University of Crete
Department of Physics

Bachelor's Diploma Thesis

Hall effect in 2D spin-orbit and topological materials with Van Hove singularities

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Contents

Introduction

As we know, the classical Hall effect arises when we apply an external magnetic field perpendicular to the direction of the current density. One can define the Hall coefficient $R_H = E_y/(j_x B_z)$, where E_y is the induced electric field. For a single band the Hall number depends only on the sign and density of carriers. That is why, at low temperatures, it can be used to determine the number of electrons or holes in electron and hole-like pockets respectively. However, in systems with multiple bands or near Fermi-surface topological transitions, the Hall coefficient starts to deviate from its exclusive counting nature. This anomalous behavior can be attributed to singularities in the density of states, called Van Hove singularities [1, 2]. Most materials show logarithmic-type Van Hove singularities, which correspond to a logarithmic divergence of the density of states at the Lifshitz transition. In our work, we are interested in higher order Van Hove singularities, which arise from more complicated Lifshitz transitions. Such singularities can be observed, for example, in $Sr_3Ru_2O_7$ which exhibits a more complicated singularity when an external magnetic field is applied [3, 4]. Another example is highly overdoped graphene and twisted bilayer graphene [5–8].

In the first section of this thesis (Chapters 1-2), we extend the well-known Chambers' formula [9] to the case of a time-dependent electric field as well as a band with non-zero Berry Curvature. To this end, we study the solution of the Boltzmann equation to first order in the magnetic field B and to quadratic order in the electric field E^2. We note that the equations of motion used in this work are valid to leading order in the electric and magnetic fields. We also neglect Zeeman splitting and spin orbit coupling which would result in more complicated expressions. We then apply our generalized Chambers' formula to study three different systems. The first is a simple rectangular lattice [10], the second is a simple model for highly doped graphene and the last is the Haldane model as an example of a system with non-zero Berry Curvature. In all of the above cases, we study the effect of the high order Van Hove singularities to the Hall coefficient of the system.

In the second section of the work (Chapter 3), we study the Reactive Hall response [11] ($T \to 0$, $\omega \to 0$) for the case of a rectangular lattice, to which we have added a spin-orbit coupling in the form of Rashba interaction. To do this, we consider a magnetic field in the y-direction modulated by a one component wave vector q. Furthermore, we study the "screening" (or slow) response, by first taking the $\omega \to 0$ limit and then the $q \to 0$ limit. Through our analysis, we analytically derive an equation that connects the Hall constant R_H with the Drude weight D of the system. We then apply our formula to study the behavior of the Hall constant of our system for different values of the hopping elements. This gives us a qualitative image of the sign change of the carriers close to a metal-insulator transition. Once again, we observe that closely to the Van Hove singularity points, the behavior of the Hall constant rapidly changes.

Chapter 1

Chambers' formula for time dependent Electric fields

1.1 Boltzmann equation and solution

The semiclassical equations of motion for a single particle Hamiltonian $H(\mathbf{k})$ to leading order in the electric and magnetic field can be written as:

$$\frac{d\mathbf{r}}{dt} = \nabla_{\mathbf{k}}\varepsilon(\mathbf{k}) \tag{1.1}$$

$$\frac{d\mathbf{k}}{dt} = -e\mathbf{E}(\mathbf{r},t) - e\nabla_{\mathbf{k}}\varepsilon(\mathbf{k}) \times \mathbf{B}(\mathbf{r},t) \tag{1.2}$$

In the relaxation time approximation, the Boltzmann equation is given by

$$\frac{\partial}{\partial t} f(\mathbf{k},\mathbf{r},t) + \mathbf{w}\cdot\nabla\left[f(\mathbf{k},\mathbf{r},t)\right] = \frac{f_0(\varepsilon(\mathbf{k})) - f(\mathbf{k},\mathbf{r},t)}{\tau_S(\mathbf{k})} \tag{1.3}$$

where τ_S is the relaxation time, $\mathbf{w} = \left(\frac{d\mathbf{r}}{dt}, \frac{d\mathbf{k}}{dt}\right)$ and $\nabla = (\nabla_{\mathbf{r}}, \nabla_{\mathbf{k}})$. If we consider our electric and magnetic fields to be spatially uniform the above equation reduces to the following:

$$\frac{\partial f}{\partial t} + \frac{d\mathbf{k}}{dt}\cdot\frac{\partial f}{\partial \mathbf{k}} = -\frac{1}{\tau_S(\mathbf{k})}\left[f - f_0(\varepsilon(\mathbf{k}))\right] \tag{1.4}$$

This equation can be analytically solved and its solution is given by

$$f(\mathbf{k},t) = f(\mathbf{k}_0,t_0)\,exp\left(-\int_{t_0}^{t}\frac{ds}{\tau_S(\mathbf{k}(s))}\right) + \int_{t_0}^{t} ds\frac{f_0(\varepsilon(\mathbf{k}(s)))}{\tau_S(\mathbf{k}(s))}exp\left(-\int_{s}^{t}\frac{dt'}{\tau_S(\mathbf{k}(t'))}\right) \tag{1.5}$$

If the intial time is set to $t_0 \to -\infty$ then the solution becomes

$$f(\mathbf{k},t) = \int_{-\infty}^{t} ds\frac{f_0(\varepsilon(\mathbf{k}(s)))}{\tau_S(\mathbf{k}(s))}exp\left(-\int_{s}^{t}\frac{dt'}{\tau_S(\mathbf{k}(t'))}\right) \tag{1.6}$$

The distribution function of the electrons can now be used in order to calculate the current density in our 2D case. We have that

$$J_a(t) = -e\int\frac{d^2k}{(2\pi)^2}\frac{dr_a}{dt}f(\mathbf{k},t) \tag{1.7}$$

1.2 Derivation of the formula

To find the current to first and second order in the electric field we write: $\mathbf{E}(t) = \lambda \mathbf{E}(t)$ with $\lambda = 1$. The solution of our equations can now be written as an analytic asymptotic series of the form

$$\mathbf{k}(t) = \mathbf{k}_0(t) + \lambda \mathbf{k}_1(t) + \lambda^2 \mathbf{k}_2(t) + ... \tag{1.8}$$

with $\mathbf{k}_1(t_0) = \mathbf{k}_2(t_0) = ... = 0$. From the equations of motion we obtain to 0^{th} and to 1^{st} order in λ that

$$\frac{d\mathbf{k}_0(t)}{dt} = -e\nabla_{\mathbf{k}}\varepsilon(\mathbf{k}_0(t)) \times \mathbf{B} \tag{1.9}$$

$$\frac{d\mathbf{k}_1(t)}{dt} = -e\left[\sum_a k_{1a}(t)\frac{\partial}{\partial k_a}\nabla_{\mathbf{k}}\varepsilon(\mathbf{k}_0(t))\right] \times \mathbf{B} - e\mathbf{E}(t) \tag{1.10}$$

In exactly analogous way we can write

$$\varepsilon(t) = \varepsilon_0(t) + \lambda\varepsilon_1(t) + \lambda^2\varepsilon_2(t) + ... \tag{1.11}$$

with $\varepsilon_1(t_0) = \varepsilon_2(t_0) = ... = 0$. We note here that

$$\frac{d\varepsilon(\mathbf{k}(t))}{dt} = \nabla_{\mathbf{k}}\varepsilon(\mathbf{k}(t)) \cdot \frac{d\mathbf{k}(t)}{dt} = -e\nabla_{\mathbf{k}}\varepsilon(\mathbf{k}(t)) \cdot [\mathbf{E}(t) + \nabla_{\mathbf{k}}\varepsilon(\mathbf{k}(t)) \times \mathbf{B}(\mathbf{r},t)] \Rightarrow$$

$$\frac{d\varepsilon(\mathbf{k}(t))}{dt} = -e\nabla_{\mathbf{k}}\varepsilon(\mathbf{k}(t)) \cdot \mathbf{E}(t) \tag{1.12}$$

To order λ and λ^2 we have that

$$\varepsilon_1(t) = -e\int_{t_0}^{t} \nabla_{\mathbf{k}}\varepsilon(\mathbf{k}_0(s)) \cdot \mathbf{E}(s)\, ds \tag{1.13}$$

$$\varepsilon_2(t) = -e\sum_{a,\beta}\int_{t_0}^{t} E_\beta(s)\, k_{1a}(s)\frac{\partial^2}{\partial k_a \partial k_\beta}\varepsilon(\mathbf{k}_0(s))\, ds \tag{1.14}$$

Before we Taylor expand our solution we first note that

$$\int_{-\infty}^{t} dt' \frac{1}{\tau_S(\mathbf{k}(t'))} exp\left(-\int_{t'}^{t}\frac{ds}{\tau_S(\mathbf{k}(s))}\right) = 1 \tag{1.15}$$

so we do need to expand this term to obtain the correct formula to first order in the electric field. Expanding our solution we can write:

$$f(\mathbf{k},t) = \int_{-\infty}^{t} dt' \frac{f_0(\varepsilon(\mathbf{k}(t')))}{\tau_S(\mathbf{k}(t'))} exp\left(-\int_{t'}^{t}\frac{ds}{\tau_S(\mathbf{k}(s))}\right) \simeq$$

$$\simeq f_0(\varepsilon(\mathbf{k}_0(t))) + \frac{\partial f_0(\varepsilon(\mathbf{k}_0(t)))}{\partial\varepsilon}\int_{-\infty}^{t} dt' \frac{\varepsilon_1(t')}{\tau_S(\mathbf{k}_0(t'))} exp\left(-\int_{t'}^{t}\frac{ds}{\tau_S(\mathbf{k}_0(s))}\right) \tag{1.16}$$

The first term does not contribute to the current density, since we cannot have current without an electric field. The current density can now be written as follows:

$$J_a(t) = -e\int \frac{d^2k}{(2\pi)^2}\nabla_{k_a}\varepsilon(\mathbf{k}_0)\left[\frac{\partial f_0(\varepsilon(\mathbf{k}_0(t)))}{\partial\varepsilon}\int_{-\infty}^{t} dt' \frac{\varepsilon_1(t')}{\tau_S(\mathbf{k}_0(t'))} exp\left(-\int_{t'}^{t}\frac{ds}{\tau_S(\mathbf{k}_0(s))}\right)\right] \quad (1.17)$$

Integrating the last term by parts and using the fact that $\frac{d\varepsilon_1(t)}{dt} = -e\nabla_{\mathbf{k}}\varepsilon(\mathbf{k}_0(t))\cdot\mathbf{E}(t)$ we get

$$J_a(t) = -e^2\int \frac{d^2k}{(2\pi)^2}\nabla_{k_a}\varepsilon(\mathbf{k}_0)\left[\frac{\partial f_0(\varepsilon(\mathbf{k}_0(t)))}{\partial\varepsilon}\int_{-\infty}^{t} dt'\nabla_{\mathbf{k}}\varepsilon(\mathbf{k}_0(t'))\cdot\mathbf{E}(t')\,\eta(t;t')\right] \quad (1.18)$$

where we have also defined $\eta(t;t') \equiv exp\left(-\int_{t'}^{t}\frac{ds}{\tau_S(\mathbf{k}(s))}\right)$ for convenience. The conductivity tensor components can now be written in the following form:

$$\sigma_{\alpha\beta} = -e^2\int \frac{d^2k}{(2\pi)^2}\nabla_{k_a}\varepsilon(\mathbf{k}_0)\left[\frac{\partial f_0(\varepsilon(\mathbf{k}_0(t)))}{\partial\varepsilon}\int_{-\infty}^{t} dt'\nabla_{k_\beta}\varepsilon(\mathbf{k}_0(t'))\,exp(-i\omega(t-t'))\,\eta(t;t')\right] \quad (1.19)$$

In the limit $T\to 0$ and for $\tau_S(\mathbf{k}) = \tau_S$ we obtain

$$\sigma_{\alpha\beta} = \frac{e^3B}{(2\pi)^2}\int_0^{\tau} dt u_a(t)\int_{-\infty}^{t} dt' u_\beta(t')\,exp\left(-\left[i\omega + \frac{1}{\tau_S}\right](t-t')\right) \quad (1.20)$$

where τ is the period of a single orbit. In the limit $\omega\to 0$ the last formula reduces to Chambers' formula. The same procedure can be used to obtain the 2^{nd} order correction term to the current density. We know that:

$$f(\mathbf{k},t) = \int_{-\infty}^{t} dt' \frac{f_0([\varepsilon_0+\varepsilon_1+\varepsilon_2](t'))}{\tau_S([\mathbf{k}_0+\mathbf{k}_1+\mathbf{k}_2](t'))} exp\left(-\int_{t'}^{t}\frac{ds}{\tau_S([\mathbf{k}_0+\mathbf{k}_1+\mathbf{k}_2](s))}\right) \quad (1.21)$$

To order E^2 the terms that contribute to the current density are

$$\begin{aligned} f(\mathbf{k},t) \simeq\ & \frac{\partial f_0(\varepsilon(\mathbf{k}_0(t)))}{\partial\varepsilon}\int_{-\infty}^{t} dt'\frac{\varepsilon_1(t')+\varepsilon_2(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t') \\ & +\frac{1}{2}\frac{\partial^2 f_0(\varepsilon(\mathbf{k}_0(t)))}{\partial\varepsilon^2}\int_{-\infty}^{t} dt'\frac{\varepsilon_1^2(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t') \\ & -\frac{\partial f_0(\varepsilon(\mathbf{k}_0(t)))}{\partial\varepsilon}\int_{-\infty}^{t} dt'\frac{\varepsilon_1(t')}{\tau_S(\mathbf{k}_0(t'))}\frac{\nabla_{\mathbf{k}}\tau_S(\mathbf{k}_0(t'))\cdot\mathbf{k}_1(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t') \\ & +\frac{\partial f_0(\varepsilon(\mathbf{k}_0(t)))}{\partial\varepsilon}\int_{-\infty}^{t} dt'\frac{\varepsilon_1(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t')\int_{t'}^{t} dl\frac{\nabla_{\mathbf{k}}\tau_S(\mathbf{k}_0(l))\cdot\mathbf{k}_1(l)}{\tau_S^2(\mathbf{k}_0(l))} \end{aligned} \quad (1.22)$$

Integrating the last term by parts we get that

$$\int_{-\infty}^{t} dt' \frac{\varepsilon_1(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t') \int_{t'}^{t} dl \frac{\nabla_{\mathbf{k}}\tau_S(\mathbf{k}_0(l))\cdot\mathbf{k}_1(l)}{\tau_S^2(\mathbf{k}_0(l))} =$$

$$= -\int_{-\infty}^{t} dt' \frac{d\varepsilon_1(t)}{dt}\eta(t;t') \int_{t'}^{t} dl \frac{\nabla_{\mathbf{k}}\tau_S(\mathbf{k}_0(l))\cdot\mathbf{k}_1(l)}{\tau_S^2(\mathbf{k}_0(l))} + \int_{-\infty}^{t} dt' \frac{\varepsilon_1(t')}{\tau_S(\mathbf{k}_0(t'))}\frac{\nabla_{\mathbf{k}}\tau_S(\mathbf{k}_0(t'))\cdot\mathbf{k}_1(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t') \tag{1.23}$$

Our solution now reduces into:

$$\begin{aligned} f(\mathbf{k},t) \simeq{}& \frac{\partial f_0(\varepsilon(\mathbf{k}_0(t)))}{\partial\varepsilon}\int_{-\infty}^{t} dt' \frac{\varepsilon_1(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t') \\ &+ \frac{\partial f_0(\varepsilon(\mathbf{k}_0(t)))}{\partial\varepsilon}\int_{-\infty}^{t} dt' \frac{\varepsilon_2(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t') \\ &+ \frac{1}{2}\frac{\partial^2 f_0(\varepsilon(\mathbf{k}_0(t)))}{\partial\varepsilon^2}\int_{-\infty}^{t} dt' \frac{\varepsilon_1^2(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t') \\ &- \frac{\partial f_0(\varepsilon(\mathbf{k}_0(t)))}{\partial\varepsilon}\int_{-\infty}^{t} dt' \frac{d\varepsilon_1(t)}{dt}\eta(t;t')\int_{t'}^{t} dl \frac{\nabla_{\mathbf{k}}\tau_S(\mathbf{k}_0(l))\cdot\mathbf{k}_1(l)}{\tau_S^2(\mathbf{k}_0(l))} \end{aligned} \tag{1.24}$$

