A stochastic mass conserved reaction-diffusion equation

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Abstract

We prove the existence and uniqueness of weak solution of the initial boundary value problem for a stochastic mass conserved Allen-Cahn equation with nonlinear diffusion together with a homogeneous Neumann boundary condition in an open bounded domain of $\mathbb{R}^n$ with a smooth boundary. We suppose that the additive noise is induced by a Q-Brownian motion.

We decompose our original problem into two problems: a nonlinear stochastic heat equation with homogeneous initial condition, and a stochastic nonlocal reaction diffusion equation with nonlinear reaction but without the noise term. We will prove the existence of solution for these two problems by applying a Galerkin method, which amounts to establishing suitable a priori estimates that we need to get weak compactness of the approximate solution, namely convergence along a subsequence to a limit. The main problem is then to identify the limit of the diffusion term and the reaction term, which we do by means of the so-called monotonicity method. We also prove the uniqueness of the weak solution.

Περίληψη

Στην παρούσα εργασία μελετάμε την ύπαρξη και μοναδικότητα ασθενών λύσεων για το αρχικό συνοριακό πρόβλημα στοχαστικής Allen-Cahn εξίσωσης με διατήρηση μάζας και μη γραμμικής διάχυσης με ομογενείς συνοριακές συνθήκες Neumann σε ενα ανοιχτό φραγμένο σύνολο με ομαλό σύνορο. Ο προσθετικός θόρυβος αποτελείται από μια Q-Brownian κίνηση.

Αποσυνθέτουμε το αρχικό μας πρόβλημα σε δύο προβλήματα: σε μια μη γραμμική στοχαστική εξίσωση θερμότητας με ομογενή αρχική κατάσταση και σε μια στοχαστική μη τοπική εξίσωση αντίδρασης διάχυσης με μη γραμμική αντίδραση αλλά χωρίς τον όρο θορύβου. Αποδεικνύουμε την ύπαρξη λύσης για αυτά τα δύο προβλήματα με την εφαρμογή μιας μεθόδου Galerkin, η οποία ισοδυναμεί με την απόδειξη κατάλληλων εκ των προτέρων εκτιμήσεων για να επιτύχουμε ασθενή συμπάγεια για την κατά προσέγγιση λύση, δηλαδή σύγκλιση μιας υποακολουθίας σε ένα όριο. Το κύριο πρόβλημα τότε είναι να προσδιορίσουμε το όριο του όρου διάχυσης και του όρου αντίδρασης, που κάνουμε μέσω της λεγόμενης μεθόδου μονοτονίας. Αποδεικνύουμε επίσης τη μοναδικότητα της ασθενής λύσης.

Contents

Chapter 1

Introduction

1.1 The problem

We study the initial boundary value problem for the stochastic nonlocal reaction-diffusion equation with nonlinear diffusion together with null-flux boundary condition in an open bounded domain of $\mathbb{R}^n$ with smooth boundary and additive noise which induced by a Q-Brownian motion. This is the stochastic mass conserved Allen-Cahn equation which is mentioned in [7],

$$(P) \quad \begin{cases} \dfrac{\partial \varphi}{\partial t} = \operatorname{div}(A(\nabla\varphi)) + f(\varphi) - \dfrac{1}{|D|}\displaystyle\int_D f(\varphi)dx + \dfrac{\partial W}{\partial t}, & x \in D,\ t \geq 0, \\ A(\nabla\varphi)\cdot\nu = 0, & \text{on } \partial D \times \mathbb{R}^+, \\ \varphi(x,0) = \varphi_0(x), & x \in D, \end{cases}$$

where D is an open bounded set of $\mathbb{R}^n$ with a smooth boundary ∂D, ν is the outer normal vector to ∂D and φ_0 is the initial function such that $\varphi_0 \in L^2(D)$.

We suppose that the nonlinear function f is a smooth function which satisfies the following properties:

(F_1) There exist positive constants C_1 and C_2 such that

$$f(a+b)a \leq -C_1 a^{2p} + f_2(b), \qquad |f_2(b)| \leq C_2(b^{2p}+1), \text{ for all } a, b \in \mathbb{R}.$$

(F_2) There exist positive constants C_3 and $\tilde{C_3}$ such that

$$|f(s)| \leq C_3|s-M|^{2p-1} + \tilde{C_3}(M)$$

(F_3) There exists a positive constant C_4 such that

$$f'(s) \leq C_4.$$

We will check in the Appendix A that the function $f(s) = \sum_{r=0}^{2p-1} b_r s^r$ with $b_{2p-1} < 0, p \geq 2$ satisfies the properties $(F_1) - (F_3)$.

We also assume that $A = \nabla_v \Psi(v) : \mathbb{R}^n \to \mathbb{R}^n$ for some strictly convex function $\Psi \in C^{1,1}$ (i.e $\Psi(v) \in C^1(\mathbb{R}^n)$ and $\nabla\Psi(v)$ is Lipschitz continuous) satisfying

$$\begin{cases} A(0) = \nabla\Psi(0) = 0, \Psi(0) = 0 \\ \left\|D^2\Psi\right\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})} \leq c_1, \end{cases} \tag{1.1}$$

for some constant $c_1 > 0$. We remark that (1.1) implies that

$$|A(a) - A(b)| \leq C|a - b| \tag{1.2}$$

for all $a, b \in \mathbb{R}^n$, where C is a positive constant, and that the strict convexity of Ψ implies that A is strictly monotone, namely there exists a positive constant C_0 such that

$$(A(a) - A(b))(a - b) \geq C_0|a - b|^2, \tag{1.3}$$

for all $a, b \in \mathbb{R}^n$.

Remark. *If A is the identity matrix, the nonlinear diffusion operator $-\mathrm{div}(A(\nabla u))$ reduces to the linear operator $-\Delta u$.*

1.2 Physical Background

The Allen-Cahn equation, also called Model A in the theory of dynamics of critical phenomena, is a reaction-diffusion equation of mathematical physics which describes the process of phase separation in multi-component alloy systems, including order-disorder transitions. It is a second-order nonlinear parabolic partial differential equation and describes the evolution of a non-conserved order field during anti-phase domain coarsening. It is closely related to the Cahn-Hilliard equation, which is a fourth-order nonlinear parabolic partial differential equation and describes the process of phase separation of a conserved order field. It was first introduced in 1979 by John W. Cahn and Sam Allen in their work [2], in material science to study the behavior of an interface separating two different iron and aluminium alloys. In the last thirty years, Allen-Cahn equation has been widely used in many complicated moving interface problems in material science, fluid dynamics, image analysis and mean curvature flow.

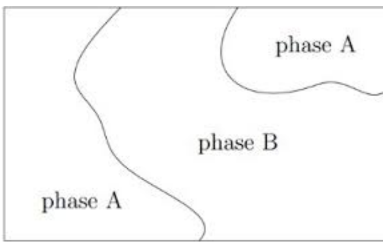

Figure 1.1: Binary mixture.

The deterministic equation (i.e., when $\frac{\partial W}{\partial t} = 0$) in the case of linear diffusion with A being the identity matrix, namely

$$\frac{\partial \varphi}{\partial t} = \Delta\varphi + f(\varphi) - \frac{1}{|D|}\int_D f(\varphi)dx, \quad x \in D,\ t \geq 0,$$

was first studied by Rubinstein and Sternberg [11] as a model for phase separation in a binary mixture. The well-posedness and stabilization of the solution for large times for the corresponding Neumann problem

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = \Delta\varphi + f(\varphi) - \dfrac{1}{|D|}\displaystyle\int_D f(\varphi)dx, & x \in D,\ t \geq 0, \\ \nabla\varphi \cdot \nu = 0, & \text{on } \partial D \times \mathbb{R}^+, \\ \varphi(x, 0) = \varphi_0(x), & x \in D, \end{cases}$$

were proved by Boussaïd, Hilhorst and Nguyen [5]. They assumed that the initial function was bounded in $L^\infty(D)$ and proved the existence of the solution in an invariant set using a Galerkin approximation together with a compactness method.

Alikakos, Chen and Fusco [1] analyzed the problem's long-time dynamics and established existence of stable sets of solutions corresponding to the motion of a small, almost semicircular interface (droplet) intersecting the boundary of the domain and moving towards a point of locally maximal curvature. Later, Bates and Jin [4] established the existence of a global invariant manifold of droplet states using the approximation given in [1].

A singular limit of the stochastic mass conserved equation with linear diffusion, i.e.,

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = \Delta\varphi + f(\varphi) - \dfrac{1}{|D|}\displaystyle\int_D f(\varphi)dx + \dfrac{\partial W}{\partial t}, & x \in D,\ t \geq 0, \\ \nabla\varphi \cdot \nu = 0, & \text{on } \ \partial D \times \mathbb{R}^+, \\ \varphi(x,0) = \varphi_0(x), & x \in D, \end{cases}$$

has been studied by Antonopoulou, Bates, Blömker and Karali, [3], to model the motion of a droplet.

The interfacial evolution process corresponding to a second order mass conserved Allen-Cahn equation shares many properties with the fourth order Cahn-Hilliard equation as discussed [11]. Da Prato and Debussche proved the existence and the uniqueness of the solution of a stochastic Cahn-Hilliard equation [6] with an additive space-time white noise. Funaki and Yokoyama [8] derive a sharp interface limit for a stochastically perturbed mass conserved Allen-Cahn equation with a sufficiently mild additive noise.

In (1), the unknown φ denotes the evolution of the concentration one of the species of the alloy, known as the phase state between materials. For a binary mixture, of two components A and B, with concentration $\varphi_A(x,t)$ and $\varphi_B(x,t)$ respectively, we can assume that $\varphi_A(x,t) + \varphi_B(x,t) = 1$, and therefore only use one concentration for the description:

$$\varphi_A(x,t) := \varphi(x,t) \quad \text{and} \quad \varphi_B(x,t) := 1 - \varphi(x,t).$$

Within this context, φ stands for

$$\varphi(x,t) = \frac{m_A - m_B}{m_A + m_B}$$

that is the difference between the concentrations of the two components in a mixture, where m_A and m_B are the masses of phases A and B.

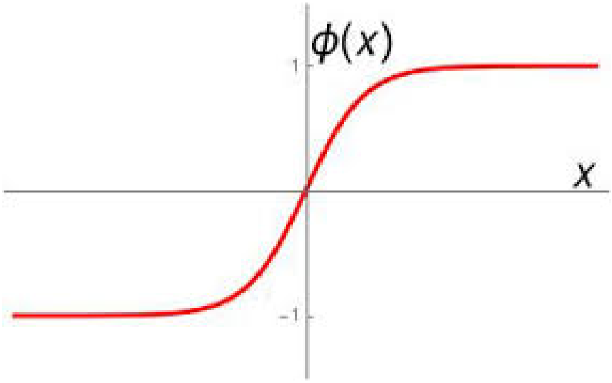

Figure 1.2: Concentration function in a neighborhood of a layer.

The nonlinear term $f(\varphi) = F'(\varphi)$ is the reaction term whose antiderivative is a potential $F(\varphi)$. The type of potential $F(\varphi)$ which has been considered most in the literature is the convex quartic double-well potential

$$F(\varphi) = \frac{1}{4}(1-\varphi^2)^2.$$

In the case of the quartic double-well potential, $f(\varphi) = \varphi - \varphi^3$ represents the bi-stable non-linearity. It favors layered functions that take values close to its minima ± 1. The zero level sets of such a function are called interfaces and the values close to ± 1 are called states.

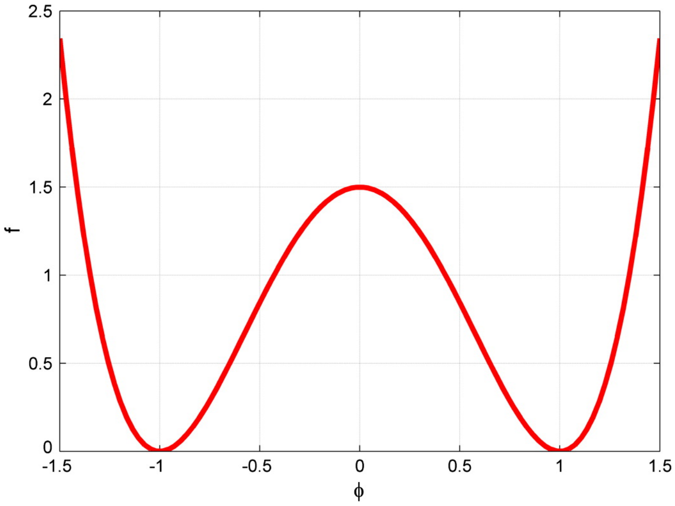

Figure 1.3: The double well potential $F(\varphi) = \frac{1}{4}(1-\varphi^2)^2$.

The deterministic part of the equation is the L^2 gradient flow of free energy functional

$$E(\varphi) \;=\; \int_D \frac{|\nabla \varphi|^2}{2} + F(\varphi)\, dx.$$

This functional behave like Lyapunov functions, that is functions that monotonically change in time. Given an energy $E(\varphi)$ the associated gradient flow is given by the equation

$$\frac{\partial \varphi}{\partial t} = -\frac{\partial E}{\partial \varphi}.$$

In other words, φ decreases along the gradient of E. The notation $\frac{\partial E}{\partial \varphi}$ denotes the so-called functional derivative of E to φ, which generalizes the 'gradient' notion for functions.

Since the classical AC equation

$$\frac{\partial \varphi}{\partial t} = \Delta \varphi + f(\varphi) \tag{1.4}$$

does not conserve the initial volume, Rubinstein and Sternberg [11] added a Lagrange multiplier $\beta(t) = \frac{1}{|D|}\int_D f(\varphi) dx$ to (1.4) in order to impose the conservation of volume.

Similarly the equation which we study here satisfies the mass conservation:

$$\begin{aligned} \frac{d}{dt}\int_D \varphi\, dx &= \int_D \frac{\partial \varphi}{\partial t}\, dx \\ &= \int_D \left[\operatorname{div}(A(\nabla\varphi)) + f(\varphi) - \frac{1}{|D|}\int_D f(\varphi)\, dx + \frac{\partial W}{\partial t}\right] dx \end{aligned}$$

$$
\begin{aligned}
&\overset{\text{(Div. Thm)}}{=} \int_{\partial D} A(\nabla\varphi)\cdot\nu\,ds + \int_D f(\varphi)\,dx - \frac{1}{|D|}\int_D f(\varphi)\,dx \int_D dx + \int_D \frac{\partial W}{\partial t}\,dx \\
&= \int_D f(\varphi)\,dx - \frac{1}{|D|}\int_D f(\varphi)\,dx|D| + \int_D \frac{\partial W}{\partial t}\,dx \\
&= \int_D \frac{\partial W}{\partial t}\,dx.
\end{aligned}
$$

Taking the expectation of above, we have

$$
\mathbb{E}[\frac{d}{dt}\int_D \varphi\,dx] = \mathbb{E}[\int_D \frac{\partial W}{\partial t}\,dx] = 0.
$$

Due to mass conservation, a phase separation begins either by spinodal decomposition, or as in our case as the mass is very asymmetrical by nucleation.

1.3 Assumptions on the noise

The noise term $\frac{\partial W}{\partial t}$ is defined as the formal derivative of a Wiener process, which is given by a Fourier series with coefficients being independent Brownian motions in time.

We consider a symmetric positive linear operator Q in Hilbert space $L^2(D)$, as the covariance operator. Then there exists an orthonormal basis (formed by eigenfunctions of Q) $\{e_l\}_{l\geq 1}$ for $L^2(D)$ and by a (bounded) sequence of nonnegative real numbers (eigenvalues of Q) $\{\lambda_l\}_{l\geq 1}$ such that

$$
Qe_l = \lambda_l e_l
$$

for all $l = 1, 2, ...$ We will always assume that the trace of the operator Q is finite, i.e.,

$$
\begin{aligned}
TrQ &= \sum_{l=1}^{\infty} \langle Qe_l, e_l\rangle_{L^2(D)} \\
&= \sum_{l=1}^{\infty} \int_D Qe_l \cdot e_l\,dx \qquad (Qe_l = \lambda_l e_l) \\
&= \sum_{l=1}^{\infty} \int_D \lambda_l e_l \cdot e_l\,dx \\
&= \sum_{l=1}^{\infty} \lambda_l \|e_l\|^2_{L(D)}\,dx \qquad (\|e_l\|^2_{L(D)} = 1,\ e_l \text{ orthonormal basis}) \\
&= \sum_{l=1}^{\infty} \lambda_l \leq \Lambda_0, \qquad (1.5)
\end{aligned}
$$

for some positive constant Λ_0. We suppose furthermore that $e_l \in H^1(D)\cap L^\infty(D)$ for $l = 1, 2, ...$ and that there exist positive constants Λ_1 and Λ_2 such that

$$
\sum_{l=1}^{\infty} \lambda_l \|e_l\|^2_{L^\infty(D)} \leq \Lambda_1, \tag{1.6}
$$

and

$$
\sum_{l=1}^{\infty} \lambda_l \|\nabla e_l\|^2_{L^2(D)} \leq \Lambda_2. \tag{1.7}
$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A stochastic process $W(t)$, taking values in $L^2(D)$, for $t \geq 0$ is called a Wiener process with covariance operator Q, or Q-Wiener process, if

(i) $W(0) = 0$

(ii) W has continuous trajectories (sample paths)

(iii) W has independent increments, and

(iv) $W(t) - W(s) \sim N(0, (t-s)Q),\ t \geq s \geq 0.$

Hence, $W(t) \sim N(0, tQ)$, i.e., $\mathbb{E}[W(t)] = 0$ and $\text{Cov}(W(t)) = tQ$. Also W is given as the Fourier series

$$\begin{aligned} W(x,t) &:= \sum_{l=1}^{\infty} \beta_l(t) Q^{\frac{1}{2}} e_l(x) \qquad (Qe_l = \lambda e_l) \\ &= \sum_{l=1}^{\infty} \sqrt{\lambda_l} \beta_l(t) e_l(x) \end{aligned} \tag{1.8}$$

where

$$\beta_l(t) = \frac{1}{\sqrt{\lambda_l}} \langle W(t), e_l \rangle, \quad j = 1, 2, ...$$

are real valued Brownian motions, that is,

$$\beta_l(t) \sim N(0, t),$$

$$\mathbb{E}[\beta_l(t)] = 0,$$

$$\mathbb{E}[\beta_l(t)^2] = t \text{ and}$$

$$\mathbb{E}[\beta_l(t)\beta_l(s)] = \min\{t, s\}.$$

The infinite series (1.8) converges in $L^2(D)$.

If we equipped a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ and assumed that

(i) $\beta(t)$ is $\mathcal{F}_t$-measurable

(ii) $\beta(t) - \beta(s)$ is independent of $\mathcal{F}_s, \forall 0 \leq s < t$,

we say that $\beta(t)$ is a Q-Wiener process with respect to $\{\mathcal{F}_t\}_{t \geq 0}$, or $\beta(t)$ is (F_t)-Brownian motion.

1.4 Motivation - Goal

In this work, we introduce a nonlinear stochastic heat equation, perform a change of functions in order to maintain a "deterministic style" mass conserved equation by hiding the noise term and prove the existence of the solution in suitable Sobolev spaces similar to those in [6]. Funaki and Yokoyama [8] derive a sharp interface limit for a stochastically perturbed mass conserved Allen-Cahn equation with sufficiently mild additive noise. This is different from the stochastic term in this paper which is not smooth. A singular limit of a rescaled version of Problem (P) with linear diffusion has been studied by Antonopoulou, Bates, Blömker and Karali [3] to model the motion of a droplet. However, they left open the problem of proving the existence and uniqueness of the solution. The problem [7] is more general than the one in [3] since it has a nonlinear diffusion term. The proof is based on a Galerkin method together with a monotonicity argument similar to that used in [10] for a deterministic reaction-diffusion equation, and that in [9] for a stochastic problem.

My Master Thesis is based mainly in [7] and is organized as follows:

First, we introduce an auxiliary problem, more precisely the nonlinear stochastic heat equation, with nonlinear diffusion, and prove the existence and uniqueness of the solution W_A.

Next, we prove the existence of a solution of Problem (P). To that purpose we perform the change of function $u(t) = \varphi(t) - W_A$, as the new unknown function, to obtain an equation without the noise term. This change is simplifies the use of Galerkin method, which yields uniform bounds for the approximate solution in $L^\infty(0,T;L^2(\Omega\times D))$, $L^2(\Omega\times(0,T);H^1(D))$ and in $L^{2p}(\Omega\times(0,T)\times D)$. Applying the Galerkin method, we search for suitable a priori estimates. We deduce that the approximate weak solution u_m weakly converges along a subsequence to a limits. The main problem is then to identify the limit of the elliptic term, $\dim(A(\nabla(u_m + W_A)))$, and the reaction term, $f(u_m + W_A)$, as $m \to \infty$, which we do by means of the so-called monotonicity method.

Finally, we prove the uniqueness of the weak solution which in turn implies the convergence of the whole sequence.

Chapter 2

Existence and uniqueness of the stochastic Allen-Cahn equation

2.1 Functions spaces

Throughout this thesis we work with the following functions spaces:

$$H = \Big\{ v \in L^2(D), \int_D v = 0 \Big\}, \qquad V = H^1(D) \cap H \quad \text{and} \;\; Z = V \cap L^{2p}(D)$$

and denote with $\|\cdot\|$ the norm corresponding to the space H. We also define $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{Z^*, Z}$ as the duality product between Z and its dual space $Z^* = V^* + L^{\frac{2p}{2p-1}}(D)$.

2.2 An auxiliary problem

We consider the Neumann boundary value problem for the stochastic nonlinear heat equation

$$(P_1) \begin{cases} \dfrac{\partial W_A}{\partial t} = \operatorname{div}(A(\nabla W_A)) + \dfrac{\partial W}{\partial t}, & x \in D, \; t \geq 0, \\ A(\nabla W_A) \cdot \nu = 0, & x \in \partial D, \; t \geq 0, \\ W_A(x, 0) = 0, & x \in D. \end{cases}$$

We define a solution of Problem $(P1)$ as follows:

Definition 2.1. **(Strong solution)** *We say that W_A is a strong solution of Problem (P_1) if :*

(i) $W_A \in L^\infty(0, T; L^2(\Omega \times D)) \cap L^2(\Omega \times (0, T); H^1(D))$;

(ii) $W_A \in L^2(\Omega; C([0, T]; L^2(D)))$;

(iii) $\operatorname{div}(A(\nabla W_A)) \in L^2(\Omega \times (0, T); L^2(D))$;

(iv) W_A *satisfies a.s. for all* $t \in (0, T)$ *the problem*

$$\begin{cases} W_A(t) = \displaystyle\int_0^t \operatorname{div}(A(\nabla W_A(s)))ds + W(t), \quad \text{in } L^2(D), \\ \\ A(\nabla W_A(t)) \cdot n = 0, \;\; \text{in a suitable sense of trace on } \partial D. \end{cases} \tag{2.1}$$

We will show the existence and uniqueness of the strong solution W_A of Problem (P_1). Moreover we will prove that

$$W_A \in L^\infty(0,T;L^q(\Omega \times D)) \quad \text{for all } q \in [2,\infty). \tag{2.2}$$

2.3 Existence of a solution of Problem (P_1)

We study the solution W_A of problem (P_1) using the Galerkin approximation. At first we derive an approximate solution $W_A^{m,n}$ of (P_1) and a priori estimates, which we need to show that the solution is global in time. A priori estimates imply that the elliptic term $\mathrm{div}(A(\nabla W_A))$ is bounded in $L^2(D)$. We also remark that there are no reaction terms i.e., $f_i = 0$ for i from 1 to n and that the noise is additive.

We prove the following result.

Theorem 2.1. *There exists a unique solution of problem (P_1).*

Proof. **Step 1:** *Solution of the approximate problem.* To begin with, we approximate the function Ψ by a sufficiently smooth function Ψ^n such that

$$\Psi^n \to \Psi \quad \text{in} \quad C^1(\mathbb{R}^n) \tag{2.3}$$

and

$$\|D^2\Psi^n\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})} \le c_1,\ \nabla\Psi^n(0) = 0.$$

It turns out that the upper bounds which we find do not depend on n. We define an approximate solution $W_A^{m,n}$ of (P_1) by

$$W_A^{m,n}(t) = \int_0^t P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))]ds + \sum_{l=1}^m P_m(\sqrt{\lambda_l}e_l)\beta_l(t) \tag{2.4}$$

a.s., where for $v \in L^2(D)$

$$P_m v := \sum_{j=1}^m (\int_D v w_j) w_j$$

and

$$P_m : H^1(D) \to H_m = span\{w_1, \dots, w_m\}, \quad m \in \mathbb{N}$$

is the continuous operator defined by

$$\|a - P_m a\|^2_{H^1(D)} = \inf_{v \in H_m} \|a - v\|^2_{H^1(D)}, a \in H^1(D).$$

Note that

$$\|P_m a\|_{H^1(D)} \le \|a\|_{H^1(D)}. \tag{2.5}$$

and that

$$P_m a \to a, \quad \text{in } H^1(D) \text{ as } m \to \infty. \tag{2.6}$$

This implies in particular that

$$P_m a \to a, \quad \text{in } L^2(D) \text{ as } m \to \infty. \tag{2.7}$$

In addition, we have that

$$\int_D u_m P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]\,dx = -\int_D \nabla u_m \nabla\Psi^n(\nabla W_A^{m,n})\,dx. \tag{2.8}$$

Indeed,

$$
\begin{aligned}
\int_D u_m P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]\,dx &= \\
\int_D u_m \sum_{j=1}^m (\int_D \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))w_j\,dx)w_j\,dx &= \\
\sum_{j=1}^m \int_D u_m w_j \int_D \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))w_j\,dx\,dx &= \\
\int_D \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n})) \sum_{j=1}^m (\int_D u_m w_j\,dx)w_j\,dx &= \\
\int_D \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n})) \sum_{j=1}^m \langle u_m, w_j\rangle w_j\,dx &= \\
\int_D \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))u_m\,dx &\overset{\text{(int. by parts)}}{=\!=} \\
-\int_D \nabla\Psi^n(\nabla W_A^{m,n})\nabla u_m\,dx &
\end{aligned}
$$

2.3.1 A priori estimates

Step 2: *A priori estimates for $W_A^{m,n}$.* In what follows, we derive a priori estimates for the function $W_A^{m,n}$.We will use these estimates to prove the boundedness of $W_A^{m,n}$.

