Cosmic Structures in Alternative Gravity Models

By

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Abstract

The present thesis is a study of cosmic structures in modified gravity models. In particular, we are interested in Brans-Dicke like, scalar-tensor theories in the presence of a positive cosmological constant Λ. We discuss the validity of the no-hair theorem in the context of such theories. We first prove that regular, stationary black-hole solutions exist if and only if the scalar field is constant and the Brans-Dicke parameter, ω, is infinite. These solutions coincide with the General Relativity ones. We also prove perturbatively that, in the absence of the stationary cosmological horizon, black-holes exist with non-trivial $\phi-$hair. In addition, it is shown that the presence of a stationary cosmological event horizon rules out any regular spherical stationary solution, appropriate for the description of a star. Thus, to describe a star one has to assume deviation from de Sitter asymptotics. Under this assumption generic cosmic structures are studied perturbatively and shown than only for $\omega > 0$ or $\omega \lesssim -5$ their predicted maximum sizes are consistent with observations. Finally, we use the perturbative solutions to solve numerically the full (unperturbed) equations and calculate the maximum turnaround radius of different structures, for different values of ω.

Περίληψη

Η παρούσα εργασία είναι μια μελέτη κοσμικών δομών σε τροποποιημένα βαρυτικά μοντέλα. Συγκεκριμένα, ενδιαφερόμαστε για θεωρίες τύπου Brans-Dicke που έχουν ένα βαθμωτό και ένα τανυστικό πεδίο με την παρουσία μιας θετικής κοσμολογικής σταθεράς Λ. Μελετήσαμε την εγκυρότητα του θεωρήματος no-hair στο πλαίσιο αυτών των θεωριών. Αρχικά, αποδείξαμε ότι ομαλές, στάσιμες λύσεις για μαύρες τρύπες υπάρχουν μόνο αν το βαθμωτό πεδίο είναι σταθερό και η παράμετρος Brans-Dicke, ω, είναι άπειρη. Οι λύσεις στην περίπτωση αυτή, συμπίπτουν με αυτές τις Γενικής Σχετικότητας. Αποδείξαμε επίσης, ότι διαταρακτικά και ελλείψη ενός στάσιμου κοσμολογικού ορίζοντα, υπάρχουν μαύρες τρύπες με μη τετριμμένη συνεισφορά από το βαθμωτό πεδίο ϕ. Επιπρόσθετα, δείξαμε ότι η παρουσία ενός στάσιμου κοσμολογικού ορίζοντα γεγονότων αποκλείει κάθε ομαλή, σφαιρικά συμμετρική λύση, κατάλληλη για την περιγραφή αστέρων. Έτσι, για να περιγράψει κάποιος ένα αστέρι πρέπει να υποθέσει ότι ο χωρόχρονος δεν είναι ασυμπτωτικά de Sitter.Υπό αυτή την προϋπόθεση, μελετούμε διαταρακτικά κοσμικές δομές και δείχνουμε ότι, για $\omega > 0$ ή $\omega \lesssim -5$, το προβλεπόμενο μέγιστο μέγεθός τους είναι συνεπές με τις παρατηρήσεις. Τέλος, χρησιμοποιούμε τις διαταρακτικές αυτές λύσεις για να λύσουμε αναλυτικά τις εξισώσεις και υπολογίζουμε την μέγιστη πιθανή ακτίνα δομών, για διαφορετικές τιμές της παραμέτρου ω.

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Notation and Conventions

Symbol	Meaning
$g_{\mu\nu}$ $(g^{\mu\nu})$	metric (inverse) tensor
$\delta^{\mu}{}_{\nu} = g^{\mu\alpha} g_{\alpha_\nu}$	Kronecker delta
$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$	line element
$g = det(g_{\mu\nu})$	determinant of the metric tensor
$\eta_{\mu\nu} = diag(-1, 1, 1, 1)$	Minkowski metric
$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\mu\beta,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta} \right)$	Christoffel symbols
$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta;\gamma} - \Gamma^{\alpha}{}_{\beta\gamma;\delta} + \Gamma^{\alpha}{}_{\gamma\rho} \Gamma^{\rho}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\delta\rho} \Gamma^{\rho}{}_{\beta\gamma}$	Riemann curvature tensor
$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta} = g^{\mu\nu} R_{\alpha\mu\beta\nu}$	Ricci tensor
$R = g^{\alpha\beta} R_{\alpha\beta}$	Ricci (curvature) scalar
$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$	Einstein tensor
$\Box = \nabla_\mu \nabla^\mu = g^{\mu\nu} \nabla_\mu \nabla_\nu$	box/D'Alembertian operator in curved spacetimes
$\kappa = 8\pi G/c^4$	constant factor that appears in the action
$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$	2-sphere