Doing a final integration by parts and substituting the result into the current density equation we are left with our final expression

$$\begin{aligned} J_a^{(2)} ={}& -e^2\int \frac{d^2k}{(2\pi)^2}\nabla_{k_a}\varepsilon(\mathbf{k})\frac{\partial f_0(\varepsilon(\mathbf{k}))}{\partial\varepsilon}\int_{-\infty}^{t} dt' \nabla_{\mathbf{k}}\varepsilon(\mathbf{k}_0(t'))\cdot\mathbf{E}(t')\eta(t;t') \\ &- e^2\int \frac{d^2k}{(2\pi)^2}\nabla_{k_a}\varepsilon(\mathbf{k})\frac{\partial f_0(\varepsilon(\mathbf{k}))}{\partial\varepsilon}\int_{-\infty}^{t} dt' \sum_{\beta,\gamma} E_\beta(t')\,k_{1\gamma}\frac{\partial^2}{\partial k_\gamma\partial k_\beta}\varepsilon(\mathbf{k}_0(t'))\eta(t;t') \\ &+ e^3\int \frac{d^2k}{(2\pi)^2}\nabla_{k_a}\varepsilon(\mathbf{k})\frac{\partial^2 f_0(\varepsilon(\mathbf{k}))}{\partial\varepsilon^2}\int_{-\infty}^{t} dt' \nabla_{\mathbf{k}}\varepsilon(\mathbf{k}_0(t'))\cdot\mathbf{E}(t')\eta(t;t')\int_{-\infty}^{t'} dl \nabla_{\mathbf{k}}\varepsilon(\mathbf{k}_0(l))\cdot\mathbf{E}(l) \\ &- e^2\int \frac{d^2k}{(2\pi)^2}\nabla_{k_a}\varepsilon(\mathbf{k})\frac{\partial f_0(\varepsilon(\mathbf{k}))}{\partial\varepsilon}\int_{-\infty}^{t} dt' \nabla_{\mathbf{k}}\varepsilon(\mathbf{k}_0(t'))\cdot\mathbf{E}(t')\eta(t;t')\int_{t'}^{t} dl \frac{\nabla_{\mathbf{k}}\tau_S(\mathbf{k}_0(l))\cdot\mathbf{k}_1(l)}{\tau_S^2(\mathbf{k}_0(l))} \end{aligned} \tag{1.25}$$

For our applications we will use only the order **E** formula.

1.3 Application to the Rectangular lattice

In this section we use the frequency dependent Chambers' formula to calculate the conductivity tensor components for the following dispersion:

$$\varepsilon(k) = -2t_x cos(k_x) - 2t_y cos(k_y), \ (t_y > t_x) \tag{1.26}$$

For this dispersion we have the following topologies:

$$\begin{cases} -\mu_0 < \mu < -\mu_c & electron\,pockets \\ -\mu_c < \mu < \mu_c & open\,Fermi\,surface \\ \mu_c < \mu < \mu_0 & hole\,pockets \end{cases} \tag{1.27}$$

where we have defined $\mu_0 \equiv 2(t_x + t_y)$ and $\mu_c \equiv 2(t_y - t_x)$. The components of the conductivity are derived in detail in Appendix A and are found to be

$$\sigma^i_{xx} = \frac{2\sigma_0}{K}\sum_n \frac{[1+i\omega\tau_S]\, sech^2\left[\frac{n\pi K'}{2K}\right] sin^2\left[\frac{n\pi u_i}{2K}\right]}{[1+i\omega\tau_S]^2 + [n\omega_c\tau_S]^2} \tag{1.28}$$

$$\sigma^i_{yy} = \frac{\sigma_0\delta_{i,o}}{K}\frac{1}{1+i\omega\tau_S} + \frac{2\sigma_0}{K}\sum_n \frac{[1+i\omega\tau_S]\, sech^2\left[\frac{n\pi K'}{2K}\right] cos^2\left[\frac{n\pi u_i}{2K}\right]}{[1+i\omega\tau_S]^2 + [n\omega_c\tau_S]^2} \tag{1.29}$$

$$\sigma^i_{xy} = (\delta_{i,o} + \delta_{i,e} - \delta_{i,h})\frac{\sigma_0}{K}\sum_n \frac{[n\omega_c\tau_S]\, sech^2\left[\frac{n\pi K'}{2K}\right] sin\left[\frac{n\pi u_i}{K}\right]}{[1+i\omega\tau_S]^2 + [n\omega_c\tau_S]^2} \tag{1.30}$$

where e stands for electron orbits, h stands for hole orbits and o stands for open orbits. We have also defined $\kappa \equiv \sqrt{\frac{\mu_0^2-\mu^2}{\mu_0^2-\mu_c^2}}$, $\omega_0 \equiv eB\sqrt{4t_xt_y}$, $m_0 \equiv \frac{1}{\sqrt{4t_xt_y}}$, $\sigma_0 \equiv e^2\tau_S\sqrt{4t_xt_y}$ and $K(\kappa)$ is the complete elliptic integral of the first kind. For closed surfaces $K \equiv K(\kappa)$ and $K' \equiv K\left(\sqrt{1-\kappa^2}\right)$. For open surfaces we substitute $K(\kappa) \to \frac{1}{\kappa}K(1/\kappa)$ and similarly for K'. In addition

$$\omega_c \equiv \begin{cases} \frac{\pi\omega_0}{2K(\kappa)} & open\,orbits \\ \frac{\pi\kappa\omega_0}{2K(1/\kappa)} & closed\,orbits \end{cases} \tag{1.31}$$

Finally, u_i and u_0 are defined via Jacobian elliptic functions as follows:

$$sn(u_e, \kappa) = \sqrt{\frac{\mu_0 - \mu_c}{\mu_0 - \mu}} \tag{1.32}$$

$$sn(u_h, \kappa) = \sqrt{\frac{\mu_0 - \mu_c}{\mu_0 + \mu}} \tag{1.33}$$

$$sn(\kappa u_o, 1/\kappa) = \sqrt{\frac{\mu_0 + \mu}{\mu_0 + \mu_c}} \tag{1.34}$$

For closed Fermi surfaces the sums are over positive odd integers, whereas for open surfaces the sums are over even integers. The hall number is found to be independent of frequency both in the high and low magnetic field limits, as we will show in Appendix A. Here we present our numerical results for two intermediate cases.

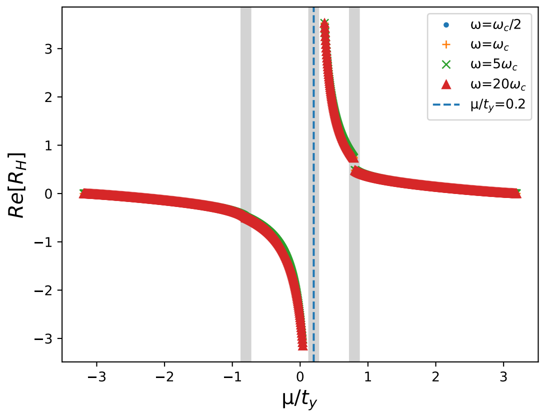

Figure 1.1: Real part of Hall coefficient for $\omega_c \tau_S = 0.01$

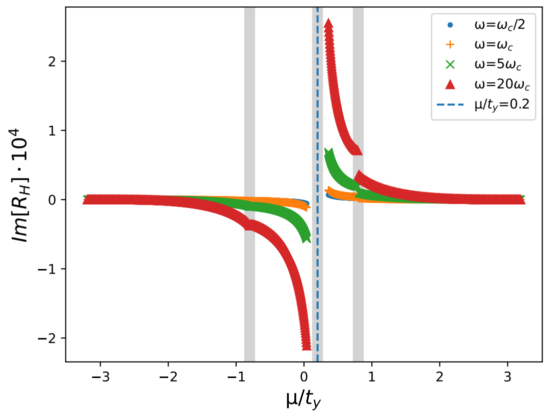

Figure 1.2: Imaginary part of Hall coefficient for $\omega_c \tau_S = 0.01$

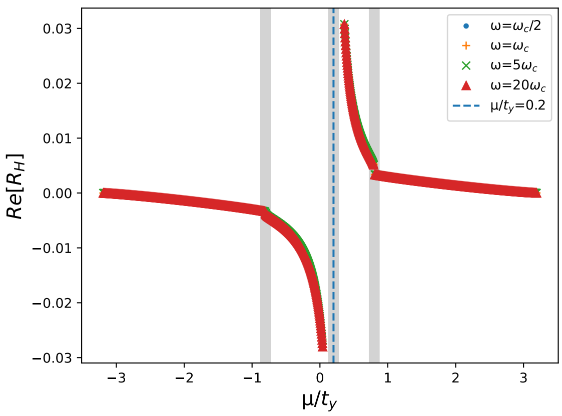

Figure 1.3: Real part of Hall coefficient for $\omega_c \tau_S = 1$

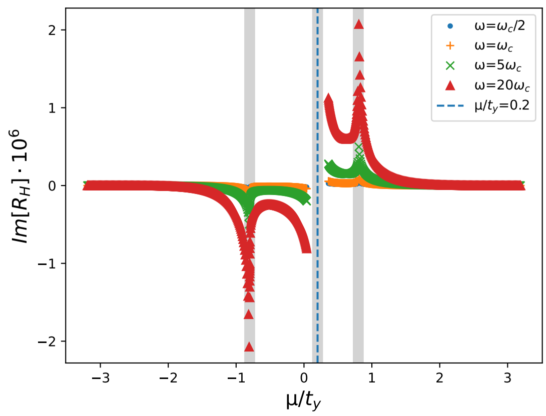

Figure 1.4: Imaginary part of Hall coefficient for $\omega_c \tau_S = 1$

The first case is $\omega_c \tau_S = 0.01 \ll 1$ and the second one $\omega_c \tau_S = 1$. For both cases the magnetic field is $eB = 0.01$ in units of hc/a^2 where a is the lattice constant. For convenience we will work with $h = c = a = 1$. We have also chosen $t_y/t_x = 5/3$. One can observe that at $\mu/t_y = 0.2$ there is a regular Van Hove singularity. Around this singularity the Hall coefficient changes sign from electron to hole-like. At the values $\mu/t_y = \pm 0.8$ the hall constant displays a discontinuity, which is due to transition from electron pockets to open Fermi surface, or from open Fermi surface to hole pockets respectively. Even though the real part of the conductivity does not show any significant dependence on the electric field frequency, its imaginary part does. Namely, the above characteristics become much more pronounced as the frequency grows. Finally, away from the Van Hove singularities the imaginary part of the Hall coefficient drops to zero. It is also worth emphasizing that our results are valid outside the grey areas. Close to the Van Hove singularities quantum effects become important and our semiclassical approach cannot be applied.

1.4 Application to highly doped graphene model

For our graphene model we will consider the following Hamiltonian on a hexagonal lattice:

$$H = \begin{pmatrix} 0 & f(\mathbf{k}) \\ f^*(\mathbf{k}) & 0 \end{pmatrix} \tag{1.35}$$

where

$$f(\mathbf{k}) = -t\left[e^{-ik_y} + e^{\frac{i}{2}\left(\sqrt{3}k_x+k_y\right)} + e^{\frac{i}{2}\left(-\sqrt{3}k_x+k_y\right)}\right] - c\left[e^{2ik_y} + e^{-i\left(\sqrt{3}k_x+k_y\right)} + e^{-i\left(-\sqrt{3}k_x+k_y\right)}\right] \tag{1.36}$$

The energy dispersion of this Hamiltonian is:

$$\varepsilon_{\pm}(\mathbf{k}) = \pm\sqrt{f^*(\mathbf{k})\, f(\mathbf{k})} \tag{1.37}$$

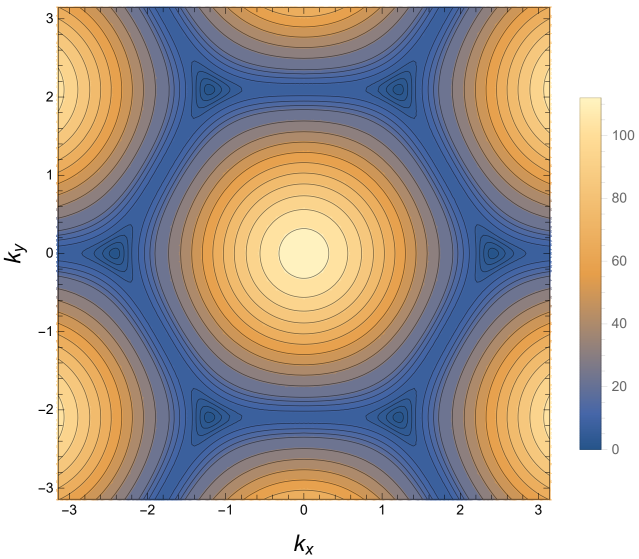

Figure 1.5: Dispersion for graphene with $t = 30$ and $c = t/4$

This leads to a higher order Van Hove singularity in both bands at wave vector $\mathbf{G} = \frac{2\pi}{3}(0,1)$ for $c = t/4$. For this value of c and for a magnetic field $eB = 0.01$ we present numerical results of the conductivity components and hall coefficient for two different values of $\omega_c\tau_S$. Here, ω_c is defined as $\omega_c = \frac{eB}{m^*c}$, with the effective mass $m^* \sim t^{-1}$. In each case we present our results for $\omega = 0$ and also for three non-zero electric field frequencies. The results for $\omega_c\tau_S = 0.3$ are shown below. One can see that around the Van Hove singularity, which occurs at $\mu/t = 0.25$ for $c = t/4$, the Hall coefficient changes sign. At the Van Hove singularity, many discontinuities can be observed in the conductivity components and the Hall coefficients. In addition, the frequency dependence shows itself on the imaginary parts. For the imaginary part of the Hall coefficient, this frequency dependence occurs around the Van Hove singularity and becomes more pronounced as the frequency increases. Away from the van Hove singularity the imaginary part of the Hall coefficient again goes to zero.

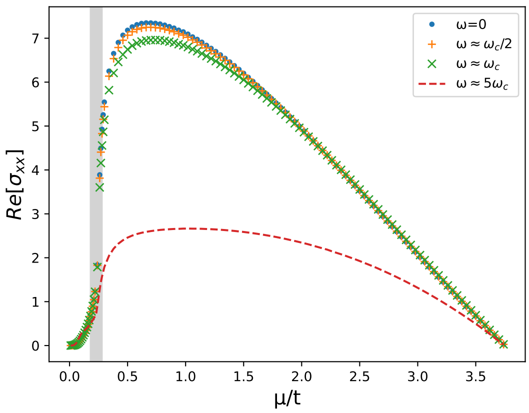

Figure 1.6: Real part of σ_{xx} for $\omega_c \tau_S = 0.3$

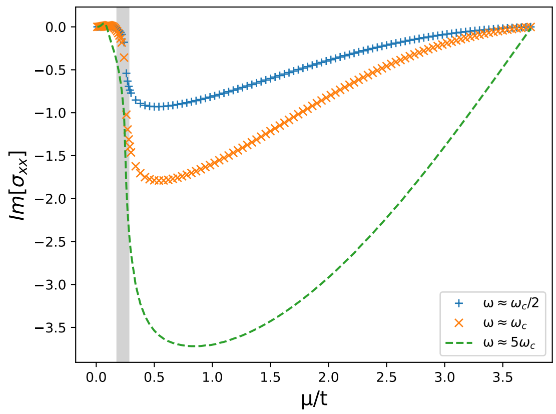

Figure 1.7: Imaginary part of σ_{xx} for $\omega_c \tau_S = 0.3$

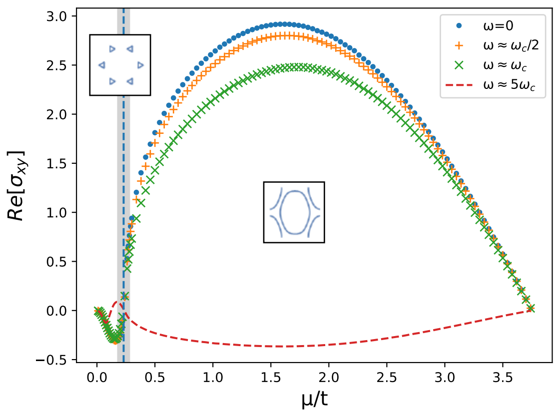

Figure 1.8: Real part of σ_{xy} for $\omega_c \tau_S = 0.3$. The different Fermi surface topologies are also shown and the topological transition is evident.

