GRAVITATIONAL WAVES IN A "GRAVITY-DRIVEN" COSMOLOGICAL MODEL

A thesis presented

by

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Table of Contents

1 Prologue

The purpose of this thesis is to analyze the behaviour of gravitational waves generated around the last stage of the inflationary era in a purely gravitational cosmological model. In Sections 2 and 3 we make a very brief review of the gravitational and cosmological basics needed for our purpose. Section 4 provides the physical motivation for considering such models; it also describes the construction of a simple "gravity-driven" cosmological model. Section 5 is the main part of the thesis and examines the spectrum of the gravitational waves produced.

I would like to thank my advisor, Prof. Nikolaos Tsamis, for suggesting the problem and his guidance, Prof. Richard Woodard, and Prof. Georgios Kofinas.

2 The Background: Gravitation

Gravity is the dominant force on large scales and plays the dominant role in shaping the large scale structure of the universe, although is by far the weakest of all. The fundamental interactions are divided into four classes: the strong and weak nuclear interactions, electromagnetism and gravity. Of these, the strong and weak nuclear interactions have a very short range $\sim 10^{-13} cm$. Although electromagnetism is a long-range interaction and the ratio of the gravitational to the electric force between two electrons is about 10^{-40}, for macroscopic bodies the repulsion of like charges is very nearly balanced by the attraction of opposite charges. On the other hand, gravity appears to be always attractive. Thus, the gravitational field for sufficiently large bodies dominates over all other forces, because it is a force which affects every particle in the same way. Since Cosmology is the scientific study of the large scale properties of the universe, gravity plays a very important role for Cosmology.

- **The General Theory of Relativity**

In 1916 Einstein developed his General Theory of Relativity which he proposed as a new theory of gravity. General Relativity generalizes Newton's theory of gravity, which is valid only for bodies at rest or moving very slow compared to the speed of light. General Relativity, on the other hand, is valid for bodies in motion as well as bodies at rest. The key idea of Einstein's theory is that gravity can be described by a field directly connected with distortions of space and time itself. In one sentence, matter tells space how to curve, and space tells matter how to move. The theory was able to account, for instance, for peculiarities in the orbit of Mercury and the bending of light by the Sun, both unexplained by Newton's theory of gravity.

- **The Principle of Equivalence**

The Principle of Equivalence of Gravitation and Inertia is based on the equality of gravitational and inertial mass and tell us how an arbitrary physical system responds to an external gravitational field. Einstein in his classic elevator thought experiment said that no external static, homogeneous gravitational field could be detected in a freely falling elevator, because the observer and the elevator itself would respond to the field with the same acceleration. This cancellation of gravitational by inertial forces is obtained for *all* freely falling systems.

Therefore, the Equivalence Principle states that at every spacetime point in an arbitrary gravitational field it is possible to choose a locally inertial coordinate system such that the laws of nature take the same form as in an unaccelerated Cartesian coordinate system in the absence of gravitation. That is in a freely falling (non-rotating) system the laws of physics are those of Special Relativity. Although in an inhomogeneous or time dependent gravitational field, inertial forces do not exactly cancel gravitational forces, we can still expect an approximate cancellation if we restrict ourselves to such a small region of space and time that the field changes very litle over the region.

- **Dynamical Variables and Equations of Motion**

The dynamical variable of gravity is the symmetric tensor field $g_{\mu\nu}(x)$ – known as the metric – which describes the geometry of spacetime. It translates the coordinate labels of points $x^\mu = (ct, \mathbf{x})$ into physical distances and angles. For instance, the square of the distance between x^μ and an infinitesimally close point $x^\mu + dx^\mu$ is given by the invariant interval ds^2:

$$ds^2 \;\equiv\; g_{\mu\nu}(x)\, dx^\mu\, dx^\nu \quad . \tag{1}$$

The equations of motion can be derived from the Lagrangian density of the theory:

$$\mathcal{L}_{\rm GR} = \frac{1}{16\pi G}\Big(-2\Lambda + R\,\Big)\,\sqrt{-g} \quad , \tag{2}$$

where Λ is the cosmological constant and $R(x)$ the Ricci scalar. In general the theory, besides the purely gravitational sector (2) will contain a matter sector. In General Relativity, we describe the distribution of mass-energy in a covariant way by specifying a symmetric rank-2 stress-energy tensor $T^{\mu\nu}$, which acts as a source for the gravitational field. The stress-energy tensor $T^{\mu\nu}$ has the units of energy per volume and the physical meaning of its components is:

T^{00} : energy density
T^{i0} : momentum density in the i-direction
T^{0i} : energy flux in the i-direction
T^{ij} : rate of flow of the i-component of momentum flux in the j-direction

If $T_{\mu\nu}(x)$ is the matter stress-energy tensor, the equations of motion take the form:

$$G_{\mu\nu} \;\equiv\; R_{\mu\nu} \,-\, \frac{1}{2} g_{\mu\nu}\, R \;=\; -\Lambda\, g_{\mu\nu} \,+\, 8\pi G\, T_{\mu\nu} \quad . \tag{3}$$

Here $G_{\mu\nu}(x)$ is the Einstein tensor formed out of the Ricci tensor $R_{\mu\nu}(x)$ and Ricci scalar $R(x)$. The stress-energy tensor is covariantly conserved:

$$D_\mu \, T^{\mu\nu}(x) \;=\; 0 \;\; . \tag{4}$$

$\triangleright$ *Notation:* Hellenic indices take on spacetime values while Latin indices take on space values. Our metric tensor has spacelike signature $(- + + +)$ and our curvature tensor equals $R^\alpha{}_{\beta\mu\nu} \equiv \Gamma^\alpha{}_{\nu\beta,\mu} + \Gamma^\alpha{}_{\mu\rho} \Gamma^\rho{}_{\nu\beta} - (\mu \leftrightarrow \nu)$.

3 The Background: Cosmology

Cosmology is the scientific study of the large-scale properties of the universe as a whole. It strives to understand the origin, evolution and ultimate fate of the entire universe. It involves the construction of theories about the universe which make specific predictions for phenomena that can be tested with observations. Depending on the outcome of the observations the theories will need to be abandoned, revised or extended to accomodate the data. Cosmology rests on two ideas: General Relativity, which was already discussed, and the Cosmological Principle.

- **The Cosmological Principle**

The most important property of the universe is that it is homogeneous and isotropic on large scales. This property is known as the Cosmological Principle; it ignores local features and assumes that the universe does not change in moving between spatial points at the same time or by looking in any special direction. The first property is known as homogeneity; the second as isotropy.

The Cosmological Principle is the theoretical basis for Cosmology and leads to very specific predictions for observable properties of the universe. We can use it and General Relativity to obtain the corresponding distortion of spacetime, due to gravitational effects of the matter. With a simplifying assumption [1] the invariant interval of a homogenenous and isotropic universe can be written as:

$$ds^2 = -c^2 dt^2 + a^2(t)\, d\mathbf{x} \cdot d\mathbf{x} \quad . \tag{5}$$

It is apparent from (5) that t measures physical time the same way as in the Minkowski geometry. However, the spatial 3-vector $\mathbf{x}$ must be multiplied by $a(t)$ to give physical distances. For this reason $a(t)$ is known as the scale factor. A spacetime of this general form is called Friedmann-Robertson-

[1] We have regarded spatial curvature as a type of stress-energy as opposed to the proper way of viewing spatial curvature which is as an additional parameter in the homogeneous and isotropic metric (5). At any rate, the measured value of spatial curvature is consistent with zero.

Walker (FRW) spacetime and its components are:

$$g_{\mu\nu}(t, \mathbf{x}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix}$$

- **Fundamental Cosmological Parameters**

The time variation of the scale factor gives the instantaneous values of the *Hubble parameter* $H(t)$ – a measure of the cosmic expansion rate – and the *deceleration parameter* $q(t)$ – a measure of the cosmic acceleration:

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} = \frac{d}{dt} \ln a(t) \quad , \tag{6}$$

$$q(t) \equiv -\frac{\dot{a}(t)\,\ddot{a}(t)}{\dot{a}^2(t)} = -1 - \frac{\dot{H}(t)}{H^2(t)} \equiv -1 + \epsilon(t) \quad . \tag{7}$$

Various kinds of stress-energy comprise the content of the universe. For *any* such kind we define its dimensionless contribution Ω_{any} to the total content as: [2]

$$\Omega_{any} \equiv \frac{\rho_{any}}{\rho_{cr}}\Big|_{t=t_0} \qquad , \qquad \rho_{cr} \equiv \frac{3c^2 H_0^2}{8\pi G} \quad . \tag{8}$$

In (8) ρ_{any} is the energy density of the corresponding kind and ρ_{cr} is the critical density.

Gravity makes tiny inhomogeneities grow. It is believed that even the largest objects in today's universe had their origin in tiny quantum fluctuations of magnitude $\frac{\Delta\rho}{\rho} \simeq 10^{-5}$ which occured in the last stage of the inflationary era in the very early universe. The imprint of these fluctuations in the cosmic microwave background has been imaged with excellent accuracy by the WMAP satellite. The basic quantities that characterize this imprint are the dimensionless *scalar power spectrum* $\Delta^2_{\mathcal{R}}(k)$:

$$\Delta^2_{\mathcal{R}}(k) \simeq \frac{1}{\pi} \frac{GH^2(t_k)}{1 + q(t_k)} \quad , \tag{9}$$

and *tensor power spectrum* $\Delta^2_h(k)$:

$$\Delta^2_h(k) \simeq \frac{16}{\pi} GH^2(t_k) \quad , \tag{10}$$

[2] The subscript 0 in various parameters indicates their current values. Further details concerning basic cosmological variables can be found in the Appendix.

as well as the respective indices; the *scalar spectral index* n_s:

$$\Delta^2_{\mathcal{R}}(k) \ \equiv \ \Delta^2_{\mathcal{R}}(k_0) \times \left(\frac{k}{k_0}\right)^{n_s-1} \quad , \tag{11}$$

and the *tensor scalar index* n_t:

$$\Delta^2_{h}(k) \ \equiv \ \Delta^2_{h}(k_0) \times \left(\frac{k}{k_0}\right)^{n_t} \quad . \tag{12}$$

Here t_k is the time at which first horizon crossing occured for the wavenumber k. [3] The "reference" wavenumber $k_0 \equiv 0.002\,(Mpc)^{-1}$ corresponds to a wavelength of about half the size of the currently observed universe.

- **Dynamical Variables and Equations of Motion**

The homogeneity and isotropy restricted the metric tensor $g_{\mu\nu}(x)$ to only one function of time, the scale factor $a(t)$. Similarly, they restrict the matter stress-energy tensor to only an energy density $\rho(t)$ and a pressure $p(t)$:

$$T_{00} \ = \ -\rho(t)\, g_{00} \qquad , \qquad T_{0i} \ = \ 0 \qquad , \qquad T_{ij} \ = \ p(t)\, g_{ij} \ . \tag{13}$$

The dynamical variable is $a(t)$. In this geometry the evolution equations (3) take the form:

$$3H^2 \ = \ 8\pi G\, c^{-2}\, \rho \ , \tag{14}$$

$$-2\dot{H}^2 \ - \ 3H^2 \ = \ 8\pi G\, c^{-2}\, p \ , \tag{15}$$

The energy density and pressure are the sources; stress-energy conservation (4) gives a relation among them:

$$\dot{\rho} \ = \ -3H\,(\rho + p) \ . \tag{16}$$

- **Cosmological Evolution Epochs**

A constant equation of state $p(t) \equiv w\,\rho(t)$ suffices to describe the actual phases during the evolution of the universe. From the conservation equation (16) we can express the energy density in terms of the scale factor:

$$\rho(t) \ = \ \rho_I \left[\frac{a(t)}{a_I}\right]^{-3(1+w)} \quad , \tag{17}$$

[3] When $t = t_k$ we have: $k = H(t)\, a(t)$.

for some initial values ρ_I and a_I. We then substitute (17) in the equation of motion (14) to obtain as its solution:

$$a(t) \; = \; a_I \left[1 \, + \, \frac{3}{2} (1 + w) \, H_I \, (t - t_I) \right]^{\frac{2}{3(1+w)}} \quad . \tag{18}$$

The cosmologically relevant cases are:

Stress-Energy	w	Energy Density	Scale Factor
Radiation	$w = \frac{1}{3}$	$\rho \propto a^{-4}$	$a(t) \propto (H_I t)^{\frac{1}{2}}$
Non-Relativistic Matter	$w = 0$	$\rho \propto a^{-3}$	$a(t) \propto (H_I t)^{\frac{2}{3}}$
Spatial Curvature	$w = -\frac{1}{3}$	$\rho \propto a^{-2}$	$a(t) \propto H_I t$
Vacuum Energy	$w = -1$	$\rho \propto 1$	$a(t) \propto e^{H_I t}$

The actual universe seems to be composed of at least three of these types, so the actual scale factor does not have a simple time dependence. However, as long as each type is separately conserved, we can use (17) to conclude:

$$\rho(t) \; = \; \frac{\rho_{\rm rad}}{a^4(t)} \, + \, \frac{\rho_{\rm mat}}{a^3(t)} \, + \, \frac{\rho_{\rm cur}}{a^2(t)} \, + \, \rho_{\rm vac} \quad . \tag{19}$$

The cosmology in which a radiation dominated universe evolves to matter domination is a feature of what is known as the Big Bang scenario. Although strongly supported by observation, the composition of ρ at the start of radiation domination ($t = t_r$ and $a = a_r$) does not seem natural:

$$\rho_{rad} \, a_r^{-4} \; \gg \; \rho_{vac} \; \gg \; \rho_{cur} \, a_r^{-2} \quad . \tag{20}$$

The natural expectation would be for all three terms in (20) to be comparable in which case the universe would become quickly dominated by vacuum energy. There is no accepted explanation for the first inequality in (20). However, the second inequality in (20) finds a natural explanation in the context of inflation.