Lemma 2.1. *There exists a positive constant K such that*

$$\mathbb{E}\int_0^T \int_D (W_A^{m,n})^2\,dxdt \;\le\; K, \tag{2.9}$$

$$\mathbb{E}\int_0^T \int_D |\nabla(W_A^{m,n})|^2\,dxdt \;\le\; K, \tag{2.10}$$

$$\mathbb{E}\int_0^T \|P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))\|^2_{L^2(D)}dt \;\le\; K, \tag{2.11}$$

$$\sup_{t\in(0,T)} \mathbb{E}\int_D (W_A^{m,n})^2 \;\le\; K. \tag{2.12}$$

Proof. We first prove the last estimate. We recall Itô's formula, and is applicable to systems of stochastic ordinary differential equations.

Lemma 2.2. *For a smooth vector function h and an adapted process $(g(t), t \ge 0)$ with $\int_0^T |g(t)| \le \infty$ almost surely, for all $T > 0$ set*

$$X(t) := \int_0^t g(s)ds + \int_0^t hdW(s), \quad 0 \le t \le T,$$

where h is a vector of components $h_l, l = 1, \dots, m$ and dW is a vector of components $d\beta_l$, $l = 1, \dots, m$ with β_l a one-dimensional Brownian motion. Then, for F twice continuously differentiable in X and continuously differentiable in t, one has

$$
\begin{aligned}
F(X(t),t) &= F(X(0),0) + \int_0^t F_t(X(s),s)ds + \int_0^t F_x(X(s),s)g(s)ds \\
&\quad + \int_0^t F_x(X(s),s)hdW(s) + \frac{1}{2}\sum_{l=1}^m \int_0^t F_{xx}(X(s),s)h_l^2 ds.
\end{aligned} \tag{2.13}
$$

Next we apply Lemma 2.2 to (2.4) with $hdW = \sum_{l=1}^{m} P_m \sqrt{\lambda_l} e_l d\beta_l(s)$ and $h_l = P_m \sqrt{\lambda_l} e_l$, supposing that F does not depend on time and setting

$$\begin{aligned}
&X(t) = W_A^{m,n}(t),\\
&F(X(t)) = (X(t))^2 = (W_A^{m,n}(t))^2,\\
&F'(X(t)) = 2X(t) = 2W_A^{m,n}(t),\\
&F''(X(t)) = 2,\\
&g(s) = P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s))).
\end{aligned}$$

We remark that in this case F does not depend on t, so $F_t(X(s), s) = 0$ and $F(X(0), 0) = 0$. Substituting on (2.13) we have

$$\begin{aligned}
W_A^{m,n}(x,t)^2 \;\; &= \;\; 2\int_0^t W_A^{m,n} P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))ds \\
&\quad + 2\sum_{l=1}^{m}\int_0^t W_A^{m,n} P_m\sqrt{\lambda_l} e_l d\beta_l(s) + \int_0^t \sum_{l=1}^{m} (P_m\sqrt{\lambda_l} e_l)^2 ds.
\end{aligned}$$

After integrating on D, we obtain almost surely, for all $t \in [0,T]$,

$$\begin{aligned}
\int_D W_A^{m,n}(x,t)^2\, dx \;\; &= \;\; 2\int_0^t \int_D W_A^{m,n} P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))\, dxds \\
&\quad + 2\sum_{l=1}^{m}\int_0^t \int_D W_A^{m,n} P_m \sqrt{\lambda_l} e_l\, dxd\beta_l(s) \\
&\quad + \int_0^t \sum_{l=1}^{m} \int_D (P_m \sqrt{\lambda_l} e_l)^2\, dxds \qquad (2.14)
\end{aligned}$$

or equivalently

$$\begin{aligned}
\|W_A^{m,n}(t)\|^2_{L^2(D)} \;\; &= \;\; 2\int_0^t \int_D W_A^{m,n} P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))\, dxds \\
&\quad + 2\sum_{l=1}^{m}\int_0^t \int_D W_A^{m,n} P_m \sqrt{\lambda_l} e_l dxd\beta_l(s) \\
&\quad + \int_0^t \sum_{l=1}^{m} \|P_m \sqrt{\lambda_l}\|^2_{L^2(D)} ds. \qquad (2.15)
\end{aligned}$$

Using (2.8), we have

$$\int_D W_A^{m,n} P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))dx = -\int_D \nabla W_A^{m,n} \nabla\Psi^n(\nabla W_A^{m,n}(s))dx.$$

Substituting in (2.15), we take

$$\begin{aligned}
&\|W_A^{m,n}(t)\|^2_{L^2(D)} + 2\int_0^t \int_D \nabla W_A^{m,n} \nabla\Psi^n(\nabla W_A^{m,n}(s))dxds \\
&= \; 2\sum_{l=1}^{m}\int_0^t \int_D W_A^{m,n} P_m \sqrt{\lambda_l} e_l dxd\beta_l(s) + \int_0^t \sum_{l=1}^{m} \|P_m \sqrt{\lambda_l}\|^2_{L^2(D)} ds. \qquad (2.16)
\end{aligned}$$

Taking the expectation, we obtain

$$\mathbb{E}\|W_A^{m,n}(t)\|^2_{L^2(D)} + 2\mathbb{E}\int_0^t \int_D \nabla W_A^{m,n} \nabla\Psi^n(\nabla W_A^{m,n}(s))dxds$$

$$= \mathbb{E}\int_0^t \sum_{l=1}^m \|P_m\sqrt{\lambda_l}e_l\|^2_{L^2(D)}ds, \tag{2.17}$$

where we have used the fact that $2\mathbb{E}[\sum_{l=1}^m \int_0^t \int_D W_A^{m,n} P_m\sqrt{\lambda_l}e_l dxd\beta_l(s)] = 0$. We deduce from (2.5) that

$$\begin{aligned}
\sum_{l=1}^m \|P_m\sqrt{\lambda_l}\|^2_{L(D)} &\leq \sum_{l=1}^m \|P_m\sqrt{\lambda_l}\|^2_{H^1(D)} \\
&= \sum_{l=1}^m \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)} + \|\nabla(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)} \\
&\leq \Lambda_0 + \Lambda_1
\end{aligned} \tag{2.18}$$

Returning to (2.17), we have

$$\begin{aligned}
\mathbb{E}\|W_A^{m,n}(t)\|^2_{L^2(D)} &= \mathbb{E}\int_0^t \sum_{l=1}^m \|P_m\sqrt{\lambda_l}\|^2_{L^2(D)}ds \\
&\quad -2\mathbb{E}\int_0^t \int_D \nabla W_A^{m,n}\nabla\Psi^n(\nabla W_A^{m,n}(s))dxds,
\end{aligned}$$

implies that

$$\mathbb{E}\|W_A^{m,n}(t)\|^2_{L^2(D)} \leq \mathbb{E}\int_0^t \sum_{l=1}^m \|P_m\sqrt{\lambda_l}\|^2_{L^2(D)}ds.$$

Taking the supremum of the above equation we obtain

$$\begin{aligned}
\sup_{t\in(0,T)} \mathbb{E}\|W_A^{m,n}(t)\|^2_{L^2(D)} &\leq \sup_{t\in(0,T)} \mathbb{E}\int_0^t \sum_{l=1}^m \|P_m\sqrt{\lambda_l}\|^2_{L^2(D)}ds \\
&= \sup_{t\in(0,T)} \mathbb{E}\sum_{l=1}^m t\|P_m\sqrt{\lambda_l}\|^2_{L^2(D)} \\
&\overset{(2.18)}{\leq} T(\Lambda_0 + \Lambda_1) \\
&\leq K.
\end{aligned}$$

This completes the proof of (2.9). In order to obtain an H^2-type estimate for W_A^m ((2.9), (2.10), (2.11) estimates), we take the gradient of the equation (2.4).

$$\begin{aligned}
\nabla W_A^{m,n}(t) &= \int_0^t \nabla\{P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]\}ds + \sum_{l=1}^m \nabla\{P_m[\sqrt{\lambda_l}e_l]\}\beta_l(t) \\
&= \int_0^t \nabla\{P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]\}ds + \sum_{l=1}^m \int_0^t \nabla P_m[\sqrt{\lambda_l}e_l]d\beta_l(s)
\end{aligned} \tag{2.19}$$

We fix $x \in D$ and apply below for a second time Itô's formula Lemma 2.2 to the integral equation (2.19) where in this case $hdW = \sum_{l=1}^m \nabla\{P_m\sqrt{\lambda_l}e_l\}d\beta_l(s)$ and $h_l = \nabla\{P_m\sqrt{\lambda_l}e_l\}$ with:

$$X(t) = \nabla W_A^{m,n}(x,t),$$
$$F(X(t)) = \Psi^n(\nabla W_A^{m,n}(x,t)),$$

$$F'(X(t)) = \nabla\Psi^n(\nabla W_A^{m,n}(x,t)),$$
$$F''(X(t)) = D^2\Psi^n(\nabla W_A^{m,n}(x,t)),$$
$$g(s) = \nabla\{P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(x,s)))\}.$$

We remark that F does not depend on t, so $F_t(X(t)) = 0$ and $F(X(0)) = 0$. Substituting all above in (2.19), we obtain

$$\begin{aligned}\Psi^n(\nabla W_A^{m,n}(x,t)) \quad = \quad & \int_0^t \nabla\Psi^n(\nabla W_A^{m,n}(x,s))\nabla\{P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))\}ds \\ & + \int_0^t \nabla\Psi^n(\nabla W_A^{m,n}(x,s))\sum_{l=1}^m \nabla\{P_m\sqrt{\lambda_l}e_l\}d\beta_l(s) \\ & + \frac{1}{2}\sum_{l=1}^m \int_0^t D^2\Psi^n(\nabla W_A^{m,n}(x,t))|\nabla P_m(\sqrt{\lambda_l}e_l)|^2 ds.\end{aligned}$$

After integrating over D, we obtain almost surely, for all $t \in [0,T]$,

$$\begin{aligned} & \int_D \Psi^n(\nabla W_A^{m,n}(x,t))dx \\ & = \int_0^t \int_D \nabla\Psi^n(\nabla W_A^{m,n}(x,s))\nabla\{P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))\}dxds \\ & \quad + \sum_{l=1}^m \int_0^t \int_D \nabla\Psi^n(\nabla W_A^{m,n}(x,s))\nabla\{P_m\sqrt{\lambda_l}e_l\}dxd\beta_l(s) \\ & \quad + \frac{1}{2}\sum_{l=1}^m \int_0^t \int_D D^2\Psi^n(\nabla W_A^{m,n}(x,s))|\nabla P_m(\sqrt{\lambda_l}e_l)|^2 dxds. \end{aligned} \tag{2.20}$$

By (2.8) the second term of the above equation becomes

$$\begin{aligned} & \int_0^t \int_D \nabla\Psi^n(\nabla W_A^{m,n}(x,s))\nabla\{P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))\}dxds \\ & = -\int_0^t \int_D P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))dxds \\ & = -\int_0^t \int_D [P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))]^2 dxds \\ & = -\int_0^t \|P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))\|^2_{L^2(D)}ds. \end{aligned}$$

Substituting in (2.20) we obtain

$$\begin{aligned} \int_D \Psi^n(\nabla W_A^{m,n}(x,t))dx \quad & = \\ -\int_0^t \|P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))\|^2_{L^2(D)}ds & \\ + \sum_{l=1}^m \int_0^t \int_D \nabla\Psi^n(\nabla W_A^{m,n}(x,s))\nabla\{P_m\sqrt{\lambda_l}e_l\}dxd\beta_l(s) & \\ + \frac{1}{2}\|D^2\Psi^n(\nabla W_A^{m,n}(s))\|^2_{L^2(D)}\sum_{l=1}^m \int_0^t \int_D |\nabla P_m(\sqrt{\lambda_l}e_l)|^2 dxds \quad & \overset{(1.1)}{\leq} \\ -\int_0^t \|P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))\|^2_{L^2(D)}ds & \end{aligned}$$

$$+\sum_{l=1}^{m}\int_0^t\int_D \nabla\Psi^n(\nabla W_A^{m,n}(x,s))\nabla\{P_m\sqrt{\lambda_l}e_l\}dxd\beta_l(s)$$

$$+\frac{1}{2}c_1\sum_{l=1}^{m}\int_0^t\int_D |\nabla P_m(\sqrt{\lambda_l}e_l)|^2 dxds \quad =$$

$$-\int_0^t \|P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))\|^2_{L^2(D)}ds$$

$$+\sum_{l=1}^{m}\int_0^t\int_D \nabla\Psi^n(\nabla W_A^{m,n}(x,s))\nabla\{P_m\sqrt{\lambda_l}e_l\}dxd\beta_l(s)$$

$$+\frac{c_1}{2}\sum_{l=1}^{m}\int_0^t \|\nabla P_m(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)}ds. \tag{2.21}$$

Thus taking the expectation of (2.21) we obtain

$$\begin{aligned}\mathbb{E}\int_D \Psi^n(\nabla W_A^{m,n}(x,t))dx \leq &-\mathbb{E}\int_0^t \|P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))\|^2_{L^2(D)}ds\\ &+\mathbb{E}\sum_{l=1}^{m}\int_0^t\int_D \nabla\Psi^n(\nabla W_A^{m,n}(x,s))\nabla\{P_m\sqrt{\lambda_l}e_l\}dxd\beta_l(s)\\ &+\frac{c_1}{2}\mathbb{E}\sum_{l=1}^{m}\int_0^t \|\nabla P_m(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)}ds.\end{aligned}$$

Using the fact that

$$\mathbb{E}[\sum_{l=1}^{m}\int_0^t\int_D \nabla\Psi^n(\nabla W_A^{m,n}(x,s))\nabla\{P_m(\sqrt{\lambda_l}e_l)\}dxd\beta_l(s)] = 0,$$

we have

$$\mathbb{E}\int_D \Psi^n(\nabla W_A^{m,n}(x,t))dx + \mathbb{E}\int_0^t \|P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))\|^2_{L^2(D)}ds \quad \leq$$

$$\frac{c_1}{2}\mathbb{E}\sum_{l=1}^{m}\int_0^t \|\nabla P_m(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)}ds. \tag{2.22}$$

Adding (2.17) and (2.22), using (2.5), (1.6) and (1.7) we obtain

$$\begin{aligned}&\mathbb{E}\int_D \Psi^n(\nabla W_A^{m,n}(x,t))dx + \mathbb{E}\int_0^t \|P_m \mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))\|^2_{L^2(D)}ds\\ &\quad + \mathbb{E}\|W_A^{m,n}(t)\|^2_{L^2(D)} + 2\mathbb{E}\int_0^t\int_D \nabla W_A^{m,n}\nabla\Psi^n(\nabla W_A^{m,n}(s))dxds \quad \leq\\ &\frac{c_1}{2}\mathbb{E}\sum_{l=1}^{m}\int_0^t \|\nabla P_m(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)}ds + \mathbb{E}\int_0^t\sum_{l=1}^{m}\|P_m\sqrt{\lambda_l}e_l\|^2_{L^2(D)}ds \quad \overset{(\frac{c_1}{2},1\leq c_0)}{\leq}\\ &c_0\mathbb{E}\sum_{l=1}^{m}\int_0^t \|\nabla P_m(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)}ds + c_0\mathbb{E}\int_0^t\sum_{l=1}^{m}\|P_m\sqrt{\lambda_l}e_l\|^2_{L^2(D)}ds \quad =\\ &c_0\mathbb{E}\sum_{l=1}^{m}\int_0^t \|\nabla P_m(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)} + \|P_m\sqrt{\lambda_l}e_l\|^2_{L^2(D)}ds \quad =\\ &c_0\mathbb{E}\sum_{l=1}^{m}\int_0^t \|P_m\sqrt{\lambda_l}e_l\|^2_{H^1(D)}ds \quad \overset{(2.5)}{\leq}\end{aligned}$$

$$
\begin{aligned}
c_0\mathbb{E}\sum_{l=1}^{m}\int_0^t \|\sqrt{\lambda_l}e_l\|^2_{H^1(D)}ds &= \\
c_0\mathbb{E}\sum_{l=1}^{m}\int_0^t \|\nabla\sqrt{\lambda_l}e_l\|^2_{L^2(D)}+\|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}ds &= \\
c_0\mathbb{E}\int_0^t \Big(\sum_{l=1}^{m}\lambda_l\|\nabla e_l\|^2_{L^2(D)}+\lambda_l\|e_l\|^2_{L^2(D)}\Big)ds &= \\
c_0\mathbb{E}\int_0^t ds\Big(\sum_{l=1}^{m}\lambda_l\|\nabla e_l\|^2_{L^2(D)}+\lambda_l\|e_l\|^2_{L^2(D)}\Big) &= \\
c_0\mathbb{E}\Big(\sum_{l=1}^{m}\lambda_l\|\nabla e_l\|^2_{L^2(D)}+\lambda_l\|e_l\|^2_{L^2(D)}\Big)t &\overset{(1.6),(1.7)}{\leq} \\
c_0T(\Lambda_0+\Lambda_1), \qquad (2.23)
\end{aligned}
$$

where $c_0 = max(1, \frac{c_1}{2})$. In view of (1.3) we obtain,

$$
\begin{aligned}
2\mathbb{E}\int_0^t\int_D \nabla W_A^{m,n}\nabla\Psi^n(\nabla W_A^{m,n}(s))dxds &= 2C_0\mathbb{E}\int_0^t\int_D |\nabla W_A^{m,n}(s)|^2dxds \\
&= 2C_0\mathbb{E}\int_0^t \|\nabla W_A^{m,n}(s)\|^2_{L^2(D)}ds.
\end{aligned}
$$

Substituting in (2.23) we obtain,

$$
\begin{aligned}
&\mathbb{E}\int_D \Psi^n(\nabla W_A^{m,n}(x,t))dx+\mathbb{E}\int_0^t \|P_m\text{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))\|^2_{L^2(D)}ds \\
&\quad+\mathbb{E}\|W_A^{m,n}(t)\|^2_{L^2(D)}+2C_0\mathbb{E}\int_0^t \|\nabla W_A^{m,n}(s)\|^2_{L^2(D)}ds \\
&\leq c_0T(\Lambda_0+\Lambda_1)\leq K,
\end{aligned}
$$

which completes the proof of (2.9), (2.10) and (2.11). □

According to the a priori estimates, we see that the sequences are bounded.
Step 3: *Passing to the limit.* Consequently there exist a subsequence which we denote again by $W_A^{m,n}$ and a function $W_A \in L^2(\Omega\times(0,T);H^1)\cap L^\infty(0,T;L^2(\Omega\times D))$ such that

$$W_A^{m,n} \rightharpoonup W_A \text{ weakly in } L^2(\Omega\times(0,T);H^1(D)) \tag{2.24}$$

$$W_A^{m,n} \rightharpoonup W_A \text{ weakly star in } L^\infty(0,T;L^2(\Omega\times D)) \tag{2.25}$$

$$P_m\text{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s))) \rightharpoonup \tilde{\Phi} \text{ weakly in } L^2(\Omega\times(0,T);L^2(D)) \tag{2.26}$$

as $m,n\to\infty$.
In addition, one can show the following result.

Lemma 2.3.

$$\sum_{l=1}^{m}P_m(\sqrt{\lambda_l}e_l)\beta_l(t)\overset{m\to\infty}{\longrightarrow}\sum_{l=1}^{m}\sqrt{\lambda_l}e_l\beta_l(t), \quad in\ L^\infty((0,T);L^2(\Omega;L^2(D))). \tag{2.27}$$

Proof. For all $t\in[0,T]$,

$$\mathbb{E}\int_D\Big|\sum_{l=1}^{\infty}\sqrt{\lambda_l}e_l\beta_l(t)-\sum_{l=1}^{m}P_m(\sqrt{\lambda_l}e_l)\beta_l(t)\Big|^2dx$$

$$
\begin{aligned}
&= \mathbb{E}\int_D \Big| \sum_{l=1}^{\infty} \sqrt{\lambda_l} e_l \beta_l(t) - \sum_{l=1}^{m} P_m(\sqrt{\lambda_l} e_l)\beta_l(t) \\
&\quad \pm \sum_{l=1}^{m} \sqrt{\lambda_l} e_l \beta_l(t) \Big|^2 dx \\
&= \mathbb{E}\int_D \Big| \sum_{l=1}^{\infty} \sqrt{\lambda_l} e_l \beta_l(t) - \sum_{l=1}^{m} \sqrt{\lambda_l} e_l \beta_l(t) \\
&\quad + \sum_{l=1}^{m} \sqrt{\lambda_l} e_l \beta_l(t) - \sum_{l=1}^{m} P_m(\sqrt{\lambda_l} e_l)\beta_l(t) \Big|^2 dx \\
\big(\scriptstyle (a+b)^2 \le 2(a^2+b^2)\big) \quad &\le 2\mathbb{E}\int_D \Big| \sum_{l=1}^{\infty} \sqrt{\lambda_l} e_l \beta_l(t) - \sum_{l=1}^{m} \sqrt{\lambda_l} e_l \beta_l(t) \Big|^2 dx \\
&\quad + 2\mathbb{E}\int_D \Big| \sum_{l=1}^{m} \sqrt{\lambda_l} e_l \beta_l(t) - \sum_{l=1}^{m} P_m(\sqrt{\lambda_l} e_l)\beta_l(t) \Big|^2 dx \\
&= 2\mathbb{E}\int_D \Big| \sum_{l=m+1}^{\infty} \sqrt{\lambda_l} e_l \beta_l(t) \Big|^2 dx \\
&\quad + 2\mathbb{E}\int_D \Big| \sum_{l=1}^{m} [P_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l]\beta_l(t) \Big|^2 dx \\
&= W_1 + W_2.
\end{aligned}
$$

We deduce that $W_1 \to 0$ in $C([0,T])$ as $m \to \infty$. For W_2, by the properties of the Brownian motion $\beta_l(t)$, ($\beta_l(t) \sim (0,t)$), we have that

$$
\begin{aligned}
&2\int_D \mathbb{E}\Big| \sum_{l=1}^{m} [P_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l]\beta_l(t) \Big|^2 dx \\
&= 2\sum_{l=1}^{m} \int_D [P_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l]^2 dx\, \mathbb{E}[\beta_l^2(t)] \\
\big(\scriptstyle \mathrm{Var}(\beta_l(t)) = \mathbb{E}[\beta_l^2(t)] = t\big) \quad &= 2\sum_{l=1}^{m} \|P_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|^2_{L^2(D)} t \\
&\le 2T \sum_{l=1}^{\infty} \|P_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|^2_{L^2(D)}.
\end{aligned} \tag{2.28}
$$

In order to prove that the right-hand side of (2.28) tends to zero as $m \to \infty$, we use (2.5) and (2.7) to deduce that

$$
\begin{aligned}
&\sum_{l=1}^{\infty} \|P_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|^2_{L^2(D)} \\
= \quad &\sum_{l=1}^{K} \|P_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|^2_{L^2(D)} + \sum_{l=K+1}^{\infty} \|P_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|^2_{L^2(D)} \\
\le \quad &\sum_{l=1}^{K} \|P_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|^2_{L^2(D)} + \sum_{l=K+1}^{\infty} 2\|P_m(\sqrt{\lambda_l} e_l)\|^2_{L^2(D)} + 2\|\sqrt{\lambda_l} e_l\|^2_{L^2(D)} \\
\le \quad &\sum_{l=1}^{K} \|P_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|^2_{L^2(D)} + \sum_{l=K+1}^{\infty} 2\|P_m(\sqrt{\lambda_l} e_l)\|_{H^1(D)} + 2\|\sqrt{\lambda_l} e_l\|_{H^1(D)} \\
\overset{(2.5)}{\le} \quad &\sum_{l=1}^{K} \|P_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|^2_{L^2(D)} + \sum_{l=K+1}^{\infty} 2\|\sqrt{\lambda_l} e_l\|^2_{H^1(D)} + 2\|\sqrt{\lambda_l} e_l\|^2_{H^1(D)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{l=1}^{K} \|P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l\|^2_{L^2(D)} + 4\sum_{l=K+1}^{\infty} \|\sqrt{\lambda_l}e_l\|^2_{H^1(D)} \\
&= \sum_{l=1}^{K} \|P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l\|^2_{L^2(D)} + 4\sum_{l=K+1}^{\infty} \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)} + \|\nabla\sqrt{\lambda_l}e_l\|^2_{L^2(D)} \\
&= \sum_{l=1}^{K} \|P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l\|^2_{L^2(D)} + 4\sum_{l=K+1}^{\infty} \lambda_l + \lambda_l\|\nabla e_l\|^2_{L^2(D)} \\
&\leq P_1 + P_2. \qquad (2.29)
\end{aligned}
$$

Let $\epsilon > 0$ be arbitrary. We choose K such that $P_2 \leq \frac{\epsilon}{2}$. For a fixed K, we choose m sufficiently large such that $P_1 \leq \frac{\epsilon}{2}$. Therefore,

$$
\sum_{l=1}^{\infty} \|P_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l\|^2_{L^2(D)} \leq \epsilon, \qquad (2.30)
$$

so that $W_2 \to 0$ in $C([0,T])$ as $m \to \infty$. □

Let y be an arbitrary bounded random variable, and let ψ be an arbitrary bounded function on $(0,T)$. Next we multiply the equation (2.4) by the product $y\psi$, integrate on D between 0 and T and take the expectation to obtain

$$
\begin{aligned}
\mathbb{E}\int_0^T \int_D y\psi(t)W_A^{m,n} w_j dxdt &= \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle P_m(\mathrm{div}[\nabla\Psi^n(\nabla W_A^{m,n})]), w_j\rangle ds\}dt \\
&\quad + \mathbb{E}\int_0^T y\psi(t)\{\int_D \sum_{l=1}^{m} P_m(\sqrt{\lambda_l}e_l)\beta_l(t)w_j dx\}dt.
\end{aligned}
$$

