for the $P-$ function is satisfied.

So we consider the case where $e_1 < 0 < e_2$. In this case e_1 and e_2 can be written explicitly,

$$e_1 = \frac{\Delta u - \sqrt{\mathcal{D}}}{2} \quad \text{and} \quad e_2 = \frac{\Delta u + \sqrt{\mathcal{D}}}{2} \tag{5.55}$$
$$\text{where } \mathcal{D} = (\Delta u)^2 - 4det(\nabla^2 u) = (u_{xx} - u_{yy})^2 + 4u_{xy}^2 > 0$$

as solutions of the characteristic polynomial of the matrix $(\nabla^2 u)$.

Next, we can calculate explicitly R^- in Theorem 5.3.7.

$$R^- = \lambda \frac{\partial e_2}{\partial u_{xy}} u_{xy} + \Lambda \frac{\partial e_1}{\partial u_{xy}} u_{xy}$$

and

$$\frac{\partial e_1}{\partial u_{xy}} = -\frac{2u_{xy}}{\sqrt{\mathcal{D}}} \quad \text{and} \quad \frac{\partial e_2}{\partial u_{xy}} = \frac{2u_{xy}}{\sqrt{\mathcal{D}}}$$

thus

$$R^- = 2(\lambda - \Lambda)\frac{u_{xy}^2}{\sqrt{\mathcal{D}}} \leq 0$$

and $F' \geq 0$, therefore the condition $R^- F' \leq 0$ is satisfied and by Theorem 5.3.7 above, we conclude that

$$\frac{1}{2}|\nabla u|^2 \leq \frac{F(u)}{\lambda}$$

□

Note: The gradient bound in Theorem 5.3.9 also holds for $\mathcal{M}^+(\nabla^2 u) = F'(u)$ if we assume that $F'(u) \leq 0$ (instead of $F' \geq 0$).

5.3.2 Gradient Bounds for entire solutions by the Examples of subsection 2.1

Next, we observe that in Example (2) above, for solutions of (5.22),

$$P(u, |\nabla u|^2) = \frac{|\nabla u|^4}{2} - 2\int_0^u (\int_0^y \sqrt{-f(z)f'(z)}dz)^2 dy \quad , \text{ if } f(t)f'(t) \leq 0 \tag{5.56}$$

satisfies $P(u, 0) \leq 0$.

Therefore, we have

Corollary 5.3.1. Let u be a smooth and bounded entire solution to

$$\begin{gathered}\Delta u = f(u) \\ \text{where } \ f \in C^{1,\alpha}(\mathbb{R}^n) \ \text{ and } \ f(t)f'(t) \leq 0.\end{gathered} \tag{5.57}$$

Then

$$\frac{|\nabla u|^4}{4} \leq \int_0^u (\int_0^y \sqrt{-f(z)f'(z)}dz)^2 dy \tag{5.58}$$

Proof. By elliptic regularity theory (see [14]) we have that $u \in C^{2,\alpha}(\mathbb{R}^n)$ and that $|\nabla u|$ is bounded in $\mathbb{R}^n$. It suffices to prove that P defined in (5.56) is a $P-$function of (5.57) and then the conclusion is direct application of Theorem 5.3.5.

We have that P satisfies (5.23),

$$P(s,t) = \frac{t^2}{2} - 2\int_0^s (\int_0^y \sqrt{-f(z)f'(z)}dz)^2 dy = \frac{t^2}{2} + q(s)$$

so $P_t = t > 0$ for $t > 0$, $P_{tt} \geq 0$ and $\mu = P_t(u, |\nabla u|^2)|\nabla u|^2 = \frac{1}{2}|\nabla u|^4$.

Finally,

$$t^2 P_{ss}(s,t^2) + P_s(s,t^2)f(s) + 2P_t(s,t^2)t^2 f'(s) = 2t^4 f'(s) + t^2 q''(s) + q'(s)f(s) \geq 0$$

since the above polynomial has zero discriminant. □

For Example (3), we have the following gradient bound

Corollary 5.3.2. Let u be a non negative entire solution of

$$F(|\nabla u|^2 - cu, \nabla^2 u) = 0 \tag{5.59}$$

that satisfy (5.29), where F satisfies the ellipticity condition (5.8) and assume $\sum_{i,j} F_{a_{ij}} u_{x_i x_j} \leq \dfrac{\lambda c}{2}$ for some $c > 0$.

Then

$$|\nabla u|^2 \leq cu \tag{5.60}$$

Proof. We have that the function

$$P(s,t) = t - cs$$

satisfy the condition (5.16) and also, $P(u, 0) = -cu \leq 0$ since u is non negative by assumption. Therefore we conclude by the Theorem 5.3.5. □

The following gradient bound that we derive from Theorem 5.3.5, via Example (4), is a quite more general form of the gradient bound in [4].

Corollary 5.3.3. Let u be an entire solution of

$$div(\Phi'(|\nabla u|^2)\nabla u) = \rho(|\nabla u|^2)F'(u) \ \ , \ F \geq 0 \tag{5.61}$$

that satisfy assumption (5.29), with $\Phi'(t), \rho(t) > 0$ and $\Phi'(t)+2t\Phi''(t) > 0$, $\forall t \geq 0$.

Then

$$\begin{gathered}|\nabla u|^2 \leq \Psi(u) \ \ , \text{ where } \ \Psi(u) = Q^{-1}(2F(u)) \\ \text{and } \ Q(t) = \int_0^t \frac{\Phi'(y) + 2y\Phi''(y)}{\rho(y)} dy\end{gathered} \tag{5.62}$$

Proof. By Theorem 1 in [18], we have that $P(u, |\nabla u|^2) = Q(|\nabla u|^2) - 2F(u)$ is a $P-$function of (5.61) with $\mu(t) > 0$, $\forall\, t \geq 0$ and satisfies $P(u, 0) \leq 0$ since $F \geq 0$. Thus we apply Theorem 5.3.5 and we conclude. □

5.4 Rigidity results and properties of entire solutions of fully nonlinear equations

In this section we will see that if an equation admits a $P-$function of the form $P = |\nabla u|^2$, then the solutions that satisfy assumption (5.29) are constant. As a result we have a Liouville theorem for Pucci's equations and special cases are some of the well-known results of J. Serrin in [19].

In particular we have

Theorem 5.4.10. Let u be an entire solution of

$$F(u, \nabla u, \nabla^2 u) = 0 \tag{5.63}$$

that satisfy assumption (5.29) where F satisfies the condition (5.8) and assume $F_u \leq 0$.

Then u is a constant

Proof. The proof of Theorem 5.4.10 is a consequence of Theorem 5.3.5 and Lemma 5.2.3 by considering $P = |\nabla u|^2$. □

Remark 5.4.11. **(1)** If $F = F(x, u, \nabla u, \nabla^2 u)$ and assume in addition that $\nabla_x F \cdot \nabla u \leq 0$, then the conclusion of Theorem 5.4.10 still holds by Theorem 5.3.5 and Lemma 5.2.3. Note also that Theorem 1, Theorem 6 and Theorem 8 in [19] are recovered.
(2) We note that, in the special case where equation (5.63) takes the form

$$div(\Phi'(|\nabla u|^2 \nabla u) = f'(u) \tag{5.64}$$

then the condition $F_u \leq 0$ reads $f''(u) \geq 0$ which implies stability of the solutions, i.e. the second variation of the associated energy functional $J(u) = \int(\frac{1}{2}\Phi(|\nabla u|^2) + f(u))dx$ is non negative (see also Theorem 4.5 in [4]).

In addition, we have a Liouville-type result for Pucci's equation as a direct application of Theorem 5.4.10.

Corollary 5.4.1. Let u be an entire solution of

$$\mathcal{M}^-(\nabla^2 u) = F'(u) \tag{5.65}$$

that satisfy (5.29) and assume that $F''(u) \geq 0$.
Then u is a constant.

Another consequence of Theorem 5.3.5 is the following Liouville-type result

Theorem 5.4.12. Let u be an entire solution of (5.30) that satisfies assumption (5.29) and P is a $P-$function from 5.2.4 that satisfies $P(u, 0) \leq 0$. If there exists $x_0 \in \mathbb{R}^n$ such that $B(u(x_0)) = 0$, then $u \equiv$ const. in $\mathbb{R}^n$.

Proof. We argue as in the proof of Theorem 1.8 in [4] with slight modifications. For the convenience of the reader we provide the details.

Suppose that $B(u(x_0)) = 0$, let $u_0 = u(x_0)$ and consider the set

$$V = \{x \in \mathbb{R}^n \mid u(x) = u_0\} \tag{5.66}$$

V is a closed set and by the assumption, non empty. Let $x_1 \in V$ and consider the function $\phi(t) = u(x_1 + t\omega) - u_0$, where $|\omega| = 1$ is arbitrarily fixed. We have $|\phi'(t)| = |\nabla u(x_1 + t\omega)|$. By the gradient bound in Corollary 5.3.1 we have,

$$|\nabla u|^2 \leq \Psi(u) \ \ , \text{ where } \ \Psi(s) = A^{-1}(-B(s)) \tag{5.67}$$

Since $\Psi \in C^2(\mathbb{R})$ and $\Psi(u_0) = 0$, we have $\Psi(u) = O(|u-u_0|^2)$, as $|u-u_0| \to 0$. So, we conclude from (5.67) that $|\phi'(t)| \leq C|\phi(t)|$ for t small enough. Since $\phi(0) = 0$, we must have $\phi \equiv 0$ on $[-\delta, \delta]$, for some $\delta > 0$. Thus V is open, which gives that $V = \mathbb{R}^n$. $\square$

Also, we have a De Giorgi type property for solutions that attain the equality at a point in the gradient bound obtained in Corollary 5.3.1.

Theorem 5.4.13. Let u be an entire solution of

$$F(u, \nabla u, \nabla^2 u) = 0 \tag{5.68}$$

that satisfy assumption (5.29) and let $P = P(u, |\nabla u|^2)$ be a $P-$function of (5.68) obtained in Lemma 5.2.4 that satisfies $P(u, 0) \leq 0$. If there exists $x_0 \in \mathbb{R}^n$ such that

$$P(u(x_0), |\nabla u(x_0)|^2) = 0 \tag{5.69}$$

then there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} \text{either} \quad & u(x) = g(a \cdot x + b) \ , \ a \in \mathbb{R}^n \ \text{ with } \ |a| = 1, \ , \ b \in \mathbb{R} \\ \text{or} \quad & u(x) = g(|x - z_0| + c) \ \ , \ z_0 \in \mathbb{R}^n \ \text{ and } \ c \in \mathbb{R} \end{aligned} \tag{5.70}$$

Proof. By Corollary 5.3.1, we have that $P(u, |\nabla u|^2) \leq 0$.

We begin by considering the set

$$\mathscr{A} = \{x \in \mathbb{R}^n \ : \ P(u, |\nabla u|^2) = 0\} \tag{5.71}$$

$\mathscr{A}$ is closed and by the assumption $\mathscr{A} \neq \emptyset$. We are going to prove that $\mathscr{A}$ is open.

Let $x_1 \in \mathscr{A}$, if $\nabla u(x_1) = 0$, we obtain by the form $P(s,t) = A(t) + B(s)$ that $P(u(x_1), 0) = -B(u(x_1)) = 0$. By Theorem 5.4.12, we conclude that $u \equiv u(x_1)$ and $\nabla u \equiv 0$ and hence $P \equiv 0$.

On the other hand, if $\nabla u(x_1) \neq 0$, we have $\inf_{\overline{B}_{\delta_1}(x_1)} |\nabla u| > 0$ for some $\delta_1 > 0$ and by Theorem 5.2.1 we conclude that $P(u, |\nabla u|^2) \equiv 0$ in $B_{\delta_1}(x_1)$ and therefore $\mathscr{A}$ is open.

By connectedness, we have that $\mathscr{A} = \mathbb{R}^n$, that is,

$$P(u, |\nabla u|^2) \equiv 0 \;\; , \; \forall\, x \in \mathbb{R}^n \tag{5.72}$$

and $P_t = A'(t) > 0$, thus

$$|\nabla u|^2 = \Psi(u) \quad , \quad \text{in } \mathbb{R}^n \quad , \text{ where } \Psi(u) = A^{-1}(-B(u)) \tag{5.73}$$

Now, if there exists $x_2 \in \mathbb{R}^n$ such that $\Phi(u(x_2)) = 0$, so $|\nabla u(x_2)| = 0$, again by Theorem 5.4.12 we have that $u \equiv u(x_2)$.

If, on the other hand $\Psi(u(x)) > 0 \;\; , \; \forall\, x \in \mathbb{R}^n$, we set

$$\begin{gathered} v = G(u) \quad , \quad \text{where } G'(s) = \frac{1}{\Psi(s)} \\ \text{and } |\nabla v|^2 = 1 \quad \text{in } \mathbb{R}^n \end{gathered} \tag{5.74}$$

Therefore, by the result in [3], we have that

$$\begin{gathered} \text{either } v(x) = a \cdot x + b \;\; , \; a \in \mathbb{R}^n \text{ with } |a| = 1 \text{ and } b \in \mathbb{R} \\ \text{or } v(x) = |x - z_0| + c \;\; , \; z_0 \in \mathbb{R}^n \text{ and } c \in \mathbb{R} \end{gathered} \tag{5.75}$$

So we conclude that

$$\begin{gathered} \text{either} \quad u(x) = g(a \cdot x + b) \;\; , \; a \in \mathbb{R}^n \text{ with } |a| = 1 \; , \;\; b \in \mathbb{R} \text{ where } g(s) = G^{-1}(s) \\ \text{or} \quad u(x) = g(|x - z_0| + c) \;\; , \; z_0 \in \mathbb{R}^n \text{ and } c \in \mathbb{R} \end{gathered} \tag{5.76}$$

□

Remark 5.4.14. Note that if $u : \Omega \to \mathbb{R}$ where Ω is an open and connected domain in $\mathbb{R}^n$ and $P = P(u, |\nabla u|^2) = A(|\nabla u|^2) + B(u)$ with $A' > 0$ that attains its maximum at a point then u will be a solution of the Eikonal equation $|\nabla u|^2 = \Psi(u)$. If in addition $u_{x_n} > 0$ and consider $F_i = \frac{u_{x_i}}{u_{x_n}}$, by Proposition 2.1 in [11], the function $F = (F_1, ..., F_{n-1})$ will satisfy the Isobaric Euler equation.

5.5 A Harnack-type inequality and Local Estimates for the gradient

5.5.1 Estimates for Quasi-Linear equations

We will establish a Harnack inequality and local estimates for the gradient of solutions to Quasi-linear equations in a domain $\Omega \subset \mathbb{R}^n$. Properties of entire solutions for such equations have been studied in [4, 6, 9] among others. These estimates can be extended for some fully nonlinear equations such as for the Monge-Ampère equation.

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth solution of

$$div(\Phi'(|\nabla u|^2)\nabla u) = f'(u) \tag{5.77}$$

or equivalently,

$$\begin{gathered}\sum_{i,j} a_{ij}(\nabla u)u_{x_i x_j} = f'(u) \\ \text{where } \ a_{ij}(\sigma) = 2\Phi''(|\sigma|^2)\sigma_i\sigma_j + \Phi'(|\sigma|^2)\delta_{ij}\end{gathered} \tag{5.78}$$

and we assume that a_{ij} satisfy the ellipticity condition

$$\theta|\xi|^2 \leq \sum_{i,j} a_{ij}\xi_i\xi_j \leq \Theta|\xi|^2 \tag{5.79}$$

Some important examples are

(1) The $p-$Laplacian

$$div(|\nabla u|^{p-2}\nabla u) = f'(u) \quad , \ \text{for} \ \ p > 1 \tag{5.80}$$

(2) The Allen-Cahn equation (for $p = 2$)

$$\Delta u = f'(u) \tag{5.81}$$

(3) The minimal surface equation

$$div(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}) = f'(u) \tag{5.82}$$

Assume also that $a_{ij} \in L^\infty(\Omega)$. First, we have Local Boundedness on the gradient of solutions.

Theorem 5.5.15. Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth solution of (5.77) and assume (5.79). Suppose $f''(u)|\nabla u|^2 \in L^q(\Omega)$ for some $q > \frac{n}{2}$.

Then for any $B_R \subset \Omega$, any $0 < r < R$ and any $p \geq 1$

$$\sup_{B_r} |\nabla u|^2 \leq C[\frac{||\nabla u||^2_{L^{2p}(B_R)}}{(R-r)^{n/p}} + R^{2-n/q}||f''(u)|\nabla u|^2||_{L^q(B_R)}] \tag{5.83}$$

where $C = C(n, \theta, \Theta, p, q)$ is a positive constant.