It was suggested that the Big Bang scenario was proceded by a period of vacuum energy domination, or inflation, following which the vacuum energy changed almost completely into radiation. If all types of stress-energy are equally represented at some very early time, we see from (19) that the total energy density rapidly becomes dominated by vacuum energy, following which the scale factor grows exponentially with a constant Hubble parameter H_I.

The duration of inflation in units of H_I^{-1} is known as the number of inflationary e-foldings N_I. Viable models must have $N_I \geq 50$ to address issues of causality. Furthermore, if at start of inflation $\rho_{\rm cur}\, a_I^{-2} \sim \rho_{\rm vac}$, equation (19) shows that the curvature is negligible at the end of inflation:

$$\frac{\rho_{\rm cur}\ a_r^{-2}}{\rho_{\rm vac}} \sim \left(\frac{a_I}{a_r}\right)^2 = e^{-2N_I} \leq 10^{-44} \ . \tag{21}$$

Inflation makes the other types of stress-energy even smaller and assumes the existence of a natural mechanism through which vacuum energy can be converted into radiation, a process known as reheating.

4 The Background: "Gravity-Driven" Cosmological Models

The standard inflationary cosmology is based on the assumption that the dynamical degree of freedom responsible for driving the inflationary phase is a scalar. Although this can be realized in a quite successful way it is by no means the simplest nor the most natural way for reasons that will be described later in this Section. In "gravity-driven" models of cosmological evolution the inflationary phase is due to the graviton – *the* particle associated with the cosmologically dominant force – and the conditions that govern evolution are *far more* natural than those of "scalar-driven" inflation. We now turn to the main physical steps that lead to the construction of a simple model with reasonable time evolution at least during the early part of the history of the universe.

- **Effective Field Theories Description**

Consider the effective four-dimensional gravitational theory that emerges from the full – and yet unknown – quantum gravitational theory when we restrict physical processes to scales well below the Planck scale. It is governed, among other things, by general coordinate invariance and can be written as a series of local terms of increasing dimensionality in the curvature. Of these terms, only the lowest dimensionality ones are relevant in the range of scales of interest: the cosmological constant and the Ricci scalar. On the quantum level this effective theory is not renormalizable but, nonetheless, is BPHZ renormalizable:

$$\mathcal{L}_{\rm QG} = \frac{1}{16\pi G}\Big(-2\Lambda + R\,\Big)\,\sqrt{-g}\;+\;({\rm counterterms})\;. \tag{22}$$

Each of the BPHZ counterterms contains an infinite and a finite part. The infinite parts are fixed by having to absorb the ultraviolet divergences that are generated order by order. Of the finite parts, only the lowest dimensionality ones are known and are fixed from the measured values of the expansion rate for Λ and the Newtonian force for G. All the remaining finite parts are unknown and can only be determined from the full theory. However, since cosmology is determined by the infrared sector of the theory – where only the lowest dimensionality finite parts dominate – it is insensitive to these unknown parts.

In the above gravitational effective theory and if Λ is assumed to be positive, the "no-hair" theorems imply that – classically – the local geometry approaches the maximally symmetric solution at late times. This solution is de Sitter spacetime and, thus, Λ-driven inflation is intrinsic to (2) and commences naturally in a way that scalar-driven inflation cannot.

- **De Sitter Inflation**

A locally de Sitter geometry provides the simplest paradigm for inflation. To see why, consider the general homogeneous, isotropic and spatially flat geometry (5). The nonzero components of the Riemann tensor are:

$$R^{0}{}_{i0j} = -qH^2\, g_{ij} \qquad , \qquad R^{i}{}_{jk\ell} = H^2\Big(\delta^i_k\, g_{j\ell} - \delta^i_\ell\, g_{jk}\Big) \ . \tag{23}$$

Inflation is defined as positive expansion ($H(t) > 0$) with negative deceleration ($q(t) < 0$). On the other hand, stability – in the form of the weak energy condition – implies $q(t) \geq -1$. At the limit of $q = -1$ we see from (23) that the Riemann tensor assumes the locally de Sitter form:

$$\lim_{q=-1} R^{\rho}{}_{\sigma\mu\nu} \;=\; H^2\Big(\delta^\rho_\mu\, g_{\sigma\nu} - \delta^\rho_\nu\, g_{\sigma\mu}\Big) \ . \tag{24}$$

It follows from (7) that the Hubble parameter is actually constant – which we denote by H_I and which satisfies $3H_I^2 = \Lambda$ – and that the zero of time can be chosen to make the scale factor take the simple exponential form:

$$\textit{de Sitter Inflation} \quad \Longrightarrow \quad a(t) \;=\; e^{H_I t} \ . \tag{25}$$

- **Real Particle Production**

The homogeneity of spacetime expansion in (5) does not change the fact that particles have constant wave vectors $\vec{k}$, but it does alter their physical meaning. In particular, the energy of a particle with mass m and wave number k becomes time dependent:

$$E(t,k) \;=\; \sqrt{m^2c^4 + \frac{c^2k^2}{a^2(t)}} \qquad , \qquad k \equiv \|\vec{k}\| \ . \tag{26}$$

This results in an interesting change in the energy-time uncertainty principle which restricts how long a virtual pair of such particles with wave vectors $\pm\vec{k}$ can exist. If the pair was created at time t, it can last up to a time Δt provided that:

$$\int_t^{t+\Delta t} dt'\; E(t',k) \;\lesssim\; 1 \ . \tag{27}$$

Just as in flat space, particles with the smallest masses persist longest. For the fully massless case and for de Sitter spacetime, the integral is simple to evaluate:

$$\int_t^{t+\Delta t} dt' \, E(t',k) \Big\vert_{m=0} = \Big[1 - e^{-H_I \Delta t} \Big] \frac{k}{H_I \, a(t)} \quad . \tag{28}$$

We, therefore, conclude that any massless virtual particle which happens to emerge from the vacuum during inflation can persist *forever* provided: [4]

$$\textit{Unbounded Lifetime} \quad \Longrightarrow \quad k \le H_I \, a(t) \quad . \tag{29}$$

- **Conformal Invariance Implications**

Most massless particles possess conformal invariance. A simple change of variables defines a conformal time η in terms of which the invariant element (5) is just a conformal factor times that of flat space:

$$ds^2 \, = \, -dt^2 + a^2(t) \, d\mathbf{x} \cdot d\mathbf{x} \, = \, a^2(\eta) \Big(-d\eta^2 + d\mathbf{x} \cdot d\mathbf{x} \Big) \quad , \quad d\eta \equiv \frac{dt}{a(t)} \quad . \tag{30}$$

In the $(\eta, \mathbf{x})$ coordinates, conformally invariant theories are locally identical to their flat space counterparts. The rate at which virtual particles emerge from the vacuum per unit conformal time must be the same constant – call it Γ – as in flat space. Hence, the rate of emergence per unit physical time is:

$$\frac{dN}{dt} = \frac{dN}{d\eta} \frac{d\eta}{dt} = \frac{\Gamma}{a(t)} \quad . \tag{31}$$

Consequently – although any sufficiently long wavelength, massless and conformally invariant particle emerging from the vacuum can persist forever during inflation – very few such particles will actually emerge.

For particles which do not possess conformal invariance, the rates are generically different in de Sitter than in flat spacetime.

- **Quantum Enhancement**

Gravitons and minimally coupled scalars are two kinds of massless particles which do not possess conformal invariance. To see that – unlike massless conformally invariant particles – the production of these two kinds of particles is not suppressed during inflation, note that each polarization and wave

[4] The inequality (27) is eventually violated as Δt grows for the other three relevant cosmological spacetimes: radiation, matter, spatial curvature.

number behaves like a harmonic oscillator with time dependent mass and frequency:

$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 \qquad , \qquad m(t) = a^3(t) \quad \& \quad \omega(t) = \frac{k}{a(t)} \; . \tag{32}$$

The Heisenberg equation of motion can be exactly solved:

$$\ddot{q} + 3H \, \dot{q} + \frac{k^2}{a^2} q = 0 \qquad \Longrightarrow \qquad q(t) = u(t,k) \, \alpha + u^*(t,k) \, \alpha^\dagger \; , \tag{33}$$

where the mode functions u and the commutation relations obeyed by the operators α and $\alpha^\dagger$ are given by:

$$u(t,k) = \frac{H_I}{\sqrt{2k^3}} \left[1 - \frac{ik}{H_I \, a(t)} \right] \exp\left(\frac{ik}{H_I \, a(t)} \right) \qquad , \qquad [\, \alpha \, , \, \alpha^\dagger \,] = 1 \; . \tag{34}$$

The co-moving energy operator for this system is:

$$E(t) = \frac{1}{2} m(t) \, \dot{q}^2(t) + \frac{1}{2} m(t) \, \omega^2(t) \, q^2(t) \; . \tag{35}$$

Owing to the time dependent mass and frequency, there are no stationary states for this system. At any given time the minimum eigenstate of $E(t)$ has energy $\frac{1}{2}\omega(t)$, but which state this is changes for each value of time. The state $|\Omega\rangle$ which is annihilated by α has minimum energy in the distant past. The expectation value of the energy operator in its presence is:

$$\Big\langle \Omega \Big| \, E(t) \, \Big| \Omega \Big\rangle = \frac{1}{2} a^3(t) \, |\dot{u}(t,k)|^2 + \frac{1}{2} a(t) \, k^2 \, |u(t,k)|^2 = \frac{k}{2a} + \frac{H_I^2 \, a}{4k} \; . \tag{36}$$

The first term is just the – properly redshifted – minimum energy; the second term is the result of particle production. A typical mode begins at $t = 0$ with the first term dominant. The second term becomes comparable at “horizon crossing” and dominates thereafter. This is the source of inflationary particle creation, and the onset of this enormous growth is what distinguishes infrared and ultraviolet modes. Horizon crossing of a mode occurs when its physical wave number equals the horizon:

$$\textit{Horizon Crossing} \quad \Longrightarrow \quad k_{\rm phys} = k \; a^{-1}(t) = H_I \; , \tag{37}$$

and provides the physical separation between infrared and ultraviolet modes:

$$\textit{Infrared} \quad \Longrightarrow \quad H_I < k < H_I\, e^{H_I t} \;\; , \tag{38}$$

$$\textit{Ultraviolet} \quad \Longrightarrow \quad k > H_I\, e^{H_I t} \;\; . \tag{39}$$

If we think of each particle as having energy $k\, a^{-1}(t)$, it follows that the number of particles N with any polarization and wave number k grows as the square of the inflationary scale factor:

$$N(t,k) \;=\; \left[\frac{H_I\, a(t)}{2k} \right]^2 \;\; . \tag{40}$$

As expected, it is only infrared gravitons that are produced:

$$\textit{Infrared} \quad \Longrightarrow \quad N^{dS}_{\rm IR}(t,\mathbf{k}) \;\sim\; \left(\frac{H_I\, e^{H_I t}}{2k} \right)^2 \;\gg\; 1 \;\; , \tag{41}$$

$$\textit{Ultraviolet} \quad \Longrightarrow \quad N^{dS}_{\rm UV}(t,\mathbf{k}) \;\sim\; 0 \;\; . \tag{42}$$

To get a sense of the density of infrared gravitons present per causal volume, we express the number of Hubble volumes in terms of the initial condition $H(0) = H_I$:

$$N_H(t) \;=\; \left[\frac{H(t)\; a(t)}{H_I} \right]^3 \;\; . \tag{43}$$

For the inflationary geometry, we trivially get:

$$N^{dS}_H(t) \;=\; e^{3 H_I t} \;\; , \tag{44}$$

so that – within one Hubble volume – the number of any infrared gravitons at time t is given by:

$$N^{dS}_{\rm IR}(t) \;=\; \frac{1}{N^{dS}_H(t)} \times \frac{1}{2\pi^2 H_I^3} \int_{H_I}^{H_I e^{H_I t}} dk\; k^2 \times N^{dS}_{\rm IR}(t,\mathbf{k}) \;=\; \frac{1}{8\pi^2} \;\; . \tag{45}$$

The presence of about one infrared graviton in each Hubble volume implies that the initial vacuum choice and perturbation theory is an excellent approximation during the inflationary regime. Such a low density cannot by itself drive a significant infrared screening effect; the coherent superposition from all infrared gravitons does.

