Correlation Functions in AdS/CFT

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Undergraduate Thesis

Abstract

This thesis explores correlation functions in the AdS/CFT correspondence. Using the supergravity approximation we compute the two-point functions of scalar and vector fields. The same calculation is performed for a general Conformal Field Theory. Furthermore, we compute Witten diagrams that provide first and second order contributions to conformal correlators in the large N limit. The Witten diagrams are written in the Mellin representation and their associated Mellin amplitude is calculated.

Contents

1 Introduction

The AdS/CFT Correspondence is the conjecture that two very different physical theories are equivalent descriptions of an underlying physical system. More specifically, certain Conformal Field Theories are thought to be dual to certain String Theories. The string theory is considered on Anti-de Sitter space times a compact manifold. When the length scale or "radius" of AdS is much bigger than the string length scale the graviton looks like a point particle and the string theory can be approximated by classical supergravity. The CFT on the other hand is usually a gauge theory of rank N and we take the limit where N is large. This is why AdS/CFT is sometimes called Gauge-Gravity duality.

In this dual picture, the string coupling is related to the rank of the gauge group $g \sim 1/N$. The effective coupling for the gauge theory is $\lambda = g_{YM}^2 N$. This coupling constant is proportional to $R/l_s \sim \lambda^a$ with $a > 0$, with R and l_s being the space and string length scales. This suggests that in the areas where one theory is weakly coupled the corresponding theory is strongly coupled. It is believed that the conjecture holds in all orders of perturbation theory, however, it is typically very hard to do calculations beyond the first terms in the expansion. This is why even though there is a lot of evidence for AdS/CFT, it is stil a conjecture.

This thesis is organized in the following way: the second section is a small introduction to Anti-de Sitter geometry and representations of AdS. The third section provides a small introduction to Conformal Field Theories in three or more dimensions. The fourth section follows the paper by E. Witten titled "Anti-de Sitter space and holography" [1]. In that section we calculate the two-point functions of massless scalar fields and gauge fields, using the tools of classical gravity. The respective calculations are performed in the CFTs by enforcing the conformal constraints.

The final section follows the paper by J. Penedones titled "Writing CFT correlation functions as AdS scattering amplitudes" [2]. It is devoted to Witten diagrams. They are a diagrammatical way to represent and calculate scattering in AdS space. These diagrams provide corrections to conformal correlation functions in the large N limit. For scalar fields, these diagrams can be written in the Mellin representation and each one has an associated Mellin amplitude. We calculate these quantities for an n-point contact diagram and a four-point scalar exchange diagram.

2 Anti-de Sitter Spacetime

Anti-de Sitter space is a homogeneous and isotropic Lorentzian manifold with constant negative scalar curvature. It is a solution of Einstein's field equations for a universe with negative cosmological constant.

Ads geometry

For reviews on AdS see [3], [4]. AdS_{d+1} is the hyperboloid

$$-X_0^2 - X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 = -R^2 \tag{2.1}$$

Embedded in $d+2$ dimensional Minkowski space $\mathbb{R}^{d,2}$ with metric

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2 \tag{2.2}$$

We can solve this equation by setting

$$\begin{aligned} X^0 &= R\,\text{cos}t\;\text{cosh}\rho \\ X^{d+1} &= R\,\text{sin}t\;\text{cosh}\rho \\ X^\mu &= R\,\Omega^\mu\;\text{sinh}\rho \end{aligned} \tag{2.3}$$

where Ω^μ parametrizes a unit $d-1$ sphere. These are called global coordinates and give the metric on AdS_{d+1}:

$$ds^2 = R^2\left(-\text{cosh}^2\rho\, d\tau^2 + d\rho^2 + \text{sinh}^2\rho\, d\Omega_{d-1}^2\right) \tag{2.4}$$

If we now set $\text{tan}\theta = \text{sinh}\rho$ where here θ must take values $0 \le \theta < \pi/2$ we find:

$$ds^2 = \frac{R^2}{\text{cos}^2\theta}\left(-d\tau^2 + d\theta^2 + \text{sin}^2\theta\, d\Omega_{d-1}^2\right) \tag{2.5}$$

Where we used $\text{cosh}\left(\text{sinh}^{-1}\left(\text{tan}\theta\right)\right) = \sqrt{\text{tan}^2\theta + 1}$. If we rescale by $R^{-2}\text{cos}^2\theta$ we find

$$ds'^2 = -d\tau^2 + d\theta^2 + \text{sin}^2\theta\, d\Omega_{d-1}^2 \tag{2.6}$$

This is the metric of the Einstein static universe $\mathbb{R} \times S^d$. For $\theta = 0$ we have the "south" pole of S^d and for $\theta = \pi/2$ the metric is that of a d-sphere S^d. However since $0 \le \theta < \pi/2$ and θ does not take values up to $\theta = \pi$, this maps AdS_{d+1} to half of the Einstein static universe. An analogous conformal compactification can be performed for d-dimensional Minkowski space and it turns out to be identical to this one. This fact, along with the symmetry group similarities that we will see shortly, provides a first hint at the AdS/CFT correspondence.

Wick Rotation

Consider flat Minkowski space in 4 dimensions:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2.7}$$

If we let time take imaginary values $t = -i\tau$ we can transform this metric into Euclidian space in 4 dimensions

$$ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2 \tag{2.8}$$

This is called Wick rotation and transforms $\mathbb{R}^{1,3} \to \mathbb{R}^4$. If we use this on our initial representation of AdS (2.1) and let $X^{d+1} \to -iX^{d+1}$ we find

$$-X_0^2 + \sum_{i=1}^{d+1} X_i^2 = -R^2 \tag{2.9}$$

This is called Euclidean AdS and is manifestly $SO\left(d+1,1\right)$ invariant. We can also introduce Poincaré coordinates:

$$\begin{aligned} X^0 &= R\,\frac{1 + x^2 + z^2}{2z} \\ X^\mu &= R\,\frac{x^\mu}{z} \\ X^{d+1} &= R\,\frac{1 - x^2 - z^2}{2z} \end{aligned} \tag{2.10}$$

Where $z > 0$ and by x^2 we mean $x_\mu x^\mu$. In these coordinates the metric reads

$$ds^2 = R^2\,\frac{dz^2 + \delta_{\mu\nu}dx^\mu dx^\nu}{z^2} \tag{2.11}$$

These coordinates show that after conformally rescaling, AdS is mapped to $\mathbb{R}^+ \times \mathbb{R}^d$. At z=0 the boundary is simply $\mathbb{R}^d$. In this form of the metric the subgroups $ISO\,(d)$ and $SO\,(1,1)$ of the $SO\,(d+1,1)$ isometry are apparent. $ISO\,(d)$ are Poincaré transformations

$$\begin{aligned} &x^\mu \to \Lambda^\mu_\nu x^\nu \quad \text{Lorentz Transformations} \\ &x^\mu \to x^\mu + a^\mu \quad \text{Spacetime Translations} \end{aligned} \tag{2.12}$$

$SO\,(1,1)$ is called dilatation. It is simply a coordinate re-scaling $z \to cz$, $x^\mu \to cx^\mu$ for some $c > 0$. As we will see in the next chapter these symmetries correspond to the conformal group, the group of CFTs.

3 Conformal Field Theory

A Conformal Field Theory is a Quantum Field Theory that is scale-invariant. Meaning that at whatever length scale we "look" at the theory the underlying physics is the same. A CFT is invariant under transformations that preserve angles, but not necessarily lengths. Combining this scale invariance symmetry with the Poincaré symmetry leads to further symmetries. All these transformations form the *Conformal Group* which we will review in this chapter. This is a Lie Group which means that we can write an element of the group using the infinitesimal generators. It is important to stress that the 2-dimensional case is special as there are infinite generators. However, it will not be relevant for our purposes. From here on, we assume that we are talking about a CFT in $d > 2$. For comprehensive notes on CFTs in $d > 2$ see [5],[6].

Let's take a look at the generators of the different subgroups of the Conformal Group:
(i) Momentum:

$$P_\mu = \partial_\mu \tag{3.13}$$

This generates spacetime translations, a^μ.
(ii) Lorentz:

$$L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \tag{3.14}$$

This generates boosts and rotations, $\omega^{\mu\nu}$.
(iii) Dilatation:

$$D = x_\mu \partial_\mu \tag{3.15}$$

This generates scale transformations, σ.
(iv) Special conformal generator:

$$K_\mu = x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu \tag{3.16}$$

This generates a new parameter, b^μ.

Primary Field Transformations

Before we look at how these tranformations act on Scalar Primary Fields we shall first carry out some dimensional analysis to define the notion of the Weight Δ of a field. From the massless scalar field action:

$$S = \int d^d x \; \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \tag{3.17}$$

Using units where $\hbar = c = 1$, the action S is dimensionless. The derivatives have dimension $[\partial_\mu] = [L]^{-1}$ and the differential has $\left[d^d x\right] = [L]^d$. Putting all this together we find

$$[\phi] = [L]^{\,\frac{2-d}{2}} \tag{3.18}$$

We also know from $[\hbar] = [E] \cdot [T]$ that $[E] = [T]^{-1}$ and of course from c that $[L] = [T]$. So in energy units the field has dimensions $[\phi] = [E]^{\frac{d-2}{2}}$. We usually assume that we are using energy (or mass) dimensions so we write

$$[\phi] = \frac{d-2}{2} \implies \Delta_\phi = \frac{d-2}{2} \tag{3.19}$$

This is the classical definition of the Weight Δ_ϕ. Based on this discussion we can see that under a dilatation of the form

$$x^\mu \to x'^\mu = \lambda x^\mu \tag{3.20}$$

a scalar field transforms as:

$$\phi'(x') = \lambda^{-\Delta} \phi(x) \tag{3.21}$$

The Weight Δ_ϕ will be important to us in the calculation of correlation functions and in relating our findings in the CFT with the calculations in AdS.