"When I heard the learn'd astronomer
When the proofs, the figures, were ranged in columns before me
When I was shown the charts and the diagrams, to add, divide, and measure them
When I, sitting, heard the astronomer, where he lectured with much applause in the lecture-room,
How soon, unaccountable, I became tired and sick
Till rising and gliding out, I wander'd off by myself,
In the mystical moist night-air, and from time to time, Look'd up in perfect silence at the stars."

Walt Whitman

To my family.

Contents

Chapter 1

Introduction

In this chapter we will introduce the basic concepts of Einstein's theory of relativity. We will define the action that describes the theory and derive the equations of motion, we'll discuss how the latter change if we add a cosmological constant and provide some identities that follow the theory. Moreover, we'll present why a modification of gravity is important and how is this possible. Finally, we will describe the ΛCDM cosmological model, which nowadays is the most acceptable according to observations and also we'll introduce the notion of the turnaround radius and calculate its maximum value in the framework of this model.

1.1 General Relativity

On November 1915, Einstein presented the General Theory of Relativity (GR) to the Prussian Academy of Sciences[1]. The radical idea in this theory is that gravity is not just an attractive force, as it was believed since Newton. Gravity is an intrinsic property of the geometry of space-time and it arises from its curvature.

Einstein's equations in vacuum can be easily derived, by varying the, very well known, Einstein-Hilbert (EH) action with respect to the metric. If, in this action, we add a matter action, we get the action of GR

$$\mathcal{S}_{GR} = \mathcal{S}_{EH} + \mathcal{S}_m = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa}R + \mathcal{L}_m\right) \tag{1.1}$$

where $\kappa = 8\pi G c^{-4}$. By varying this action with respect to the metric, we obtain the Einstein's equation under the presence of matter

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.2}$$

where $T_{\mu\nu}$ is the stress-energy tensor of the matter fields and is defined as

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}} = -2\frac{\delta\mathcal{L}_m}{\delta g_{\mu\nu}} + g_{\mu\nu}\mathcal{L}_m\,. \tag{1.3}$$

[1]The Royal Prussian Academy of Sciences was an academic academy established in Berlin on 11 July 1700, four years after the "Arts Academy", to which "Berlin Academy" may also refer.

Since the metric tensor is a 2-rank tensor, in d dimensions it contains d^2 components. Its symmetry constrains $d(d-1)/2$ of them, thus ending up with $d(d+1)/2$ independent components. Hence the Einstein equations in 4 dimensions are 10. These are not all independent and can be constrained with the four Bianchi identities, ending up with 6 independent non-linear partial differential equations. The physical interpretation the of equations (1.2) is that matter/energy, $T_{\mu\nu}$, tells space-time, $G_{\mu\nu}$, how to curve and respectively space-time tells matter/energy how to move.

If we add by hand a constant term in the action (1.1), this will become

$$\mathcal{S}_{GR} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,(R-2\Lambda) + \mathcal{S}_m \tag{1.4}$$

and the corresponding equations will be

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.5}$$

This term has the meaning of a cosmological constant and if its value is positive it explains the acceleration of the universe.