- **A Perturbative Result**

Quantum field theoretic effects are driven by essentially classical physics operating in response to the source of virtual particles implied by quantization. On the basis of (40), one might expect inflation to dramatically enhance quantum effects from massless, minimally coupled scalars and gravitons. This has been confirmed explicitly and the oldest results are the cosmological perturbations induced by scalar inflatons and by gravitons. The more recent result which motivated the present analysis is that the gravitational backreaction from the inflationary production of gravitons induces an ever greater slowing in the expansion rate.

In the two-parameter effective gravitational theory (2), the corresponding mass scales are:

$$M_{\rm Pl}^2 \equiv \frac{1}{G} \quad , \quad M \equiv \left(\frac{\Lambda}{8\pi G}\right)^{\frac{1}{4}} \quad , \tag{46}$$

and the dimensionless coupling constant of the theory is $\varepsilon \equiv G\Lambda$. Perturbation theory is valid if and only if:

$$\varepsilon \equiv G\Lambda < 1 \quad \Leftrightarrow \quad \left(\frac{M}{M_{\rm Pl}}\right)^4 < 1 \ , \tag{47}$$

and can accomodate a quite wide range of scales M.

The imprint of geometrically significant differences between classical and quantum backgrounds can be attributed to a quantum-induced stress tensor defined from the deficit by which the *quantum* background fails to obey the *classical* equations of motion:

$$8\pi G\, T_{\mu\nu}[g] \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda \ . \tag{48}$$

The first non-trivial results were obtained about a locally de Sitter background on the manifold $T^3 \times \Re$ and in the presence of a state which is free Bunch-Davies vacuum at $t = 0$. The quantum-induced energy density and pressure, and the expansion rate – at late observation times – are:

$$\rho_{dS}[g](t) = -\varepsilon H_I^4 [\, \# \, (H_I\, t) + O(1)\,] + O(\varepsilon^2) \ , \tag{49}$$

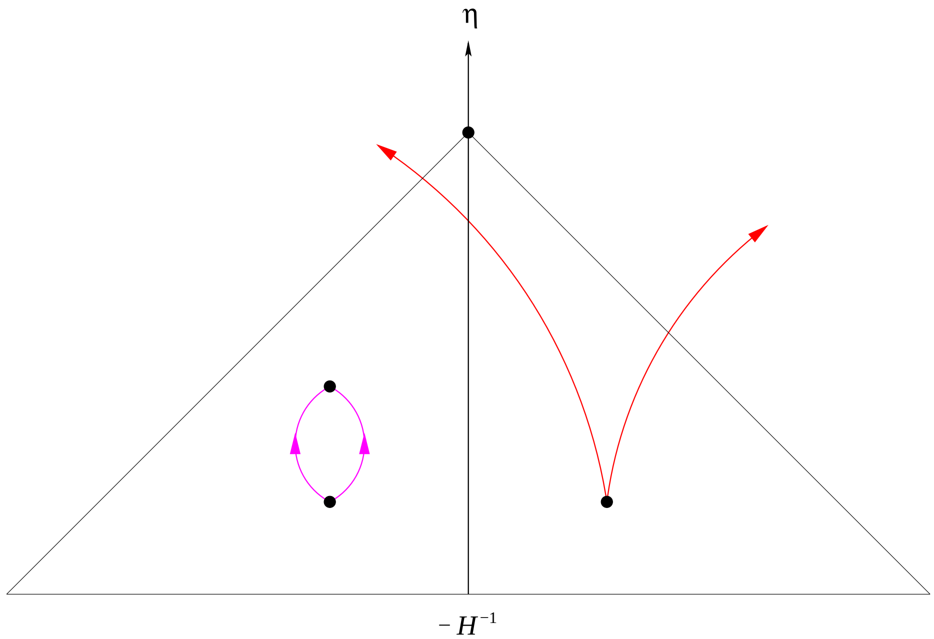

Figure 1: Short wavelength – $\lambda_{\rm phys} < H^{-1}$ – graviton pairs *(violet)* recombine while long wavelength – $\lambda_{\rm phys} > H^{-1}$ – ones *(red)* cannot.

$$p_{dS}(t) \;\; = \;\; \varepsilon H_I^4 \left[\, \# \, (H_I \, t) \; + \; O(1) \,\right] + \; O(\varepsilon^2) \;\; , \tag{50}$$

$$H_{dS}(t) \;\; = \;\; H_I \left\{ 1 - \varepsilon^2 \left[\, \#' \, (H_I \, t) \; + \; O(1) \,\right] + \; O(\varepsilon^3) \right\} \;\; . \tag{51}$$

It becomes apparent that the rate of expansion decreases by an amount which becomes non-perturbatively large at late times. This occurs when the effective dimensionless coupling constant becomes of order one and gives a rough estimate of the number of inflationary e-foldings:

$$\varepsilon^2 \, H_I \, t_1 \sim 1 \quad \Rightarrow \quad N_1 \equiv H_I \, t_1 \sim \left(\frac{M_{\rm Pl}}{M} \right)^8 \gg 50 \;\; . \tag{52}$$

The physical process responsible for the expansion diminishing quantum-induced stress tensor, are the correlated interactions among inflationary particles produced throughout the past lightcone of the observer. At the onset of inflation and in the vacuum state, all modes are virtual. As time evolves, infrared graviton pairs continuously appear and – in contradistinction to ultraviolet pairs – cannot recombine to annihilate because they get pulled by the rapid expansion of spacetime *(see Figure 1)*. While these long wavelength gravitons get separated and become causally disconnected, their long-range gravitational potentials persist since a potential exists everywhere in the forward lightcone of its source. The addition of the potentials of each receding infrared graviton pair is a secular effect and provides the negative gravitational interaction energy responsible for the reduction of the expansion rate.

- **The Newtonian Physical Picture**

It is possible to construct a simple model which shows the above physical mechanism within Newtonian gravity. There are three ingredients needed:

▷ *The classical background,* which is taken to be de Sitter spacetime on $T^3 \times \Re$.

▷ *The free infrared graviton kinematics,* which are characterized by the polarization and the wavenumber $\mathbf{k}$ such that $H_I \leq k \leq H_I \exp(H_I\, t)$.

▷ *The free infrared graviton dynamics,* which are similar to those of a massless minimally coupled scalar.

The physical energy of infrared mode $\mathbf{k}$ at time t is given by (35). From it, we can determine the Newtonian energy density and potential of mode $\mathbf{k}$ at time t:

$$\rho_{\mathbf{k}} \;=\; \frac{E_{\mathbf{k}} \,-\, \frac{1}{2}\, k\, e^{-H_I\, t}}{V_3(t)} \;=\; \frac{H_I^4}{4k}\; e^{-2H_I\, t} \qquad \Rightarrow \qquad \varphi_{\mathbf{k}} \;=\; -\frac{\pi\, G H_I^5}{k^3} \quad . \tag{53}$$

The total infrared newtonian energy density and potential at t are obtained by integrating over all infrared modes:

$$\rho_{\rm IR} \;=\; \frac{1}{\pi^2 H_I^3} \int_{H_I}^{H_I\, \exp[H_I\, t]} dk\, k^2\, \rho_{\mathbf{k}} \;=\; \frac{H_I^4}{8\pi^2} \quad , \tag{54}$$

$$\varphi_{\rm IR} \;=\; \frac{1}{\pi^2 H_I^3} \int_{H_I}^{H_I\, \exp[H_I\, t]} dk\, k^2\, \varphi_{\mathbf{k}} \;=\; -\frac{G H_I^2}{\pi}\; H_I\, t \quad . \tag{55}$$

Finally, the total infrared Newtonian *interaction* energy density at t is:

$$\rho_{\rm newton} \;=\; \rho_{\rm IR} \times \varphi_{\rm IR} \;=\; -\frac{G H_I^6}{8\pi^3}\; H_I\, t \quad , \tag{56}$$

and is negative and ever-increasing. Although $\rho_{\rm newton}$ starts much smaller than $\rho_{\rm IR}$, it eventually dominates:

$$\frac{\rho_{\rm newton}}{\rho_{\rm IR}} \;\sim\; G H_I^2\, (H_I\, t) \qquad , \qquad G H_I^2 \;\leq\; 10^{-12} \quad . \tag{57}$$

There is clear qualitative similarity with the quantum gravitational perturbative result (51).

- **Advantages of "Gravity-Driven" Cosmology**

There are distinct advantages when the inflationary regime of cosmological evolution is solely handled by the pure gravitational sector of the theory (the graviton) as opposed to the scalar sector (the inflaton):

▷ *Initial Conditions*

As long as the matter stress-energy is finite and obeys the weak energy condition, pre-inflationary expansion redshifts the initial matter stress-energy until it is dominated by the cosmological constant $\Lambda > 0$. By contrast, scalar-driven inflation is triggered by a random field fluctuation which must be homogeneous over more than a Hubble volume. This condition is so unlikely that it has not happened even once in the observed history of the universe.

▷ *Potential Issues*

To achieve the desired phenomenology, the inflaton potential must be flat over an extended interval, a requirement which places stringent and unnatural constraints on the coupling constants of the potential. On the other hand, the cosmological constant Λ *is* a constant and no such problem exists for "gravity-driven" inflation.

Furthermore, the inflaton potential must be arranged to be very close to zero at its minimum and this entails the severe fine-tunings associated with the cosmological constant. In the gravity case, we have already seen how – at least in perturbation theory – Λ can be screened.

Finally, the inflaton potential is arbitrary and no fundamental physical principle restricts it. Again, this is not the case with gravity where the theory at the infrared is well established.

▷ *Reheating Issues*

The cold universe that the inflationary era created must be reheated so that the currently observed structures are formed later in its evolution. In standard scalar-driven inflation, the inflaton transfers its energy to the matter sector via its couplings to matter. However there are different couplings to different matter fields and, more importantly, quantum matter corrections generate effective inflaton potentials that must be added to the bare inflaton potential and can either destabilize the theory or require further fine-tuning.

The gravitational sector has a universal coupling to matter – which is an advantage – but this coupling is extremely weak – which is a disadvantage – and adequate reheating cannot be generated. There is an exception to this if – besides the zero-mode – *all* infrared graviton modes participate in the reheating. In this case, gravity can provide naturally potent reheating. The simple model that follows has this property.

- **Gravitationally Induced Stress-Energy**

During the inflationary era infrared gravitons are produced out of the vacuum because of the accelerated expansion of spacetime. The interaction stress among the gravitons produced – an inherently non-local effect – can lead to a non-trivial quantum gravitational back-reaction on inflation. Non-local models of cosmology have been much studied because they can avoid the problem that de Sitter must be a solution for any local, stable theory, and because non-local couplings between different times can "ease" fine tuning problems. In this section we propose a phenomenological model which can provide evolution beyond perturbation theory. We constructed an *effective* conserved stress-energy tensor $T_{\mu\nu}[g]$ which modifies the gravitational equations of motion:

$$G_{\mu\nu} \;\equiv\; R_{\mu\nu} \,-\, \frac{1}{2} g_{\mu\nu}\, R \;=\; -\Lambda\, g_{\mu\nu} \,+\, 8\pi G\, T_{\mu\nu}[g] \;\; . \tag{58}$$

and which, we hope, contains the most cosmologically significant part of the full effective quantum gravitational equations.