A scalar primary field with weight Δ, by definition, transforms under a general conformal transformation as:

$$\phi'(x') = \Omega^{-\Delta}(x) \phi(x) \tag{3.22}$$

We would like to write this equation infinitesimally. We start by writting:

$$x'^\mu = x^\mu + \xi^\mu(x) \quad \text{and} \quad \Omega = 1 + \kappa(x) \tag{3.23}$$

where the ξ and κ are infinitesimal. Under this translation the field transforms as: $\phi(x) \to \phi(x) - \xi^\mu \partial_\mu \phi(x)$ Inserting this into (3.22) we find:

$$\begin{aligned} \phi'(x') &= (1 - \Delta\kappa(x))(\phi(x) - \xi^\mu \partial_\mu \phi(x)) \\ \phi'(x') - \phi(x) &\equiv \delta\phi(x) = -\Delta\kappa(x)\phi(x) - \xi^\mu \partial_\mu \phi(x) \end{aligned} \tag{3.24}$$

Where we discard terms of order $\mathcal{O}(\xi^2), \mathcal{O}(\kappa^2), \mathcal{O}(\xi\kappa)$. The ξ and κ are found by solving the conformal Killing equation:

$$2\kappa(x)\eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{3.25}$$

The general solution (for $d > 2$) is:

$$\begin{aligned} \xi^\mu(x) =& a^\mu + \omega^\mu_\nu x^\nu + \sigma x^\mu + b^\mu x^2 - 2b^\nu x_\nu x^\mu \\ \kappa(x) =& \frac{1}{d}\partial_\mu \xi^\mu = \sigma - 2b^\nu x_\nu \end{aligned} \tag{3.26}$$

Substituting this into (3.24) we find:

$$\begin{aligned} \delta\phi(x) = -\Delta\sigma\phi(x) + \Delta 2b^\nu x_\nu \phi(x) - a^\mu \partial_\mu \phi(x) - \omega^\mu_\nu x^\nu \partial_\mu \phi(x) - \sigma x^\mu \partial_\mu \phi(x) \\ - b^\mu x^2 \partial_\mu \phi(x) + 2b^\nu x_\nu x^\mu \partial_\mu \phi(x) \end{aligned} \tag{3.27}$$

Now by reading of the coefficients of the parameters $a^\mu, b^\mu, \sigma, \omega^{\mu\nu}$ we find the action of their respective infinitesimal generators on scalar primary fields.

$$\begin{aligned} &\delta_{P_\mu} = -\partial_\mu \quad \text{for} \quad a^\mu \\ &\delta_{L_{\mu\nu}} = x_\nu \partial_\mu - x_\mu \partial_\nu \quad \text{for} \quad \omega^{\mu\nu} \\ &\delta_D = -(\Delta + x^\mu \partial_\mu) \quad \text{for} \quad \sigma \\ &\delta_{K_\mu} = 2\Delta x_\mu - x^2 \partial_\mu + 2x_\mu x^\nu \partial_\nu \quad \text{for} \quad b^\mu \end{aligned} \tag{3.28}$$

Conformal Charges

Noether's Theorem

A continuous symmetry of the Lagrangian gives rise to a conserved current $j^\mu(x)$. A conserved current means that

$$\partial_\mu j^\mu (x) = 0 \tag{3.29}$$

Translation invariance of the Lagrangian leads to a conserved current for each dimension, all expressed in the energy-momentum tensor T^μ_ν, that satisfies $\partial_\mu T^\mu_\nu = 0$. Noethers theorem also implies the existence of a conserved charge Q, defined as:

$$Q_\xi (\Sigma) = -\int_\Sigma dS_\mu \xi_\nu (x) T^{\mu\nu} (x) \tag{3.30}$$

Here Σ is a surface with boundary S. In the case of canonical QFT this would be an integral over d^3x. ξ^μ is a Killing Vector in flat space:

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0 \tag{3.31}$$

To see why Q is indeed conserved in (3.30) we take the derivative:

$$\partial_\mu Q_\xi (\Sigma) = -\int_\Sigma dS_\mu \partial_\mu (\xi_\nu (x) T^{\mu\nu} (x)) \tag{3.32}$$

$$\begin{aligned} &\partial_\mu (\xi_\nu (x) T^{\mu\nu} (x)) = \\ &(\partial_\mu \xi_\nu) T^{\mu\nu} + \xi_\nu (\partial_\mu T^{\mu\nu}) = \\ &\frac{1}{2} (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) T^{\mu\nu} = 0 \end{aligned} \tag{3.33}$$

Where we used the fact that $\partial_\mu T^\mu_\nu = 0$ and the fact that the energy-momentum tensor is symmetric: $T^{\mu\nu} = T^{\nu\mu}$.

The important takeaway is that for each of our generators (3.13)-(3.16) there is an associated conserved charge which we denote with a tilde e.g.

$$\tilde{P}_\mu = Q_{P_\mu} \tag{3.34}$$

Taking the commutators of these charges differs from the commutators of the generators by a minus sign

$$[Q_a, Q_b] = -Q_{[a,b]} \tag{3.35}$$

So the commutator of two conformal charges is opposite to the conserved charge corresponding the commutator of the two charge generators. The action of a charge on a local operator is given by the commutator

$$[Q_a, \mathcal{O}(x)] = -\delta_a \mathcal{O} \tag{3.36}$$

So for example $\left[\tilde{P}_\mu, \mathcal{O}(x)\right] = \partial_\mu \mathcal{O}$

Finally, it is important to mention how to think about CFTs with respect to regular Quantum Field Theories. A QFT can be thought of as the renormalization group (RG) flow between CFTs. For example, consider a non relativistic theory for a scalar field with a ϕ^4 interaction term:

$$S = \int d^3x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + g\phi^4 \right) \tag{3.37}$$

By dimensional analysis as we did in (3.17) we find $[\phi] = 1/2$, $[g] = 1$ and of course $[m] = 1$. So at high energies the kinetic term dominates and we say the theory is free in the UV. In the low energy limit the IR theory will be a CFT for a special value of g^2/m^2. This will be an interacting CFT that is not exactly solvable as the free theory.

4 AdS/CFT

As mentioned in the introduction the AdS/CFT Correspondence refers to two equivalent descriptions of the same underlying physics. The more precise relation that defines the correspondence is the folowing

$$Z_{CFT} = Z_{AdS} \tag{4.1}$$

where Z is the generating functional (or partition function). In QFT it is defined as:

$$Z[J] = \int \mathcal{D}\phi \exp\left[i \int d^d x \left[\mathcal{L} + J(x)\phi(x)\right]\right] \tag{4.2}$$

This a path integral, an integral over all field configurations. $J(x)$ is called a source term and its insertion allows us to produce correlation functions by taking the functional derrivative and setting $J = 0$.

$$\langle 0| \, |T(\phi(x_1)\phi(x_2))| \, |0\rangle = Z[J]^{-1}\left(-i\,\frac{\delta}{\delta J(x_1)}\right)\left(-i\,\frac{\delta}{\delta J(x_2)}\right) z[J]\bigg|_{J=0} \tag{4.3}$$

It is convenient to work in Euclidian signature so we consider the Wick-rotated version of (4.2):

$$Z[J] = \int \mathcal{D}\phi \exp\left[-\int d^d x \left[\mathcal{L} - J(x)\phi(x)\right]\right] \tag{4.4}$$

This gives the Euclidian correlator

$$\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle \tag{4.5}$$

Here we assumed scalar fields but the analysis is simmilar for vector and tensor fields. The relation (4.1) tells us that the boundary value of a field ϕ_0 is the source J for a scalar operator $\mathcal{O}$ in the CFT.

$$\left\langle exp \int_{S^d} \phi_0 \mathcal{O} \right\rangle_{CFT} = Z_S(\phi_0) \tag{4.6}$$

In this section we provide examples of this relation for scalar and vector fields. This section follows the paper by E. Witten titled "Anti-de Sitter space and holography" [1].

Scalar Field

The classical supergravity action for a massless scalar field is

$$I(\phi) = \frac{1}{2}\int_{B_{d+1}} d^{d+1}x\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{4.7}$$

Where $\sqrt{g}$ is the determinant of the metric. We assume that the boundary value ϕ_0 of ϕ is the source of an operator $\mathcal{O}$ in the conformal theory. Our task now is to write ϕ in terms of ϕ_0. To do this we look for a Green's function $K(x - x')$ that is a solution of the Laplace equation on AdS_{d+1} such that

$$\nabla_\mu \nabla^\mu K(x - x') = \delta(x - x') \tag{4.8}$$

To find this function we shall use the representation of AdS_{d+1} in Poincare coordinates (2.11). We simply relabel $z = x_0$ and set $R = 1$.

$$ds^2 = \frac{1}{x_0^2}\sum_{i=0}^{d}(dx_i)^2 \tag{4.9}$$

Here $x_0 > 0$ and the boundary is at $x_0 = \infty$. Notice that the boundary conditions are independant of the x_i. In addition, if we consider a translation of the x_i ie $x_i' = x_i + a_i$ we can see that the metric is invariant under this translation since $d(x_i + a_i) = dx_i$. So the Green's function K will have this symmetry and as a result will be a function only of x_0. Before we find $K(x_0)$ we shall first prove that the action (4.7) indeed gives a laplace equation for ϕ.

$$\mathcal{L} = \frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{4.10}$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = 0\,, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \sqrt{g}g^{\mu\nu}\partial_\nu\phi \tag{4.11}$$

So from the Euler-Lagrange equation we find:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right) = 0 \tag{4.12}$$

Which is equivalent to the Laplace equation in curved space:

$$\Box\phi = \nabla_\mu\nabla^\mu\phi = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right) = 0 \tag{4.13}$$

In order to solve the equation for $K(x_0)$ we need to find the inverse of the metric $g^{\mu\nu}$, this is easy to do since the metric in our representation (4.9) is block diagonal and so we only need to take the inverse of the elements on the diagonal.

$$g_{\mu\nu} = \begin{pmatrix} x_0^{-2} & 0 & \cdots & 0 \\ 0 & x_0^{-2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_0^{-2} \end{pmatrix} \rightarrow g^{\mu\nu} = \begin{pmatrix} x_0^{2} & 0 & \cdots & 0 \\ 0 & x_0^{2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_0^{2} \end{pmatrix} \tag{4.14}$$

To find the determinant of the metric above we can start with $d = 1$ and for a 2x2 matrix calculate $det(g) = x_0^{-4} = x_0^{-2(d+1)}$. For $d = 2$ we find $det(g) = x_0^{-6} = x_0^{-2(d+1)}$ so we have found the pattern for d dimensions and we have $\sqrt{g} = x_0^{-d-1}$. If we substitute this into the Laplace equation (4.13) we find for K

$$\frac{d}{dx_0}\left(x_0^{-d-1}x_0^2\frac{d}{dx_0}K(x_0)\right) = 0 \tag{4.15}$$

Where we can multiply by the factor $\sqrt{g}$ (since $x_0 > 0$) to get rid of the $\frac{1}{\sqrt{g}}$ in (4.13). Now by using the product rule the above equation takes the form:

$$\begin{aligned} &(1-d)\,x_0^{-d}K'(x_0) + x_0^{1-d}K''(x_0) = 0 \;\Rightarrow \\ &(1-d)\,K'(x_0) + x_0K''(x_0) = 0 \end{aligned} \tag{4.16}$$

To solve this linear differential equation we let $K(x_0) = (x_0)^s$. With this substitution (4.16) reads:

$$\begin{aligned} &(1-d)\,sx_0^{s-1} + x_0s(s-1)\,x_0^{s-2} = 0 \\ &(1-d)\,s + s(s-1) = 0 \\ &1-d = 1-s \quad \text{or} \quad s = 0 \\ &s = d \quad \text{or} \quad s = 0 \end{aligned} \tag{4.17}$$

So the general solution for K is

$$K(x_0) = cx_0^d + c' \tag{4.18}$$

However we want the solution to approach zero as $x_0 \to 0$ so we take $c' = 0$ and our final sollution is $K(x_0) = cx_0^d$. This grows at infinity and the singularity is a delta function. To show this we make an $SO(1, d+1)$ transformation that maps the point at $x_0 = \infty$ to a finite point.

$$x_i \to \frac{x_i}{x_0^2 + \sum_{j=1}^{d} x_j^2} \tag{4.19}$$

Under this transformation $K(x_0)$ transforms as

$$K(x_0) = \frac{cx_0^d}{\left(x_0^2 + \sum_{j=1}^{d} x_j^2\right)^d} \tag{4.20}$$

