Infinite Ramanujan graphs and the Kadison-Singer Problem

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Chapter 1

Introduction

Here I will provide an outline of the proof and the path we follow in order to prove the Kadison-Singer problem

Chapter 2

Preliminaries

The basic result of this chapter is to prove that there is an infinite sequence of bipartite Ramanujan graphs and in addition an infinite sequence of irregular bipartite Ramanujan graphs. The steps we follow in order to derive a simple proof of this statement are the following: Firstly we will provide definitions about the matching polynomial and the path tree. Attributes that make the matching polynomial really useful, are also stated. We will continue to state some theorems about the real rootedness of certain polynomials.

2.1 Some basic definitions

Definition 2.1. Biregular bipartite graph
A (c, d)- biregular bipartite graph is a graph whose vertices can be divided into two disjoint and independent sets U and V such that every vertex in U has exactly c neighbors in V and none in U, and every vertex in V has exactly d neighbors in U and none in V.

Definition 2.2. Adjacency matrix of a graph
Let G be a graph with n vertices. We define a matrix A with n rows and columns, such that in its (i, j) entry we put the number of edges that connect the vertices i and j.

Definition 2.3. Ramanujan graph
Let G be a graph with adjacency matrix A_G and maximum degree d. We say that this graph is Ramanujan if all the eigenvalues of its adjacency matrix lie in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$

Definition 2.4. Biregular Ramanujan graph
Let G be a (c, d)-bipartite graph with adjacency matrix A_G. We call this graph Ramanujan, should all the eigenvalues of A_G be bounded in absolute value by $\sqrt{c-1} + \sqrt{d-1}$.

Definition 2.5. Matching polynomial of a graph
Let $G = (V, E)$ be a graph with n vertices . Also let m_i denote the number of matchings on the graph with i edges. We define the matching polynomial to be $Q_x(G) = \sum_{i \geq 0} (-1)^i m_i x^{n-2i}$

Definition 2.6. Path tree
Let again, $G = (V, E)$ be a graph with n vertices. Also chose a vertex a of G. For this vertex, the path tree, written $T_a G$ is a tree whose vertices correspond to all paths in G that start at $a \in G$ and do not contain any vertex twice. One path is connected to another if one extends the other by one vertex.

Definition 2.7. Universal cover
Let $G = (V, E)$ be a graph. The universal cover of a this graph, written as U, is the infinite tree T such that every connected lift of G is a quotient of the tree. It can be defined concretely by first fixing a "root" vertex $v_0 \in G$ and then placing a vertex in U for every non backtracking walk $(u_0, u_1, ..., u_l)$ of any length $l \in \mathbb{N}$, where a walk is non backtracking if $u_{i-1} \neq u_{i+1}$ for all i's. Two vertices are adjacent if and only if the walk corresponding to one can be obtained by appending one vertex to the walk corresponding to the other. That is, the edges of T are all of the form $(v_0, v_1, ..., v_l) \sim (v_0, .., v_l, v_{l+1}))$.

The following definition is crucial.

Definition 2.8. Interlacing zeros of polynomials
Let f be a polynomial that has exactly n zeros at the points $a_1, ..., a_n$. And also let g be a polynomial with its only roots being at the points $b_1, b_2, .., b_{n-1}$. We say that g interlaces f if we have the following relation

$$a_1 \leq b_1 \leq a_2 \leq ... \leq b_{n-1} \leq a_n$$

Definition 2.9. (Spectral radius)
Let G be a graph with adjacency matrix A. Then its spectral radius is the absolute value of its maximun eigenvalue, which we denote by $\rho(G)$

Chapter 3

Theorems about the matching polynomial

Now we can state our first theorem. It was first proved by E.Lieb and O.Heilemann in their paper [1].

Theorem 3.1. *For every graph $G = (V, E)$ the roots of its matching polynomial are all real.*

In order to prove this theorem we need first to establish the validity of the following theorems.

Theorem 3.2. ***Recurrence relation for the matching polynomial*** *The matching polynomial obeys the recurrence relation:*

$$Q_x(G) = Q_x(G-i) - \sum_{j \sim i} Q_x(G-i-j)$$

where $G - i$ is the graph with the i-th vertex deleted.

Proof. Let's denote by M the set of all matchings of our graph. We fix a vertex i and divide the matchings of G into two classes: those that involve vertex i and those that do not. The number of matchings of size k that do not involve i is $m_k(G-i)$. On the other hand, those that do involve i connect

i to one of its neighbors. To count these we enumerate the neighbors j of i. A matching of size k that includes the edge (i, j) can be written as the union of (i, j) and a matching of size $k-1$ in $G-i-j$. So, the number of matchings that involve i is

$$\sum_{j\sim i} m_{k-1}(G-i-j)$$

So,

$$m_k(G) = m_k(G-i) + \sum_{j\sim i} m_{k-1}(G-i-j)$$

To turn this into a reccurence for $Q_x(G)$, we can write:

$$x^{n-2k}(-1)^k m_k(G) = x x^{n-1-2k}(-1)^k m_k(G-i) - x^{n-2-2(k-1)}(-1)^{k-1} \sum_{j\sim i} m_{k-1}(G-i-j)$$

this proves the desired reccurence relation. □

Now we will review a relation that arises when studying trees

Lemma 3.3. A relation about trees
If G is a tree then its characteristic polynomial is the same as its matching polynomial

Proof. Let A_G be the adjacency matrix of our tree. And let $\chi_G(x)$ denote its characteristic polynomial. Then, by definition:

$$\begin{aligned}\chi_G(x) = det(xI - A_{Gs}) &= \sum_{\sigma\in S_n} sign(\sigma) \prod_{i=1}^{n} (xI - A_G)_{i,\sigma(i)} \\ &= \sum_{\sigma\in S_n} sign(\sigma) x^{|i:i=\sigma(i)|} \prod_{i\neq\sigma(i)} (-A_G)_{i,\sigma(i)}\end{aligned}$$

We will prove that the only permutations that contribute to this sum are the ones for which $\sigma\sigma(a) = a$. They correspond to matchings. If σ is a permutation for which there is an a such that $\sigma(\sigma(a)) \neq a$ then a k

must exist such that $\exists a_1, .., a_k$ such that $\sigma(a_i) = a_{i+1}$ $\forall i \in (1, .., k-1)$ and moreover $\sigma(a_k) = a_1$. For the term $(-A_s)_{i,\sigma(i)}$ to contribute it must be the case that $(A_G)_{i,\sigma(i)} = 1$ for all i's, and of course $(A_G)_{a_k,a_1} = 1$. Now lets remind ourselves that G is a tree. But, the only cycles that can occur in a tree with non repeating vertices are cycles of length 2. Thus the number of permutations with k cycles of length 2 is equal to the number of matchings with k edges. As the sign of a permutation with k cycles of length 2 is $(-1)^k$, the coefficient of x^{n-2k} is $(-1)^k m_k(G)$

□

In the next theorem we will derive a somewhat strange equality:

Theorem 3.4. *For every graph G and vertex $a \in G$,*

$$\frac{Q_x(G-a)}{Q_x(G)} = \frac{Q_x(T_a(G)-a)}{Q_x(T_a(G))}$$

Let us first note that the upper-right hand side is a little odd. It is a forrest obtained by removing the root of the tree $T_a(G)$. We may write it as a disjoint union of trees as:

$$T_a(G) - a = \bigcup_{b \sim a} T_b(G-a)$$

Proof. If G is a tree, then the left and right sides are identical, and so the equality holds. As the only graphs with less than 3 vertices are trees, the theorem holds for all graphs on at most 2 vertices. We will now prove it by induction on the number of vertices. We will use the recurrence relation of our matching polynomial to expand the reciprocal of the left hand-side.

$$\frac{Q_x(G)}{Q_x(G-a)} = \frac{xQ_x(G-a) - \sum_{b\sim a} Q_x(G-a-b)}{Q_x(G-a)} = x - \sum_{b\sim a} \frac{Q_x(G-a-b)}{Q_x(G-a)}$$

By applying the inductive hypothesis to $G-a$, we see that this equals:

$$x - \sum_{b\sim a} \frac{Q_x(T_b(G-a)-b)}{Q_x(T_b(G-a))} \tag{3.1}$$

To simplify this expression, we examine these graphs carefully. By the observation we made before the proof,

$$T_b(G-a)-b=\bigcup_{c\sim b,c\neq a}T_c(G-a-b).$$

Similarly,

$$T_a(G)-a=\bigcup_{c\sim a}T_c(G-a),$$

which implies

$$Q_x(T_a(G-a)=\prod_{c\sim a}Q_x(T_c(G-a))$$

Now let ab be the vertex in $T_a(G)$ corresponding to the path from a to b. We also have

$$T_a(G-a-ab)=\bigcup_{c\sim a,c\neq b}T_c(G-a)\cup\bigcup_{c\sim b,c\neq a}T_c(G-a-b)=$$

$$\bigcup_{c\sim a,c\neq b}T_c(G-a)\cup T_b(G-a-b)$$

which implies

$$Q_x(T_a(G-a-ab))=\prod_{c\sim a,c\neq b}Q_x(T_c(G-a))Q_x((T_b(G-a)-b))$$

Thus,

$$\frac{Q_x(T_a(G-a-ab))}{Q_x(T_a(G)-a)}=\frac{\prod_{c\sim a,c\neq b}Q_x(T_c(G-a))Q_x((T_b(G-a)-b))}{\prod_{c\sim a}Q_x(T_x(G-a))}$$