Our stress-energy tensor must be a *non-local* functional of the metric and our physical ansatz consisted of parametrizing $T_{\mu\nu}[g]$ as a "perfect fluid":

$$T_{\mu\nu}[g] \;=\; (\rho + p)\, u_\mu\, u_\nu \,+\, p\, g_{\mu\nu} \;\; , \tag{59}$$

In order to completely determine it we provide :
(i) the energy density ρ as a functional of the metric tensor $\rho[g](x)$,
(ii) the pressure p as a functional of the metric tensor $p[g](x)$,
(iii) the 4-velocity field u_μ as a functional of the metric tensor $u_\mu[g](x)$, chosen to be timelike and normalized:

$$g^{\mu\nu}\, u_\mu u_\nu = -1 \qquad \Longrightarrow \qquad u^\mu\, u_{\mu;\nu} = 0 \;\; . \tag{60}$$

Because of the normalization (60), only three of the components of u_μ are algebraically independent. Thus, $T_{\mu\nu}[g]$ contains five independent quantities in total. Stress-energy conservation:

$$D^\mu T_{\mu\nu} \;=\; 0 \;\; , \tag{61}$$

provides four equations and allows us to determine any four of these quantities in terms of any one. It turns out to be more convenient to specify the induced pressure functional $p[g]$ and then use conservation to obtain the form of the induced energy density $\rho[g]$ and 4-velocity $u_\mu[g]$ up to their initial value data.

- **Requirements on the Induced Pressure**

▷ *The initial value requirement*
Our gravitationally induced source should not disturb the basic nature of the pure gravitational equations (58). The latter can be evolved from the initial spacelike surface knowing only the metric and its first time derivative. This property of gravity must be retained in the presence of the source and constrains both the local and non-local parts of its functional form; for instance, any local parts in $T_{\mu\nu}[g]$ can contain at most second time derivatives of the metric.

▷ *The non-locality requirement*
We argued that the physical effect responsible for gravitationally inducing $T_{\mu\nu}[g]$ is inherently non-local and, therefore, our source must be non-local. It is important to mention that this conclusion can also be reached by noting that no local modification of pure gravity can prevent de Sitter spacetime from being a solution of the field equations eternally. Any local modification simply changes the initial Hubble constant H_0 and can be absorbed by the cosmological constant counterterm $\delta\Lambda$ to leave no change and de Sitter spacetime as a solution for all time. Thus, the important part of the induced stress-energy tensor *must* be non-local.

▷ *The simplicity requirement*
A simple non-local operator at our disposal is the inverse of the scalar d'Alembertian: [5]

$$\square \;\equiv\; \frac{1}{\sqrt{-g}}\,\partial_\mu\Big(\, g^{\mu\nu}\sqrt{-g}\;\partial_\nu \Big) \;\; , \tag{62}$$

and a simple scalar it can act on is the curvature scalar R. Hence, we shall

[5] Our scalar d'Alembertian is defined with retarded boundary conditions.

explore *ansatze* in which the pressure is a function of the quantity $X[g]$:

$$X \equiv \frac{1}{\Box} R \ . \tag{63}$$

$\triangleright$ *The correspondence requirement*
Our gravitationally induced source should reproduce the perturbative results obtained in de Sitter spacetime.

- **A Simple Model for Cosmological Evolution**

The physical requirements and correspondence limits lead us to the following *ansatz* for the gravitationally induced pressure $p[g](x)$ in a general geometry:

$$p[g](x) \ = \ \Lambda^2 f[-G\Lambda X](x) \qquad , \qquad X \equiv \frac{1}{\Box} R \ , \tag{64}$$

where the function f grows without bound and satisfies:

$$f[-G\Lambda\, X] \ = \ -G\Lambda\, X \ + \ O[G\Lambda)^2] \ . \tag{65}$$

The homogeneous and isotropic evolution of this model – using a combination of numerical and analytical methods – revealed the following basic features: [6]

– After the onset and during the era of inflation, the source $X(t)$ grows while the curvature scalar $R(t)$ and Hubble parameter $H(t)$ decrease.

– Inflationary evolution dominates roughly until we reach a critical point X_{cr} defined by:

$$1 - 8\pi G\Lambda\, f[-G\Lambda\, X_{cr}] \ \equiv \ 0 \ . \tag{66}$$

– The epoch of inflation ends close to but before the universe evolves to the critical time. This is most directly seen from the deceleration parameter since initially $q(t = 0) = -1$ while at criticality $q(t = t_{cr}) = +\frac{1}{2}$.

– Oscillations in $R(t)$ become significant as we approach the end of inflation; they are centered around $R = 0$, their frequency equals:

$$\omega \ = \ G\Lambda H_0 \sqrt{72\pi\, f'_{cr}} \ , \tag{67}$$

and their envelope is linearly falling with time.

[6] The analytical results were obtained for any function f satisfying (65) and growing without bound; the numerical results for the choice: $f(x) = \exp(x) - 1$.

– During the oscillations era, although there is net expansion, the oscillations of $H(t)$ take it to small negative values for short time intervals – a feature conducive to rapid reheating; The oscillations of $\dot{H}(t)$ take it to positive values for about half the time; and those of $a(t)$ are centered around a linear increase with time.

A novel feature of this class of models is the existence of an oscillatory regime of short duration which commences towards the very end of the inflationary era. During this period $\dot{H}(t)$ is positive about half the time, which represents a violation of the weak energy condition. Such a violation cannot occur in classical stable theories but it can be driven by quantum effects of the type we seek to model without endangering stability. It is therefore a very distinctive feature of this model. The purpose of this study is to determine whether this oscillatory regime leaves its signature on the observable tensor power spectrum. We shall, therefore, obtain the amplitude and frequency of two kinds of gravitational waves and examine their evolution under the expansion history that this class of models predicts. The first kind of waves is now of cosmological scale and originated during inflation while the second kind was on the verge of experiencing first horizon crossing when the epoch of oscillations began. Because we shall be relating scales from the very early universe to current measurements, we first focus on presenting the basic equations and relevant relations, and then we apply them for our purpose.

5 Gravitational Waves in the Simple Model

The analysis of tensor perturbations in this class of models is much simpler than that of scalar perturbations. The reason is that – unlike the case of scalar perturbations – the non-local nature of the model does not alter the basic equation which the tensor perturbations h_{ij}^{TT} satisfy at linearized order:

$$\square \, h_{ij}^{TT}(t, \mathbf{x}) = 0 \quad \Longrightarrow \quad \left[\frac{\partial^2}{\partial t^2} + 3H(t) \frac{\partial}{\partial t} - \frac{\nabla^2}{a^2(t)} \right] h_{ij}^{TT}(t, \mathbf{x}) = 0 \ . \tag{68}$$

where the tensor perturbations h_{ij}^{TT} are defined by:

$$h_{ij}^{TT}(t, \mathbf{x}) \equiv \sqrt{2} \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda} \Big\{ u(t,k) \, e^{i\mathbf{k}\cdot\mathbf{x}} \, \epsilon_{ij}(\mathbf{k}, \lambda) \, \alpha(\mathbf{k}, \lambda) + u^*(t,k) \, e^{-i\mathbf{k}\cdot\mathbf{x}} \, \epsilon_{ij}^*(\mathbf{k}, \lambda) \, \alpha^\dagger(\mathbf{k}, \lambda) \Big\} \ . \tag{69}$$

In equation (69) $u(t,k)$ are the mode functions, $\epsilon_{ij}(\mathbf{k}, \lambda)$ is the polarization tensor and $\alpha(\mathbf{k}, \lambda)$ the annihilation operator.

Therefore, up to sub-dominant corrections coming from the exact form of the mode functions before and after first horizon crossing, the resulting power spectrum Δ_h^2 will have the usual form (10):

$$\Delta_h^2(k) \simeq \frac{16}{\pi} G H^2(t_k) \ , \tag{70}$$

where the Hubble parameter H is evaluated at the time t_k of first horizon crossing of the mode with wavenumber k:

$$k = H(t_k) \, a(t_k) \ . \tag{71}$$

Moreover, the tensor spectral index n_T defined by (12) equals:

$$n_T \equiv \frac{d}{d \ln k} \ln[\Delta_h^2(k)] \ . \tag{72}$$

For the power spectrum (70) the index n_T takes the form:

$$\begin{aligned} n_T &= \frac{dk}{d\ln k} \frac{d}{dk} \ln[\Delta_h^2(k)] \\ &\simeq \frac{dk}{d\ln k} \frac{dt_k}{dk} \frac{d}{dt_k} \left[\frac{16}{\pi} G H^2(t_k) \right] \\ &\simeq k \frac{dt_k}{dk} \frac{2\dot{H}(t_k)}{H(t_k)} \ . \end{aligned} \tag{73}$$

From (71) we can evaluate the second term in (73):

$$1 \;=\; \frac{d}{dk}k \;=\; \frac{d}{dk}\Big[\,H(t_k)\,a(t_k)\,\Big] \;=\; \frac{dt_k}{dk}\,\frac{d}{dt_k}\Big[\,H(t_k)\,a(t_k)\,\Big]$$
$$=\; \frac{dt_k}{dk}\,H^2 a\left(1+\frac{\dot{H}}{H^2}\right)\Bigg|_{t_k} \;=\; \frac{dt_k}{dk}\,H^2 a\,(1-\epsilon)\Big|_{t_k} \quad , \qquad (74)$$

to obtain:

$$n_T \;\simeq\; -\frac{2\epsilon(t_k)}{1-\epsilon(t_k)} \;\simeq\; -2\epsilon(t_k) \quad , \qquad (75)$$

where the last step assumes that $\epsilon(t_k) \ll 1$. It is apparent that knowledge of the relevant scale factor $a(t)$ suffices to compute the tensor power spectrum and spectral index.

- **Assumptions about the Expansion History**

We divide cosmological history into three epochs:

– *Primordial Inflation.* The most convenient time parameter for the epoch of primordial inflation is the number N of e-foldings before criticality:

$$a(t) \;\equiv\; a_{\rm cr}\,e^{-N} \quad . \qquad (76)$$

The important cosmological parameters during this phase are:

$$H^2(t) \;\simeq\; \frac{1}{9}\,\omega^2\left(4N+\frac{4}{3}\right) \quad , \qquad (77)$$
$$\epsilon(t) \;\simeq\; \frac{2}{4N+\frac{4}{3}} \quad . \qquad (78)$$

At the end of inflation the scale factor is about $a_{\rm cr}$ and the Hubble parameter is about ω.

– *Oscillations.* The distinctive feature of our model is the epoch of oscillations. The most convenient time parameter during this era is the co-moving time after criticality:

$$\Delta t \;\equiv\; t - t_{\rm cr} \quad . \qquad (79)$$

The important cosmological parameters during this phase are:

$$a(t) \;\simeq\; a_{\rm cr}\Big[\,\omega\,\Delta t \,+\, 1 \,+\, \sqrt{2}\Big(\cos(\omega\,\Delta t) - 1\Big)\Big] \quad , \qquad (80)$$

$$H(t) \simeq \frac{\omega \left[1 - \sqrt{2} \sin(\omega \Delta t) \right]}{\omega \Delta t + (1-\sqrt{2}) + \sqrt{2} \cos(\omega \Delta t)} \quad , \tag{81}$$

$$\epsilon(t) \simeq \frac{\sqrt{2} \left[\omega \Delta t + (1-\sqrt{2}) \right] \cos(\omega \Delta t) + 3 - 2\sqrt{2} \sin(\omega \Delta t)}{\left[1 - \sqrt{2} \sin(\omega \Delta t) \right]^2} \quad . \tag{82}$$

The epoch of oscillations is terminated by the flow of energy density to the matter sector from the vast reservoir of super-horizon scalar modes, all of which begin to oscillate with frequency ω. We believe this should lead to very rapid reheating. Let us call the number of oscillatory e-foldings ΔN. Then, at the end of the oscillations era:

$$a \simeq a_{\rm cr} \, e^{\Delta N} \qquad , \qquad H \simeq \omega \, e^{-\Delta N} \quad . \tag{83}$$

– *ΛCDM.* The ΛCDM cosmology after the epoch of oscillations is standard, and we do not require explicit forms for the three geometrical parameters. To compare quantities from the first two eras with their redshifted descendants at present time it is useful to express the energy density ρ_R at the onset of the ΛCDM epoch in terms of the reheating temperature T_R and the number $n \approx 10^3$ of relativistic species:

$$\rho_R \simeq \frac{3c^2 \omega^2 \, e^{-2\Delta N}}{8\pi G} \simeq n \times \frac{\pi^2}{30} \frac{(k_B T_R)^4}{(\hbar c)^3} \quad . \tag{84}$$