For $x_0 \to 0$ K vanishes everywhere except at $x_1 = x_2 = ... = x_d = 0$ where $K(x_0) \sim x_0^{-d}$. So K becomes a delta function with the infinity at $x_1 = x_2 = ... = x_d = 0$. The solution for ϕ can now be written as:

$$\phi(x_0, x_i) = \int d^d x' K(x - x') \phi_0(x_i') \tag{4.21}$$

To see this one can act with the Laplacian on the previous equation and then use the definition of the Green's function (4.8). When we substitute K the equation above reads:

$$\phi(x_0, x_i) = c \int d^d x' \frac{x_0^d}{\left(x_0^2 + \sum_{i=1}^{d} (x_i - x_i')^2\right)^d} \phi_0(x_i') \tag{4.22}$$

Using the notation $\sum_{j=1}^{d} x_j^2 \equiv |\mathbf{x}|^2$ we rewrite (4.22) as

$$\phi(x_0, x_i) = c \int d^d x' \frac{x_0^d}{\left(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2\right)^d} \phi_0(x') \tag{4.23}$$

for $x_0 \to 0$

$$\phi(x_0, x_i) \sim c \int d^d x' \frac{x_0^d}{|\mathbf{x} - \mathbf{x}'|^{2d}} \phi_0(x') \;\to\; \frac{\partial \phi}{\partial x_0} \sim c d x_0^{d-1} \int d^d x' \frac{x_0^d}{|\mathbf{x} - \mathbf{x}'|^{2d}} \phi_0(x') \tag{4.24}$$

Now we integrate the action (4.7) by parts and find

$$I(\phi) = \frac{1}{2} \int d^{d+1} x \partial_\nu \left(\sqrt{g} g^{\mu\nu} \partial_\mu \phi \cdot \phi\right) - \frac{1}{2} \int d^{d+1} x \partial_\nu \left(\sqrt{g} g^{\mu\nu} \partial_\mu \phi\right) \phi \tag{4.25}$$

The second term here vanishes as it satisfies the Laplace equation. The first term can be written as a surface integral using the divergence theorem. We also take the limit of $x_0 = \epsilon$ and $I(\phi)$ is

$$I(\phi) = \lim_{\epsilon \to 0} \frac{1}{2} \int d^d x \sqrt{h} \phi \left(\hat{n} \cdot \vec{\nabla} \phi\right) \tag{4.26}$$

where h is the induced metric and $\hat{n}$ is the unit vector on the x_0 plane. $\hat{n} = \hat{x_0} N$, the direction of $\hat{n}$ is clear but we need to find N for its length to be 1:

$$\hat{n} \cdot \hat{n} = \frac{1}{x_0^2} N^2 = 1 \Rightarrow |N| = x_0 \tag{4.27}$$

Where we used the euclidian signature metric (4.9) for the dot product. The induced metric is one dimension smaller than g since $dx_0 = 0$ so we find $\sqrt{h} = x_0^{-d}$. Putting all this together along with (4.24) we find:

$$\begin{aligned} I(\phi) &= \frac{1}{2} \int d^d x \cdot x_0^{-d} \phi \cdot x_0 \frac{\partial \phi}{\partial x_0} \Rightarrow \\ I(\phi) &= \frac{cd}{2} \int d^d x d^d x' \frac{\phi_0(x) \phi_0(x')}{|\mathbf{x} - \mathbf{x}'|^{2d}} \end{aligned} \tag{4.28}$$

According to our formulation of the conjecture we can get the two point function of the operator $\mathcal{O}$ by taking the functional derivative $\frac{\delta J(y)}{\delta J(x)} \equiv \delta^{(d)}(x-y)$. So we find that

$$\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\rangle = \frac{c_{12}}{|x_1-x_2|^{2d}}, \quad (c_{12}=cd/2) \tag{4.29}$$

As we will see soon, this is the same result one obtains by calculating the correlation functions in the Conformal Field Theory for a scalar operator of conformal dimension d. In principle, one can calculate all the correlation functions for a specific CFT using Feynman diagrams. However, in practice this is very difficult. Therefore, we calculate the correlation functions by using the general properties and symmetries of our theory to constrain the form of the functions. As we will see, this is enough to fully fix the two-point function.

Conformal Constraints

We will now perform the same calculation in the CFT to calculate the two point function of a scalar operator. Using the Ward Identity one can show that

$$\langle \delta\mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\rangle + \ldots + \langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\ldots\delta\mathcal{O}_n(x_n)\rangle = 0 \tag{4.30}$$

Where δ is the action of a conserved charge. In other words, if a symmetry of the Lagrangian transforms an operator $\mathcal{O}\to\mathcal{O}'=\mathcal{O}+\delta\mathcal{O}$, we don't expect the correlation functions which are realted to observable quantities to change:

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\rangle = \langle \mathcal{O}_1'(x_1)\,\mathcal{O}_2'(x_2)\ldots\mathcal{O}_n'(x_n)\rangle \tag{4.31}$$

We now apply this to the two point function

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\rangle = f(x_1,x_2) \tag{4.32}$$

Using (4.30) and the action of the infinitesimal generators (3.28) we impose the different symmetries.

Translations

$$\begin{aligned} (\partial_1+\partial_2)\,f(x_1^\mu,x_2^\mu)=0 &\Longrightarrow \frac{\partial f}{\partial x_1^\nu}\frac{\partial x_1^\mu}{\partial x_1^\nu}+\frac{\partial f}{\partial x_2^\nu}\frac{\partial x_2^\mu}{\partial x_2^\nu}=0 \\ \frac{\partial f}{\partial x_1^\nu}\eta^{\mu\nu}+\frac{\partial f}{\partial x_2^\nu}\eta^{\mu\nu}=0 &\Longrightarrow \frac{\partial f}{\partial x_1^\nu}=-\frac{\partial f}{\partial x_2^\nu} \end{aligned} \tag{4.33}$$

This means that f must be a function of $f\left((x_1-x_2)^\mu\right)$.

Lorentz

The two point function of scalar operators transforms as a scalar under the Lorentz Group. This means that f must transform as a scalar and so can't have any vector induces. This further constrains the form of the function

$$f\left((x_1-x_2)^\mu\,(x_1-x_2)_\mu\right) = f\left(|x_1-x_2|^2\right) \tag{4.34}$$

This analysis so far holds for any relativistic QFT. In a CFT however we have two additional symmetry sub groups to impose.

Dilatation

$$(x_1^\mu\partial_{1\mu}+\Delta_1+x_2^\mu\partial_{2\mu}+\Delta_2)\,f\left(|x_1-x_2|^2\right)=0 \tag{4.35}$$

For the derivatives we find

$$\begin{aligned}\partial_{1\mu} f\left((x_1-x_2)^\mu (x_1-x_2)_\mu\right) &= 2(x_1-x_2)_\mu f'\left(|x_1-x_2|^2\right)\\ \partial_{2\mu} f\left((x_1-x_2)^\mu (x_1-x_2)_\mu\right) &= -2(x_1-x_2)_\mu f'\left(|x_1-x_2|^2\right)\end{aligned} \tag{4.36}$$

Plugging this back to the previous equation we find:

$$2(x_1-x_2)_\mu f'\left(|x_1-x_2|^2\right)(x_1-x_2)^\mu + (\Delta_1+\Delta_2) f\left(|x_1-x_2|^2\right) = 0 \tag{4.37}$$

If we set $y = |x_1-x_2|^2$ we have

$$\begin{aligned}&2yf'(y) + (\Delta_1+\Delta_2) f(y) = 0\\ &\frac{f'(y)}{f(y)} = \frac{-(\Delta_1+\Delta_2)}{2y} \Longrightarrow \ln f(y) = \frac{-(\Delta_1+\Delta_2)}{2}\ln y + c\\ &f(y) = c_{12} y^{(-\Delta_1-\Delta_2)/2}\end{aligned} \tag{4.38}$$

And if we reintroduce x we conclude that the correlation function must be of the form

$$f = \frac{c_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} \tag{4.39}$$

Special Conformal Transformation

$$\left[-2x_{1\mu}\Delta_1 + x_1^2\partial_{1\mu} - 2x_{1\mu}x_1^\nu\partial_{1\nu} - 2x_{2\mu}\Delta_2 + x_2^2\partial_{2\mu} - 2x_{2\mu}x_2^\nu\partial_{2\nu}\right] \frac{c_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} = 0 \tag{4.40}$$

To simplify the calculation we define the differential operators $K_{i\mu}$

$$K_{i\mu} = x_i^2\partial_{i\mu} - 2x_{i\mu}x_i^\nu\partial_{i\nu} \tag{4.41}$$

As a first step we look at how this operator acts on the invariant $|x_1-x_2|^2 \equiv x_{12}^2$.

$$\begin{aligned}(K_{1\mu}+K_{2\mu})\, x_{12}^2 &= 2x_1^2(x_1-x_2)_\mu - 4x_{1\mu}x_1^\nu(x_1-x_2)_\nu + (x_1 \leftrightarrow x_2)\\ &= -2x_1^2x_{1\mu} - 2x_1^2x_{2\mu} + 4x_{1\mu}x_1^\nu x_{2\nu} - 2x_2^2x_{2\mu} - 2x_2^2x_{1\mu} + 4x_{2\mu}x_2^\nu x_{1\nu}\\ &= -2(x_{1\mu}+x_{2\mu})\left(x_1^2 - 2x_1\cdot x_2 + x_2^2\right) = -2(x_{1\mu}+x_{2\mu})(x_1-x_2)^2\end{aligned} \tag{4.42}$$

However $K_{i\mu}$ is a differential operator:

$$\begin{aligned}(K_{1\mu}+K_{2\mu})\left(x_{12}^2\right)^{\frac{-\Delta_1-\Delta_2}{2}} &= \frac{-\Delta_1-\Delta_2}{2}\left(x_{12}^2\right)^{\frac{-\Delta_1-\Delta_2}{2}-1}(K_{1\mu}+K_{2\mu})\, x_{12}^2\\ &= \frac{-\Delta_1-\Delta_2}{2}\left(x_{12}^2\right)^{\frac{-\Delta_1-\Delta_2}{2}-1}(-2)(x_{1\mu}+x_{2\mu})(x_1-x_2)^2\\ &= (\Delta_1+\Delta_2)(x_{1\mu}+x_{2\mu})(x_{12})^{-\Delta_1-\Delta_2}\end{aligned} \tag{4.43}$$

Plugging this back to (4.40) we find

$$(\Delta_1-\Delta_2)(x_{2\mu}-x_{1\mu})\frac{c_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} = 0 \tag{4.44}$$

So we have $\Delta_1 = \Delta_2 = \Delta$ or $c_{12} = 0$. So we conclude that the only form of the two point function compatible with the conformal symmetries is

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\rangle = \frac{c_{12}}{|x_1-x_2|^{2\Delta}} \tag{4.45}$$

We can even take $c_{12} = 1$ by renormalizing the operators. This is the result we obtained in our gravity calculation (4.29). Moreover, for massless fields in AdS the weights of the scalar operators in the CFT are given by the dimension d of the AdS_{d+1} space we are considering.