Passing to the limit when $m,n \to \infty$, using (2.24)-(2.26) and (2.27), and remembering that the linear combinations of w_j are dense in $H^1(D)$, yields

$$
\begin{aligned}
\mathbb{E}\int_0^T \int_D y\psi(t)W_A \tilde{w} dxdt &= \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle \Phi, \tilde{w}\rangle ds\}dt \\
&\quad + \mathbb{E}\int_0^T y\psi(t)\{\int_D \sum_{l=1}^{\infty}(\sqrt{\lambda_l}e_l)\beta_l(t)\tilde{w} dx\}dt,
\end{aligned}
$$

for all $\tilde{w} \in H^1(D)$. Therefore, we deduce that

$$
W_A(t) = \int_0^t \tilde{\Phi}(s)ds + \sum_{l=1}^{\infty} \sqrt{\lambda_l}e_l\beta_l(t) \text{ on } \Omega \times (0,T) \times D. \qquad (2.31)
$$

We will prove below, using the **monotonicity method**, that

$$
\tilde{\Phi} = \mathrm{div}(\nabla\Psi(\nabla W_A)).
$$

2.3.2 Monotonicity argument

Step 1: Let w be such that $w \in L^2(\Omega\times(0,T); H^1(D))$ and let c be a positive constant. We define

$$
\begin{aligned}
O_{mn} &= \mathbb{E}\Big[\int_0^T e^{-cs}\{2\langle P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))] - P_m[\mathrm{div}(\nabla\Psi^n(\nabla w))], W_A^{m,n} - w\rangle \\
&\quad - c\|W_A^{m,n} - w\|^2\}ds\Big]
\end{aligned}
$$

$$= \quad J_1 + J_2,$$

where

$$J_1 \quad = \quad \mathbb{E}\int_0^T e^{-cs} 2\langle P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))] - P_m[\mathrm{div}(\nabla\Psi^n(\nabla w))], W_A^{m,n} - w\rangle ds$$

and

$$J_2 \quad = \quad -\mathbb{E}\int_0^T e^{-cs} c\|W_A^{m,n} - w\|^2 ds.$$

Step 2: We will check the following result

Lemma 2.4.

$$O_{mn} \leq 0.$$

Proof. Using (2.8) and (1.3) we have that

$$\begin{aligned}
J_1 &= \mathbb{E}\int_0^T e^{-cs} 2\langle P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))] - P_m[\mathrm{div}(\nabla\Psi^n(\nabla w))], W_A^{m,n} - w\rangle ds \\
&= 2\mathbb{E}\int_0^T e^{-cs} \int_D P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))] - P_m[\mathrm{div}(\nabla\Psi^n(\nabla w))](W_A^{m,n} - w) \\
&\overset{(2.8)}{=} -2\mathbb{E}\int_0^T e^{-cs} \int_D [\nabla\Psi^n(\nabla W_A^{m,n}) - \nabla\Psi^n(\nabla w)]\nabla(W_A^{m,n} - w) dxds \\
&\overset{(1.3)}{\leq} -2\mathbb{E}\int_0^T e^{-cs} \int_D C_0|\nabla(W_A^{m,n} - w)|^2 dxds \\
&= -2C_0\mathbb{E}\int_0^T e^{-cs}\|\nabla(W_A^{m,n} - w)\|^2_{L(D)} ds \\
&\leq 0
\end{aligned}$$

and

$$J_2 \quad = \quad -\mathbb{E}\int_0^T e^{-cs} c\|W_A^{m,n} - w\|^2 ds \leq 0,$$

which complete the proof. □

Step 3: We have to find the $\lim_{m,n\to\infty} \sup O_{mn}$.
We write O_{mn} in the form $O_{mn} = O^1_{mn} + O^2_{mn}$ where

$$O^1_{mn} = \mathbb{E}\int_0^T e^{-cs}\{2\langle P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))], W_A^{m,n}\rangle - c\|W_A^{m,n}\|^2\} ds \qquad (2.32)$$

and O^2_{mn} is defined by the difference. We apply Itô's formula Lemma 2 on (2.4) where $h_l dW = \sum_{l=1}^m P_m(\sqrt{\lambda_l} e_l) d\beta_l(s)$ and $h_l = P_m(\sqrt{\lambda_l} e_l)$ with

$$\begin{aligned}
&X(t) = W_A^{m,n}(x,t), \\
&F(X,t) = e^{-ct}(X)^2 = e^{-ct}(W_A^{m,n})^2, \\
&F_t = -ce^{-ct}(X)^2 = -ce^{-ct}(W_A^{m,n})^2, \\
&F' = 2e^{-ct}(X) = 2e^{-ct}(W_A^{m,n}), \\
&F'' = 2e^{-ct}, \\
&g(s) = P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}(x,s)))],
\end{aligned}$$

we obtain

$$
\begin{aligned}
e^{-ct}(W_A^{m,n})^2 = &- c\int_0^t e^{-ct}(W_A^{m,n})^2 ds + 2\int_0^t e^{-ct}W_A^{m,n}P_m[\operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]ds \\
&+ 2\int_0^t e^{-ct}W_A^{m,n}\sum_{l=1}^m P_m(\sqrt{\lambda_l}e_l)d\beta_l(s)ds \\
&+ \frac{1}{2}\sum_{l=1}^m\int_0^t 2e^{-ct}|P_m(\sqrt{\lambda_l}e_l)|^2 ds.
\end{aligned}
$$

After integrating over D, we obtain almost surely, for $t \in [0,T]$,

$$
\begin{aligned}
\int_D e^{-cT}(W_A^{m,n})^2 dx \quad = \quad & -c\int_0^T e^{-cs}\int_D (W_A^{m,n})^2 dt \\
& +2\int_0^T e^{-cs}\int_D W_A^{m,n}P_m[\operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]dxdt \\
& +2\sum_{l=1}^m\int_0^T e^{-cs}\int_D W_A^{m,n}P_m(\sqrt{\lambda_l}e_l)d\beta_l(s)dxdt \\
& +\int_0^T e^{-cs}\sum_{l=1}^m\int_D |P_m(\sqrt{\lambda_l}e_l)|^2 dxdt
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
e^{-cT}\|W_A^{m,n}\|^2_{L^2(D)} \quad = \quad & -c\int_0^T e^{-cs}\|W_A^{m,n}\|^2_{L^2(D)}dt \\
& +2\int_0^T e^{-cs}\int_D W_A^{m,n}P_m[\operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]dxdt \\
& +2\sum_{l=1}^m\int_0^T e^{-cs}\int_D W_A^{m,n}P_m(\sqrt{\lambda_l}e_l)d\beta_l(s)dxdt \\
& +\int_0^T e^{-cs}\sum_{l=1}^m \|P_m(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)}dt.
\end{aligned}
$$

Taking the expectation, we have that

$$
\begin{aligned}
\mathbb{E}[e^{-cT}\|W_A^{m,n}\|^2_{L^2(D)}] \quad = \quad & -c\mathbb{E}\int_0^T e^{-cs}\|W_A^{m,n}\|^2_{L^2(D)}dt \\
& +2\mathbb{E}\int_0^T e^{-cs}\int_D W_A^{m,n}P_m[\operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]dxdt \\
& +2\mathbb{E}\sum_{l=1}^m\int_0^T e^{-cs}\int_D W_A^{m,n}P_m(\sqrt{\lambda_l}e_l)d\beta_l(s)dxdt \\
& +\int_0^T e^{-cs}\sum_{l=1}^m \|P_m(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)}dt. \qquad (2.33)
\end{aligned}
$$

From the fact that

$$
\mathbb{E}\sum_{l=1}^m\int_0^T e^{-cs}\int_D W_A^{m,n}P_m(\sqrt{\lambda_l}e_l)d\beta_l(s)dxdt = 0,
$$

(2.33) becomes

$$
\mathbb{E}[e^{-cT}\|W_A^{m,n}(T)\|^2_{L^2(D)}] \quad = \quad -c\mathbb{E}\int_0^T e^{-cs}\|W_A^{m,n}\|^2_{L^2(D)}dt
$$

$$
\begin{aligned}
&+2\mathbb{E}\int_0^T e^{-cs}\int_D W_A^{m,n}P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]dxdt\\
&+\int_0^T e^{-cs}\sum_{l=1}^m \|P_m(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)}dt.
\end{aligned}
$$

Substituting the above into (2.32) we find that

$$
\begin{aligned}
O^1_{mn} &= \mathbb{E}\int_0^T e^{-cs}\{2\langle P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))], W_A^{m,n}\rangle - c\|W_A^{m,n}\|^2_{L^2(D)}\}dt\\
&= 2\mathbb{E}\int_0^T e^{-cs}\langle P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))], W_A^{m,n}\rangle dt\\
&\quad -c\mathbb{E}\int_0^T e^{-cs}\|W_A^{m,n}\|^2_{L^2(D)}dt\\
&= 2\mathbb{E}\int_0^T e^{-cs}\langle P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))], W_A^{m,n}\rangle dt + \mathbb{E}[e^{-cT}\|W_A^{m,n}\|^2_{L^2(D)}]\\
&\quad -2\mathbb{E}\int_0^T e^{-cs}\int_D W_A^{m,n}P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]dxdt\\
&\quad -\int_0^T e^{-cs}\sum_{l=1}^m \|P_m(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)}dt\\
&= \mathbb{E}[e^{-cT}\|W_A^{m,n}(T)\|^2_{L^2(D)}] - \int_0^T e^{-cs}\sum_{l=1}^m \|P_m(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)}dt.
\end{aligned}
$$

We have that

$$
\begin{aligned}
&\Big|\int_0^T e^{-cs}\{\sum_{l=1}^m \|P_m\sqrt{\lambda_l}e_l\|^2_{L^2(D)} - \sum_{l=1}^\infty \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}\}dt\Big|\\
\overset{\left(\pm\sum_{l=1}^m\|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}\right)}{=}\;&\Big|\int_0^T e^{-cs}\sum_{l=1}^m \|P_m\sqrt{\lambda_l}e_l\|^2_{L^2(D)} - \sum_{l=1}^m \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}\\
&+\sum_{l=1}^m \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)} - \sum_{l=1}^\infty \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}dt\Big|\\
\overset{\text{(triangle ineq.)}}{\leq}\;&\Big|\int_0^T e^{-cs}\sum_{l=1}^m \|P_m\sqrt{\lambda_l}e_l\|^2_{L^2(D)} - \sum_{l=1}^m \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}dt\Big|\\
&+\Big|\int_0^T e^{-cs}\sum_{l=1}^m \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)} - \sum_{l=1}^\infty \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}dt\Big|\\
\leq\;&\int_0^T e^{-cs}\Big|\sum_{l=1}^m \|P_m\sqrt{\lambda_l}e_l\|^2_{L^2(D)} - \sum_{l=1}^m \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}\Big|dt\\
&+\int_0^T e^{-cs}\Big|\sum_{l=1}^m \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)} - \sum_{l=1}^\infty \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}\Big|dt\\
\leq\;&\int_0^T e^{-cs}\sum_{l=1}^m \Big|\|P_m\sqrt{\lambda_l}e_l\|^2_{L^2(D)} - \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}\Big|dt\\
&+\int_0^T e^{-cs}\sum_{l=m+1}^\infty \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}dt\\
\overset{\left(\|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}=\lambda_l\right)}{\leq}\;&T\sum_{l=1}^\infty \Big|\|P_m\sqrt{\lambda_l}e_l\|^2_{L^2(D)} - \|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}\Big| + T\sum_{l=m+1}^\infty \lambda_l
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad T\sum_{l=1}^{\infty}\|P_m\sqrt{\lambda_l}e_l-\sqrt{\lambda_l}e_l\|^2_{L^2(D)}+T\sum_{l=m+1}^{\infty}\lambda_l \\
&\leq \quad \epsilon
\end{aligned}
$$

which, in view of (2.30) and (1.5), tends to zero as $m\to\infty$. Thus,

$$
\lim_{m\to\infty}\int_0^T e^{-cs}\sum_{l=1}^{m}\|P_m\sqrt{\lambda_l}e_l\|^2_{L^2(D)}dt=\int_0^T e^{-cs}\sum_{l=1}^{\infty}\|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}dt.
$$

Letting m and n tend to infinity in (2.34), we deduce that

$$
\begin{aligned}
\lim_{m,n\to\infty}\sup O^1_{mn} \quad &= \quad \lim_{m,n\to\infty}\sup\Big[\mathbb{E}[e^{-cT}\|W_A^{m,n}(T)\|^2_{L^2(D)}] \\
&\quad -\int_0^T e^{-cs}\sum_{l=1}^{m}\|P_m\sqrt{\lambda_l}e_l\|^2_{L^2(D)}dt\Big] \\
\left(\pm\mathbb{E}[e^{-cT}\|W_A(T)\|^2_{L^2(D)}]\right) \quad &= \quad \lim_{m,n\to\infty}\sup\Big[\mathbb{E}[e^{-cT}\|W_A^{m,n}(T)\|^2_{L^2(D)}] \\
&\quad -\int_0^T e^{-cs}\sum_{l=1}^{\infty}\|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}dt \\
&\quad +\mathbb{E}[e^{-cT}\|W_A(T)\|^2_{L^2(D)}]-\mathbb{E}[e^{-cT}\|W_A(T)\|^2_{L^2(D)}] \\
&= \quad \mathbb{E}[e^{-cT}\|W_A(T)\|^2_{L^2(D)}]-\int_0^T e^{-cs}\sum_{l=1}^{\infty}\lambda_l dt+\delta e^{-cT} \qquad (2.34)
\end{aligned}
$$

where

$$
\delta=\lim_{m,n\to\infty}\sup\mathbb{E}[\|W_A^{m,n}(T)\|^2_{L^2(D)}]-\mathbb{E}[\|W_A(T)\|^2_{L^2(D)}]\geq 0.
$$

On the other hand, the equation (2.31) implies that a.s. in $L^2(D)$

$$
W_A(t)=\int_0^t\tilde{\Phi}(s)ds+\int_0^t dW(s),\ \forall t\in[0,T]. \qquad (2.35)
$$

Next we recall a simplified form of the Itô's formula, which will suffice for our purpose. We do so since the Itô's formula given in Lemma 2.2 only applies to finite dimensional problems.

Lemma 2.5. *Let h be an $L^2(D)$ valued progressively measurable Bochner integrable process. Consider the following well defined process :*

$$
X(t)=\int_0^t h(s)ds+W(t),\ t\in[0,T].
$$

Assume that a function $F:[0,T]\times L^2(D)\to\mathbb{R}$ and its partial derivatives F_t, F_x, F_{xx} are uniformly continuous on bounded subsets of $[0,T]\times L^2(D)$, and that $F(X(0),0)=0$. Then, a.s., for all $t\in[0,T]$,

$$
\begin{aligned}
F(X(t),t) \quad &= \quad \int_0^t F_t(X(s),s)ds+\int_0^t\langle F_x(X(s),s),h(s)\rangle_{L^2(D)}ds \qquad (2.36) \\
&\quad +\int_0^t\langle F_x(X(s),s),dW(s)\rangle_{L^2(D)}+\frac{1}{2}\int_0^t Tr[F_{xx}(X(s),s)Q]ds \qquad (2.37)
\end{aligned}
$$

where

$$
Tr[F_{xx}(X(s))Q]=\sum_{l=1}^{\infty}\langle F_{xx}(X(s),s)Qe_l,e_l\rangle_{L^2(D)}
$$

and

$$\langle u,v\rangle_{L^2(D)}=\int_D u(x)v(x)dx,$$

where we note that $TrA=\sum_{l=1}^{\infty}\langle Ae_l,e_l\rangle_{L^2(D)}$ *is bounded linear operator on* $L^2(D)$.

Applying Lemma 2.5 to (2.4) with

$$\begin{aligned}
&X=W_A,\\
&F(X,t)=e^{-ct}\|X\|^2_{L^2(D)}=e^{-ct}\|W_A\|^2_{L^2(D)},\\
&F_t(X,t)=-ce^{-ct}\|X\|^2_{L^2(D)}=-ce^{-ct}\|W_A\|^2_{L^2(D)},\\
&F_x(X,t)=2e^{-ct}X=2e^{-ct}W_A,\\
&F_{xx}(X,t)=2e^{-ct}I,\\
&h=\tilde{\Phi}.
\end{aligned}$$

Substituting in (2.36), we take

$$\begin{aligned}
e^{-ct}\|W_A\|^2_{L^2(D)} &= \int_0^t -ce^{-ct}\|W_A\|^2_{L^2(D)}ds+2\int_0^t e^{-ct}\langle\tilde{\Phi},W_A\rangle_{L^2(D)}ds\\
&\quad+\int_0^t\langle 2e^{-ct}W_A,dW(s)\rangle_{L^2(D)}+\frac{1}{2}\int_0^t Tr[2e^{-ct}IQ]ds\\
&= -c\int_0^t e^{-ct}\|W_A\|^2_{L^2(D)}ds+2\int_0^t e^{-ct}\langle\tilde{\Phi},W_A\rangle_{L^2(D)}ds\\
&\quad+2\int_0^t\int_D e^{-ct}W_A\sum_{l=1}^{\infty}\sqrt{\lambda_l}e_l d\beta_l(s)dxds\\
&\quad+\frac{1}{2}\int_0^t\sum_{l=1}^{\infty}\langle 2e^{-ct}Qe_l,e_l\rangle_{L^2(D)}ds\\
&\overset{(1.5)}{=\!=} -c\int_0^t e^{-ct}\|W_A\|^2_{L^2(D)}ds+2\int_0^t e^{-ct}\langle\tilde{\Phi},W_A\rangle_{L^2(D)}ds\\
&\quad+2\sum_{l=1}^{\infty}\int_0^t\int_D e^{-ct}W_A\sqrt{\lambda_l}e_l d\beta_l(s)dxds\\
&\quad+\frac{1}{2}\int_0^t\sum_{l=1}^{\infty}2e^{-ct}\lambda_l ds\\
&= -c\int_0^t e^{-ct}\|W_A\|^2_{L^2(D)}ds+2\int_0^t e^{-ct}\langle\tilde{\Phi},W_A\rangle_{L^2(D)}ds\\
&\quad+2\sum_{l=1}^{\infty}\int_0^t\int_D e^{-ct}W_A\sqrt{\lambda_l}e_l d\beta_l(s)dxds\\
&\quad+\int_0^t e^{-ct}\sum_{l=1}^{\infty}\lambda_l ds.
\end{aligned}$$

After taking the expectation, we deduce that

$$\begin{aligned}
\mathbb{E}[e^{-ct}\|W_A\|^2_{L^2(D)}] &= -c\mathbb{E}\int_0^t e^{-ct}\|W_A\|^2_{L^2(D)}ds+2\mathbb{E}\int_0^t e^{-ct}\langle\tilde{\Phi},W_A\rangle_{L^2(D)}ds\\
&\quad+2\mathbb{E}\sum_{l=1}^{\infty}\int_0^t\int_D e^{-ct}W_A\sqrt{\lambda_l}e_l d\beta_l(s)dxds
\end{aligned}$$

$$+\int_0^t e^{-ct}\sum_{l=1}^{\infty}\lambda_l ds.$$

From the fact that $\mathbb{E}\sum_{l=1}^{\infty}\int_0^t\int_D e^{-ct}W_A\sqrt{\lambda_l}e_l d\beta_l(s)dxds=0$, we take

$$\begin{aligned}\mathbb{E}[e^{-ct}\|W_A\|^2_{L^2(D)}] &= -c\mathbb{E}\int_0^t e^{-ct}\|W_A\|^2_{L^2(D)}ds+2\mathbb{E}\int_0^t e^{-ct}\langle\tilde{\Phi},W_A\rangle_{L^2(D)}ds\\ &\quad+\int_0^t e^{-ct}\sum_{l=1}^{\infty}\lambda_l ds,\end{aligned}$$

which we combine with (2.34) to deduce that

$$\begin{aligned}\lim_{m,n\to\infty}\sup O^1_{mn} &= \mathbb{E}[e^{-cT}\|W_A(T)\|^2_{L^2(D)}]-\int_0^T e^{-cs}\sum_{l=1}^{\infty}\lambda_l ds+\delta e^{-cT}\\ &= -c\mathbb{E}\int_0^T e^{-cs}\|W_A\|^2_{L^2(D)}ds+2\mathbb{E}\int_0^T e^{-cs}\langle\tilde{\Phi},W_A\rangle_{L^2(D)}ds\\ &\quad+\int_0^T e^{-cs}\sum_{l=1}^{\infty}\lambda_l ds-\int_0^T e^{-cs}\sum_{l=1}^{\infty}\lambda_l ds+\delta e^{-cT}\\ &= 2\mathbb{E}\int_0^T e^{-cs}\langle\tilde{\Phi},W_A\rangle_{L^2(D)}ds-c\mathbb{E}\int_0^T e^{-cs}\|W_A\|^2_{L^2(D)}ds+\delta e^{-cT}\end{aligned} \tag{2.38}$$

It remains to compute the limit of O^2_{mn}:

$$\begin{aligned}O^2_{mn} &= O_{mn}-O^1_{mn}\\ &= \mathbb{E}\Big[\int_0^T e^{-cs}\{2\langle P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]-P_m[\mathrm{div}(\nabla\Psi^n(\nabla w))],W_A^{m,n}-w\rangle\}\\ &\quad-c\|W_A^{m,n}-w\|^2_{L^2(D)}ds\Big]-\mathbb{E}\Big[\int_0^T e^{-cs}\{2\langle P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))],W_A^{m,n}\rangle\\ &\quad-c\|W_A^{m,n}\|^2_{L^2(D)}\}ds\Big]\\ &= \mathbb{E}\int_0^T e^{-cs}\{2\int_D\Big(P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]\\ &\quad-P_m[\mathrm{div}(\nabla\Psi^n(\nabla w))]\Big)(W_A^{m,n}-w)dx\}\\ &\quad-c(\|W_A^{m,n}\|^2_{L^2(D)}-2\langle W_A^{m,n},w\rangle_{L^2(D)}+\|w\|^2_{L^2(D)})\\ &\quad-\{2\int_D P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]W_A^{m,n}dx\}+c\|W_A^{m,n}\|^2_{L(D)}ds\\ &= \mathbb{E}\int_0^T e^{-cs}\{2\int_D P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]W_A^{m,n}+P_m[\mathrm{div}(\nabla\Psi^n(\nabla w))]w\\ &\quad-P_m[\mathrm{div}(\nabla\Psi^n(\nabla w))]W_A^{m,n}-P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]w\\ &\quad-P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]W_A^{m,n}dx\}\\ &\quad-c\|W_A^{m,n}\|^2_{L^2(D)}+2c\langle W_A^{m,n},w\rangle_{L^2(D)}-c\|w\|^2_{L^2(D)}+c\|W_A^{m,n}\|^2_{L^2(D)}ds\\ &= \mathbb{E}\int_0^T e^{-cs}\{2\int_D -P_m[\mathrm{div}(\nabla\Psi^n(\nabla w))]W_A^{m,n}+P_m[\mathrm{div}(\nabla\Psi^n(\nabla w))]w\\ &\quad-P_m[\mathrm{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]wdx\}+2c\langle W_A^{m,n},w\rangle_{L^2(D)}-c\|w\|^2_{L^2(D)}ds\end{aligned}$$

$$
\begin{aligned}
= \; & \mathbb{E}\int_0^T e^{-cs}\{-2\langle P_m[\operatorname{div}(\nabla\Psi^n(\nabla w))], W_A^{m,n}\rangle \\
& + 2\langle P_m[\operatorname{div}(\nabla\Psi^n(\nabla w))], w\rangle - 2\langle P_m[\operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n}))], w\rangle \\
& + 2c\langle W_A^{m,n}, w\rangle_{L^2(D)} - c\|w\|^2_{L^2(D)}\}ds.
\end{aligned}
$$

In view of (2.24), (2.26), using (2.3) and (2.8) we deduce that

$$
\begin{aligned}
\lim_{m,n\to\infty} O^2_{mn} \quad = \quad & \lim_{m,n\to\infty}\mathbb{E}\int_0^T e^{-cs}\{-2\langle P_m[\operatorname{div}(\nabla\Psi^n(\nabla w))], W_A^{m,n}\rangle \\
& + 2\langle P_m[\operatorname{div}(\nabla\Psi^n(\nabla w))], w\rangle \\
& - 2\langle P_m[\operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n}))], w\rangle \\
& + 2c\langle W_A^{m,n}, w\rangle_{L^2(D)} - c\|w\|^2_{L^2(D)} ds\} \\
= \quad & \lim_{m,n\to\infty}\mathbb{E}\int_0^T e^{-cs}\{-2\int_D P_m[\operatorname{div}(\nabla\Psi^n(\nabla w))]W_A^{m,n}dx \\
& + 2\int_D P_m[\operatorname{div}(\nabla\Psi^n(\nabla w))]wdx \\
& - 2\int_D P_m[\operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]wdx \\
& + 2c\langle W_A^{m,n}, w\rangle_{L^2(D)} - c\|w\|^2_{L^2(D)}\}ds \\
\overset{(2.8)}{=\!=} \quad & \lim_{m,n\to\infty}\mathbb{E}\int_0^T e^{-cs}\{2\int_D \nabla\Psi^n(\nabla w)\nabla W_A^{m,n}dx \\
& - 2\int_D \nabla\Psi^n(\nabla w)\nabla wdx \\
& - 2\int_D P_m[\operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]wdx \\
& + 2c\langle W_A^{m,n}, w\rangle_{L^2(D)} - c\|w\|^2_{L^2(D)}\}ds \\
\overset{\text{(int. by parts)}}{=\!=} \quad & \lim_{m,n\to\infty}\mathbb{E}\int_0^T e^{-cs}\{-2\int_D \operatorname{div}(\nabla\Psi^n(\nabla w))W_A^{m,n}dx \\
& + 2\int_D \operatorname{div}(\nabla\Psi^n(\nabla w))wdx \\
& - 2\int_D P_m[\operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n}))]wdx \\
& + 2c\langle W_A^{m,n}, w\rangle_{L^2(D)} - c\|w\|^2_{L^2(D)}\}ds \\
= \quad & \mathbb{E}\int_0^T e^{-cs}\{-2\int_D \operatorname{div}(\nabla\Psi(\nabla w))W_A dx \\
& + 2\int_D \operatorname{div}(\nabla\Psi(\nabla w))wdx - 2\int_D \tilde{\Phi}wdx \\
& + 2c\langle W_A, w\rangle_{L^2(D)} - c\|w\|^2_{L^2(D)}\}ds \\
= \quad & \mathbb{E}\int_0^T e^{-cs}\{-2\langle \operatorname{div}(\nabla\Psi(\nabla w)), W_A\rangle \\
& + 2\langle \operatorname{div}(\nabla\Psi(\nabla w)), w\rangle - 2\langle\tilde{\Phi}, w\rangle \\
& + 2c\langle W_A, w\rangle_{L^2(D)} - c\|w\|^2_{L^2(D)}\}ds \\
= \quad & \mathbb{E}\int_0^T e^{-cs}\{-2\langle \operatorname{div}(\nabla\Psi(\nabla w)), W_A\rangle
\end{aligned}
$$

$$
\begin{aligned}
&-2\langle\tilde\Phi-\operatorname{div}(\nabla\Psi(\nabla w)),w\rangle\\
&+2c\langle W_A,w\rangle_{L^2(D)}-c\|w\|^2_{L^2(D)}\}ds. \qquad (2.39)
\end{aligned}
$$