The current energy density $\rho_{\rm now}$ can be written in terms of its tiny radiation fraction $\Omega_r \approx 8.5 \times 10^{-4}$ and the corresponding temperature $T_{\rm now} \approx 2.726 K$ of that radiation:

$$\rho_{\rm now} = \frac{3c^2 H_{\rm now}}{8\pi G} \simeq \frac{2}{\Omega_r} \times \frac{\pi^2}{30} \frac{(k_B T_{\rm now})^4}{(\hbar c)^3} \quad . \tag{85}$$

Dividing (85) by (84) gives a relation between current conditions and those prevailing at the end of inflation:

$$\left(\frac{T_{\rm now}}{T_R} \right)^4 \simeq \frac{n \, \Omega_r}{2} \, e^{2\Delta N} \left(\frac{H_{\rm now}}{\omega} \right)^2 \quad . \tag{86}$$

We define $N_{\rm now}$ as the number of e-foldings from criticality to the present time. Using the relation (86) and:

$$\frac{T_{\rm now}}{T_R} \simeq \frac{a_R}{a_{\rm now}} \simeq \frac{a_{\rm cr} \, e^{\Delta N}}{a_{\rm now}} \quad , \tag{87}$$

we see that $N_{\rm now}$ equals:

$$N_{\rm now} \;\simeq\; \Delta N \,+\, \ln\Big[\frac{T_R}{T_{\rm now}}\Big] \;=\; \frac{1}{2}\ln\Big[\frac{\omega}{H_{\rm now}}\Big] \,+\, \frac{1}{2}\,\Delta N \,-\, \frac{1}{4}\ln\Big[2n\,\Omega_r\Big] \;\;. \quad (88)$$

We shall later show that the measured value of the scalar power spectrum $\Delta^2_{\mathcal{R}}$, and the current limit on the tensor-to-scalar ratio r, together imply the restriction $\omega \lesssim 10^{55} H_{\rm now}$. Hence we conclude:

$$N_{\rm now} \;\lesssim\; 63 \,+\, \frac{1}{2}\,\Delta N \;\;. \quad (89)$$

There are two interesting problems concerning the relation between late times and early times:

01. Given a physical wave number $K_{\rm now}$ at the current time, find the e-folding $N_{\rm hor}$ when it experienced first horizon crossing during inflation. To solve this problem, we first use the horizon crossing condition (71) to express $K_{\rm now}$ in terms of $N_{\rm hor}$:

$$K_{\rm now} \;=\; \frac{k}{a_{\rm now}} \;=\; \frac{k}{a(t_k)} \times \frac{a(t_k)}{a_{\rm cr}} \times \frac{a_{\rm cr}}{a_{\rm now}} \quad (90)$$

$$\approx\; \frac{1}{3}\,\omega\,\sqrt{4N + \frac{4}{3}} \,\times\, e^{-N_{\rm hor}} \,\times\, e^{-N_{\rm now}} \;\;. \quad (91)$$

Now invert (91) to solve for $N_{\rm hor}$: [7]

$$N_{\rm hor} \;\approx\; \ln\Big[\frac{\omega}{K_{\rm now}}\Big] \,-\, N_{\rm now} \,+\, \frac{1}{2}\,\ln\Big[\frac{4}{9}N + \frac{4}{27}\Big] \quad (92)$$

$$\approx\; \frac{1}{2}\,\ln\Big[\frac{\omega H_{\rm now}}{c^2 K^2_{\rm now}}\Big] \,-\, \frac{1}{2}\,\Delta N \,+\, \frac{1}{4}\,\ln\Big[2n\,\Omega_r\Big] \,+\, \frac{1}{2}\,\Big[\frac{2}{9}\ln\Big(\frac{\omega H_{\rm now}}{c^2 K^2_{\rm now}}\Big)\Big] \quad (93)$$

For the ℓ-th partial wave contribution to the anisotropies of the cosmic ray microwave background, the corresponding number N_ℓ of e-foldings before the end of inflation is:

$$K_{\rm now} \;\approx\; \frac{\ell}{2} \times \frac{H_{\rm now}}{c} \quad \Longrightarrow \quad (94)$$

$$N_\ell \;\approx\; -\ln\Big(\frac{\ell}{2}\Big) + \frac{1}{2}\ln\Big(\frac{\omega}{H_{\rm now}}\Big) - \frac{1}{2}\Delta N + \frac{1}{4}\ln\Big(2n\,\Omega_r\Big) + \frac{1}{2}\ln\Big[\frac{2}{9}\ln\Big(\frac{\omega}{H_{\rm now}}\Big)\Big]$$

[7]The inversion was done under the assumption that the Hubble parameter at the end of inflation is much bigger than its present value: $\omega\, H^{-1}_{\rm now} \gg 1$.

The restriction $\omega \lesssim 10^{55} H_{\rm now}$ then implies:

$$N_\ell \;\lesssim\; 65 \;-\; \frac{1}{2}\ln\Big(\frac{\ell}{2}\Big) \;-\; \frac{1}{2}\,\Delta N \quad . \tag{95}$$

We cannot hope to detect a signal outside the range $2 \leq \ell \lesssim 100$, so the interesting values of N_ℓ lie within a band of only four e-foldings.

The second general problem is in some ways the inverse of the first:

02. Given a physical wave number K_N from the epoch of inflation, find its physical wave number now. To achieve this, we express the current physical wave number in terms of K_N and N:

$$\begin{aligned} K_{\rm now} \;&=\; \frac{k}{a_{\rm now}} \;=\; \frac{k}{a(t)} \times \frac{a(t)}{a_{\rm cr}} \times \frac{a_{\rm cr}}{a_{\rm now}} \;=\; K_N \times e^{-N} \times e^{-N_{\rm now}} \\ &\approx\; \sqrt{\frac{K_N^2\, H_{\rm now}}{\omega}}\; \Big(\frac{n\,\Omega_r}{2}\Big)^{\frac{1}{4}}\; e^{-N - \frac{1}{2}\Delta N} \quad , \end{aligned} \tag{96}$$

where in the last step we used (88) for $N_{\rm now}$. An important special case is the oscillation frequency $f_{\rm now}$ for the wave vector $K_N = \frac{\omega}{c}$ at $N = 0$. In this situation, (96) gives:

$$f_{\rm now} \;\equiv\; \left(\frac{ck}{a_{\rm now}}\right)_{K_0 = \frac{\omega}{c}} \;\approx\; \sqrt{\omega\, H_{\rm now}}\; e^{-\frac{1}{2}\Delta N} \quad . \tag{97}$$

Imposing the restriction $\omega \lesssim 10^{55} H_{\rm now}$ and using the current value of $H_{\rm now} \approx 71\, km\, s^{-1} (Mpc)^{-1} \approx 3.2 \times 10^{-18} Hz$, implies:

$$f_{\rm now} \;\lesssim\; \Big(10^{10}\, Hz\Big)\, e^{-\frac{1}{2}\Delta N} \quad . \tag{98}$$

Finally, we deduce the restriction on ω coming from the measured value of the scalar power spectrum $\Delta^2_{\mathcal{R}}$:

$$\Delta^2_{\mathcal{R}}(k_0) \;\approx\; 2.44 \times 10^{-9} \qquad , \qquad k_0 \;\equiv\; 0.0002\; (Mpc)^{-1} \quad , \tag{99}$$

and the 95% confidence bound on the tensor-to-scalar ratio: $r(k_0) \lesssim 0.22$. Employing expressions (70) and (81) we get:

$$\Delta^2_h(k_0) \;=\; r(k_0)\, \Delta^2_{\mathcal{R}}(k_0) \;\simeq\; \frac{16}{9\pi}\, G\omega^2 \Big(4N_0 + \frac{4}{3}\Big) \tag{100}$$

$$\lesssim\; \Big[0.22\Big] \times \Big[2.44 \times 10^{-9}\Big] \quad . \tag{101}$$

Now the wave number k_0 and its associated number of e-foldings N_0 correspond to the $\ell = 2$ partial wave so that – under the asumption that the Hubble parameter at the end of inflation is much bigger than its present value ($\omega H_{\rm now}^{-1} \gg 1$) and that the duration of the oscillations era is very short ($\Delta N < 10$) – equation (94) implies: $N_0 = N_{\ell=2} \lesssim 60$. Thus, expressions (100-101) reduce to $\omega\sqrt{G} \lesssim 2 \times 10^{-6}$ and when we convert to Hz we get:

$$\omega \;\lesssim\; 2 \times 10^{-6} \sqrt{\frac{c^5}{G\hbar}} \;\approx\; 3.7 \times 10^{37} Hz \quad \Longrightarrow \qquad \omega \;\lesssim\; 10^{55}\, H_{\rm now} \;\; , \quad (102)$$

where we used $H_{\rm now} \approx 3.2 \times 10^{-18} Hz$ for the current value of the Hubble parameter.

- **Overview of Gravitational Waves in the Oscillating Regime**

In terms of the mode functions $u(t, k)$ the basic equation (68) takes the form:

$$\ddot{u} \,+\, 3H\,\dot{u} \,+\, \frac{k^2}{a^2}\, u \;=\; 0 \;\; . \qquad (103)$$

We do not possess exact forms for the two, linearly independent solutions during the oscillatory regime. [8] Even if we had these solutions, we would not know the linear combination of them that gives "the" mode function $u(t, k)$, which we define to be the coefficient of the annihilation operator in the free field expansion of the graviton. It makes sense to first develop a reasonable approximation for the linearly independent solutions and then consider which combination of them occurs in the actual mode function $u(t, k)$. In approximating the solutions it also makes sense to first include the effect of the overall linear expansion – for which exact solutions exist – and then numerically superimpose the effect of the oscillations.

- **The Case of Linear Expansion**

During the oscillatory epoch the scale factor (80) consists of a linear expansion plus an oscillatory term which causes the Hubble parameter (81) to become negative for brief periods. Because we wish to quantify the potential enhancement from these periods of negative $H(t)$, it is useful to factor out the behaviour that would arise from the linear growth, without the oscillatory term:

$$\bar{a}(t) \;=\; a_{\rm cr}\Big[1 + \omega\Delta t\Big] \;\; . \qquad (104)$$

[8] The exact solution $u(t, k)$ includes the full evolution before $t_{\rm cr}$ and the oscillations.

Then, the Hubble parameter $\bar{H}$ can be expressed in terms of the scale factor $\bar{a}$ as follows:

$$\bar{H}(t) \;=\; \frac{\dot{\bar{a}}}{\bar{a}} \;=\; \frac{\omega}{1+\omega\Delta t} \;=\; \frac{\omega\, a_{\rm cr}}{\bar{a}(t)} \quad . \tag{105}$$

The canonically normalized, Bunch-Davies mode function for the linear expansion is:

$$\bar{u}(t,k) \;=\; \frac{1}{\sqrt{2\,\sqrt{c^2k^2-\omega^2 a_{\rm cr}^2}}} \times \frac{1}{\bar{a}(t)}\, \exp\!\left(-\,i\,\sqrt{\frac{c^2k^2}{\omega^2 a_{cr}^2}-1}\;\ln\!\Big[\frac{\bar{a}(t)}{a_{\rm cr}}\Big]\right) \tag{106}$$

and already indicates some (integrable) enchancement since $\bar{u}(t,k)$ becomes singular when $c^2k^2 = \omega^2 a_{\rm cr}^2$. Since the product $\bar{H}(t)\times\bar{a}(t)$ is constant, there is no horizon crossing during linear expansion. Modes which are sub-horizon at criticality ($k > \omega a_{\rm cr}$) remain sub-horizon, and modes which are super-horizon at criticality ($k < \omega a_{\rm cr}$) also remain super-horizon.