U(1) Gauge Theory

Our next example of the AdS/CFT correspondence is in free $U\left(1\right)$ gauge theory. As before, we look for a Green's function, in this case, a solution of Maxwell's equations on AdS_{d+1}, using the metrtic representation (4.9). A natural language to solve this problem is the *differential forms* formalism*. In this formalism Maxwell's equations are compactly written as:

$$\begin{aligned} dF &= 0 \\ d*F &= \mu_0 J \end{aligned} \tag{4.46}$$

Where F is the electromagnetic 2-form analogus to the E/M tensor $F_{\mu\nu}$.

$$F = \frac{1}{2}F_{\alpha\beta}dx^{\alpha}\wedge dx^{\beta} \tag{4.47}$$

And J is the current 3-form: $J = -J_a * dx^{\alpha}$. In the free theory $J = 0$. By d we denote the exterior derivative. For a p-form $\phi = gdx^I = gdx^1\wedge dx^2\wedge ...\wedge dx^p$ it is defined as:

$$d\phi = \frac{\partial g}{\partial x^i}dx^i\wedge dx^I \tag{4.48}$$

Where $i = 1, 2, .., d$ for a d-dimensional space. From this definition one can check that a second order exterior derivative is always zero $d\left(d\phi\right) = 0$ for any ϕ. This leads us to write $F = dA$ which is a consequence of the first equation in (4.46).

A is the potential 1-form $A = A_{\alpha}dx^{\alpha}$. Note that if we make a transformation $A' = A + d\lambda$ we find $dA' = dA$. The gauge invariance of A also comes out naturally in this formalism.
Finaly, the " $*$ " symbol is the Hodge star operator that maps a p-form to a (d-p)-form:

$$\left(*A\right)_{\mu_1...\mu_{d-p}} = \frac{\sqrt{g}}{p!}\epsilon^{\nu_1...\nu_p}_{\mu_1...\mu_{d-p}}A_{\nu_1...\nu_p} \tag{4.49}$$

where ϵ is the totaly antisymmetric Levi-Civita symbol. With this small introduction out of the way we can look for our Green's function. We write the second equation in (4.46) as:

$$d\left(*dA\right) = 0 \tag{4.50}$$

Just like the scalar case we expect our one-form A to only depend on x_0

$$A = f\left(x_0\right)dx^i \quad \text{for some} \quad i \neq 0 \tag{4.51}$$

Taking the exterior derivative (4.48) since A only depends on x_0 we find

$$dA = f'\left(x_0\right)dx^0\wedge dx^i \tag{4.52}$$

If we now act with the Hodge star (4.49) we find

$$*\, dA = \sqrt{g}\epsilon^{i0}_{12...d}f'\left(x_0\right)dx^1\wedge dx^2\wedge ...\wedge \widehat{dx^i}\wedge ...\wedge dx^d \tag{4.53}$$

The notation $\widehat{dx^i}$ means we do not include that differential in the wedge product. We lower the 2 upper indexes of ϵ by acting with the metric:

$$\epsilon^{i0}_{12...d} = g^{0\nu}g^{i\mu}\epsilon_{\nu\mu12..d} = g^{00}g^{ii}\epsilon_{0i12..d} = x_0^4\left(-1\right)^i \tag{4.54}$$

*A calculation in covariant form is performed in Appendix A

Here we used the inverse of the metric (4.14) and the fact that the Levi-Civita symbol is +1 for even permutations of $(1,2,...,d)$ while -1 for odd permutations of $(1,2,...,d)$. Using also $\sqrt{g} = x_0^{-d-1}$ we have for our $d-2$-form:

$$* \, dA = \frac{1}{x_0^{d-3}} \left(-1\right)^i f'\left(x_0\right) dx^1 \wedge dx^2 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^d \tag{4.55}$$

Maxwell's equations $d\left(*dA\right) = 0$ give:

$$\begin{aligned} &\frac{\partial}{\partial x_0}\left(\frac{1}{x_0^{d-3}} f'\left(x_0\right)\right)\left(-1\right)^i dx^0 \wedge dx^1 \wedge dx^2 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^d = 0 \Longrightarrow \\ &\frac{1}{x_0^{d-3}} f'\left(x_0\right) = c \Longrightarrow f\left(x_0\right) = \frac{c x_0^{d-2}}{d-2} + c' \end{aligned} \tag{4.56}$$

As before, we want $f\left(0\right) \to 0$ so we take $c' = 0$. We find

$$A = c x_0^{d-2} dx^i \tag{4.57}$$

Now we make an inversion $x_i \to \frac{x_i}{\left(x_0^2 + \sum_{i=1}^n x_i^2\right)} \equiv \frac{x_i}{\left(x_0^2 + |\mathbf{x}|^2\right)}$ and find

$$A = c\left(\frac{x_0}{x_0^2 + |\mathbf{x}|^2}\right)^{d-2} d\left(\frac{x_i}{x_0^2 + |\mathbf{x}|^2}\right) \tag{4.58}$$

And we can also make a gauge transformation

$$A \to A + d\left(\frac{-1}{d-2} \frac{x_0^{d-2} x_i}{\left(x_0^2 + |\mathbf{x}|^2\right)^{d-1}}\right) \tag{4.59}$$

This exterior derivative is equal to:

$$\begin{aligned} &\frac{-1}{d-2} \frac{\partial}{\partial x_0}\left(\frac{x_0^{d-2} x_i}{\left(x_0^2 + |\mathbf{x}|^2\right)^{d-1}}\right) dx^0 + \frac{-1}{d-2} \frac{\partial}{\partial x_i}\left(\frac{x_0^{d-2} x_i}{\left(x_0^2 + |\mathbf{x}|^2\right)^{d-1}}\right) dx^i = \\ &- \frac{1}{d-2} \frac{\left(d-2\right) x_0^{d-3} x_i - 2 x_0^{d-1} x_i \left(d-1\right)\left(x_0^2 + |\mathbf{x}^2|\right)^{-1}}{\left(x_0^2 + |\mathbf{x}^2|\right)^{d-1}} dx^0 \\ &- \frac{1}{d-2} \frac{x_0^{d-2} - 2 x_0^{d-2} x_i^2 \left(x_0^2 + |\mathbf{x}^2|\right)^{-1}}{\left(x_0^2 + |\mathbf{x}^2|\right)^{d-1}} dx^i \end{aligned} \tag{4.60}$$

Where we used $\frac{\partial}{\partial x_i} \sum_{j=1}^n x_j^2 = 2x_i$. Now we also calculate the exterior derivative of (4.58) and find:

$$A = c\left(- \frac{2 x_0^{d-1} x_i}{\left(x_0^2 + |\mathbf{x}^2|\right)^d} dx^0 - \frac{x_0^{d-2} 2 x_i^2}{\left(x_0^2 + |\mathbf{x}^2|\right)^d} dx^i + \frac{x_0^{d-2}}{\left(x_0^2 + |\mathbf{x}^2|\right)^{d-1}} dx^i\right) \tag{4.61}$$

We can set the constant $c = \frac{d-1}{d-2}$ and notice that the second terms the fractions of dx^0 and dx^i in (4.60) cancel with the first and second terms in (4.61). Grouping together the remaining terms leads to:

$$A = \frac{x_0^{d-2}}{\left(x_0^2 + |\mathbf{x}^2|\right)^{d-1}} dx^i - \frac{x_0^{d-3} x_i}{\left(x_0^2 + |\mathbf{x}^2|\right)^{d-1}} dx^0 \tag{4.62}$$

This is the final form of the Green's function. Using it we can write our solution of Maxwell's equations:

$$A\left(x_0, \mathbf{x}\right) = \int d^d x' \frac{x_0^{d-2}}{\left(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2\right)^{d-1}} a_i\left(x'\right) dx^i - x_0^{d-3} dx^0 \int d^d x' \frac{\left(x - x'\right)^i a_i\left(x'\right)}{\left(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2\right)^{d-1}} \tag{4.63}$$

As before, our goal is to compute the action at the boundary limit of AdS. To do this need to calculate $F = dA$:

$$\begin{aligned} F = &(d-2)\, x_0^{d-3} dx^0 \int d^d x' \, \frac{dx^i a_i(x')}{\left(x_0^2 + |\mathbf{x}-\mathbf{x}'|^2\right)^{d-1}} \\ &- 2(d-1)\, x_0^{d-1} dx^0 \int d^d x' \, \frac{dx^i a_i(x')}{\left(x_0^2 + |\mathbf{x}-\mathbf{x}'|^2\right)^{d}} \\ &- 2(d-1)\, x_0^{d-3} dx^0 \int d^d x' \, \frac{(x_i - x_i')\, dx^i a_k(x')\,(x-x')^k}{\left(x_0^2 + |\mathbf{x}-\mathbf{x}'|^2\right)^{d}} \\ &+ 2(d-1)\, x_0^{d-2} \int d^d x' \, \frac{(x_i - x_i')\, dx^i a_k(x')\, dx^k}{\left(x_0^2 + |\mathbf{x}-\mathbf{x}'|^2\right)^{d}} \end{aligned} \tag{4.64}$$

where a wedge product is implied between the differentials. Now we integrate the action by parts:

$$I(A) = \frac{1}{2}\int F \wedge *F = \frac{1}{2}\int d(A \wedge *F) - \frac{1}{2}\int A \wedge d(*F) \tag{4.65}$$

The second term vanishes by virtue of $d(*dA) = 0$ and the first term becomes a surface integral over T_ϵ for $x_0 = \epsilon$ by the divergence theorem.

$$I(A) = \frac{1}{2}\lim_{\epsilon\to 0}\int_{T_\epsilon} A \wedge *F \tag{4.66}$$

Using the above expressions for F and A and the hodge operation as we did for dA, this integral gives:

$$I = \int dx dx' a_i(x)\, a_j(x') \left(\frac{\delta_{ij}}{|\mathbf{x}-\mathbf{x}'|^{2d-2}} - \frac{2(x-x')_i\,(x-x')_j}{|\mathbf{x}-\mathbf{x}'|^{2d}} \right) \tag{4.67}$$

Taking the functional derivative of this result gives us the two point function of a conserved, primary, spin-1 field in the Conformal Field Theory. We make use of $\frac{\partial J(y)}{\partial J(x)} = \delta^{(d)}(x-y)$ and find:

$$\langle J_i(x_1)\, J_j(x_2) \rangle = \frac{\delta_{ij}}{|x_1 - x_2|^{2d-2}} - \frac{2(x_1-x_2)_i\,(x_1-x_2)_j}{|x_1-x_2|^{2d}} \tag{4.68}$$

In the following subsection will shall derive this result in the CFT by making use of the embedding space formalism and light cone coordinates. This will hopefully demonstrate the usefullness of light-cone coordinates in solving the conformal symmetry constraints. Furthermore, we shall make use of this formalism in Section 5 to compute higher point functions.

Light Cone Coordinates and Embedding Space Formalism

It is convenient to think of the conformal boundary of AdS as the space of light rays

$$-\left(P^0\right)^2 + \left(P^1\right)^2 + ... + \left(P^{d+1}\right)^2 = 0 \tag{4.69}$$

Where we use light cone coordinates

$$P^A = \left(P^+, P^-, P^a\right) \tag{4.70}$$

And the metric is given by

$$P^A \cdot P^B = -P^+ P^- + \delta_{ab} P^a P^b \tag{4.71}$$

A section of the light cone is the d-dimendional space of the CFT. We can go back to the Poincare section by setting

$$P^0(x) = \frac{1+x^2}{2}, \quad P^\mu(x) = x^\mu, \quad P^{d+1}(x) = \frac{1-x^2}{2} \tag{4.72}$$

This gives $(dx^\mu)^2 = 0$ where $\mu = 1, 2, .., d$ so this is simply $\mathbb{R}^d$. It is natural to extend primary operators to the light cone by imposing the homogeneity property:

$$\mathcal{O}(\lambda P) = \lambda^{-\Delta}\mathcal{O}(P) \tag{4.73}$$

This allows us to calculate correlation functions in the embedding space and then translating the results to physical space. In addition, we extend this to vector and tensor operators by imposing a transversality condition:

$$P^A \cdot J_A(P) = 0 \tag{4.74}$$

The physical operator is obtained by projecting to the Poincare section

$$\mathcal{O}_\mu(x) = \frac{\partial P^A}{\partial x^\mu}\mathcal{O}_A(P) \tag{4.75}$$

With this in mind we can go back to calculating correlation functions.