Combining (2.38) and (2.39), we obtain

$$
\begin{aligned}
\lim_{m,n\to\infty}O_{mn} &= \lim_{m,n\to\infty}O^1_{mn}+\lim_{m,n\to\infty}O^2_{mn}\\
&= 2\mathbb{E}[\int_0^T e^{-cs}\langle\tilde\Phi,W_A\rangle ds]-c\mathbb{E}[\int_0^T e^{-cs}\|W_A\|^2_{L^2(D)}ds]+\delta e^{-cT}\\
&\quad -2\mathbb{E}[\int_0^T e^{-cs}\langle\operatorname{div}(\nabla\Psi(\nabla w)),W_A\rangle ds]-c\mathbb{E}[\int_0^T e^{-cs}\|w\|^2_{L^2(D)}ds]\\
&\quad -2\mathbb{E}[\int_0^T e^{-cs}\langle\tilde\Phi-\operatorname{div}(\nabla\Psi(\nabla w)),w\rangle ds]+2c\mathbb{E}[\langle W_A,w\rangle ds]\\
&= 2\mathbb{E}[\int_0^T e^{-cs}\int_D\tilde\Phi W_A dxds]-c\mathbb{E}[\int_0^T e^{-cs}\|W_A\|^2_{L^2(D)}ds]+\delta e^{-cT}\\
&\quad -2\mathbb{E}[\int_0^T e^{-cs}\int_D\operatorname{div}(\nabla\Psi(\nabla w))W_A dxds]-c\mathbb{E}[\int_0^T e^{-cs}\|w\|^2_{L^2(D)}ds]\\
&\quad -2\mathbb{E}[\int_0^T e^{-cs}\int_D(\tilde\Phi-\operatorname{div}(\nabla\Psi(\nabla w)))wdxds]+2c\mathbb{E}[\int_D W_A wdxds]\\
&= 2\mathbb{E}[\int_0^T e^{-cs}\int_D(\tilde\Phi-\operatorname{div}(\nabla\Psi(\nabla w)))W_A dxds]\\
&\quad -2\mathbb{E}[\int_0^T e^{-cs}\int_D(\tilde\Phi-\operatorname{div}(\nabla\Psi(\nabla w)))wdxds]\\
&\quad -c\mathbb{E}[\int_0^T e^{-cs}(\|W_A\|^2_{L^2(D)}-2\int_D W_A wdx+\|w\|^2_{L^2(D)})ds]+\delta e^{-cT}\\
&= 2\mathbb{E}[\int_0^T e^{-cs}\int_D(\tilde\Phi-\operatorname{div}(\nabla\Psi(\nabla w)))(W_A-w)dxds]\\
&\quad -c\mathbb{E}[\int_0^T e^{-cs}\|W_A-w\|^2_{L^2(D)}ds]+\delta e^{-cT}\\
&= \mathbb{E}\int_0^T e^{-cs}[2\int_D(\tilde\Phi-\operatorname{div}(\nabla\Psi(\nabla w)))(W_A-w)dx\\
&\quad -c\|W_A-w\|^2_{L^2(D)}]ds+\delta e^{-cT}\\
&= \mathbb{E}\int_0^T e^{-cs}[2\langle\tilde\Phi-\operatorname{div}(\nabla\Psi(\nabla w)),W_A-w\rangle\\
&\quad -c\|W_A-w\|^2_{L^2(D)}]ds+\delta e^{-cT}.
\end{aligned}
$$

Remembering that $O_{mn}\le 0$, yields

$$
\mathbb{E}\int_0^T e^{-cs}[2\langle\tilde\Phi-\operatorname{div}(\nabla\Psi(\nabla w)),W_A-w\rangle-c\|W_A-w\|^2_{L^2(D)}]ds+\delta e^{-cT}\le 0.
$$

Since $\delta\ge 0$, it follows from that

$$
\mathbb{E}\int_0^T e^{-cs}[2\langle\tilde\Phi-\operatorname{div}(\nabla\Psi(\nabla w)),W_A-w\rangle-c\|W_A-w\|^2_{L^2(D)}]ds\le 0.
$$

Step 4: Let $\tilde v\in L^2(\Omega\times(0,T);H^1(D))$ be arbitrary and set

$$
w=W_A-\lambda\tilde v,\ \text{ with }\lambda\in\mathbb{R}_+.
$$

Substituting in above inequality, we see that

$$\mathbb{E}\int_0^T e^{-cs}[2\langle\tilde{\Phi} - \operatorname{div}(\nabla\Psi(\nabla W_A - \lambda\tilde{v})), \lambda\tilde{v}\rangle - c\|\lambda\tilde{v}\|^2_{L^2(D)}]ds \leq 0.$$

Dividing by λ, we find that

$$\begin{aligned}&\Big(\frac{1}{\lambda}\Big)\mathbb{E}\int_0^T e^{-cs}[2\langle\tilde{\Phi} - \operatorname{div}(\nabla\Psi(\nabla W_A - \lambda\tilde{v})), \lambda\tilde{v}\rangle - c\|\lambda\tilde{v}\|^2_{L^2(D)}]ds \leq 0 \Rightarrow \\ &\mathbb{E}\int_0^T e^{-cs}[2\langle\tilde{\Phi} - \operatorname{div}(\nabla\Psi(\nabla W_A - \lambda\tilde{v})), \tilde{v}\rangle - c\lambda\|\tilde{v}\|^2_{L^2(D)}]ds \leq 0\end{aligned}$$

Letting λ tend to zero, we have that

$$\mathbb{E}\int_0^T e^{-cs}\langle\tilde{\Phi} - \operatorname{div}(\nabla\Psi(\nabla W_A)), \tilde{v}\rangle ds \quad \leq \quad 0.$$

Since $\tilde{v}$ is arbitrary, it follows that

$$\begin{aligned}\mathbb{E}\int_0^T \langle\tilde{\Phi} - \operatorname{div}(\nabla\Psi(\nabla W_A)), \tilde{v}\rangle ds &= 0 \\ \mathbb{E}\int_0^T \langle\tilde{\Phi}, \tilde{v}\rangle ds \;&=\; \mathbb{E}\int_0^T \langle\operatorname{div}(\nabla\Psi(\nabla W_A)), \tilde{v}\rangle ds,\end{aligned}$$

for all $\tilde{v} \in L^2(\Omega \times (0,T); H^1(D))$, that is

$$\tilde{\Phi} = \operatorname{div}(\nabla\Psi(\nabla W_A)) \tag{2.40}$$

a.s. a.e. in $D \times (0,T)$. One finally concludes that W_A satisfies Definition 2.1 .

Next, we prove below the boundedness of W_A in $L^\infty(0,T; L^q(\Omega \times D))$, for all $q \geq 2$.

Theorem 2.2. *Let W_A be a solution of Problem (P_1); then $W_A \in L^\infty(0,T; L^q(\Omega \times D))$, for all $q \geq 2$.*

Proof. For each positive constant k, denote by $\Phi_k : \mathbb{R} \to \mathbb{R}$ the function

$$\Phi_k(\xi) \;=\; \begin{cases} |\xi|^q, & \text{if } |\xi| < k, \\ \frac{q}{2}(q-1)k^{q-2}\xi^2 - q(q-2)k^{q-1}|\xi| + (\frac{q}{2}-1)(q-1)k^q, & \text{if } k \leq |\xi|. \end{cases}$$

Φ_k is convex C^2 function and Φ_k' is a Lipschitz-continuous function with $\Phi_k'(0) = 0$. The function Φ_k satisfies the inequalities $0 \leq \Phi_k'(\xi) \leq c(k)\xi$ and $0 \leq \Phi_k(\xi) = \displaystyle\int_0^\xi \Phi_k'(\zeta)d\zeta \leq \frac{c(k)}{2}\xi^2$ for all $\xi \in \mathbb{R}^+$. This yields in view of Definition 2.1 (i) that,

$$\mathbb{E}\int_D \Phi_k(W_A(x,t))dx \leq \frac{c(k)}{2}\mathbb{E}\int_D W_A^2(x,t)dx \leq \bar{c}(k) \text{ for a.e } t \in [0,T].$$

Lemma 2.6. (i) *One has $0 \leq \Phi_k''(\xi) \leq c_k$ for all $\xi \in \mathbb{R}$ where c_k is a positive constant depending on k.*
(ii) *One has $0 \leq \Phi_k''(\xi) \leq q(q-1)(1+\Phi_k(\xi))$, for all $\xi \in \mathbb{R}$.*

Proof. (i) For each positive constant k, denote by $\Phi_k : \mathbb{R} \to \mathbb{R}$ the function

$$\Phi_k(\xi) \;=\; \begin{cases} |\xi|^q, & \text{if } |\xi| < k, \\ \frac{q}{2}(q-1)k^{q-2}\xi^2 - q(q-2)k^{q-1}|\xi| + (\frac{q}{2}-1)(q-1)k^q, & \text{if } k \le |\xi|. \end{cases}$$

The first derivative of function is

$$\Phi_k'(\xi) \;=\; \begin{cases} q|\xi|^{q-1}, & \text{if } |\xi| < k, \\ q(q-1)k^{q-2}\xi - q(q-2)k^{q-1}, & \text{if } k \le \xi, \end{cases}$$

and the second derivative is

$$\Phi_k''(\xi) \;=\; \begin{cases} q(q-1)|\xi|, & \text{if } |\xi| < k, \\ q(q-1)k^{q-2}, & \text{if } k \le \xi, \end{cases}$$

- if $|\xi| < k \Rightarrow \Phi_k''(\xi) \le q(q-1)k$,
- if $k \le \xi \Rightarrow \Phi_k''(\xi) = q(q-1)k^{q-2}$.

Thus,

$$\Phi_k''(\xi) = q(q-1)k^{q-2} =: c_k.$$

(ii) if $|\xi| \le k,\ \Phi_k''(\xi) = q(q-1)|\xi|^{q-2}$,

- $1 \le |\xi| < k,\ |\xi|^{q-2} \le |\xi|^q$,we have

$$\Phi_k''(\xi) = q(q-1)|\xi|^{q-2} \le q(q-1)|\xi|^q \le q(q-1)\Phi_k(\xi) < q(q-1)(1+\Phi_k(\xi)).$$

- if $0 \le |\xi| < 1,\ |\xi|^q \le |\xi|^{q-2} < 1$ them $|\xi|^{q-2} \le 1 + |\xi|^q$, we have

$$\Phi_k'' = q(q-1)|\xi|^{q-2} < q(q-1)(1+|\xi|^q) = q(q-1)(1+\Phi_k(\xi)).$$

If $|\xi| \ge k,\ \Phi_k''(\xi) = q(q-1)k^{q-2}$ the problem then reduces to prove that

$$H(\xi) = 1 + \frac{q}{2}(q-1)k^{q-2}\xi^2 - q(q-2)k^{q-1}|\xi| + (\frac{q}{2}-1)(q-1)k^q - k^{q-2} \ge 0.$$

Let us consider the function $H(\xi) = F(\xi) + G$ where $F(\xi) = \frac{q}{2}(q-1)k^{q-2}\xi^2 - q(q-2)k^{q-1}|\xi|$ and $G = (\frac{q}{2}-1)(q-1)k^q - k^{q-2} + 1$.

- if $\xi \ge k, H'(\xi) = F'(\xi) \ge 0$ and $H(k) \le 0$ for all $k > 0$, thus $H(\xi) \ge H(k) \ge 0$ for all $\xi \ge k$.
- if $\xi \le -k$ then $H(-\xi) = F(-\xi) + G \ge 0$. Therefore

$$H(\xi) \ge H(-\xi) \ge 0.$$

□

Next we apply Lemma 2.5 to (2.1), supposing that F does not depend on time, i.e., $F_t(X(t), t) = 0$, and setting

$$\begin{aligned} &X(t) = W_A(t), \\ &F(X(t)) = \int_D \Phi_k(X(t))dx = \int_D \Phi_k(W_A(t))dx \\ &F'(X(t)) = \Phi_k'(X(t)) = \Phi_k'(W_A(t)) \\ &h = div(A(\nabla W_A)), \end{aligned}$$

$$F''(X(t)) = \Phi_k''(X(t)) = \Phi_k''(W_A(t)),$$

$$\begin{aligned}
\int_D \Phi_k(W_A(t))dx &= \int_0^t \langle \Phi_k'(W_A(s)), div(A(\nabla W_A(s)))\rangle_{L^2(D)} ds \\
&\quad + \int_0^t \langle \Phi_k'(W_A(s)), dW(s)\rangle_{L^2(D)} ds \\
&\quad + \frac{1}{2}\int_0^t Tr[\Phi_k''(W_A(s))Q]ds \\
&= \int_0^t \int_D div(A(\nabla W_A(s)))\Phi_k'(W_A(s))dxds \\
&\quad + \int_0^t \int_D \Phi_k'(W_A(s))dW(s)dxds \\
&\quad + \frac{1}{2}\int_0^t \sum_{l=1}^{\infty} \langle \Phi_k''(W_A(s))Qe_l, e_l\rangle_{L^2(D)} ds \\
&= \int_0^t \int_D div(A(\nabla W_A))\Phi_k'(W_A(s))dxds \\
&\quad + \int_0^t \int_D \Phi_k'(W_A(s))dW(s)dxds \\
\left(Qe_l=\lambda e_l\right) \qquad &\quad + \frac{1}{2}\int_0^t \sum_{l=1}^{\infty} \int_D \Phi_k''(W_A(s))\lambda(e_l)^2 dxds \\
&\overset{\text{(int. by parts)}}{\leq} \int_0^t \int_D A(\nabla W_A(s))\Phi_k''(W_A(s))\nabla W_A(s)dxds \\
&\quad + \int_0^t \int_D \Phi_k'(W_A(s))dW(s)dxds \\
\left(\textstyle\int_D (e_l)^2 dx=\|e_l\|^2\right) \qquad &\quad + \frac{1}{2}\sum_{l=1}^{\infty} \lambda \|e_l\|_{L^\infty}^2 \int_0^t \int_D \Phi_k''(W_A(s))dxds \qquad (2.41)
\end{aligned}$$

From the coercivity property (1.3) and from (1.6) it results

$$\begin{aligned}
\int_D \Phi_k(W_A(t))dx &\leq -C_0 \int_0^t \int_D \Phi_k''(W_A)|\nabla W_A|^2 dxds \qquad \text{(from (1.3))} \\
&\quad + \int_0^t \int_D \Phi_k'(W_A(s))dW(s)dxds \\
&\quad + \frac{1}{2}\Lambda_1 \int_0^t \int_D \Phi_k''(W_A(s))dxds \qquad \text{(from (1.6))}
\end{aligned}$$

Taking the expectation, we deduce from the fact that $\mathbb{E}\int_0^t \int_D \Phi_k'(W_A)dW(s) = 0$, that

$$\mathbb{E}\int_D \Phi_k(W_A(t))dx \leq -C_0 \mathbb{E}\int_0^t \int_D \Phi_k''(W_A)|\nabla W_A|^2 dxds + \frac{1}{2}\Lambda_1 \mathbb{E}\int_0^t \int_D \Phi_k''(W_A(s))dxds.$$

Using the fact that $\Phi_k'' \geq 0$, we obtain

$$\mathbb{E}\int_D \Phi_k(W_A(t))dx \leq \frac{1}{2}\Lambda_1 \mathbb{E}\int_0^t \int_D \Phi_k''(W_A(s))dxds.$$

Then using Lemma 2.6 (ii) we obtain, defining $C(q) = \frac{1}{2}q(q-1)$,

$$\begin{aligned}
\mathbb{E}\int_D \Phi_k(W_A(t))dx &\leq \frac{1}{2}\Lambda_1\mathbb{E}\int_0^t\int_D \Phi_k''(W_A(s))dxds \\
&\overset{(\text{Lemma } 2.6\,(ii))}{\leq} \frac{1}{2}\Lambda_1\mathbb{E}\int_0^t\int_D q(q-1)(1+\Phi_k(W_A(s)))dxds \\
&\leq \frac{1}{2}q(q-1)\Lambda_1\mathbb{E}\int_0^t\int_D (1+\Phi_k(W_A(s)))dxds \\
&\leq C(q)\Lambda_1\mathbb{E}[\int_0^t\int_D 1dxds + \int_0^t\int_D \Phi_k(W_A(s)))dxds] \\
&\leq C(q)\Lambda_1\mathbb{E}[t|D| + \int_0^t\int_D \Phi_k(W_A(s)))dxds] \\
&\leq C(q)\Lambda_1 t|D| + C(q)\Lambda_1\mathbb{E}\int_0^t\int_D \Phi_k(W_A(s)))dxds.
\end{aligned}$$

Using Gronwall's Lemma B.1, with $u(t) = \mathbb{E}\int_D \Phi_k(W_A(t))dx$, $a(t) = C(q)\Lambda_1 t|D|$, $\beta(s) = C(q)\Lambda_1$, we obtain

$$\begin{aligned}
\mathbb{E}\int_D \Phi_k(W_A(t))dx &\leq C(q)\Lambda_1 t|D|e^{\int_0^t C(q)\Lambda_1 ds} \\
&= C(q)\Lambda_1 t|D|e^{C(q)\Lambda_1\int_0^t 1ds} \\
&= C(q)\Lambda_1 t|D|e^{C(q)\Lambda_1 t}.
\end{aligned}$$

Thus, $\mathbb{E}\int_D \Phi_k(W_A)dx$ is bounded independently of k.

Finally, since $\Phi_k(W_A(x,t))$ converges to $|W_A(x,t)|^q$ for a.e. x and t when k goes to infinity, if follows from Fatou's Lemma that

$$\begin{aligned}
\mathbb{E}\int_D |W_A(x,t)|^q dx &= \mathbb{E}\int_D \lim_{k\to\infty}\Phi_k(W_A(x,t))dx \\
&= \mathbb{E}\int_D \lim_{k\to\infty}\inf\Phi_k(W_A(x,t))dx \\
&\overset{(\text{Fatou's Lemma})}{\leq} \lim_{k\to\infty}\inf\mathbb{E}\int_D \Phi_k(W_A(x,t))dx \\
&\leq C(q)\Lambda_1 t|D|e^{C(q)\Lambda_1 t}
\end{aligned}$$

for all $t > 0$. Therefore, $W_A \in L^\infty(0,T,L^q(\Omega\times D))$ for all $q \geq 2$. □

2.4 Uniqueness of the solution W_A

Let ω be given such that two pathwise solutions of Problem (P_2), $W_A^1 = W_A^1(\omega,x,t)$ and $W_A^2 = W_A^2(\omega,x,t)$ satisfy

$$\begin{aligned}
&u_i(\cdot,\cdot,\omega) \in L^\infty(0,T;L^2(D)) \cap L^2(0,T;H^1(D)), \\
&\operatorname{div}(A(\nabla(u_i + W_A)) \in L^2((0,T);L^2(D))
\end{aligned}$$

for $i = 1,2$. The difference of the two solutions satisfies the equation

$$W_A^1 - W_A^2 = \int_0^t \operatorname{div}(A(\nabla W_A^1(s)))ds + W(t) - \int_0^t \operatorname{div}(A(\nabla W_A^2(s)))ds - W(t)$$

$$= \int_0^t \operatorname{div}\{A(\nabla W_A^1(s)) - A(\nabla W_A^2(s))\}ds$$

in $L^2((0,T) \times D)$. We take the duality product of this equation with $W_A^1 - W_A^2 \in L^2((0,T); H^1(D))$. In view of (1.3) we obtain

$$\begin{aligned}
\langle W_A^1 - W_A^2, W_A^1 - W_A^2\rangle_{L^2(D)} &= \\
\int_D (W_A^1 - W_A^2)(W_A^1 - W_A^2)dx &= \\
\int_D \int_0^t div\{A(\nabla W_A^1(s)) - A(\nabla W_A^2(s))\}(W_A^1 - W_A^2)dsdx &\overset{\text{(int. by parts)}}{=\!=} \\
-\int_D \int_0^t [A(\nabla W_A^1(s)) - A(\nabla W_A^2(s))]\nabla(W_A^1 - W_A^2)dsdx &\overset{(1.3)}{\leq} \\
-C_0 \int_0^t \|\nabla(W_A^1 - W_A^2)\|^2_{L^2(D)}ds.
\end{aligned}$$

Which in turn implies that

$$\|W_A^1 - W_A^2\|^2_{L^2(D)} \leq 0.$$

Nevertheless

$$\|W_A^1 - W_A^2\|^2_{L^2(D)} \geq 0.$$

Therefore

$$\|W_A^1 - W_A^2\|^2_{L^2(D)} = 0$$

or equivalently

$$W_A^1 = W_A^2 \text{ a.e in } D \times (0,T).$$

□

2.5 Change of functions

To begin with, we perform the change of functions

$$u(t) := \varphi(t) - W_A(t);$$

then φ is a solution of (P) if and only if u satisfies:

$$(P_2)\begin{cases}
\dfrac{\partial u}{\partial t} = \operatorname{div}(A(\nabla(u + W_A)) - A(\nabla W_A)) + f(u + W_A) \\
\qquad\qquad -\dfrac{1}{|D|}\displaystyle\int_D f(u + W_A)dx, & x \in D,\ t \geq 0, \\
A(\nabla(u + W_A)) \cdot \nu = 0, & x \in \partial D, t \geq 0, \\
u(x,0) = \varphi_0(x), & x \in D.
\end{cases}$$

Remark. The problem (P_2) has the form of deterministic problem; however it is stochastic since the random function W_A appears in the parabolic equation for u.

Definition 2.2. *We say that u is a solution of Problem (P_2) if :*

(i) $u \in L^\infty(0,T; L^2(\Omega \times D)) \cap L^2(\Omega \times (0,T); H^1(D)) \cap L^{2p}(\Omega \times (0,T) \times D)$;

(ii) $\operatorname{div}[A(\nabla(u + W_A))] \in L^2(\Omega \times (0,T); H^1(D)')$;

(iii) u satisfies almost surely the problem : for all $t \in [0,T]$

$$
\begin{cases} u(t) = \varphi_0 + \int_0^t \operatorname{div}[A(\nabla(u+W_A)) - A(\nabla W_A)]ds + \int_0^t f(u+W_A)ds \\ \qquad - \int_0^t \frac{1}{|D|}\int_D f(u+W_A)dxds, \quad \textit{in the sense of distributions}, \\ A(\nabla(u+W_A))\cdot\nu = 0, \quad \textit{in the sense of distributions on } \partial D\times\mathbb{R}^+. \end{cases} \tag{2.42}
$$

In order to check the conservation of mass property, namely that

$$
\int_D u(x,t)dx = \int_D \varphi_0 dx, \text{ a.s. for a.e. } t\in\mathbb{R}^+,
$$

we recall that $Z^* = V^* + L^{\frac{2p}{2p-1}}(D)$ and take the duality product of (2.2) with 1 for a.e. t and ω.

2.6 Existence of the solution of Problem (P_2)

In this section we apply the Galerkin method to prove the existence of a solution of Problem (P_2).

The main result is the following

Theorem 2.3. *There exists a unique solution of Problem (P_2).*

Proof. **Step 1:** *Solution of the approximate problem.* We select a sequence of $w_k, k = 0, ...,$ constituting an orthonormal basis of $L^2(D), w_k$ are the eigenfunctions correspond to the eigenvalues $0 < \gamma_1 < \gamma_2 \leq ... \leq \gamma_{\tilde{k}} \leq ...$ of the operator $-\Delta$. Note that they are smooth functions.