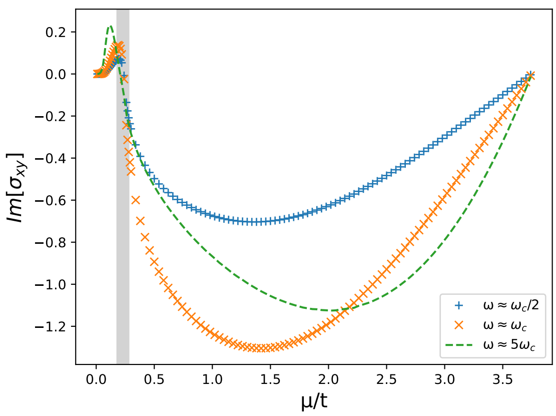

Figure 1.9: Imaginary part of σ_{xy} for $\omega_c \tau_S = 0.3$

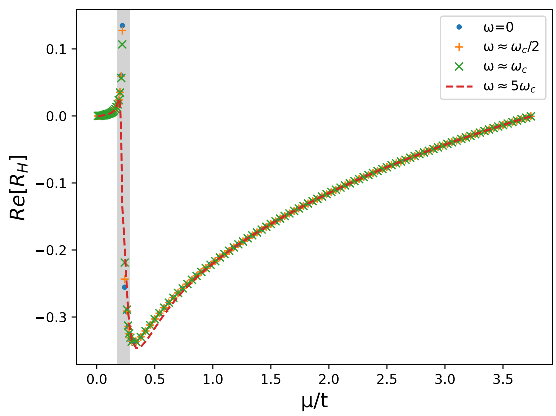

Figure 1.10: Real part of Hall coefficient for $\omega_c \tau_S = 0.3$

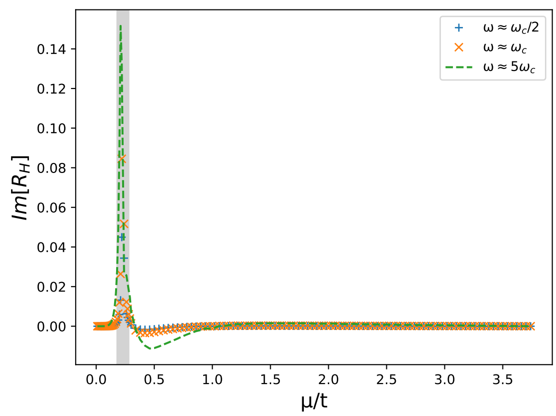

Figure 1.11: Imaginary part of Hall coefficient for $\omega_c \tau_S = 0.3$

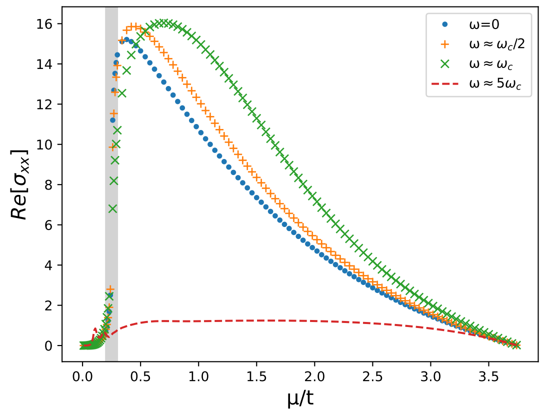

Figure 1.12: Real part of σ_{xx} for $\omega_c \tau_S = 1$

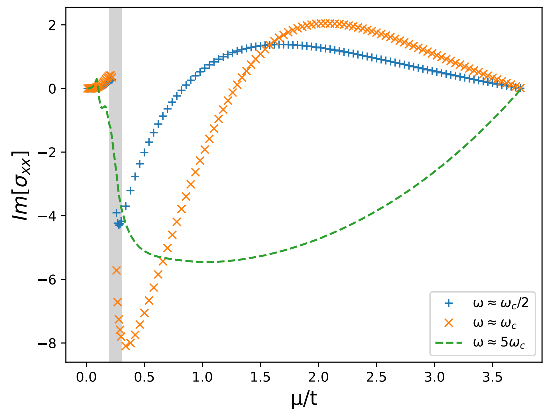

Figure 1.13: Imaginary part of σ_{xx} for $\omega_c \tau_S = 1$

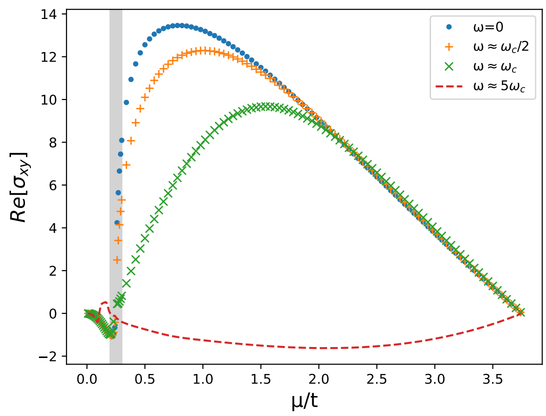

Figure 1.14: Real part of σ_{xy} for $\omega_c \tau_S = 1$

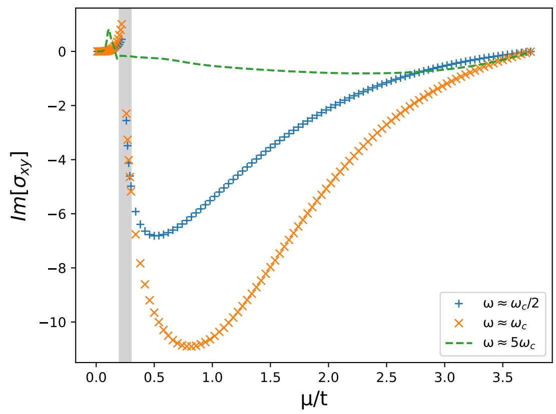

Figure 1.15: Imaginary part of σ_{xy} for $\omega_c \tau_S = 1$

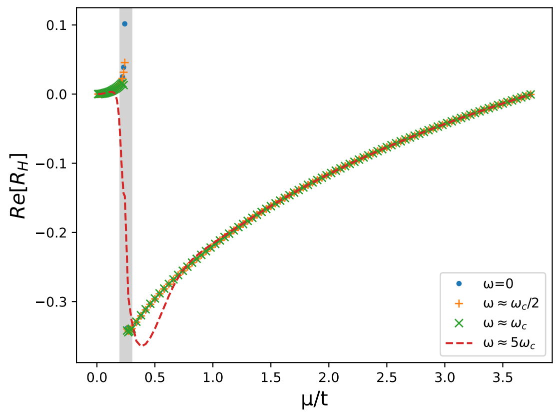

Figure 1.16: Real part of Hall coefficient for $\omega_c \tau_S = 1$

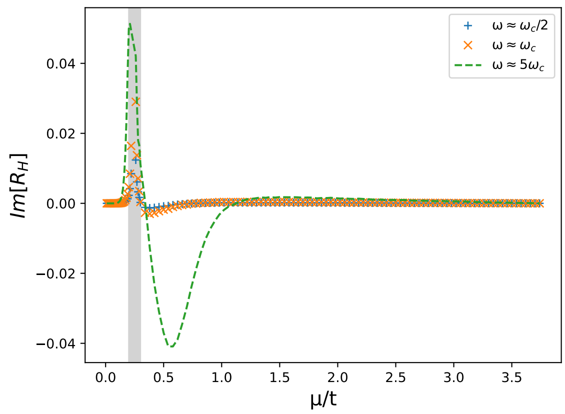

Figure 1.17: Imaginary part of Hall coefficient for $\omega_c \tau_S = 1$

Chapter 2

Chambers' formula for bands with Berry Curvature

2.1 Boltzmann equation and solution

The semiclassical equations of motion for a single particle Hamiltonian $H(\mathbf{k})$ to leading order in the electric and magnetic field for a band with non-zero Berry curvature can be written as:

$$\frac{d\mathbf{r}}{dt} = D^{-1}(\mathbf{r},\mathbf{k},t)\left[\nabla_{\mathbf{k}}\varepsilon_M(\mathbf{k}) + e\mathbf{E}(\mathbf{r},t)\times\Omega(\mathbf{k})\right] + e\left[\Omega(\mathbf{k})\cdot\nabla_{\mathbf{k}}\varepsilon_M(\mathbf{k})\right]\mathbf{B}(\mathbf{r},t) \tag{2.1}$$

$$\frac{d\mathbf{k}}{dt} = -D^{-1}(\mathbf{r},\mathbf{k},t)\left[e\mathbf{E}(\mathbf{r},t) + e\nabla_{\mathbf{k}}\varepsilon_M(\mathbf{k})\times\mathbf{B}(\mathbf{r},t)\right] + e^2\left[\mathbf{B}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)\right]\Omega(\mathbf{k}) \tag{2.2}$$

where we have introduced $D(\mathbf{r},\mathbf{k},t) = 1 + e\mathbf{B}(\mathbf{r},t)\cdot\Omega(\mathbf{k})$. The berry curvature is defined as the pseudovector $\Omega(\mathbf{k}) = \nabla_{\mathbf{k}}\times\mathcal{A}(\mathbf{k})$, where $\mathcal{A}(\mathbf{k}) = i\langle u(\mathbf{k})|\nabla_{\mathbf{k}}|u(\mathbf{k})\rangle$. Here $|u(\mathbf{k})\rangle$ is an eigenstate of the Hamiltonian. Finally $\varepsilon_M(\mathbf{k}) = \varepsilon(\mathbf{k}) - \mathbf{m}(\mathbf{k})\cdot\mathbf{B}(\mathbf{r},t)$, with $\mathbf{m}(\mathbf{k}) = -i\frac{e}{2\hbar}\langle\nabla_{\mathbf{k}}u(\mathbf{k})|\times[H(\mathbf{k}) - \varepsilon(\mathbf{k})]|\nabla_{\mathbf{k}}u(\mathbf{k})\rangle$. In our 2D case the berry curvature is perpendicular to the xy plane so our previous equations reduce into the following ones:

$$\frac{d\mathbf{r}}{dt} = D^{-1}(\mathbf{r},\mathbf{k},t)\left[\nabla_{\mathbf{k}}\varepsilon_M(\mathbf{k}) + e\mathbf{E}(\mathbf{r},t)\times\Omega(\mathbf{k})\right] \tag{2.3}$$

$$\frac{d\mathbf{k}}{dt} = -D^{-1}(\mathbf{r},\mathbf{k},t)\left[e\mathbf{E}(\mathbf{r},t) + e\nabla_{\mathbf{k}}\varepsilon_M(\mathbf{k})\times\mathbf{B}(\mathbf{r},t)\right] \tag{2.4}$$

In the relaxation time approximation, the Boltzmann equation is given by

$$\frac{\partial}{\partial t}f(\mathbf{k},\mathbf{r},t) + \mathbf{w}\cdot\nabla\left[f(\mathbf{k},\mathbf{r},t)\right] = \frac{f_0(\varepsilon(\mathbf{k})) - f(\mathbf{k},\mathbf{r},t)}{\tau_S(\mathbf{k})} \tag{2.5}$$

where τ_S is the relaxation time, $\mathbf{w} = \left(\frac{d\mathbf{r}}{dt}, \frac{d\mathbf{k}}{dt}\right)$ and $\nabla = (\nabla_{\mathbf{r}}, \nabla_{\mathbf{k}})$. If we consider our electric and magnetic fields to be spatially uniform the above equation reduces to the following:

$$\frac{\partial f}{\partial t} + \frac{d\mathbf{k}}{dt}\cdot\frac{\partial f}{\partial\mathbf{k}} = -\frac{1}{\tau_S(\mathbf{k})}\left[f - f_0(\varepsilon(\mathbf{k}))\right] \tag{2.6}$$

This equation can be analytically solved and its solution is given by

$$f(\mathbf{k},t) = f(\mathbf{k}_0,t_0)\,exp\left(-\int_{t_0}^{t}\frac{ds}{\tau_S(\mathbf{k}(s))}\right) + \int_{t_0}^{t}dt'\frac{f_0(\varepsilon(\mathbf{k}(t')))}{\tau_S(\mathbf{k}(t'))}exp\left(-\int_{t'}^{t}\frac{ds}{\tau_S(\mathbf{k}(s))}\right) \tag{2.7}$$

If the initial time is set to $t_0 \to -\infty$ then the solution becomes

$$f(\mathbf{k},t) = \int\limits_{-\infty}^{t} dt' \frac{f_0(\varepsilon(\mathbf{k}(t')))}{\tau_S(\mathbf{k}(t'))} exp\left(-\int\limits_{t'}^{t} \frac{ds}{\tau_S(\mathbf{k}(s))}\right) \tag{2.8}$$

The distribution function of the electrons can now be used in order to calculate the current density in our 2D case. We have that

$$J_a(t) = -e\int \frac{d^2k}{(2\pi)^2} D(\mathbf{k}) \frac{dr_a}{dt} f(\mathbf{k},t) \tag{2.9}$$

2.2 Derivation of the formula

To find the current to first and second order in the electric field we write: $\mathbf{E}(t) = \lambda\mathbf{E}(t)$ with $\lambda = 1$. The solution of our equations can now be written as an analytic asymptotic series of the form

$$\mathbf{k}(t) = \mathbf{k}_0(t) + \lambda\mathbf{k}_1(t) + \lambda^2\mathbf{k}_2(t) + ... \tag{2.10}$$

with $\mathbf{k}_1(t_0) = \mathbf{k}_2(t_0) = ... = 0$. From the equations of motion we obtain to 0^{th} and to 1^{st} order in λ that

$$\frac{d\mathbf{k}_0(t)}{dt} = -eD^{-1}(\mathbf{k}_0(t))\left[\nabla_{\mathbf{k}}\varepsilon_M(\mathbf{k}(t)) \times \mathbf{B}\right]_{\mathbf{k}_0} \tag{2.11}$$

$$\frac{d\mathbf{k}_1(t)}{dt} = -e\sum_a k_{1a}(t)\left[\frac{\partial}{\partial k_a}\left[D^{-1}(\mathbf{k}(t))\left[\nabla_{\mathbf{k}}\varepsilon_M(\mathbf{k}(t)) \times \mathbf{B}\right]\right]\right]_{\mathbf{k}_0} - eD^{-1}(\mathbf{k}_0(t))\mathbf{E}(t) \tag{2.12}$$

For convenience we introduce $\mathbf{u}_0(\mathbf{k}) = D^{-1}(\mathbf{k})\nabla_{\mathbf{k}}\varepsilon_M(\mathbf{k})$. In exactly analogous way we can write

$$\varepsilon_M(t) = \varepsilon_0(t) + \lambda\varepsilon_1(t) + \lambda^2\varepsilon_2(t) + ... \tag{2.13}$$

with $\varepsilon_1(t_0) = \varepsilon_2(t_0) = ... = 0$. We note here that

$$\frac{d\varepsilon_M(\mathbf{k}(t))}{dt} = \nabla_{\mathbf{k}}\varepsilon_M(\mathbf{k}(t)) \cdot \frac{d\mathbf{k}(t)}{dt} = -eD^{-1}(\mathbf{k}(t))\nabla_{\mathbf{k}}\varepsilon_M(\mathbf{k}(t)) \cdot \left[\mathbf{E}(t) + \nabla_{\mathbf{k}}\varepsilon_M(\mathbf{k}(t)) \times \mathbf{B}(\mathbf{r},t)\right] \Rightarrow$$

$$\frac{d\varepsilon_M(\mathbf{k}(t))}{dt} = -eD^{-1}(\mathbf{k}(t))\nabla_{\mathbf{k}}\varepsilon_M(\mathbf{k}(t)) \cdot \mathbf{E}(t) \tag{2.14}$$