In addition, if $f'' \geq 0$, the assumption on $f''(u)|\nabla u|^2$ is dismissed and we have

$$\sup_{B_r} |\nabla u|^2 \leq \frac{C}{(R-r)^{n/p}}||\nabla u||^2_{L^{2p}(B_R)} \tag{5.84}$$

Proof. Arguing as in Lemma 5.2.3 with $g = 0$, we have

$$\sum_{i,j} a_{ij} P_{x_i x_j} \geq 2\theta |Hes\, u|^2 + 2 \sum_{i,j,k} a_{ij} u_{x_k} u_{x_k x_i x_j} \tag{5.85}$$

Differentiating (5.78) over x_k, multiplying with u_{x_k} and summing over k,

$$\sum_{i,j,k} (a_{ij} u_{x_i x_j})_{x_k} u_{x_k} = f''(u)|\nabla u|^2 \tag{5.86}$$

Now since $\dfrac{\partial a_{ij}}{\partial \sigma_k} = \dfrac{\partial a_{ki}}{\partial \sigma_j}$ we have

$$\begin{aligned} &\sum_{i,j,k} (a_{ij} u_{x_k x_i})_{x_j} u_{x_k} = f''(u)|\nabla u|^2 \\ \Rightarrow &\sum_{i,j,k} a_{ij} u_{x_k x_i x_j} u_{x_k} = f''(u)|\nabla u|^2 - \sum_{i,j,k} (a_{ij})_{x_j} u_{x_k x_i} u_{x_k} \end{aligned} \tag{5.87}$$

and thus (5.85) becomes

$$\sum_{i,j} (a_{ij} P_{x_i})_{x_j} \geq 2\theta |Hes\, u|^2 + 2 f''(u)|\nabla u|^2 \tag{5.88}$$

Finally, by the local boundedness for elliptic equations we conclude (see for example Theorem 4.14 in [15]).

In the case where $f'' \geq 0$ holds, we have that $-\sum_{i,j}(a_{ij}P_{x_i})_{x_j} \leq 0$ and similarly we have (5.84). $\square$

Additionally, a Harnack-type inequality is established with similar arguments.

Theorem 5.5.16. Let $u : \Omega \to \mathbb{R}$ be a smooth solution of (5.77) and assume (5.79). Suppose $|Hes\, u|^2, f''(u)|\nabla u|^2 \in L^q(\Omega)$ for some $q > \frac{n}{2}$. Then for any $B_R \subset \Omega$ there holds that for any $0 < p < \frac{n}{n-2}$ and any $0 < \eta < \tau < 1$

$$C(\frac{1}{R^n}\int_{B\tau R} |\nabla u|^{2p})^{\frac{1}{p}} \leq \inf_{B_{\eta R}} |\nabla u|^2 + R^{2-\frac{n}{p}}||Hes\, u||^2_{L^{2q}(B_R)} + ||f''(u)|\nabla u|^2||_{L^q(B_R)} \tag{5.89}$$

where $C = C(n, p, q, \theta, \Theta, \eta, \tau)$.

Moreover, if $f'' \leq 0$, the assumption on $f''(u)|\nabla u|^2$ is dismissed and we have

$$C(\frac{1}{R^n}\int_{B\tau R} |\nabla u|^{2p})^{\frac{1}{p}} \leq \inf_{B_{\eta R}} |\nabla u|^2 + R^{2-\frac{n}{p}}||Hes\, u||^2_{L^{2q}(B_R)} \tag{5.90}$$

Proof. As in the proof of Theorem 5.5.15, together with the ellipticity of a_{ij} there holds

$$\sum_{i,j}(a_{ij}P_{x_i})_{x_j} \leq 2\Theta|Hes\, u|^2 + 2f''(u)|\nabla u|^2 \tag{5.91}$$

and by the Harnack inequality applied in $P = |\nabla u|^2$ we conclude (see for example Theorem 4.15 in [15]).

If $f'' \leq 0$, (5.91) becomes

$$-\sum_{i,j}(a_{ij}P_{x_i})_{x_j} \geq -2\Theta|Hes\, u|^2 \tag{5.92}$$

and similarly we conclude. $\square$

5.5.2 Estimates for Fully-Nonlinear Elliptic equations

The local estimates and the Harnack-type inequality can be extended to fully nonlinear elliptic equations. By Lemma 5.2.3 with $g = 0$ or by Lemma 5.2.4 with $B = 0$, we can utilize the elliptic inequality (5.15) or (5.21) respectively and assuming that $F_{a_{ij}} \in L^\infty(\Omega)$, we can apply Local Estimates for subsolutions to general elliptic operators to the $P-$function and obtain similar local estimates for the gradient of solutions to fully nonlinear equations.

So, we consider solutions of the equation

$$F(x, u, \nabla u, \nabla^2 u) = 0 \tag{5.93}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ is a continuous function and satisfies the ellipticity condition (5.8). We denote $F = F(x, s, q, A)$, $F_{q_i} = \dfrac{\partial F}{\partial q_i}$, $F_{a_{ij}} = \dfrac{\partial F}{\partial a_{ij}}$.

In this subsection we will assume the bound

$$|F_{a_{ij}}|, |F_{q_i}| \le M \tag{5.94}$$

We first establish a local pointwise estimate for the gradient of solutions.

Theorem 5.5.17. Let $u : \Omega \to \mathbb{R}$ be a smooth solution of (5.93) and assume $\nabla_x F \cdot \nabla u \le 0$ and $F_u|\nabla u|^2 \in L^n(\Omega)$.

Then for any $B_{2R} \subset \Omega$ and any $p \ge 1$

$$\sup_{B_R} |\nabla u|^2 \le C[\frac{||\nabla u||^2_{L^{2p}(B_R)}}{|B_{2R}|^{1/p}} + \frac{R}{\lambda}||F_u(u)|\nabla u|^2||_{L^n(B_{2R})}] \tag{5.95}$$

where $C = C(n, p, \lambda, \Lambda, MR^2)$ is a positive constant.

In addition, if $F_u \le 0$, the assumption on $F_u(u)|\nabla u|^2$ is dismissed and we have

$$\sup_{B_R} |\nabla u|^2 \le \frac{C}{|B_{2R}|^{1/p}}||\nabla u||^2_{L^{2p}(B_{2R})} \tag{5.96}$$

Proof. Set $P = |\nabla u|^2$, by Lemma 5.2.3 and the assumption $\nabla_x F \cdot \nabla u \le 0$ we have

$$\sum_{i,j} d_{ij} P_{x_i x_j} + \nabla_q F \nabla_x P \ge -2F_u|\nabla u|^2 \tag{5.97}$$

where $d_{ij} = F_{a_{ij}}$. That is

$$L\, P \le 2F_u|\nabla u|^2 \tag{5.98}$$

Thus, by classical local pointwise estimates for subsolutions of elliptic equations (see for example Theorem 9.20 in [14]) we obtain (5.95).

In the case where $F_u \le 0$ we have $L\, P \le 0$ and similarly we conclude. $\square$

As in the previous subsection, we obtain the following Harnack estimate

Theorem 5.5.18. Let $u : \Omega \to \mathbb{R}$ be a smooth solution of (5.93). Assume $\nabla_x F \cdot \nabla u \geq 0$, $|Hes\, u|^2, F_u(u)|\nabla u|^2 \in L^n(\Omega)$.

Then for any $B_{2R} \subset \Omega$ there holds that for any $p \geq 1$,

$$(\frac{1}{|B_R|}\int_{B_R} |\nabla u|^{2p})^{\frac{1}{p}} \leq C(\inf_{B_R} |\nabla u|^2 + \frac{R}{\lambda}||Hes\, u||^2_{L^{2n}(B_{2R})} + \frac{R}{\lambda}||F_u(u)|\nabla u|^2||_{L^n(B_{2R})}) \tag{5.99}$$

where $C = C(n, p, \lambda, \Lambda, MR^2)$.

Moreover, if $F_u \geq 0$, the assumption on $F_u(u)|\nabla u|^2$ is dismissed and we have

$$(\frac{1}{|B_R|}\int_{B_R} |\nabla u|^{2p})^{\frac{1}{p}} \leq C(\inf_{B_R} |\nabla u|^2 + \frac{R}{\lambda}||Hes\, u||^2_{L^{2n}(B_{2R})}) \tag{5.100}$$

Proof. Set $P = |\nabla u|^2$. We argue as in the proof of Lemma 5.2.3 and by the assumption $\nabla_x F \cdot \nabla u \geq 0$ to obtain

$$\sum_{i,j} d_{ij} P_{x_i x_j} + \nabla_q F \nabla_x P \leq -2F_u|\nabla u|^2 + 2\Lambda|Hes\, u|^2 \tag{5.101}$$

So, by the Harnack inequality for supersolutions of elliptic equations (see Theorem 9.22 in [14] for instance), we obtain (5.99).

If in addition $F_u \geq 0$, we have $LP \geq -2\Lambda|Hes\, u|^2$, where $L = -\sum_{i,j} d_{ij}\partial_{x_i x_j} - \sum_i F_{q_i}\partial x_i$ and we conclude.

□

Last but not least, we have the following estimates for the Monge-Ampère equation

Corollary 5.5.1. Let $u : \Omega \to \mathbb{R}$ be a smooth and convex solution of

$$det(\nabla^2 u) = f(u, \nabla u) \tag{5.102}$$

where $f > 0$ and assume $|f_{q_i}|, |adj^T(\nabla^2 u)_{ij}| \leq M$.

(1) If $f_u \geq 0$, then for any $B_{2R} \subset \Omega$ and any $p \geq 1$ we have

$$\sup_{B_R} |\nabla u|^2 \leq \frac{C}{R^{n/p}}||\nabla u||^2_{L^{2p}(B_{2R})} \tag{5.103}$$

(2) If $f_u \leq 0$ and $|Hes\, u|^2 \in L^n(\Omega)$, then for any $B_{2R} \subset \Omega$ and any $p \geq 1$ we have

$$(\frac{1}{|B_R|}\int_{B_R} |\nabla u|^{2p})^{\frac{1}{p}} \leq C(\inf_{B_R} |\nabla u|^2 + \frac{R}{\lambda}||Hes\, u||^2_{L^{2n}(B_{2R})}) \tag{5.104}$$

where $C = C(n, p, \lambda, \Lambda, MR^2)$.

Proof. The proof (1) is a consequence of Theorem 5.5.17 and the proof of (2) is a consequence of Theorem 5.5.18 and since $\frac{\partial F}{\partial a_{ij}}(A) = adj^T(A)_{ij}$, for $F(A) = det(A)$ by Jacobi's formula. $\square$

Note: If $\Omega = \mathbb{R}^n$, the fact that $|\nabla u|^2$ is a $P-$function of (5.102) in view of Theorem 5.4.10 states that there is no solution of (5.102) that satisfies the assumption (5.29). Indeed, if we assume that u is an entire solution of (5.102) that satisfies (5.29), then by Theorem 5.4.10 we have that u is constant in $\mathbb{R}^n$, which contradicts the fact that it's Hessian has positive determinant. We can also see this as follows, if $|\nabla u|$ is bounded in $\mathbb{R}^n$, then ∇u can not be a global diffeomorphism and thus $det(\nabla^2 u)$ can not be strictly positive in $\mathbb{R}^n$.

5.6 Higher order nonlinear equations

In this last section, we will provide examples of $P-$functions for higher order nonlinear equations and their applications. In particular, an analogous version of Theorems 5.3.5 and 5.4.10, allow us to obtain properties and pointwise estimates of entire solutions even in this case. Moreover, we establish local pointwise estimates for nonlinear equations of order greater than two, through the mean value properties of the $P-$functions or with analogous arguments to that of section 5, applied in higher order equations. This method can be applied to many other classes of higher order nonlinear equations.

We begin by stating the analogous Theorem 5.3.5 for equations of general order.

Assumption

$$\begin{gathered} u \in C^m(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \;\; , \;\; \nabla^{m-1}u \in C^\alpha_{loc}(\mathbb{R}^n) \;\text{ for some }\; \alpha \in (0,1) \\ \text{and there exists }\; C > 0 \;\text{ such that }\; |\nabla^l u| \leq C \;\; , \;\; l = 1, ..., m-1. \end{gathered} \tag{5.105}$$

Theorem 5.6.19. Let u be an entire solution or subsolution of

$$F(x, u, \nabla u, ..., \nabla^m u) = 0 \tag{5.106}$$

that satisfies assumption (5.105) and let $P = P(u, ..., \nabla^{m-1}u) = P(u; x)$ be a $P-$function of (5.106) such that one of the following holds:
(i) $\mu = \mu(g(\nabla^k u))$ for some $g : \mathbb{R}^{n^k} \to \mathbb{R}$, $g(z) > 0$, $\forall\, z \neq 0$, $g((0, ..., 0)) = 0$, $\mu(t) > 0$, $\forall\, t > 0$ and $P(u; x) \leq 0$, when $\nabla^k u = (0, ..., 0)$, $k \in \{1, ..., m-1\}$,
(ii) $\mu = \mu(g(\nabla^k u))$ for some $g : \mathbb{R}^{n^k} \to \mathbb{R}$, $g(z) > 0$, $\forall\, z \neq 0$, $g((0, ..., 0)) = 0$, $\mu(t) > 0$, $\forall\, t > 0$, $P(u; x) \leq 0$, when $\nabla^l u = (0, ..., 0)$, $k \neq l$, $k, l \in \{1, ..., m-1\}$ and $g(\nabla^k u) > 0$, $\forall\, x \in \mathbb{R}^n$.

Then $P(u, ..., \nabla^{m-1}u) \leq 0 \;\; \forall\, x \in \mathbb{R}^n$.

Proof. The proof is similar to that of Theorem 5.3.5 with minor modifications. □

We now provide the generalization of Theorem 5.4.10 in the higher order case.

Theorem 5.6.20. Let u be an entire solution of

$$F(x, u, \nabla u, ..., \nabla^m u) = 0 \tag{5.107}$$

and let $P = P(u, ..., \nabla^{m-1}u) = P(u; x)$ be a $P-$function of (5.106) such that $\mu = \mu(g(\nabla^k u))$ for some $g : \mathbb{R}^{n^k} \to \mathbb{R}$, $g(z) > 0$, $\forall\, z \neq 0$, $g((0, ..., 0)) = 0$, $\mu(t) > 0$, $\forall\, t > 0$ and

$$\begin{gathered} P = H(\nabla^k u) \;\; , \text{ where } \; H \; : \mathbb{R}^{n^k} \to [0, +\infty) \\ \text{and } \; \{H = 0\} = \{0 \in \mathbb{R}^{n^k}\} \;\; , \; k \in \{1, ..., m-1\} \end{gathered} \tag{5.108}$$

Then $\nabla^{k-1}u$ is a constant.

Proof. The proof is direct consequence of Theorem 5.6.19 since $P = H(\nabla^k u) = 0$ when $\nabla^k u$ vanish which gives $P \equiv 0$ in $\mathbb{R}^n$. □

5.6.1 Local and Global Pointwise estimates

The arguments of section 5, can be applied for higher order nonlinear equations. In this case, we extract local and global estimates for higher order of derivatives of u, such as for the Laplacian.

Proposition 5.6.21. Let $u : \Omega \to \mathbb{R}$ be a smooth and convex subsolution of

$$\Delta^2 u - F(x, u, \nabla u, \nabla^2 u, \nabla^3 u) = 0 \ \text{, with } \ F \geq 0 \tag{5.109}$$

Then for any $B_R \subset \Omega$, any $0 < r < R$ and any $p \geq 1$,

$$\sup_{B_r} (\Delta u)^2 \leq \frac{C}{(R-r)^{n/p}} ||\Delta u||^2_{L^{2p}(B_R)} \tag{5.110}$$

where $C = C(n, p)$.

Proof.

$$\begin{aligned} P_{x_i} &= 2\Delta u \Delta u_{x_i} \\ P_{x_i x_i} &= 2(\Delta u_{x_i})^2 + 2\Delta u \Delta u_{x_i x_i} \\ \Rightarrow \Delta P &= 2|\nabla \Delta u|^2 + 2\Delta u \Delta^2 u \geq 2|\nabla \Delta u|^2 + F\Delta u \geq 0 \end{aligned} \tag{5.111}$$

Therefore, by classical estimates for subsolutions of elliptic equations he conclude (see for example Theorem 4.14 in [15]). □

Furthermore, we give some examples of $P-$functions of the form $P = P(u, |\nabla u|, \Delta u)$ related to forth order nonlinear equations together with applications.

Proposition 5.6.22. Let u be a smooth solution of

$$\begin{gathered} a(\Delta u)[|\nabla u|^2 \Delta^2 u - \Delta u(\nabla u \cdot \nabla \Delta u)] = b(u)|\nabla u|^4 \\ \text{where } \ a, b \ : \mathbb{R} \to \mathbb{R} \ \text{ and } \ a > 0 \ , \ a' \geq 0 \end{gathered} \tag{5.112}$$

and set $P(s, t) = A(t) - B(s)$ such that $A' = a$ and $B'' = b$.