Lemma 2.7. *The functions $\{w_j\}$ are an orthonormal basis of $L^2(D)$ and satisfy :*

$$
\int_D w_j w_0 dx = 0, \textit{ for all } j\neq 0 \textit{ and } w_0 = \frac{1}{\sqrt{|D|}}.
$$

Proof. We check below that $\int_D w_j(x)dx = 0$ for all $j \neq 0$. We have

$$
\begin{cases} -\Delta w_j &= \gamma_j w_j \quad \text{in } D \\ \frac{\partial w_j}{\partial\nu} &= 0 \quad \text{on } \partial D. \end{cases}
$$

Integration over D and using Green's theorem, we take

$$
\begin{aligned}
\int_D w_j dx &= -\frac{1}{\gamma_j}\int_D \Delta w_j dx \\
&= -\frac{1}{\gamma_j}\int_{\partial D} \frac{\partial w_j}{\partial\nu}dx \qquad \text{(Green's theorem)} \\
&= 0, \qquad \text{(from boundary conditions)}
\end{aligned} \tag{2.43}
$$

which implies that $\int_D w_j w_0 dx = 0$ for all $j \neq 0$. Moreover, it is standard that the eigenfunctions corresponding to different eigenvalues are orthogonal. □

We look for an approximate solution of the form

$$
u_m(x,t) - M = \sum_{i=1}^m u_{im}(t)w_i = \sum_{i=1}^m \langle u_m(t), w_i\rangle w_i,
$$

where $M = \frac{1}{|D|}\int_D \varphi_0(x)dx$. Multiplying (P_2) with w_j and integrate over D, we obtain

$$\begin{aligned}
\int_D \frac{\partial}{\partial t}(u_m(x,t) - M)w_j dx &= \int_D \big[\mathrm{div}(A(\nabla(u_m - M + W_A)) - A(\nabla W_A)) \\
&\quad + f(u_m + W_A) - \frac{1}{|D|}\int_D f(u_m + W_A)\big] w_j dx \\
&= \int_D \mathrm{div}(A(\nabla(u_m - M + W_A)) - A(\nabla W_A)) w_j dx \\
&\quad + \int_D f(u_m + W_A) w_j dx \\
&\quad - \frac{1}{|D|}\int_D (\int_D f(u_m + W_A)dx) w_j dx \\
&\overset{\text{(int. by parts)}}{=} - \int_D [A(\nabla(u_m - M + W_A)) - A(\nabla W_A)]\nabla w_j dx \\
&\quad + \int_D f(u_m + W_A) w_j dx \\
&\quad - \frac{1}{|D|}\int_D (\int_D f(u_m + W_A)dx) w_j dx \qquad (2.44)
\end{aligned}$$

for all w_j, $j = 1, \ldots, m$. We remark that $u_m(x,0) = M + \sum_{i=1}^{m}(\varphi_0, w_i)w_i$ converges strongly to φ_0 in $L^2(D)$ as $m \to \infty$.

Problem (2.44) is an initial value problem for a system of m ordinary differential equations with the unknown functions $u_{im}(t), i = 1, \ldots, m$ so that it has a unique solution u_m on some interval $(0, T_m), T_m > 0$; in fact the following a priori estimates show that this solution is global in time.
First we remark that the contribution of the nonlocal term vanishes. Indeed for all $j = 1, \ldots, m$

$$\begin{aligned}
-\frac{1}{|D|}\int_D \Big(\int_D f(u + W_A(t))dx\Big) w_j dy &= -\frac{1}{|D|}\Big(\int_D f(u + W_A(t))dx\Big)\Big(\int_D w_j dy\Big) \\
&= 0.
\end{aligned}$$

Therefore (2.44) reduces to the equation:

$$\begin{aligned}
\int_D \frac{\partial}{\partial t}(u_m(x,t) - M)w_j dx &= -\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla W_A)]\nabla w_j dx \\
&\quad + \int_D f(u_m + W_A) w_j dx. \qquad (2.45)
\end{aligned}$$

We multiply (2.45) by $u_{jm} = u_{jm}(t)$ and add these relations for $j = 1, \ldots, m$

$$\begin{aligned}
\sum_{j=1}^{m}\int_D \frac{\partial}{\partial t}(u_m(x,t) - M)u_{jm}(t) w_j dx &= \\
-\sum_{j=1}^{m}\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla W_A)] u_{jm}(t)\nabla w_j dx & \\
+\sum_{j=1}^{m}\int_D f(u + W_A) u_{jm}(t) w_j dx & \qquad \Rightarrow
\end{aligned}$$

$$\int_D \frac{\partial}{\partial t}(u_m(x,t)-M)\sum_{j=1}^{m} u_{jm}(t)w_j dx = $$
$$-\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla W_A)]\sum_{j=1}^{m} u_{jm}(t)\nabla w_j dx$$
$$+\int_D f(u+W_A)\sum_{j=1}^{m} u_{jm}(t)w_j dx$$

We have denoted $\sum_{j=1}^{m} u_{jm}(t)w_j dx = u_m - M$, so it results

$$\begin{aligned}&\int_D \frac{\partial}{\partial t}(u_m(x,t)-M)(u_m-M)dx\\ &= -\int_D [A(\nabla(u-M+W_A)) - A(\nabla W_A)]\nabla(u_m-M)dx\\ &\quad+\int_D f(u_m+W_A)(u_m-M)dx. \end{aligned} \tag{2.46}$$

Integrating the first term, we take

$$\begin{aligned}\int_D \frac{1}{2}\frac{d}{dt}(u_m-M)^2 dx &= -\int_D [A(\nabla(u-M+W_A)) - A(\nabla W_A)]\nabla(u_m-M)dx\\ &\quad+\int_D f(u_m+W_A)(u_m-M)dx.\end{aligned}$$

Next we apply the monotonicity property of A (1.3) to bound the generalized Laplacian term, which yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_D (u_m-M)^2 dx &\leq -C_0\int_D |\nabla(u_m-M)|^2 dx\\ &\quad+\int_D f(u_m+W_A)(u_m-M)dx.\end{aligned} \tag{2.47}$$

Using the property (F_1) we deduce that

$$\begin{aligned}\int_D f(u_m+W_A(t))(u_m-M)dx &\overset{(\pm M)}{=\!=}\\ \int_D f(u_m-M+M+W_A(t))(u_m-M)dx &\overset{(F_1)}{\leq}\\ \int_D -C_1(u_m-M)^{2p}+f_2(M+W_A(t))dx &\leq\\ \int_D -C_1(u_m-M)^{2p}+C_2[(M+W_A(t))^{2p}+1]dx &=\\ \int_D -C_1(u_m-M)^{2p}dx+\int_D C_2[(M+W_A(t))^{2p}+1]dx.&\end{aligned}$$

By identity $|M+W_A|^{2p} \leq C(|M|^{2p}+|W_A|^{2p}))$, the last term above becomes

$$\int_D C_2[(M+W_A(t))^{2p}+1]dx \leq \int_D C_2[C(M^{2p}+|W_A(t)|^{2p})+1]dx.$$

So we take

$$\begin{aligned}\int_D f(u_m+W_A(t))(u_m-M)dx &\leq \int_D -C_1(u_m-M)^{2p}dx \\ &+\int_D C_2[C(M^{2p}+|W_A(t)|^{2p})+1]dx \\ &= \int_D -C_1(u_m-M)^{2p}dx+\int_D C_2C|W_A(t)|^{2p}dx \\ &+\int_D C_2[CM^{2p}+1]dx \\ &= \int_D -C_1(u_m-M)^{2p}dx+\int_D C_2C|W_A(t)|^{2p}dx \\ &+C_2[CM^{2p}+1]\int_D 1dx \\ &= -\int_D C_1(u_m-M)^{2p}dx+\tilde{C_2}\int_D |W_A(t)|^{2p}dx \\ &+\tilde{C_2}(M)|D|\end{aligned}$$

where we have set $\tilde{C_2}=C_2C$, $\tilde{C_2}(M)=C_2[CM^{2p}+1]$. We substitute in (2.47) to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_D(u_m-M)^2dx+C_0\int_D|\nabla(u_m-M)|^2dx+C_1\int_D(u_m-M)^{2p}dx \\ &\leq C_2\int_D|W_A(t)|^{2p}dx+\tilde{C_2}(M)|D|. \qquad (2.48)\end{aligned}$$

2.6.1 A priori estimates

Step 2: *A priori estimates for u_m.* In what follows, we derive a priori estimates for the function u_m. With these estimates we show later that there is a subsequence of our solutions u_m of the approximate problems (2.44) which converges to a weak solution of (P_2).

Lemma 2.8. *There exists a positive constant C such that*

$$\sup_{t\in[0,T]}\mathbb{E}\int_D(u_m-M)^2dx \leq C, \qquad (2.49)$$

$$\mathbb{E}\int_0^T\int_D|\nabla(u_m-M)|^2dxdt \leq C, \qquad (2.50)$$

$$\mathbb{E}\int_0^T\int_D(u_m-M)^{2p}dxdt \leq C, \qquad (2.51)$$

$$\mathbb{E}\int_0^T\int_D(f(u_m+M))^{\frac{2p}{2p-1}}dxdt \leq C, \qquad (2.52)$$

$$\mathbb{E}\int_0^T\|\text{div}A(\nabla(u_m+W_A))\|^2_{(H^1(D))'}dt \leq C. \qquad (2.53)$$

Proof. To prove the first three a priori estimates we integrate (2.48) from 0 to t

$$\int_0^t\Big(\frac{1}{2}\frac{d}{dt}\int_D(u_m-M)^2dx+C_0\int_D|\nabla(u_m-M)|^2dx+C_1\int_D(u_m-M)^{2p}dx\Big)ds \leq \int_0^t\Big(C_2\int_D|W_A(t)|^{2p}dx+\tilde{C_2}(M)|D|\Big)ds \Rightarrow$$

$$\frac{1}{2}\int_0^t \frac{d}{dt}\int_D (u_m - M)^2 dxds + C_0 \int_0^t \int_D |\nabla(u_m - M)|^2 dxds$$
$$+C_1 \int_0^t \int_D (u_m - M)^{2p} dxds \;\leq$$
$$C_2 \int_0^t \int_D |W_A(t)|^{2p} dxds + \int_0^t \tilde{C_2}(M)|D|ds \;\Rightarrow$$
$$\frac{1}{2}\int_D [(u_m - M)^2(s)]_0^t dx + C_0 \int_0^t \int_D |\nabla(u_m - M)|^2 dxds$$
$$+C_1 \int_0^t \int_D (u_m - M)^{2p} dxds \;\leq$$
$$C_2 \int_0^t \int_D |W_A(t)|^{2p} dxds + \int_0^t \tilde{C_2}(M)|D|ds \;\Rightarrow$$
$$\frac{1}{2}\int_D (u_m - M)^2(t)dx + C_0 \int_0^t \int_D |\nabla(u_m - M)|^2 dxds + C_1 \int_0^t \int_D (u_m - M)^{2p} dxds \;\leq$$
$$\frac{1}{2}\int_D (u_m(0) - M)^2 dx + C_2 \int_0^t \int_D |W_A(t)|^{2p} dxds + \tilde{C_2}(M)|D|t.$$

Taking the expectation we deduce that for all $t \in [0,T]$

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\int_D (u_m - M)^2(t)dx + C_0\mathbb{E}\int_0^t \int_D |\nabla(u_m - M)|^2 dxds \\
&\quad +C_1\mathbb{E}\int_0^t \int_D (u_m - M)^{2p} dxds \\
\leq\;& \frac{1}{2}\mathbb{E}\int_D (u_m(0) - M)^2 dx + C_2\mathbb{E}\int_0^t \int_D |W_A(t)|^{2p} dxds + \mathbb{E}[\tilde{C_2}(M)|D|t] \\
\leq\;& \frac{1}{2}\int_D (u_m(0) - M)^2 dx + C_2\mathbb{E}\int_0^t \int_D |W_A(t)|^{2p} dxds + \tilde{C_2}(M)|D|T \\
\leq\;& \frac{1}{2}\|u_0 - M\|^2_{L^2(D)} + C_2\mathbb{E}\int_0^t cds + \tilde{C_2}(M)|D|T \\
\leq\;& \frac{1}{2}\|u_0 - M\|^2_{L^2(D)} + C_2 c\mathbb{E}[t] + \tilde{C_2}(M)|D|T \\
\leq\;& \frac{1}{2}\|u_0 - M\|^2_{L^2(D)} + C_2 cT + \tilde{C_2}(M)|D|T \\
=\;& \frac{1}{2}\|u_0 - M\|^2_{L^2(D)} + c_2 T + \tilde{C_2}(M)|D|T \\
\leq\;& K
\end{aligned}$$

where we have used (2.2), namely $W_A \in L^\infty(0,T;L^q(\Omega\times D))$ and set $c = \int_D |W_A(t)|^{2p} dx$ $= \|W_A(t)\|^{2p}_{L^{2p}(D)}$ and $c_2 = C_2 c$. We deduce that :

$$\frac{1}{2}\mathbb{E}\int_D (u_m - M)^2(t)dx \leq K \;\;\Rightarrow\;\; \mathbb{E}\int_D (u_m - M)^2(t)dx \;\leq\; 2K,$$

for all $t \in [0,T]$,

$$C_0\mathbb{E}\int_0^T \int_D |\nabla(u_m - M)|^2 dxds \leq K \;\;\Rightarrow\;\; \mathbb{E}\int_0^T \int_D |\nabla(u_m - M)|^2 dxds \leq \frac{K}{C_0},$$
$$C_1\mathbb{E}\int_0^T \int_D (u_m - M)^{2p} dxds \leq K \;\;\Rightarrow\;\; \mathbb{E}\int_0^T \int_D (u_m - M)^{2p} dxds \leq \frac{K}{C_1}.$$

Therefore u_m is bounded independently of m in

$$L^\infty(0,T,L^2(\Omega\times D))\cap L^2(\Omega\times(0,T);H^1(D))\cap L^{2p}(\Omega\times(0,T)\times D)).$$

Now we prove the other two estimates. Using the property (F_2) we deduce that

$$\begin{aligned}
\mathbb{E}\,\|f(u_m+W_A)\|^{\frac{2p}{2p-1}}_{L^{\frac{2p}{2p-1}}((0,T)\times D)} &= \\
\mathbb{E}\int_0^T\int_D|f(u_m+W_A)|^{\frac{2p}{2p-1}}dxdt &\overset{(F_2)}{\leq} \\
\mathbb{E}\int_0^T\int_D[C_3|u_m+W_A-M|^{2p-1}+\tilde{C_3}(M)]^{\frac{2p}{2p-1}}dxdt &\overset{\text{(triangle ineq.)}}{\leq} \\
\mathbb{E}\int_0^T\int_D[C_3(|u_m-M|+|W_A|)^{2p-1}+\tilde{C_3}(M)]^{\frac{2p}{2p-1}}dxdt.
\end{aligned}$$

Using the inequality $|a+b|^{2p}\leq C(|a|^{2p}+|b|^{2p})$, with $a=C_3(|u_m-M|+|W_A|)^{2p-1}$ and $b=\tilde{C_3}(M)$, we obtain

$$\begin{aligned}
\mathbb{E}\int_0^T\int_D[C_3(|u_m-M|+|W_A|)^{2p-1}+\tilde{C_3}(M)]^{\frac{2p}{2p-1}}dxdt &\leq \\
2^{\frac{2p}{2p-1}-1}\mathbb{E}\int_0^T\int_D[C_3(|u_m-M|+|W_A|)^{2p-1}]^{\frac{2p}{2p-1}}dxdt+\mathbb{E}\int_0^T\int_D\tilde{C_3}(M)^{\frac{2p}{2p-1}}dxdt.
\end{aligned}$$

Setting $C_5=C_3^{\frac{2p}{2p-1}}$ and $\tilde{C_5}=\tilde{C_3}^{\frac{2p}{2p-1}}(M)$, we take

$$\begin{aligned}
2^{\frac{2p}{2p-1}-1}\mathbb{E}\int_0^T\int_D[C_3(|u_m-M|+|W_A|)^{2p-1}]^{\frac{2p}{2p-1}}dxdt+\mathbb{E}\int_0^T\int_D\tilde{C_3}(M)^{\frac{2p}{2p-1}}dxdt &\leq \\
2^{\frac{2p}{2p-1}-1}\mathbb{E}\int_0^T\int_D C_5[(|u_m-M|+|W_A|)^{2p-1}]^{\frac{2p}{2p-1}}dxdt+\tilde{C_5}\mathbb{E}\int_0^T\int_D dxdt &\leq \\
2^{\frac{2p}{2p-1}-1}\mathbb{E}\int_0^T\int_D C_5[(|u_m-M|+|W_A|)^{2p-1}]^{\frac{2p}{2p-1}}dxdt+\tilde{C_5}|D|T.
\end{aligned}$$

Using again the inequality $|a+b|^{2p}\leq C(|a|^{2p}+|b|^{2b})$, with $a=|u_m-M|$ and $b=|W_A|$, we obtain

$$\begin{aligned}
2^{\frac{2p}{2p-1}-1}\mathbb{E}\int_0^T\int_D C_5[(|u_m-M|+|W_A|)^{2p-1}]^{\frac{2p}{2p-1}}dxdt+\tilde{C_5}|D|T &\leq \\
2^{\frac{2p}{2p-1}-1}\mathbb{E}\int_0^T\int_D C_5[c(|u_m-M|^{2p-1}+|W_A|^{2p-1})]^{\frac{2p}{2p-1}}dxdt+\tilde{C_5}|D|T &\leq \\
2^{\frac{2p}{2p-1}-1}\mathbb{E}\int_0^T\int_D C_5c^{\frac{2p}{2p-1}}[|u_m-M|^{2p-1}+|W_A|^{2p-1}]^{\frac{2p}{2p-1}}dxdt+\tilde{C_5}|D|T.
\end{aligned}$$

Setting $c_3=2^{\frac{2p}{2p-1}-1}C_5c^{\frac{2p}{2p-1}}$, we take

$$\begin{aligned}
2^{\frac{2p}{2p-1}-1}C_5c^{\frac{2p}{2p-1}}\mathbb{E}\int_0^T\int_D[|u_m-M|^{2p-1}+|W_A|^{2p-1}]^{\frac{2p}{2p-1}}dxdt+\tilde{C_5}|D|T &\leq \\
c_3\mathbb{E}\int_0^T\int_D[|u_m-M|^{2p-1}+|W_A|^{2p-1}]^{\frac{2p}{2p-1}}dxdt+\tilde{C_5}|D|T.
\end{aligned}$$

Using again the inequality $|a+b|^{2p} \le C(|a|^{2p}+|b|^{2b})$, with $a = |u_m - M|^{2p-1}$ and $b = |W_A|^{2p-1}$, we obtain

$$\begin{aligned}
c_3\mathbb{E}\int_0^T\int_D [|u_m - M|^{2p-1} + |W_A|^{2p-1}]^{\frac{2p}{2p-1}} dxdt + \tilde{C_5}|D|T \quad &\le \\
c_3\mathbb{E}\int_0^T\int_D [|u_m - M|^{2p-1}]^{\frac{2p}{2p-1}} dxdt + c_3\mathbb{E}\int_0^T\int_D [|W_A|^{2p-1}]^{\frac{2p}{2p-1}} dxdt + \tilde{C_5}|D|T \quad &= \\
c_3\mathbb{E}\int_0^T\int_D |u_m - M|^{2p} dxdt + c_3\mathbb{E}\int_0^T\int_D |W_A|^{2p} dxdt + \tilde{C_5}|D|T \quad &\le \\
&K_1,
\end{aligned}$$

by (2.51) and (2.2), with c_3 a positive constant. Then we have show that

$$\mathbb{E}\,\|f(u_m + W_A)\|^{\frac{2p}{2p-1}}_{L^{\frac{2p}{2p-1}}((0,T)\times D)} \;\le\; K.$$

Finally we show that the elliptic term is bounded in $(H^1(D))'$. We have that

$$\begin{aligned}
\mathbb{E}\int_0^T \|\mathrm{div} A(\nabla(u_m + W_A))\|^2_{H^1(D))'}\, dt \quad &= \\
\mathbb{E}\int_0^T (\sup_{v\in H^1, \|v\|_{H^1}\le 1} |\langle \mathrm{div}(A(\nabla(u_m+W_A))), v\rangle|)^2 dt \quad &= \\
\mathbb{E}\int_0^T (\sup_{v\in H^1, \|v\|_{H^1}\le 1} |\int_D \mathrm{div}(A(\nabla(u_m+W_A)))v dx|)^2 dt \quad &\overset{\text{(int. by parts)}}{=\!=} \\
\mathbb{E}\int_0^T (\sup_{v\in H^1, \|v\|_{H^1}\le 1} |-\int_D A(\nabla(u_m+W_A))\nabla v dx|)^2 dt \quad &\le \\
\mathbb{E}\int_0^T (\sup_{v\in H^1, \|v\|_{H^1}\le 1} \int_D |A(\nabla(u_m+W_A))\nabla v| dx)^2 dt \quad &\overset{\text{(Hölder ineq.)}}{\le} \\
\mathbb{E}\int_0^T \{\sup_{v\in H^1, \|v\|_{H^1}\le 1} (\int_D |A(\nabla(u_m+W_A))|^2 dx)^{\frac{1}{2}} (\int_D |\nabla v|^2 dx)^{\frac{1}{2}}\}^2 dt \quad &\le \\
\mathbb{E}\int_0^T (\sup_{v\in H^1, \|v\|_{H^1}\le 1} \int_D |A(\nabla(u_m+W_A))|^2 dx \int_D |\nabla v|^2 dx) dt \quad &\overset{(\int_D |\nabla v|^2 dx \le 1)}{\le} \\
\mathbb{E}\int_0^T\int_D |A(\nabla(u_m+W_A))|^2 dx. &
\end{aligned} \tag{2.54}$$

Next we use (1.2) and (1.1) to estimate the term on the right-hand-side of (2.54)

$$\begin{aligned}
\mathbb{E}\int_0^T\int_D |A(\nabla(u_m+W_A))|^2 dx \quad &\overset{(1.2)}{\le} \\
C\mathbb{E}\int_0^T\int_D |\nabla(u_m+W_A)|^2 dxdt \quad &\overset{(|a+b|^2\le 2(|a|^2+|b|^2))}{\le} \\
2C(\mathbb{E}\int_0^T\int_D |\nabla u_m|^2 dxdt + \mathbb{E}\int_0^T\int_D |\nabla W_A|^2 dxdt) \quad &\le \\
&K_2.
\end{aligned}$$

The last line follows from the a priori estimates and the regularity of the solution of Problem (P_1). □

We showed the a priori estimates, namely we proved that the sequences are bounded at the respective functional spaces.

Step 3: *Passing to the limit.* Hence there exist a subsequence which we denote again by $\{u_m - M\}$ and a function $u - M \in L^2(\Omega \times (0,T); V) \cap L^{2p}(\Omega \times (0,T) \times D) \cap L^\infty(0,T; L^2(\Omega \times D))$ such that

$$u_m - M \rightharpoonup u - M \quad \text{weakly in } L^2(\Omega \times (0,T); V) \text{ and } L^{2p}(\Omega \times (0,T) \times D) \tag{2.55}$$

$$u_m - M \rightharpoonup u - M \quad \text{weakly star in } L^\infty(0,T; L^2(\Omega \times D)) \tag{2.56}$$

$$f(u_m + W_A) \rightharpoonup \chi \quad \text{weakly in } L^{\frac{2p}{2p-1}}(\Omega \times (0,T) \times D) \tag{2.57}$$

$$\mathrm{div}(A(\nabla(u_m + W_A))) \rightharpoonup \Phi \quad \text{weakly in } L^2(\Omega \times (0,T); (H^1)') \tag{2.58}$$

as $m \to \infty$.

Next, we pass to the limit as $m \to \infty$. To that purpose we integrate in time the equation (2.45) to obtain

$$\begin{aligned}
\int_0^t \int_D \frac{\partial}{\partial t}(u_m(x,t) - M) w_j dxds &= \\
-\int_0^t \int_D [A(\nabla(u_m - M + W_A)) - A(\nabla W_A)]\nabla w_j dxds + \int_0^t \int_D f(u_m + W_A) w_j dx &\Rightarrow \\
\int_D [(u_m(x,t) - M)w_j - (u_m(0) - M)w_j] dx &= \\
\int_0^t \langle div[A(\nabla(u_m - M + W_A)) - A(\nabla W_A)], w_j\rangle ds + \int_0^t \int_D f(u_m + W_A) w_j dx &\Rightarrow \\
\int_D (u_m(x,t) - M) w_j dx &= \\
\int_D (u_m(0) - M) w_j dx + \int_0^t \langle div[A(\nabla(u_m - M + W_A)) - A(\nabla W_A)], w_j\rangle ds & \\
+ \int_0^t \int_D f(u_m + W_A) w_j dxds, &
\end{aligned} \tag{2.59}$$

for all $j = 1, \ldots, m$.