▷ *Super-horizon mode functions*

For super-horizon modes we have the condition:

$$\frac{c^2k^2}{\omega^2 a_{cr}^2} \ll 1 \quad , \tag{107}$$

so that the exponential in (106) becomes:

$$\begin{aligned}
\exp\!\left(-\,i\,\sqrt{\frac{c^2k^2}{\omega^2 a_{cr}^2}-1}\;\ln\!\Big[\frac{\bar{a}(t)}{a_{\rm cr}}\Big]\right) \;&=\; \exp\!\left(\pm\sqrt{1-\frac{c^2k^2}{\omega^2 a_{cr}^2}}\;\ln\!\Big[\frac{\bar{a}(t)}{a_{\rm cr}}\Big]\right)\\
&=\; \exp\!\left(\ln\!\Big[\frac{a_{\rm cr}}{\bar{a}(t)}\Big]^{\pm\sqrt{\left(1-\frac{c^2k^2}{\omega^2 a_{cr}^2}\right)}}\right) \;=\; \left[\frac{a_{\rm cr}}{\bar{a}(t)}\right]^{\pm\sqrt{\left(1-\frac{c^2k^2}{\omega^2 a_{cr}^2}\right)}} \quad ,
\end{aligned} \tag{108}$$

and the super-horizon mode functions fall off like :

$$\bar{u}(t,k) \;\simeq\; \frac{1}{\bar{a}(t)}\left[\frac{a_{\rm cr}}{\bar{a}(t)}\right]^{\pm\sqrt{\left(1-\frac{c^2k^2}{\omega^2 a_{cr}^2}\right)}} \;=\; \left[\frac{a_{\rm cr}}{\bar{a}(t)}\right]^{1\,\pm\sqrt{\left(1-\frac{c^2k^2}{\omega^2 a_{cr}^2}\right)}} \quad . \tag{109}$$

To identify two linearly independent solutions that exhibit the two different behaviours we define the convenient variable: [9]

$$x \;\equiv\; \sqrt{\left(1-\frac{c^2k^2}{\omega^2 a_{cr}^2}\right)}\;\ln\!\Big[\frac{a_{\rm cr}}{\bar{a}(t)}\Big] \quad , \tag{110}$$

[9] For super-horizon modes, due to condition (107), we have: $e^x \sim a$.

and re-organize the two linearly independent solutions:

$$\bar{u}_{rl}(t,k) \simeq \frac{1}{\bar{a}(t)} \cosh x \simeq \frac{1}{\bar{a}(t)} \frac{1}{2}(e^{x} + e^{-x}) \ , \tag{111}$$

$$\bar{u}_{im}(t,k) \simeq \frac{1}{\bar{a}(t)} \sinh x \simeq \frac{1}{\bar{a}(t)} \frac{1}{2}(e^{x} - e^{-x}) \ , \tag{112}$$

by forming their sum and difference:

$$\bar{u}_{+} = \bar{u}_{rl} + \bar{u}_{im} = \frac{1}{\bar{a}(t)} e^{x} \simeq \text{constant} \ , \tag{113}$$

$$\bar{u}_{-} = \bar{u}_{rl} - \bar{u}_{im} = \frac{1}{\bar{a}(t)} e^{-x} \simeq \frac{1}{\bar{a}^2(t)} \ . \tag{114}$$

It is clear from (113-114) that, given enough time evolution, the u_+ solution dominates. Note that this assumes $a(t)$ increases – as it does for linear expansion and for $H > 0$. But if $a(t)$ starts decreasing – as it would for $H < 0$ – the u_- solution starts contributing. This is what happens when we add the oscillatory term to the exactly soluble scale factor (104).

▷ *Sub-horizon mode functions*
For sub-horizon modes we have the condition:

$$\frac{c^2 k^2}{\omega^2 a_{cr}^2} \gg 1 \ . \tag{115}$$

A similar analysis is possible by defining a convenient variable:

$$x \equiv \sqrt{\left(\frac{c^2 k^2}{\omega^2 a_{cr}^2} - 1\right)} \ \ln\Big[\frac{a_{\rm cr}}{\bar{a}(t)}\Big] \ , \tag{116}$$

and noting that the two linearly independent solutions are:

$$\bar{u}_{rl}(t,k) \simeq \frac{1}{\bar{a}(t)} \cos x = \frac{1}{\bar{a}(t)} \frac{1}{2}(e^{ix} + e^{-ix}) \ , \tag{117}$$

$$\bar{u}_{im}(t,k) \simeq \frac{1}{\bar{a}(t)} \sin x = \frac{1}{\bar{a}(t)} \frac{1}{2}(e^{ix} - e^{-ix}) \ . \tag{118}$$

It is already evident from expression (106 that sub-horizon mode functions oscillate and redshift, with the period of oscillations also redshifting.

- **Initial Conditions**

We still have to include the effect of oscillations. That defines a mode function $\widetilde{u}(t,k)$ which obeys equation (103) with the full oscillating geometry (80-81). We construct these mode functions to agree initially with those of the phase of linear expansion:

$$\widetilde{u}(t_{\rm cr},k) \;=\; \bar{u}(t_{\rm cr},k) \qquad , \qquad \dot{\widetilde{u}}(t_{\rm cr},k) \;=\; \dot{\bar{u}}(t_{\rm cr},k) \;\; . \tag{119}$$

The actual mode function $u(t,k)$, by which we mean the coefficient of the annihilation operator in the free field expansion, is neither $\widetilde{u}(t,k)$ nor $\widetilde{u}^*(t,k)$. The general solution is the linear superposition of the two solutions:

$$u(t,k) \;=\; \alpha\, \widetilde{u}(t,k) \,+\, \beta\, \widetilde{u}^*(t,k) \;\; . \tag{120}$$

We can solve for the combination coefficients in terms of the values of $u(t,k)$ and $\dot{u}(t,k)$ at criticality. The Wronskian of the two linearly independent solutions $\widetilde{u}(t,k)$ and $\widetilde{u}^*(t,k)$ is:

$$W(x) \;\equiv\; \widetilde{u}\, \dot{\widetilde{u}}^* \,-\, \dot{\widetilde{u}}\, \widetilde{u}^* \;\; . \tag{121}$$

Since the Wronskian is independent of time, we can evaluate it at criticality where the mode functions are known from (119):

$$W \;\equiv\; \widetilde{u}\, \dot{\widetilde{u}}^* \,-\, \dot{\widetilde{u}}\, \widetilde{u}^* \;=\; \bar{u}\, \dot{\bar{u}}^* \,-\, \dot{\bar{u}}\, \bar{u}^* \;=\; \frac{i}{a_{\rm cr}^3} \;\; . \tag{122}$$

Thus the combination coefficients are :

$$\alpha \;=\; -i\, a_{\rm cr}^3 \Big[u(t_{\rm cr},k)\, \dot{\bar{u}}^*(t_{cr},k) \,-\, \dot{u}(t_{\rm cr},k)\, \bar{u}^*(t_{cr},k) \Big] \;\; , \tag{123}$$

$$\beta \;=\; -i\, a_{\rm cr}^3 \Big[\dot{u}(t_{\rm cr},k)\, \bar{u}(t_{cr},k) \,-\, u(t_{\rm cr},k)\, \dot{\bar{u}}(t_{cr},k) \Big] \;\; . \tag{124}$$

Although we do not know precisely what these values are, some reasonable guesses can be made. For example, a far super-horizon mode, which experienced first horizon crossing $N_{\rm hor}$ e-foldings before criticality, should have:

$$u(t_{\rm cr},k) \;\approx\; \frac{H_{N_{\rm hor}}}{\sqrt{2k^3}} \;\; , \tag{125}$$

$$\dot{u}(t_{\rm cr},k) \;\approx\; -\frac{H_{N_{\rm hor}}^2}{\sqrt{2k^3}} \left(\frac{k}{H_{N_{\rm hor}}\; a_{\rm cr}} \right)^2 \left[1 \,+\, \frac{ik}{H_{N_{\rm hor}}\; a_{\rm cr}} \right] \;\; . \tag{126}$$

Recall that the potentially observable modes in the cosmic microwave background correspond to $N_{\rm hor} \approx 60$, which implies:

$$\frac{k}{H_{N_{\rm hor}}\; a_{\rm cr}} \sim 10^{-26} \quad . \tag{127}$$

There is absolutely no point in retaining such small numbers. So during the oscillatory phase after criticality, the mode function of a cosmologically observable wave number would be unchanged from (125), for all practical purposes. That had to be true because, for far super-horizon wave numbers, (103) simplifies to:

$$\ddot{u}(t,k) \,+\, 3H(t)\,\dot{u}(t,k) \;\approx\; 0 \;\; . \tag{128}$$

and $u(t,k) = constant$ remains a solution – independent of $a(t)$ – for as long as it is valid to neglect the last term of (103).

- **Gravitational Waves Enhancement**

Now let us consider the effect of adding the oscillatory term to the scale factor. It is obvious that one gets a significant response at resonance; then, the natural time scale of the mode function is close to the inverse of the oscillatory frequency ω. Whether or not this occurs depends upon two things: the wave number k and the values of $u(t,k)$ and $\dot{u}(t,k)$ at the start of the oscillatory period. The reason the initial condition matters is that there are always two, linearly independent solutions to the mode equation and they can have vastly different natural time scales. Initial conditions define the coefficients of the two linearly independent solutions, thus we can see which of the two solution dominates. There are three interesting wave number regimes:

$*$ The *Far Super-horizon,* with $c\,k \ll \omega\, a_{\rm cr}$.
From expression (109) it is evident that, without the oscillatory term, super-horizon modes fall off with time scales:

$$T_{\pm} \;\simeq\; \frac{\omega^{-1}}{1 \pm \sqrt{1 - \frac{c^2k^2}{\omega^2 a_{\rm cr}^2}}} \quad . \tag{129}$$

For $ck \ll \omega a_{\rm cr}$ one of these is – within a factor of two – close to ω while the other is vastly longer. Numerical analysis shows – see Figure 2 – that the oscillations amplify the solution with the shorter time scale by about a factor of four. As might be expected, the solution with the longer time scale experiences no significant amplification. Because the natural initial conditions (125-126) imply the mode enters the oscillatory epoch almost entirely in the long time scale solution, the effect is that far super-horizon modes experience no significant enhancement from the oscillation.

∗ The *Far Sub-horizon,* with $c\,k \gg \omega\, a_{\rm cr}$.
Far sub-horizon modes also receive no substantial enhancement, but for a different reason. For far sub-horizon modes the natural frequencies of both solutions are about $ck\, a_{\rm cr}^{-1}$ which is much bigger than ω, so neither solution experiences much enhancement and it does not matter much what the initial condition is.

∗ The *Near-horizon,* with $c\,k \approx \omega\, a_{\rm cr}$.
As one might expect, it is the near-horizon modes which experience the greatest enhancement. Figures 3 and 4 present numerical simulations for the case of $ck = \frac{11}{10}\omega a_{\rm cr}$, giving the ratios of the actual mode functions – evolved with the oscillatory term – compared with the solution (106) which starts from the same initial condition but is evolved without the oscillatory term. [10] In the near-horizon regime one expects both solutions to be present with about the same amplitude, so a reasonable estimate of the total enhancement is by adding the two solutions in quadrature and taking the ratio with, and without the oscillatory term:

$$Q \;\equiv\; \frac{|\,\widetilde{u}(t,k)\,|}{|\,\bar{u}(t,k)\,|} \;=\; \frac{\sqrt{({\rm Re}\widetilde{u})^2 + ({\rm Im}\widetilde{u})^2}}{\sqrt{({\rm Re}\bar{u})^2 + ({\rm Im}\bar{u})^2}} \quad . \tag{130}$$

From Figure 5 one can see that the enhancement factor is about $Q \approx 10$. By comparison with the Hubble parameter – see Figure 6 – we see that almost all the enhancement derives from the first oscillation. Figures 7 and 8 give the total enhancement factor for the cases of $ck = 2\omega a_{\rm cr}$ and $ck = 5\omega a_{\rm cr}$, respectively.
Modes which are slightly super-horizon are quite similar to those which are slightly sub-horizon. Figure 9 gives the total enhancement factor for the case of $ck = \frac{9}{10}\omega a_{\rm cr}$. However, decreasing the wave number much more rapidly reaches the factor of four enhancement which is concentrated on the solution

[10] The enhancement factor $Q1$ is associated with the real part: $Q1 \equiv {\rm Re}[\widetilde{u}(t,k)] \div {\rm Re}[\bar{u}(t,k)]$ while $Q2$ with the imaginary part: $Q2 \equiv {\rm Im}[\widetilde{u}(t,k)] \div {\rm Im}[\bar{u}(t,k)]$.

that is not likely to be present. Figures 10 and 11 give the behaviours for $ck = \frac{1}{2}\omega a_{\rm cr}$ and $ck = \frac{1}{5}\omega a_{\rm cr}$, respectively. Finally, Figures 12 and 13 display the near-horizon enhancement by wave numbers very close on either side.