Then $P = P(u, \Delta u) = A(\Delta u) - B(u)$ is a $P-$function of (5.112).
In addition, if u satisfies (5.105) with $m = 4$, $B(u) \geq 0$ and $u_{x_n} > 0$, then

$$\Delta u \leq \Gamma(u) \quad \forall\, x \in \mathbb{R}^n, \ \text{ where } \ \Gamma(u) = A^{-1}(B(u)). \tag{5.113}$$

Proof. We have

$$P_{x_i} = P_s u_{x_i} + P_t \Delta u_{x_i} \tag{5.114}$$

and so,

$$\begin{aligned} \Delta u(\nabla P \cdot \nabla u) &= P_s |\nabla u|^2 \Delta u + P_t \Delta u \sum_{i=1}^{n} u_{x_i} \Delta u_{x_i} \\ \Leftrightarrow -B'(u)|\nabla u|^2 \Delta u &= \Delta u(\nabla P \cdot \nabla u) - A'(\Delta u)\Delta u \sum_{i=1}^{n} u_{x_i} \Delta u_{x_i} \end{aligned} \tag{5.115}$$

on the other hand we have

$$\begin{aligned} P_{x_i x_i} &= P_{ss} u_{x_i}^2 + 2P_{st} u_{x_i} \Delta u_{x_i} + P_{tt}(\Delta u_{x_i})^2 + P_s u_{x_i x_i} + P_t \Delta u_{x_i x_i} \\ \Rightarrow \Delta P &= (-B''(u))|\nabla u|^2 + A''(\Delta u) \sum_{i=1}^{n} (\Delta u_{x_i})^2 - B'(u)\Delta u + A'(\Delta u)\Delta^2 u \end{aligned} \tag{5.116}$$

and by (5.115) and the assumptions of A and B, (5.116) becomes

$$|\nabla u|^2 \Delta P - \Delta u(\nabla P \cdot \nabla u) \geq a(\Delta u)[|\nabla u|^2 \Delta^2 u - \Delta u(\nabla u \cdot \nabla \Delta u)] - b(u)|\nabla u|^4 = 0 \tag{5.117}$$

For the bound of the Laplacian, we have $P(u,0) = -B(u) \leq 0$ and $\mu = |\nabla u|^2 > 0 \ \forall\, x \in \mathbb{R}^n$ since $u_{x_n} > 0$, so the assumption (i) in Theorem 5.6.19 is satisfied and we conclude. □

Proposition 5.6.23. Let u be a smooth solution of

$$\begin{gathered} |Hes\, u|^2 = F(u, |\nabla u|^2, \Delta u) + \frac{u}{2}\Delta^2 u \\ \text{where } \ F : \mathbb{R}^3 \to \mathbb{R} \text{ is such that } \ F(s,t,w) \geq \frac{1}{2} w^2. \end{gathered} \tag{5.118}$$

Then $P = P(u, |\nabla u|^2, \Delta u) = |\nabla u|^2 - u\Delta u$ is a $P-$function of (5.120). In addition, if u is non negative, convex solution of (5.120) that satisfies assumption (5.105), then

$$|\nabla u|^2 \leq u\Delta u \quad , \ \forall\, x \in \mathbb{R}^n. \tag{5.119}$$

Proof. We have that

$$P_{x_i} = 2\sum_{j=1}^{n} u_{x_j} u_{x_j x_i} - u_{x_i}\Delta u - u\Delta u_{x_i}$$

and

$$\Delta P = 2|Hes\, u|^2 + 2\nabla u \nabla \Delta u - (\Delta u)^2 - 2\nabla u \nabla \Delta u - u\Delta^2 u$$

so by (5.120),

$$\Delta P = 2F(u, |\nabla u|^2, \Delta u) - (\Delta u)^2 \geq 0$$

For the gradient bound we see that $P(u, 0, \Delta u) = -u\Delta u \leq 0$ since u is non negative and convex, so the assumption (i) of Theorem 5.6.19 is satisfied and we conclude. □

As a result, we have the following pointwise estimate

Corollary 5.6.1. Let $u : B_2 \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth solution of

$$\begin{gathered} |Hes\, u|^2 = F(u, |\nabla u|^2, \Delta u) + \frac{u}{2}\Delta^2 u \\ \text{where } F : \mathbb{R}^3 \to \mathbb{R} \text{ is such that } F(s,t,w) \geq \frac{1}{2}w^2. \end{gathered} \tag{5.120}$$

Then

$$\begin{gathered} |\nabla u(x)|^2 - u(x)\Delta u(x) \leq C(||u||_{H^1(B_2)} + ||\Delta u||_{L^2(B_2)}) \ , \\ \forall\, x \in B_1 = \{y \in \mathbb{R}^n \ : \ |y| < 1\} \ , \text{ and } C \text{ depends only on } n. \end{gathered} \tag{5.121}$$

Proof. By Proposition 5.6.23, we have that $P = |\nabla u|^2 - u\Delta u = P(u; x)$ is subharmonic. Therefore we have

$$P(u; x) \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} P(u; y)dy \ \ , \ \forall\, B(x,r) \subset B_2 \tag{5.122}$$

Also, $P \leq |\nabla u|^2 + \frac{1}{2}(u^2 + (\Delta u)^2)$.

So,

$$\int_{B(x,r)} P(u; y)dy \leq ||u||_{H^1(B_2)} + ||\Delta u||_{L^2(B_2)} \ , \ \forall\, B(x,r) \subset B_2 \tag{5.123}$$

Thus, for any $x \in B_1$ (since $B(x,1) \subset B_2$), we have

$$P(u; x) \leq \frac{1}{|B_1|}(||u||_{H^1(B_2)} + ||\Delta u||_{L^2(B_2)}) \tag{5.124}$$

□

Remark 5.6.24. Note that if $F(u, |\nabla u|^2, \Delta u) = \frac{1}{2}(\Delta u)^2$, we have a reduction of order result, that is, if u is a smooth and bounded entire solution of

$$2|Hes\, u|^2 = (\Delta u)^2 + u\Delta^2 u \tag{5.125}$$

such that $\nabla u, \Delta u \in L^\infty(\mathbb{R}^n)$, then u satisfies $u\Delta u = |\nabla u|^2 + c$ for some $c \in \mathbb{R}$. We can see this from the proof of Proposition 5.6.23, where $P = |\nabla u|^2 - u\Delta u$ will be harmonic for this particular equation. Also, $|P| \leq M$ for some $M = M(||u||_{L^\infty(\mathbb{R}^n)}, ||\nabla u||_{L^\infty(\mathbb{R}^n)}, ||\Delta u||_{L^\infty(\mathbb{R}^n)}) > 0$ and thus $P \equiv$constant.

A special case of Corollary 5.6.1 is the following estimate

Corollary 5.6.1. Let $u : B_2 \subset \mathbb{R}^2 \to \mathbb{R}$ be a smooth solution of

$$\begin{gathered} det(\nabla^2 u) = G(u, |\nabla u|^2) - \frac{u}{4}\Delta^2 u \\ \text{where } \ G : \mathbb{R}^2 \to [0, +\infty) \end{gathered} \tag{5.126}$$

Then

$$|\nabla u(x)|^2 - u(x)\Delta u(x) \leq C(||u||_{H^1(B_2)} + ||\Delta u||_{L^2(B_2)}) \tag{5.127}$$

Proof. We write $det(\nabla^2 u) = \dfrac{1}{2}((\Delta u)^2 - |Hesu|^2)$ since u is defined a domain in the plane and then the proof is a consequence of Corollary 5.6.1 for $F = G(u, |\nabla u|^2) + (\Delta u)^2$, $G \geq 0$. □

5.6.2 A Liouville theorem and a De Giorgi-type property

A direct consequence of Theorem 5.6.20 is the following

Corollary 5.6.2. Let u be a convex entire subsolution of

$$\sum_{i,j} a_{ij}(x, u, \nabla u, \nabla^2 u, \nabla^3 u)\Delta u_{x_i x_j} - F(x, u, \nabla u, \nabla^2 u, \nabla^3 u) = 0 \tag{5.128}$$

that satisfies assumption (5.105) with $m = 4$ and assume a_{ij} satisfy the ellipticity condition (5.8) and $F \geq 0$.

Then u is constant.

Proof. Consider $P = P(u, \nabla u, \Delta u) = (\Delta u)^2$, so as in the proof of Proposition 5.6.21 we calculate

$$\begin{aligned} P_{x_i x_j} &= 2\Delta u_{x_i} \Delta u_{x_j} + 2\Delta u \Delta u_{x_i x_j} \\ \Rightarrow \sum_{i,j} a_{ij} P_{x_i x_j} &\geq 2\lambda |\nabla \Delta u|^2 + F\Delta u \geq 0 \end{aligned} \tag{5.129}$$

and $P(u, \nabla u, 0) = 0$ with $\mu = 1$, so by Theorem 5.6.20 we obtain $\Delta u \equiv 0$ in $\mathbb{R}^n$ and u is bounded by (5.105), so u is constant. □

Finally, we have a De Giorgi-type property

Proposition 5.6.25. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a smooth and bounded solution of

$$F(u, \nabla u, \nabla^2 u, \nabla^3 u, \nabla^4 u) = 0 \tag{5.130}$$

such that $u_y > 0$ and assume $P = P(u, \Delta u)$ is a $P-$function of (5.130), such that $P_t > 0$
$(P = P(s,t))$ with $\mu = \mu(|\nabla u|)$, $\mu(t) > 0$, $\forall\, t > 0$.

If there exists $x_0 \in \mathbb{R}^2$ such that

$$P(u(x_0), \Delta u(x_0)) = \sup_{\mathbb{R}^n} P(u, \Delta u) < +\infty \tag{5.131}$$

then there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$u(x) = g(ax + by) \quad , \quad \text{for } \ a, b \in \mathbb{R} \tag{5.132}$$

Proof. Arguing as in the proof of Theorem 5.4.13 we obtain that

$$P(u, \Delta u) \equiv c_0 \quad , \quad \text{where } \ c_0 = \sup_{\mathbb{R}^n} P(u, \Delta u) \tag{5.133}$$

since $P_t > 0$ we have

$$\Delta u = f(u) \quad , \quad \text{for some } \ f : \mathbb{R} \to \mathbb{R} \tag{5.134}$$

and u is bounded entire solution of (5.134) such that $u_y > 0$.

Therefore, by Theorem 1.1 in [12], we conclude that

$$u(x) = g(ax + by) \quad , \quad \text{for some } \ g : \mathbb{R} \to \mathbb{R} \tag{5.135}$$

□

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5.7 Appendix A: Some additional examples of $P-$functions and their gradient bounds

(1) The following example can be found in [17]. Let u be a solution of

$$\Delta u = u(k|\nabla u|^2 + \lambda e^{-cu^2})$$
$$\text{and let } P(s,t) = \begin{cases} te^{-ks^2} + \frac{\lambda}{k+c} e^{-s^2(k+c)} & , \ k \neq -c \\ te^{cs^2} - \lambda s^2 & , \ k = -c \end{cases} \tag{5.136}$$

Then $P = P(u, |\nabla u|^2)$ is a $P-$function of (5.136).

Corollary 5.7.1. Let u be an entire solution of

$$\Delta u = u(k|\nabla u|^2 + \lambda e^{-cu^2}) \tag{5.137}$$

that satisfy (5.29).

Then

$$|\nabla u|^2 \leq \begin{cases} -\frac{\lambda}{k+c} e^{-cu^2} & , \text{ if } \lambda(k+c) < 0 \\ \lambda u^2 e^{-cu^2} & , \text{ if } k = -c \text{ and } \lambda \geq 0 \end{cases} \tag{5.138}$$

Proof. By [17], we have that

$$P(s,t) = \begin{cases} te^{-ks^2} + \frac{\lambda}{k+c} e^{-s^2(k+c)} & , \ k \neq -c \\ te^{cs^2} - \lambda s^2 & , \ k = -c \end{cases} \tag{5.139}$$

is a $P-$function of (5.137) with $\mu(t) > 0$, $\forall\, t \geq 0$ and $P(s,0) \leq 0$ in both cases since either $\lambda(k+c) < 0$ or $k = -c$ and $\lambda \geq 0$. Therefore by Theorem 5.3.5 we conclude that $P(u, |\nabla u|^2) \leq 0 \ \ \forall x \in \mathbb{R}^n$ and we obtain the gradient bound (5.138). $\square$

Remark 5.7.26. For $\lambda = 0$, Corollary 5.7.1 says that $|\nabla u| \equiv 0$ and thus u is a constant. That is a Liouville-type result and can also be obtained either as an application of Theorem 5.63 or by Liouville's theorem by setting $v = g(u)$, where $g(z) = \int_0^y e^{-kz^2}dz$ and then $\Delta v = 0$ and v is bounded since u is bounded.

(2) Consider the equation

$$div(|\nabla u|^2\nabla u) = f'(u) \tag{5.140}$$

where f is such that $f \geq 0$ and $f'' \leq 0$.

This is a quasi-linear equation of the form (5.77) with $\Phi(t) = t^2$. In view of Theorem 5.4.10, the assumption $f'' \leq 0$ is reasonable since the entire solutions would be constant in the case where $f'' \geq 0$.

By Lemma 5.2.4 we have that

$$P(u, |\nabla u|^2) = |\nabla u|^2 - \frac{6}{\lambda}f(u) \tag{5.141}$$

is a $P-$function of (5.140), where λ is the ellipticity constant.

Indeed, we calculate J in (5.16) with $A(t) = t$, $B(s) = -\frac{6}{\lambda}f(u)$

$$\begin{aligned} J &= (\lambda B''(u) + 2f''(u))|\nabla u|^2 + B'(u)\sum_{i,j,k}\frac{\partial a_{ij}}{\partial \sigma_k}(\nabla u)u_{x_ix_j}u_{x_k} \\ &\quad + B'(u)\sum_{i,j} a_{ij}(\nabla u)u_{x_ix_j} + \frac{\lambda}{2}(B'(u))^2 \\ &\geq -4f''(u)|\nabla u|^2 - \frac{6}{\lambda}f'(u)\sum_{i,j,k}\frac{\partial a_{ij}}{\partial \sigma_k}(\nabla u)u_{x_ix_j}u_{x_k} - \frac{6}{\lambda}(f'(u))^2 + \frac{18}{\lambda}(f'(u))^2 \end{aligned} \tag{5.142}$$

where $a_{ij} = 4\sigma_i\sigma_j + 2|\sigma|^2\delta_{ij}$ as defined in (5.78).

Therefore we have

$$\begin{aligned} &\frac{\partial a_{ij}}{\partial \sigma_k} = 4\sigma_i\delta_{jk} + 4\sigma_j\delta_{ik} + 4\sigma_k\delta_{ij} \\ \Rightarrow &\sum_{i,j,k}\frac{\partial a_{ij}}{\partial \sigma_k}(\nabla u)u_{x_k}u_{x_ix_j} = 2f'(u) \end{aligned}$$

So,

$$J \geq -4f''(u)|\nabla u|^2 \geq 0$$

since $f'' \leq 0$.

In addition, $P(u,0) = -\frac{6}{\lambda}f(u) \leq 0$, since f is non negative. Thus, by Theorem 5.3.5 we obtain the bound

Corollary 5.7.1. Let u be an entire solution of

$$div(|\nabla u|^2 \nabla u) = f'(u) \tag{5.143}$$

that satisfy (5.29), where f is such that $f \geq 0$ and $f'' \leq 0$.

Then

$$|\nabla u|^2 \leq \frac{6}{\lambda} f(u) \tag{5.144}$$

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Chapter 6

On the Stability and Convergence of Physics Informed Neural Networks

Abstract

Physics Informed Neural Networks is a numerical method which uses neural networks to approximate solutions of partial differential equations. It has received a lot of attention and is currently used in numerous physical and engineering problems. The mathematical understanding of these methods is limited, and in particular, it seems that, a consistent notion of stability is missing. Towards addressing this issue we consider model problems of partial differential equations, namely linear elliptic and parabolic PDEs. We consider problems with different stability properties, and problems with time discrete training. Motivated by tools of nonlinear calculus of variations we systematically show that coercivity of the energies and associated compactness provide the right framework for stability. For time discrete training we show that if these properties fail to hold then methods may become unstable. Furthermore, using tools of $\Gamma-$convergence we provide new convergence results for weak solutions by only requiring that the neural network spaces are chosen to have suitable approximation properties.

6.1 Introduction

6.1.1 PDEs and Neural Networks

In this work we consider model problems of partial differential equations (PDEs) approximated by deep neural learning (DNN) algorithms. In particular we focus on linear elliptic and parabolic PDEs and Physics Informed Neural Networks, i.e., algorithms where the discretisation is based on the minimisation of the L^2 norm of the residual over a set of neural networks with a given architecture. Standard tools of numerical analysis assessing the quality and performance of an algorithm are based on the notions of stability and approximability. Typically, in problems arising in scientific applications another important algorithmic characteristic is the preservation of key qualitative properties of the simulating system at the discrete level. In important classes of problems, stability and structural consistency are often linked. Our aim is to introduce a novel notion of stability for the above DNN algorithms approximating solutions of PDEs. In addition, we show convergence provided that the set of DNNs has the right approximability properties and the training of the algorithm produces stable approximations.

In the area of machine learning for models described by partial differential equations, at present, there is intense activity at multiple fronts: developing new methods for solving differential equations using neural networks, designing special neural architectures to approximate families of differential operators (operator learning), combination of statistical and machine learning techniques for related problems in uncertainty quantification and statistical functional inference. Despite the progress at all these problems in the last years, basic mathematical, and hence algorithmical, understanding is still under development.