Let $y = y(\omega)$ be an arbitrary bounded random variable, and let ψ be an arbitrary bounded function on $(0,T)$. We multiply the equation (2.59) by the product $y\psi$, integrate between 0 and T and take the expectation to deduce

$$\begin{aligned}
&\mathbb{E}\int_0^T \int_D y\psi(t)(u_m(x,t) - M) w_j dxdt \\
&= \mathbb{E}\int_0^T \int_D y\psi(t)(u_m(0) - M) w_j dxdt \\
&\quad + \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle div[A(\nabla(u_m - M + W_A)) - A(\nabla W_A)], w_j\rangle\} dsdt \\
&\quad + \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \int_D f(u_m + W_A) w_j dxds\} dt
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\int_0^T \int_D y\psi(t)(u_m(0)-M)w_j dxdt \\
&+\mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle div[A(\nabla(u_m - M + W_A))], w_j\rangle\} dsdt \\
&-\mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle div[A(\nabla W_A)], w_j\rangle\} dsdt \\
&+\mathbb{E}\int_0^T y\psi(t)\{\int_0^t \int_D f(u_m + W_A)w_j dxds\} dt
\end{aligned} \tag{2.60}
$$

for all $j = 1, \ldots, m$.
Next we pass to the limit in (2.60). We will use the Lebesgue-dominated convergence theorem, so we have to prove at first that every term in (2.60) is bounded. First we give the proof of convergence for the last term , using the a priori estimates and Hölder inequality. For boundedness we have

$$
\begin{aligned}
& \Big|\mathbb{E}\Big(y\psi(t)\int_0^t \int_D f(u_m + W_A)w_j dxds\Big)\Big| \\
= \quad & \Big|\psi(t)\mathbb{E}\Big(y\int_0^t \int_D f(u_m + W_A)w_j dxds\Big)\Big| \\
\leq \quad & \Big|\mathbb{E}[y]\Big|\psi(t)\Big|\mathbb{E}\int_0^t \int_D f(u_m + W_A)w_j dxds\Big| \\
\leq \quad & \|y\|_{L^\infty(\Omega)}\|\psi(t)\|_{L^\infty(0,T)}\mathbb{E}\int_0^t \int_D \Big|f(u_m + W_A)w_j\Big| dxds \\
\overset{\text{(Hölder ineq.)}}{\leq} \quad & \|y\|_{L^\infty(\Omega)}\|\psi(t)\|_{L^\infty(0,T)} \\
& \cdot\Big(\mathbb{E}\int_0^t \int_D |f(u_m + W_A)|^{\frac{2p}{2p-1}} dxds\Big)^{\frac{2p-1}{2p}}\Big(\mathbb{E}\int_0^t \int_D |w_j|^{2p} dxds\Big)^{\frac{1}{2p}}.
\end{aligned}
$$

Using (2.52) and $\mathbb{E}\int_0^t \int_D |w_j|^{2p} dxds \leq C$ we have that

$$
\Big|\psi(t)\mathbb{E}\Big(y\int_0^t \int_D f(u_m + W_A)w_j dxds\Big)\Big| \quad \leq \quad \|y\|_{L^\infty(\Omega)}\|\psi(t)\|_{L^\infty(0,T)}\bar{C}.
$$

This shows that $\Big|\psi(t)\mathbb{E}\Big(y\int_0^t \int_D f(u_m + W_A)w_j dxds\Big)\Big|$ is uniformly bounded by a function belonging to $L^1(0,T)$. In addition using (2.57) we have that

$$
\psi(t)\mathbb{E}\Big(y\int_0^t \int_D f(u_m + W_A)w_j dxds\Big) \quad \to \quad \psi(t)\mathbb{E}\Big(y\int_0^t \int_D \chi w_j dxds\Big)
$$

for a.e. $t \in (0,T)$. Applying Lebesgue-dominated convergence theorem we deduce that:

$$
\begin{aligned}
\lim_{m\to\infty}\int_0^T \psi(t)\mathbb{E}\Big(y\int_0^t \int_D f(u_m + W_A)w_j dxds\Big) dt & \overset{\text{(Lebesgue dom. theorem)}}{=} \\
\int_0^T \lim_{m\to\infty}\psi(t)\mathbb{E}\Big(y\int_0^t \int_D f(u_m + W_A)w_j dxds\Big) dt & = \\
\int_0^T \psi(t)\mathbb{E}\Big(y\int_0^t \int_D \chi w_j dxds\Big) dt & =
\end{aligned}
$$

$$\mathbb{E}\int_0^T y\psi(t)\Big(\int_0^t\int_D \chi w_j dxds\Big)dt.$$

We perform a similar proof for the three first terms in (2.60), we show first that every term is uniformly bounded by a function belonging to $L^1(0,T)$ and then we pass to the limit by using Lebesgue-dominated convergence theorem. For the first term of (2.60) we have

$$\begin{aligned}
&\Big|\mathbb{E}\int_D y\psi(t)(u_m-M)w_j dxdt\Big| \\
\leq\ & \|y\|_{L^\infty(\Omega)}|\psi(t)|\mathbb{E}\int_0^t\int_D |(u_m-M)w_j|dxds \\
\overset{(\text{Hölder ineq.})}{\leq}\ & \|y\|_{L^\infty(\Omega)}\|\psi(t)\|_{L^\infty(0,T)}\Big(\mathbb{E}\int_0^t\int_D |u_m-M|^2dxds\Big)^{\frac{1}{2}}\Big(\mathbb{E}\int_0^t\int_D |w_j|^2dxds\Big)^{\frac{1}{2}} \\
\overset{(2.49)}{\leq}\ & \|y\|_{L^\infty(\Omega)}\|\psi(t)\|_{L^\infty(0,T)}\bar{C}
\end{aligned}$$

Using (2.56) and Lebesgue-dominated convergence theorem we have

$$\begin{aligned}
\lim_{m\to\infty}\mathbb{E}\int_0^T\int_D y\psi(t)(u_m-M)w_j dxdt &\overset{(\text{Lebesgue dom. theorem})}{=} \\
\mathbb{E}\int_0^T\int_D \lim_{m\to\infty} y\psi(t)(u_m-M)w_j dxdt &= \\
\mathbb{E}\int_0^T\int_D y\psi(t)(u-M)w_j dxdt.&
\end{aligned}$$

For the second term

$$\begin{aligned}
&\Big|\mathbb{E}\int_D y\psi(t)(u_m(0)-M)w_j dxdt\Big| \\
\leq\ & \|y\|_{L^\infty(\Omega)}|\psi(t)|\mathbb{E}\int_0^t\int_D |(u_m(0)-M)w_j|dxds \\
\overset{(\text{Hölder ineq.})}{\leq}\ & \|y\|_{L^\infty(\Omega)}\|\psi(t)\|_{L^\infty(0,T)} \\
& \cdot\Big(\mathbb{E}\int_0^t\int_D |u_m(0)-M|^2dxds\Big)^{\frac{1}{2}}\Big(\mathbb{E}\int_0^t\int_D |w_j|^2dxds\Big)^{\frac{1}{2}} \\
\overset{(2.49)}{\leq}\ & \|y\|_{L^\infty(\Omega)}\|\psi(t)\|_{L^\infty(0,T)}\bar{C}.
\end{aligned}$$

Using (2.56) and Lebesgue-dominated convergence theorem we have

$$\begin{aligned}
\lim_{m\to\infty}\mathbb{E}\int_0^T\int_D y\psi(t)(u_m(0)-M)w_j dxdt &\overset{(\text{Lebesgue dom. theorem})}{=} \\
\mathbb{E}\int_0^T\int_D \lim_{m\to\infty} y\psi(t)(u_m(0)-M)w_j dxdt &= \\
\mathbb{E}\int_0^T\int_D y\psi(t)(\varphi_0-M)w_j dxdt.&
\end{aligned}$$

For the third term

$$\Big|\mathbb{E}\Big(y\psi(t)\int_0^t\langle \mathrm{div}[A(\nabla(u_m-M+W_A))],w_j\rangle dsdt\Big)\Big|$$

$$
\begin{aligned}
&\leq \quad \|y\|_{L^\infty(\Omega)} |\psi(t)| \mathbb{E}\int_0^t \int_D |(\mathrm{div}[A(\nabla(u_m + W_A)))w_j| dxds \\
&\overset{\text{(Hölder ineq.)}}{\leq} \quad \|y\|_{L^\infty(\Omega)} \|\psi(t)\|_{L^\infty(0,T)} \\
&\qquad \cdot \Big(\mathbb{E}\int_0^t \int_D |\mathrm{div}[A(\nabla(u_m + W_A))|^2 dxds\Big)^{\frac{1}{2}} \Big(\mathbb{E}\int_0^t \int_D |w_j|^2 dxds\Big)^{\frac{1}{2}} \\
&\overset{(2.53)}{\leq} \quad \|y\|_{L^\infty(\Omega)} \|\psi(t)\|_{L^\infty(0,T)} \overline{C}.
\end{aligned}
$$

Using (2.57) and Lebesgue-dominated convergence theorem we have

$$
\begin{aligned}
\lim_{m\to\infty} \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle \mathrm{div}[A(\nabla(u_m - M + W_A))], w_j\rangle ds\}dt &\overset{\text{(Lebesgue dom. theorem)}}{=} \\
\mathbb{E}\int_0^T \lim_{m\to\infty} y\psi(t)\{\int_0^t \langle \mathrm{div}[A(\nabla(u_m + W_A))], w_j\rangle ds\}dt &= \\
\mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle \Phi, w_j\rangle ds\}dt.&
\end{aligned}
$$

This yields

$$
\begin{aligned}
\mathbb{E}\int_0^T \int_D y\psi(t)(u(t) - M)w_j dxdt \quad = \quad & \mathbb{E}\int_0^T \int_D y\psi(t)(\varphi_0 - M)w_j dxdt \\
+ \quad & \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle \Phi, w_j\rangle ds\}dt \\
- \quad & \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle \mathrm{div}[A(\nabla W_A)], w_j\rangle ds\}dt \\
+ \quad & \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \int_D \chi w_j dxds\}dt \\
= \quad & \mathbb{E}\int_0^T \int_D y\psi(t)(\varphi_0 - M)w_j dxdt \\
+ \quad & \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle \Phi - \mathrm{div}[A(\nabla W_A), w_j\rangle ds\}dt \\
+ \quad & \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \int_D \chi w_j dxds\}dt, \qquad (2.61)
\end{aligned}
$$

for all $j = 1, \ldots, m$. We remark that the linear combinations of w_j are dense in $V \cap L^{2p}(D)$, so that

$$
\begin{aligned}
\mathbb{E}\int_0^T \int_D y\psi(t)(u(t) - M)\tilde{w} dxdt \quad = \quad & \mathbb{E}\int_0^T \int_D y\psi(t)(\varphi_0 - M)\tilde{w} dxdt \\
+ \quad & \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle \Phi - \mathrm{div}[A(\nabla W_A), \tilde{w}\rangle ds\}dt \\
+ \quad & \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \int_D \chi \tilde{w} dxds\}dt
\end{aligned}
$$

or equivalently

$$
\mathbb{E}\int_0^T y\psi(t)\langle u(t) - M, \tilde{w}\rangle dt \quad = \quad \mathbb{E}\int_0^T y\psi(t)\langle \varphi_0 - M, \tilde{w}\rangle dt
$$

$$+ \quad \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle \Phi + \chi - \text{div}[A(\nabla W_A), \tilde{w}\rangle ds\}dt,$$

for all $\tilde{w} \in V \cap L^{2p}(D), y \in L^\infty(\Omega)$ and $\psi \in L^\infty(0,T)$. This implies that for a.e $(t,\omega) \in (0,T) \times \Omega$

$$\langle u(t) - M, \tilde{w}\rangle \;=\; \langle \varphi_0 - M, \tilde{w}\rangle + \int_0^t \langle \Phi + \chi - \text{div}(A(\nabla W_A)), \tilde{w}\rangle ds \qquad (2.62)$$

for all $\tilde{w} \in V \cap L^{2p}(D)$.

Lemma 2.9. *The function u is such that $u \in C([0,T]; L^2(D))$ a.s.*

Proof.

$$Z \subseteq H \subseteq Z^*.$$

Since $u - M \in L^2(0,T;Z)$ a.s and $\frac{du}{dt} \in L^2(0,T;V^*) + L^2(0,T;L^{\frac{2p}{2p-1}}(D)) = L^2(0,T;Z^*)$ a.s it follows (by applying Lemma 1.2 p.260 in [12]) that $u - M \in C(0,T;H)$ a.s. □

It remains to prove that:

$$\langle \Phi + \chi, \tilde{w}\rangle = \langle \text{div}(A(\nabla(u + W_A))) + f(u + W_A(t)), \tilde{w}\rangle, \text{ for all } \tilde{w} \in V \cap L^{2p}(D).$$

We do so by means of the **monotonicity method**.

2.6.2 Monotonicity argument

Step 1: Let w be such that $w - M \in L^2(\Omega \times (0,T); V) \cap L^{2p}(\Omega \times D \times (0,T))$ and let c be a positive constant which will be fixed later. We define

$$\begin{aligned} O_m &= \mathbb{E}[\int_0^T e^{-cs}\{2\langle \text{div}(A(\nabla(u_m - M + W_A)) - A(\nabla W_A)) \\ &\quad - \text{div}(A(\nabla(w - M + W_A)) - A(\nabla W_A)), u_m - M - (w - M)\rangle_{Z^*,Z} \\ &\quad + 2\langle f(u_m + W_A) - f(w + W_A), u_m - M - (w - M)\rangle_{Z^*,Z} \\ &\quad - c\,\|u_m - M - (w - M)\|^2_{L^2(D)}\}dt] \\ &= J_1 + J_2 + J_3 \end{aligned} \qquad (*)$$

where

$$J_1 \;=\; \mathbb{E}\int_0^T e^{-cs}\{2\langle \text{div}(A(\nabla(u_m - M + W_A))) - \text{div}(A(\nabla(w - M + W_A))), u_m - M - (w - M)\rangle_{Z^*,Z}\}dt,$$

$$J_2 \;=\; \mathbb{E}\int_0^T e^{-cs}\{2\langle f(u_m + W_A) - f(w + W_A), u_m - w\rangle_{Z^*,Z}\}dt$$

and

$$J_3 \;=\; \mathbb{E}\int_0^T e^{-cs}\{-c\,\|u_m - w\|^2_{L^2(D)}\}dt.$$

Step 2: We prove below the following result

Lemma 2.10.

$$O_m \;\leq\; 0.$$

Proof. First we estimate J_1 and apply (1.3)

$$\begin{aligned}
J_1 &= \mathbb{E}\int_0^T e^{-cs}\{2\langle \operatorname{div}(A(\nabla(u_m - M + W_A)))\\
&\quad - \operatorname{div}(A(\nabla(w - M + W_A))), u_m - M - (w-M)\rangle_{Z^*,Z}\}dt\\
&= 2\mathbb{E}\int_0^T e^{-cs}\int_D \big[\operatorname{div}(A(\nabla(u_m - M + W_A)))\\
&\quad - \operatorname{div}(A(\nabla(w - M + W_A)))\big](u_m - M - (w-M))dxdt\\
&= 2\mathbb{E}\int_0^T e^{-cs}\int_D \operatorname{div}(A(\nabla(u_m - M + W_A)))(u_m - M - (w-M))\\
&\quad - \operatorname{div}(A(\nabla(w - M + W_A)))(u_m - M - (w-M))dxdt\\
&\overset{\text{(int. by parts)}}{=} 2\mathbb{E}\int_0^T e^{-cs}\Big[-\int_D A(\nabla(u_m - M + W_A))\nabla(u_m - M - (w-M))dx\\
&\quad + \int_D A(\nabla(w - M + W_A))\nabla(u_m - M - (w-M))dx\Big]dt\\
&\overset{(\pm W_A)}{=} 2\mathbb{E}\int_0^T e^{-cs}\int_D \big[-A(\nabla(u_m - M + W_A)) + A(\nabla(w - M + W_A))\big]\\
&\quad \cdot\big[\nabla(u_m - M - W_A + W_A - (w-M)\big]dxdt\\
&= -2\mathbb{E}\int_0^T e^{-cs}\int_D \big[A(\nabla(u_m - M + W_A)) - A(\nabla(w - M + W_A))\big]\\
&\quad \cdot\big[\nabla(u_m - M + W_A) + \nabla(w - M + W_A)\big]dxdt\\
&\overset{(1.3)}{\leq} -2\mathbb{E}\int_0^T e^{-cs}\int_D C_0|\nabla(u_m - M + W_A) - \nabla(w - M + W_A)|^2dxdt\\
&\leq -2C_0\mathbb{E}\int_0^T e^{-cs}\int_D |\nabla(u_m - w)|^2dxdt\\
&\leq -2C_0\mathbb{E}\int_0^T e^{-cs}\left\|\nabla(u_m - w)\right\|^2_{L^2(D)}dt\\
&\leq 0.
\end{aligned}$$

Now we estimate J_2 and apply (F_3) and the mean value theorem (MVT)

$$\begin{aligned}
J_2 &= \mathbb{E}\int_0^T e^{-cs}\{2\langle f(u_m + W_A) - f(w + W_A), u_m - M - (w-M)\rangle_{Z^*,Z}\}dt\\
&= 2\mathbb{E}\int_0^T e^{-cs}\int_D (f(u_m + W_A) - f(w + WA))(u_m - w)dxdt\\
&\overset{\left(\cdot\frac{(u_m-w)}{(u_m-w)}\right)}{=} 2\mathbb{E}\int_0^T e^{-cs}\int_D \frac{(f(u_m + W_A) - f(w + W_A))}{(u_m - w)}(u_m - w)^2dxdt\\
&\overset{\text{(MVT)}}{=} 2\mathbb{E}\int_0^T e^{-cs}\int_D f'(s)(u_m - w)^2dxdt\\
&\overset{(F_3)}{\leq} 2\mathbb{E}\int_0^T e^{-cs}C_4\left\|u_m - w\right\|^2_{L^2(D)}dt.
\end{aligned}$$

Choosing $c \geq 2C_4$, we have

$$J_2 \leq \mathbb{E}\int_0^T e^{-cs}c\left\|u_m - w\right\|^2_{L^2(D)}dt.$$

Replacing the inequalities for J_1, J_2 in $(*)$, it turns out that

$$\begin{aligned} O_m &\leq -2C_0\mathbb{E}\int_0^T e^{-cs}\left\|\nabla(u_m - w)\right\|^2_{L^2(D)} dxdt \\ &\quad +\mathbb{E}\int_0^T e^{-cs}c\left\|u_m - w\right\|^2_{L^2(D)} dt - \mathbb{E}\int_0^T e^{-cs}c\left\|u_m - w\right\|^2_{L^2(D)} dt \\ &= -2C_0\mathbb{E}\int_0^T e^{-cs}\left\|\nabla(u_m - w)\right\|^2_{L^2(D)} dxdt \\ &\leq 0. \end{aligned}$$

□

Step 3: We have to find the $\lim_{m\to\infty}\sup O_m$.
We represent O_m in the form $O_m = O_m^1 + O_m^2$ where

$$\begin{aligned} O_m^1 &= \mathbb{E}[\int_0^T e^{-cs}\{2\langle \operatorname{div}(A(\nabla(u_m - M + W_A)) - A(\nabla W_A)), u_m - M\rangle_{Z^*,Z} \\ &\quad + 2\langle f(u_m + W_A), u_m - M\rangle_{Z^*,Z} - c\left\|u_m - M\right\|^2_{L^2(D)}\}]dt \end{aligned} \tag{2.63}$$

and O_m^2 is defined by the difference. We integrate the equation (2.45) between 0 and T to obtain

$$\begin{aligned} &\int_0^T\int_D \frac{\partial}{\partial t}(u_m(x,t) - M)w_j dxdt \\ &= -\int_0^T\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))]\nabla w_j dxdt \\ &\quad + \int_0^T\int_D f(u_m + W_A)w_j dxdt \end{aligned}$$

or equivalently

$$\begin{aligned} \int_D (u_m(x,T) - M)w_j dx &= \int_D (u_m(0) - M)w_j dx \\ &\overset{\text{(int. by parts)}}{+} \int_0^T \langle \operatorname{div}[A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))], w_j\rangle_{Z^*,Z} dt \\ &\quad + \int_0^T\int_D f(u_m + W_A)w_j dxdt, \end{aligned} \tag{2.64}$$

for all $j = 1, \ldots, m$.
Next we recall a chain rule formula, which can be viewed as a simplified Itôs formula.

Proposition 2.1. *Let X be a real valued function such that*

$$X(t) = X(0) + \int_0^t h(s)ds, \quad 0 \leq s \leq t,$$

and suppose that h is measurable in time such that $h \in L^1(0,T)$. Suppose that the function $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ and its partial derivatives $\frac{\partial F}{\partial t}$ and $\frac{\partial F}{\partial X}$ are continuous on $[0,T] \times \mathbb{R}$. Then for all $t \in [0,T]$

$$F(t, X(t)) = F(0, X(0)) + \int_0^t \frac{\partial F}{\partial t}(s, X(s))ds + \int_0^t \frac{\partial F}{\partial X}(s, X(s))h(s)ds. \tag{2.65}$$

Applying (2.65) to the m equation in (2.64) with

$$\begin{aligned}
X_j &= \int_D (u_m - M) w_j dx,\ j = 1, \dots, m,\ F(s,q) = e^{-cs} q^2, \\
h(s) &= \langle \mathrm{div}[A(\nabla(u_m - M + W_A)) - A(\nabla W_A)] + f(u_m + W_A), w_j \rangle_{Z^*,Z}, \\
F(t, X(t)) &= e^{-ct} X(t)^2 = e^{-ct} \Big(\int_D (u_m(x,t) - M) w_j dx \Big)^2, \\
F(0, X(0)) &= X(0)^2 = \Big(\int_D (u_m(0) - M) w_j dx \Big)^2, \\
\frac{\partial F}{\partial t}(s, X(s)) &= -c e^{-cs} X^2(s) = -c e^{-cs} \Big(\int_D (u_m - M) w_j dx \Big)^2,
\end{aligned}$$

and

$$\frac{\partial F}{\partial X}(s, X(s)) = e^{-cs} \frac{\partial X^2(s)}{\partial X} = 2e^{-cs} X(s) = 2e^{-cs} \int_D (u_m - M) w_j dx.$$

Substituting all above in (2.65), we deduce that

$$\begin{aligned}
& e^{-cT} \Big(\int_D (u_m(x,T) - M) w_j dx \Big)^2 = \\
& \Big(\int_D (u_m(0) - M) w_j dx \Big)^2 - c \int_0^T e^{-cs} \Big(\int_D (u_m - M) w_j dx \Big)^2 dt \\
& + \int_0^T 2e^{-cs} \Big(\int_D (u_m - M) w_j dx \Big) \langle \mathrm{div}[A(\nabla(u_m - M + W_A)) - A(\nabla W_A)] \\
& \qquad + f(u_m + W_A), w_j \rangle_{Z^*,Z} dt = \\
& \Big(\int_D (u_m(0) - M) w_j dx \Big)^2 - c \int_0^T e^{-cs} \Big(\int_D (u_m - M) w_j dx \Big)^2 dt \\
& + 2 \int_0^T e^{-cs} \{ \int_D (u_m - M) w_j dx \} \langle \mathrm{div}[A(\nabla(u_m - M + W_A)) - A(\nabla W_A)], w_j \rangle_{Z^*,Z} dt \\
& + 2 \int_0^T e^{-cs} \{ \int_D (u_m - M) w_j dx \} \langle f(u_m + W_A), w_j \rangle_{Z^*,Z} dt,
\end{aligned} \tag{2.66}$$

for all $j = 1, \dots, m$. In what follows, we will use the identity

Lemma 2.11. *Let $F \in Z^*$ and $B_m = \sum_{j=1}^{m} \langle B_m, w_j \rangle w_j$. Then*

$$\sum_{j=1}^{m} \langle F, w_j \rangle \langle B_m, w_j \rangle = \langle F, B_m \rangle. \tag{2.67}$$

Proof.

$$\begin{aligned}
\sum_{j=1}^{m} \langle F, w_j \rangle \langle B_m, w_j \rangle &= \sum_{j=1}^{m} \langle F, \langle B_m, w_j \rangle w_j \rangle \qquad (\langle F, w_j \rangle\ \lambda = \langle F, \lambda w_j \rangle, \lambda \in \mathbb{R}) \\
&= \langle F, \sum_{j=1}^{m} \langle B_m, w_j \rangle w_j \rangle \\
&= \langle F, B_m \rangle.
\end{aligned}$$

□

Summing (2.66) on $j = 1, \dots, m$ and applying the identity (2.67) yields

$$\sum_{j=1}^{m} e^{-cT}\Big(\int_D (u_m(x,T) - M) w_j dx\Big)^2 =$$

$$\sum_{j=1}^{m}\Big(\int_D (u_m(0) - M) w_j dx\Big)^2 - \sum_{j=1}^{m} c\int_0^T e^{-cs}\Big(\int_D (u_m - M) w_j dx\Big)^2 dt$$

$$+ 2\sum_{j=1}^{m}\int_0^T e^{-cs}\{\int_D (u_m - M) w_j dx\}\langle \mathrm{div}[A(\nabla(u_m - M + W_A))$$

$$- A(\nabla W_A)], w_j\rangle_{Z^*,Z} dt$$

$$+ 2\sum_{j=1}^{m}\int_0^T e^{-cs}\{\int_D (u_m - M) w_j dx\}\langle f(u_m + W_A), w_j\rangle_{Z^*,Z} dt \Rightarrow$$

$$e^{-cT}\sum_{j=1}^{m}\int_D (u_m(x,T) - M)^2 w_j^2 dx =$$

$$\sum_{j=1}^{m}\int_D (u_m(0) - M)^2 w_j^2 dx - c\int_0^T e^{-cs}\sum_{j=1}^{m}\int_D (u_m - M)^2 w_j^2 dx dt$$

$$+ 2\int_0^T e^{-cs}\sum_{j=1}^{m}\{\int_D (u_m - M) w_j dx\}\langle \mathrm{div}[A(\nabla(u_m - M + W_A))$$

$$- A(\nabla W_A)], w_j\rangle_{Z^*,Z} dt$$

$$+ 2\int_0^T e^{-cs}\sum_{j=1}^{m}\{\int_D (u_m - M) w_j dx\}\langle f(u_m + W_A), w_j\rangle_{Z^*,Z} dt \Rightarrow$$

$$e^{-cT}\left\|u_m(T) - M\right\|^2_{L^2(D)}\sum_{j=1}^{m}\left\|w_j\right\|^2_{L^2(D)} =$$

$$\left\|u_m(0) - M\right\|^2_{L^2(D)}\sum_{j=1}^{m}\left\|w_j\right\|^2_{L^2(D)} - c\int_0^T e^{-cs}\left\|u_m - M\right\|^2_{L^2(D)}\sum_{j=1}^{m}\left\|w_j\right\|^2_{L^2(D)} dt$$

$$+ 2\int_0^T e^{-cs}\sum_{j=1}^{m}\langle u_m - M, w_j\rangle\langle \mathrm{div}[A(\nabla(u_m - M + W_A)) - A(\nabla W_A)], w_j\rangle_{Z^*,Z} dt$$

$$+ 2\int_0^T e^{-cs}\sum_{j=1}^{m}\langle u_m - M, w_j\rangle\langle f(u_m + W_A), w_j\rangle_{Z^*,Z} dt$$

or equivalently

$$e^{-cT}\left\|u_m(T) - M\right\|^2_{L^2(D)} \overset{\text{(Lemma 2.11)}}{\underset{(\|w_j\|^2_{L^2(D)^2}=1)}{=\!=}}$$

$$\left\|u_m(0) - M\right\|^2_{L^2(D)} - c\int_0^T e^{-cs}\left\|u_m - M\right\|^2_{L^2(D)} dt$$

$$+ 2\int_0^T e^{-cs}\langle \mathrm{div}[A(\nabla(u_m - M + W_A)) - A(\nabla W_A)], u_m - M\rangle_{Z^*,Z} dt$$

$$+ 2\int_0^T e^{-cs}\langle f(u_m + W_A), u_m - M\rangle_{Z^*,Z} dt. \tag{2.68}$$

Taking the expectation of the equation (2.68) yields

$$\mathbb{E}[e^{-cT}\left\|u_m(T) - M\right\|^2_{L^2(D)}]$$

$$
\begin{aligned}
&= \mathbb{E}[\|u_m(0)-M\|^2_{L^2(D)}] - c\mathbb{E}\int_0^T e^{-cs}\,\|u_m(s)-M\|^2_{L^2(D)}\,ds \\
&\quad + 2\mathbb{E}\int_0^T e^{-cs}\langle \mathrm{div}[A(\nabla(u_m - M + W_A)) - A(\nabla W_A)], u_m - M\rangle_{Z^*,Z} ds \\
&\quad + 2\mathbb{E}\int_0^T e^{-cs}\langle f(u_m + W_A), u_m - M\rangle_{Z^*,Z} ds. \qquad (2.69)
\end{aligned}
$$

It follows from (2.63) and (2.69) that

$$
O_m^1 \;=\; \mathbb{E}[e^{-cT}\,\|u_m(T)-M\|^2_{L^2(D)}] - \mathbb{E}[\|u_m(0)-M\|^2_{L^2(D)}].
$$

From this we obtain

$$
\lim_{m\to\infty}\sup O_m^1 \;=\; \mathbb{E}[e^{-cT}\,\|u(T)-M\|^2_{L^2(D)}] - \mathbb{E}[\|u(0)-M\|^2_{L^2(D)}] + \delta e^{-cT}, \qquad (2.70)
$$

where

$$
\delta \;=\; \lim_{m\to\infty}\sup \mathbb{E}[\|u_m(T)-M\|^2] - \mathbb{E}[\|u(T)-M\|^2] \geq 0.
$$

On the other hand, the equation (2.62) implies that

$$
u(t) - M \;=\; \varphi_0 - M + \int_0^t \Phi - \mathrm{div}(A(\nabla W_A)) + \int_0^t \chi,\;\; \forall t\in[0,T] \qquad (2.71)
$$

a.s in $Z^* = V^* + L^{\frac{2p}{2p-1}}(D)$. Next we recall a second variant of the chain rule formula, which can be viewed as a simplified Itô's formula, and involves different function spaces. Consider the Gelfand triple

$$
Z \subset H \subset Z^*,
$$

where $Z = V \cap L^{2p}(D)$ and Z^* are defined in the introduction.