Scalar Two Point Function

Based on the homegeneity condition (4.73) we know that the function must be of order $-\Delta_1$ in $\Phi_1(P_1)$ and of order $-\Delta_2$ in $\Phi_2(P_2)$. Also, the function can only depend on $P_1 \cdot P_2$ due to Lorentz invariance. Since P_1 and P_2 must always apear together this means that $\Delta_1 = \Delta_2 = \Delta$ and we have

$$\langle \Phi_1(P_1)\Phi_2(P_2)\rangle = \frac{c}{(P_{12})^\Delta} \tag{4.76}$$

Where we define $P_{ij} \equiv -2P_i \cdot P_j$. We choose the light cone section

$$P = \left(1, x^2, x^\mu\right) \tag{4.77}$$

This gives

$$P_{ij} \equiv -2P_i \cdot P_j = \left(x^i\right)^2 + \left(x^j\right)^2 - \delta_{ij}x^i x^j = (x_i - x_j)^2 \tag{4.78}$$

Plugging this back to (4.76) gives our well known result (4.29).

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{c}{|x_1 - x_2|^{2\Delta}} \tag{4.79}$$

Vector Two Point Function

We now consider the two-point function of a primary vector field:

$$G_{AB}(P_1, P_2) = \langle J_A(P_1)J_B(P_2)\rangle \tag{4.80}$$

G must be a second rank tensor under the Lorentz Group $SO(d+1, 1)$. The most general such tensor will be a linear combination of:

$$G_{AB}(P_1, P_2) = c_1\left(\eta_{AB} - \frac{P_{1B}P_{2A}}{P_1 \cdot P_2}\right) + c_2\frac{P_{1A}P_{2B}}{P_1 \cdot P_2} \tag{4.81}$$

This is because these satisfy the transversality conditions for P_1,P_2:

$$P_1^A G_{AB}(P_1, P_2) = 0\ , \quad P_2^B G_{AB}(P_1, P_2) = 0 \tag{4.82}$$

$$P_1^A \eta_{AB} - P_1^A\ \frac{P_{1B}P_{2A}}{P_1 \cdot P_2} = P_{1B} - P_{1B} = 0 \tag{4.83}$$

$$P_1^A\ \frac{P_{1A}P_{2B}}{P_1 \cdot P_2} = 0\ , \quad \left(P^2 = 0\right) \tag{4.84}$$

And similarly for P_2^B. The c_1, c_2 must be scalars of the form $P_1 \cdot P_2$. However, as before the homogeneity condition mandates $\Delta_1 = \Delta_2 = \Delta$ and we have:

$$G_{AB}(P_1, P_2) = \frac{1}{(P_{12})^{\Delta}} \left[c_1 \left(\eta_{AB} - \frac{P_{1B}P_{2A}}{P_1 \cdot P_2}\right) + c_2\ \frac{P_{1A}P_{2B}}{P_1 \cdot P_2}\right] \tag{4.85}$$

Now we project this result to physical space using (4.75)

$$g_{ij}(x) = \frac{\partial P_1^A}{\partial x^i}\ \frac{\partial P_2^B}{\partial x^j} G_{AB}(P_{1x}, P_{2x}) \tag{4.86}$$

From (4.77) we see that the projection operators take the from

$$\frac{\partial P^A}{\partial x^i} = (0, 2x_i, \delta_i^a) \tag{4.87}$$

The first term projects to

$$\frac{\partial P_1^A}{\partial x^i}\ \frac{\partial P_2^B}{\partial x^j}\eta_{AB} = \frac{\partial P_1^A}{\partial x^i}\ \frac{\partial P_{2A}}{\partial x^j} = (0, 2x_{1i}, \delta_i^a)(0, 2x_{2i}, \delta_i^a) = \delta_{ai}\delta_j^a = \delta_{ij} \tag{4.88}$$

$$\begin{aligned} \frac{\partial P_2^B}{\partial x^j} P_{1B} &= \left(0, 2x_{2j}, \delta_j^b\right)\left(1, x_1^2, x_{1b}\right) = -\ \frac{0 \cdot x_1^2}{2} - \frac{2x_{2j}}{2} + \delta_j^b x_{1b} = -x_{2j} + x_{1j} \\ \frac{\partial P_1^A}{\partial x^i} P_{2A} &= (0, 2x_{1i}, \delta_i^a)\left(1, x_2^2, x_{2a}\right) = -\ \frac{0 \cdot x_2^2}{2} - \frac{2x_{1i}}{2} + \delta_i^a x_{2a} = -x_{1i} + x_{2i} \end{aligned} \tag{4.89}$$

Putting this all together we have

$$\frac{\partial P_1^A}{\partial x^i}\ \frac{\partial P_2^B}{\partial x^j}\left(\eta_{AB} - \frac{P_{1B}P_{2A}}{P_1 \cdot P_2}\right) = \delta_{ab} - 2\ \frac{(x_1 - x_2)_i (x_1 - x_2)_j}{(x_1 - x_2)^2} \tag{4.90}$$

Where we used $P_1 \cdot P_2 = -\ \frac{1}{2}(x_i - x_j)^2$. For the second term we have

$$\frac{\partial P_1^A}{\partial x^i} P_{1A} = (0, 2x_{1i}, \delta_i^a)\left(1, x_1^2, x_{1a}\right) = -\ \frac{0 \cdot x_1^2}{2} - \frac{2x_{1i}}{2} + \delta_i^a x_{1a} = -x_{1i} + x_{1i} = 0 \tag{4.91}$$

So the second term projects to zero and is simply a redundancy of the light cone. So we obtain:

$$\langle J_i(x_1)\, J_j(x_2)\rangle = \frac{c_1}{(x_1 - x_2)^{2\Delta}} \left(\delta_{ab} - 2\ \frac{(x_1 - x_2)_i (x_1 - x_2)_j}{(x_1 - x_2)^2}\right) \tag{4.92}$$

Which the result we obtained in our gravity calculation for $d = \Delta + 1$ (4.68) . For more on the topic of Light Cone Coordinates for CFT correlation functions see [7],[8].

5 Witten diagrams in the Mellin representation

The more modern way of thinking about the AdS/CFT correspondence is in terms of AdS scattering amplitudes and correlation functions. In this picture an n-point contact diagram in AdS gives the leading contribution of an n-operator correlation function in the CFT. An exchange diagram would give a small correction to this amplitude, a one-loop diagram an even smaller correction and so on. The Mellin representation makes this analogy apparent and allows for some physical interpretation of the perturbative corrections. This section follows the paper by J. Penedones titled "Writing CFT correlation functions as AdS scattering amplitudes" [2]

Mellin Amplitude

A correlation function of primary scalar fields can be written as: (G. Mack [9])

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle = \frac{\mathcal{N}}{(2\pi i)^{n(n-3)/2}}\int d\delta_{ij} M(\delta_{ij})\prod_{i<j}^{n}\Gamma(\delta_{ij})\left(x_{ij}^2\right)^{-\delta_{ij}} \tag{5.1}$$

The integration measure $d\delta_{ij} = d\delta_{12}d\delta_{23}...$ is related to the Mandelstam invariants as we will see shortly. $M(\delta_{ij})$ is the Mellin amplitude and $\Gamma(\delta_{ij})$ are gamma functions with simple poles at non-positive integers. We integrate parallel to the imaginary axis so that poles of the gamma functions stay on one side of the contour. We think of δ_{ij} as the product of two "momentum" vectors

$$\delta_{ij} = k_i \cdot k_j = \frac{\Delta_i + \Delta_j - s_{ij}}{2} \tag{5.2}$$

Where $s_{ij} = -(k_i + k_j)^2$ is analogus to the Mandelstam invariant in an n-particle scattering process. s corresponds to the total center of mass energy squared. The k_i are subject to momentum conservation

$$\sum_{i=1}^{n} k_i = 0 \Rightarrow \sum_{i=1}^{n} k_i \cdot k_j = 0 \Rightarrow \sum_{i=1}^{n}\delta_{ij} = 0 \tag{5.3}$$

We think of the Δ_i as the masses of the particles and require them to satisfy the "mass-shell" relation

$$-k_i^2 = \Delta_i \tag{5.4}$$

Putting these constraints together we notice that we have $n(n-3)/2$ independent integration variables. This is in analogy to the $n(n-3)/2$ independent Mandelstam invariants required to capture the kinematics of an n-particle scattering process. For example, in the $2 \to 2$ scattering process we have two independent variables usually s,t while the third u is fixed by the relation

$$s + t + u = \sum_{i=1}^{4} m_i^2 \tag{5.5}$$

As we have motivated in the previous section the correlation function in (5.1) can be calculated from an AdS scattering process with amplitude $A(x_i)$

$$A(x_i) = \langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle \tag{5.6}$$

In this section we compute Witten diagrams and show that they can be written in the form (5.1). We calculate the Mellin amplitude $M(\delta_{ij})$ which comes out to be a simple algebraic function.

Witten Diagrams

The computation of Witten diagrams is simplified by the use of Embedding space formalism. We consider Euclidean AdS_{d+1} embedded in $d+2$ dimensional Minkowski space. So we have the bulk geometry

$$-X_0^2+\sum_{i=1}^{d+1}X_i^2=-R^2 \tag{5.7}$$

And at the boundary we have

$$-(P_0)^2+\sum_{i=1}^{d+1}(P_i)^2=0 \tag{5.8}$$

In QFT the basic object required to calculate Feynman diagrams is the Feynman propagator. Similarly, for Witten diagrams the essential object is the propagator. In this case we are interested in the boundary limit of AdS so we require the Bulk to Boundary propagator. It is given by:

$$G_{B\partial}(X,P)=\frac{C_\Delta}{R^{(d-1)/2}(-2P\cdot X/R)^\Delta} \tag{5.9}$$

Where

$$C_\Delta=\frac{\Gamma(\Delta)}{2\pi^h\Gamma(\Delta-h+1)}\,,\quad h=\frac{d}{2} \tag{5.10}$$

This is obtained by taking the limit of the bulk to bulk propagator. If we take the limit of the bulk point X to a boundary point P_2 we get the boundary to boundary propagator:

$$G_{\partial\partial}(P_1,P_2)=\frac{C_\Delta}{R^{(d-1)/2}(-2P_1\cdot P_2/R)^\Delta} \tag{5.11}$$

Which gives the known expression of the two-point function

$$G_{\partial\partial}(P_1,P_2)=\langle\mathcal{O}_\Delta(P_1)\,\mathcal{O}_\Delta(P_2)\rangle=\frac{C_\Delta}{(-2P_1\cdot P_2)^\Delta} \tag{5.12}$$

For our purposes we will need the integral representation of (5.9) which we can write using a gamma function identity

$$\int_0^\infty dt\,t^b e^{-at}=\frac{\Gamma(b+1)}{a^{b+1}} \tag{5.13}$$

One can check that this holds by a simple change of variables $t'=at$. Using this we rewrite the propagator as

$$G_{B\partial}(X,P)=\frac{C_\Delta}{R^{(d-1)/2}\Gamma(\Delta)}\int_0^\infty\frac{dt}{t}\,t^\Delta e^{2tP\cdot X/R} \tag{5.14}$$

We shall now apply this formalism to the calculation of a tree level n-point contact interaction and a tree level scalar exchange.