Partial Differential Equations (PDEs) has been proven an area of very important impact in science and engineering, not only because many physical models are described by PDEs, but crucially, methods and techniques developed in this field contributed to the scientific development in several areas where very few scientists would have guessed as possible. Numerical solution of PDEs utilising neural networks is at an early stage and has received a lot of attention. Such methods have significantly different characteristics compared to more traditional methods, and have been proved quite effective, e.g., in solving problems in high-dimensions, or when methods combining statistical approaches and PDEs are needed. Physics Informed Neural Networks is one of the most successful numerical methods which uses neural networks to approximate solutions of PDEs, see e.g., [39], [33]. Residual based methods were considered in [29], [6], [40], [46] and their references. Other neural network methods for differential equations and related problems include, for example, [41], [18], [27], [48], [12], [20], [23]. The term *Physics Informed Neural Networks* was introduced in the highly influential paper [39]. It was then used extensively in numerous physical and engineering problems; for a broader perspective of the related methodologies and the importance of the NN methods for scientific applications, see e.g., [26]. Despite progress at some fronts, see [46], [3], [44], [45], [35, 36], the mathematical understanding of these methods is limited. In particular, it seems that, a consistent notion of stability is missing. Stability is an essential tool, in a priori error analysis and convergence of the algorithms, [30]. It provides valuable information for fixed values of the discretisation parameters, i.e., in the pre-asymptotic regime, and it is well known that unstable methods have poor algorithmic performance. On the other hand, stability is a problem dependent notion and not always easy to identify. Towards addressing this issue we consider model problems of partial differential equations, namely linear elliptic and parabolic PDEs. We consider PDEs with different stability properties, and parabolic problems with time discrete training. Since, apparently, the training procedure influences the behaviour of the method in an essential manner, but, on the other hand, complicates the analysis considerably, we have chosen as a first step in this work to consider time discrete only training. Motivated by tools of nonlinear calculus of variations we systematically show that coercivity of the energies and associated compactness provide the right framework for stability. For time discrete training we show that if these properties fail to hold then methods become unstable and it seems that they do not converge. Furthermore, using tools of $\Gamma-$convergence we provide new convergence results for weak solutions by only requiring that the neural network spaces are chosen to have suitable approximation properties.

6.1.2 Model problems and their Machine Learning approximations

In this work we consider linear elliptic and parabolic PDEs. To fix notation, we consider simple boundary value problems of the form,

$$\begin{cases} L\,u = f & \text{in } \mathit{\Omega} \\ u = 0 & \text{on } \partial\mathit{\Omega} \end{cases} \tag{6.1}$$

where $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$, Ω is an open, bounded set with smooth enough boundary, $f \in L^2(\Omega)$ and L a self-adjoint elliptic operator of the form

$$Lu := -\sum_{1\le i,j\le d} \left(a_{ij}u_{x_i}\right)_{x_j} + cu$$

$$\text{where } \sum_{i,j} a_{ij}(x)\xi_i\xi_j \ge \theta|\xi|^2 \text{ for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^n, \quad \text{for some } \theta > 0 \tag{6.2}$$

also, $a_{ij} = a_{ji} \in C^1(\overline{\Omega}), b_i, c \in L^\infty(\Omega)$ and hence bounded in $\overline{\Omega}$. Further assumptions on L will be discussed in the next sections. Dirichlet boundary conditions were selected for simplicity. The results of this work can be extended to other boundary conditions with appropriate technical modifications.

We shall study the corresponding parabolic problem as well. We use the compact notation $\Omega_T = \Omega \times (0, T]$, $\partial\Omega_T = \partial\Omega \times (0, T]$ for some fixed time $T > 0$. We consider the initial-boundary value problem

$$\begin{cases} u_t + Lu = f, & \text{in } \Omega_T, \\ u = 0, & \text{on } \partial\Omega \times (0, T], \\ u = u^0, & \text{in } \Omega\,, \end{cases} \tag{6.3}$$

where $f \in L^2(\Omega_T)$, $u^0 \in H^1_0(\Omega)$ and L is as in (6.2). In the sequel we shall use the compact operator notation $\mathscr{L}$ for either $u_t + Lu$ or Lu for the parabolic or the elliptic case correspondingly. The associated energies used will be the L^2-residuals

$$\mathcal{E}(v) = \int_{\Omega_D} |\mathscr{L}v - f|^2 \mathrm{d}\overline{x} + \mu \int_\Omega |v - u^0|^2\, \mathrm{d}x + \tau \int_{\partial\Omega_T} |v|^2\, \mathrm{d}\overline{S} \tag{6.4}$$

defined over smooth enough functions and domains Ω_D being Ω_T or Ω (with measures $d\overline{x}$) for the parabolic or the elliptic case correspondingly. Clearly, the coefficient $\mu \geq 0$ of the initial condition is set to zero in the elliptic case.

It is typical to consider regularised versions of $\mathcal{E}(v)$ as well. Such functionals have the form

$$\mathcal{E}_{reg}(v) = \mathcal{E}(v) + \lambda \mathcal{J}(v)\,, \tag{6.5}$$

where the regularisation parameter $\lambda = \lambda_{reg} > 0$ is in principle small and $\mathcal{J}(v)$ is an appropriate functional (often a power of a semi-norm) reflecting the qualitative properties of the regularisation. The formulation of the method extends naturally to nonlinear versions of the generic operator $\mathscr{L}v - f$, whereby in principle both $\mathscr{L}$ and f might depend on v.

6.1.3 Discrete Spaces generated by Neural Networks

We consider functions u_θ defined through neural networks. Notice that the structure described is indicative and it is presented in order of fix ideas. Our results do not depend on particular neural network architectures but only on their approximation ability. A deep neural network maps every point $\overline{x} \in \mathit{\Omega}_D$ to a number $u_\theta(\overline{x}) \in \mathbb{R}$, through

$$u_\theta(\overline{x}) = C_L \circ \sigma \circ C_{L-1} \cdots \circ \sigma \circ C_1(\overline{x}) \quad \forall \overline{x} \in \mathit{\Omega}_D. \tag{6.6}$$

The process

$$\mathcal{C}_L := C_L \circ \sigma \circ C_{L-1} \cdots \circ \sigma \circ C_1 \tag{6.7}$$

is in principle a map $\mathcal{C}_L : \mathbb{R}^m \to \mathbb{R}^{m'}$; in our particular application, $m = d$ (elliptic case) or $m = d + 1$ (parabolic case) and $m' = 1$. The map $\mathcal{C}_L$ is a neural network with L layers and activation function σ. Notice that to define $u_\theta(\overline{x})$ for all $\overline{x} \in \Omega_D$ we use the same $\mathcal{C}_L$, thus $u_\theta(\cdot) = \mathcal{C}_L(\cdot)$. Any such map $\mathcal{C}_L$ is characterised by the intermediate (hidden) layers C_k, which are affine maps of the form

$$C_k y = W_k y + b_k, \qquad \text{where } W_k \in \mathbb{R}^{d_{k+1} \times d_k}, b_k \in \mathbb{R}^{d_{k+1}}. \tag{6.8}$$

Here the dimensions d_k may vary with each layer k and $\sigma(y)$ denotes the vector with the same number of components as y, where $\sigma(y)_i = \sigma(y_i)$. The index θ represents collectively all the parameters of the network $\mathcal{C}_L$, namely W_k, b_k, $k = 1, \dots, L$. The set of all networks $\mathcal{C}_L$ with a given structure (fixed $L, d_k, k = 1, \dots, L$) of the form (6.6), (6.8) is called $\mathcal{N}$. The total dimension (total number of degrees of freedom) of $\mathcal{N}$, is $\dim \mathcal{N} = \sum_{k=1}^{L} d_{k+1}(d_k + 1)$. We now define the space of functions

$$V_\mathcal{N} = \{u_\theta : \Omega_D \to \mathbb{R}, \text{ where } u_\theta(\overline{x}) = \mathcal{C}_L(\overline{x}), \text{ for some } \mathcal{C}_L \in \mathcal{N}\}. \tag{6.9}$$

It is important to observe that $V_\mathcal{N}$ is not a linear space. We denote by

$$\Theta = \{\theta \, : u_\theta \in V_\mathcal{N}\}. \tag{6.10}$$

Clearly, Θ is a linear subspace of $\mathbb{R}^{\dim \mathcal{N}}$.

6.1.4 Discrete minimisation on $V_\mathcal{N}$

Physics Informed Neural networks are based on the minimisation of residual-type functionals of the form (6.5) over the discrete set $V_\mathcal{N}$:

Definition 6.1.7. Assume that the problem

$$\min_{v \in V_\mathcal{N}} \mathcal{E}(v) \tag{6.11}$$

has a solution $v^\star \in V_\mathcal{N}$. We call $v^\star$ a deep-$V_\mathcal{N}$ minimiser of $\mathcal{E}$.

A key difficulty in studying this problem lies on the fact that $V_\mathcal{N}$ is not a linear space. Computationally, this problem can be equivalently formulated as a minimisation problem in $\mathbb{R}^{\dim \mathcal{N}}$ by considering θ as the parameter vector to be identified through

$$\min_{\theta \in \Theta} \mathcal{E}(u_\theta). \tag{6.12}$$

Notice that although (6.12) is well defined as a discrete minimisation problem, in general, this is non-convex with respect to θ even though the functional $\mathcal{E}(v)$ is convex with respect to v. This is the source of one of the main technical difficulties in machine learning algorithms.

6.1.5 Time discrete Training

To implement such a scheme we shall need computable discrete versions of the energy $\mathcal{E}(u_\theta)$. This can be achieved through different ways. A common way to achieve this is to use appropriate quadrature for integrals over Ω_D (Training through quadrature). Just to fix ideas such a quadrature requires a set K_h of discrete points $z \in K_h$ and corresponding nonnegative weights w_z such that

$$\sum_{z \in K_h} w_z \, g(z) \approx \int_{\Omega_D} g(\overline{x}) \, \mathrm{d}\overline{x}. \tag{6.13}$$

Then one can define the discrete functional

$$\mathcal{E}_{Q,h}(g) = \sum_{z \in K_h} w_z \, |\mathscr{L} v(z) - f(z)|^2 \, . \tag{6.14}$$

In the case of the parabolic problem a similar treatment should be done for the term corresponding to the initial condition $\int_\Omega |v - u^0|^2 dx$. Notice that both deterministic and probabilistic (Monte-Carlo, Quasi-Monte-Carlo) quadrature rules are possible, yielding different final algorithms. In this work we shall not consider in detail the influence of the quadrature (and hence of the training) to the stability and convergence of the algorithms. This requires a much more involved technical analysis and it will be the subject of future research. However, it will be instrumental for studying the notion of stability introduced herein, to consider a hybrid algorithm where quadrature (and discretisation) is applied only to the time variable of the parabolic problem. This approach is instrumental in the design and analysis of time-discrete methods for evolution problems, and we believe that it is quite useful in the present setting.

To apply a quadrature in the time integral only we proceed as follows: Let $0 = t^0 < t^1 < \cdots < t^N = T$ define a partition of $[0, T]$ and $I_n := (t^{n-1}, t^n]$, $k_n := t^n - t^{n-1}$. We shall denote by $v^m(\cdot)$ and $f^m(\cdot)$ the values $v(\cdot, t^m)$ and $f(\cdot, t^m)$. Then we define the discrete in time quadrature by

$$\sum_{n=1}^{N} k_n \, g(t^n) \approx \int_o^T g(t) \, \mathrm{d}t. \tag{6.15}$$

We proceed to define the time-discrete version of the functional (6.5) as follows

$$\mathcal{G}_{k,IE}(v) = \sum_{n=1}^{N} k_n \int_\Omega \Big| \frac{v^n - v^{n-1}}{k_n} + L v^n - f^n \Big|^2 \, \mathrm{d}x + \int_\Omega |v - u^0|^2 \mathrm{d}x \tag{6.16}$$

We shall study the stability and convergence properties of the minimisers of the problems:

$$\min_{v \in V_\mathcal{N}} \mathcal{G}_{k,IE}(v) \, . \tag{6.17}$$

It will be interesting to consider a seemingly similar (from the point of view of quadrature and approximation) discrete functional:

$$\mathcal{G}_{k,EE}(v) = \sum_{n=1}^{N} k_n \int_\Omega \Big|\frac{v^n - v^{n-1}}{k_n} + Lv^{n-1} - f^{n-1}\Big|^2 \,\mathrm{d}x + \sigma \int_\Omega |v - u^0|^2 dx, \quad (6.18)$$

and compare its properties to the functional $\mathcal{G}_{k,IE}$, and the corresponding $V_\mathcal{N}$ minimisers.

6.2 Our results

In this section we discuss our main contributions. Our goal is twofold: to suggest a consistent notion of stability and a corresponding convergence framework for the methods considered.

Equi-Coercivity and Stability.

Equi-Coercivity is a key notion in the $\Gamma-$convergence analysis which drives compactness and the convergence of minimisers of the approximate functionals. Especially, in the case of discrete functionals (denoted below by $\mathcal{E}_\ell$, ℓ stands for a discretisation parameter) stability is a prerequisite for compactness and convergence. Our analysis is driven by two key properties which are roughly stated as follows:

[S1] If energies $\mathcal{E}_\ell$ are uniformly bounded

$$\mathcal{E}_\ell[u_\ell] \le C,$$

then there exists a constant $C_1 > 0$ and $\ell-$dependent norms V_ℓ such that

$$\|u_\ell\|_{V_\ell} \le C_1. \quad (6.19)$$

[S2] Uniformly bounded sequences in $\|u_\ell\|_{V_\ell}$ have convergent subsequences in H,

where H is a normed space (typically a Sobolev space) which depends on the form of the discrete energy considered. Property [S1] requires that $\mathcal{E}_\ell[v_\ell]$ is coercive with respect to (possibly ℓ-dependent) norms (or semi-norms). Further, [S2], implies that, although $\|\cdot\|_{V_\ell}$ are ℓ-dependent, they should be such that, from uniformly bounded sequences in these norms, it is possible to extract convergent subsequences in a weaker topology (induced by the space H).

We argue that these properties provide the right framework for stability. Although, in principle, the use of discrete norms is motivated from a nonlinear theory, [21], [9], [22], in order to focus on ideas rather than on technical tools, we started our study in this work on simple linear problems. To this end, we consider four different problems, where [S1] and [S2] are relevant: Two elliptic problems with distinct regularity properties: namely elliptic operators posed on convex and non-convex Lipschitz domains. In addition, we study linear parabolic problems and their time-discrete only version. The last example highlights that training is a key factor in algorithmic design, since it influences not only the accuracy, but crucially, the stability properties of the algorithm. In fact, we provide evidence that functionals related to time discrete training of the form (6.87), which fail to satisfy the stability criteria [S1] and [S2], produce approximations with unstable behaviour.

Section 3 is devoted to elliptic problems and Section 4 to parabolic. In Section 3.1 and Section 3.2 we consider the same elliptic operator but posed on convex and non-convex Lipschitz domains respectively. It is interesting to compare the corresponding stability results, Propositions 6.3.2 and 6.3.6 where in the second case the stability is in a weaker norm as expected. Similar considerations apply to the continuous formulation (without training) of the parabolic problem, Proposition 6.4.9. Here an interesting feature appears to be that a maximal regularity estimate is required for the parabolic problem. In the case of time-discrete training, Proposition 6.4.12, [S1] holds with an $\ell-$ dependent norm. Again it is interesting to observe that a discrete maximal regularity estimate is required in the proof of Proposition 6.4.12. Although we do not use previous results, it is interesting to compare to [28], [31], [2].

Let us mention that for simplicity in the exposition we assume that the discrete energies are defined on spaces where homogenous Dirichlet conditions are satisfied. This is done only to highlight the ideas presented herein without extra technical complications. It is clear that all results can be extended when these conditions are imposed weakly through the loss functional. It is interesting to note, that in certain cases, however, the choice of the form of the boundary terms in the discrete functional might affect how strong is the norm of the underlined space H in [S1], [S2], see Remark 6.3.3.

Convergence – $\liminf - \limsup$ *framework.*

We show convergence of the discrete minimisers to the solutions of the underlined PDE under minimal regularity assumptions. For certain cases, see Theorem 6.3.4 for example, it is possible by utilising the stability of the energies and the linearity of the problem, to show direct bounds for the errors and convergence. This is in particular doable in the absence of training. In the case of regularised fuctionals, or when time discrete training is considered one has to use the liminf-limsup framework of De Giorgi, see Section 2.3.4 of [14], and e.g., [10], used in the $\Gamma-$convergence of functionals arising in non-linear PDEs, see Theorems

6.3.5, 6.3.8, (regularised functionals) and Theorem 6.4.13 (time-discrete training). These results show that stable functionals in the sense of [S1], [S2], yield neural network approximations converging to the weak solutions of the PDEs, under no extra assumptions. This analytical framework combined with the stability notion introduced above provides a consistent and flexible toolbox, for analysing neural network approximations to PDEs. It can be extended to various other, possibly nonlinear, problems. Furthermore, it provides a clear connection to PDE well posedness and discrete stability when training is taking place.