To order λ and λ^2 we have that

$$\varepsilon_1(t) = -e\int\limits_{-\infty}^{t} D^{-1}(\mathbf{k}_0(t))\left[\nabla_{\mathbf{k}}\varepsilon_M(\mathbf{k}(s)) \cdot \mathbf{E}(s)\right]_{\mathbf{k}_0} ds \tag{2.15}$$

$$\varepsilon_2(t) = -e\sum_{a,\beta}\int\limits_{-\infty}^{t} E_\beta(s)\, k_{1a}(s)\frac{\partial}{\partial k_a}\left[D^{-1}(\mathbf{k}(t))\frac{\partial}{\partial k_\beta}\varepsilon_M(\mathbf{k}(s))\right]_{\mathbf{k}_0} ds \tag{2.16}$$

Before we Taylor expand our solution we first note that

$$\int_{-\infty}^{t} dt' \frac{1}{\tau_S\left(\mathbf{k}\left(t'\right)\right)} exp\left(-\int_{t'}^{t} \frac{ds}{\tau_S\left(\mathbf{k}\left(s\right)\right)}\right) = 1 \tag{2.17}$$

so we do need to expand this term to obtain the correct formula to first order in the electric field. Expanding our solution we can write:

$$f\left(\mathbf{k},t\right) = \int_{-\infty}^{t} dt' \frac{f_0\left(\varepsilon_M\left(\mathbf{k}\left(t'\right)\right)\right)}{\tau_S\left(\mathbf{k}\left(t'\right)\right)} \eta\left(t;t'\right) \simeq$$

$$\simeq f_0\left(\varepsilon_M\left(\mathbf{k}_0\left(t\right)\right)\right) + \frac{\partial f_0\left(\varepsilon_M\left(\mathbf{k}_0\left(t\right)\right)\right)}{\partial\varepsilon} \int_{-\infty}^{t} dt' \frac{\varepsilon_1\left(t'\right)}{\tau_S\left(\mathbf{k}_0\left(t'\right)\right)} \eta\left(t;t'\right) \tag{2.18}$$

The first term does not contribute to the current density, since we cannot have current without an electric field. The current density can now be written as follows:

$$J_a\left(t\right) = -e\int \frac{d^2k}{\left(2\pi\right)^2} D\left(\mathbf{k}_0\right)\left[u_{0a}\left(\mathbf{k}_0\right) + eD^{-1}\left(\mathbf{k}_0\right)\left[\mathbf{E}\left(t\right)\times\Omega\left(\mathbf{k}_0\right)\right]_a\right]\cdot$$

$$\cdot\left[f_0\left(\varepsilon_M\left(\mathbf{k}_0\left(t\right)\right)\right) + \frac{\partial f_0\left(\varepsilon\left(\mathbf{k}_0\left(t\right)\right)\right)}{\partial\varepsilon} \int_{-\infty}^{t} dt' \frac{\varepsilon_1\left(t'\right)}{\tau_S\left(\mathbf{k}_0\left(t'\right)\right)} \eta\left(t;t'\right)\right] \tag{2.19}$$

Dropping the 0^{th} order term in the electric field, since it does not contribute to the current density, and keeping only the 1^{st} order terms we get:

$$J_a^{(1)}\left(t\right) = -e^2\int \frac{d^2k}{\left(2\pi\right)^2} f_0\left(\varepsilon_M\left(\mathbf{k}_0\left(t\right)\right)\right)\left[\mathbf{E}\left(t\right)\times\Omega\left(\mathbf{k}_0\right)\right]_a$$

$$-\,e\int \frac{d^2k}{\left(2\pi\right)^2} D\left(\mathbf{k}_0\right) u_{0a}\left(\mathbf{k}_0\right) \frac{\partial f_0\left(\varepsilon\left(\mathbf{k}_0\left(t\right)\right)\right)}{\partial\varepsilon} \int_{-\infty}^{t} dt' \frac{\varepsilon_1\left(t'\right)}{\tau_S\left(\mathbf{k}_0\left(t'\right)\right)} \eta\left(t;t'\right) \tag{2.20}$$

Integrating the last term by parts and using the fact that $\frac{d\varepsilon_1(t)}{dt} = -eD^{-1}\left(\mathbf{k}_0\left(t\right)\right)\left[\nabla_{\mathbf{k}}\varepsilon_M\left(\mathbf{k}\left(t\right)\right)\cdot\mathbf{E}\left(t\right)\right]$ we get

$$J_a^{(1)}\left(t\right) = -e^2\int \frac{d^2k}{\left(2\pi\right)^2} f_0\left(\varepsilon_M\left(\mathbf{k}_0\left(t\right)\right)\right)\left[\mathbf{E}\left(t\right)\times\Omega\left(\mathbf{k}_0\right)\right]_a$$

$$-\,e^2\int \frac{d^2k}{\left(2\pi\right)^2} D\left(\mathbf{k}_0\right) u_{0a}\left(\mathbf{k}_0\right) \frac{\partial f_0\left(\varepsilon\left(\mathbf{k}_0\left(t\right)\right)\right)}{\partial\varepsilon} \int_{-\infty}^{t} dt' \mathbf{u}_0\left(t'\right)\cdot\mathbf{E}\left(t'\right)\eta\left(t;t'\right) \tag{2.21}$$

In the limit $T \to 0$ and for $\tau_S\left(\mathbf{k}\right) = \tau_S$ the conductivity tensor components become

$$\sigma_{\alpha\beta} = -e^2\varepsilon_{\alpha\beta}\int \frac{d^2k}{\left(2\pi\right)^2} f_0\left(\varepsilon_M\left(\mathbf{k}_0\left(t\right)\right)\right)\Omega\left(\mathbf{k}_0\right)$$

$$+\,\frac{e^3B}{\left(2\pi\right)^2}\int_{0}^{\tau} dt D^2\left(t\right) u_{0a}\left(t\right)\int_{-\infty}^{t} dt' u_{0\beta}\left(t'\right) exp\left(-\left[i\omega + \frac{1}{\tau_S}\right]\left(t-t'\right)\right) \tag{2.22}$$

where τ is the period of a single orbit. In the limit $\Omega(\mathbf{k}) \to 0$ we recover equation (1.20). The same procedure can be used to obtain the 2^{nd} order correction term to the current density. We have previously shown that to second order in the electric field

$$f(\mathbf{k},t) \simeq f_0(\varepsilon_M(\mathbf{k}_0(t))) + \frac{\partial f_0(\varepsilon_M(\mathbf{k}_0(t)))}{\partial\varepsilon}\int\limits_{-\infty}^{t} dt'\frac{\varepsilon_1(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t')$$
$$+\frac{\partial f_0(\varepsilon_M(\mathbf{k}_0(t)))}{\partial\varepsilon}\int\limits_{-\infty}^{t} dt'\frac{\varepsilon_2(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t')$$
$$+\frac{1}{2}\frac{\partial^2 f_0(\varepsilon_M(\mathbf{k}_0(t)))}{\partial\varepsilon^2}\int\limits_{-\infty}^{t} dt'\frac{\varepsilon_1^2(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t')$$
$$-\frac{\partial f_0(\varepsilon_M(\mathbf{k}_0(t)))}{\partial\varepsilon}\int\limits_{-\infty}^{t} dt'\frac{d\varepsilon_1(t)}{dt}\eta(t;t')\int\limits_{t'}^{t} dl\frac{\nabla_{\mathbf{k}}\tau_S(\mathbf{k}_0(l))\cdot\mathbf{k}_1(l)}{\tau_S^2(\mathbf{k}_0(l))} \tag{2.23}$$

The current density up to second order in the electric field equals:

$$J_a^{(2)}(t) = -e\int\frac{d^2k}{(2\pi)^2}D(\mathbf{k}_0)\left[u_{0a}(\mathbf{k}_0) + eD^{-1}(\mathbf{k}_0)\left[\mathbf{E}(t)\times\Omega(\mathbf{k}_0)\right]_a\right]f_0(\varepsilon_M(\mathbf{k}_0(t)))$$

$$-e\int\frac{d^2k}{(2\pi)^2}D(\mathbf{k}_0)\left[u_{0a}(\mathbf{k}_0) + eD^{-1}(\mathbf{k}_0)\left[\mathbf{E}(t)\times\Omega(\mathbf{k}_0)\right]_a\right]\frac{\partial f_0(\varepsilon_M(\mathbf{k}_0(t)))}{\partial\varepsilon}\int\limits_{-\infty}^{t} dt'\frac{\varepsilon_1(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t')$$
$$-e\int\frac{d^2k}{(2\pi)^2}D(\mathbf{k}_0)u_{0a}(\mathbf{k}_0)\frac{\partial f_0(\varepsilon_M(\mathbf{k}_0(t)))}{\partial\varepsilon}\int\limits_{-\infty}^{t} dt'\frac{\varepsilon_2(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t')$$
$$-\frac{e}{2}\int\frac{d^2k}{(2\pi)^2}D(\mathbf{k}_0)u_{0a}(\mathbf{k}_0)\frac{\partial^2 f_0(\varepsilon_M(\mathbf{k}_0(t)))}{\partial\varepsilon^2}\int\limits_{-\infty}^{t} dt'\frac{\varepsilon_1^2(t')}{\tau_S(\mathbf{k}_0(t'))}\eta(t;t')$$
$$+e\int\frac{d^2k}{(2\pi)^2}D(\mathbf{k}_0)u_{0a}(\mathbf{k}_0)\frac{\partial f_0(\varepsilon_M(\mathbf{k}_0(t)))}{\partial\varepsilon}\int\limits_{-\infty}^{t} dt'\frac{d\varepsilon_1(t)}{dt}\eta(t;t')\int\limits_{t'}^{t} dl\frac{\nabla_{\mathbf{k}}\tau_S(\mathbf{k}_0(l))\cdot\mathbf{k}_1(l)}{\tau_S^2(\mathbf{k}_0(l))} \tag{2.24}$$

Dropping now the lower order terms which are included in equation (2.21) and performing some integration by parts we are left with:

$$J_a^{(2)}(t) = -e^3\varepsilon_{\alpha\beta}\int\frac{d^2k}{(2\pi)^2}E_\beta(t)\Omega(\mathbf{k}_0)\frac{\partial f_0(\varepsilon_M(\mathbf{k}_0(t)))}{\partial\varepsilon}\int\limits_{-\infty}^{t} dt'\mathbf{u}_0(t')\cdot\mathbf{E}(t')\eta(t;t')$$

$$-e^2\int\frac{d^2k}{(2\pi)^2}D(\mathbf{k}_0)u_{0a}(\mathbf{k}_0)\frac{\partial f_0(\varepsilon_M(\mathbf{k}_0(t)))}{\partial\varepsilon}\int\limits_{-\infty}^{t} dt'E_\beta(t')k_{1\gamma}(t')\frac{\partial}{\partial k_\gamma}\left[D^{-1}(\mathbf{k}_0(t'))\frac{\partial}{\partial k_\beta}\varepsilon_M(\mathbf{k}_0(t'))\right]\eta(t;t')$$

$$+e^3\int\frac{d^2k}{(2\pi)^2}D(\mathbf{k}_0)u_{0a}(\mathbf{k}_0)\frac{\partial^2 f_0(\varepsilon_M(\mathbf{k}_0(t)))}{\partial\varepsilon^2}\int\limits_{-\infty}^{t} dt'\mathbf{u}_0(t')\cdot\mathbf{E}(t')\eta(t;t')\int\limits_{-\infty}^{t'}\mathbf{u}_0(s)\cdot\mathbf{E}(s)\,ds$$

$$-e^2\int\frac{d^2k}{(2\pi)^2}D(\mathbf{k}_0)u_{0a}(\mathbf{k}_0)\frac{\partial f_0(\varepsilon_M(\mathbf{k}_0(t)))}{\partial\varepsilon}\int\limits_{-\infty}^{t} dt'\mathbf{u}_0(t')\cdot\mathbf{E}(t')\eta(t;t')\int\limits_{t'}^{t} dl\frac{\nabla_{\mathbf{k}}\tau_S(\mathbf{k}_0(l))\cdot\mathbf{k}_1(l)}{\tau_S^2(\mathbf{k}_0(l))} \tag{2.25}$$

2.3 Application to the Haldane Model

The Haldane model Hamiltonian is given by $H(\theta) = d_0(\theta) + \mathbf{d}(\theta) \cdot \sigma$, where

$$\begin{gathered} d_0(\theta) = -2t_2 \left[cos\theta_1 + cos\theta_2 + cos\left(\theta_1 + \theta_2\right)\right] cos\varphi \\ d_x(\theta) = -t_1 \left[1 + cos\theta_1 + cos\theta_2\right] \\ d_y(\theta) = t_1 \left[sin\theta_1 - sin\theta_2\right] \\ d_z(\theta) = m - 2t_2 \left[sin\theta_1 + sin\theta_2 - sin\left(\theta_1 + \theta_2\right)\right] sin\varphi \end{gathered} \tag{2.26}$$

and σ denotes the Pauli matrices. We have also defined $\theta_1 \equiv \frac{1}{2}\left(k_x + \sqrt{3}k_y\right)$ and $\theta_2 \equiv \frac{1}{2}\left(k_x - \sqrt{3}k_y\right)$. The energy eigenvalues for this Hamiltonian are given by $\varepsilon_\pm(\mathbf{k}) = d_0 \pm \sqrt{d_x^2 + d_y^2 + d_z^2}$. For simplicity we choose $\varphi = \pi/2$, $m = 0$ and consider ε_+ for $t_1 = 50$ and $t_2 = 25$ with a non trivial Chern number of -1. A contour plot of this dispersion is shown below.

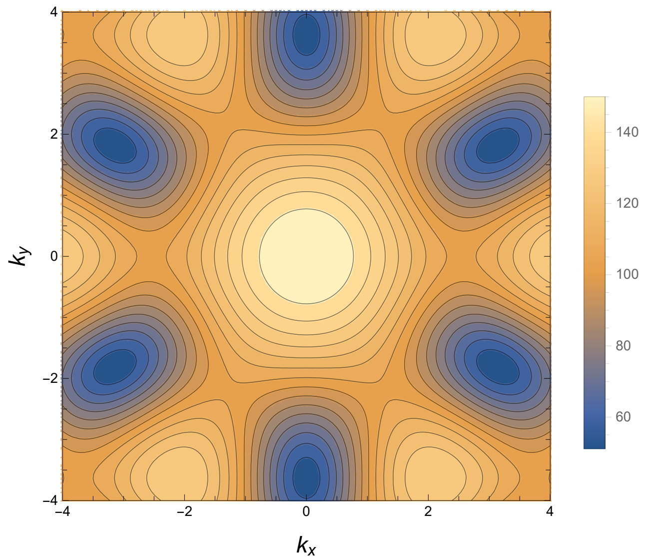

Figure 2.1: Dispersion for Haldane model with $t_1 = 50$ and $t_2 = 25$

We present numerical results for the case where $eB = 0.01$ and $\omega_c \tau_S = 0.3$. Once again ω_c is defined as $\omega_c \equiv \frac{eB}{m^* c}$ with $m^* \sim t_1^{-1}$. The contributions from Eq. (2.22) are also presented separately. We note here that a constant $-\frac{1}{2\pi}$ has been added t the total conductivity, due to the contribution from the lower band. We see that the conductivity is dominated by the second term of Eq. (2.22). In Figure (2.4) the different Fermi surface regimes are shown, making clear the type of topological transitions that occur in the system. The Hall conductivity displays a different behavior in each regime.