Plugging this in to equation 3.1 we get:

$$\frac{Q_x(G)}{Q_x(G-a)}=x-\sum_{b\sim a}\frac{Q_x(T_a(G)-a-ab)}{Q_x(T_a(G)-a)}=$$

$$\frac{xQ_x(T_a(G-a))-\sum_{b\sim a}Q_x(T_a(G)-a-ab)}{Q_x(T_a(G)-a)}=$$

$$\frac{Q_x(T_a(G))}{Q_x(T_a(G)-a)}$$

We obtain the equality claimed in the theorem by taking the reciprocals of both sides. □

We will now use theorem 3.4 to prove that the matching polynomial of a graph divides the matching polynomial of its path tree (the path tree starts from a random vertex $a \in G$)

Theorem 3.5. *For every vertex $a \in G$, the polynomial $Q_x(G)$ divides the polynomial $Q_x(T_a(G))$*

Proof. We again prove this by induction in the number of vertices in G, using as our base case graphs with at most 2 vertices. Now by induction for $b \sim a$, $Q_x(G-a)$ divides $Q_x(T_b(G-a))$
As

$$T_a(G) - a = \bigcup_{b \sim a} T_b(G-a)$$

we can easily obtain that

$$Q_x(G-a) \mid Q_x(T_a(G) - a)$$

and so

$$\frac{Q_x(T_a(G)-a)}{Q_x(G-a)}$$

is a polynomial with variable x. To finish the proof we apply theorem 3.4, which gives out

$$Q_x(T_a(G)) = Q_x(T_a(G)-a)\frac{Q_x(G)}{Q_x(G-a)} =$$

$$Q_x(G)\frac{Q_x(T_a(G)-a)}{Q_x(G-a)}$$

which finally implies that the matching polynomial of our graph divides the matching polynomial of its path tree. □

Now in order to complete the proof of theorem 3.1 we just need to show that the roots of the matching polynomial of the path tree are all real. In other words, since we proved that the matching polynomial of any tree is the same as the characteristic polynomial of its adjacency matrix, we just need to show that the eigenvalues of the adjacency matrix of any tree are all real. Since the matching polynomial is of degree n and divides the matching

polynomial of its path tree, it can easily be concluded that it has n real roots. Well the characteristic polynomial of any tree is a real symmetric matrix (of any graph in general), therefore it is a Hermitian matrix and as a result has only real eigenvalues.
We now continue to another very significant theorem that bounds the roots of the matching polynomial.

Theorem 3.6. *Let G be a graph whose maximum degree is d. Then the roots of the matching polynomial are bounded by $2\sqrt{d-1}$.*

Proof. Lets first make the observation that the maximum degree of a path tree of G, lets say $T_a(G)$ would also be d. Therefore if we show the theorem for trees then since the matching polynomial of a graph divides the matching polynomial of its path tree, we can generalize it for arbitrary graphs. So lets take a tree T and its adjacency matrix A. Choose a random vertex to be the root of the tree, and define its height to be 0. For every other vertex α, define $h(a)$, to be its distance to the root. Now define D to be the diagonal matrix whose (a, a) entry is

$$D(a, a) = (\sqrt{d-1})^{h(a)}$$

Recall that the eigenvalues of A are the same as the eigenvalues of DAD^{-1}. We will use the fact that all eigenvalues of a non-negative matrix are upper bounded in absolute value by its maximum row sum.
So we need to prove that all row sums of DAD^{-1} are at most $2\sqrt{d-1}$. There are three types of vertices to consider. First, the row of the root has up to d entries that are all $\frac{1}{\sqrt{d-1}}$. And for $d \geq 2$ we know that $\frac{d}{\sqrt{d-1}} \leq 2\sqrt{d-1}$. Every leaf only has one non-zero entry in its row, and that entry equals $\sqrt{d-1}$. The intermediate vertices have one entry in their row that equals $\sqrt{d-1}$, and up to $d-1$ entries that are equal to $\frac{1}{\sqrt{d-1}}$. If we sum that up we get $2\sqrt{d-1}$.
Therefore the roots of the matching polynomial of the path tree of our graph are bounded by $2\sqrt{d-1}$, therefore our matching polynomial has zeros that do not exceed in absolute value $2\sqrt{d-1}$. □

Lemma 3.7. Let G be a graph and let U be its universal cover. Then the roots of $Q_x(G)$ are bounded in absolute value by $\rho(U)$

3.1 2-lifts and the matching polynomial

Bilu and Linial [2] suggested constructing Ramanujan graphs through a sequence of 2-lifts of a base graph. Given a graph $G = (V, E)$, a 2-lift is a graph that has two vertices for each vertex in V. This pair of vertices is called the fiber of the original vertex. Every edge in E corresponds to two edges in the 2-lift. If (u, v) is an edge in E, $\{u_0, u_1\}$ is the fiber of u, and $\{v_0, v_1\}$ is the fiber of v, then the 2-lift can either contain the pair of edges:

1. $\{(u_0, v_0), (u_1, v_1)\}$

2. $\{(u_0, v_1), (u_1, v_0)\}$

To analyze the eigenvalues of a 2-lift, Bilu and Linial study signings $s : E \to \{\pm 1\}$ of the edges of G. They place signings in one-to-one correspondence with 2-lifts by setting $s(u, v) = 1$ if edges of type (1) appear in the 2-lift and $s(u, v) = -1$ if edges of type (2) appear. Then they define the signed adjacency matrix A_s to be the same as the adjacency matrix of G, except that the entries corresponding to an edge (u, v) are $s(u, v)$.
The central theorem of this section is stated bollow. This is the first use of the probabilistic method in this paper.

Theorem 3.8. *Let G be a graph and A_s the adjacency matrix with entries $s(u, v)$. We define $f_s(x) = \det(xI - A_s)$ for an arbitrary 2-lift s. Then by letting s to be uniformly random we get*

$$\mathbb{E}_{s \in \{\pm 1\}^m} f_s(x) = Q_x(G)$$

Proof. Let $sym(S)$ denote the set of permutations of a set S and let $|\pi|$ denote the number of inversions of a permutation π. Expanding the determinant as a sum over permutations $\sigma \in sym([n])$, we have:

$$\begin{aligned}
&\mathbb{E}_s[det(xI - A_s)] = \\
&= \mathbb{E}_s \sum_{\sigma \in sym[n]} (-1)^{|\sigma|} \prod_{i=1}^{n} (xI - A_s)_{i,\sigma(i)} \\
&= \mathbb{E}_s \sum_{\sigma \in sym[n]} (-1)^{|\sigma|} x^{|\{i:i=\sigma(i)|} \prod_{i \neq \sigma(i)} (-A_s)_{i,\sigma(i)} \\
&= \sum_{\sigma \in sym[n]} (-1)^{|\sigma|} x^{|\{i:i=\sigma(i)|\}} \mathbb{E}_s \prod_{i \neq \sigma(i)} (-s_{i,\sigma(i)})
\end{aligned}$$

Since $s_{i,j}$ are independent with $\mathbb{E}[s_{i,j}]$, only those products which contain even powers (0 or 2) of the $s_{i,j}$ survive. Thus, we may restrict our attention to the permutations σ which contain only orbits of size two, or in other words in permutations that can be written as the product of disjoint 2-cycles. These are the k-matchings on S. Now we change the index of the summation. The index was the permutation itself, now we change it to k, which counts the number of 2-cycles this permutation has. So if $|\{i : i = \sigma(i)| = n - 2k$ then the sign of the permutation would be $(-1)^k$ and the coefficient of $(-1)^k x^{n-2k}$ would be $m_k(G)$, since the number of permutations with k-cycles correspond to the number of k-matchings. As a result:

$$\mathbb{E}_s[\det(xI - A_s)] = Q_x(G)$$

□

In order to demonstrate the importance of the signed matrix of a graph G, namely A_s the following theorem must be stated.

Lemma 3.9. Let A be the adjacency matrix of a graph G, and A_s the signed adjacency matrix associated with a 2-lift $\hat{G}$. Then every eigenvalue of A and every eigenvalue of A_s are eigenvalues of $\hat{G}$. Furthermore, the multiplicity of each eigenvalue of $\hat{G}$ is the sum of its multiplicities in A and A_s.