Previous works.

Previous works on the analysis of methods based on residual minimisation over neural network spaces for PDEs include [46], [3], [44], [45], [35], [25], [36]. In [46] convergence was established for smooth enough classical solutions of a class of nonlinear parabolic PDEs, without considering training of the functional. Convergence results, under assumptions on the discrete minimisers or the NN space, when Monte-Carlo training was considered, were derived in [44], [45], [25]. In addition, in [45], continuous stability of certain linear operators is used in the analysis. The results of [3], [35], [36] were based on estimates where the bounds are dependent on the discrete minimisers and their derivatives. These bounds imply convergence only under the assumption that these functions are uniformly bounded in appropriate Sobolev norms. The results in [25] with deterministic training, are related, in the sense that they are applicable to NN spaces where by construction high-order derivatives are uniformly bounded in appropriate norms. Conceptually related is the recent work on Variational PINNs (the residuals are evaluated in a weak-variational sense), [8], where the role of quadrature was proven crucial in the analysis of the method.

As mentioned, part of the analysis is based on Γ-convergence arguments. Γ-convergence is a very natural framework which is used in nonlinear energy minimisation. In [37] Γ-convergence was used in the analysis of deep Ritz methods without training. In the recent work [32], the $\liminf - \limsup$ framework was used in general machine learning algorithms with probabilistic training to derive convergence results for global and local discrete minimisers. For recent applications to computational methods where the discrete energies are rather involved, see [5], [21], [9], [22]. It seems that these analytical tools coming from nonlinear PDEs provide very useful insight in the present neural network setting, while standard linear theory arguments are rarely applicable due to the nonlinear character of the spaces $V_{\mathcal{N}}$.

6.3 Elliptic problems

We consider the problem

$$Lu = f \tag{6.20}$$

where $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$, Ω is an open, bounded set with Lipschitz boundary, $f \in L^2(\Omega)$ and L the elliptic operator as in (6.2).

For smooth enough v now define the energy as follows

$$\mathcal{E}(v) = \int_\Omega |Lv - f|^2 \,\mathrm{d}x + \int_{\partial\Omega} |v|^2 \,\mathrm{d}x \tag{6.21}$$

Define now the linear space $\mathcal{H}_L = \{v \in H^1(\Omega) : \; Lv \in L^2(\Omega)\,\}$. We consider now the minimisation problem:

$$\min_{u \in \mathcal{H}_L} \mathcal{E}(u)\,. \tag{6.22}$$

We show next that the (unique) solution of (6.22) is the weak solution of the PDE (6.20). The Euler-Lagrange equations for (6.22) are

$$\int_\Omega (Lu - f)\, Lv \;\mathrm{d}x + \int_{\partial\Omega} u\, v \,\mathrm{d}x = 0 \qquad \text{for all } v \in \mathcal{H}_L\,. \tag{6.23}$$

Let $w \in H^1_0(\Omega)$ be given but arbitrary. Consider $\overline{v}$ to be the solution of $L\overline{v} = w$ with zero boundary conditions. Hence $\overline{v} \in H^1_0(\Omega)$. Then there holds,

$$\int_\Omega (Lu - f)\, w \;\mathrm{d}x + \int_{\partial\Omega} u\, \overline{v} \,\mathrm{d}x = \int_\Omega (Lu - f)\, w \;\mathrm{d}x = 0 \qquad \text{for all } w \in H^1_0(\Omega)\,. \tag{6.24}$$

Hence, $Lu = f$ in the sense of distributions. We turn now to (6.23) and observe that $\int_{\partial\Omega} u\, v \,\mathrm{d}x = 0$ for all $v \in \mathcal{H}_L$. We conclude therefore that $u = 0$ on $\partial\Omega$ and the claim is proved.

In this section we assume that if we select the networks appropriately, as we increase their complexity we may approximate any w in H^2 . To this end, we select a sequence of spaces $V_\mathcal{N}$ as follows: for each $\ell \in \mathbb{N}$ we correspond a DNN space $V_\mathcal{N}$, which is denoted by V_ℓ with the following property: For each $w \in H^2_0(\Omega)$ there exists a $w_\ell \in V_\ell$ such that,

$$\|w_\ell - w\|_{H^2(\Omega)} \leq \; \beta_\ell\,(w), \qquad \text{and} \;\; \beta_\ell\,(w) \to 0, \;\; \ell \to \infty\,. \tag{6.25}$$

If in addition, $w \in H^m(\Omega) \cap H^2_0(\Omega)$ is in higher order Sobolev space then

$$\|w_\ell - w\|_{H^2(\Omega)} \leq \; \tilde{\beta}_\ell\, \|w\|_{H^m(\Omega)}, \qquad \text{and} \;\; \tilde{\beta}_\ell \to 0, \;\; \ell \to \infty\,. \tag{6.26}$$

We do not need specific rates for $\tilde{\beta}_\ell$, but only the fact that the right-hand side of (6.26) has an explicit dependence of Sobolev norms of w. This assumption is a reasonable one in view of the available approximation results of neural network spaces, see for example [48], [13, 24, 43, 16, 7], and their references.

Remark 6.3.1. Due to higher regularity needed by the loss functional one has to use smooth enough activation functions, such as tanh or ReLU^k, that is, $\sigma(y) = (\max\{0, y\})^k$, see e.g., [48], [15]. In general, the available results so far do not provide enough information on specific architectures required to achieve specific bounds with rates. Since the issue of the approximation properties is an important but independent problem, we have chosen to require minimal assumptions which can be used to prove convergence.

6.3.1 Convex domains

Next, we study first the case where elliptic regularity bounds hold. Consider the sequence of energies

$$\mathcal{E}_\ell(u_\ell) = \begin{cases} \mathcal{E}(u_\ell) & , \quad u_\ell \in V_\ell \cap H_0^2(\Omega) \\ +\infty & , \quad \text{otherwise} \end{cases} \tag{6.27}$$

where V_ℓ are chosen to satisfy (6.25).

Stability

Now we have equicoercivity of $\mathcal{E}_\ell$ as a corollary of the following result.

Proposition 6.3.2 (Stability/Equi-coercivity)**.** Assume that Ω is convex. Let (u_ℓ) be a sequence of functions in V_ℓ such that for a constant $C > 0$ independent of ℓ, it holds that

$$\mathcal{E}_\ell(u_\ell) \le C. \tag{6.28}$$

Then there exists a constant $C_1 > 0$ such that

$$||u_\ell||_{H^2(\Omega)} \le C_1\,. \tag{6.29}$$

Proof. Since $\mathcal{E}_\ell(u_\ell) \le C$, from the definition of $\mathcal{E}_\ell$, it holds that $\mathcal{E}(u_\ell) \le C$. We have that

$$\mathcal{E}(u) = \int_\Omega (|Lu|^2 - 2f\, Lu + |f|^2)\, \mathrm{d}x \le C\,. \tag{6.30}$$

From Hölder's inequality we have, since $f \in L^2(\Omega)$,

$$||Lu||_{L^2(\Omega)} \leq C_1 \,. \tag{6.31}$$

Finally, since $u|_{\partial\Omega} = 0$, by the global elliptic regularity in H^2 theorem (see Theorem 4, p.334 in [19]) we have

$$||u||_{H^2(\Omega)} \leq C_2(||Lu||_{L^2(\Omega)} + ||u||_{L^2(\Omega)}) \tag{6.32}$$

where C_2 depends only on Ω and the coefficients of L. Now since $0 \notin \Sigma$ (Σ is the spectrum of L), by Theorem 6 in [19] (p.324), we have

$$||u||_{L^2(\Omega)} \leq C_3||Lu||_{L^2(\Omega)} \tag{6.33}$$

where C_3 depends only on Ω and the coefficients of L. Thus by (6.31), (6.32) and (6.33) we conclude

$$||u||_{H^2(\Omega)} \leq \tilde{C} \,. \tag{6.34}$$

□

Remark 6.3.3 (Boundary loss)**.** As mentioned in the introduction, in order to avoid the involved technical issues related to boundary conditions we have chosen to assume throughout that homogenous Dirichlet conditions are satisfied. It is evident that that our results are valid when the boundary conditions are imposed weakly through the discrete loss functional under appropriate technical modifications. In the case where the loss is

$$\int_\Omega |Lv - f|^2 \mathrm{d}x + \tau \int_{\partial\Omega} |v|^2 \,\mathrm{d}S \tag{6.35}$$

the assumption $\mathcal{E}_\ell(u_\ell) \leq C$ provides control of the $\|v\|_{L^2(\partial\Omega)}$ which is not enough to guarantee that elliptic regularity estimates will hold up to the boundary, see e.g., [11], [42], for a detailed discussion of subtle issues related to the effect of the boundary conditions on the regularity. Since the choice of the loss is at our disposal during the algorithm design, it will be interesting to consider more balanced choices of the boundary loss, depending on the regularity of the boundary. This is beyond the scope of the present work. Alternatively, one might prefer to use the framework of [47] to exactly satisfy the boundary conditions. As noted in this paper, there are instances where the boundary loss of (6.35) is rather weak to capture accurately the boundary behaviour of the approximations. The above observations is yet another indication that our stability framework is consistent and able to highlight possible imbalances at the algorithmic design level.

Convergence of the minimisers

In this subsection, we discuss the convergence properties of the discrete minimisers. Given the regularity properties of the elliptic problem and in the absence of training, it is possible to show the following convergence result.

Theorem 6.3.4 (Estimate in H^2)**.** Let $\mathcal{E}_\ell$ be the energy functionals defined in (6.27) and let (u_ℓ), $u_\ell \in V_\ell$, be a sequence of minimisers of $\mathcal{E}_\ell$. Then, if u is the exact solution of (6.1),

$$\|u - u_\ell\|_{H^2(\Omega)} \leq C \inf_{\varphi \in V_\ell} \|u - \varphi\|_{H^2(\Omega)} . \tag{6.36}$$

and furthermore,

$$u_\ell \to u, \quad \text{in } H^2(\Omega) , \qquad \ell \to \infty . \tag{6.37}$$

Proof. Let $u \in H^2_0(\Omega)$ be the unique solution of (6.20). Consider the sequence of minimisers (u_ℓ) . Obviously,

$$\mathcal{E}_\ell(u_\ell) \leq \mathcal{E}_\ell(v_\ell), \qquad \text{for all } v_\ell \in V_\ell .$$

Then,

$$\mathcal{E}_\ell(u_\ell) = \int_\Omega |Lu_\ell - f|^2 = \int_\Omega |L(u_\ell - u)|^2 \geq \beta \|u - u_\ell\|^2_{H^2(\Omega)}, \tag{6.38}$$

by Proposition 6.3.2, which proves the first claim. For the second, let $u \in H^2_0(\Omega)$ be the unique solution of (6.20). Consider the sequence of minimisers (u_ℓ) . Obviously,

$$\mathcal{E}_\ell(u_\ell) \leq \mathcal{E}_\ell(v_\ell), \qquad \text{for all } v_\ell \in V_\ell .$$

In particular,

$$\mathcal{E}_\ell(u_\ell) \leq \mathcal{E}_\ell(\tilde{u}_\ell),$$

where $\tilde{u}_\ell$ is the recovery sequence corresponding to u by assumption (6.25). Then $\tilde{u}_\ell \to u$ in $H^2(\Omega)$ and

$$\mathcal{E}_\ell(\tilde{u}_\ell) = ||L\tilde{u}_\ell - f||^2_{L^2(\Omega)} = ||L(\tilde{u}_\ell - u)||^2_{L^2(\Omega)} , \tag{6.39}$$

and the proof is complete in view of (6.38). □

In the present smooth setting, the above proof hinges on the fact that $\mathcal{E}(u) = 0$ and on the linearity of the problem. In the case of regularised functional

$$\mathcal{E}_{reg}(v) = \mathcal{E}(v) + \lambda \mathcal{J}(v) , \tag{6.40}$$

the proof is more involved. We need certain natural assumptions on the functional $\mathcal{J}(v)$ to conclude the convergence. We shall work with convex functionals $\mathcal{J}(v)$ that are $\mathcal{H}$ consistent, i.e., they satisfy the properties:

$$\begin{aligned} &(i) \quad \mathcal{J}(v) \geq 0, \\ &(ii) \quad \mathcal{J}(v) \leq \liminf_{\ell\to\infty} \mathcal{J}(v_\ell) \text{ for all weakly convergent sequences } v_\ell \rightharpoonup v \in \mathcal{H}, \\ &(iii) \quad \mathcal{J}(w) = \lim_{\ell\to\infty} \mathcal{J}(w_\ell) \text{ for all convergent sequences } w_\ell \to w \in \mathcal{H}, \end{aligned} \tag{6.41}$$

where $\mathcal{H}$ is an appropriate Sobolev (sub)space which will be specified in each statement.

The proof of the next theorem is very similar to the (more complicated) proof of the Theorem 6.3.8 and it is omitted.

Theorem 6.3.5 (Convergence for the regularised functional)**.** Let $\mathcal{E}_{reg}$, $\mathcal{E}_{reg,\ell}$ be the energy functionals defined in (6.40) and

$$\mathcal{E}_{reg,\ell}(u_\ell) = \begin{cases} \mathcal{E}_{reg}(u_\ell), & u_\ell \in V_\ell \cap H_0^2(\Omega) \\ +\infty, & \text{otherwise}. \end{cases} \tag{6.42}$$

Assume that the convex functional $\mathcal{J}(v)$ is $H^2(\Omega)$ consistent. Let (u_ℓ), $u_\ell \in V_\ell$, be a sequence of minimisers of $\mathcal{E}_\ell$, i.e.

$$\mathcal{E}_{reg,\ell}(u_\ell) = \inf_{v_\ell \in V^\ell} \mathcal{E}_{reg,\ell}(v_\ell)\,. \tag{6.43}$$

Then,

$$u_\ell \to u^{(\lambda)}, \quad \text{in } H^1(\Omega)\,, \qquad \ell \to \infty\,, \tag{6.44}$$

where $u^{(\lambda)}$ is the exact solution of the regularised problem

$$\mathcal{E}_{reg}(u^{(\lambda)}) = \min_{v \in H_0^2(\Omega)} \mathcal{E}_{reg}(v)\,. \tag{6.45}$$

Proof. We assume there is a sequence, still denoted by v_ℓ, such that $\mathcal{E}_\ell(v_\ell) \leq C$ uniformly in ℓ, otherwise $\mathcal{E}(u) \leq \liminf_{\ell\to\infty} \mathcal{E}_\ell(v_\ell) = +\infty$. From Proposition 6.3.2 the uniform bound $\mathcal{E}_\ell(v_\ell) \leq C$ implies that $\|v_\ell\|_{H^2(\Omega)}$ are uniformly bounded. Since $H^2(\Omega)$ is reflexive Banach space, therefore weakly compact, we conclude that there exists a $v \in H^2(\Omega)$

$$\nabla^2 v_\ell \rightharpoonup \nabla^2 v \ \text{ weakly in } \ L^2(\Omega) \tag{6.46}$$

hence $Lv_\ell - f \rightharpoonup Lv - f$. The convexity of $\int_\Omega |Lu_\ell - f|^2$ implies weak lower semicontinuity, that is

$$\int_\Omega |Lv - f|^2 \leq \liminf_{\ell\to\infty} \int_\Omega |Lv_\ell - f|^2 \tag{6.47}$$

and therefore we conclude that $\mathcal{E}(v) \leq \liminf_{\ell\to\infty} \mathcal{E}_\ell(v_\ell)$ for each such sequence (v_ℓ).