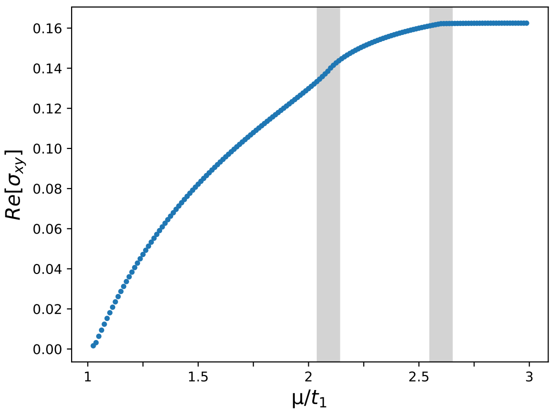

Figure 2.2: Real part of the first term of Eq. (2.22)

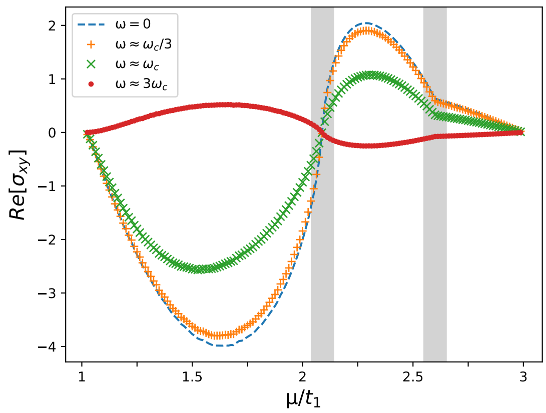

Figure 2.3: Real part of the second term of Eq. (2.22)

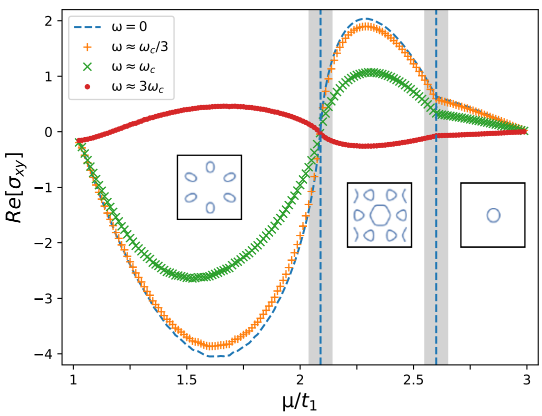

Figure 2.4: Real part of the Hall conductivity. A constant of $-1/(2\pi)$ has been included due to the lower band contribution.

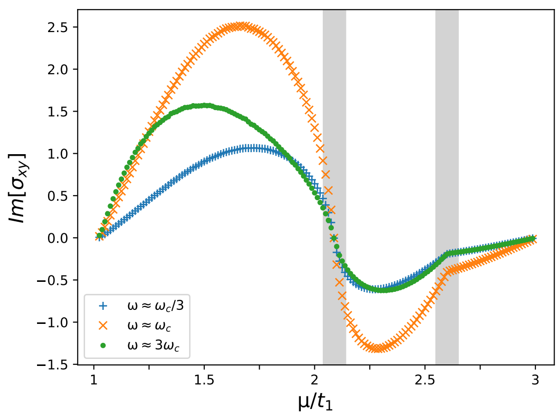

Figure 2.5: Imaginary part of the Hall conductivity. This comes from the second term in Eq. (2.22)

Chapter 3

Reactive Hall Response with spin orbit coupling

3.1 Calculation of the Hall Coefficient

In this section we will consider a simple tight binding Hamiltonian in a rectangular lattice, to which we have included a spin-orbit coupling term in the form of Rashba interaction. In the absence of an electric and a magnetic field this Hamiltonian be written as:

$$H_0 = -t \sum_{l,m,\sigma} c^{\dagger}_{l+1,m,\sigma} c_{l,m,\sigma} + H.c. - t' \sum_{l,m,\sigma} c^{\dagger}_{l,m+1,\sigma} c_{l,m,\sigma} + H.c.$$

$$- t_{SO} \sum_{l,m} \left[c^{\dagger}_{l+1,m,\downarrow} c_{l,m,\uparrow} - c^{\dagger}_{l+1,m,\uparrow} c_{l,m,\downarrow} \right] + H.c. - t'_{SO} \sum_{l,m} \left[i c^{\dagger}_{l,m+1,\downarrow} c_{l,m,\uparrow} + i c^{\dagger}_{l,m+1,\uparrow} c_{l,m,\downarrow} \right] + H.c. \tag{3.1}$$

Now we add a magnetic field along the z direction modulated by a one component wave vector q along the y direction, generated by the vector potential A_m. We also add electric fields in the x,y directions modulated by the time dependent vector potentials $\phi^{x,y}(t)$.

$$H = -t \sum_{l,m,\sigma} e^{i\phi^x(t)} e^{iA_m} c^{\dagger}_{l+1,m,\sigma} c_{l,m,\sigma} + H.c. - t' \sum_{l,m,\sigma} e^{i\phi^y_{m+1/2}(t)} c^{\dagger}_{l,m+1,\sigma} c_{l,m,\sigma} + H.c.$$

$$-t_{SO} \sum_{l,m} e^{i\phi^x(t)} e^{iA_m} \left[c^{\dagger}_{l+1,m,\downarrow} c_{l,m,\uparrow} - c^{\dagger}_{l+1,m,\uparrow} c_{l,m,\downarrow} \right] + H.c.$$

$$- t'_{SO} \sum_{l,m} e^{i\phi^y_{m+1/2}(t)} \left[i c^{\dagger}_{l,m+1,\downarrow} c_{l,m,\uparrow} + i c^{\dagger}_{l,m+1,\uparrow} c_{l,m,\downarrow} \right] + H.c. \tag{3.2}$$

Here $A_m = e^{iqm} \frac{iB}{2sin(q/2)} \simeq e^{iqm} \frac{iB}{q}$ is the vector potential that produces the magnetic field and it satisfies $B_{m+1/2} = -(A_{m+1} - A_m) = Be^{iq(m+1/2)}$. The vector potentials of the electric fields satisfy $\phi^{x,y}(t) = \frac{E^{x,y}(t)}{iz}$ and $\phi^y_{m+1/2}(t) = e^{iq(m+1/2)} \phi^y(t)$, where we have also defined $E^x(t) = E^x e^{-izt}$, $E^y(t) = iE^y e^{-izt}$. Current densities can now be calculated through derivatives in the Hamiltonian:

$$J^x = -\frac{\partial H}{\partial \phi^x} = t \sum_{l,m,\sigma} \left(i e^{i\phi^x(t)} e^{iA_m} c^{\dagger}_{l+1,m,\sigma} c_{l,m,\sigma} + H.c. \right)$$

$$+ t_{SO} \sum_{l,m} \left(i e^{i\phi^x(t)} e^{iA_m} \left[c^{\dagger}_{l+1,m,\downarrow} c_{l,m,\uparrow} - c^{\dagger}_{l+1,m,\uparrow} c_{l,m,\downarrow} \right] + H.c. \right) \tag{3.3}$$

and

$$J_q^y = -\frac{\partial H}{\partial \phi^y} = t' \sum_{l,m,\sigma} e^{iq(m+1/2)} \left(ie^{i\phi^y_{m+1/2}(t)} c^\dagger_{l,m+1,\sigma} c_{l,m,\sigma} + H.c. \right)$$

$$+ t'_{SO} \sum_{l,m} e^{iq(m+1/2)} \left(ie^{i\phi^y_{m+1/2}(t)} \left[ic^\dagger_{l,m+1,\downarrow} c_{l,m,\uparrow} + ic^\dagger_{l,m+1,\uparrow} c_{l,m,\downarrow} \right] + H.c. \right) \tag{3.4}$$

Expanding in ϕ and keeping only the linear terms we identify a diamagnetic and a paramagnetic part in the current density.

$$j^x = t \sum_{l,m,\sigma} \left(ie^{iA_m} c^\dagger_{l+1,m,\sigma} c_{l,m,\sigma} + H.c. \right)$$

$$+ t_{SO} \sum_{l,m} \left(ie^{iA_m} \left[c^\dagger_{l+1,m,\downarrow} c_{l,m,\uparrow} - c^\dagger_{l+1,m,\uparrow} c_{l,m,\downarrow} \right] + H.c. \right) \tag{3.5}$$

$$j^{x,d} = -t \sum_{l,m,\sigma} \left(e^{iA_m} c^\dagger_{l+1,m,\sigma} c_{l,m,\sigma} + H.c. \right) \phi^x (t)$$

$$- t_{SO} \sum_{l,m} \left(e^{iA_m} \left[c^\dagger_{l+1,m,\downarrow} c_{l,m,\uparrow} - c^\dagger_{l+1,m,\uparrow} c_{l,m,\downarrow} \right] + H.c. \right) \phi^x (t) \tag{3.6}$$

where j^x is the paramagnetic part and $j^{x,d}$ is the diamagnetic part. Similarly we have for the y components that:

$$j_q^y = t' \sum_{l,m,\sigma} e^{iq(m+1/2)} \left(ic^\dagger_{l,m+1,\sigma} c_{l,m,\sigma} + H.c. \right)$$

$$+ t'_{SO} \sum_{l,m} e^{iq(m+1/2)} \left(i \left[ic^\dagger_{l,m+1,\downarrow} c_{l,m,\uparrow} + ic^\dagger_{l,m+1,\uparrow} c_{l,m,\downarrow} \right] + H.c. \right) \tag{3.7}$$

$$j_q^{y,d} = -t' \sum_{l,m,\sigma} e^{2iq(m+1/2)} \left(c^\dagger_{l,m+1,\sigma} c_{l,m,\sigma} + H.c. \right) \phi^y (t)$$

$$- t'_{SO} \sum_{l,m} e^{2iq(m+1/2)} \left(\left[ic^\dagger_{l,m+1,\downarrow} c_{l,m,\uparrow} + ic^\dagger_{l,m+1,\uparrow} c_{l,m,\downarrow} \right] + H.c. \right) \phi^y (t) \tag{3.8}$$

Expanding our initial Hamiltonian in ϕ and keeping only the linear terms we see that it can be written a follows:

$$H = H_0 - j^x \phi^x (t) - j_q^y \phi^y (t) \tag{3.9}$$

Using the Kubo formula we get that:

$$\langle j^x \rangle = \langle j^x \rangle_0 + \chi_{j^x j^x} \phi^x (t) + \chi_{j^x j_q^y} \phi^y (t) \tag{3.10}$$

$$\left\langle j_q^y \right\rangle = \left\langle j_q^y \right\rangle_0 + \chi_{j_q^y j^x} \phi^x (t) + \chi_{j_q^y j_q^y} \phi^y (t) \tag{3.11}$$

where we have defined

$$\chi_{AB} \equiv i \int_0^\infty dt e^{izt} \left\langle \left[A(t), B \right] \right\rangle \tag{3.12}$$

Furthermore considering that without an electric field all currents are zero we must have $\langle j^x \rangle_0 = \left\langle j^y_q \right\rangle_0 = 0$. For the diamagnetic parts we can observe that

$$\left\langle j^{x,d} \right\rangle = -\left\langle \frac{\partial^2 H}{\left(\partial \phi^x\right)^2} \right\rangle \phi^x (t) \tag{3.13}$$

$$\left\langle j^{y,d}_q \right\rangle = -\left\langle \frac{\partial^2 H}{\left(\partial \phi^y\right)^2} \right\rangle \phi^y (t) \tag{3.14}$$

Combining the above equations along with the fact that $\frac{\partial^2 H}{\partial \phi^x \partial \phi^y} = 0$ we can write the total current density mean values in the following form:

$$\langle J^x \rangle = \sigma_{j^x j^x} E^x (t) + \sigma_{j^x j^y_q} E^y (t) \tag{3.15}$$

$$\left\langle J^y_q \right\rangle = \sigma_{j^y_q j^x} E^x (t) + \sigma_{j^y_q j^y_q} E^y (t) \tag{3.16}$$

where we have defined

$$\sigma_{j^\alpha j^\beta} \equiv \frac{i}{z} \left(\left\langle \frac{\partial^2 H}{\partial \phi^\alpha \partial \phi^\beta} \right\rangle - \chi_{j^\alpha j^\beta} \right) \tag{3.17}$$

We will now proceed by converting the current-current to current-density correlations. The following equations are derived in detail in Appendix B. We have that

$$\langle J^x \rangle = \sigma_{j^x j^x} E^x (t) + \frac{1}{q} \chi_{j^x n_q} E^y (t) \tag{3.18}$$

$$\left\langle J^y_q \right\rangle = -\frac{1}{q} \chi_{j^x n_q} E^x (t) + \frac{i}{z} \left(\frac{z}{q} \right)^2 \chi_{n_q n_q} E^y (t) \tag{3.19}$$

where we have defined $n_q = \sum_{l,m,\sigma} \left(-ie^{iqm}\right) c^\dagger_{l,m,\sigma} c_{l,m,\sigma}$. Now, in contrast to the usual Hall coefficient derivation, we will first take the $\omega \to 0$ limit and then the $q \to 0$. The previous expressions now become

$$\langle J^x \rangle_0 = \sigma''_{j^x j^x} (\omega \to 0) \left[i E^x (t) \right] + \frac{1}{q} \chi'_{j^x n_q} (\omega \to 0) E^y (t) \tag{3.20}$$

$$\left\langle J^y_q \right\rangle_0 = -\frac{1}{q} \chi'_{j^x n_q} (\omega \to 0) E^x (t) + \frac{1}{\omega} \left(\frac{\omega}{q} \right)^2 \chi'_{n_q n_q} (\omega \to 0) \left[i E^y (t) \right] \tag{3.21}$$

where the subscript zero implies the $\omega \to 0$ limit. We also denote with $'$ the real part of a quantity and with $''$ its imaginary part. Now, let $|n\rangle, E_n$ be the eigenstates and eigenvalues respectively of the Hamiltonian in the presence of the magnetic field.

$$\chi_{AB} (\omega \to 0) = \lim_{z \to 0} i \int_0^\infty dt e^{izt} \langle [A(t), B] \rangle = \lim_{z \to 0} i \int_0^\infty dt e^{izt} \langle A(t) B - B A(t) \rangle =$$

$$= \lim_{z \to 0} \sum_m i \int_0^\infty dt e^{izt} \left[\langle m | \hat{\rho} A(t) B | m \rangle - \langle m | \hat{\rho} B A(t) | m \rangle \right] =$$

$$= \lim_{z \to 0} \sum_{m,n} i \int_0^\infty dt e^{izt} \frac{e^{-\beta E_m}}{Z} \left[\langle m | A(t) | n \rangle \langle n | B | m \rangle - \langle m | B | n \rangle \langle n | A(t) | m \rangle \right] =$$

$$= \lim_{z\to 0}\sum_{m,n} i\int_0^\infty dt e^{izt}\frac{e^{-\beta E_m}}{Z}\left[e^{i\omega_{mnt}t}\langle m|A|n\rangle\langle n|B|m\rangle - e^{i\omega_{nm}t}\langle m|B|n\rangle\langle n|A|m\rangle\right]$$

Since we are working at $T \to 0$ we see that we are going to have contribution only from $m = 0$. Additionally we dont have any contributions from $n = m$. Therefore

$$\chi_{AB}(\omega\to 0) = \lim_{z\to 0}\sum_{n>0} i\int_0^\infty dt e^{izt}\left[e^{i\omega_{0n}t}\langle 0|A|n\rangle\langle n|B|0\rangle - e^{i\omega_{n0}t}\langle 0|B|n\rangle\langle n|A|0\rangle\right] =$$

$$= -\lim_{z\to 0}\sum_{n>0}\left[\frac{\langle 0|A|n\rangle\langle n|B|0\rangle}{z+\omega_{0n}} - \frac{\langle 0|B|n\rangle\langle n|A|0\rangle}{z+\omega_{n0}}\right] \Rightarrow$$

$$\chi_{AB}(\omega\to 0) = \sum_{n>0}\frac{\langle 0|A|n\rangle\langle n|B|0\rangle + H.c.}{E_n - E_0} \tag{3.22}$$