Proof. It is not hard to see that the adjacency matrix of $\hat{G}$ is $\hat{A} =$

$$\begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}$$

where A_1 is the adjacency matrix of $(V, s^1(1))$ and A_2 the adjacency matrix of $(V, s^{-1}(-1))$. So $A = A_1 + A_2$, $A_s = A_1 - A_2$. Let v be an eigenvector of A with eigenvalue μ. It is easy to check that $\hat{v} = (v, v)$ is an eigenvector of $\hat{A}$ with eigenvalue μ. Similarly, if u is an eigenvector of A_s with eigenvalue λ, then $\hat{u} = (u, -u)$ is an eigenvector of $\hat{A}$ with eigenvalue λ. As the $\hat{v}$'s and $\hat{u}$'s are perpendicular and $2n$ in number, they span all the eigenvectors of $\hat{A}$.

□

Chapter 4

Interlacing families

Definition 4.1. We say that the polynomial $g(x) = \prod_{i=1}^{n-1}(x - a_i)$ interlaces a polynomial $f(x) = \prod_{i=1}^{n}(x - b_i)$ if

$$b_1 \leq a_1 \leq b_2 \leq ... \leq a_{n-1} \leq b_n$$

We say that the polynomials $f_1, f_2, ..., f_k$ have a common interlacing if there is a polynomial g so that g interlaces f_i for each i.

Lemma 4.2. Let $f_1, .., f_k$ be polynomials of the same degree that are real-rooted and have positive leading coefficients. Define

$$f_\emptyset = \sum_{i=1}^{k} f_i$$

If $f_1, .., f_k$ have a common interlacing, then there exists an i such that

$$maxroot f_i \leq maxroot f_\emptyset$$

Proof. Let the polynomials be of degree n. Let g be a polynomial that interlaces all the f_i and let a_{n-1} be the largest root of g. As each f_i has a positive leading coefficient, it is positive for sufficiently large x. As each f_i has exactly one root that is at least a_{n-1}, each f_i is non positive at a_{n-1}. So, $f_\emptyset$ is also non-positive at a_{n-1}, and eventually becomes positive. This tells us that $f_\emptyset$ has a root that is at least a_{n-1}, so the largest root is at least a_{n-1}. Let b_n be this root. As $f_\emptyset$ is the sum of the f_i, there must be some i for which $f_i(b_n) \geq 0$. As f_i has at most one root that is at least a_{n-1}, and $f_i(a_{n-1}) \leq 0$, the largest root that is at least a_{n-1} and $f_i(a_{n-1}) \leq 0$, the largest root of f_i is at least a_{n-1} and at most b_n. □

Now lets introduce a significant definition.

Definition 4.3. Interlacing Family
Let $S_1, .., S_m$ be finite sets and for every assignment $s_1, \ldots, s_m \in S_1 \times \cdots \times S_m$, let $f_{s_1,\ldots,s_m}$ be a real-rooted degree n polynomial with positive leading coefficients. For a partial assignment $s_1, \ldots, s_k \in S_1 \times \cdots \times S_k$ with $k < m$ define

$$f_{s_1,\ldots,s_k} = \sum_{s_{k+1}\in S_{k+1},\ldots,s_m\in S_m} f_{s_1,\ldots,s_k,s_{k+1},\ldots,s_m}$$

as well as

$$f_\emptyset = \sum_{s_1,\ldots,s_m} f_{s_1,\ldots,s_m}$$

We say that the polynomials $\{f_{s_1,\ldots,s_m}\}$ form an interlacing family if for all $k = 0, \ldots, m-1$ and all $s_1, \ldots, s_k \in S_1 \bigtimes \ldots \bigtimes S_k$, the polynomials

$$\{f_{s_1,\ldots,s_k,t}\}_{t\in S_{k+1}}$$

have a common interlacing.

Theorem 4.4. *Let $S_1, \ldots, S_m$ be finite sets and let $\{f_{s_1,\ldots,s_m}\}$ be an interlacing family of polynomials. Then, there exists some $s_1, \ldots, s_m \in S_1 \times \cdots \times S_m$ such that $maxroot f_{s_1,\ldots,s_m} \leq maxroot f_\emptyset$.*

Proof. It is an easy consequence of the previously stated lemma. □

Lemma 4.5. Let $\epsilon > 0$ and f a real-rooted polynomial of degree n. Then the polynomial

$$f_\epsilon = (I - \epsilon\partial)^n f$$

is real rooted and has simple roots.

Proof. Lets first take the operator $T_\epsilon = I - \epsilon\partial$. It suffices to prove that $T_\epsilon f$ is real rooted with $n-1$ distinct roots. If this statement holds, then one can repeat this process n times. Let $\{x_i\}_1^n$ be all the real roots of f. Lets suppose that the polynomial $T_\epsilon f = f - \epsilon\dot{f}$ has a non-real root z_0. Take a sufficiently small contour around this root, and a sufficiently small ϵ, such that

$$|\epsilon \sum_i \frac{1}{z - x_i}| < 1 \iff |\epsilon \frac{\dot{f}}{f}| < 1 \iff |-\epsilon \dot{f}| < |f|$$

on this contour. Using Rouche's theorem we can deduce that inside this contour f and $T_\epsilon f$ have the same number of zeros. Therefore T_ϵ is real rooted. To continue, lets assume that f has a zero z_0 of multiplicity m. Namely $f(z) = (z - z_0)^m g(z)$, and z_0 is not a root of $g(z)$. Then

$$T_\epsilon f = (z - z_0)^{m-1}((z - z_0)g(z) - \epsilon m g(z) - \dot{g}(z)(z - z_0))$$

One can easily see that the multiplicity of z_0 has dropped by 1. □

Lemma 4.6. Suppose $f_1, \ldots, f_m$ are monic, real-rooted polynomials of degree n. Let $\lambda_k(f_j)$ denote the k-th largest root of f_j and let μ be any probability measure on the set $\{1, \ldots, m\}$. If $f_1, \ldots, f_m$ have a common interlacing, then for all $k \in \{1, \ldots, n\}$:

$$\min_i \lambda_k(f_i) \leq \lambda_k(\mathbb{E}_{I \sim \mu}(f_I)) \leq \max_i \lambda_k(f_i)$$

Proof. Fix $k \in \{1, \ldots, n\}$. Let $a_n \leq, \ldots, \leq a_1$ be the roots of the common interlacing of the polynomials $f_1, \ldots, f_m$ i.e. $a_{k+1} \leq \lambda_k(f_i) \leq a_k$. The polynomials (f_i) are all monic, so they all have the same sign at a_{k+1} and the same opposite sign at a_k. Hence their average changes sign in the interval $[a_{k+1}, a_k]$, so it has to vanish in the same interval by the intermediate value theorem. It is therefore real-rooted by a simple counting argument. Moreover it is easy to see that the root is sandwiched between the smallest and the largest root of the f_i's on the interval $[a_{k+1}, a_k]$. □

Lemma 4.7. Let $f_1, \ldots, f_k$ be (univariate) polynomials of the same degree with positive leading coefficients. Then $f_1, \ldots, f_k$ have a common interlacing if and only if $\sum_{i=1}^k \lambda_i f_i$ is real rooted for all convex combinations $\lambda_i \geq 0$, $\sum_{i=1}^k \lambda_i = 1$.

Proof. Instead of proving this theorem for an arbitrary k, we will prove it for $k = 2$, and the generalized one follows by induction. Namely, we will prove that the following two statements are equivalent

1. f and g have a common interlacing.

2. The polynomial $h_t = tf + (1-t)g$ is real-rooted for all $t \in [0, 1]$.

The statement (1. $\Rightarrow$ 2.) follows from the previous lemma. Assume as a first step that f and g have no common roots and that their roots are simple. Under these assumptions the roots of h_t trace n different intervals I_i on the real line as t varies from 0 to 1, starting from the roots of g and ending at the roots of f. Each one of these intervals contains exactly one root of f and one root of g. Otherwise, (taking g as an example) there would exist a $t \neq 0$ and $z \in \mathbb{R}$ such that $h_t(z) = g(z) = 0$ which would imply that $f(z) = 0$ which in turn contradicts the no-common-roots assumption. Therefore one can choose subintervals $J_i \subset I_i$ with pairwise disjoint interiors containing one root of f and one root of g only, hence establishing interlacing. To prove the general case, notice that the no-common-roots assumption is not problematic since one can always factor the common roots out and put them back at the end. One could easily get an interlacing sequence for f and g from an interlacing sequence of the factored-out. $\square$

Chapter 5

Real Stable polynomials

In this chapter we will establish the real-rootedness of a class of polynomials. We will do this by considering a multivariate generalization of real-rootedness called real-stability. In particular, we will show that the univariate polynomials we are interested in are the images, under well-behaved linear transformation, of a multivariate real stable polynomial.