We will now prove the limsup inequality. And in fact we will show, $\mathcal{E}(w) = \lim_{\ell\to\infty} \mathcal{E}_\ell(w_\ell)$ for a recovery sequence (w_ℓ). Let $w \in H_0^2(\Omega)$ arbitrary. By (6.25), there exists $w_\ell \in V_\ell$ such that $w_\ell \to u$ in $H^2(\Omega)$ (and in particular $w_\ell \to w$ in $H^1(\Omega)$). In particular, we have $w_\ell \to w$ in $H^2(\Omega)$, so $a_{ij}\partial_{x_ix_j} w_\ell \to a_{ij}\partial_{x_ix_j} w$ in $L^2(\Omega)$ and thus, $Lw_\ell - f \to Lw - f$ in $L^2(\Omega)$, which gives

$$||Lw_\ell - f||_{L^2(\Omega)} \to ||Lw - f||_{L^2(\Omega)}\,, \tag{6.48}$$

and

$$\mathcal{E}_\ell(w_\ell) \to \mathcal{E}(w). \tag{6.49}$$

Next, let $u \in H_0^2(\Omega)$ be the unique solution of (6.20). Consider the sequence of minimisers (u_ℓ). Obviously,

$$\mathcal{E}_\ell(u_\ell) \leq \mathcal{E}_\ell(v_\ell), \qquad \text{for all } v_\ell \in V_\ell\,.$$

In particular,

$$\mathcal{E}_\ell(u_\ell) \leq \mathcal{E}_\ell(\tilde{u}_\ell),$$

where $\tilde{u}_\ell$ is the recovery sequence corresponding to u by assumption (6.25) . Thus the discrete energies are uniformly bounded. Then the stability result, Proposition 6.3.2, implies that

$$\|u_\ell\|_{H^2(\Omega)} < C, \tag{6.50}$$

uniformly. By the Rellich-Kondrachov theorem, [19], and the lim inf argument above, there exists $u \in H_0^2(\Omega)$ such that $u_h \to u$ in $H^1(\Omega)$ up to a subsequence not re-labeled here. Next we show that u is a global minimiser of $\mathcal{E}$. We combine the lim inf and lim sup inequalities as follows: Let $w \in H_0^2(\Omega)$, then the lim sup inequality implies the existence of $w_\ell \in V_\ell$ such that $||Lw_\ell - f||_{L^2(\Omega)} \to ||Lw - f||_{L^2(\Omega)}$, Therefore, since $u_\ell \to u$ in $H^1(\Omega)$ the lim inf inequality and the fact that u_ℓ are minimisers of the $\mathcal{E}_\ell$, imply that

$$\mathcal{E}(u) \leq \liminf_{\ell\to\infty} \mathcal{E}_\ell(u_\ell) \leq \limsup_{\ell\to\infty} \mathcal{E}_\ell(u_\ell) \leq \limsup_{\ell\to\infty} \mathcal{E}_\ell(w_\ell) = \mathcal{E}(w), \tag{6.51}$$

for all $wH_0^2(\Omega)$. Therefore u is a minimiser of $\mathcal{E}$, and hence the unique solution of (6.20). □

6.3.2 Non-convex Lipschitz domains

In this subsection we discuss the case on non-convex Lipschitz domains, i.e., elliptic regularity bounds are no longer valid, and solutions might form singularities and do not belong in general to $H^2(\Omega)$. We will see that the stability notion discussed in [S1] and [S2] is still relevant but in a weaker topology than in the previous case.

In the analysis below we shall use the bilinear form associated to the elliptic operator L, denoted $B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$. In particular,

$$B(u,v) = \int_\Omega \Big(\sum_{i,j=1}^{n} a_{ij} u_{x_i} v_{x_j} + cuv \Big)\, \mathrm{d}x\,. \tag{6.52}$$

In the sequel, we shall assume that the coefficients a_{ij}, c are smooth enough and satisfy the required positivity properties for our purposes. We have the following stability result:

Proposition 6.3.6. The functional $\mathcal{E}$ defined in (6.5) is stable with respect to the H^1-norm: Let (u_ℓ) be a sequence of functions in V_ℓ such that for a constant $C > 0$ independent of ℓ, it holds that

$$\mathcal{E}_\ell(u_\ell) \le C. \tag{6.53}$$

Then there exists a constant $C_1 > 0$ such that

$$\|u_\ell\|_{H^1(\Omega)} \le C_1\,. \tag{6.54}$$

Proof. We show that, if $\mathcal{E}_\ell(v) \le C$ for some $C > 0$, then $\|v\|_{H^1(\Omega)} \le \tilde{C}$ for some $\tilde{C} > 0$. Indeed the positivity properties of the coefficients imply, for any $v \in H_0^1(\Omega)$,

$$\theta \|\nabla v\|^2_{L^2(\Omega)} \le B(v,v)\,. \tag{6.55}$$

Also, if $Lu \in L^2(\Omega)$,

$$B(v,v) = \int_\Omega vLv\, \mathrm{d}x \le \|v\|_{L^2(\Omega)} \|Lv\|_{L^2(\Omega)}\,, \tag{6.56}$$

and the claim follows by applying Hölder and Poincaré inequalities. ∎

The convergence proof below relies on a crucial lim sup inequality which is proved in the next Theorem 6.3.8.

Theorem 6.3.7 (Convergence in H^1)**.** Let $\mathcal{E}_\ell$ be the energy functionals defined in (6.27) and let (u_ℓ), $u_\ell \in V_\ell$, be a sequence of minimisers of $\mathcal{E}_\ell$, where Ω is a possibly non-convex Lipschitz domain. Then, if u is the exact solution of (6.1),

$$u_\ell \to u, \quad \text{in } H^1(\Omega)\,, \qquad \ell \to \infty\,. \tag{6.57}$$

Proof. Let $u \in \mathcal{H}_L$ be the unique solution of (6.20). Consider the sequence of minimisers (u_ℓ). Obviously,

$$\mathcal{E}_\ell(u_\ell) \le \mathcal{E}_\ell(v_\ell), \qquad \text{for all } v_\ell \in V_\ell\,.$$

By the proof of Proposition 6.3.6, we have, for $c_0 > 0$,

$$\mathcal{E}_\ell(u_\ell) = \int_\Omega |Lu_\ell - f|^2 = \int_\Omega |L(u_\ell - u)|^2 \ge c_0 \|u - u_\ell\|^2_{H^1(\Omega)}\,. \tag{6.58}$$

Furthermore, let $\tilde{u}_\ell$ be the recovery sequence corresponding to u constructed in the proof of Theorem 6.3.8. Since

$$\mathcal{E}_\ell(u_\ell) \le \mathcal{E}_\ell(\tilde{u}_\ell),$$

and

$$\lim_{\ell\to\infty} \mathcal{E}_\ell(\tilde{u}_\ell) = \mathcal{E}(u) = 0,$$

the proof follows. □

Next, we utilise the standard lim inf-lim sup framework of Γ-convergence, to prove that the sequence of discrete minimisers (u_ℓ) of the regularised functionals converges to a global minimiser of the continuous regularised functional.

Theorem 6.3.8 (Convergence of the regularised functionals)**.** Let $\mathcal{E}_{reg}$, $\mathcal{E}_{reg,\ell}$ be the energy functionals defined in (6.40) and (6.42) respectively, where Ω is a possibly non-convex Lipschitz domain. Assume that the convex functional $\mathcal{J}(v)$ is $\mathcal{H}_L$ consistent. Let (u_ℓ), $u_\ell \in V_\ell$, be a sequence of minimisers of $\mathcal{E}_{reg,\ell}$. Then,

$$u_\ell \to u^{(\lambda)}, \quad \text{in } L^2(\Omega), \quad u_\ell \rightharpoonup u^{(\lambda)}\,, \quad \text{in } H^1(\Omega), \qquad \ell \to \infty\,. \tag{6.59}$$

where $u^{(\lambda)}$ is the exact solution of the regularised problem

$$\mathcal{E}_{reg}(u^{(\lambda)}) = \min_{v\in\mathcal{H}_L(\Omega)} \mathcal{E}_{reg}(v)\,. \tag{6.60}$$

Proof. We start with a lim inf inequality: We assume there is a sequence, still denoted by u_ℓ, such that $\mathcal{E}_\ell(u_\ell) \le C$ uniformly in ℓ, otherwise $\mathcal{E}(u) \le \liminf_{\ell\to\infty} \mathcal{E}_\ell(u_\ell) = +\infty$. The above stability result, Proposition 6.3.6, implies that $||u_\ell||_{H^1(\Omega)}$ are uniformly bounded. Therefore, up to subsequences, there exists a $v \in H^1(\Omega)$, such that $u_\ell \rightharpoonup v$ in H^1 and $u_\ell \to u$ in L^2, thus $u_\ell \rightharpoonup u$ in H^1. Also, from the energy bound we have that $||Lu_\ell||_{L^2(\Omega)} \le C$ and therefore $Lu_\ell \rightharpoonup w$. Next we shall show that $w = Lu$. Indeed, we have

$$\lim_{\ell\to\infty} \int_\Omega Lu_\ell \phi \, \mathrm{d}x = \int_\Omega w\phi \, \mathrm{d}x \quad , \; \forall\, \phi \in C_0^\infty(\Omega)\,, \tag{6.61}$$

and

$$\lim_{\ell\to\infty}\int_\Omega Lu_\ell\phi\,\mathrm{d}x=\lim_{\ell\to\infty}B(u_\ell,\phi)=B(u,\phi),\quad \text{since } u_\ell\rightharpoonup u \text{ in } H^1(\Omega)\,,\tag{6.62}$$

hence,

$$B(u,\phi)=\int_\Omega w\phi\,\mathrm{d}x,\tag{6.63}$$

for all test functions. That is, $Lu=w$ weakly. The convexity of $\int_\Omega|Lu_\ell-f|^2$ implies weak lower semicontinuity, that is

$$\int_\Omega|Lv-f|^2\le\liminf_{\ell\to\infty}\int_\Omega|Lv_\ell-f|^2\tag{6.64}$$

and since $\mathcal{J}(v)$ is $\mathcal{H}_L$ consistent, (ii) of (6.41) implies that $\mathcal{E}_{reg}(v)\le\liminf_{\ell\to\infty}\mathcal{E}_{reg,\ell}(v_\ell)$ for each such sequence (v_ℓ).

Let $w\in\mathcal{H}_L$ be arbitrary; we will show the existence of a recovery sequence (w_ℓ), such that $\mathcal{E}(w)=\lim_{\ell\to\infty}\mathcal{E}_\ell(w_\ell)$. For each $\delta>0$ we can select a smooth enough mollifier $w_\delta\in H^2_0(\Omega)\cap C^m_0(\Omega)$, $m>2$, such that

$$\begin{aligned}&\|w-w_\delta\|_{H^1(\Omega)}+\|Lw-Lw_\delta\|_{L^2(\Omega)}\lesssim\delta\,,\quad\text{and,}\\&|w_\delta|_{H^s(\Omega)}\lesssim\frac{1}{\delta^s}|w|_{H^1(\Omega)}.\end{aligned}\tag{6.65}$$

For w_δ, (6.26), there exists $w_{\ell,\delta}\in V_\ell$ such that

$$\|w_{\ell,\delta}-w_\delta\|_{H^2(\Omega)}\le\ \tilde\beta_\ell\,\|w_\delta\|_{H^s(\Omega)}\le\ \tilde\beta_\ell\frac{1}{\delta^s}\,\|w\|_{H^1(\Omega)},\qquad\text{and }\ \tilde\beta_\ell\,(w)\to0,\quad\ell\to\infty\,.$$

Choosing δ appropriately as function of $\tilde\beta_\ell$ we can ensure that $w_\ell=w_{\ell,\delta}$ satisfies,

$$||Lw_\ell-f||_{L^2(\Omega)}\to||Lw-f||_{L^2(\Omega)}\,,\tag{6.66}$$

since $\mathcal{J}(v)$ is $\mathcal{H}_L$ consistent, (iii) of (6.41) implies that $\mathcal{J}(w_\ell)\to\mathcal{J}(w)$ and hence

$$\mathcal{E}_{reg,\ell}(w_\ell)\to\mathcal{E}_{reg}(w).\tag{6.67}$$

Next, let $u^{(\lambda)}\in\mathcal{H}_L$ be the unique solution of (6.60) and consider the sequence of the discrete minimisers (u_ℓ). Clearly,

$$\mathcal{E}_{reg,\ell}(u_\ell)\le\mathcal{E}_{reg,\ell}(v_\ell),\qquad\text{for all } v_\ell\in V_\ell\,.$$

In particular, $\mathcal{E}_{reg,\ell}(u_\ell)\le\mathcal{E}_{reg,\ell}(\tilde u_\ell)$, where $\tilde u_\ell$ is the recovery sequence constructed above corresponding to $w=u^{(\lambda)}$. Thus the discrete energies are uniformly bounded. Then the stability result Proposition 6.3.6, implies that

$$\|u_\ell\|_{H^1(\Omega)}<C,\tag{6.68}$$

uniformly. By the Rellich-Kondrachov theorem, [19], and the $\liminf$ argument above, there exists $\tilde{u} \in \mathcal{H}_L$ such that $u_\ell \to u$ in $L^2(\Omega)$ up to a subsequence not re-labeled here. Next we show that $\tilde{u}$ is a global minimiser of $\mathcal{E}_{reg}$. We combine the $\liminf$ and $\limsup$ inequalities as follows: Let $w \in \mathcal{H}_L$, and $w_\ell \in V_\ell$ be its recovery sequence such that $||Lw_\ell - f||_{L^2(\Omega)} \to ||Lw - f||_{L^2(\Omega)}$. Therefore, the $\liminf$ inequality and the fact that u_ℓ are minimisers of the $\mathcal{E}_{reg,\ell}$, imply that

$$\mathcal{E}_{reg}(\tilde{u}) \le \liminf_{\ell\to\infty} \mathcal{E}_{reg,\ell}(u_\ell) \le \limsup_{\ell\to\infty} \mathcal{E}_{reg,\ell}(u_\ell) \le \limsup_{\ell\to\infty} \mathcal{E}_{reg,\ell}(w_\ell) = \mathcal{E}_{reg}(w), \tag{6.69}$$

for all $w \in \mathcal{H}_L$. Therefore $\tilde{u}$ is a minimiser of $\mathcal{E}$, and since $u^{(\lambda)}$ is the unique global minimiser of $\mathcal{E}_{reg}$ on $\mathcal{H}_L$ we have that $\tilde{u} = u^{(\lambda)}$.

□

6.4 Parabolic problems

Let as before $\Omega \subset \mathbb{R}^d$, open, bounded and set $\Omega_T = \Omega \times (0, T]$ for some fixed time $T > 0$. We consider the parabolic problem

$$\begin{cases} u_t + Lu = f, & \text{in } \Omega_T, \\ u = 0, & \text{on } \partial\Omega \times (0, T], \\ u = u^0, & \text{on } \Omega \times \{t = 0\}\,. \end{cases} \tag{6.70}$$

In this section we discuss convergence properties of approximations of (6.70) obtained by minimisation of continuous and time-discrete energy functionals over appropriate sets of neural network functions. We shall assume that Ω is a convex Lipschitz domain. The case of a non-convex domain can be treated with the appropriate modifications.

6.4.1 Exact time integrals

So now we define $\mathcal{G} : H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^2(\Omega)) \to \overline{\mathbb{R}}$ as follows

$$\mathcal{G}(v) = \int_0^T \|v_t(t) + Lv(t) - f(t)\|_{L^2(\Omega)}^2 \mathrm{d}t + |v(0) - u^0|_{H^1(\Omega)}^2\,. \tag{6.71}$$

We use $H^1(\Omega)$ seminorm for the initial condition, since then the regularity properties of the functional are better. Of course, one can use the $L^2(\Omega)$ norm instead with appropriate modifications in the proofs.

As before, we select a sequence of spaces $V_{\mathcal{N}}$ as follows: for each $\ell \in \mathbb{N}$ we correspond a DNN space $W_{\mathcal{N}}$, which is denoted by W_ℓ such that: For each $w \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$ there exists a $w_\ell \in W_\ell$ such that,

$$\|w_\ell - w\|_{H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))} \leq \ \beta_\ell\,(w), \qquad \text{and} \ \ \beta_\ell\,(w) \to 0, \ \ \ell \to \infty\,. \tag{6.72}$$

If in addition, w has higher regularity, we assume that

$$\|(w_\ell - w)'\|_{H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))} \leq \ \tilde{\beta}_\ell\,\|w'\|_{H^m(0,T;H^2(\Omega))}, \qquad \text{and} \ \ \tilde{\beta}_\ell \to 0, \ \ \ell \to \infty\,. \tag{6.73}$$

As in the elliptic case, we do not need specific rates for $\tilde{\beta}_\ell$, but only the fact that the right-hand side of (6.73) has an explicit dependence of Sobolev norms of w. See [1] and its references where space-time approximation properties of neural network spaces are derived, see also [48], [15] and Remark 6.3.1.

In the sequel we consider the sequence of energies

$$\mathcal{G}_\ell(u_\ell) = \begin{cases} \mathcal{G}(u_\ell), & u_\ell \in W_\ell \cap L^2(0,T;H_0^1(\Omega)) \\ +\infty, & \text{otherwise} \end{cases} \tag{6.74}$$

where W_ℓ is chosen as before.

Equi-coercivity

Now we have equicoercivity of $\mathcal{G}_\ell$ as a corollary of the following result.

Proposition 6.4.9. The functional $\mathcal{G}$ defined in (6.71) is equicoercive with respect to the
$H^1(0,T;H_0^2(\Omega))$-norm. That is,

$$\begin{gathered} \text{If } \ \mathcal{G}(u) \leq C \ \text{ for some } \ C > 0\ , \text{ we have} \\ ||u||_{L^2(0,T;H^2(\Omega))} + ||u'||_{L^2(0,T;L^2(\Omega))} \leq C_1 \end{gathered} \tag{6.75}$$

Proof. As in the proof of equicoercivity for (6.5), we have

$$\mathcal{G}(u) = \int_{\Omega_T} (|u_t + Lu|^2 - 2f\,(u_t + Lu) + |f|^2) \leq C \tag{6.76}$$

Hence, one can conclude that since $f \in L^2(\Omega_T)$,

$$||u_t + Lu||_{L^2(0,T;L^2(\Omega))} \leq C_1 \tag{6.77}$$

From regularity theory for parabolic equations (see for example Theorem 5, p.382 in [19]) we have

$$\begin{aligned} &\text{ess sup}_{0\le t\le T}||u(t)||_{H_0^1(\Omega)} + ||u||_{L^2(0,T;H^2(\Omega))} + ||u'||_{L^2(0,T;L^2(\Omega))} \\ &\qquad \le \tilde{C}(||u_t + Lu||_{L^2(0,T;L^2(\Omega))} + ||u(0)||_{H_0^1(\Omega)}) \end{aligned} \tag{6.78}$$

the constant $\tilde{C}$ depending only on Ω , T and the coefficients of L. Notice that (6.78) is a maximal parabolic regularity estimate in $L^2(0,T;L^2(\Omega))$. This completes the proof. □

Compactness and Convergence of Discrete Minimizers

As in the previous section, from standard arguments in the theory of Γ-convergence, we will prove that under some boundedness hypothesis on u_ℓ, the sequence of discrete minimizers (u_ℓ) converges in $L^2(0,T;H^1(\Omega))$ to a global minimiser of the continuous functional. We will also need the well-known Aubin–Lions theorem as an analog of the Rellich-Kondrachov theorem in the parabolic case, that can be found, for example, in [49].