Now, following [12] we can express the different terms of Eqs. (3.20), (3.21) as derivatives of the ground state of the following fictitious Hamiltonian:

$$\begin{aligned} H(\phi^x,\mu_q) = &-t\sum_{l,m,\sigma} e^{i\phi^x(t)}e^{iA_m}c^\dagger_{l+1,m,\sigma}c_{l,m,\sigma} + H.c. - t'\sum_{l,m,\sigma}c^\dagger_{l,m+1,\sigma}c_{l,m,\sigma} + H.c. \\ &-t_{SO}\sum_{l,m} e^{i\phi^x(t)}e^{iA_m}\left[c^\dagger_{l+1,m,\downarrow}c_{l,m,\uparrow} - c^\dagger_{l+1,m,\uparrow}c_{l,m,\downarrow}\right] + H.c. \\ &- t'_{SO}\sum_{l,m}\left[ic^\dagger_{l,m+1,\downarrow}c_{l,m,\uparrow} + ic^\dagger_{l,m+1,\uparrow}c_{l,m,\downarrow}\right] + H.c. + \mu_q n_q \end{aligned} \tag{3.23}$$

For a general Hamiltonian depending on two parameters μ and λ, we have the following identity:

$$\varepsilon^0_{\mu\lambda} = \frac{\partial^2\varepsilon^0}{\partial\mu\partial\lambda} = \left\langle 0\left|\frac{\partial^2 H}{\partial\mu\partial\lambda}\right|0\right\rangle - \sum_{n>0}\left[\frac{\left\langle 0\left|\frac{\partial H}{\partial\mu}\right|n\right\rangle\left\langle n\left|\frac{\partial H}{\partial\lambda}\right|0\right\rangle + H.c.}{E_n - E_0}\right] \tag{3.24}$$

The above result is derived in Appendix C. With this observation, we can rewrite the current densities in the following form.

$$\langle J^x\rangle_0 = \frac{1}{\omega}\varepsilon^0_{\phi^x\phi^x}\left[iE^x(t)\right] + \frac{1}{q}\varepsilon^0_{\phi^x\mu_q}E^y(t) \tag{3.25}$$

$$\left\langle J^y_q\right\rangle_0 = -\frac{1}{q}\varepsilon^0_{\mu_q\phi^x}E^x(t) - \frac{\omega}{q^2}\varepsilon^0_{\mu_q\mu_q}\left[iE^y(t)\right] \tag{3.26}$$

We will now set $\left\langle J^y_q\right\rangle_0 = 0$ and calculate the hall constant:

$$R_H \equiv \frac{1}{B}\frac{E^y(t)}{\langle J^x\rangle_0} \Rightarrow R_H = \frac{q}{B}\frac{\varepsilon^0_{\mu_q\phi^x}}{\varepsilon^0_{\phi^x\phi^x}\varepsilon^0_{\mu_q\mu_q} + \varepsilon^0_{\mu_q\phi^x}\varepsilon^0_{\phi^x\mu_q}} \tag{3.27}$$

Neglecting the cross $\mathcal{O}(B^2)$ term $\varepsilon^0_{\mu_q\phi^x}\varepsilon^0_{\phi^x\mu_q}$ and Taylor expanding the numerator in B we can rewrite the above equation as:

$$R_H = q\frac{\frac{\partial^3\varepsilon^0}{\partial B\partial\mu_q\partial\phi^x}}{\varepsilon^0_{\phi^x\phi^x}\varepsilon^0_{\mu_q\mu_q}} = q\frac{\frac{\partial}{\partial\mu_q}\left(\frac{\partial^2\varepsilon^0}{\partial B\partial\phi^x}\right)}{\varepsilon^0_{\phi^x\phi^x}\varepsilon^0_{\mu_q\mu_q}} = -\frac{\frac{\partial D_q}{\partial\mu_q}}{D\kappa_q} \tag{3.28}$$

where in the last step we have defined:

$$j_q^x = t\sum_{l,m,\sigma}\left(-ie^{iqm}\right)\left(ic^\dagger_{l+1,m,\sigma}c_{l,m,\sigma}+H.c.\right)$$
$$+t_{SO}\sum_{l,m}\left(-ie^{iqm}\right)\left(i\left[c^\dagger_{l+1,m,\downarrow}c_{l,m,\uparrow}-c^\dagger_{l+1,m,\uparrow}c_{l,m,\downarrow}\right]+H.c.\right) \tag{3.29}$$
$$T_q^x = -t\sum_{l,m,\sigma}\left(-ie^{iqm}\right)\left(c^\dagger_{l+1,m,\sigma}c_{l,m,\sigma}+H.c.\right)$$
$$-t_{SO}\sum_{l,m}\left(-ie^{iqm}\right)\left(\left[c^\dagger_{l+1,m,\downarrow}c_{l,m,\uparrow}-c^\dagger_{l+1,m,\uparrow}c_{l,m,\downarrow}\right]+H.c.\right) \tag{3.30}$$

and

$$D_q=-\frac{q}{2}\frac{\partial^2\varepsilon^0}{\partial B\partial\phi^x}=-\frac{q}{2}\left[\left\langle 0\left|\frac{\partial^2 H}{\partial B\partial\phi^x}\right|0\right\rangle-\sum_{n>0}\left[\frac{\left\langle 0\left|\frac{\partial H}{\partial\phi^x}\right|n\right\rangle\left\langle n\left|\frac{\partial H}{\partial B}\right|0\right\rangle+H.c.}{E_n-E_0}\right]\right]\Rightarrow$$
$$D_q=\frac{1}{2}\left[\left\langle 0\left|-T_q^x\right|0\right\rangle-\sum_{n>0}\left[\frac{\langle 0\,|j^x|\,n\rangle\left\langle n\left|j_q^x\right|0\right\rangle+H.c.}{E_n-E_0}\right]\right] \tag{3.31}$$

In addition $D\equiv\frac{1}{2}\varepsilon^0_{\phi^x\phi^x}$ is the Drude weight and it is identical to D_q by the replacement of j_q^x and T^x by j^x and j_q^x respectively. We also define the comressibility as $\varepsilon^0_{\mu_q\mu_q}\equiv\kappa_q=\frac{\partial n_q}{\partial\mu_q}$. Finally, by taking the $q\to 0$ limit we are left with

$$R_H=-\frac{1}{D}\frac{\partial D}{\partial n} \tag{3.32}$$

As a quick application of our model, we will begin by calculating the Hall coefficient for $t'\to 0$ and $t'_{SO}\to 0$. In this case the Hamiltonian can be exactly diagonalized. We find the following two bands:

$$\varepsilon_\pm(k)=-2\sqrt{t^2+t_{SO}^2}cos\left(k_x-\phi^x\mp k_0\right),\ k_0=tan^{-1}\left(\frac{t_{SO}}{t}\right) \tag{3.33}$$

We can now proceed to calculate the Drude weight.

$$D=D_++D_-=\frac{1}{2L}\frac{L}{2\pi}\int_{k_0-n\pi/2}^{k_0+n\pi/2}\left(\frac{\partial^2\varepsilon_+}{\left(\partial\phi^x\right)^2}\right)_{\phi^x\longrightarrow 0}dk_x+\frac{1}{2L}\frac{L}{2\pi}\int_{-k_0-n\pi/2}^{-k_0+n\pi/2}\left(\frac{\partial^2\varepsilon_-}{\left(\partial\phi^x\right)^2}\right)_{\phi^x\longrightarrow 0}dk_x=$$
$$=\frac{\sqrt{t^2+t_{SO}^2}}{\pi}\left[\int_{k_0}^{k_0+n\pi/2}cos\left(k_x-k_0\right)dk_x+\int_{-k_0}^{-k_0+n\pi/2}cos\left(k_x+k_0\right)dk_x\right]\Rightarrow$$
$$\boxed{D=\frac{2\sqrt{t^2+t_{SO}^2}}{\pi}sin\left(\frac{n\pi}{2}\right)} \tag{3.34}$$
$$\boxed{R_H=-\frac{1}{D}\frac{\partial D}{\partial n}=-\frac{\pi}{2}\frac{1}{tan\left(\frac{n\pi}{2}\right)}} \tag{3.35}$$

We see that the Hall coefficient does not depend on the Rashba coupling, since the contributions from the two bands cancel one another. In addition close to the metallic state the Hall coefficient is equal to $R_H\approx -1/n$, whereas close to the insulating state it is equal to $R_H\approx 1/\delta$, with $\delta=2-n$.

3.2 Numerical Simulations

In this section we present our numerical results for the Hall coefficient for three different cases. In the first case we have taken $t'/t = 2$ and $t_{SO} = t'_{SO} = 0$, in the second case we have taken $t'/t = 1$, $t_{SO}/t = 2$ and $t'_{SO}/t = 2$ and in the third case we have taken $t'/t = 2$, $t_{SO}/t = 2$ and $t'_{SO}/t = 2$. In all of the graphs below we have included the Fermi surface contour plots for different topology regimes. In the Fermi surface contour plots the blue line is the lower band, whereas the red line is the upper band. The dispersion of those bands can be found by diagonalizing the full Hamiltonian. It is given by:

$$\varepsilon_{\pm}(k) = -2t\cos(k_x) - 2t'\cos(k_y) \pm 2\sqrt{(t_{SO}\sin(k_x))^2 + (t'_{SO}\sin(k_y))^2} \tag{3.36}$$

The Hall coefficient graphs for all three cases are shown below:

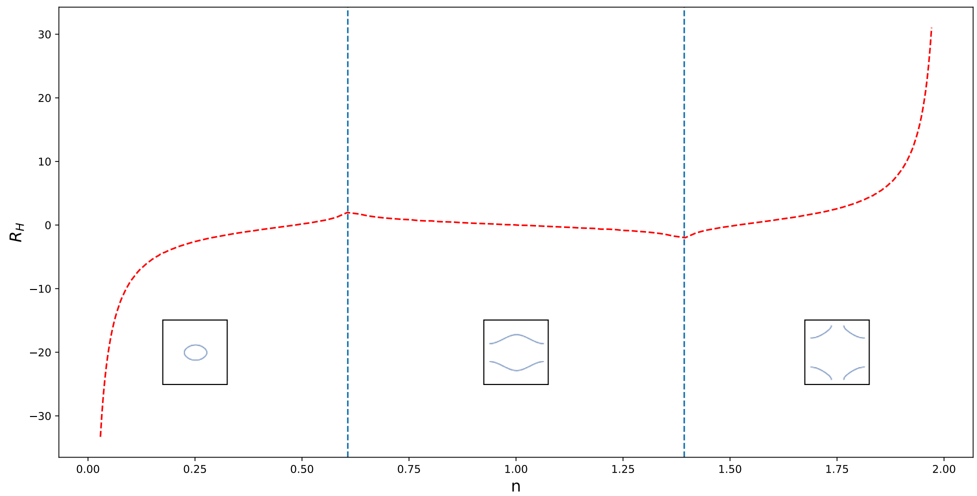

Figure 3.1: Hall coefficient for $t'/t = 2$, $t_{SO} = t'_{SO} = 0$.

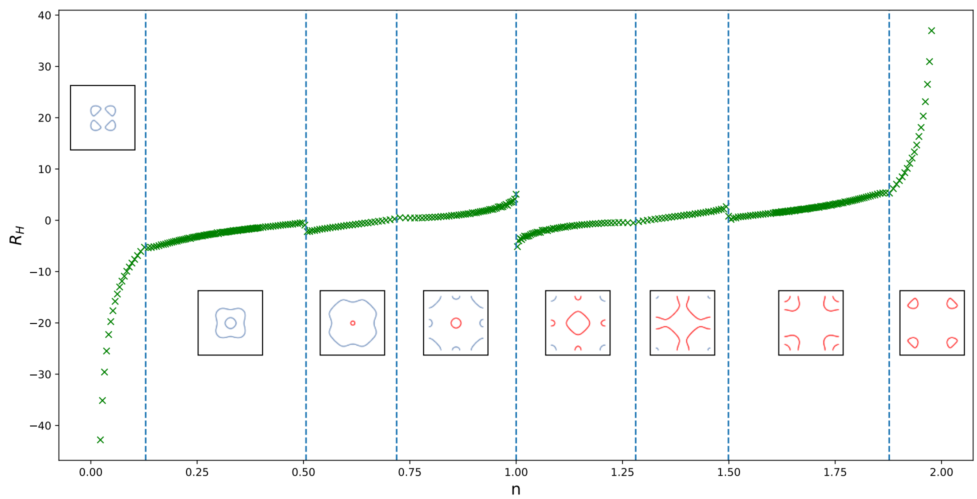

Figure 3.2: Hall coefficient for $t'/t = 1$, $t_{SO}/t = 2$, $t'_{SO}/t = 2$.

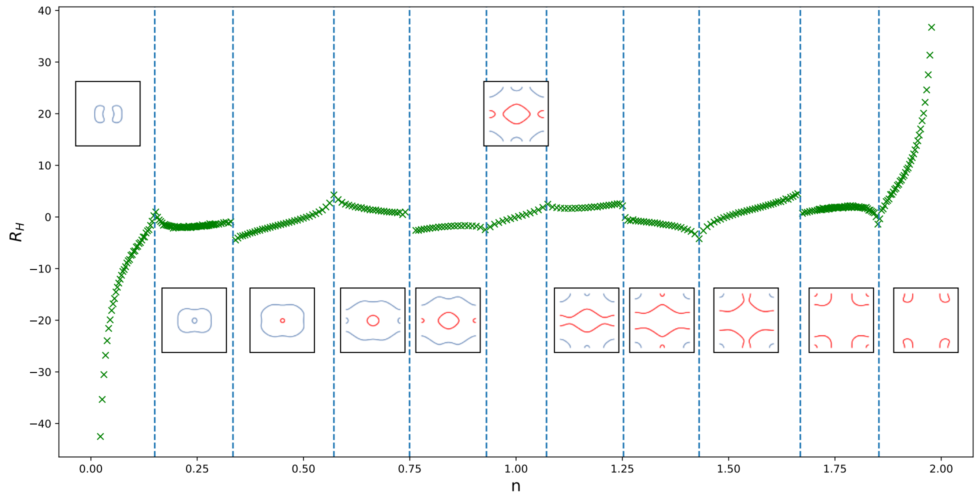

Figure 3.3: Hall coefficient for $t'/t = 2$, $t_{SO}/t = 2$, $t'_{SO}/t = 2$.

As was the case before, close to the metallic state the Hall coefficient is equal to $R_H \approx -1/n$, whereas close to the insulating state it is equal to $R_H = 1/ \approx$, with $\delta = 2 - n$. In addition, around every topological transition the behavior of the Hall coefficient changes. More specifically, when different parts of the Fermi surface connect or disconnect the Hall coefficient displays a discontinuity in its derivative. On the other hand when a part of the Fermi surface disappears from the lower band and appears on the upper band the Hall coefficient displays a discontinuity.