Definition 5.1. A multivariate polynomial $f \in \mathbb{R}[z_1, \ldots, z_n]$ is called real stable if it is the zero polynomial or if

$$f(z_1, .., z_n) \neq 0$$

whenever the imaginary part of every z_i is strictly positive.

Lemma 5.2. Let $A_1, .., A_m$ be positive semi definite matrices. Then

$$det(z_1 A_1 + ... + z_m A_m)$$

is real stable

Proof. By a standard continuity argument using Hurwitz' theorem it suffices to prove the result only in the case when all matrices $A_1, .., A_n$ are positive definite. Set $z(t) = \alpha + \lambda t$ with $\alpha \in \mathbb{R}^n$,$\lambda \in \mathbb{R}^n_+$ and $t \in \mathbb{R}$. Note that $P = \lambda_1 A_1 + ... + \lambda_n A_n$ is positive definite and thus it has a square root. Then

$$f(z(t)) = \det(P)\det(tI + P^{1/2} H P^{-1/2})$$

where $H = \alpha_1 A_1 + ... + \alpha_n A_n$. Since $f(z(t))$ is a constant multiple of the characteristic polynomial of the Hermitian matrix H, it has only real zeros. □

Lemma 5.3. Let $p, q \in \mathbb{R}^+$ and u, v variables. Let $T = 1 + p\partial_u + q\partial_v$ be an operator acting on polynomials of two complex variables. Then T preserves real stability.

Proof. Let $f(u, v)$ be a real stable polynomial of two complex variables u, v. Should we fix v we can get for $f(u, v)$, since it is a one variable complex polynomial that $f(u, v) = c_1 \prod_i (u - w_{i,1})$. On the other hand, should we fix u, we get that $f(u, v) = c_2 \prod_j (v - w_{j,2})$. Now we compute $Tf(u, v)$. We easily get that:

$$Tf(u, v) = f(u, v) + pf_u(u, v) + qf_v(u, v) = f(u, v)(1 + \sum_i \frac{p}{u - w_{i,1}} + \sum_j \frac{q}{v - w_{j,2}})$$

Since $f(u, v)$ is real stable then the imaginary parts of $w_{i,1}, w_{j,2}$ are non-positive. Should we suppose that the imaginary parts of both u, v are positive, the the imaginary part of

$$\sum_i \frac{p}{u - w_{i,1}} + \sum_j \frac{q}{v - w_{j,2}}$$

is negative. Therefore establishing that $Tf(u, v)$ is real stable. □

Before giving the proof to a very important lemma, let's first state the celebrated matrix-determinant lemma:

Lemma 5.4. Matrix determinant lemma
Let A be an invertible matrix and u, v vectors in $\mathbb{R}^n$. Then

$$\det(A + uv^T) = \det(A)(1 + v^T A^{-1} u)$$

.

Proof. If $\det(I + uv^T) = 1 + v^T u$ is shown then the result of the lemma is an easy consequence. Lets set $B = uv^T =$

$$\begin{bmatrix} u_1v_1 & \dots & u_1v_n \\ u_2v_1 & \dots & u_2v_n \\ \vdots & \ddots & \vdots \\ u_nv_1 & \dots & u_nv_n \end{bmatrix}$$

Let us start by finding the characteristic polynomial $p_B(x) = \det(B - xI)$ of B. Since B has $rank(B) \leq 1$, we know that it has at least $n-1$ eigenvectors associated to the eigenvalue 0. Since the sum of all eigenvalues must be $tr(B) = v^T u$, we see that

$$p_b(x) = \det(B - xI) = (-1)^n x^{n-1}(x - v^T u)$$

Now plug in $x = (-1)$ and deduce the required formula:

$$p_B(1) = \det(B + I) = 1 + v^T u$$

□

Lemma 5.5. Let A be an invertible matrix, and let a, b be vectors in $\mathbb{R}^n$ and $p \in [0, 1]$. Also suppose that Z_u is an operator that acts on multivariate polynomials induced by setting the u variable equal to zero. Then:

$$Z_u Z_v(1+p\partial_u+(1-p)\partial_v)\det(A+uaa^T+vbb^T) = p\det(A+aa^T)+(1-p)\det(A+bb^T)$$

Proof. Using the matrix-determinant lemma, which states that for every non-singular matrix A and every real number t that

$$\det(A + taa^T) = \det(A)(1 + ta^T A^{-1}a)$$

One consequence of this is Jacobi's formula for the derivative of the determinant:

$$\partial_t \det(A + taa^T) = \det(A)(a^T A^{-1}a)$$

This formula implies that:

$$Z_u Z_v(1+p\partial_u+(1-p)\partial_v)\det(A+uaa^T+vbb^T) = \det(A)(1+p(a^T A^{-1}a)+(1-p)(b^T A^{-1}b)).$$

By the matrix determinant lemma this equals

$$p\det(A + aa^T) + (1 - p)\det(A + bb^T)$$

. □

Using the above results we can prove our main result on real rootedness.

Theorem 5.6. *Let $a_1, \dots, a_m$ and $b_1, \dots, b_m$ be vectors in $\mathbb{R}^n$, and let $p_1, \dots, p_m$ be real numbers in $[0,1]$, and let D be a positive semidefinite matrix. Then every (univariate) polynomial of the form*

$$P(x) = \sum_{S\subset[m]} (\prod_{i\in S} p_i)(\prod_{i\notin S}(1-p_i)) \det(xI + D + \sum_{i\in S} a_i a_i^T + \sum_{i\notin S} b_i b_i^T)$$

is real rooted.

Proof. Let $u_1, \dots, u_m$ and $v_1, \dots, v_m$ be formal variables and define

$$Q(x, u_1, \dots, u_m, v_1, \dots, v_m) = \det(xI + D + \sum_i u_i a_i a_i^T + \sum_i v_i b_i b_i^T)$$

Lemma 5.2 implies that Q is real stable. We claim we can rewrite $P(x)$ as

$$P(x) = (\prod_{i=1}^m Z_{u_i} Z_{v_i} T_i) Q(x, u_1, \dots, u_m, v_1, \dots, v_m),$$

where $T_i = 1 + p_i \partial_{u_i} + (1-p_i)\partial_{v_i}$. To see this we prove by induction on k that

$$(\prod_{i=1}^k Z_{u_i} Z_{v_i} T_i) Q(x, u_1, \dots, u_m, v_1, \dots, v_m) =$$
$$\sum_{S\subset[k]} (\prod_{i\in S} p_i)(\prod_{i\in[k]\S}(1-p_i)) \det(xI + D + \sum_{i\in S} a_i a_i^T + \sum_{i\in[k]\S} b_i b_i^T + \sum_{i>k}(u_i a_i a_i^T + v_i b_i b_i^T))$$

The base case ($k = 0$) is trivially true, as it is the definition of Q. The inductive step follows from the matrix determinant lemma, stated earlier. The case $k = m$ is exactly the claimed identity. Starting with Q (a real stable polynomial) we can then apply Lemma 5.3 and the closure of real stable polynomials under the restrictions of variables to real constants to see that each of the polynomials above, including $P(x)$, is also real stable. As $P(x)$ is real stable and has one variable, it is real rooted. □

Having established the above useful theorem, we need to apply it to deduce another theorem, that will finally prove a stronger version of a conjecture of Bilu and Linial, regarding Ramanujan graphs.

Theorem 5.7. *We need to prove that the polynomial*

$$\sum_{s\in\{\pm 1\}^m} (\prod_{i:s_i=1} p_i)(\prod_{i:s_i=-1} (1-p_i)) \det(xI - A_s)$$

is real rooted.

Proof. For each vertex u, let d_u be its degree, and let $d = max_u d_u$. This is of course equivalent to proving that the following polynomial is real-rooted:

$$\sum_{s\in\{\pm 1\}^m} (\prod_{i:s_i=1} p_i)(\prod_{i:s_i=-1} (1-p_i)) \det(xI + dI - A_s) \tag{5.1}$$

as their roots only differ by d. We now define for each edge $(u, v) \in E$, the rank 1 matrices

$$L^1_{u,v} = (e_u - e_v)(e_u - e_v)^T$$
$$L^{-1}_{u,v} = (e_u + e_v)(e_u + e_v)^T$$

where e_u is the elementary unit vector in the direction u. Consider a signing s and let $s_{u,v}$ denote the sign it assigns to edge (u, v). Since the original graph has maximum degree d, we have

$$dI - A_s = \sum_{(u,v)\in E} L^{s_{u,v}}_{u,v} + D$$

where D is the diagonal matrix whose uth diagonal entry equals $d - d_u$. As the diagonal entries of D are non-negative, it is positive semidefinite. If we now set $a_{u,v} = (e_u - e_v)$ and $b_{u,v} = (e_u + e_v)$ we can express the polynomial 5.1 as

$$\sum_{s\in\{\pm 1\}^m} (\prod_{i:s_i=1} p_i)(\prod_{i:s_i=-1} (1-p_i)) \det(xI + D + \sum_{s_{u,v}=1} a_{u,v}a^T_{u,v} + \prod_{s_{u,v}=-1} b_{u,v}b^T_{u,v})$$

which we know from the previous theorem 5.6, to be real rooted. □

Before moving to the proof of some conjectures on Ramanujan graphs, we will provide a theorem that is useful in the proof of the Kadison-Singer problem. It is also a theorem about real stability.