Theorem 6.4.10 (Aubin-Lions)**.** Let B_0, B, B_1 be three Banach spaces where B_0, B_1 are reflexive. Suppose that B_0 is continuously imbedded into B, which is also continuously imbedded into B_1, and the imbedding from B_0 into B is compact. For any given p_0, p_1 with $1 < p_0, p_1 < \infty$, let

$$W = \{v \mid v \in L^{p_0}([0,T], B_0)\ ,\ v_t \in L^{p_1}([0,T], B_1)\}. \tag{6.79}$$

Then the imbedding from W into $L^{p_0}([0,T], B)$ is compact.

Theorem 6.4.11 (Convergence of discrete minimisers)**.** Let $(u_\ell) \subset W_\ell$ be a sequence of minimizers of $\mathcal{G}_\ell$, i.e.,

$$\mathcal{G}_\ell(u_\ell) = \inf_{w_\ell \in W_\ell} \mathcal{G}_\ell(w_\ell) \tag{6.80}$$

then

$$u_\ell \to u, \quad \text{in } L^2(0,T;H^1(\Omega)) \tag{6.81}$$

where u is the solution of (6.70).

Proof. We begin with the liminf inequality. We assume there is a sequence, still denoted by u_ℓ, such that $\mathcal{G}_\ell(u_\ell) \leq C$ uniformly in ℓ, otherwise $\mathcal{G}(u) \leq \liminf_{\ell\to\infty} \mathcal{G}_\ell(u_\ell) = +\infty$. From Proposition 6.4.9, the uniform bound $\mathcal{G}_\ell(u_\ell) \leq C$ implies that $||u_\ell||_{L^2(0,T;H^2(\Omega))} + ||u'_\ell||_{L^2(0,T;L^2(\Omega))}$ are uniformly bounded. This implies (we denote $u' := u_t$)

$$\nabla^2 u_\ell \rightharpoonup \nabla^2 u \ \text{ and } \ u'_\ell \rightharpoonup u' \ \text{ weakly in } \ L^2(0,T;L^2(\Omega)), \tag{6.82}$$

and hence $u'_\ell + Lu_\ell - f \rightharpoonup u' + Lu - f$. The convexity of $\int_{\Omega_T} |u'_\ell + Lu_\ell - f|^2$ implies weak lower semicontinuity, that is

$$\int_{\Omega_T} |u' + Lu - f|^2 \leq \liminf_{\ell\to\infty} \int_{\Omega_T} |u'_\ell + Lu_\ell - f|^2 \tag{6.83}$$

and therefore we conclude that $\mathcal{G}(u) \leq \liminf_{\ell\to\infty} \mathcal{G}_\ell(u_\ell)$.

Let $w \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$, by (6.72) there exists $w_\ell \in W_\ell$ such that $w_\ell \to w$ in $H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$. We can conclude that $w'_\ell + Lw_\ell \to w' + Lw$ in $L^2(0,T;L^2(\Omega))$, and hence

$$||w'_\ell + Lw_\ell - f||_{L^2(0,T;L^2(\Omega))} \to ||w' + Lw - f||_{L^2(0,T;L^2(\Omega))} \tag{6.84}$$

That is, $\mathcal{G}_\ell(w_\ell) \to \mathcal{G}(w)$. We argue as in Theorem 6.3.8 and we conclude the proof. The only difference is that we utilise Theorem 6.4.10 instead of Rellich-Kondrachov Theorem, with $B_0 = H^2(\Omega)$, $B = H^1(\Omega)$ and $B_1 = L^2(\Omega)$. □

6.4.2 Time discrete training

To apply a quadrature in the time integral only we proceed as follows: Let $0 = t^0 < t^1 < \cdots < t^N = T$ define a partition of $[0,T]$ and $I_n := (t^{n-1}, t^n]$, $k_n := t^n - t^{n-1}$. We shall denote by $v^m(\cdot)$ and $f^m(\cdot)$ the values $v(\cdot, t^m)$ and $f(\cdot, t^m)$. Then we define the discrete in time quadrature by

$$\sum_{n=1}^{N} k_n \, g(t^n) \approx \int_0^T g(t) \, \mathrm{d}t. \tag{6.85}$$

We proceed to define the time-discrete version of the functional (6.5) as follows

$$\mathcal{G}_{IE,k}(v) = \sum_{n=1}^{N} k_n \int_\Omega \Big| \frac{v^n - v^{n-1}}{k_n} + Lv^n - f^n \Big|^2 \, \mathrm{d}x + \int_\Omega |v^0 - u^0|^2_{H^1(\Omega)} \, \mathrm{d}x \, . \tag{6.86}$$

We shall study the stability and convergence properties of the minimisers of the problems:

$$\min_{v\in V_{\mathcal{N}}} \mathcal{G}_{IE,k}(v)\,. \tag{6.87}$$

Next we introduce the *time reconstruction* $\widehat{U}$ of a time dependent function U to be the piecewise linear approximation of U defined by linearly interpolating between the nodal values U^{n-1} and U^n:

$$\widehat{U}(t) := \ell_0^n(t)U^{n-1} + \ell_1^n(t)U^n, \quad t \in I_n, \tag{6.88}$$

with $\ell_0^n(t) := (t^n - t)/k_n$ and $\ell_1^n(t) := (t - t^{n-1})/k_n$. This reconstruction of the discrete solution has been proven useful in various instances, see [4], [38], [17] and for higher-order versions [34]. Correspondingly, the piecewise constant interpolant of U^j is denoted by $\overline{U}$,

$$\overline{U}(t) := U^n, \quad t \in I_n\,. \tag{6.89}$$

So now the discrete energy $\mathcal{G}_{IE,k}$ can be written as follows

$$\begin{aligned}\mathcal{G}_{IE,k}(U) =& \|\widehat{U}_t + L\overline{U} - \overline{f}\|^2_{L^2(0,T;L^2(\Omega))} + \int_\Omega |\widehat{U}^0 - u^0|^2_{H^1(\Omega)}\,\mathrm{d}x \\ =& \int_0^T \|\widehat{U}_t + L\overline{U} - \overline{f}\|^2_{L^2(\Omega)}\,\mathrm{d}t + \int_\Omega |\widehat{U}^0 - u^0|^2_{H^1(\Omega)}\,\mathrm{d}x\,.\end{aligned} \tag{6.90}$$

Stability-Equi-coercivity

Now we have equicoercivity of $\mathcal{G}_{IE,k}$ as a corollary of the following result.

Proposition 6.4.12. The functional $\mathcal{G}_{IE,k}$ defined in (6.90) is equicoercive with respect to $\widehat{U}, \overline{U}$. That is,

$$\begin{aligned}&\text{If } \ \mathcal{G}_k(U) \le C \ \text{ for some } \ C > 0\ , \text{ we have} \\ &\|\overline{U}\|_{L^2(0,T;H^2(\Omega))} + \|\widehat{U}'\|_{L^2(0,T;L^2(\Omega))} \le C_1\end{aligned} \tag{6.91}$$

Proof. As in the proof of equicoercivity for (6.5), we have

$$\int_{\Omega_T} (|\widehat{U}_t + L\overline{U}|^2 - 2\overline{f}\,(\widehat{U}_t + L\overline{U}) + |\overline{f}|^2) \le C \tag{6.92}$$

Thus we can conclude that since $f \in L^2(\Omega_T)$, we have the uniform bound

$$\|\widehat{U}_t + L\overline{U}\|_{L^2(0,T;L^2(\Omega))} \le C_1\,. \tag{6.93}$$

We shall need a discrete maximal regularity estimate in the present Hilbert-space setting. To this end we observe,

$$
\begin{aligned}
\|\widehat{U}_t + L\overline{U}\|^2_{L^2(0,T;L^2(\Omega))} &= \|\widehat{U}_t\|^2_{L^2(0,T;L^2(\Omega))} + \|L\overline{U}\|^2_{L^2(0,T;L^2(\Omega))} + 2\sum_{n=1}^{N}\int_{I_n} \langle \widehat{U}_t, L\overline{U}\rangle \ dt \\
&= \|\widehat{U}_t\|^2_{L^2(0,T;L^2(\Omega))} + \|L\overline{U}\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + 2\sum_{n=1}^{N}\int_{I_n} \langle [\frac{U^n - U^{n-1}}{k_n}], LU^n\rangle \ dt \\
&= \|\widehat{U}_t\|^2_{L^2(0,T;L^2(\Omega))} + \|L\overline{U}\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + 2\sum_{n=1}^{N} \langle \big[U^n - U^{n-1}\big], LU^n\rangle \\
&= \|\widehat{U}_t\|^2_{L^2(0,T;L^2(\Omega))} + \|L\overline{U}\|^2_{L^2(0,T;L^2(\Omega))} + \langle L\, U^N, U^N\rangle \\
&\quad + \sum_{n=1}^{N} \langle L\big[U^n - U^{n-1}\big], U^n - U^{n-1}\rangle - \langle L\, U^0, U^0\rangle\, .
\end{aligned}
\tag{6.94}
$$

Since all but the last term $\langle L\, U^0, U^0\rangle$ are positive, we conclude,

$$
\|\widehat{U}_t\|^2_{L^2(0,T;L^2(\Omega))} + \|L\overline{U}\|^2_{L^2(0,T;L^2(\Omega))} \le \|\widehat{U}_t + L\overline{U}\|^2_{L^2(0,T;L^2(\Omega))} + \langle L\, U^0, U^0\rangle\, , \tag{6.95}
$$

and the proof is complete. □

$\liminf$ **inequality**

We assume there is a sequence, still denoted by U_ℓ, such that $\mathcal{G}_{IE,\ell}(U_\ell) \le C$ uniformly in ℓ, otherwise $\liminf_{\ell\to\infty} \mathcal{G}_{IE,\ell}(U_\ell) = +\infty$. From the discrete stability estimate, the uniform bound $\mathcal{G}_{IE,\ell}(U_\ell) \le C$ implies that $\|\overline{U}_\ell\|_{L^2(0,T;H^2(\Omega))} + \|\widehat{U}_\ell'\|_{L^2(0,T;L^2(\Omega))} \le C_1$, are uniformly bounded. By the relative compactness in $L^2(0,T;L^2(\Omega))$ we have (up to a subsequence not re-labeled) the existence of $u_{(1)}$ and $u_{(2)}$ such that

$$
L\overline{U}_\ell \rightharpoonup Lu_{(1)} \ \text{ and } \ \widehat{U}_\ell' \rightharpoonup u_{(2)}' \ \text{ weakly in } \ L^2(0,T;L^2(\Omega))\, . \tag{6.96}
$$

Notice that, for any space-time test function $\varphi \in C_0^\infty$ there holds (we have set $\tilde{\varphi}^n := \frac{1}{k_n}\int_{I_n}\varphi\ dt$)

$$\begin{aligned}
-\int_0^T \langle \widehat{U}_\ell, \varphi' \rangle \mathrm{d}t &= \int_0^T \langle \widehat{U}'_\ell, \varphi \rangle \mathrm{d}t \\
&= \sum_{n=1}^N \int_{I_n} \langle [\frac{U_\ell^n - U_\ell^{n-1}}{k_n}], \varphi \rangle \ dt = \sum_{n=1}^N \langle U_\ell^n, \tilde{\varphi}^n \rangle - \langle U_\ell^{n-1}, \tilde{\varphi}^n \rangle \\
&= \sum_{n=1}^N \langle U_\ell^n, \varphi^{n-1} \rangle - \langle U_\ell^{n-1}, \varphi^{n-1} \rangle + \sum_{n=1}^N \langle U_\ell^n, \big[\tilde{\varphi}^n - \varphi^{n-1}\big] \rangle - \langle U_\ell^{n-1}, \big[\tilde{\varphi}^n - \varphi^{n-1}\big] \rangle \\
&= -\sum_{n=1}^N \langle U_\ell^n, \varphi^n - \varphi^{n-1} \rangle + \sum_{n=1}^N \langle \big[U_\ell^n - U_\ell^{n-1}\big], \big[\tilde{\varphi}^n - \varphi^{n-1}\big] \rangle \\
&= -\int_0^T \langle \overline{U}_\ell, \varphi' \rangle \mathrm{d}t + \sum_{n=1}^N \langle \big[U_\ell^n - U_\ell^{n-1}\big], \big[\tilde{\varphi}^n - \varphi^{n-1}\big] \rangle \,.
\end{aligned} \tag{6.97}$$

By the uniform bound,

$$\|\widehat{U}'_\ell\|^2_{L^2(0,T;L^2(\Omega))} = \sum_{n=1}^N \frac{1}{k_n} \|U_\ell^n - U_\ell^{n-1}\|^2_{L^2(\Omega)} \le C_1^2 \,,$$

and standard approximation properties for $\tilde{\varphi}^n - \varphi^{n-1}$ we conclude that for any fixed test function,

$$\int_0^T \langle \widehat{U}_\ell, \varphi' \rangle \mathrm{d}t - \int_0^T \langle \overline{U}_\ell, \varphi' \rangle \mathrm{d}t \to 0, \qquad \ell \to \infty \,. \tag{6.98}$$

We can conclude therefore that $u_{(1)} = u_{(2)} = u$ and thus,

$$\widehat{U}'_\ell + L\overline{U}_\ell - \overline{f} \rightharpoonup u' + Lu - f, \qquad \ell \to \infty \,. \tag{6.99}$$

The convexity of $\int_{\Omega_T} |\cdot|^2$ implies weak lower semicontinuity, that is

$$\int_{\Omega_T} |u' + Lu - f|^2 \le \liminf_{\ell\to\infty} \int_{\Omega_T} |\widehat{U}'_\ell + L\overline{U}_\ell - \overline{f}|^2 \tag{6.100}$$

and therefore we conclude that $\mathcal{G}(u) \le \liminf_{\ell\to\infty} \mathcal{G}_{IE,\ell}(U_\ell)$.