Appendix A

In Ref. [10] the authors have solved Boltzmann transport equations for this energy dispersion analytically. Then they calculated the velocities u_x and u_y and took their Fourier series expansions which are shown below:

$$u_x^i\left(t\right) = \left(1-2\delta_{i,h}\right)\frac{2\pi}{m_0 K\left(\kappa\right)}\sum_{n=1}^{\infty} sech\left[\frac{\left(2n-1\right)\pi K'}{2K}\right] sin\left[\frac{\left(2n-1\right)\pi u_i}{2K}\right] sin\left[\frac{\left(2n-1\right)\pi\omega_0 t}{2K\left(\kappa\right)}\right] \tag{A.1}$$

$$u_y^i\left(t\right) = \frac{2\pi}{m_0 K\left(\kappa\right)}\sum_{n=1}^{\infty} sech\left[\frac{\left(2n-1\right)\pi K'}{2K}\right] cos\left[\frac{\left(2n-1\right)\pi u_i}{2K}\right] cos\left[\frac{\left(2n-1\right)\pi\omega_0 t}{2K\left(\kappa\right)}\right] \tag{A.2}$$

where $i=e$ for electrons and $i=h$ for holes. For open surfaces the corresponding results are

$$u_x^o\left(t\right) = \frac{2\pi\kappa}{m_0 K\left(1/\kappa\right)}\sum_{n=1}^{\infty} sech\left[\frac{n\pi K'}{K}\right] sin\left[\frac{n\pi u_o}{K}\right] sin\left[\frac{n\pi\kappa\omega_0 t}{K\left(1/\kappa\right)}\right] \tag{A.3}$$

$$u_y^o\left(t\right) = \frac{2\pi\kappa}{m_0 K\left(1/\kappa\right)}\left\{\frac{1}{2}+\sum_{n=1}^{\infty} sech\left[\frac{n\pi K'}{K}\right] cos\left[\frac{n\pi u_o}{K}\right] cos\left[\frac{n\pi\kappa\omega_0 t}{K\left(1/\kappa\right)}\right]\right\} \tag{A.4}$$

For electron and hole pockets we can write for simplicity:

$$u_x^i\left(t\right) = \tilde{u}_x\sum_{n=1}^{\infty} a_n^x sin\left[\frac{\left(2n-1\right)\pi\omega_0 t}{2K\left(\kappa\right)}\right] \tag{A.5}$$

$$u_y^i\left(t\right) = \tilde{u}_y\sum_{n=1}^{\infty} a_n^y cos\left[\frac{\left(2n-1\right)\pi\omega_0 t}{2K\left(\kappa\right)}\right] \tag{A.6}$$

The previous equations are also true for open surfaces. But in the case of $u_y\left(t\right)$ we should write

$$u_x^o\left(t\right) = \tilde{u}_x\sum_{n=1}^{\infty} a_n^x sin\left[\frac{n\pi\kappa\omega_0 t}{K\left(1/\kappa\right)}\right] \tag{A.7}$$

$$u_y^o\left(t\right) = \tilde{u}_y\left\{\frac{1}{2}+\sum_{n=1}^{\infty} a_n^y cos\left[\frac{n\pi\kappa\omega_0 t}{K\left(1/\kappa\right)}\right]\right\} \tag{A.8}$$

We can now use the above equations to calculate the σ components for all 3 cases. We begin by calculating the σ components for closed Fermi surfaces. For convenience we define $\widetilde{\eta}\left(t;t'\right)\equiv\exp\left(-\left[1/\tau+i\omega\right]\left(t-t'\right)\right)$.

$$\sigma_{xy} = \frac{e^3 B}{\left(2\pi\right)^2}\int_0^{4K/\omega_0}\tilde{u}_x\sum_{n=1}^{\infty} a_n^x sin\left[\frac{\left(2n-1\right)\pi\omega_0 t}{2K\left(\kappa\right)}\right] dt\int_{-\infty}^{t}\tilde{u}_y\sum_{m=1}^{\infty} a_m^y cos\left[\frac{\left(2m-1\right)\pi\omega_0 t'}{2K\left(\kappa\right)}\right]\widetilde{\eta}\left(t;t'\right)dt' =$$

$$= \frac{e^3 B}{4\pi}\tilde{u}_x\tilde{u}_y\sum_{n=1}^{\infty} a_n^x a_n^y\frac{2n-1}{\left[1/\tau+i\omega\right]^2+\left[\frac{\left(2n-1\right)\pi\omega_0}{2K\left(\kappa\right)}\right]^2} \tag{A.9}$$

$$\sigma_{xx} = \frac{e^3 B}{(2\pi)^2} \int\limits_0^{4K/\omega_0} \tilde{u}_x \sum_{n=1}^{\infty} a_n^x sin\left[\frac{(2n-1)\,\pi\omega_0 t}{2K(\kappa)}\right] dt \int\limits_{-\infty}^{t} \tilde{u}_x \sum_{m=1}^{\infty} a_m^x sin\left[\frac{(2m-1)\,\pi\omega_0 t'}{2K(\kappa)}\right] \tilde{\eta}(t;t')\, dt' =$$

$$= \frac{e^3 B}{(2\pi)^2} \tilde{u}_x^2 \sum_{n=1}^{\infty} (a_n^x)^2 \frac{[1/\tau + i\omega]}{[1/\tau + i\omega]^2 + \left[\frac{(2n-1)\pi\omega_0}{2K(\kappa)}\right]^2} \frac{2K(\kappa)}{\omega_0} \tag{A.10}$$

$$\sigma_{yy} = \frac{e^3 B}{(2\pi)^2} \int\limits_0^{4K/\omega_0} \tilde{u}_y \sum_{n=1}^{\infty} a_n^y cos\left[\frac{(2n-1)\,\pi\omega_0 t}{2K(\kappa)}\right] dt \int\limits_{-\infty}^{t} \tilde{u}_y \sum_{m=1}^{\infty} a_m^y cos\left[\frac{(2m-1)\,\pi\omega_0 t'}{2K(\kappa)}\right] \tilde{\eta}(t;t')\, dt' =$$

$$= \frac{e^3 B}{(2\pi)^2} \tilde{u}_y^2 \sum_{n=1}^{\infty} (a_n^y)^2 \frac{[1/\tau + i\omega]}{[1/\tau + i\omega]^2 + \left[\frac{(2n-1)\pi\omega_0}{2K(\kappa)}\right]^2} \frac{2K(\kappa)}{\omega_0} \tag{A.11}$$

Similarly, for open surfaces we have that

$$\sigma_{xx} = \frac{e^3 B}{(2\pi)^2} \int\limits_0^{4K/\kappa\omega_0} \tilde{u}_x \sum_{n=1}^{\infty} a_n^x sin\left[\frac{n\pi\kappa\omega_0 t}{K(1/\kappa)}\right] dt \int\limits_{-\infty}^{t} \tilde{u}_x \sum_{m=1}^{\infty} a_m^x sin\left[\frac{m\pi\kappa\omega_0 t}{K(1/\kappa)}\right] \tilde{\eta}(t;t')\, dt' =$$

$$= \frac{e^3 B}{(2\pi)^2} \tilde{u}_x^2 \sum_{n=1}^{\infty} (a_n^x)^2 \frac{[1/\tau + i\omega]}{[1/\tau + i\omega]^2 + \left[\frac{n\pi\kappa\omega_0}{K(1/\kappa)}\right]^2} \frac{K(1/\kappa)}{\kappa\omega_0} \tag{A.12}$$

$$\sigma_{xy} = \frac{e^3 B}{(2\pi)^2} \int\limits_0^{4K/\kappa\omega_0} \tilde{u}_x \sum_{n=1}^{\infty} a_n^x sin\left[\frac{n\pi\kappa\omega_0 t}{K(1/\kappa)}\right] dt \int\limits_{-\infty}^{t} \tilde{u}_y \left\{\frac{1}{2} + \sum_{m=1}^{\infty} a_m^y cos\left[\frac{m\pi\kappa\omega_0 t}{K(1/\kappa)}\right]\right\} \tilde{\eta}(t;t')\, dt' =$$

$$= \frac{e^3 B}{(2\pi)^2} \tilde{u}_x \tilde{u}_y \sum_{n=1}^{\infty} a_n^x a_n^y \frac{n\pi}{[1/\tau + i\omega]^2 + \left[\frac{n\pi\kappa\omega_0}{K(1/\kappa)}\right]^2} \tag{A.13}$$

$$\sigma_{yy} = \frac{e^3 B}{(2\pi)^2} \int\limits_0^{4K/\kappa\omega_0} \tilde{u}_y \left\{\frac{1}{2} + \sum_{n=1}^{\infty} a_n^y cos\left[\frac{n\pi\kappa\omega_0 t}{K(1/\kappa)}\right]\right\} dt \int\limits_{-\infty}^{t} \tilde{u}_y \left\{\frac{1}{2} + \sum_{m=1}^{\infty} a_m^y cos\left[\frac{m\pi\kappa\omega_0 t}{K(1/\kappa)}\right]\right\} \tilde{\eta}(t;t')\, dt' =$$

$$= \frac{e^3 B}{(2\pi)^2} \frac{\tilde{u}_y^2}{[1/\tau + i\omega]} \frac{K(1/\kappa)}{\kappa\omega_0} + \frac{e^3 B}{(2\pi)^2} \tilde{u}_y^2 \sum_{n,1}^{\infty} (a_n^y)^2 \frac{[1/\tau + i\omega]}{[1/\tau + i\omega]^2 + \left[\frac{n\pi\kappa\omega_0}{K(1/\kappa)}\right]^2} \frac{K(1/\kappa)}{\kappa\omega_0}$$

Substituting the appropriate $\tilde{u}_x$,$\tilde{u}_y$,a_n^x,a_n^y, equations (1.28), (1.29), (1.30) follow. It also needs to be pointed out that for closed Fermi surfaces the sums are over positive odd integers while for open surfaces the sums are over positive even integers. We can now proceed to calculate the hall number for both high and low magnetic fields. For high magnetic fields we have that

$$\frac{1}{n_{Hall}} = \lim_{B\to\infty} \frac{1}{B} \frac{-\sigma_{xy}}{\sigma_{xx}\sigma_{yy} + \sigma_{xy}^2} \tag{A.14}$$

Using the above equation we get

$$n^e_{Hall} = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{\left(n-\frac{1}{2}\right)}sech^2\left[\left(n-\frac{1}{2}\right)\frac{\pi K'(\kappa)}{K(\kappa)}\right]sin\left[(2n-1)\frac{\pi u_e}{K(\kappa)}\right] \tag{A.15}$$

$$n^h_{Hall} = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{\left(n-\frac{1}{2}\right)}sech^2\left[\left(n-\frac{1}{2}\right)\frac{\pi K'(\kappa)}{K(\kappa)}\right]sin\left[(2n-1)\frac{\pi u_h}{K(\kappa)}\right] \tag{A.16}$$

$$n^o_{Hall} = -\frac{1}{\pi}\frac{\sum_{n=1}^{\infty}\frac{1}{n^2}sech^2\left[\frac{n\pi K'(1/\kappa)}{K(1/\kappa)}\right]sin^2\left[\frac{n\pi u_o}{K(1/\kappa)}\right]}{\sum_{n=1}^{\infty}\frac{1}{2n}sech^2\left[\frac{n\pi K'(1/\kappa)}{K(1/\kappa)}\right]sin\left[\frac{2n\pi u_o}{K(1/\kappa)}\right]} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}sech^2\left[\frac{n\pi K'(1/\kappa)}{K(1/\kappa)}\right]sin\left[\frac{2n\pi u_o}{K(1/\kappa)}\right] \tag{A.17}$$

We observe that in the high-field case the hall number does not depend on the frequency. Now for the low-field case we have

$$\frac{1}{n_{Hall}} = \lim_{B\to 0}\frac{1}{B}\frac{-\sigma_{xy}}{\sigma_{xx}\sigma_{yy}+\sigma_{xy}^2} \tag{A.18}$$

Similarly we get:

$$n^e_{Hall} = -\frac{4}{\pi}\frac{\left[\sum_{n=1}^{\infty}sech^2\left[\left(n-\frac{1}{2}\right)\frac{\pi K'}{K}\right]sin^2\left[\left(n-\frac{1}{2}\right)\frac{\pi u_e}{K}\right]\right]\left[\sum_{n=1}^{\infty}sech^2\left[\left(n-\frac{1}{2}\right)\frac{\pi K'}{K}\right]cos^2\left[\left(n-\frac{1}{2}\right)\frac{\pi u_e}{K}\right]\right]}{\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right)sech^2\left[\left(n-\frac{1}{2}\right)\frac{\pi K'}{K}\right]sin\left[\frac{(2n-1)\pi u_e}{K}\right]} \tag{A.19}$$

$$n^h_{Hall} = \frac{4}{\pi}\frac{\left[\sum_{n=1}^{\infty}sech^2\left[\left(n-\frac{1}{2}\right)\frac{\pi K'}{K}\right]sin^2\left[\left(n-\frac{1}{2}\right)\frac{\pi u_h}{K}\right]\right]\left[\sum_{n=1}^{\infty}sech^2\left[\left(n-\frac{1}{2}\right)\frac{\pi K'}{K}\right]cos^2\left[\left(n-\frac{1}{2}\right)\frac{\pi u_h}{K}\right]\right]}{\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right)sech^2\left[\left(n-\frac{1}{2}\right)\frac{\pi K'}{K}\right]sin\left[\frac{(2n-1)\pi u_h}{K}\right]} \tag{A.20}$$

$$n^o_{Hall} = -\frac{4}{\pi}\frac{\left[\sum_{n=1}^{\infty}sech^2\left[\frac{n\pi K'}{K}\right]sin^2\left[\frac{n\pi u_o}{K}\right]\right]\left[\frac{1}{2}+\sum_{n=1}^{\infty}sech^2\left[\frac{n\pi K'}{K}\right]cos^2\left[\frac{n\pi u_o}{K}\right]\right]}{\sum_{n=1}^{\infty}n\,sech^2\left[\frac{n\pi K'}{K}\right]sin\left[\frac{2n\pi u_o}{K}\right]} \tag{A.21}$$

The above results for the hall coefficient are the same with the ones calculated in [10], both in the high and the low magnetic field limits.

Appendix B

In this Section we will convert the current-current to current-density correlations. We will use the following

$$\left[c_a^\dagger c_b, c_c^\dagger c_d\right] = c_a^\dagger c_d \delta_{bc} - c_c^\dagger c_b \delta_{ad} \tag{B.1}$$

We also define:

$$\rho_q \equiv \sum_{l,m,\sigma} e^{iqm} c_{l,m,\sigma}^\dagger c_{l,m,\sigma} \tag{B.2}$$

We have that

$$\left[j_q^y, \rho_{-q}\right] = t' \sum_{l,m,\sigma,l',m',\sigma'} i e^{iq(m+1/2)} e^{-iqm'} \left\{ c_{l,m+1,\sigma}^\dagger c_{l',m',\sigma'} \delta_{l,l'} \delta_{m,m'} \delta_{\sigma,\sigma'} - c_{l',m',\sigma'}^\dagger c_{l,m,\sigma} \delta_{l,l'} \delta_{m+1,m'} \delta_{\sigma,\sigma'} + H.c. \right\}$$

$$+ t'_{SO} \sum_{l,m,l',m',\sigma'} i e^{iq(m+1/2)} e^{-iqm'} \{ i c_{l,m+1,\downarrow}^\dagger c_{l',m',\sigma'} \delta_{l,l'} \delta_{m,m'} \delta_{\uparrow,\sigma'} - i c_{l',m',\sigma'}^\dagger c_{l,m,\uparrow} \delta_{l,l'} \delta_{m+1,m'} \delta_{\downarrow,\sigma'} +$$

$$+ i c_{l,m+1,\uparrow}^\dagger c_{l',m',\sigma'} \delta_{l,l'} \delta_{m,m'} \delta_{\downarrow,\sigma'} - i c_{l',m',\sigma'}^\dagger c_{l,m,\downarrow} \delta_{l,l'} \delta_{m+1,m'} \delta_{\uparrow,\sigma'} + H.c. \} =$$

$$= t' \sum_{l,m,\sigma} i \left(e^{iq/2} - e^{-iq/2}\right) \left(c_{l,m+1,\sigma}^\dagger c_{l,m,\sigma} + H.c.\right)$$

$$+ t'_{SO} \sum_{l,m} i \left(e^{iq/2} - e^{-iq/2}\right) \left(\left[i c_{l,m+1,\downarrow}^\dagger c_{l,m,\uparrow} + i c_{l,m+1,\uparrow}^\dagger c_{l,m,\downarrow} \right] + H.c. \right)$$

We can now define the quantity

$$T^y \equiv -t' \sum_{l,m,\sigma} \left(c_{l,m+1,\sigma}^\dagger c_{l,m,\sigma} + H.c.\right) - t'_{SO} \sum_{l,m} \left(\left[i c_{l,m+1,\downarrow}^\dagger c_{l,m,\uparrow} + i c_{l,m+1,\uparrow}^\dagger c_{l,m,\downarrow} \right] + H.c. \right) \tag{B.3}$$

and since we are working with small q one can observe that

$$\sigma_{j_q^y j_q^y} = \frac{i}{z} \left(\langle -T^y \rangle - \chi_{j_q^y j_q^y} \right) \tag{B.4}$$

Finally we can conclude that:

$$\left[j_q^y, \rho_{-q}\right] = 2 sin\left(q/2\right) T^y \simeq q T^y \tag{B.5}$$