In order for the energy to be conserved, which is a physical requirement of every theory, it has to be

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1.6}$$

and thus from eq. (1.2)

$$\nabla_\mu G^{\mu\nu} = 0 \tag{1.7}$$

which yields also from the Bianchi identity

$$\nabla_\mu R^\alpha{}_{\beta\gamma\nu} + \nabla_\nu R^\alpha{}_{\beta\mu\gamma} + \nabla_\gamma R^\alpha{}_{\beta\nu\mu} = 0 \tag{1.8}$$

1.2 Modifying Gravity: why and how

Although Einstein's theory is a very successful and well-tested theory, it could be that it is not the final theory of gravity. The recent discovery of the accelerated cosmic expansion suggests that our universe may be endowed with a positive cosmological constant Λ. If we assume a homogeneous and isotropic Universe in the framework of GR (which we know from observations that is reasonable), we obtain an equation for the scale factor $a(t)$,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}\,.^{2} \tag{1.9}$$

Hence, if Λ is not positive, from the fact that the universe accelerates, we get $P < -\rho/3$. This means that we have two options, either accept a positive Λ, or add

[2] This is the second of the two independent Friedmann equations. The other one is

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G + \Lambda}{3}$$

where k is constant throughout a particular solution, but may vary from one solution to another and has to do with the shape of the Universe.

an exotic component to the energy storage of the Universe with negative pressure. This is called Dark Energy. But, even if we accept the existence of a positive cosmological constant, we then face a problem with its value. The effective energy density in Λ is

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} = M_{pl}^2 \Lambda \sim 10^{-47} GeV^4 \sim (10^{-3} eV)^4 \,. \tag{1.10}$$

but from quantum field theory we estimate a vacuum energy of

$$\rho_{vac} \sim M_{pl}^4 \sim 10^{76} GeV^4 \,. \tag{1.11}$$

This large discrepancy in the value of the cosmological constant is called "fine-tuning" problem and has led many researchers to pursue alternative explanations, by modifying GR. Furthermore, best fit cosmological model, ΛCDM, which considers GR to be the gravity theory that describes the Universe at all scales, suggests that more than eighty percent of the matter content of the Universe, is "invisible", or even worse "unknown", the so-called Dark Matter.

These two "strange things", dark matter and dark energy, are the biggest problems of modern cosmology. There have been many attempts to modify gravity from different points of view. Soon after Einstein's presentation of the theory[3], proposals had been made (Eddington, Weyl, Kaluza-Klein etc.) to incorporate it into a more general theory. Dirac was the first who proposed that Newton's constant could vary with time, because he realized that there was a relation between this and the ratio of the mass and scale of the Universe. In the early '60s Brans and Dicke, based on this argument, formulated a new theory of gravity, by introducing a scalar field to standard GR, and thus leading to what are today known as scalar-tensor theories.

Higher derivative and non-local theories of gravity, such as $f(R)$, Hořava-Lifshitz and Galileons were proposed (and are still of interest) as an alternative explanation for the accelerated expansion of the Universe (for details see e.g. [1], [2], [3], [4] and references therein). In addition to these, higher dimensional theories of gravity, such as Kaluza-Klein, Randall-Sundrum, Dvali-Gabadadze-Porrati and Gauss-Bonnet gravity are also models that have been proposed through time as alternatives to GR. Finally, there are also attempts to alternatively explain the rotation curves of galaxies by modifying the dynamics of Newton's theory at small accelerations, by introducing a new constant acceleration, under which Newton's law change. This was firstly proposed by Milgrom and is known as Modified Newtonian Dynamics (MOND). Bekenstein generalised Milgrom's idea, when he formulated a relativistic covariant field theory, TeVeS, by introducing a tensor, a vector and a scalar field. This theory has MOND characteristics in the weak acceleration limit and provides a setting for constructing cosmological models [5], [6].

[3] At the time there was no need to modify gravity, unless someone wanted to incorporate it into a more general theory. In fact, even Newton's theory was very successful and continues to be used as an excellent approximation of the effects of gravity in most applications. Einstein's motivations to formulate the theory of relativity were more philosophical and less scientific.

Summarizing, what we mean by modifying gravity is any change of the following properties: general covariance[4], universal couplings to all matter fields[5], as well as satisfaction of the Einstein's field equations[6].

1.3 Turnaround Radius in ΛCDM

Let us now briefly review the ΛCDM model and also introduce the notion of the turnaround radius of a structure.

The ΛCDM model is the so-called standard model of cosmology. It assumes that, GR, in the presence of a cosmological constant (which is usually associated with dark energy), is the correct theory of gravity on cosmological scales. Its success in describing observations, such as the existence and the structure of the cosmic microwave background (CMB), the large scale structure in the distribution of galaxies and the acceleration of the universe observed from supernovae, is remarkable. It accepts the dark matter hypothesis and, in particular, it predicts the existence of weakly interacting and slow moving particles which constitute the "cold dark matter" in the universe (CDM).