Theorem 5.8. *If $p \in \mathbb{R}[z_1, \ldots, z_m]$ is real stable, then so is*

$$(1 - \partial_{z_1})p(z_1, \ldots, z_m)$$

Proof. If we fix $z_2, \ldots z_m$ then the resulting polynomial is also real stable. Let's then study that one with fixed all variables but one, which we call z. Then:

$$(1 - \partial_z)p(z) = p(z) - \partial_z p(z)$$

We study that polynomial when $Im(z) > 0$ to check whether it has any roots. We know that $Im(z) > 0 \implies p(z) \neq 0$. Then

$$p(z) - \partial_z p(z) = p(z)(1 - \frac{\partial_z p(z)}{p(z)}) = p(z)(1 - \sum_i \frac{c}{z - z_i})$$

where z_i are the roots of $p(z)$. Since $Im(z_i) < 0$ and $Im(z) > 0$ then $Im(z - z_i) > 0 \quad \forall i$. That implies that $Im(\sum_i \frac{c}{z-z_i}) > 0$, and therefore the theorem holds. □

Chapter 6

Results on a conjecture of Bilu and Linial

Now it is time to state a theorem about the characteristic polynomials of the 2-lifts.

Theorem 6.1. *The polynomials* $\{f_s\}_{s\in\{\pm1\}^m}$ *are an interlacing family.*

Proof. We will show that for every $0 \le k \le m-1$, every partial assignment $s_1 \in \pm1, \dots, s_k \pm 1$ and every $\lambda \in [0,1]$, the polynomial

$$\lambda f_{s_1,\dots,s_k,1}(x) + (1-\lambda) f_{s_1,\dots,s_k,-1}(x)$$

is real-rooted. The theorem will follow then from Lemma 4.7. To show that the above polynomial is real-rooted, we will apply Theorem 5.7 with $p_{k+1} = \lambda, \quad p_{k+2}, \dots, \quad p_m = 1/2$, and $p_i = \frac{1+s_i}{2}$ for $1 \le i \le k$. □

Theorem 6.2. *Let G be a graph with adjacency matrix A and universal cover T. Them there is a signing s of A so that all of the eigenvalues of A_s are at most $\rho(T)$. In particular, if G is d-regular, there is a signing s so that the eigenvalues of A_s are at most $2\sqrt{d-1}$.*

Proof. The first statement follows immediately from theorems 4.4 and 6.1 and lemma 3.7. The second statement follows by noting that the universal cover of a d-regular graph is the infinite d-regular tree, which has spectral radius at most $\sqrt{d-1}$. □

Lemma 6.3. Every non-trivial eigenvalue of a complete (c, d)-biregular graph is zero.

Proof. The adjacency matrix of this graph has rank 2, so all its eigenvalues other than $\pm\sqrt{cd}$, must be zero. □

Theorem 6.4. *For every $d \geq 3$ there is an infinite sequence of d-regular bipartite Ramanujan graphs.*

Proof. By lemma 3.9 and theorem 6.2, for every d-regular bipartite Ramanujan graph G, there is a 2-lift in which every non-trivial eigenvalue is at most $2\sqrt{d-1}$. As the 2-lift of a bipartite graph is bipartite, and the eigenvalues of a bipartite graph are symmetric about 0, this 2-lift is also a regular bipartite Ramanujan graph. Thus, for every d-regular bipartite Ramanujan graph G, there is another d-regular bipartite Ramanujan graph with twice as many vertices. □

Theorem 6.5. *For every $c, d \geq 3$, there is an infinite sequence of (c, d)-biregular bipartile Ramanujan graphs.*

Proof. We know from Lemma 6.3 that the complete (c, d)-biregular graph is Ramanujan. We will use this as a base for a construction of an infinite sequense of (c, d)-biregular bipartile Ramanujan graphs. Let G be any (c, d)-biregular Ramanujan graph. As mentioned in the definitions chapter, the universal cover of G is the infinite (c, d)-biregular tree, which has spectral radius $\sqrt{c-1} + \sqrt{d-1}$. Thus Theorem 6.2 tells us that there is a 2-lift of G with all its eigenvalues at most $\sqrt{c-1}+\sqrt{d-1}$. As this graph is bipartile, all of its non-trivial eigenvalues have absolute value at most $\sqrt{c-1}+\sqrt{d-1}$. So, the resulting 2-lift is a larger (c, d)-biregular bipartile Ramanujan graph. □

Chapter 7

The Mixed Characteristic Polynomial

This is essentially the second part of this thesis, in which the Kadison-Singer problem is proved.

Lemma 7.1. For every square matrix A and random vector v, we have

$$\mathbb{E}\det(A - vv^\star) = (1 - \partial_t)\det(A + t\mathbb{E}vv^\star)|_{t=0}$$

Proof. First, we assume that A is invertible. For the general case just choose a sequence of invertible matrices that approach A. By the 'matrix determinant lemma' we have

$$\begin{aligned}\mathbb{E}\det(A - vv^\star) &= \mathbb{E}\det(A)(1 - v^\star A^{-1}v) \\ &= \mathbb{E}\det(A)(1 - Tr(A^{-1}vv^\star)) \\ &= det(A) - det(A)\mathbb{E}Tr(A^{-1}vv^\star) \\ &= det(A) - det(A)Tr(A^{-1}\mathbb{E}vv^\star)\end{aligned}$$

On the other hand, with the use of Jacobi's formula

$$(1 - \partial_t)\det(A + t\mathbb{E}vv^\star) = \det(A + t\mathbb{E}vv^\star) - \det(A)Tr(A^{-1}\mathbb{E}vv^\star)$$

The claim follows by setting $t = 0$ □

Now let's introduce the mixed characteristic polynomial and give a theorem about it.

Theorem 7.2. *Let $v_1, \ldots, v_m$ be independent random column vectors in $\mathbb{C}^d$ with finite support. For each i, let $A_i = \mathbb{E} v_i v_i^\star$. Then:*

$$\mu[A_1, \ldots, A_m](x) = \mathbb{E}\chi[\sum_{i=1}^m v_i v_i^\star](x) = \prod_{i=1}^m (1 - \partial_{z_i}) \det(xI + \sum_{i=1}^m z_i A_i)|_{z_1 = \cdots = z_m = 0}$$

Proof. The proof of this theorem relies on Lemma 7.1 We will show by induction on k that for every matrix M,

$$\mathbb{E}\det(M - \sum_{i=1}^k v_i v_i^\star) = \prod_{i=1}^k (1 - \partial_{z_i}) \det(M + \sum_{i=1}^k z_i A_i)|_{z_1 = \cdots = z_k = 0}$$

The base case $k = 0$ is trivial. Assuming the claim holds for $i < k$, we have:

$$\begin{aligned}
\mathbb{E}\det(M - \sum_{i=1}^k v_i v_i^\star) &= \mathbb{E}_{v_1, \ldots, v_{k-1}\mathbb{E}_{v_k}} \det(M - \sum_{i=1}^{k-1} v_i v_i^\star - v_k v_k^\star) \quad \textit{by independence} \\
&= \mathbb{E}_{v_1, \ldots, v_{k-1}} (1 - \partial_{z_k}) \det(M - \sum_{i=1}^{k-1} v_i v_i^\star + z_k A_k)|_{z_k = 0} \quad \textit{by Lemma 7.1} \\
&= (1 - \partial_{z_k}) \mathbb{E}_{v_1, \ldots, v_{k-1}} \det(M + z_k A_k - \sum_{i=1}^{k-1} v_i v_i^\star)|_{z_k = 0} \quad \textit{by linearity} \\
&= (1 - \partial_{z_k}) \prod_{i=1}^{k-1} (1 - \partial_{z_i}) \det(M + z_k A_k + \sum_{i=1}^{k-1} z_i A_i)|_{z_1 = \cdots = z_{k-1}} = 0|_{z_k = 0} \\
&= \prod_{i=1}^k (1 - \partial_{z_i}) \det(M + \sum_{i=1}^k z_i A_i)|_{z_1 = \cdots = z_k = 0}
\end{aligned}$$

as desired. □

Of course the mixed characteristic polynomial is real rooted.