Tree level n-point contact interaction

Our first diagram is the simple diagram in Figure 1. It gives the leading contribution (of order g) to the AdS scattering amplitude of n-scalar fields.

$$A(P_i)=g\int_{AdS}dX\prod_{i=1}^{n}G_{B\partial}(X,P_i) \tag{5.15}$$

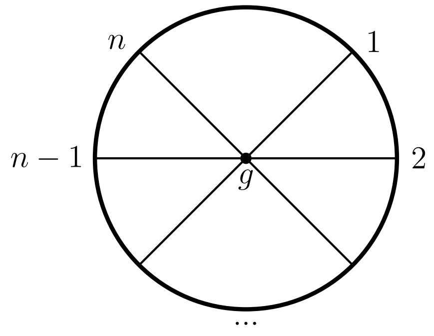

Figure 1: Witten diagram for a n-point contact interaction in AdS

As a first check note that we integrate over all of AdS and so expect our amplitude to depend only on the boundary points P_i. Using the integral representation of the propagator (5.14) we obtain:

$$A\left(P_i\right) = gR^{d+1+n(1-d)/2} \int_{AdS} d\left(X/R\right) \prod_{i=1}^{n} \frac{C_{\Delta_i}}{\Gamma\left(\Delta_i\right)} \int_0^\infty \frac{dt_i}{t_i} t_i^{\Delta_i} e^{2tP_i \cdot X/R} \tag{5.16}$$

The R^{d+1} term comes form the AdS volume element in Poincare coordinates which we will calculate shortly. This multi integral formula is usually defined as the D-function as it appears in many diagrams of this type:

$$D_{\Delta_1 \ldots \Delta_n}\left(P_i\right) = \prod_{i=1}^{n} \left(\frac{1}{\Gamma\left(\Delta_i\right)}\right) \int_0^\infty \frac{dt_1}{t_1} t_1^{\Delta_1} \ldots \int_0^\infty \frac{dt_n}{t_n} t_n^{\Delta_n} \int_{AdS} d\left(X/R\right) e^{2Q\cdot X/R} \tag{5.17}$$

Where we defined $Q = \sum_{i=1}^{n} t_i P_i$. We can calculate this amplitude and bring it to the Mellin representation form. We start from the AdS integral:

$$d\left(X\right) = \sqrt{g} dz d^d x = \left(z/R\right)^{-d-1} dz d^d x \tag{5.18}$$

Using lightcone coordinates we can take $Q = |Q|\left(1,1,0\right)$ and $X = \left(X^+, X^-, X^\mu\right)$ $= \frac{R}{z}\left(1, z^2+x^2, x^\mu\right)$. So the dot product will be

$$Q \cdot X = \frac{R}{2z}\left(-1\left(z^2+x^2\right) - 1\cdot 1\right) = -\frac{R}{2z}\left(1 + x^2 + z^2\right) \tag{5.19}$$

This gives

$$\begin{aligned} \int_{AdS} d\left(X/R\right) e^{2Q\cdot X/R} &= \int_0^\infty \frac{dz}{z} z^{-d} \int_{\mathbb{R}^d} d^d x e^{-\left(1+z^2+x^2\right)|Q|/z} \\ &= \int_0^\infty \frac{dz}{z} z^{-d} e^{\left(-1+z^2\right)|Q|/z} \int_{\mathbb{R}^d} d^d x e^{-x^2|Q|/z} \\ &= \pi^h \int_0^\infty \frac{dz}{z} \left(z\,|Q|\right)^{-h} e^{-\left(1+z^2\right)|Q|/z} \end{aligned} \tag{5.20}$$

Where in the second line we perfomed d gausssian integrals giving a factor of $\left[\frac{\pi z}{|Q|}\right]^{d/2}$. As before we set $h = d/2$. To proceed with the integral of z we let $z\,|Q| \to z$:

$$\int_{AdS} d\left(X/R\right) e^{2Q\cdot X/R} = \pi^h \int_0^\infty \frac{dz}{z} z^{-h} e^{-z+Q^2/z} \tag{5.21}$$

The + sign in the Q^2 is due to the fact that $Q^2 = -\left|Q\right|^2$. Substituting this into the original expression (5.16) and rescaling by $t_i \to t_i\sqrt{z}$ leads to

$$\begin{aligned} A\left(P_i\right) = gR^{d+1+n(1-d)/2}\pi^h \prod_{i=1}^{n} \frac{C_{\Delta_i}}{\Gamma\left(\Delta_i\right)} \int_0^\infty \frac{dt_1}{t_1}\left(t_1\sqrt{z}\right)^{\Delta_1} \ldots \int_0^\infty \frac{dt_n}{t_n}\left(t_n\sqrt{z}\right)^{\Delta_n} \\ \int_0^\infty \frac{dz}{z} z^{-h} e^{-z+Q^2} \end{aligned} \tag{5.22}$$

This gives for z integral

$$e^{Q^2}\int_0^\infty \frac{dz}{z} z^{\left(\sum_{i=1}^n \Delta_i - d\right)/2} e^{-z} = e^{Q^2}\Gamma\left(\frac{\sum_{i=1}^n \Delta_i - d}{2}\right) \tag{5.23}$$

For Q^2 we have from its definition

$$Q^2 = \left(\sum_{i=1}^n t_i P_i\right)\left(\sum_{j=1}^n t_j P_j\right) = \left(t_1P_1 + t_2P_2 + ...t_nP_n\right)\left(t_1P_1 + t_2P_2 + ...t_nP_n\right) \tag{5.24}$$

The P^2 are zero by definition and the cross-terms are of the form:

$$2t_it_jP_iP_j = -t_it_j\left(x_i - x_j\right)^2 = -t_it_jP_{ij} \tag{5.25}$$

So we obtain

$$A\left(P_i\right) = gR^{d+1+n(1-d)/2} 2\mathcal{N} \prod_{i=1}^n \int_0^\infty \frac{dt_i}{t_i} t_i^{\Delta_i} e^{-\sum_{i<j} t_it_jP_{ij}} \tag{5.26}$$

where we defined the normalization constant $\mathcal{N}$

$$2\mathcal{N} = \pi^h \Gamma\left(\frac{\sum_{i=1}^n \Delta_i - d}{2}\right)\prod_{i=1}^n \frac{C_{\Delta_i}}{\Gamma\left(\Delta_i\right)} \tag{5.27}$$

Finally, we make use of the identity (Symanzik [10])

$$2\prod_{i=1}^n \int_0^\infty \frac{dt_i}{t_i} t_i^{\Delta_i} e^{-\sum_{i<j} t_it_jQ_{ij}} = \frac{1}{\left(2\pi i\right)^{n(n-3)/2}} \int_{\Sigma_n} d\delta_{ij} \prod_{i<j}^n \Gamma\left(\delta_{ij}\right)\left(Q_{ij}\right)^{-\delta_{ij}} \tag{5.28}$$

which transforms the integral into the Mellin representation

$$A\left(x_i\right) = \frac{\mathcal{N}}{\left(2\pi i\right)^{n(n-3)/2}} \int_{\Sigma_n} d\delta_{ij} M\left(\delta_{ij}\right) \prod_{i<j}^n \Gamma\left(\delta_{ij}\right)\left(Q_{ij}\right)^{-\delta_{ij}} \tag{5.29}$$

With normalization constant (5.27) and Mellin amplitude

$$M\left(\delta_{ij}\right) = gR^{d+1+n(1-d)/2} \tag{5.30}$$

So we conclude that contact interactions in AdS have constant Mellin amplitudes.

Tree level scalar exchange

Our next diagram is the tree-level scalar exchange shown in Figure 2. It gives a contribution of order g^2 to the AdS scattering amplitude of 4-scalar fields.

$$A\left(P_i\right) = g^2\int_{AdS} dXdY G_{B\partial}\left(X, P_1\right) G_{B\partial}\left(X, P_3\right) G_{BB}\left(X, Y\right) G_{B\partial}\left(Y, P_2\right) G_{B\partial}\left(Y, P_4\right) \tag{5.31}$$

The bulk to bulk scalar propagator is given by

$$\begin{aligned} G_{BB}\left(X,Y\right) &= \frac{c_\Delta}{u^\Delta} {}_2F_1\left(\Delta, \frac{2\Delta - d + 1}{2}, 2\Delta - d + 1, -\frac{4}{u}\right) \\ &= \frac{1}{\left(4\pi\right)^{h+1/2}} \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{\Gamma\left(z\right)\Gamma\left(\Delta - z\right)\Gamma\left(1/2 - h + z\right)}{\Gamma\left(z + \Delta - 2h + 1\right)} \left(\frac{u}{4}\right)^{-z} \end{aligned} \tag{5.32}$$

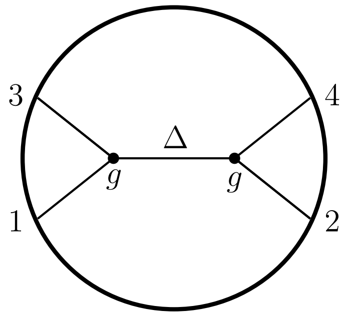

Figure 2: Witten diagram for a 4-point scalar exchange in AdS

where the second expression is given by the Barnes integral. ${}_2F_1$ is the hypergeometric function and $u = (X - Y)^2$. The harmonic space expansion of the bulk to bulk propagator is computed in appendix B. It reads

$$G_{BB}(X,Y) = \int_{-i\infty}^{+i\infty} \frac{dc}{2\pi i} \frac{1}{(\Delta - h)^2 - c^2} \Omega_c(X,Y) \tag{5.33}$$

Where

$$\Omega_c(X,Y) = N(c) \int_{\partial AdS} dP \frac{1}{(-2P \cdot X)^{h+c} (-2P \cdot Y)^{h-c}} \tag{5.34}$$

∂AdS denotes integral over the AdS boundary and N is given by

$$N(c) = \frac{\Gamma(h+c)\Gamma(h-c)}{2\pi^{2h}\Gamma(c)\Gamma(-c)} \tag{5.35}$$

We reintroduce factors of R and we use the integral representation

$$\frac{1}{(-2P \cdot X/R)^{\Delta}} = \frac{1}{\Gamma(\Delta)} \int_0^\infty \frac{dt}{t} t^{\Delta} e^{2tP\cdot X/R} \tag{5.36}$$

So $G_{BB}(X,Y)$ takes the form

$$G_{BB}(X,Y) = \frac{1}{R^{d-1}} \int_{-i\infty}^{+i\infty} dc \frac{f(c)}{2\pi i} \int_{\partial AdS} dP \int_0^\infty \frac{dt}{t} t^{h+c} e^{2tP\cdot X} \int_0^\infty \frac{d\bar{t}}{\bar{t}} \bar{t}^{h-c} e^{2\bar{t}P\cdot Y} \tag{5.37}$$

Where we write the constants compactly as

$$f(c) = \frac{1}{2\pi^{2h}\Gamma(c)\Gamma(-c)} \frac{1}{(\Delta - h)^2 - c^2} \tag{5.38}$$