From CMB observations we know that the whole universe was in a hot, dense state approximately 14 billion years ago but, today it contains stars, planets, galaxies and galaxy clusters. All these structures were formed from the fluctuations of the primordial plasma in the very early universe. According to ΛCDM the structure formation in the universe cannot last forever. Nowadays, since the dark energy is dominant in the universe ($\Omega_{\Lambda,0} \simeq 0.73$), structure formation should be almost finished [7],[8].

The acceleration of the universe is caused due to the negative pressure of the cosmological constant. From classical thermodynamics, we know that, a change in volume dV requires work done equal to a change of energy $-PdV$, where P is the pressure. The energy is equal to ρV, where ρ is the energy density and thus in the "empty" universe, when the volume increases, the energy will increase too [9]. Therefore, P is negative and specifically $P = -\rho$, which agrees also with observations.

This negative pressure means that the effect of dark energy in the universe is opposite to the gravitational attraction created by normal matter. On small scales, this attraction is dominant, due to the mass while on large scales Λ causes repulsion. As a consequence, there should be a region where these two opposite forces are neutralized. The acceleration of a test particle in this region has to be zero and thus, if this particle lies at rest outside that region, it will follow the expansion of the universe due to Λ and if it lies inside, it will fall towards the center of the

[4]As general covariance is defined the invariance of the form that the physical laws have, under arbitrary differentiable coordinate transformations. Since there exists no *a priori* correct coordinate system, a covariant physical law should have the same form in all coordinate systems.

[5]This means that independently of the number of the degrees of freedom, whatever other interactions may occur, the effect of gravity on the matter fields can only be through interactions with the rank-2 tensor, the metric tensor.

[6]If the equations of the modified gravity are any other than the Einstein equations, we consider the theory modified.

mass shell. We call this unstable equilibrium position maximum turnaround radius, because it gives a bound on the maximum sizes of the structures in the universe.

Let us calculate the maximum value of the turnaround radius in the ΛCDM model. Consider a spherical mass M embedded in a de-Sitter universe with cosmological constant $\Lambda > 0$. The metric that describes the space-time is the Schwarzschild-de-Sitter metric and is given by

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2 d\Omega^2 \tag{1.12}$$

where $F(r) = 1 - 2M/r - \Lambda r^2/3$ in units $G = c = 1$. The maximum turnaround radius is given by the maximum of the norm of the timelike Killing vector field ($= F(r)$), and thus

$$F'(r) = \frac{2M}{r^2} - \frac{2\Lambda r}{3} = 0 \Rightarrow \tag{1.13}$$

$$r_{ta} = \left(\frac{3M}{\Lambda}\right)^{1/3} . \tag{1.14}$$

For a more thorough derivation of this formula as well as for observational tests of this bound, see [10], [11] and [12].

Chapter 2

An alternative theory of gravity: Brans-Dicke theory

Brans and Dicke proposed a more general theory of gravity than this of Einstein, in order to be more satisfactory from the standpoint of Mach's principle. In this chapter, we present the formulation of the theory they proposed, as well as the generalisation of this in the presence of a cosmological constant. We discuss also the limit in which this theory reduces to GR and derive the equations of motion for the dynamical fields of the theory.

2.1 Brans-Dicke model

Newton claimed that inertial frames were determined by absolute space. But, since absolute space is not observable, Berkeley and Mach suggested that, it is more satisfactory to attempt to correlate the inertial frames with observable features of the universe. In particular, what is known as Mach's principle, says that inertial are the frames that do not accelerate relative to the "fixed stars", that is relative to a suitably defined mean of all the matter in the universe. In other words "local physical laws are determined by the large scale structure of the universe" [13].

If we assume the validity of this principle, it implies that kinematically equivalent motions must be dynamically equivalent as well. For example [14], the fact that the earth is rotating and the universe is at rest should be dynamically equivalent to the fact that the universe is rotating and the earth is at rest, which is not true if we assume absolute space. In addition, Mach's principle implies that, the inertial reaction experienced in an accelerated laboratory, relative to the distant matter of the universe, is the same as a gravitational force acting on a static laboratory, due to the presence of distant accelerated matter. Einstein tried to make general relativity compatible with this idea but as he said [14, 15], he failed to do so. In fact, Einstein showed that his field equations imply that a test-particle in an otherwise empty universe has inertial properties. That is why Brans and Dicke (and others before them) wanted to find a theory that is (more) compatible with Mach's principle.