$\limsup$ **inequality**

Let $w \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$. We will now show the existence of a recovery sequence (w_ℓ) such that $w_\ell \to w$ and $\mathcal{G}(w) = \lim_{\ell\to\infty} \mathcal{G}_{IE,\ell}(w_\ell)$. Since $C^\infty(0,T;H^2(\Omega))$ is dense in $L^2(0,T;H^2(\Omega))$ we can select a $(w_\delta) \subset C^\infty(0,T;H^2(\Omega))$ with the properties

$$\begin{aligned} &\|w - w_\delta\|_{H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))} \lesssim \delta\,, \quad \text{and,}\\ &|w_\delta'|_{H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))} \lesssim \frac{1}{\delta}|w|_{H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))}\,. \end{aligned} \tag{6.101}$$

If $w_{\delta,\ell} \in W_\ell$ is a neural network function satisfying (6.72), (6.73), we would like to show

$$||\widehat{w}'_{\delta,\ell} + L\,\overline{w}_{\delta,\ell} - \overline{f}||_{L^2(0,T;L^2(\Omega))} \to ||w' + Lw - f||_{L^2(0,T;L^2(\Omega))} \tag{6.102}$$

where $\delta = \delta(\ell)$ is appropriately selected. Then,

$$\mathcal{G}_{IE,\ell}(w_{\delta,\ell}) \to \mathcal{G}(w)\,. \tag{6.103}$$

To this end it suffices to consider the difference

$$\|\widehat{w}'_{\delta,\ell} + L\,\overline{w}_{\delta,\ell} - w' - Lw\|_{L^2(0,T;L^2(\Omega))}\,. \tag{6.104}$$

We have

$$\begin{aligned} \|\widehat{w}'_{\delta,\ell} + L\,\overline{w}_{\delta,\ell} - w' - Lw\|_{L^2(0,T;L^2(\Omega))} \le &\|\widehat{w}'_{\delta,\ell} + L\,\overline{w}_{\delta,\ell} - \widehat{w}'_\delta - L\,\overline{w}_\delta\|_{L^2(0,T;L^2(\Omega))}\\ &+ \|\widehat{w}'_\delta + L\,\overline{w}_\delta - w' - Lw\|_{L^2(0,T;L^2(\Omega))}\\ =: &\, A_1 + A_2\,. \end{aligned} \tag{6.105}$$

To estimate A_1 we proceed as follows: Let $\theta_\ell(t) := w_{\delta,\ell}(t) - w_\delta(t)$. Then,

$$\begin{aligned} \|\widehat{w}'_{\delta,\ell} - \widehat{w}'_\delta\|^2_{L^2(0,T;L^2(\Omega))} &= \sum_{n=1}^N \int_{I_n} \Big\|\frac{\theta_\ell^n - \theta_\ell^{n-1}}{k_n}\Big\|^2_{L^2(\Omega)}\, dt\\ &= \sum_{n=1}^N \frac{1}{k_n}\big\|\theta_\ell^n - \theta_\ell^{n-1}\big\|^2_{L^2(\Omega)}\\ &= \sum_{n=1}^N \frac{1}{k_n}\Big\|\int_{I_n} \theta_\ell'(t)\, dt\Big\|^2_{L^2(\Omega)}\\ &\le \sum_{n=1}^N \frac{1}{k_n}\int_{I_n} \big\|\theta_\ell'(t)\big\|^2_{L^2(\Omega)}\, dt\, k_n\\ &= \|\theta_\ell'\|^2_{L^2(0,T;L^2(\Omega))}\,. \end{aligned} \tag{6.106}$$

Similarly,

$$
\begin{aligned}
\|L\,\overline{w}_{\delta,\ell} - L\,\overline{w}_{\delta}\|_{L^2(0,T;L^2(\Omega))} &= \Big\{\sum_{n=1}^{N}\int_{I_n} \big\|L\,\theta_\ell^n\big\|^2_{L^2(\Omega)}\ dt\Big\}^{1/2}\\
&\le \Big\{\sum_{n=1}^{N} k_n\,\big\|L\,\theta_\ell^n - \frac{1}{k_n}\int_{I_n} L\,\theta_\ell(t)\mathrm{d}t\big\|^2_{L^2(\Omega)}\Big\}^{1/2} + \Big\{\sum_{n=1}^{N}\frac{1}{k_n}\,\|\int_{I_n} L\,\theta_\ell(t)\mathrm{d}t\big\|^2_{L^2(\Omega)}\Big\}^{1/2}\\
&\le \Big\{\sum_{n=1}^{N} k_n\,\big\|L\,\theta_\ell^n - \frac{1}{k_n}\int_{I_n} L\,\theta_\ell(t)\mathrm{d}t\big\|^2_{L^2(\Omega)}\Big\}^{1/2} + \Big\{\sum_{n=1}^{N}\int_{I_n}\big\|L\,\theta_\ell(t)\big\|^2_{L^2(\Omega)}\ dt\Big\}^{1/2}\\
&= \Big\{\sum_{n=1}^{N} k_n\,\big\|L\,\theta_\ell^n - \frac{1}{k_n}\int_{I_n} L\,\theta_\ell(t)\mathrm{d}t\big\|^2_{L^2(\Omega)}\Big\}^{1/2} + \|L\,\theta_\ell\|_{L^2(0,T;L^2(\Omega))}\,.
\end{aligned}
\tag{6.107}
$$

It remains to estimate,

$$
\begin{aligned}
\Big\{\sum_{n=1}^{N} k_n\,\big\|L\,\theta_\ell^n - \frac{1}{k_n}\int_{I_n} L\,\theta_\ell(t)\mathrm{d}t\big\|^2_{L^2(\Omega)}\Big\}^{1/2} &= \Big\{\sum_{n=1}^{N}\frac{1}{k_n}\,\|\int_{I_n}\big[L\,\theta_\ell^n - L\,\theta_\ell(t)\big]\,\mathrm{d}t\big\|^2_{L^2(\Omega)}\Big\}^{1/2}\\
&\le \Big\{\sum_{n=1}^{N}\frac{1}{k_n}\,\big[\int_{I_n}\big\|L\,\theta_\ell^n - L\,\theta_\ell(t)\big\|_{L^2(\Omega)}\,\mathrm{d}t\big]^2\Big\}^{1/2}\\
&\le \Big\{\sum_{n=1}^{N}\frac{1}{k_n}\,\big[\int_{I_n}\int_{I_n}\big\|L\,\theta_\ell'(s)\big\|_{L^2(\Omega)}\,\mathrm{d}s\,\mathrm{d}t\big]^2\Big\}^{1/2}\\
&= \Big\{\sum_{n=1}^{N} k_n\,\big[\int_{I_n}\big\|L\,\theta_\ell'(t)\big\|_{L^2(\Omega)}\,\mathrm{d}t\big]^2\Big\}^{1/2}\\
&\le k\,\|L\,\theta_\ell'\|_{L^2(0,T;L^2(\Omega))}\,.
\end{aligned}
\tag{6.108}
$$

We conclude therefore that, $k = \max_n k_n$,

$$
A_2 \le \|\theta_\ell'\|_{L^2(0,T;L^2(\Omega))} + \|L\,\theta_\ell\|_{L^2(0,T;L^2(\Omega))} + k\,\|L\,\theta_\ell'\|_{L^2(0,T;L^2(\Omega))}\,. \tag{6.109}
$$

On the other hand, standard time interpolation estimates yield,

$$
A_1 \le C\,k\,\big[\|w_\delta''\|_{L^2(0,T;L^2(\Omega))} + \|L\,w_\delta'\|_{L^2(0,T;L^2(\Omega))}\big]\,. \tag{6.110}
$$

Hence, we have using (6.72), (6.73), (6.101),

$$
A_1 + A_2 \le \beta_\ell(w_\delta) + \frac{k}{\delta^{m+1}}\tilde{\beta}_\ell\|w\|_{L^2(0,T;H^2(\Omega))} + C\frac{k}{\delta}\|w\|_{H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))}\,. \tag{6.111}
$$

Therefore, we conclude that (6.102) holds upon selecting $\delta = \delta(\ell, k)$ appropriately.

Convergence of the minimisers

In this subsection, we conclude the proof that the sequence of discrete minimisers (u_ℓ) converges in $L^2(0,T;H^1(\Omega))$ to the minimiser of the continuous problem.

Theorem 6.4.13 (Convergence)**.** Let $\mathcal{G}$, $\mathcal{G}_{IE,\ell}$ be the energy functionals defined in (6.71) and (6.86) respectively. Let u be the exact solution of (6.70) and let (u_ℓ), $u_\ell \in V_\ell$, be a sequence of minimisers of $\mathcal{G}_{IE,\ell}$, i.e.

$$\mathcal{G}_{IE,\ell}(u_\ell) = \inf_{v_\ell \in W_\ell} \mathcal{G}_{IE,\ell}(v_\ell)\,. \tag{6.112}$$

Then,

$$\hat{u}_\ell \to u, \quad \text{in } \; L^2(0,T;H^1(\Omega)), \tag{6.113}$$

where $\hat{u}_\ell$ is defined by (6.88).

Proof. Next, let $u \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ be the solution of (6.70). Consider the sequence of minimisers (u_ℓ) . Obviously,

$$\mathcal{G}_{IE,\ell}(u_\ell) \le \mathcal{G}_{IE,\ell}(v_\ell), \qquad \text{for all } v_\ell \in V_\ell\,.$$

In particular,

$$\mathcal{G}_{IE,\ell}(u_\ell) \le \mathcal{G}_{IE,\ell}(\tilde{u}_\ell),$$

where $\tilde{u}_\ell$ is the recovery sequence $w_{\delta,\ell}$ corresponding to $w = u$ constructed above. Hence, we conclude that the sequence $\mathcal{G}_{IE,\ell}(u_\ell)$ is uniformly bounded. The stability-equi-coercivity of the discrete functional, see Proposition 6.4.12, implies that

$$\|\overline{u}_\ell\|_{L^2(0,T;H^2(\Omega))} + \|\widehat{u}_\ell\|_{L^2(0,T;H^2(\Omega))} + \|\widehat{u}_\ell'\|_{L^2(0,T;L^2(\Omega))} \le C\,. \tag{6.114}$$

The Aubin-Lions theorem ensures that there exists $\tilde{u} \in L^2(0,T;H^1(\Omega))$ such that $\widehat{u}_\ell \to \tilde{u}$ in $L^2(0,T;H^1(\Omega))$ up to a subsequence not re-labeled. Furthermore the previous analysis shows that $L\tilde{u} \in L^2(0,T;L^2(\Omega))$. To prove that $\tilde{u}$ is the minimiser of $\mathcal{G}$, and hence $\tilde{u} = u$, we combine the results of Sections 6.4.2 and 6.4.2: Let $w \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$. We did show the existence of a recovery sequence (w_ℓ) such that $w_\ell \to w$ and

$$\mathcal{G}(w) = \lim_{\ell\to\infty} \mathcal{G}_{IE,\ell}(w_\ell).$$

Therefore, the lim inf inequality and the fact that u_ℓ are minimisers of the discrete problems imply that

$$\mathcal{G}(\tilde{u}) \le \liminf_{\ell\to\infty} \mathcal{G}_{IE,\ell}(u_\ell) \le \limsup_{\ell\to\infty} \mathcal{G}_{IE,\ell}(u_\ell) \le \limsup_{\ell\to\infty} \mathcal{G}_{IE,\ell}(w_\ell) = \mathcal{G}(w), \tag{6.115}$$

for all $w \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$. Therefore $\tilde{u}$ is the minimiser of $\mathcal{G}$, hence $\tilde{u} = u$ and the entire sequence satisfies

$$\hat{u}_\ell \to u, \quad \text{in } \; L^2(0,T;H^1(\Omega)).$$

Therefore the proof is complete. ∎

Explicit time discrete training

It will be interesting to consider a seemingly similar (from the point of view of quadrature and approximation) discrete functional:

$$\mathcal{G}_{k,EE}(v) = \sum_{n=1}^{N} k_n \int_\Omega \big|\frac{v^n - v^{n-1}}{k_n} + Lv^{n-1} - f^{n-1}\big|^2 \, \mathrm{d}x + \int_\Omega |v - u^0|^2 dx, \quad (6.116)$$

and compare its properties to the functional $\mathcal{G}_{k,IE}(v)$ and the corresponding $V_\mathcal{N}$ minimisers. The functional (6.116) is related to *explicit* Euler discretisation in time as opposed to the *implicit* Euler discretisation in time for $\mathcal{G}_{k,IE}(v)$. Clearly, in the discrete minimisation framework, both energies are fully implicit, since the evaluation of the minimisers involves the solution of global space-time problems. It is therefore rather interesting that these two energies result in completely different stability properties.

Let us first note that it does not appear possible that a discrete coercivity such as (6.91) can be proved. Indeed, an argument similar to (6.97) is possible but with the crucial difference that the second to last term of this relation will be negative instead of positive. This is a fundamental point directly related to the (in)stability of the forward Euler method. Typically for finite difference forward Euler schemes one is required to assume a strong CFL condition of the form $k \leq Ch^2$ where h is the spatial discretisation parameter to preserve stability. It appears that a phenomenon of similar nature is present in our case as well. Although we do not show stability bounds when spatial training is taking place, the numerical experiments show that the stability behaviour of the explicit training method deteriorates when we increase the number of spatial training points while keeping k constant. These stability considerations are verified by the numerical experiments we present below. Indeed, these computations provide convincing evidence that coercivity bounds similar to (6.91) are necessary for stable behaviour of the approximations. In the computations we solve the one dimensional heat equation with zero boundary conditions and two different initial values plotted in black. All runs were performed using the package *DeepXDE*, [33], with random spatial training and constant time step.

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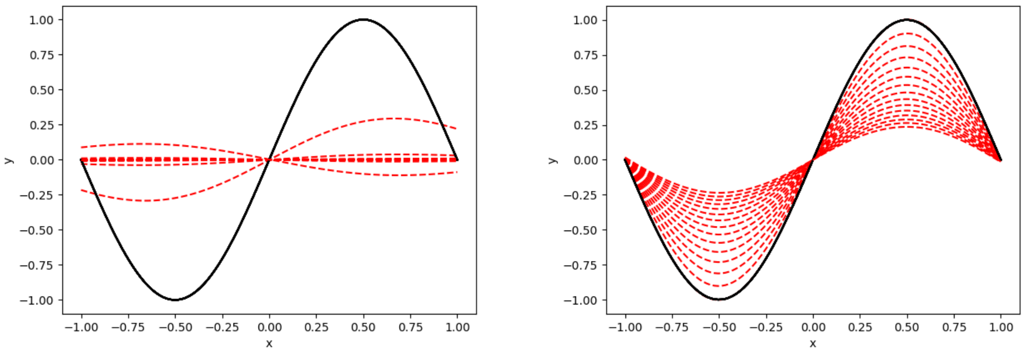

Figure 6.1: *Explicit time discrete training. Left: time step* 0.4 : *the approximate solution seems that diverge. Right: time step* 0.01 : *with much smaller time step the approximate solution has stable behaviour.*

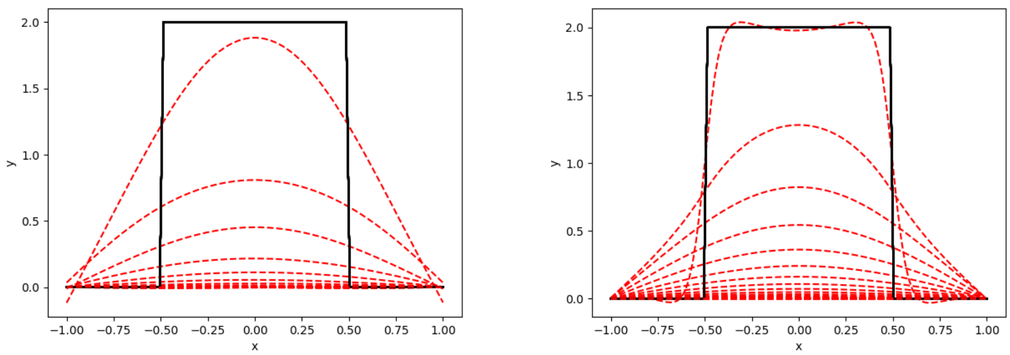

Figure 6.2: *The approximations at times* $t_n = n(0.2)$, $n = 1, 2, \dots,$ *are displayed with red and the initial condition with black. Left: Explicit time discrete training with time step* 0.2 *and* 16 *training points. The approximate solution seems that diverge. Left: Implicit time discrete training with time step* 0.2 *and* 16 *training points. The approximate solution seems reasonable.*

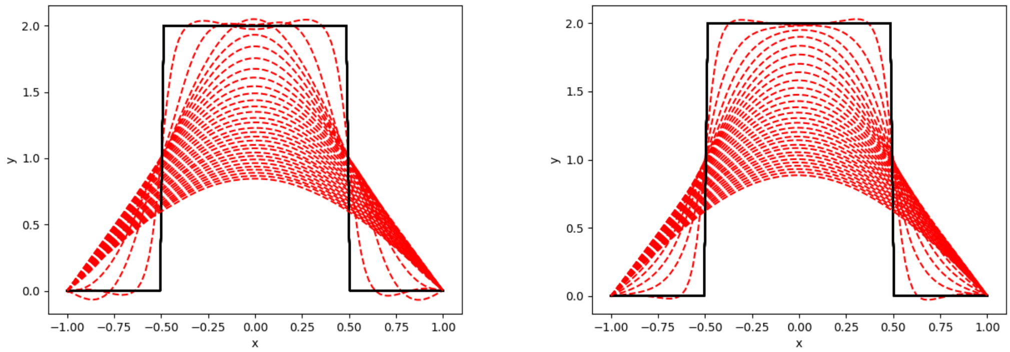

Figure 6.3: *Left: Explicit time discrete training with time step* 0.01 *and* 16 *training points. The approximate solution seems reasonable due to the much smaller time step. Left: Implicit time discrete training with time step* 0.01 *and* 16 *training points.*

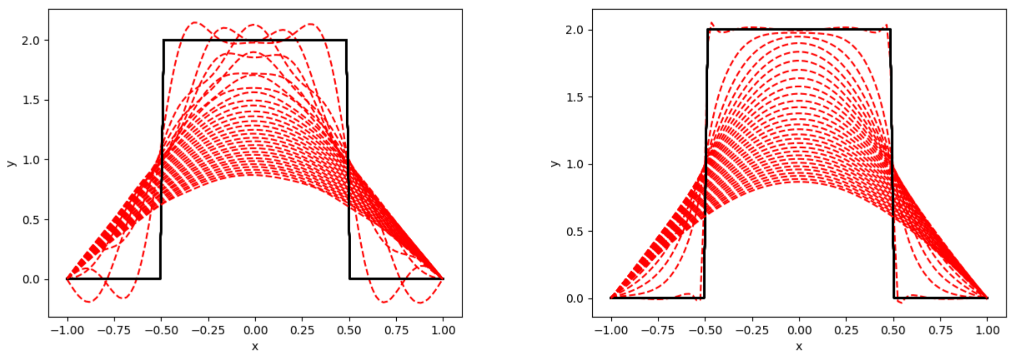

Figure 6.4: *Left: Explicit time discrete training with time step* 0.01 *and* 100 *training points. Initial instabilities are again evident, due to the higher number of spatial training points while the time step is the same as in Figure 3. Left: Implicit time discrete training with time step* 0.01 *and* 100 *training points.*

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