Finally, we use the real rootedness of mixed characteristic polynomials to show that every sequence of independent finitely supported random vectors

$v_1, \ldots, v_m$ define an interlacing family. Let l_i be the size of the support of the random vector v_i, and let v_i take the values $\omega_{i,1}, \ldots, \omega_{i,l_i}$, with probabilities $p_{i,1}, \ldots, p_{i,l_i}$. Where: $\omega_{i,1} = (u_i, 0^d, \ldots, 0^d), \quad \omega_{i,2} = (0^d, u_2, 0^d, \ldots, 0^d), \quad \ldots$

where $u_1, \ldots, u_m \in \mathbb{C}^d$ vectors.

Now for $j_1 \in [l_1], \ldots, j_m \in [l_m]$, define:

$$q_{j_1,\ldots,j_m} = \prod_{i=1}^{m} p_{i,j_i} \chi[\sum_{i=1}^{m} \omega_{i,j_i} \omega_{i,j_i}^{\star}](x)$$

Theorem 7.3. *The polynomials $q_{j_1,\ldots,j_m}$ form an interlacing family.*

Proof. For $1 \leq k \leq m$ and $j_1 \in [l_1], \ldots, j_k \in [l_k]$, define the partial assignment

$$q_{j_1,\ldots,j_k}(x) = \prod_{i=1}^{k} p_{i,j_i} \mathbb{E}_{v_{k+1},\ldots,v_m} \chi[\sum_{i=1}^{k} \omega_{i,j_i} \omega_{i,j_i}^{\star} + \sum_{k+1}^{m} v_i v_i^{\star}](x)$$

Also let

$$q_{\emptyset}(x) = \mathbb{E}_{v_1,\ldots,v_m} \chi[\sum_{i=1}^{m} v_i v_i^{\star}](x)$$

We need to prove that for every partial assignment $j_1, \ldots, j_k$ (possibly empty), the polynomials

$$\{q_{j_1,\ldots,j_k,t}(x)\}_{t=1,\ldots,l_{k+1}}$$

have a common interlacing. By Lemma 4.7, it suffices to prove that for every nonnegative $\lambda_1, \ldots, \lambda_{l_{k+1}}$ summing to one, the polynomial

$$\sum_{t=1}^{l_{k+1}} \lambda_t q_{j_1,\ldots,j_k,t}(x)$$

is real rooted. To show this, let u_{k+1} be a random vector that equals $\omega_{k+1,t}$, with probability λ_t.

Then, the above polynomial equals:

$$\prod_{i=1}^{k} p_{i,j_i} \mathbb{E}_{v_{k+1},\ldots,v_m} \chi[\sum_{i=1}^{k} \omega_{i,j_i} \omega_{i,j_i}^{\star} + u_{k+1} u_{k+1}^{\star} + \sum_{i=k+2}^{m} v_i v_i^{\star}](x)$$

which is a multiple of a mixed characteristic polynomial and therefore real rooted. □

Chapter 8

The Multivariate Barrier Argument

In this chapter we will prove an upper bound on the roots of the mixed characteristic polynomial $\mu[A_1, \ldots, A_m](x)$ as a function A_i, in the case of interest $\sum_{i=1}^m A_i = I$. Our main theorem:

Theorem 8.1. *Suppose $A_1, \ldots, A_m$ are Hermitian positive semidefinite matrices satisfying $\sum_{i=1}^m A_i = I$ and $Tr(A_i) \le \epsilon$ for all i. Then:*

$$\textit{maxroot } \mu[A_1, \ldots, A_m](x) \le (1 + \sqrt{\epsilon})^2$$

We begin by deriving a slightly different expression for $\mu[A_1, \ldots, A_m](x)$ that allows us to reason separetely about the effect of each A_i on the roots.

Lemma 8.2. Let $A_1, \ldots, A_m$ be Hermitian positive semidefinite matrices. If $\sum_i A_i = I$, then

$$\mu[A_1, \ldots, A_m](x) = \prod_{i=1}^m (1 - \partial_{y_i}) \det(\sum_{i=1}^m y_i A_i)|_{y_1 = \cdots = y_m = x}$$

Proof. For any differentiable funtion f, we have

$$\partial_{y_i}(f(y_i))|_{y_i = z_i + x} = \partial_{z_i} f(z_i + x)$$

So, the lemma follows by substituting $y_i = z_i + x$ into the previous product and observing that it produces the expression on the right hand side of Theorem 7.2. □

We will often use the polynomial

$$Q(y_1, \dots, y_m) = \prod_{i=1}^{m}(1 - \partial_{y_i}) \det(\sum_{i=1}^{m} y_i A_i)|_{y_1,\dots,y_m}$$

To be more specific $\mu[A_1, \dots, A_m](x) = Q(x, \dots, x)$ holds, as seen in lemma 8.2

Definition 8.3. Above the roots
Let $p(z_1, \dots, z_m)$ be a multivariate polynomial. We say that $z \in \mathbb{R}^m$ is above the roots of p if

$$p(z + t) > 0 \quad \textit{for all} \quad t = (t_1, \dots, t_m) \in \mathbb{R}^m, t_i \geq 0$$

i.e. if p is postive on the nonnegative orthant with origin at z. We will denote the set of points which are above the roots of p by Ab_p

Definition 8.4. Barrier function
Given a real stable polynomial p and a point $z = (z_1, \dots, z_m) \in Ab_p$, the barrier function of p in direction i at z is

$$\Phi_p^i(z) = \frac{\partial_{z_i} p(z)}{p(z)} = \partial_{z_i} \log p(z)$$

Equivalently, we may define Φ_p^i by:

$$\Phi_p^i(z_1, \dots, z_m) = \frac{\dot{q}_{z,i}(z_i)}{q_{z,i}(z_i)} = \sum_{j=1}^{r} \frac{1}{z_i - \lambda_j}$$

where the univariate restriction

$$q_{z,i}(t) = p(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_m)$$

has roots $\lambda_1, \dots, \lambda_r$ which are real.

Now lets give out first theorem about the barrier functions.

Lemma 8.5. Convexity of the barrier function
Let p real stable polynomial of m variables. Then:

$$(-1)^k \frac{\partial^k}{\partial z_j^k} \Phi_p^i(x) \geq 0$$

whenever $k = 0, 1, 2, \dots$ and x lies above the roots of p. In particular the function $t \mapsto \Phi_p(x + te_j)$ is non-negative, non increasing, and convex for non-negative t, where e_j is the vector with zeros except a 1 in the j-th entry.

Proof. **The first case** is when $i = j$. Since we are only interested in z such that $z \in Ab_p$, we can study $p(z) = p(z_1, \dots, z_i, \dots, z_m)$ for all variables fixed and real, except z_i, which would just vary on the real line. We get:

$$(-1)^k \frac{\partial^k}{\partial z_i^k} \Phi_p^i(z) = (-1)^k \frac{\partial^{k+1}}{\partial z_i^{k+1}} \log p(z) = k! \sum_{j=1}^{d} (\frac{1}{z_i - y_j})^{k+1}$$

where y_i are all real roots. Since $z_i - y_j$ is positive when $z \in Ab_p$ the claim follows.
Of course, if $k = 0$, then by fixing all variables except one, let's suppose it is x_1, we get that:

$$\Phi_p(x_1) = \sum_{i=1}^{d} \frac{1}{x_1 - y_i}$$

We know that $x_1 - y_i > 0$, so $\Phi_p > 0$

Now for **the second case** $i \neq j$. By freezing all the other variables and relabeling, we may assume that there are only two variables z_1 and z_2, which we call from now on x_1, x_2, since they are real and $x = (x_1, x_2)$. Thus $p(x_1, x_2)$ is a real stable polynomial. The task is to show

$$(-1)^k \frac{\partial^k}{\partial x_2^k} \Phi_p^1(x) \geq 0, \quad \forall k \in \mathbb{N}$$

This is equivalent to

$$(-1)^k \frac{\partial^k}{\partial x_2^k} \frac{\partial}{\partial x_1} \log p(x) = \frac{\partial}{\partial x_1} (-1)^k \frac{\partial^k}{\partial x_2^k} \log p(x)$$

So it suffices to show that $(-1)^k \frac{\partial^k}{\partial x_2^k} \log p(x)$ is non-decreasing in the x_1 direction. By continuity, it suffices to do this for generic x_1 (thus we may exclude a finite number of exceptional x_1 if we wish)

For fixed x_1, the univariate polynomial $p_{x_1} : x_2 \mapsto p(x_1, x_2)$ is real stable, and thus has real roots, which we denote as $y_1(x_1), \ldots, y_d(x_1)$. For generic x_1, the number d of roots does not depend on x_1, and the $y_i(x_1)$ can be chosen to vary smoothly in x_1, and the multiplicity of each root $y_i(x_1)$ is locally constant in x_1. We then have:

$$(-1)^k \frac{\partial^k}{\partial x_2^k} \log p(x) = -(k-1)! \sum_{i=1}^{d} (\frac{1}{x_2 - y_i(x_1)})^k$$