Proposition 2.2. *Let $X \in L^2(0,T;V)\cap L^{2p}(0,T;L^{2p}(D))$ and $Y \in L^2(0,T;V^*) + L^{\frac{2p}{2p-1}}(0,T;L^{\frac{2p}{2p-1}}(D))$ be such that*

$$
X(t) := X_0 + \int_0^t Y(s)ds,\; t\in[0,T].
$$

Suppose that the function $F:[0,T]\times Z$. Then for all $t\in[0,T]$

$$
F(t,X(t)) = F(0,X(0)) + \int_0^t \frac{\partial F}{\partial t}(s,X(s))ds + \int_0^t \langle Y(s), \frac{\partial F}{\partial X}(s,X(s))\rangle_{Z^*,Z} ds. \qquad (2.72)
$$

Applying Proposition 2.2 to the equation (2.71), we set

$$
\begin{aligned}
&X(t) = u(t) - M,\\
&F(t,X(t)) = e^{-ct}\|X(t)\|^2,\\
&F(0,X(0)) = \|u(0)-M\|^2,\\
&\frac{\partial F}{\partial t}(t,X(t)) = -ce^{-ct}\,\|X(t)\|^2 = -ce^{-ct}\,\|u(t)-M\|^2,\\
&Y(t) = \Phi - \mathrm{div}(A(\nabla W_A)) + \chi,\\
&\frac{\partial F}{\partial X}(t,X(t)) = 2e^{-ct}X(t) = 2e^{-ct}(u(t)-M),
\end{aligned}
$$

in (2.72), from [0, T], to deduce that

$$
e^{-cT}\,\|u(T)-M\|^2 \;=\; \|u(0)-M\|^2 - c\int_0^T e^{-ct}\,\|u(s)-M\|^2\,dt
$$

$$
\begin{aligned}
& + \int_0^T \langle \Phi - \mathrm{div}(A(\nabla W_A)) + \chi, 2e^{-ct}|u-M|\rangle_{Z^*,Z} dt \\
= \ & \|u(0)-M\|^2 - c\int_0^T e^{-ct}\|u(s)-M\|^2\,dt \\
& + 2\int_0^T e^{-cs}\langle \Phi - \mathrm{div}(A(\nabla W_A)), u-M\rangle_{Z^*,Z} dt \\
& + 2\int_0^T e^{-cs}\langle \chi, u-M\rangle_{Z^*,Z} dt
\end{aligned}
$$

Taking the expectation we deduce that

$$
\begin{aligned}
\mathbb{E}[e^{-cT}\|u(T)-M\|^2] \ = \ & \mathbb{E}[\|u(0)-M\|^2] - c\mathbb{E}[\int_0^T e^{-ct}\|u(t)-M\|^2\,dt] \\
& + 2\mathbb{E}[\int_0^T e^{-ct}\langle \Phi - \mathrm{div}(A(\nabla W_A)), u-M\rangle_{Z^*,Z} dt] \\
& + 2\mathbb{E}[\int_0^T e^{-ct}\langle \chi, u-M\rangle_{Z^*,Z} dt],
\end{aligned}
$$

which we combine with (2.70) to deduce that

$$
\begin{aligned}
\lim_{m\to\infty}\sup O_m^1 \ = \ & 2\mathbb{E}[\int_0^T e^{-ct}\langle \Phi - \mathrm{div}(A(\nabla W_A)), u-M\rangle_{Z^*,Z} dt] \\
& + 2\mathbb{E}[\int_0^T e^{-ct}\langle \chi, u-M\rangle_{Z^*,Z} dt] \\
& - c\mathbb{E}[\int_0^T e^{-ct}\|u(t)-M\|^2\,dt] + \delta e^{-cT}. \qquad (2.73)
\end{aligned}
$$

It remains to compute the limit of O_m^2 :

$$
\begin{aligned}
O_m^2 \ = \ & O_m - O_m^1 \\
= \ & \mathbb{E}[\int_0^T e^{-ct}\{2\langle \mathrm{div}(A(\nabla(u_m - M + W_A)) - A(\nabla W_A)) \\
& - \mathrm{div}(A(\nabla(w - M + W_A)) - A(\nabla W_A)), u_m - M - (w-M)\rangle_{Z^*,Z}\} \\
& + 2\langle f(u_m + W_A) - f(w+W_A), u_m - M - (w-M)\rangle_{Z^*,Z} \\
& - c\|u_m - M - (w-M)\|^2\}dt] \\
& - \mathbb{E}[\int_0^T e^{-ct}\{2\langle \mathrm{div}(A(\nabla(u_m - M + W_A)) - A(\nabla W_A)), u_m - M\rangle_{Z^*,Z} \\
& + 2\langle f(u_m + W_A), u_m - M\rangle_{Z^*,Z} - c\|u_m - M\|^2\}dt] \\
= \ & \mathbb{E}\int_0^T e^{-ct}\{2\langle \mathrm{div}(A(\nabla(u_m - M + W_A)) - A(\nabla W_A)), u_m - M\rangle_{Z^*,Z} \\
& - 2\langle \mathrm{div}(A(\nabla(u_m - M + W_A)) - A(\nabla W_A)), w - M\rangle_{Z^*,Z} \\
& - 2\langle \mathrm{div}(A(\nabla(w - M + W_A)) - A(\nabla W_A)), u_m - M\rangle_{Z^*,Z} \\
& + 2\langle \mathrm{div}(A(\nabla(w - M + W_A)) - A(\nabla W_A)), w - M\rangle_{Z^*,Z} \\
& + 2\langle f(u_m + W_A), u_m - M\rangle_{Z^*,Z} - 2\langle f(w + W_A), u_m - M\rangle_{Z^*,Z} \\
& - 2\langle f(u_m + W_A), w - M\rangle_{Z^*,Z} + 2\langle f(w + W_A), w - M\rangle_{Z^*,Z} \\
& - c\|u_m - M\|^2 - c\|w - M\|^2 + 2c\langle u_m - M, w - M\rangle_{Z^*,Z} \\
& - 2\langle \mathrm{div}(A(\nabla(u_m - M + W_A)) - A(\nabla W_A)), u_m - M\rangle_{Z^*,Z}
\end{aligned}
$$

$$
\begin{aligned}
& - 2\langle f(u_m + W_A), u_m - M\rangle_{Z^*,Z} + c\|u_m - M\|^2\}dt \\
= \;& \mathbb{E}\int_0^T e^{-ct}\{-2\langle \mathrm{div}(A(\nabla(u_m - M + W_A)) - A(\nabla W_A)), w - M\rangle_{Z^*,Z} \\
& - 2\langle \mathrm{div}(A(\nabla(w - M + W_A)) - A(\nabla W_A)), u_m - M\rangle_{Z^*,Z} \\
& + 2\langle \mathrm{div}(A(\nabla(w - M + W_A)) - A(\nabla W_A)), w - M\rangle_{Z^*,Z} \\
& - 2\langle f(w + W_A), u_m - M\rangle_{Z^*,Z} \\
& - 2\langle f(u_m + W_A), w - M\rangle_{Z^*,Z} + 2\langle f(w + W_A), w - M\rangle_{Z^*,Z} \\
& - c\|w - M\|^2 + 2c\langle u_m - M, w - M\rangle_{Z^*,Z}\}dt \\
= \;& \mathbb{E}\int_0^T e^{-ct}\{-2\langle \mathrm{div}(A(\nabla(w - M + W_A)) - A(\nabla W_A)), u_m - M\rangle_{Z^*,Z} \\
& - 2\langle \mathrm{div}[A(\nabla(u_m - M + W_A)) - A(\nabla W_A)] \\
& - \mathrm{div}[A(\nabla(w - M + W_A)) - A(\nabla W_A)], w - M\rangle_{Z^*,Z} \\
& - 2\langle f(w + W_A), u_m - M\rangle_{Z^*,Z} - 2\langle f(u_m + W_A) - f(w + W_A), w - M\rangle_{Z^*,Z} \\
& - c\|w - M\|^2 + 2c\langle u_m - M, w - M\rangle_{Z^*,Z}\}dt
\end{aligned}
$$

In view of (2.56), (2.57) and (2.58), we deduce that

$$
\begin{aligned}
& \lim_{m\to\infty} O_m^2 \\
& = \mathbb{E}\int_0^T e^{-ct}\{-2\langle \mathrm{div}(A(\nabla(w - M + W_A)) - A(\nabla W_A)), u - M\rangle_{Z^*,Z} \\
& \quad - 2\langle \Phi - \mathrm{div}(A(\nabla W_A)) - \mathrm{div}A(\nabla(w - M + W_A)) - A(\nabla W_A), w - M\rangle_{Z^*,Z} \\
& \quad - 2\langle f(w + W_A), u - M\rangle_{Z^*,Z} - 2\langle \chi - f(w + W_A), w - M\rangle_{Z^*,Z} \\
& \quad - c\|w - M\|^2 + 2c\langle u - M, w - M\rangle_{Z^*,Z}\}dt.
\end{aligned} \tag{2.74}
$$

Combining (2.73) and (2.74), we find

$$
\begin{aligned}
\lim_{m\to\infty}\sup O_m \;=\;& \lim_{m\to\infty}\sup O_m^1 + \lim_{m\to\infty}\sup O_m^2 \\
=\;& 2\mathbb{E}\int_0^T e^{-ct}\langle \Phi - \mathrm{div}(A(\nabla W_A)), u - M\rangle_{Z^*,Z}dt \\
& + 2\mathbb{E}\int_0^T e^{-ct}\langle \chi, u - M\rangle_{Z^*,Z}dt \\
& - c\mathbb{E}\int_0^T e^{-ct}\,\|u(t) - M\|^2\,dt + \delta e^{-cT} \\
& - 2\mathbb{E}\int_0^T e^{-ct}\langle \mathrm{div}(A(\nabla(w - M + W_A)) - A(\nabla W_A)), u - M\rangle_{Z^*,Z}dt \\
& - 2\mathbb{E}\int_0^T e^{-ct}\langle \Phi - \mathrm{div}(A(\nabla W_A)) \\
& - \mathrm{div}(A(\nabla(w - M + W_A)) - A(\nabla W_A)), w - M\rangle_{Z^*,Z}dt \\
& - 2\mathbb{E}\int_0^T e^{-ct}\langle f(w + W_A), u - M\rangle_{Z^*,Z}dt \\
& - 2\mathbb{E}\int_0^T e^{-ct}\langle \chi - f(w + W_A), w - M\rangle_{Z^*,Z}dt \\
& - c\mathbb{E}\int_0^T e^{-ct}\|w - M\|^2 dt + 2c\mathbb{E}\int_0^T e^{-ct}\langle u - M, w - M\rangle_{Z^*,Z}dt
\end{aligned}
$$

$$
\begin{aligned}
= \;& \mathbb{E}\int_0^T e^{-ct} 2\langle \Phi - \operatorname{div}(A(\nabla W_A)), u - M\rangle_{Z^*,Z} \\
& + 2\langle \chi, u - M\rangle_{Z^*,Z} - c\|u-M\|^2 \\
& - 2\langle \operatorname{div}[A(\nabla(w - M + W_A)) - A(\nabla W_A)], u - M\rangle_{Z^*,Z} \\
& - 2\langle \Phi - \operatorname{div}(A(\nabla W_A)) \\
& - \operatorname{div}(A(\nabla(w - M + W_A))) - A(\nabla W_A)), w - M\rangle_{Z^*,Z} \\
& - 2\langle f(w + W_A), u - M\rangle_{Z^*,Z} - 2\langle \chi - f(w+W_A), w - M\rangle_{Z^*,Z} \\
& - c\|w - M\|^2 dt + 2c\langle u - M, w - M\rangle_{Z^*,Z} + \delta e^{-cT} \\
= \;& \mathbb{E}\int_0^T e^{-ct} 2\langle \Phi - \operatorname{div}(A(\nabla W_A)) - \operatorname{div}(A(\nabla(w - M + W_A))) \\
& + \operatorname{div}(A(\nabla W_A)), u - M - (w - M)\rangle_{Z^*,Z} \\
& + 2\langle \chi - f(w + W_A), u - M\rangle_{Z^*,Z} - 2\langle \chi - f(w+W_A), w - M\rangle_{Z^*,Z} \\
& - c\|u - M\|^2 - c\|w - M\|^2 + 2c\langle u - M, w - M\rangle_{Z^*,Z} dt + \delta e^{-cT} \\
= \;& \mathbb{E}\int_0^T e^{-ct} 2\langle \Phi - \operatorname{div}[A(\nabla(w - M + W_A))], u - M - (w - M)\rangle_{Z^*,Z} \\
& + 2\langle \chi - f(w + W_A), u - M - (w - M)\rangle_{Z^*,Z} \\
& - c\|u - M - (w - M)\|^2 dt + \delta e^{-cT}.
\end{aligned}
$$

Remembering that $O_m \leq 0$, yields

$$
\begin{aligned}
\mathbb{E}\int_0^T e^{-ct} 2\langle \Phi - \operatorname{div}[A(\nabla(w - M + W_A))], u - M - (w - M)\rangle_{Z^*,Z} \\
+ 2\langle \chi - f(w + W_A), u - M - (w - M)\rangle_{Z^*,Z} \\
- c\|u - M - (w - M)\|^2 dt + \delta e^{-cT} \leq 0.
\end{aligned}
$$

Since $\delta \geq 0$, it follows from that

$$
\begin{aligned}
\mathbb{E}\int_0^T e^{-ct} 2\langle \Phi - \operatorname{div}[A(\nabla(w - M + W_A))], u - M - (w - M)\rangle_{Z^*,Z} \\
+ 2\langle \chi - f(w + W_A), u - M - (w - M)\rangle_{Z^*,Z} \\
- c\|u - M - (w - M)\|^2 dt \leq 0.
\end{aligned}
$$

Step 4: Let $v \in L(\Omega \times (0,T); V) \cap L^{2p}(\Omega \times (0,T) \times D)$ be arbitrary and set

$$
w - M = u - M - \lambda v, \text{ with } \lambda \in \mathbb{R}_+.
$$

We obtain the inequality :

$$
\begin{aligned}
\mathbb{E}\int_0^T e^{-ct}\{2\langle \Phi - \operatorname{div}[A(\nabla(u - \lambda v - M + W_A))], \lambda v\rangle_{Z^*,Z} \\
+ 2\langle \chi - f(u - \lambda v + W_A), \lambda v\rangle_{Z^*,Z} - c\|\lambda v\|^2\} dt &= \\
\mathbb{E}\int_0^T e^{-ct}\{2\langle \Phi - \operatorname{div}[A(\nabla(u - \lambda v - M + W_A))] + \chi - f(u - \lambda v + W_A), \lambda v\rangle_{Z^*,Z} \\
- c\|\lambda v\|^2\} dt &\leq \\
0.
\end{aligned}
$$

Dividing by λ, we see that

$$
\mathbb{E}\int_0^T e^{-ct}\langle \Phi + \chi - \operatorname{div}[A(\nabla(u - \lambda v - M + W_A))] - f(u - \lambda v + W_A), v\rangle_{Z^*,Z}
$$

$$- c\lambda\| \, v\|^2\}dt \ \leq \ 0.$$

Letting $\lambda \to 0$, we find that:

$$\mathbb{E}\int_0^T e^{-ct}\langle \Phi + \chi - \operatorname{div}[A(\nabla(u - M + W_A))] - f(u + W_A), v\rangle_{Z^*,Z} dt \quad \leq \quad 0$$

or equivalently

$$\mathbb{E}\int_0^T \langle \Phi + \chi - \operatorname{div}[A(\nabla(u - M + W_A))] - f(u + W_A), v\rangle_{Z^*,Z} dt \quad \leq \quad 0,$$

since e^{-ct} is positive. Since v is arbitrary, it follows that

$$\mathbb{E}\int_0^T \langle \Phi + \chi, v\rangle_{Z^*,Z} dt \quad = \quad \mathbb{E}\int_0^T \langle \operatorname{div}[A(\nabla(u - M + W_A))] + f(u + W_A), v\rangle_{Z^*,Z} dt,$$

for all $v \in L^2(\Omega \times (0,T); V) \cap L^{2p}(\Omega \times (0,T) \times D)$, or else

$$\Phi + \chi \quad = \quad \operatorname{div}[A(\nabla(u - M + W_A))] + f(u + W_A) + \theta(t,\omega), \tag{2.75}$$

a.s a.e in $D \times (0,T)$. Taking the duality product of (2.75) with $\tilde{w} \in V \cap L^{2p}(D)$ we obtain that

$$\begin{aligned}
\langle \Phi + \chi, \tilde{w}\rangle_{Z^*,Z} \quad &= \quad \langle \operatorname{div}[A(\nabla(u - M + W_A))] + f(u + W_A) + \theta(t,\omega), \tilde{w}\rangle_{Z^*,Z} \\
&= \quad \langle \operatorname{div}[A(\nabla(u - M + W_A))] + f(u + W_A), \tilde{w}\rangle_{Z^*,Z}.
\end{aligned} \tag{2.76}$$

Substituting (2.76) in (2.62) we deduce that for a.e $(t,\omega) \in (0,T) \times \Omega$

$$\begin{aligned}
\langle u(t) - M, \tilde{w}\rangle \quad &= \quad \langle \varphi_0 - M, \tilde{w}\rangle + \int_0^t \langle \Phi + \chi - \operatorname{div}(A(\nabla W_A)), \tilde{w}\rangle_{Z^*,Z} ds \\
&= \quad \langle \varphi_0 - M, \tilde{w}\rangle + \int_0^t \langle \Phi + \chi, \tilde{w}\rangle_{Z^*,Z} - \langle \operatorname{div}(A(\nabla W_A)), \tilde{w}\rangle_{Z^*,Z} ds \\
&= \quad \langle \varphi_0 - M, \tilde{w}\rangle + \int_0^t \langle \operatorname{div}[A(\nabla(u - M + W_A))] + f(u + W_A) \\
&\quad\quad - \operatorname{div}(A(\nabla W_A)), \tilde{w}\rangle_{Z^*,Z} ds
\end{aligned} \tag{2.77}$$

for all $\tilde{w} \in V \cap L^{2p}(D)$.
This completes the identification of the limit terms by the monotonicity method.
Next, we prove that u satisfies the equation (2.42) in Definition 2.2. We define

$$\mathcal{V} = H^1(D) \cap L^{2p}(D).$$

The equation (2.77) implies that a.s in $\mathcal{V}^* = (H^1(D))' + L^{\frac{2p}{2p-1}}(D)$

$$\begin{aligned}
u(t) \quad &= \quad \varphi_0 + \int_0^t \operatorname{div}[A(\nabla(u - M + W_A))] - \operatorname{div}(A(\nabla W_A)) ds + \int_0^t f(u + W_A) ds \\
&\quad + \int_0^t \lambda(s) ds,
\end{aligned} \tag{2.78}$$

for all $t \in [0,T]$.
In order to identify the last term of (2.78), we take its duality product $\langle .,. \rangle_{V^*,V}$ with 1

$$\langle u(t), 1\rangle_{V^*,V} \quad = \quad \langle \varphi_0, 1\rangle_{V^*,V} + \langle \int_0^t f(u + W_A) ds, 1\rangle_{V^*,V} + \langle \int_0^t \lambda(s) ds, 1\rangle_{V^*,V}$$

$$+ \langle \int_0^t \mathrm{div}[A(\nabla(u - M + W_A))] - \mathrm{div}(A(\nabla W_A))ds, 1\rangle_{V^*,V}$$

or equivalently

$$\begin{aligned} \int_D u(t)dx &= \int_D \varphi_0 dx + \int_D \int_0^t f(u + W_A)dsdx + \int_D \int_0^t \lambda(s)dsdx \\ &+ \int_0^t \int_D \mathrm{div}(A(\nabla(u - M + W_A))) - \mathrm{div}(A(\nabla W_A))dxds. \end{aligned}$$

Using Divergence Theorem and applying Neumann Boundary Conditions, we have

$$\begin{aligned} \int_D u(t)dx - \int_D \varphi_0 dx &= \int_D \int_0^t f(u + W_A)dsdx + \int_D \int_0^t \lambda(s)dsdx \\ &+ \int_0^t \int_D \mathrm{div}(A(\nabla(u - M + W_A)))dxds \\ &- \int_0^t \int_D \mathrm{div}(A(\nabla W_A))dxds \\ &\overset{\text{(Divergence Thm)}}{=} \int_D \int_0^t f(u + W_A)dsdx + \int_0^t \lambda(s)ds|D| \\ &+ \int_0^t \int_{\partial D} A(\nabla(u - M + W_A)).\nu\, dSds \\ &- \int_0^t \int_{\partial D} A(\nabla W_A).\nu\, dSds \\ &\overset{\text{(Neumann B.C.)}}{=} \int_D \int_0^t f(u + W_A)dsdx + \int_0^t \lambda(s)ds|D|. \end{aligned}$$

Remembering that the equation is mass conserved, i.e.,

$$\int_D u(t)dx = \int_D \varphi_0 dx,$$

we obtain

$$\int_D \int_0^t f(u + W_A)dsdx + \int_0^t \lambda(s)ds|D| = \int_D u(t)dx - \int_D \varphi_0 dx = 0. \tag{2.79}$$

Thus,

$$\int_0^t \lambda(s)ds = -\frac{1}{|D|} \int_D \int_0^t f(u + W_A)dxds,$$

so that also

$$\lambda(t) = -\frac{1}{|D|} \int_D f(u(x,t) + W_A(x,t))dx.$$

The equation (2.78) becomes

$$\begin{aligned} u(t) &= \varphi_0 + \int_0^t \mathrm{div}[A(\nabla(u - M + W_A))] - \mathrm{div}(A(\nabla W_A))ds + \int_0^t f(u + W_A)ds \\ &- \int_0^t \frac{1}{|D|} \int_D f(u(x,s) + W_A(x,s))dxds, \end{aligned}$$

for all $t \in [0, T]$.