Plugging this back to (5.31) we find

$$\begin{aligned} A(P_i) =& g^2 R^{2(1-d)} \prod_{i=1}^{4} \left(\frac{C_{\Delta_i}}{\Gamma(\Delta_i)} \int_0^\infty \frac{dt_i}{t_i} t_i^{\Delta} \right) \int_{-i\infty}^{+i\infty} \frac{dc}{2\pi i} f(c) \int_0^\infty \frac{dt d\bar{t}}{t\bar{t}} t^{h+c} \bar{t}^{h-c} \\ & \int_{\partial AdS} dP \int_{AdS} d\left(\frac{X}{R}\right) e^{2(t_1P_1 + t_3P_3 + tP)X/R} \int_{AdS} d\left(\frac{Y}{R}\right) e^{2(t_2P_2 + t_4P_4 + \bar{t}P)Y/R} \end{aligned} \tag{5.39}$$

The AdS volume element is

$$d\left(\frac{X}{R}\right) = \sqrt{g} dz d^d x = z^{-d-1} R^{d+1} dz d^d x \tag{5.40}$$

We can let

$$t_1P_1 + t_3P_3 + tP = Q\,, \quad t_2P_2 + t_4P_4 + \bar{t}P = \bar{Q} \tag{5.41}$$

With this in mind we perform the integrals over AdS using (5.21) and find

$$\begin{aligned} A\left(P_i\right) =& g^2 R^{(5-d)} \prod_{i=1}^{4}\left(\frac{C_{\Delta_i}}{\Gamma\left(\Delta_i\right)}\int_0^\infty \frac{dt_i}{t_i}t_i^{\Delta}\right)\int_{-i\infty}^{+i\infty}\frac{dc}{2\pi i}f\left(c\right)\int_0^\infty \frac{dt d\bar{t}}{t\bar{t}}t^{h+c}\bar{t}^{h-c} \\ & \pi^{2h}\int_0^\infty \frac{dz d\bar{z}}{z\bar{z}}\left(z\bar{z}\right)^{-h}e^{-z-\bar{z}}\int_{\partial AdS} dP e^{Q^2/z+\bar{Q}^2/\bar{z}} \end{aligned} \tag{5.42}$$

We now rescale

$$\begin{aligned} t_1 \to t_1\sqrt{z}, \quad t_3 \to t_3\sqrt{z}, \quad t \to t\sqrt{z} \\ t_2 \to t_1\sqrt{\bar{z}}, \quad t_4 \to t_3\sqrt{\bar{z}}, \quad \bar{t} \to \bar{t}\sqrt{\bar{z}} \end{aligned} \tag{5.43}$$

The t and z integrals transform to

$$\int_0^\infty \frac{dt d\bar{t}}{t\bar{t}}t^{h+c}\bar{t}^{h-c}\int_0^\infty z^{(h+c/2)}\bar{z}^{(-h-c)/2}\frac{dz d\bar{z}}{z\bar{z}}\left(z\bar{z}\right)^{-h}e^{-z-\bar{z}} \tag{5.44}$$

And the t_i integrals contribute

$$z^{\Delta_1/2}\, z^{\Delta_3/2}\, \bar{z}^{\Delta_2/2}\, \bar{z}^{\Delta_4/2}\prod_{i=1}^{n}\int_0^\infty \frac{dt_i}{t_i}t_i^{\Delta_i} \tag{5.45}$$

So we find

$$\int_0^\infty \frac{dz}{z}z^{(-h+c+\Delta_1\Delta_3)/2}e^{-z}\int_0^\infty \frac{d\bar{z}}{\bar{z}}\bar{z}^{(-h+c+\Delta_1\Delta_3)/2}e^{-\bar{z}} \tag{5.46}$$

These contribute two gamma functions

$$\Gamma\left(\frac{-h+c+\Delta_1+\Delta_3}{2}\right)\Gamma\left(\frac{-h-c+\Delta_2+\Delta_4}{2}\right) \tag{5.47}$$

As for the integral over the boundary, the exponents read

$$\begin{aligned} \left(t_1P_1+t_3P_3+tP\right)^2 &= -t_1t_3P_{13}+2Pt\left(t_1P_1+t_3P_3\right) \\ \left(t_2P_2+t_4P_4+\bar{t}P\right)^2 &= -t_2t_4P_{24}+2P\bar{t}\left(t_2P_2+t_4P_4\right) \end{aligned} \tag{5.48}$$

Putting all this together we find

$$\begin{aligned} A\left(P_i\right) =& g^2R^{(5-d)}\pi^{2h}\prod_{i=1}^{4}\left(\frac{C_{\Delta_i}}{\Gamma\left(\Delta_i\right)}\int_0^\infty \frac{dt_i}{t_i}t_i^{\Delta_i}\right)e^{-t_1t_3P_{13}-t_2t_4P_{24}} \\ & \int_{-i\infty}^{+i\infty}\frac{dc}{2\pi i}f\left(c\right)\Gamma\left(\frac{-h+c+\Delta_1+\Delta_3}{2}\right)\Gamma\left(\frac{-h-c+\Delta_2+\Delta_4}{2}\right) \\ & \int_0^\infty \frac{dt d\bar{t}}{t\bar{t}}t^{h+c}\bar{t}^{h-c}\int_{\partial AdS} dP e^{2P\left(t\left(t_1P_1+t_3P_3\right)+\bar{t}\left(t_2P_2+t_4P_4\right)\right)} \end{aligned} \tag{5.49}$$

The integral in the last line is of the same form as the one calculated in appendix B. It is given by

$$2\pi^h\int_0^\infty \frac{dt d\bar{t}}{t\bar{t}}t^{h+c}\bar{t}^{h-c}e^{\left(t\left(t_1P_1+t_3P_3\right)+\bar{t}\left(t_2P_2+t_4P_4\right)\right)^2} \tag{5.50}$$

So we find

$$\begin{aligned} A\left(P_i\right) =& g^2R^{(5-d)}\pi^{3h}\prod_{i=1}^{4}\left(\frac{C_{\Delta_i}}{\Gamma\left(\Delta_i\right)}\right)\int_{-i\infty}^{+i\infty}\frac{dc}{2\pi i}f\left(c\right) \\ & \int_0^\infty \frac{dt d\bar{t}}{t\bar{t}}t^{h+c}\bar{t}^{h-c}\Gamma\left(\frac{-h+c+\Delta_1+\Delta_3}{2}\right)\Gamma\left(\frac{-h-c+\Delta_2+\Delta_4}{2}\right) \\ & \int_0^\infty \prod_{i=1}^{4}\frac{dt_i}{t_i}t_i^{\Delta_i}e^{-\left(1+t^2\right)t_1t_3P_{13}-\left(1+\bar{t}^2\right)t_2t_4P_{24}-t\bar{t}\left(t_1t_2P_{12}+t_1t_4P_{14}+t_2t_3P_{23}+t_3t_4P_{34}\right)} \end{aligned} \tag{5.51}$$

Where we used $-2P_i \cdot P_j = P_{ij}$ and $P_i^2 = 0$. We can now make use of the identity (Symanzik [10])

$$2\prod_{i=1}^{n}\int_0^\infty \frac{dt_i}{t_i} t_i^{\Delta_i} e^{-\sum_{i<j} t_i t_j Q_{ij}} = \frac{1}{(2\pi i)^{n(n-3)/2}} \int_{\Sigma_n} d\delta_{ij} \prod_{i<j}^{n} \Gamma\left(\delta_{ij}\right)\left(Q_{ij}\right)^{-\delta_{ij}} \tag{5.52}$$

where we notice that

$$Q_{13} = \left(1+t^2\right) P_{13}\,, \quad Q_{24} = \left(1+\bar{t}^2\right) P_{24}\,, \quad Q_{ij} = t\bar{t} P_{ij} \tag{5.53}$$

We also insert the normalization constant (5.27) to bring the integral to the Mellin representation

$$2\mathcal{N} = \pi^h \Gamma\left(\frac{\sum_{i=1}^{n}\Delta_i - d}{2}\right)\prod_{i=1}^{n}\frac{C_{\Delta_i}}{\Gamma\left(\Delta_i\right)} \tag{5.54}$$

Putting all this together we find

$$\begin{aligned} A\left(P_i\right) =& g^2 R^{(5-d)} \pi^{2h} \frac{2\mathcal{N}}{(2\pi i)^2} \frac{1}{\Gamma\left(\frac{\sum_{i=1}^{4}\Delta_i - d}{2}\right)} \int_{\Sigma_n} d\delta_{ij} \\ & \int_{-i\infty}^{+i\infty} \frac{dc}{2\pi i} f(c)\, \Gamma\left(\frac{-h+c+\Delta_1+\Delta_3}{2}\right)\Gamma\left(\frac{-h-c+\Delta_2+\Delta_4}{2}\right) \\ & \int_0^\infty \frac{dt d\bar{t}}{t\bar{t}} t^{h+c} \bar{t}^{h-c} \left(1+t^2\right)^{-\delta_{13}} \left(1+\bar{t}^2\right)^{-\delta_{24}} \left(t\bar{t}\right)^{-\delta_{12}-\delta_{14}-\delta_{34}-\delta_{23}} \prod_{i<j}^{4} \Gamma\left(\delta_{ij}\right)\left(P_{ij}\right)^{-\delta_{ij}} \end{aligned} \tag{5.55}$$

So we find that the Mellin amplitude of this diagram is

$$\begin{aligned} M\left(\delta_{ij}\right) =& \frac{g^2 R^{(5-d)} 2\pi^{2h}}{\Gamma\left(\frac{\sum_{i=1}^{4}\Delta_i - d}{2}\right)} \int_{-i\infty}^{+i\infty} \frac{dc}{2\pi i} f(c)\, \Gamma\left(\frac{-h+c+\Delta_1+\Delta_3}{2}\right)\Gamma\left(\frac{-h-c+\Delta_2+\Delta_4}{2}\right) \\ & \int_0^\infty \frac{dt d\bar{t}}{t\bar{t}} t^{h+c} \bar{t}^{h-c} \left(1+t^2\right)^{-\delta_{13}} \left(1+\bar{t}^2\right)^{-\delta_{24}} \left(t\bar{t}\right)^{-\delta_{12}-\delta_{14}-\delta_{34}-\delta_{23}} \end{aligned} \tag{5.56}$$

The integral in the last line gives

$$\frac{\Gamma\left(\frac{h+c-s_{13}}{2}\right)\Gamma\left(\frac{\Delta_2+\Delta_4+c-h}{2}\right)\Gamma\left(\frac{h-c-s_{13}}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_3-c-h}{2}\right)}{\Gamma\left(\frac{\Delta_1+\Delta_3-s_{13}}{2}\right)\Gamma\left(\frac{\Delta_2+\Delta_4-s_{13}}{2}\right)} \tag{5.57}$$

Where we used the definition of the Mandelstam invariant $s_{ij} = -2\delta_{ij} + \Delta_i + \Delta_j$. This gives the final form the Mellin amplitude for this process

$$M\left(s_{ij}\right) = \frac{g^2 R^{5-d}}{\Gamma\left(\frac{\sum_i \Delta_i}{2} - h\right)\Gamma\left(\frac{\Delta_2+\Delta_4-s_{13}}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_3-s_{13}}{2}\right)} \int_{-i\infty}^{i\infty} \frac{dc}{2\pi i} \frac{l(c) l(-c)}{(\Delta-h)^2 - c^2} \tag{5.58}$$

With

$$l(c) = \frac{\Gamma\left(\frac{h+c-s_{13}}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_3+c-h}{2}\right)\Gamma\left(\frac{\Delta_2+\Delta_4+c-h}{2}\right)}{2\Gamma(c)} \tag{5.59}$$

We conclude that the Mellin amplitude of the scalar exchange only depends on s_{13}. It has simple poles at

$$s_{13} = \Delta + 2m\,, \quad m = 0, 1, ... \tag{5.60}$$

This has the interpretation of exchanged "mometum" going on shell, in analogy with scattering in QFT.