- **Enhanced Waves Energy Density and Frequency**

It remains to estimate the current energy density and frequency of gravitons which are produced during the epoch of oscillations. Suppose we regard the enhancement factor as $Q = 10$ for modes within the range $\frac{2}{3}\omega a_{\rm cr} < ck < \frac{3}{2}\omega a_{\rm cr}$, and zero outside this band. This is superimposed on the mode functions (106) of linear expansion. A reasonable estimate for the extra energy – above the 0-point – in a single wave vector $\mathbf{k}$ within the band of enhancement is:

$$E(t_{\rm cr}, k) \;\sim\; \hbar c^2 k^2 \Bigl| Q\, \bar{u}(t_{\rm cr}, k) \Bigr|^2 \;=\; \frac{Q^2\, \hbar c^2 k^2}{2\sqrt{|c^2 k^2 - \omega^2 a_{\rm cr}^2|}} \quad . \tag{131}$$

The associated energy density comes from integrating over all modes within the band of enhancement:

$$\rho(t_{\rm cr}) \;\sim\; \int \frac{d^3 k}{(2\pi)^3}\, \theta\Bigl(\frac{3}{2}\,\omega a_{\rm cr} - ck\Bigr)\, \theta\Bigl(ck - \frac{2}{3}\,\omega a_{\rm cr}\Bigr)\, E(t_{\rm cr}, k) \quad , \tag{132}$$

$$\sim\; \frac{Q^2\, \hbar \omega^4 a_{\rm cr}^4}{4\pi^2 c^3} \quad . \tag{133}$$

After the end of oscillations these gravitons are sub-horizon so their energy density redshifts like radiation. Its current value is:

$$\rho(t_{\rm now}) \;\sim\; \frac{Q^2\, \hbar\omega^4}{4\pi^2 c^3} \left(\frac{a_{\rm cr}}{a_{\rm now}}\right)^4 \;\sim\; \frac{Q^2\, \hbar\omega^4}{4\pi^2 c^3}\; e^{-4N_{\rm now}} \quad , \tag{134}$$

$$\sim\; \frac{2Q^2}{3\pi} \times \left(\frac{\omega}{H_{\rm now}}\right)^2 \times \frac{G\, \hbar\omega^2}{c^5} \times e^{-4N_{\rm now}} \times \left(\frac{3c^2 H_{\rm now}^2}{8\pi G}\right) \quad . \tag{135}$$

Using the restrictions (89, 102) gives the following fraction of the current total energy density:

$$\rho(t_{\rm now}) \;\lesssim\; 10^{-5} \times \left(\frac{3c^2 H_{\rm now}^2}{8\pi G}\right) \quad . \tag{136}$$

The predicted frequency is given by (98):

$$f_{\rm now} \;\lesssim\; 10^{10} Hz \quad . \tag{137}$$

It would be challenging – but perhaps not impossible – to detect gravitational waves of such frequency. Note also that (136) is of about the right magnitude to be regarded as another relativistic species, despite the fact that these gravitons are far from thermal equilibrium.

6 Epilogue

We considered a simple model within a class of cosmological models of purely gravitational origin. These models have important advantages over conventional inflationary paradigms since they do not contain, or need, scalar field degrees of freedom. Their time evolution leads – after the end of inflation – to a short oscillating period. Since *all* modes oscillate during that period it is worthwhile to investigate whether there is any enhancement of gravitational waves generated around that era. We showed that for near-horizon modes there is such an enhancement by approximately an order of magnitude. It remains to be seen whether such an enhancement can be measured by high frequency gravitational wave detectors in the future.

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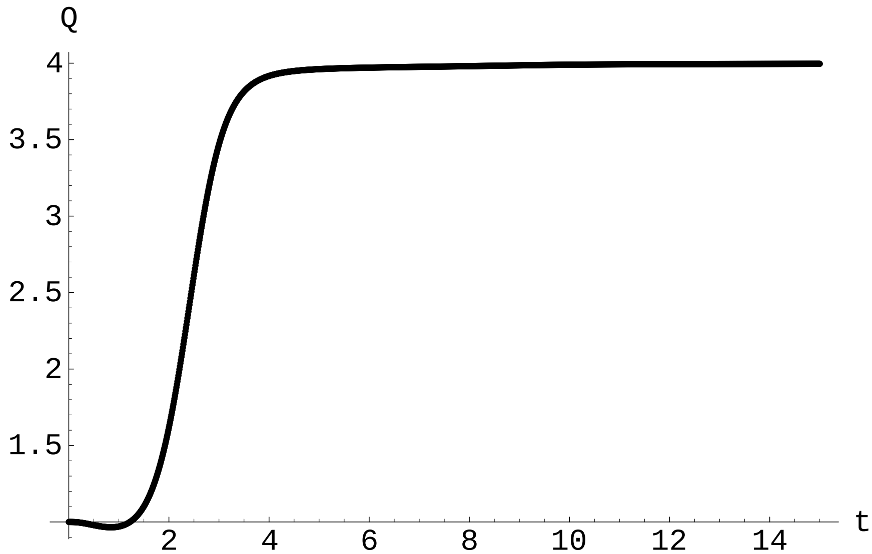

Figure 2: The enhancement factor Q *versus* co-moving time (in units of ω^{-1}) for a super-horizon mode with $ck = 0.01 \times \omega a_{\rm cr}$.

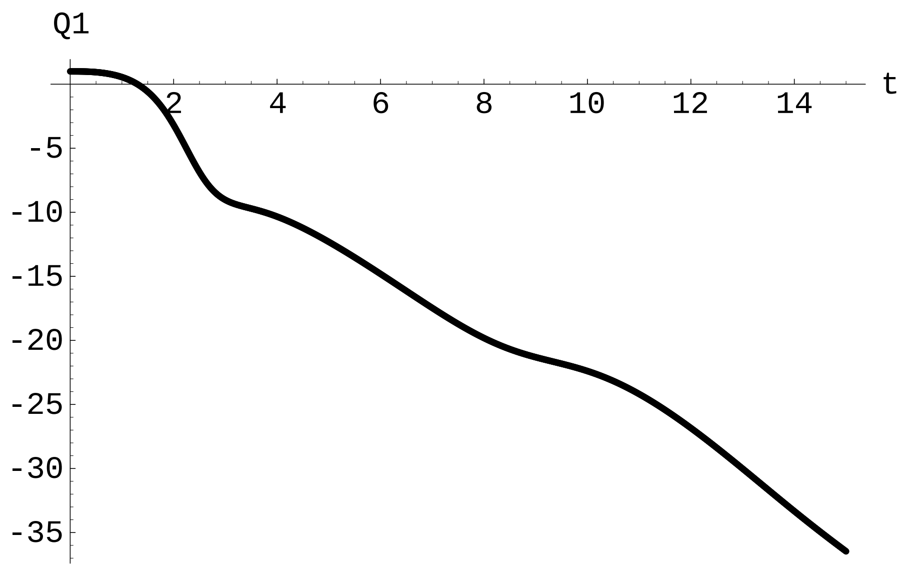

Figure 3: The enhancement factor $Q1$ *versus* co-moving time (in units of ω^{-1}) for a near-horizon mode with $ck = 1.1 \times \omega a_{\rm cr}$.

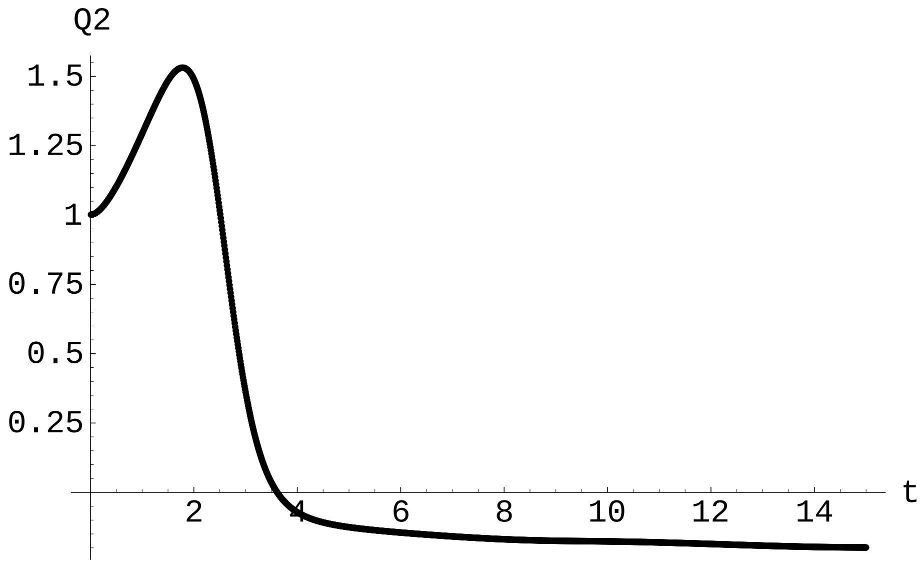

Figure 4: The enhancement factor $Q2$ *versus* co-moving time (in units of ω^{-1}) for a near-horizon mode with $ck = 1.1 \times \omega a_{\mathrm{cr}}$.

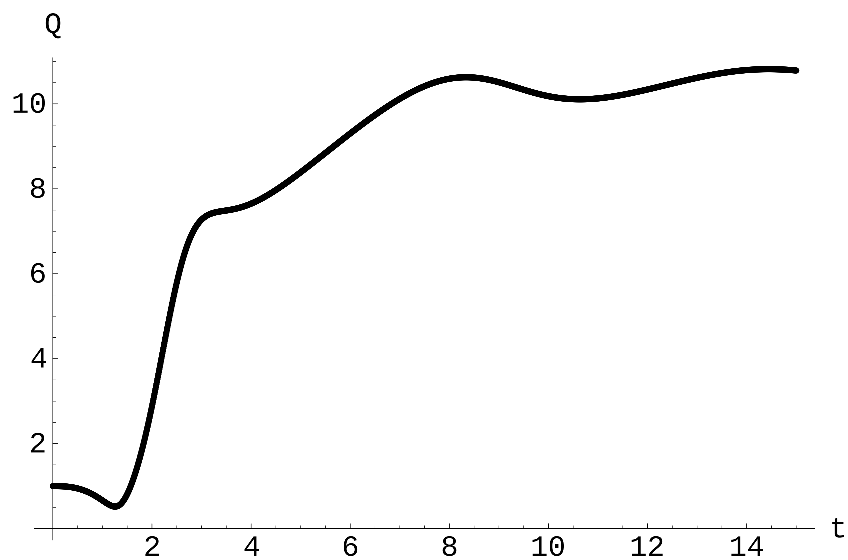

Figure 5: The enhancement factor Q *versus* co-moving time (in units of ω^{-1}) for a near-horizon mode with $ck = 1.1 \times \omega a_{\mathrm{cr}}$.

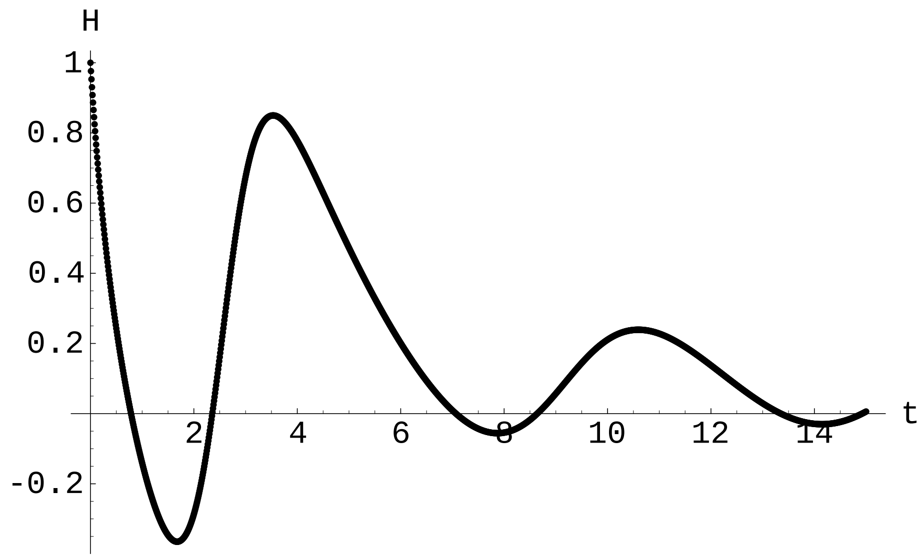

Figure 6: The Hubble paramemter H *versus* co-moving time (in units of ω). The first period of $H < 0$ coincides with the largest growth in Q on Figure 5.