2.7 Uniqueness of the solution of Problem (P_2)

Let ω be given such that two pathwise solutions of Problem (P_2), $u_1 = u_1(\omega, x, t)$ and $u_2 = u_2(\omega, x, t)$ satisfy

$$\begin{aligned} & u_i(\cdot,\cdot,\omega) \in L^\infty(0,T,L^2(D)) \cap L^2(0,T;H^1(D)) \cap L^{2p}((0,T)\times D), \\ & f(u_i + W_A) \in L^{\frac{2p}{2p-1}}((0,T)\times D), \\ & \operatorname{div}(A(\nabla(u_i + W_A)) \in L^2((0,T);(H^1(D))') \end{aligned}$$

for $i = 1, 2$ and $u_1(\cdot, 0) = u_2(\cdot, 0) = \varphi_0$. Then

$$\begin{aligned} u_1(x,t) &= u_1(x,0) + \int_0^t \operatorname{div}(A(\nabla(u_1+W_A))) - \operatorname{div}(A(\nabla W_A))ds + \int_0^t f(u_1+W_A)ds \\ &- \frac{1}{|D|}\int_0^t\int_D f(u_1+W_A)dxds, \\ u_2(x,t) &= u_2(x,0) + \int_0^t \operatorname{div}(A(\nabla(u_2+W_A))) - \operatorname{div}(A(\nabla W_A))ds + \int_0^t f(u_2+W_A)ds \\ &- \frac{1}{|D|}\int_0^t\int_D f(u_2+W_A)dxds, \end{aligned}$$

so that the difference $u_1 - u_2$ satisfies the equation

$$\begin{aligned} u_1(t) - u_2(t) &= u_1(x,0) + \int_0^t \operatorname{div}(A(\nabla(u_1+W_A))) - \operatorname{div}(A(\nabla W_A))ds \\ &+ \int_0^t f(u_1+W_A)ds - \frac{1}{|D|}\int_0^t\int_D f(u_1+W_A)dxds \\ &- u_2(x,0) + \int_0^t \operatorname{div}(A(\nabla(u_2+W_A))) - \operatorname{div}(A(\nabla W_A))ds \\ &+ \int_0^t f(u_2+W_A)ds - \frac{1}{|D|}\int_0^t\int_D f(u_2+W_A)dxds \\ &= \int_0^t \operatorname{div}[A(\nabla(u_1+W_A)) - A(\nabla(u_2+W_A))]ds \\ &+ \int_0^t f(u_1+W_A) - f(u_2+W_A)ds \\ &- \frac{1}{|D|}\int_0^t\int_D f(u_1+W_A) - f(u_2+W_A)dxds \end{aligned}$$

in $L^2((0,T);V^*) + L^{\frac{2p}{2p-1}}((0,T)\times D)$.
We take the duality product of the equation of the difference $u_1 - u_2$ with $u_1 - u_2 \in L^2((0,T);V^*) \cap L^{\frac{2p}{2p-1}}((0,T)\times D)$, to deduce that

$$\begin{aligned} \|u_1 - u_2\|^2_{L^2(D)} &= \int_0^t \langle \operatorname{div}(A(\nabla(u_1+W_A)) - A(\nabla(u_2+W_A))), u_1 - u_2\rangle_{Z^*,Z} ds \\ &+ \int_0^t \langle f(u_1+W_A) - f(u_2+W_A), u_1-u_2\rangle_{Z^*,Z} ds \\ &- \frac{1}{|D|}\int_0^t \langle \int_D f(u_1+W_A) - f(u_2+W_A)dx, u_1-u_2\rangle_{Z^*,Z} ds \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \int_D \operatorname{div}(A(\nabla(u_1+W_A)) - A(\nabla(u_2+W_A)))(u_1-u_2)dxds \\
&\quad + \int_0^t \int_D (f(u_1+W_A) - f(u_2+W_A))(u_1-u_2)dxds \\
&\quad - \frac{1}{|D|}\int_0^t \int_D \int_D (f(u_1+W_A) - f(u_2+W_A))(u_1-u_2)dxdxds \\
&\overset{\text{(int. by parts)}}{=} - \int_0^t \int_D (A(\nabla(u_1+W_A) - A(\nabla(u_2+W_A)))\nabla(u_1-u_2)dxds \\
&\quad + \int_0^t \int_D (f(u_1+W_A) - f(u_2+W_A))(u_1-u_2)dxds \\
&\quad - \frac{1}{|D|}\int_0^t [\int_D (f(u_1+W_A) - f(u_2+W_A))dx \int_D (u_1-u_2)dx]ds \\
&= - \int_0^t \int_D (A(\nabla(u_1+W_A) - A(\nabla(u_2+W_A)))\nabla(u_1-u_2)dxds \\
&\quad + \int_0^t \int_D (f(u_1+W_A) - f(u_2+W_A))(u_1-u_2)dxds
\end{aligned} \tag{2.80}
$$

where we remark that since $\int_D u_1(x,t)dx = \int_D u_2(x,t)dx = \int_D \varphi_0(x)dx$, the nonlocal term vanishes. In view of (1.3), i.e.,

$$
\begin{aligned}
&- \int_0^t \int_D (A(\nabla(u_1+W_A) - A(\nabla(u_2+W_A)))\nabla(u_1-u_2)dxds \\
&\leq - C_0 \int_0^t \int_D |\nabla(u_1-u_2)|^2 dxds,
\end{aligned}
$$

(2.80) becomes

$$
\begin{aligned}
\|u_1-u_2\|^2_{L^2(D)} \quad \leq \quad & \int_0^t \int_D (f(u_1+W_A) - f(u_2+W_A))(u_1-u_2)dxds \\
& - C_0 \int_0^t \int_D |\nabla(u_1-u_2)|^2 dxds,
\end{aligned} \tag{2.81}
$$

for all $t \in (0,T)$. In addition, the property (F_3) implies that

$$
\begin{aligned}
(f(u_1+W_A) - f(u_2+W_A))(u_1-u_2) \quad &= \quad \frac{f(u_1+W_A) - f(u_2+W_A)}{(u_1-u_2)}(u_1-u_2)^2 \\
&\leq \quad C_4(u_1-u_2)^2.
\end{aligned} \tag{2.82}
$$

Substituting (2.82) in (2.81) yields

$$
\begin{aligned}
\int_D (u_1-u_2)^2(x,t)dx \quad &= \quad \int_D (u_1-u_2)^2(x,t)dx \\
&\leq \quad \int_0^t \int_D (f(u_1+W_A) - f(u_2+W_A))(u_1-u_2)dxdt \\
&\quad\quad - C_0 \int_0^t \int_D |\nabla(u_1-u_2)|^2 dxds
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^t \int_D (f(u_1 + W_A) - f(u_2 + W_A))(u_1 - u_2)dxdt \\
&\le C_4 \int_0^t \int_D (u_1 - u_2)^2(x,t)dxds,
\end{aligned}
$$

for all $t \in (0,T)$, which in turn implies by Gronwall's Lemma B.1 $\Big($with $a = 0$, $b = 0$, $u(t) = \displaystyle\int_D (u_1 - u_2)^2(x,t)dx\Big)$ that

$$
\int_D (u_1 - u_2)^2(x,t)dx \le 0,
$$

or equivalently

$$
u_1 = u_2 \qquad \text{a.e. in } D \times (0,T).
$$

□

Appendix A

In this appendix we prove the properties $(F_1), (F_2)$ and (F_3) for the nonlinear function

$$f(s) = \sum_{r=0}^{2p-1} b_r s^r \qquad \text{with } b_{2p-1} < 0,\ p \geq 2.$$

(F_1) There exist positive constants C_1 and C_2 such that

$$f(a+b)a \leq -C_1 a^{2p} + f_2(b), \qquad \|f_2(b)\| \leq C_2(b^{2p}+1), \qquad \text{for all a,b } \in \mathbb{R}$$

Proof. For simplicity we suppose that $b_j = 1$ for all $j = 0, \dots, 2p-2$ and that $b_{2p-1} = -1$.

$$\begin{aligned} f(a+b) &= \sum_{j=0}^{2p-1} b_j (a+b)^j \\ &= b_0(a+b)^0 + b_1(a+b)^1 + b_2(a+b)^2 + \dots + b_{2p-2}(a+b)^{2p-2} \\ &\quad + b_{2p-1}(a+b)^{2p-1} \\ &= 1 + (a+b) + (a+b)^2 + \dots + (a+b)^{2p-2} - (a+b)^{2p-1} \end{aligned}$$

hence

$$\begin{aligned} f(a+b)a &= a + (a+b)a + (a+b)^2 a + \dots + (a+b)^{2p-2} a - (a+b)^{2p-1} a \\ &= L_0 + L_1 + L_2 + \dots + L_{2p-2} + L_{2p-1}. \end{aligned} \tag{A.1}$$

For the last the term L_{2p-1}, we apply the binomial expansion,

$$\begin{aligned} L_{2p-1} &= -(a+b)^{2p-1} a \\ &= -a^{2p} - C_{2p-1}^1 a^{2p-1} b - C_{2p-1}^2 a^{2p-2} b^2 - \dots - C_{2p-1}^{2p-3} a^3 b^{2p-3} - C_{2p-1}^{2p-2} a^2 b^{2p-2} \\ &\quad - ab^{2p-1}, \end{aligned} \tag{A.2}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$. Next, we consider the second term on the right-hand-side of (A.2). Using the Hölder inequality

$$\alpha\beta \leq \frac{\varepsilon|\alpha|^t}{t} + \frac{|\beta|^s}{\varepsilon s}, \tag{A.3}$$

with conjugate exponents $s = 2p$, $t = \frac{s}{s-1} = \frac{2p}{2p-1}$ (so that $\frac{1}{t} + \frac{1}{s} = 1$), and $\alpha = a^{2p-1}$, $\beta = b$, we obtain

$$C_{2p-1}^1 a^{2p-1} b \leq C_{2p-1}^1 \frac{\varepsilon(2p-1)|a|^{2p}}{2p} + C_{2p-1}^1 \frac{|b|^{2p}}{2p\varepsilon}.$$

For the last term on the right-hand-side of (A.2), we apply Hölder inequality (A.3) with conjugate exponents $t = 2p$, $s = \frac{2p}{2p-1}$ (so that $\frac{1}{t} + \frac{1}{s} = 1$), and $\alpha = a$, $\beta = b^{2p-1}$, to get

$$ab^{2p-1} \leq \frac{\varepsilon|a|^{2p}}{2p} + \frac{(2p-1)|b|^{2p}}{2p\varepsilon}.$$

Similarly, we apply Hölder inequality (A.3) also for all the other terms of (A.2) to obtain

$$\begin{aligned} L_{2p-1} \;=\; -(a+b)^{2p-1}a \;\le\; & -a^{2p} + C^1_{2p-1}\frac{\varepsilon(2p-1)|a|^{2p}}{2p} + C^1_{2p-1}\frac{|b|^{2p}}{2p\varepsilon} + \dots + \varepsilon\frac{|a|^{2p}}{2p} \\ & + \frac{(2p-1)|b|^{2p}}{2p\varepsilon} \\ \le\; & (-1+\varepsilon C'(p))|a|^{2p} + \frac{1}{\varepsilon}C_3(p)|b|^{2p} + \frac{1}{\varepsilon}C_4(p), \end{aligned} \tag{A.3}$$

where $C'(p) = C^1_{2p-1}\frac{2p-1}{2p} + \frac{1}{2p} + \dots$, $C_3(p) = C^1_{2p-1}\frac{1}{2p} + \frac{2p-1}{2p} + \dots$, and $C_4(p) = \dots$.

A quite similar argument lead to the following Lemma.

Lemma A.1.

$$L_q \;\le\; \varepsilon C_1(p)|a|^{2p} + \frac{C_3(p)}{\varepsilon}|b|^{2p} + \frac{1}{\varepsilon}C_4(p), \tag{A.4}$$

for all $q \in \{1, \dots, 2p-2\}$.

Proof. By induction, we first prove that (A.4) is true for $q = 1$. Using Hölder inequality (A.3) with conjugate exponents $t = p$, $s = \frac{p}{p-1}$ (so that $\frac{1}{t} + \frac{1}{s} = 1$), and $\alpha = a^2$, $\beta = 1$ for the first term, and $t = 2p$, $s = \frac{2p}{2p-1}$ (so that $\frac{1}{t} + \frac{1}{s} = 1$), and $\alpha = a$, $\beta = b$ fro the second, we deduce that

$$\begin{aligned} L_1 \;=\; (a+b)a \;=\; a^2 + ab \;&\le\; \frac{\varepsilon}{p}|a|^{2p} + \frac{p-1}{\varepsilon p} + \frac{\varepsilon}{2p}|a|^{2p} + \frac{2p-1}{2p\varepsilon}|b|^{\frac{2p}{2p-1}} \\ &\le\; \frac{3\varepsilon}{2p}|a|^{2p} + \frac{p-1}{\varepsilon p} + \frac{2p-1}{2p\varepsilon}\Big(\frac{1}{2p-1}|b|^{2p} + \frac{2p-2}{2p-1}\Big) \\ &\le\; \frac{3\varepsilon}{2p}|a|^{2p} + \frac{p-1}{\varepsilon p} + \frac{2p-1}{2p\varepsilon}\,\frac{1}{2p-1}|b|^{2p} + \frac{2p-1}{2p\varepsilon}\,\frac{2p-2}{2p-1} \\ &=\; \frac{3\varepsilon}{2p}|a|^{2p} + \frac{p-1}{\varepsilon p} + \frac{1}{2p\varepsilon}|b|^{2p} + \frac{p-1}{p\varepsilon} \\ &=\; \frac{3\varepsilon}{2p}|a|^{2p} + \frac{1}{2p\varepsilon}|b|^{2p} + \frac{2p-2}{p\varepsilon}, \end{aligned}$$

where we used, at the second line, Hölder inequality (A.3) with $\varepsilon = 1$, conjugate exponents $t = \frac{2p-1}{2p-2}$, $s = 2p-1$ (so that $\frac{1}{t} + \frac{1}{s} = 1$), and $\alpha = 1$, $\beta = b^{\frac{2p}{2p-1}}$.

We suppose that (A.4) is true for $q = 2p-3$ and prove that it remains true for $q = 2p-2$: Using Hölder inequality, we obtain

$$\begin{aligned} L_{2p-2} &= (a+b)^{2p-2}a \\ &= a^{2p-1} + C^1_{2p-2}a^{2p-2}b + \dots + C^{2p-3}_{2p-2}a^2b^{2p-3} + ab^{2p-2} \\ &\le \frac{\varepsilon(2p-1)}{2p}|a|^{2p} + \frac{1}{2p\varepsilon} + \dots + \frac{\varepsilon}{2p}|a|^{2p} + \frac{2p-1}{2p\varepsilon}|b|^{\frac{(2p-2)2p}{2p-1}} \\ &\le \varepsilon C_1(p)|a|^{2p} + \frac{1}{\varepsilon}C_3(p)|b|^{2p} + \frac{1}{\varepsilon}C_4(p). \end{aligned} \tag{A.5}$$

□

Combining (A.1)-(A.4) and choosing $\varepsilon < \dfrac{1}{2(C'(p)+C_1(p))}$ yields

$$\begin{aligned} f(a+b)a \;&\le\; [-1 + \varepsilon(C'(p)+C_1(p))]|a|^{2p} \\ &\quad + \frac{1}{\varepsilon}C_3(p)|b|^{2p} + C_4(p) \\ &\le\; \Big(-1 + \frac{1}{2(C'(p)+C_1(p))}(C'(p)+C_1(p))\Big)|a|^{2p} \end{aligned}$$

$$
\begin{aligned}
&\quad + C_2(b^{2p}+1)\\
&= (-1+\frac{1}{2})|a|^{2p} + C_2(b^{2p}+1)\\
&= -\frac{1}{2}|a|^{2p} + C_2(b^{2p}+1),
\end{aligned}
$$

with $C_2 = \max(\frac{1}{\varepsilon}C_3(p), C_4(p))$. □

(F_2) There exists a positive constant C_3 such that

$$
|f(s)| \le C_3|s-M|^{2p-1} + \tilde{C_3}(M).
$$

Proof. Again, we suppose that $b_j = 1$ for all $j = 0, \dots, 2p-2$ and that $b_{2p-1} = -1$.

$$
\begin{aligned}
f(s) &= \sum_{j=0}^{2p-1} b_j s^j = b_0 s^0 + b_1 s^1 + b_2 s^2 + \dots + b_{2p-2}s^{2p-2} + b_{2p-1}s^{2p-1}\\
&= 1 + s + s^2 + \dots + s^{2p-2} - s^{2p-1}.
\end{aligned} \tag{A.6}
$$

We estimate the leading term of (A.6)

$$
\begin{aligned}
|s|^{2p-1} &= |s-M+M|^{2p-1}\\
&= |s-M|^{2p-1} + C^1_{2p-1}|s-M|^{2p-2}M + \dots + C^{2p-2}_{2p-1}|s-M|M^{2p-2} + M^{2p-1}.
\end{aligned}
$$

By Hölder inequality (A.3), there holds

$$
\begin{aligned}
|s-M|^{2p-2}M &\le \frac{\varepsilon(2p-2)}{2p-1}|s-M|^{2p-1} + \frac{M^{2p-1}}{\varepsilon(2p-1)}, \dots\\
|s-M|M^{2p-2} &\le \frac{\varepsilon|s-M|^{2p-1}}{2p-1} + \frac{|M|^{2p-1}(2p-2)}{\varepsilon(2p-1)},
\end{aligned}
$$

so that

$$
\begin{aligned}
|s|^{2p-1} &= |s-M|^{2p-1} + C^1_{2p-1}|s-M|^{2p-2}M + \dots + C^{2p-2}_{2p-1}|s-M|M^{2p-1} + |M|^{2p-1}\\
&\le |s-M|^{2p-1} + C^1_{2p-1}\Big(\frac{\varepsilon(2p-2)}{2p-1}|s-M|^{2p-1} + \frac{|M|^{2p-1}}{\varepsilon(2p-1)}\Big) + \dots\\
&\quad + C^{2p-2}_{2p-1}\Big(\frac{\varepsilon|s-M|^{2p-1}}{2p-1} + \frac{|M|^{2p-1}(2p-2)}{\varepsilon(2p-1)}\Big) + |M|^{2p-1}\\
&= |s-M|^{2p-1} + C^1_{2p-1}\frac{\varepsilon(2p-2)}{2p-1}|s-M|^{2p-1} + C^1_{2p-1}\frac{|M|^{2p-1}}{\varepsilon(2p-1)} + \dots\\
&\quad + C^{2p-2}_{2p-1}\frac{\varepsilon|s-M|^{2p-1}}{2p-1} + C^{2p-2}_{2p-1}\frac{|M|^{2p-1}(2p-2)}{\varepsilon(2p-1)} + |M|^{2p-1}\\
&= \Big(1 + \varepsilon\Big(C^1_{2p-1}\frac{2p-2}{2p-1} + C^{2p-2}_{2p-1}\frac{1}{2p-1}\Big)\Big)|s-M|^{2p-1}\\
&\quad + \Big(1 + \frac{1}{\varepsilon}\Big(C^1_{2p-1}\frac{1}{2p-1} + C^{2p-2}_{2p-1}\frac{2p-2}{2p-1}\Big)\Big)|M|^{2p-1}\\
&= (1+\varepsilon C(p))|s-M|^{2p-1} + (1 + \frac{1}{\varepsilon}\tilde{C}(p))|M|^{2p-1}.
\end{aligned} \tag{A.7}
$$

Next, we estimate the last term on the right-hand-side of (A.6). It follows from Hölder inequality (A.3) that

$$
|s| = |s-M+M| \le |s-M| + |M|
$$

$$
\begin{aligned}
&\leq \frac{\varepsilon}{2p-1}|s-M|^{2p-1} + \frac{2p-2}{\varepsilon(2p-1)} + |M| \\
&= \varepsilon C_1(p)|s-M|^{2p-1} + \frac{1}{\varepsilon}\tilde{C}_1(p) + |M|. \qquad \text{(A.8)}
\end{aligned}
$$

Computing all the other terms of (A.6) similarly and substituting (A.7) and (A.8) in (A.6) we obtain

$$
\begin{aligned}
|f(s)| &= |-s^{2p-1} + s^{2p-2} + \cdots + s^2 + s| \\
&\leq |s|^{2p-1} + |s|^{sp-2} + \cdots + |s|^2 + |s| \\
&\leq (1+\varepsilon C(p))|s-M|^{2p-1} + \Big(1 + \frac{1}{\varepsilon}\tilde{C}(p)\Big)|M|^{2p-1} \\
&\quad + \cdots + \varepsilon C_1(p)|s-M|^{2p-1} + \frac{1}{\varepsilon}\tilde{C}_1(p) + |M| \\
&\leq C_3|s-M|^{2p-1} + \tilde{C}_3(M).
\end{aligned}
$$

□

(F_3) There exists a positive constant C_4 such that

$$
f'(s) \leq C_4.
$$

Proof. From (A.6)

$$
f(s) = -s^{2p-1} + s^{2p-2} + \cdots + s^2 + s + 1
$$

we have

$$
f'(s) = -(2p-1)s^{2p-2} + (2p-2)s^{2p-3} + \cdots + 2s + 1. \qquad \text{(A.9)}
$$

By Hölder inequality (A.3)

$$
|s|^{2p-3} \leq \frac{\varepsilon(2p-3)}{2p-2}|s|^{2p-2} + \frac{1}{(2p-2)\varepsilon}, \cdots, \qquad \text{(A.10)}
$$

$$
|s| \leq \frac{\varepsilon}{2p-2}|s|^{2p-2} + \frac{2p-3}{(2p-2)\varepsilon}. \qquad \text{(A.11)}
$$

We compute all the other terms similarly, and substitute them in (A.9) to obtain

$$
\begin{aligned}
f'(s) &= -(2p-1)s^{2p-2} + (2p-2)s^{2p-3} + \cdots + 2s + 1 \\
&\leq -(2p-1)|s|^{2p-2} + (2p-2)\Big[\frac{\varepsilon(2p-3)}{2p-2}|s|^{2p-2} + \frac{1}{(2p-2)\varepsilon}\Big] + \ldots \\
&\quad + 2\Big[\frac{\varepsilon}{2p-2}|s|^{2p-2} + \frac{2p-3}{(2p-2)\varepsilon}\Big] + 1 \\
&= -(2p-1)|s|^{2p-2} + \varepsilon(2p-3)|s|^{2p-2} + \frac{1}{\varepsilon} + \cdots + \frac{\varepsilon|s|^{2p-2}}{p-1} + \frac{2p-3}{(p-1)\varepsilon} + 1 \\
&= \Big[-(2p-1) + \varepsilon\Big((2p-3) + \frac{1}{p-1} + \ldots\Big)|s|^{2p-2}\Big] + \Big[1 + \frac{2p-3}{p-1} + \ldots\Big]\frac{1}{\varepsilon} + 1 \\
&= [-(2p-1) + \varepsilon C(p)]|s|^{2p-2} + \frac{\tilde{C}(p)}{\varepsilon} + 1. \qquad \text{(A.4)}
\end{aligned}
$$

We may bound the right hand side of (A.4) by choosing $\varepsilon \leq \frac{2p-1}{2C(p)}$. Indeed, we have

$$
\begin{aligned}
f'(s) &\leq [-(2p-1) + \varepsilon C(p)]|s|^{2p-2} + \frac{\tilde{C}(p)}{\varepsilon} + 1 \\
&\leq \Big(-(2p-1) + \frac{2p-1}{2C(p)}C(p)\Big)|s|^{2p-2} + \tilde{C}(p)\frac{2C(p)}{2p-2} + 1
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(-(2p-1) + \frac{2p-1}{2} \Big) |s|^{2p-2} + \frac{\tilde{C}(p)C(p)}{p-1} + 1 \\
&= \Big(\frac{-2(2p-1) + (2p-1)}{2} \Big) |s|^{2p-2} + \frac{\tilde{C}(p)C(p)}{p-1} + 1 \\
&= -\frac{2p-1}{2} |s|^{2p-2} + \frac{\tilde{C}(p)C(p)}{p-1} + 1 \\
&\leq \frac{\tilde{C}(p)C(p)}{p-1} + 1 \\
&\leq C_4.
\end{aligned}
$$

□

Appendix B

B.1 Brownian motion

Brownian motion is a natural phenomenon, which describes the random motion of particles suspended in a liquid or a gas resulting from their collision with the fast-moving molecules in the fluid or gas. This motion is named after the botanist Robert Brown, who first described the phenomenon in 1827, while looking through a microscope at pollen of the plant Clarkia pulchella immersed in water. In mathematics, Brownian motion is described by the Wiener process, a continuous-time stochastic process named in honor of Norbert Wiener.

The Wiener process W_t is characterized by four facts:

- $W_0 = 0$.
- W_t is almost surely continuous.
- W_t has independent increments, i.e., if $0 \leq s_1 < t_1 \leq s_2 < t_2$ then $W_{t_1} - W_{s_1}$ and $W_{t_2} - W_{s_2}$ are independent random variables.
- $W_t - W_s \sim \mathcal{N}(0, t-s)$ (for $0 \leq s \leq t$).

$\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2.

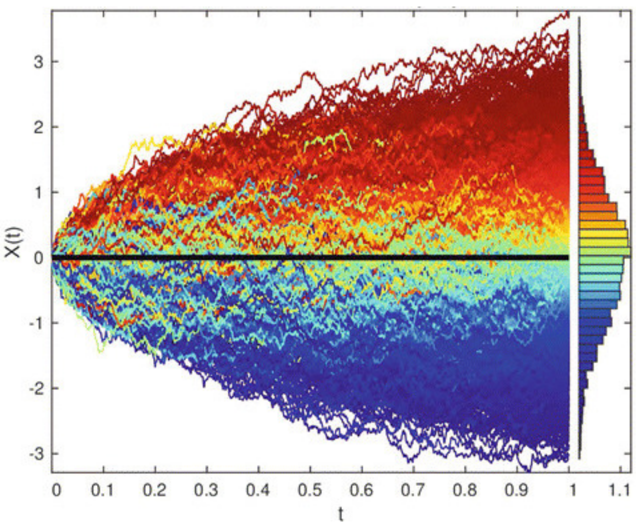

Figure B.1: Standard Brownian motion, where $X(t) = W_t$ is plotted on the vertical axis.

B.2 Adapted Process

In the study of stochastic processes, an adapted process (also referred to as a non-anticipating or non-anticipative process) is one that cannot "see into the future". An informal interpretation is that X is adapted if and only if, for every realization and every n, X_n is known at time n.

Definition B.1. *Let*

- $(\Omega, \mathcal{F}, \mathbb{P})$; *be a probability space;*
- I *be an index set with a total order* $\leq$ *(often* , I *is* $\mathbb{N}, \mathbb{N}_0, [0,T]$ *or* $[0,+\infty)$;
- $\mathcal{F}_{\cdot} = (\mathcal{F}_i)_{i \in I}$ *be a filtration of the sigma algebra* $\mathcal{F}$;
- (S, Σ) *be a measurable space, the state space;*
- $X : I \times \Omega \to S$ *be a stochastic process.*

The process X is said to be adapted to the filtration $(\mathcal{F}_i)_{i \in I}$ if the random variable $X_i : \Omega \to S$ is a $(\mathcal{F}_i, \Sigma)$-measurable function for each $i \in I$.

B.3 Grönwall's inequality

In mathematics, Grönwall's inequality (also called Grönwall's lemma) allows one to bound a function that is known to satisfy a certain differential or integral inequality by the solution of the corresponding differential or integral equation. There are two forms of the lemma, a differential form and an integral form. For the latter there are several variants. Grönwall's inequality is an important tool to obtain various estimates in the theory of ordinary and stochastic differential equations. In particular, it provides a comparison theorem that can be used to prove uniqueness of a solution to the initial value problem.

Lemma B.1. (Integral form for continuous functions.) *Let I denote an interval of the real line of the form $[a,\infty)$ or $[a,b]$ or $[a,b)$ with $a < b$. Let α, β and u be real-valued functions defined on I. Assume that β and u are continuous and that the negative part of β is integrable on every closed and bounded subinterval of I.*

If the function α is non-decreasing and if u satisfies the integral inequality

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds, \qquad \forall t \in I,$$

then

$$u(t) \leq \alpha(t) \exp\Big(\int_a^t \beta(s)ds\Big), \qquad t \in I.$$

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