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Appendix

A Maxwell's equations in AdS

Maxwell's equations in curved space in covariant form are

$$\frac{1}{\sqrt{g}}\partial_\mu \left(\sqrt{g}g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}\right) = j^\nu \tag{5.61}$$

In our case we are interested in the free theory so we get an equation for each $\nu = 0, 1, ..., d$

$$\partial_\mu \left(\sqrt{g}g^{\mu\rho}g^{\nu\sigma}\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right)\right) = 0 \tag{5.62}$$

We expect that the vector potential is only a function of $A^\mu = A^\mu\left(x_0\right)$ so the only non-vanishing terms are:

$$\frac{d}{dx_0}\left(\sqrt{g}g^{00}g^{\nu\sigma}\left(\partial_0 A_\sigma - \partial_\sigma A_0\right)\right) = 0 \tag{5.63}$$

If $\nu = 0$ we have a trivial solution. For $\nu \neq 0$ we find

$$\frac{d}{dx_0}\left(\sqrt{g}g^{00}g^{\nu\nu}\left(\partial_0 A_\nu\right)\right) = 0 \tag{5.64}$$

Using our Poincare metric representation (4.9) we have the determinant $\sqrt{g} = x_0^{-d-1}$ and the inverse of the metric $g^{ii} = x_0^2$. So we find

$$x_0^{-d+3}A'_\nu\left(x_0\right) = c \tag{5.65}$$

This gives

$$A_\nu\left(x_0\right) = \frac{cx_0^{d-2}}{d-2} + c' \tag{5.66}$$

For $x_0 \to 0$ we want $A \to 0$ so $c' = 0$. Using the relation $A = A_\mu dx^\mu$ we get the previous result of our one-form (4.57)

$$A = cx_0^{d-2}dx^i \tag{5.67}$$

As a final note, we can calculate the form of the E/M tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.68}$$

We have

$$F_{0\nu} = \partial_0 A_\nu = cx_0^{d-3} \tag{5.69}$$

$F_{\mu\nu}$ is antisymmetric and we have $F_{\mu\nu} = -F_{\nu\mu}$. So in matrix form $F_{\mu\nu}$ is

$$F_{\mu\nu} = x_0^{d-3}\begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -1 & \cdots & 0 & 0 \end{pmatrix} \tag{5.70}$$

This corresponds to zero magnetic field $B^\mu = 0$ and equal electric field in all directions, $E^\mu\left(x_0\right) = x_0^{d-3}$.

B Bulk to Bulk Propagator

The purpose of this appendix is to prove that the bulk to bulk propagator is indeed given as an expansion in the basis of harmonic functions. See J. Penedones [2] [11].

$$G_{BB}\left(X,Y\right)=\int_{-i\infty}^{+i\infty}\frac{dc}{2\pi i}\,\frac{1}{\left(\Delta-h\right)^{2}-c^{2}}\Omega_{c}\left(X,Y\right) \tag{5.71}$$

Where

$$\Omega_{c}\left(X,Y\right)=N\left(c\right)\int_{\partial AdS}dP\,\frac{1}{\left(-2P\cdot X\right)^{h+c}\left(-2P\cdot Y\right)^{h-c}} \tag{5.72}$$

Where ∂AdS denotes integral over the AdS boundary and $N\left(c\right)$ is given by

$$N\left(c\right)=\frac{\Gamma\left(h+c\right)\Gamma\left(h-c\right)}{2\pi^{2h}\Gamma\left(c\right)\Gamma\left(-c\right)} \tag{5.73}$$

We start by using the integral representation

$$\frac{1}{\left(-2P\cdot X\right)^{\Delta}}=\frac{1}{\Gamma\left(\Delta\right)}\int_{0}^{\infty}\frac{dt}{t}t^{\Delta}e^{2tP\cdot X} \tag{5.74}$$

So the function Ω_c reads

$$\Omega_{c}\left(X,Y\right)=\frac{1}{2\pi^{2h}\Gamma\left(c\right)\Gamma\left(-c\right)}\int_{\partial AdS}dP\int_{0}^{\infty}\frac{dtd\bar{t}}{t\bar{t}}t^{h+c}\bar{t}^{h-c}e^{2tP\cdot X+2\bar{t}P\cdot Y} \tag{5.75}$$

Where we used the explicit form of the gamma functions. To proceed we use Poincare section $P=(1,x^2,x^\mu)$ and pick a basis where $T=tX+\bar{t}Y=|T|\,(1,1,0)$. In Poincare coordinates the boundary volume element is simply $dP=d^dx$. So we find

$$\int_{\partial AdS}dPe^{2T\cdot P}=\int_{\mathbb{R}^{d}}d^{d}xe^{-|T|\left(1+x^{2}\right)}=e^{-|T|}\left(\frac{\pi}{|T|}\right)^{d/2} \tag{5.76}$$

Where we used $T\cdot P=\frac{-1\cdot 1}{2}+\frac{-1\cdot x^2}{2}$ and performed d gaussian integrals. So the integral over t and $\bar{t}$ reads

$$\int_{0}^{\infty}\frac{dtd\bar{t}}{t\bar{t}}t^{h+c}\bar{t}^{h-c}\,\frac{\pi^{h}}{\left|tX+\bar{t}Y\right|^{h}}e^{-|tX+\bar{t}Y|} \tag{5.77}$$

We now insert the delta function

$$\int_{0}^{\infty}ds\delta\left(s-t-\bar{t}\right)=1 \tag{5.78}$$

We make a change of variables $t\to st$, $\bar{t}\to s\bar{t}$ and find

$$\begin{aligned}&\pi^{h}\int_{0}^{\infty}ds\int_{0}^{\infty}\frac{dtd\bar{t}}{t\bar{t}}t^{h+c}\bar{t}^{h-c}\,\frac{s^{h+c}s^{h-c}}{\left|s\left(tX+\bar{t}Y\right)\right|^{h}}e^{-s|tX+\bar{t}Y|}\delta\left(s\left(1-t-\bar{t}\right)\right)\\&=\pi^{h}\int_{0}^{\infty}\frac{ds}{s}s^{h}\int_{0}^{\infty}\frac{dtd\bar{t}}{t\bar{t}}t^{h+c}\bar{t}^{h-c}e^{s(tX+\bar{t}Y)^{2}}\delta\left(1-t-\bar{t}\right)\end{aligned} \tag{5.79}$$

Where we used the delta function identity $\delta\left(\lambda x\right)=\frac{1}{|\lambda|}\delta\left(x\right)$. We now change variables once more to perform the s integral $t\to t/\sqrt{s}$, $\bar{t}\to\bar{t}/\sqrt{s}$. This gives

$$2\pi^{h}\int_{0}^{\infty}\frac{dtd\bar{t}}{t\bar{t}}t^{h+c}\bar{t}^{h-c}e^{(tX+\bar{t}Y)^{2}} \tag{5.80}$$

Now we notice that we can write the exponent in the exponential as

$$\begin{aligned}
& - ut\bar{t} + \left(t+\bar{t}\right)^2 = \left(tX+\bar{t}Y\right)^2 \ , \qquad u = (X-Y)^2 \\
& = t\bar{t} + 2X\cdot Y t\bar{t} + t\bar{t} - t^2 - 2t\bar{t} - \bar{t}^2 \\
& = 2X\cdot Y t\bar{t} - t^2 - \bar{t}^2
\end{aligned} \tag{5.81}$$

Where we used $X^2 = Y^2 = -1$. So the bulk to bulk propagator takes the form

$$G_{BB}\left(X,Y\right) = 2\pi^h \int_{-\infty}^{i\infty} \frac{dc}{2\pi i} f\left(c\right) \int_0^\infty \frac{dt d\bar{t}}{t\bar{t}} t^{h+c}\bar{t}^{h-c} e^{-(t+\bar{t})^2 - ut\bar{t}} \tag{5.82}$$

We now make use of the Cahen–Mellin integral which gives the inverse of the Mellin transform for the exponential function

$$e^{-ut\bar{t}} = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i}\Gamma\left(z\right)\left(ut\bar{t}\right)^{-z} \tag{5.83}$$

So we find

$$G_{BB}\left(X,Y\right) = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i}\Gamma\left(z\right)u^{-z} \int_{-\infty}^{i\infty} \frac{dc}{2\pi i} f\left(c\right) \int_0^\infty \frac{dt d\bar{t}}{t\bar{t}} t^{h+c-z}\bar{t}^{h-c-z} e^{-(t+\bar{t})^2} \tag{5.84}$$

We can now perform the last integral

$$\int_0^\infty \frac{dt d\bar{t}}{t\bar{t}} t^{h+c-z}\bar{t}^{h-c-z} e^{-(t+\bar{t})^2} = \frac{\Gamma(h-z)\Gamma(h-z+c)\Gamma(h-z-c)}{2\Gamma(2h-2z)} \tag{5.85}$$

So we find that

$$G_{BB}(X,Y) = \frac{1}{2\pi^h}\int \frac{dz}{2\pi i}\frac{\Gamma(z)\Gamma(h-z)}{\Gamma(2h-2z)}u^{-z}\int_{-i\infty}^{i\infty}\frac{dc}{2\pi i}\frac{\Gamma(h-z+c)\Gamma(h-z-c)}{\Gamma(c)\Gamma(-c)\left((\Delta-h)^2-c^2\right)} \tag{5.86}$$

Where reintroduced $f\left(c\right) = \frac{1}{2\pi^{2h}\Gamma(c)\Gamma(-c)}\frac{1}{(\Delta-h)^2-c^2}$. The last integral gives

$$\int_{-i\infty}^{i\infty}\frac{dc}{2\pi i}\frac{\Gamma(h-z+c)\Gamma(h-z-c)}{\Gamma(c)\Gamma(-c)\left((\Delta-h)^2-c^2\right)} = \frac{\Gamma\left(\frac{1}{2}+h-z\right)\Gamma\left(\frac{1}{2}-h+z\right)\Gamma(\Delta-z)}{2\pi\Gamma(z+\Delta-2h+1)} \tag{5.87}$$

Finally, we use the Legendre duplication formula of the Gamma function to write

$$\Gamma\left(h-z\right)\Gamma\left(h-z+\frac{1}{2}\right) = 2^{1-2h+2z}\sqrt{\pi}\Gamma\left(2h-2z\right) \tag{5.88}$$

This brings the propagator to its final form (5.32)

$$G_{BB}\left(X,Y\right) = \frac{1}{\left(4\pi\right)^{h+1/2}}\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{\Gamma\left(z\right)\Gamma\left(\Delta-z\right)\Gamma\left(1/2-h+z\right)}{\Gamma\left(z+\Delta-2h+1\right)}\left(\frac{u}{4}\right)^{-z} \tag{5.89}$$

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