So it suffices to show that each of the $(\frac{1}{x_2 - y_i(x_1)})^k$ is a non-increasing function of x_1. But if (x_1, x_2) lies above the roots of p, then the $y_i(x_1)$ all lie below x_2, so it suffices to show that the $y_i(x_1)$ are all generically non-increasing. If this were **not** the case, then $y_i(x_1)$ would have a positive derivative for all x_1 in an open interval. In particular, there would be a x_0 such that the root $y_i(x_1)$ has a positive derivative and a constant multiplicity m for all x_1 sufficiently close to x_0. By analytic continuation we conclude that for complex z_1 near x_0, the polynomial $z_2 \mapsto p(z_1, z_2)$ has a complex root $y_i(z_1)$ near $y_i(x_0)$. We expand the function $\phi(z_1, z_2) = z_2 - y_i(z_1)$ in Taylor series around (x_0, y_0), and we get

$$\phi(z_1, z_2) = -\dot{y}_i(x_0)(z_1 - x_0) + (z_2 - y_0) + \textit{lower order terms}$$

These lower order terms don't matter at all in our analysis. Thus we see by Taylor that we can choose $\dot{y}_i(x_0)(z_1 - x_0) = i\theta_1$ and $z_2 - y_0 = i\theta_1$, with $\theta_1 > 0$. Thus we get z_1, z_2 with positive imaginary parts such that $p(z_1, z_2) = 0$, which contradicts stability, and the claim follows. Note that this was possible, because $y_i(x_0) > 0$. □

The real purpose of the barrier function is to allow us to reason about the relationship between Ab_p and $Ab_{p-\partial_{z_i} p}$. In particular the monotonicity statement alone immediately implies the following lemma.

Lemma 8.6. Suppose that p is real stable, that $z \in Ab_p$, and that $\Phi_p^i(z) < 1$. Then $z \in Ab_{p-\partial_{z_i}p}$

Proof. Let t be a non-negative vector. As Φ is nonincreasing in each coordinate we have $\Phi_p^i(z+t) < 1$ therefore:

$$\partial_{z_i} p(z+t) < p(z+t) \implies (p - \partial_{z_i} p)(x+t) > 0$$

as desired □

The above Lemma allows us to prove that a vector is above the roots of $p - \partial_{z_i} p$. However we need the barrier function to be bounded away from 1. To remedy this, we present the following lemma.

Lemma 8.7. Suppose that $p(z_1, \ldots, z_m)$ is real stable, that $z \in Ab_p$, and that $\delta > 0$ satisfies:

$$\Phi_p^j(z) \leq 1 - \frac{1}{\delta}$$

Then for all i:

$$\Phi_{p-\partial_{z_j}}^i(z + \delta e_j) \leq \Phi_p^i(z)$$

Proof. We will write ∂_i instead of ∂_{z_i} to ease notation. We begin by the left hand-side of the wanted result.

$$\begin{aligned}
\Phi_{p-\partial_j p}^i &= \frac{\partial_i (p - \partial_j p)}{p - \partial_j p} \\
&= \frac{\partial_i ((1 - \Phi_p^j) p)}{(1 - \Phi_p^j) p} \\
&= \frac{(1 - \Phi_p^j)(\partial_i p)}{(1 - \Phi_p^j) p} + \frac{(\partial_i (1 - \Phi_p^j)) p}{(1 - \Phi_p^j) p} \\
&= \Phi_p^i - \frac{\partial_i \Phi_p^j}{1 - \Phi_p^j} \qquad\qquad (\partial_i \Phi_p^j = \partial_j \Phi_p^i) \\
&= \Phi_p^i - \frac{\partial_j \Phi_p^i}{1 - \Phi_p^j}
\end{aligned}$$

We would like to show that $\Phi^i_{p-\partial_j p}(z+\delta e_j) \leq \Phi^i_p(z)$. By the above identity this is equivalent to

$$-\frac{\partial_j \Phi^i_p(z+\delta e_j)}{1-\Phi^j_p(z+\delta e_j)} \leq \Phi^i_p(z) - \Phi^i_p(z+\delta e_j)$$

By the convexity of the barrier function we get that

$$\Phi^i_p(z+\delta e_j) \leq \Phi^i_p(z) + \delta \partial_j \Phi^i_p(z+\delta e_j) \iff \delta(-\partial_j \Phi^i_p(z+\delta e_j)) \leq \Phi^i_p(z) - \Phi^i_p(z+\delta e_j)$$

Thus it is sufficient to establish that:

$$-\frac{\partial_j \Phi^i_p(z+\delta e_j)}{1-\Phi^j_p(z+\delta e_j)} \leq \delta(-\partial_j \Phi^i_p(z+\delta e_j))$$

We know about the numerator of the above inequality that it is nonnegative from Lemma 8.5, so we may divide both sides by the numerator to obtain:

$$\frac{1}{1-\Phi^i_p(z+\delta e_j)} \leq \delta$$

Now use the monotonicity of the barrier function $\Phi^j_p(z+\delta e_j) \leq \Phi^j_p(z)$ and conclude that in order for the claim to follow we need

$$\frac{1}{1-\Phi^i_p(z)} \leq \delta$$

which we know to be true from the assumption $\Phi^j_p(z) \leq 1 - \frac{1}{\delta}$. □

Now let's give the proof of theorem 8.1 that will inevitably prove the conjecture of Kadison- Singer. We have all the tools necessary to move closer to the conjecture.

Proof. **Of theorem 8.1**
Let

$$P(y_1, \ldots, y_m) = \det(\sum_{i=1}^{m} y_i A_i)$$

and set $t = \epsilon + \sqrt{\epsilon}$.
The vector $t\mathbf{1} = t(1, \ldots, 1)$ is above the roots of P, since if we choose $\tilde{t} \in \mathbb{R}^{m,+}$, then

$$P(t\mathbf{1} + \tilde{t}) = P(t + \tilde{t_1}, \ldots, t + \tilde{t_m}) \quad = \det(\sum_{i=1}^{m}(t + \tilde{t}_i)A_i) = \det(tI + \sum_{i=1}^{m}\tilde{t}_i A_i) \geq 0$$

The last equality holds because it is the determinant of the sum of positive semidefinite matrices.

Now by the 'matrix determinant lemma', and the 'Jacobi's formula', we get that

$$\Phi_P^i(y_1, \ldots, y_m) = \frac{\partial_i P(y_1, \ldots, y_m)}{P(y_1, \ldots, y_m)} = Tr((\sum_{i=1}^{m} y_i A_i)^{-1} A_i)$$

Also,

$$\Phi_P^i(t\mathbf{1}) = \frac{Tr(A_i)}{t} \leq \frac{\epsilon}{t} = \frac{\epsilon}{\epsilon + \sqrt{\epsilon}} := \phi$$

Now let $\delta = \frac{1}{1-\phi} = 1 + \sqrt{\epsilon}$. Define for $k \in [m]$,

$$P_k(y_1, \ldots, y_m) = \prod_{i=1}^{k}(1 - \partial_{y_i})P(y_1, \ldots, y_m)$$

Note that $P_m = Q$
Set x^0 to be the all-t vector, and for $k \in [m]$ define x^k to be the vector that is $t\delta$ in the first k coordinates and t in the rest.

$$x^0 = (t, \ldots, t)$$
$$x^k = (t + \delta, \ldots, t + \delta, t, \ldots, t)$$

By inductively applying lemmata 8.6, 8.7 we get that x^k is above the roots of P_k, and that for all i:

$$\Phi_P^i(x^k) \leq \phi$$

It follows that the largest root of

$$\mu[A_1, \ldots, A_m] = P_m(x, \ldots, x)$$

is at most

$$t + \delta = 1 + \sqrt{\epsilon} + \sqrt{\epsilon} + \epsilon = (1 + \sqrt{\epsilon})^2$$

□

Chapter 9

Proof of the Kadison-Singer problem

In this chapter we will state and prove the Kadison-Singer problem, using the technics developed in this thesis.

The original question of Kadison singer was the following

Conjecture 1. Kadison-Singer Problem
Does every pure state on the (abelian) von Neumann algebra $\mathbb{D}$ of bounded diagonal operators on l_2 have a unique extension to a pure state on $B(l_2)$, the von Neumann algebra of all bounded operators on l_2?

This conjecture has been shown to be equivalent to a number of conjectures including Anderson's Paving conjecture[5, 6, 7], Weaver's discrepancy theoretic conjectures[11], the Bourgain-Tzafriri Conjecture[8, 10], the Feichtinger Conjecture and the R_ϵ-Conjecture [9].