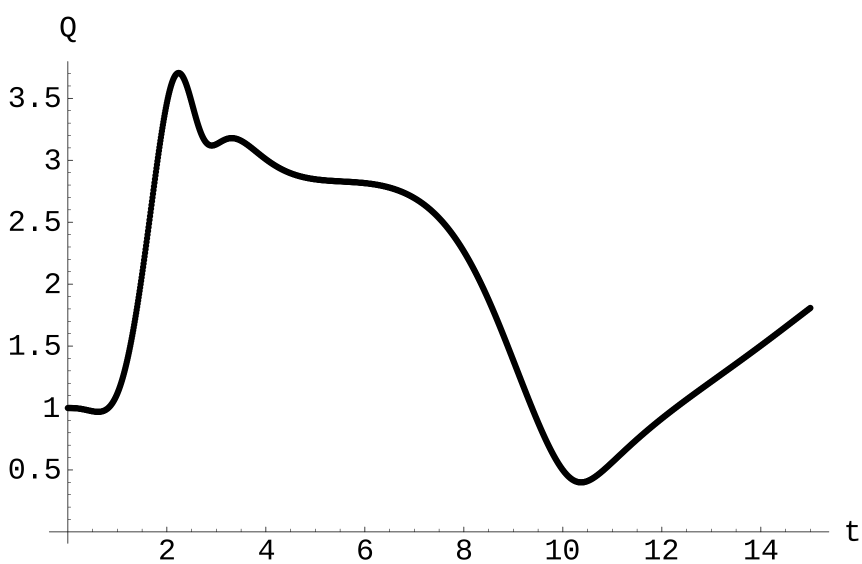

Figure 7: The enhancement factor Q *versus* co-moving time (in units of ω^{-1}) for a near-horizon mode with $ck = 2 \times \omega a_{\rm cr}$.

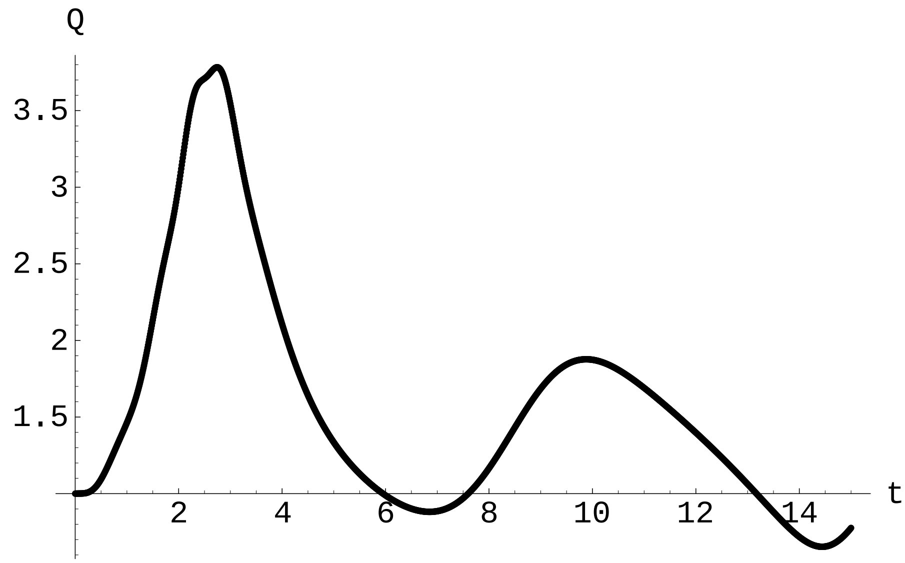

Figure 8: The enhancement factor Q *versus* co-moving time (in units of ω^{-1}) for a near-horizon mode with $ck = 5 \times \omega a_{\rm cr}$.

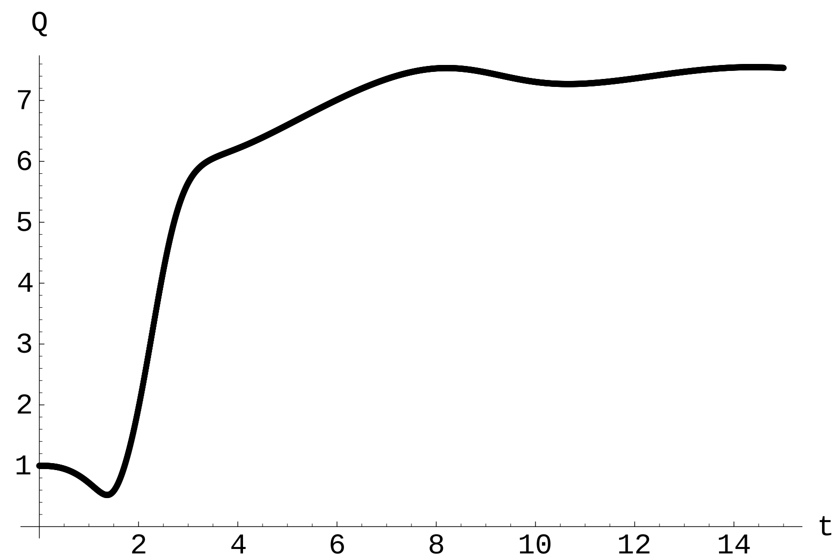

Figure 9: The enhancement factor Q *versus* co-moving time (in units of ω^{-1}) for a near-horizon mode with $ck = 0.9 \times \omega a_{\rm cr}$.

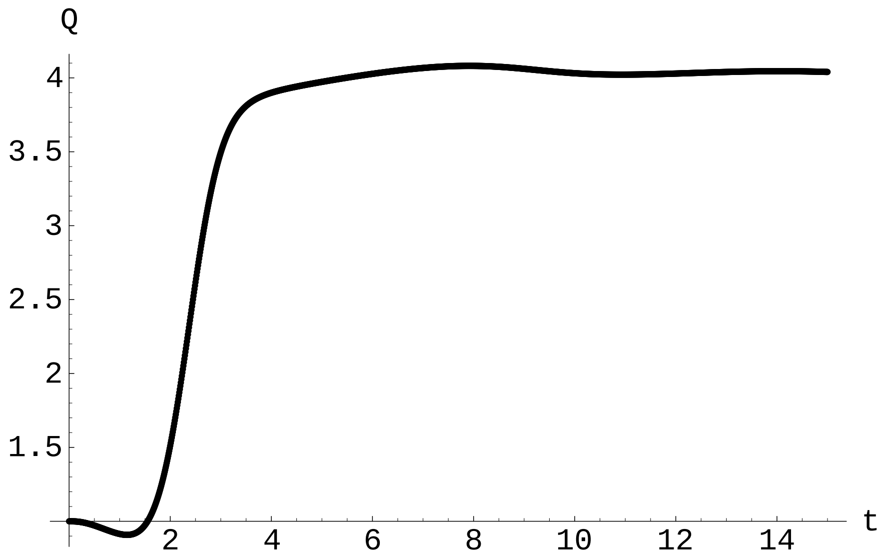

Figure 10: The enhancement factor Q *versus* co-moving time (in units of ω^{-1}) for a near-horizon mode with $ck = 0.5 \times \omega a_{\rm cr}$.

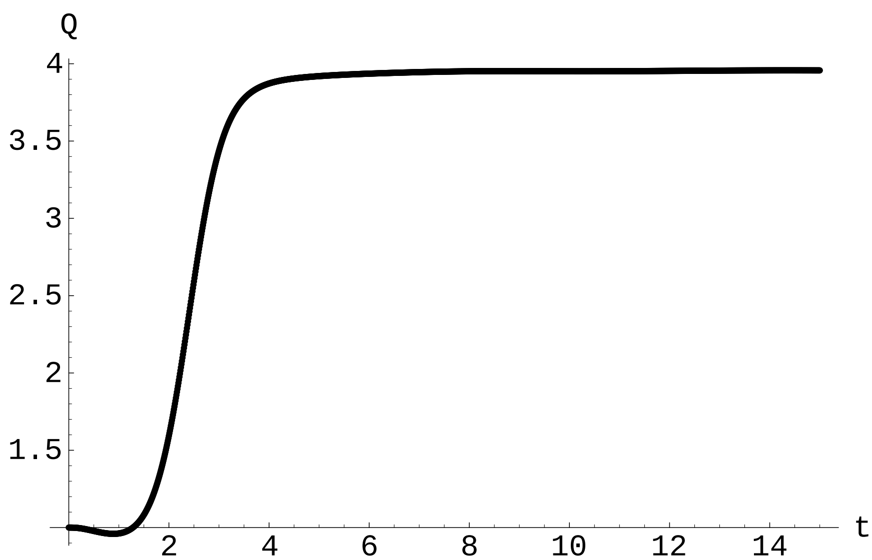

Figure 11: The enhancement factor Q *versus* co-moving time (in units of ω^{-1}) for a near-horizon mode with $ck = 0.2 \times \omega a_{\rm cr}$.

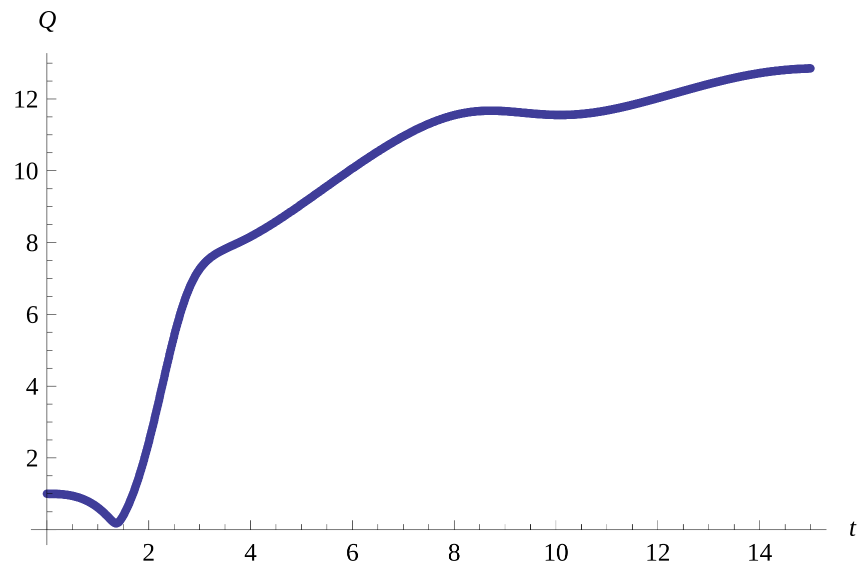

Figure 12: The enhancement factor Q *versus* co-moving time (in units of ω^{-1}) for a near-horizon mode with $ck = 0.99 \times \omega a_{\rm cr}$.

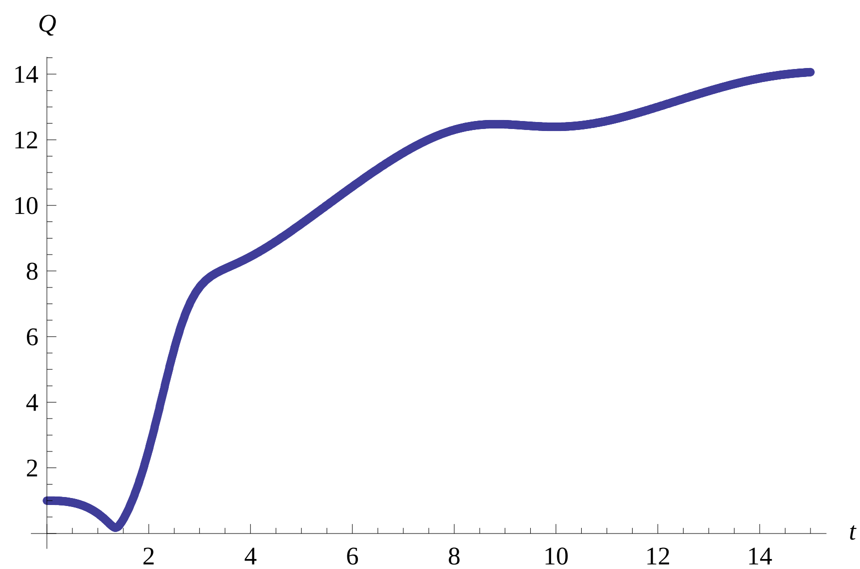

Figure 13: The enhancement factor Q *versus* co-moving time (in units of ω^{-1}) for a near-horizon mode with $ck = 1.01 \times \omega a_{\rm cr}$.