Similarly

$$[H, \rho_q] = -t \sum_{l,m,\sigma,l',m',\sigma'} e^{iqm'} \left\{ e^{i\phi^x(t)} e^{iA_m} \left(c_{l+1,m,\sigma}^\dagger c_{l',m',\sigma'} \delta_{l,l'} \delta_{m,m'} \delta_{\sigma,\sigma'} - c_{l',m',\sigma'}^\dagger c_{l,m,\sigma} \delta_{l+1,l'} \delta_{m,m'} \delta_{\sigma,\sigma'} \right) - H.c. \right\}$$

$$-t' \sum_{l,m,\sigma,l',m',\sigma'} e^{iqm'} \left\{ e^{i\phi^y_{m+1/2}(t)} \left(c_{l,m+1,\sigma}^\dagger c_{l',m',\sigma'} \delta_{l,l'} \delta_{m,m'} \delta_{\sigma,\sigma'} - c_{l',m',\sigma'}^\dagger c_{l,m,\sigma} \delta_{l,l'} \delta_{m+1,m'} \delta_{\sigma,\sigma'} \right) - H.c. \right\}$$

$$- t_{SO} \sum_{l,m,l',m',\sigma'} e^{iqm'} \{ e^{i\phi^x(t)} e^{iA_m} (c_{l+1,m,\downarrow}^\dagger c_{l',m',\sigma'} \delta_{l,l'} \delta_{m,m'} \delta_{\uparrow,\sigma'} - c_{l',m',\sigma'}^\dagger c_{l,m,\uparrow} \delta_{l+1,l'} \delta_{m,m'} \delta_{\downarrow,\sigma'} -$$

$$- c_{l+1,m,\uparrow}^\dagger c_{l',m',\sigma'} \delta_{l,l'} \delta_{m,m'} \delta_{\downarrow,\sigma'} + c_{l',m',\sigma'}^\dagger c_{l,m,\downarrow} \delta_{l+1,l'} \delta_{m,m'} \delta_{\uparrow,\sigma'}) - H.c. \}$$

$$- t'_{SO} \sum_{l,m,l',m',\sigma'} e^{iqm'} \{ e^{i\phi^y_{m+1/2}(t)} (ic^\dagger_{l,m+1,\downarrow} c_{l',m',\sigma'} \delta_{l,l'} \delta_{m,m'} \delta_{\uparrow,\sigma'} - ic^\dagger_{l',m',\sigma'} c_{l,m,\uparrow} \delta_{l,l'} \delta_{m+1,m'} \delta_{\downarrow,\sigma'} +$$

$$+ ic^\dagger_{l,m+1,\uparrow} c_{l',m',\sigma'} \delta_{l,l'} \delta_{m,m'} \delta_{\downarrow,\sigma'} - ic^\dagger_{l',m',\sigma'} c_{l,m,\downarrow} \delta_{l,l'} \delta_{m+1,m'} \delta_{\uparrow,\sigma'}) - H.c.\}$$

We see that we have no contribution from the x component. Using the fact that $\phi^{x,y}(t) \longrightarrow 0$ we get.

$$[H, \rho_q] = -t' \sum_{l,m,\sigma,l',m',\sigma'} e^{iqm'} \left\{ c^\dagger_{l,m+1,\sigma} c_{l',m',\sigma'} \delta_{l,l'} \delta_{m,m'} \delta_{\sigma,\sigma'} - c^\dagger_{l',m',\sigma'} c_{l,m,\sigma} \delta_{l,l'} \delta_{m+1,m'} \delta_{\sigma,\sigma'} - H.c. \right\}$$

$$- t'_{SO} \sum_{l,m,l',m',\sigma'} e^{iqm'} \{ ic^\dagger_{l,m+1,\downarrow} c_{l',m',\sigma'} \delta_{l,l'} \delta_{m,m'} \delta_{\uparrow,\sigma'} - ic^\dagger_{l',m',\sigma'} c_{l,m,\uparrow} \delta_{l,l'} \delta_{m+1,m'} \delta_{\downarrow,\sigma'} +$$

$$+ ic^\dagger_{l,m+1,\uparrow} c_{l',m',\sigma'} \delta_{l,l'} \delta_{m,m'} \delta_{\downarrow,\sigma'} - ic^\dagger_{l',m',\sigma'} c_{l,m,\downarrow} \delta_{l,l'} \delta_{m+1,m'} \delta_{\uparrow,\sigma'} - H.c.\} =$$

$$= -t' \sum_{l,m,\sigma} \left(e^{iqm} - e^{iq(m+1)} \right) \left\{ c^\dagger_{l,m+1,\sigma} c_{l,m,\sigma} - H.c. \right\}$$

$$-t'_{SO} \sum_{l,m} \left(e^{iqm} - e^{iq(m+1)} \right) \left\{ ic^\dagger_{l,m+1,\downarrow} c_{l,m,\uparrow} + ic^\dagger_{l,m+1,\uparrow} c_{l,m,\downarrow} - H.c. \right\}$$

It is easy to see that

$$[H, \rho_q] = 2sin\,(q/2)\, j^y_q \simeq q j^y_q \Longrightarrow \frac{\partial \rho_q}{\partial t} = iq j^y_q \tag{B.6}$$

In a similar way, we can trivially show that

$$[\rho_q, \rho_{-q}] = 0, \quad [j^x, \rho_q] = 0 \tag{B.7}$$

We now start converting the current-current to current-density correlations. We have:

$$\chi_{j^x j^y_q} = i \int_0^\infty dt e^{izt} \left\langle \left[j^x(t), j^y_q \right] \right\rangle = i \int_0^\infty dt e^{izt} \left\langle \left[j^x, j^y_q(-t) \right] \right\rangle =$$

$$= -\frac{1}{q} e^{izt} \left\langle [j^x, \rho_q(-t)] \right\rangle |_0^\infty + \frac{iz}{q} \int_0^\infty dt e^{izt} \left\langle [j^x, \rho_q(-t)] \right\rangle = \frac{iz}{q} \int_0^\infty dt e^{izt} \left\langle [j^x(t), \rho_q(t)] \right\rangle = \frac{z}{q} \chi_{j^x \rho_q}$$

Similarly

$$\chi_{j^y_q j^x} = i \int_0^\infty dt e^{izt} \left\langle \left[j^y_q(t), j^x \right] \right\rangle = \frac{1}{q} e^{izt} \left\langle [\rho_q(t), j^x] \right\rangle |_0^\infty - \frac{iz}{q} \int_0^\infty dt e^{izt} \left\langle [\rho_q(t), j^x] \right\rangle = -\frac{z}{q} \chi_{\rho_q j^x}$$

$$\chi_{j^y_q j^y_q} = i \int_0^\infty dt e^{izt} \left\langle \left[j^y_q(t), j^y_{-q} \right] \right\rangle = \frac{1}{q} e^{izt} \left\langle \left[\rho_q(t), j^y_{-q} \right] \right\rangle |_0^\infty - \frac{iz}{q} \int_0^\infty dt e^{izt} \left\langle \left[\rho_q(t), j^y_{-q} \right] \right\rangle =$$

$$= \langle -T^y \rangle - \frac{iz}{q} \int_0^\infty dt e^{izt} \left\langle \left[\rho_q, j_{-q}^y (-t) \right] \right\rangle = \langle -T^y \rangle - \frac{z}{q^2} e^{izt} \langle [\rho_q, \rho_{-q} (-t)] \rangle \,|_0^\infty + i \left(\frac{z}{q} \right)^2 \int_0^\infty dt e^{izt} \langle [\rho_q, \rho_{-q} (-t)] \rangle$$

$$= \langle -T^y \rangle + i \left(\frac{z}{q} \right)^2 \int_0^\infty dt e^{izt} \langle [\rho_q (t), \rho_{-q}] \rangle = \langle -T^y \rangle + \left(\frac{z}{q} \right)^2 \chi_{\rho_q \rho_q}$$

Instead of ρ_q we are going to use

$$n_q = -i\rho_q = \sum_{l,m,\sigma} \left(-ie^{iqm} \right) c_{l,m,\sigma}^\dagger c_{l,m,\sigma} \tag{B.8}$$

We finally get

$$\chi_{j^x j_q^y} = -\chi_{j_q^y j^x} = \frac{iz}{q} \chi_{j^x n_q} \Rightarrow \sigma_{j^x j_q^y} = -\sigma_{j_q^y j^x} = \frac{1}{q} \chi_{j^x n_q} \tag{B.9}$$

$$\chi_{j_q^y j_q^y} = \langle -T^y \rangle - \left(\frac{z}{q} \right)^2 \chi_{n_q n_q} \Rightarrow \sigma_{j_q^y j_q^y} = \frac{i}{z} \left(\frac{z}{q} \right)^2 \chi_{n_q n_q} \tag{B.10}$$

With the above convertion equations 25 and 26 become

$$\langle J^x \rangle = \sigma_{j^x j^x} E^x (t) + \frac{1}{q} \chi_{j^x n_q} E^y (t) \tag{B.11}$$

$$\left\langle J_q^y \right\rangle = -\frac{1}{q} \chi_{j^x n_q} E^x (t) + \frac{i}{z} \left(\frac{z}{q} \right)^2 \chi_{n_q n_q} E^y (t) \tag{B.12}$$

Appendix C

Let us consider a Hamiltonian depending on two parameters μ, λ, with a ground state energy ε^0.

$$\frac{\partial \varepsilon^0}{\partial \lambda} = \frac{\partial}{\partial \lambda} \langle 0 | H(\mu, \lambda) | 0 \rangle = \left\langle 0 \left| \frac{\partial H}{\partial \lambda} \right| 0 \right\rangle + \left(\frac{\partial}{\partial \lambda} \langle 0 | \right) H | 0 \rangle + \langle 0 | H \left(\frac{\partial}{\partial \lambda} | 0 \rangle \right) =$$

$$\left\langle 0 \left| \frac{\partial H}{\partial \lambda} \right| 0 \right\rangle + \varepsilon^0 \left(\frac{\partial}{\partial \lambda} \langle 0 \mid 0 \rangle \right) = \left\langle 0 \left| \frac{\partial H}{\partial \lambda} \right| 0 \right\rangle \tag{C.1}$$

We proceed to calculate the second derivative.

$$H|0\rangle = \varepsilon^0 |0\rangle \Rightarrow \frac{\partial H}{\partial \mu}|0\rangle + H\left(\frac{\partial}{\partial \mu}|0\rangle\right) = \frac{\partial \varepsilon^0}{\partial \mu}|0\rangle + \varepsilon^0 \left(\frac{\partial}{\partial \mu}|0\rangle\right) \Rightarrow$$

$$\overset{n\neq 0}{\Rightarrow} \left\langle n \left| \frac{\partial H}{\partial \mu} \right| 0 \right\rangle + E_n \langle n | \left(\frac{\partial}{\partial \mu} | 0 \rangle \right) = E_0 \langle n | \left(\frac{\partial}{\partial \mu} | 0 \rangle \right) \Rightarrow$$

$$\langle n | \left(\frac{\partial}{\partial \mu} | 0 \rangle \right) = - \frac{\left\langle n \left| \frac{\partial H}{\partial \mu} \right| 0 \right\rangle}{E_n - E_0} \; and \; \left(\frac{\partial}{\partial \mu} \langle 0 | \right) | n \rangle = - \frac{\left\langle 0 \left| \frac{\partial H}{\partial \mu} \right| n \right\rangle}{E_n - E_0} \tag{C.2}$$

$$\varepsilon^0_{\mu\lambda} = \frac{\partial^2 \varepsilon^0}{\partial \mu \partial \lambda} = \frac{\partial}{\partial \mu} \left\langle 0 \left| \frac{\partial H}{\partial \lambda} \right| 0 \right\rangle = \left\langle 0 \left| \frac{\partial^2 H}{\partial \mu \partial \lambda} \right| 0 \right\rangle + \left(\frac{\partial}{\partial \mu} \langle 0 | \right) \frac{\partial H}{\partial \lambda} | 0 \rangle + \langle 0 | \frac{\partial H}{\partial \lambda} \left(\frac{\partial}{\partial \mu} | 0 \rangle \right) =$$

$$= \left\langle 0 \left| \frac{\partial^2 H}{\partial \mu \partial \lambda} \right| 0 \right\rangle + \sum_n \left[\left(\frac{\partial}{\partial \mu} \langle 0 | \right) | n \rangle \left\langle n \left| \frac{\partial H}{\partial \lambda} \right| 0 \right\rangle + \left\langle 0 \left| \frac{\partial H}{\partial \lambda} \right| n \right\rangle \langle n | \left(\frac{\partial}{\partial \mu} | 0 \rangle \right) \right]$$

For the $n = 0$ term we have

$$\left(\frac{\partial}{\partial \mu} \langle 0 | \right) | 0 \rangle \left\langle 0 \left| \frac{\partial H}{\partial \lambda} \right| 0 \right\rangle + \left\langle 0 \left| \frac{\partial H}{\partial \lambda} \right| 0 \right\rangle \langle 0 | \left(\frac{\partial}{\partial \mu} | 0 \rangle \right) =$$

$$\left\langle 0 \left| \frac{\partial H}{\partial \lambda} \right| 0 \right\rangle \left[\left(\frac{\partial}{\partial \mu} \langle 0 | \right) | 0 \rangle + \langle 0 | \left(\frac{\partial}{\partial \mu} | 0 \rangle \right) \right] = \left\langle 0 \left| \frac{\partial H}{\partial \lambda} \right| 0 \right\rangle \frac{\partial}{\partial \mu} \langle 0 \mid 0 \rangle = 0$$

Since we have no contribution from $n = 0$ we can write

$$\varepsilon^0_{\mu\lambda} = \left\langle 0 \left| \frac{\partial^2 H}{\partial \mu \partial \lambda} \right| 0 \right\rangle + \sum_{n>0} \left[\left(\frac{\partial}{\partial \mu} \langle 0 | \right) | n \rangle \left\langle n \left| \frac{\partial H}{\partial \lambda} \right| 0 \right\rangle + \left\langle 0 \left| \frac{\partial H}{\partial \lambda} \right| n \right\rangle \langle n | \left(\frac{\partial}{\partial \mu} | 0 \rangle \right) \right] \Rightarrow$$

$$\varepsilon^0_{\mu\lambda} = \left\langle 0 \left| \frac{\partial^2 H}{\partial \mu \partial \lambda} \right| 0 \right\rangle - \sum_{n>0} \left[\frac{\left\langle 0 \left| \frac{\partial H}{\partial \mu} \right| n \right\rangle \left\langle n \left| \frac{\partial H}{\partial \lambda} \right| 0 \right\rangle + H.c.}{E_n - E_0} \right] \tag{C.3}$$

We will now define the following fictitious Hamiltonian with parameters ϕ^x and μ_q:

$$H(\phi^x, \mu_q) = -t \sum_{l,m,\sigma} e^{i\phi^x(t)} e^{iA_m} c^\dagger_{l+1,m,\sigma} c_{l,m,\sigma} + H.c. - t' \sum_{l,m,\sigma} c^\dagger_{l,m+1,\sigma} c_{l,m,\sigma} + H.c.$$

$$-t_{SO} \sum_{l,m} e^{i\phi^x(t)} e^{iA_m} \left[c^\dagger_{l+1,m,\downarrow} c_{l,m,\uparrow} - c^\dagger_{l+1,m,\uparrow} c_{l,m,\downarrow} \right] + H.c.$$

$$- t'_{SO} \sum_{l,m} \left[i c^\dagger_{l,m+1,\downarrow} c_{l,m,\uparrow} + i c^\dagger_{l,m+1,\uparrow} c_{l,m,\downarrow} \right] + H.c. + \mu_q n_q \tag{C.4}$$

We can now express the terms of Eqs. (3.20) and (3.21) as derivatives of the ground state of this Hamiltonian. Specifically:

$$\sigma_{j^x j^x}(\omega \to 0) = \frac{i}{\omega}\varepsilon^0_{\phi^x \phi^x} \tag{C.5}$$

$$\chi_{j^x n_q}(\omega \to 0) = \varepsilon^0_{\phi^x \mu_q} \tag{C.6}$$

$$\chi_{n_q j^x}(\omega \to 0) = \varepsilon^0_{\mu_q \phi^x} \tag{C.7}$$

$$\chi_{n_q n_q}(\omega \to 0) = -\varepsilon^0_{\mu_q \mu_q} \tag{C.8}$$

Using the above results, Eqs. (3.25) and (3.26) follow.

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