These conjectures are proven in this thesis by proving Weaver's Conjecture, as amended by [11, Theorem 2], and says:

Conjecture 2. There exist universal constants $\eta \geq 2$ and $\theta \geq 0$ so that the following holds. Let $\omega_1, \dots, \omega_m \in \mathbb{C}^d$ satisfy $\|\omega_i\| \leq 1$ for all i and suppose:

$$\sum_{i=1}^{m} |\langle u, \omega_i \rangle|^2 = \eta$$

for every unit vector $u \in \mathbb{C}^d$. Then there exists a partition S_1, S_2 of $\{1, \dots, m\}$ so that

$$\sum_{i \in S_j} |\langle u, \omega_i \rangle|^2 \leq \eta - \theta$$

for every unit vector $u \in \mathbb{C}^d$ and each $j \in \{1, 2\}$

A proof of Anderson's projecton paving conjecture is also given:

Conjecture 3. Paving Conjecture
For every $\epsilon > 0$ there is an $r \in \mathbb{N}$ such that for every $n \times n$ self-adjoint complex matrix T with zero diagonal, there are diagonal projections $P_1, \dots, P_r$ with $\sum_{i=1}^{r} P_i = I$ such that

$$\|P_i T P_i\| \leq \epsilon \|T\| \quad \textit{for } i = 1, \dots, r$$

The paving conjecture can be extended to infinite operators $T \in Bl_2$ by an elementary compactness argument, which then gives an immediate solution to the Kadison-Singer problem.

Now we prove two theorems, the second of which implies conjectures 2 and 3. The first will be proved using the technics developed earlier, and the second one using the first.

Theorem 9.1. *If $\epsilon > 0$ and $v_1, \dots, v_m$ are independent random vectors in $\mathbb{C}^d$ with finite support such that*

$$\sum_{i=1}^{m} \mathbb{E} v_i v_i^{\star} = I_d$$

and

$$\mathbb{E}\|v_i\|^2 \leq \epsilon \quad \textit{for all } i$$

Then:

$$\mathbb{P}[\|\sum_{i=1}^{m} v_i v_i^{\star}\| \leq (1+\sqrt{\epsilon})^2] > 0$$

Proof. We will make extensive use of the theorems presented earlier, particularly theorem 8.1.
Let $A_i = \mathbb{E} v_i v_i^{\star}$. We have

$$Tr(A_i) = \mathbb{E} Tr(v_i v_i^{\star}) = \mathbb{E} v_i^{\star} v_i = \mathbb{E}\|v_i\|^2 \leq \epsilon \quad \textit{for all } i$$

The expected characteristic polynomial of the $\sum_i v_i v_i^{\star}$ is the mixed charactersitic polynomial $\mu[A_1, \dots, A_m](x)$. Theorem 8.1 implies that the largest root of this polynomials is at most $(1+\sqrt{\epsilon})^2$.
For $i \in [m]$, let l_i be the size of the random vector v_i, and let v_i take the values $\omega_{i,1}, \dots, \omega_{i,l_i}$ with probabilities $p_{i,1}, \dots, p_{i,l_i}$ respectively. Theorem 7.3 tells us that the polynomials $q_{j_1,\dots,j_m}$ are an interlacing family. So, theorem 4.4 implies that there exist $j_1, \dots, j_m$ so that the largest root of the characteristic polynomial of

$$\sum_{i=1}^{m} \omega_{i,j_i} \omega_{i,j_i}^{\star}$$

is at most $(1+\sqrt{\epsilon})^2$

□

Corollary 9.2. Let r be a positive integer and let $u_1, \dots, u_m \in \mathbb{C}^d$ be vectors such that:

$$\sum_{i=1}^{m} u_i u_i^{\star} = I$$

and $\|u_i\|^2 \leq \delta$ for all i. Then there exists a partition $\{S_1, \dots S_r\}$ of $[m]$ such that

$$\|\sum_{i \in S_j} u_i u_i^{\star}\| \leq (\frac{1}{\sqrt{r}} + \sqrt{\delta})^2$$

Proof. For each $i \in [m]$ and $k \in [r]$, define $w_{i,j} \in \mathbb{C}^{rd}$ to be the direct sum of r vectors from $\mathbb{C}^d$, all of which are 0^d except for the k^{th} one, which is a copy of u_i. Namely:

$$w_{i,1} = \begin{bmatrix} u_i \\ 0^d \\ \vdots \\ 0^d \end{bmatrix}, \quad w_{i,2} = \begin{bmatrix} 0^d \\ u_2 \\ \vdots \\ 0^d \end{bmatrix} \quad \textit{and so on}$$

Now let $v_1, \ldots, v_m$ be independent random vectors such that v_i takes the values $\{\sqrt{r} w_{i,k}\}_{k=1}^r$ each with probability $1/r$
These vectors satisfy:

$$\mathbb{E} v_i v_i^\star = \begin{bmatrix} u_i u_i^\star & 0_{d\times d} & \ldots & 0_{d\times d} \\ 0_{d\times d} & u_i u_i^\star & \ldots & 0_{d\times d} \\ \vdots & & \ddots & \vdots \\ 0_{d\times d} & 0_{d\times d} & \ldots & u_i u_i^\star \end{bmatrix}, \quad \textit{and} \quad \|v_i\|^2 = r\|u_i\|^2 \leq r\delta$$

So,

$$\sum_{i=1}^m \mathbb{E} v_i v_i^\star = I_{rd}$$

and we can apply Theorem 9.1 with $\epsilon = r\delta$ to show that there exists an assignment of each v_i so that

$$(1+\sqrt{r\delta})^2 \geq \| \sum_{i=1}^m v_i v_i^\star \| = \| \sum_{k=1}^r \sum_{i: v_i = w_{i,k}} (\sqrt{r} w_{i,k})(\sqrt{r} w_{i,k})^\star \|$$

Therefore by setting $S_k = \{i : v_i = w_{i,k}\}$, we obtain:

$$\| \sum_{i\in S_k} u_i u_i^\star \| = \| \sum_{i \in S_k} w_{i,k} w_{i,k}^\star \| \leq \frac{1}{r} \| \sum_{k=1}^r \sum_{i: v_i = w_{i,k}} (\sqrt{r} w_{i,k})(\sqrt{r} w_{i,k})^\star \| \leq (\frac{1}{\sqrt{r}} + \sqrt{\delta})$$

and this is true for all k. □

Having established this Corollary, we can now prove the equivalent forms of the Kadison-Singer problem quite simply. If we set $r = 2$ and $\delta = 1/18$

this implies Conjecture 2 for $\eta = 18$ and $\theta = 2$. We can also get Conjecture 3 by this Corollary, however it is somewhat more involved. We will prove it in the following chapter.

Chapter 10

The Paving Conjecture, a proof

The main result of this section is the following quantitative version of conjecture 3. We will say that a matrix T can be (r, ϵ)-paved if there are coordinate projections $P_1, \ldots, P_r$ such that $\sum_{i=1}^{r} P_i = I$ and $\|P_i T P_i\| \leq \epsilon \|T\|$ for all i.
We will need a lemma presented in 3

Lemma 10.1. (*Theorem 3 of [3]*) Suppose there is a function $r : \mathbb{R}_+ \mapsto \mathbb{N}$ so that every $2n \times 2n$ projection matrix Q with diagonal entries equal to $1/2$ can be $(r(\epsilon), \frac{1+\epsilon}{2})$-paved for all $\epsilon > 0$. Then every $n \times n$ self-adjoint zero-diagonal matrix T can be $(r^2(\epsilon), \epsilon)$-paved for all $\epsilon > 0$

Now we are ready to prove the paving conjecture.

Theorem 10.2. ***The paving conjecture*** *For every $\epsilon > 0$, every zero-diagonal complex self-adjoint matrix T can be (r, ϵ)-paved with $r = (6/\epsilon)^4$.*

Proof. Let Q be an arbitrary $2n \times 2n$ projection matrix with diagonal entries equal to $1/2$. Then $Q = (u_i^{\star} u_i)_{i,j \in [2n]}$ is the gramian matrix of $2n$ vectors $u_1, \ldots u_{2n} \in \mathbb{C}^n$ with $\|u_i\| = 1/2 = \delta$. Applying Corollary 9.2 to these vectors for any r yields a partition $S_1, \ldots, S_r$ of $[2n]$. Letting P_k be the projection onto the indices in S_k, we have for each $k \in [r]$:

$$\|P_kQP_k\| = \|(u_i^\star u_j)_{i,j\in S_k}\| = \|\sum_{i\in S_k} u_i u_i^\star\| \leq (\frac{1}{\sqrt{r}} + \frac{1}{\sqrt{2}})^2 < \frac{1}{2} + \frac{3}{\sqrt{r}}$$

Thus every Q can be $(r, \frac{1+\epsilon}{2})$-paved for $r = 36/\epsilon^2$. Applying Lemma 10.1 yields the result. □

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