Sciama [16] and others, assuming that Mach's principle is valid and using dimensional arguments only[1], showed that the gravitational constant G is related to the mass distribution in a uniform expanding universe in the following way

$$\frac{GM}{Rc^2} \sim 1 \tag{2.1}$$

where M is the finite mass of the visible universe, and R its radius. From this relation we can conclude that either the ratio M/R should be fixed by the theory, or alternatively that the gravitational constant observed locally should be variable and determined by the mass distribution about the point in question. As we'll see below, Brans and Dicke consider a varying G.

General relativity is described by the Einstein-Hilbert action which is given by

$$S = \int d^4x\sqrt{-g}\left(R + \frac{16\pi G}{c^4}\mathcal{L}_m\right) \tag{2.2}$$

where as we can see the gravitational constant G is directly coupled to the matter Lagrangian. An action of this form, but with varying gravitational constant, would change the geodesic equation of test-particles and possibly violate the weak equivalence principle (for details see chapter (6)). Eötvos experiments verified the validity of the weak equivalence principle (but not the strong), so the new theory has to be compatible with it.

Brans and Dicke decided to add in the Lagrangian of GR, the Lagrangian of a scalar field ϕ, after dividing by $c^4/16\pi G$. This scalar field couples to the metric $g_{\mu\nu}$ through a dimensionless constant ω which is, in principle, arbitrary. The action of this theory is given by [17],

$$S = \frac{c^4}{16\pi}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}(\nabla\phi)^2\right] + S_M\left[g_{\mu\nu}\right] \tag{2.3}$$

where g is the determinant of the metric, R is the Ricci scalar and S_M the matter action. In this way, "we are allowing for a possible violation of the strong equivalence principle, since gravity, the universal interaction of mass, can influence local physics, not only through geometry, but also by changing the local universally coupled ϕ, thus changing internal gravitational structure" [18].

Brans and Dicke decided to work on the so-called Jordan frame, in which the scalar field ϕ, plays the role of a varying gravitational constant, like G^{-1}. As in any other scalar-tensor theory, there is also another conformally related frame, the Einstein frame in which the scalar field plays the role of a source matter field. Although the "real" frame is actively pursued and has been the issue of lively debates, which are not yet settled (see the "classification of authors" in [19]), we are considering only the case of Jordan frame, in which the theory was originally formulated and in which the particles have constant mass and move on geodesics of the physical metric, so that the physical stress-energy tensor is conserved [20].

The action (2.3) reduces to the Einstein-Hilbert action when ω goes to infinity. In that limit, the scalar kinetic term becomes dominant, implying that the only physically accepted solution for the scalar field is the constant one. Thus, $\phi = \phi_0 = 1/G_N$.

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[1] They thought that since Mach's principle has its origins in philosophy, it could be described in the absence of a theory in a qualitative way only.

2.2 Adding a Cosmological Constant

The generalisation of BD theory in the presence of a positive cosmological constant implies

$$S = \frac{c^4}{16\pi}\int d^4x\sqrt{-g}\left[\phi R - 2\Lambda - \frac{\omega}{\phi}(\nabla\phi)^2\right] + S_M\left[g_{\mu\nu}\right] \tag{2.4}$$

As we can immediately see, in the large ω limit, the action becomes

$$\begin{aligned} S &= \frac{c^4}{16\pi}\int d^4x\sqrt{-g}\phi_0\left[R - \frac{2\Lambda}{\phi_0}\right] + S_M\left[g_{\mu\nu}\right] \\ &= \frac{c^4}{16\pi G_N}\int d^4x\sqrt{-g}\left[R - 2G_N\Lambda\right] + S_M\left[g_{\mu\nu}\right] \end{aligned} \tag{2.5}$$

If we compare with Einstein-Hilbert action with cosmological constant we obtain

$$G_N\Lambda^{BD} = \Lambda^{GR} = 10^{-52}m^{-2} \tag{2.6}$$

The above value for the cosmological constant is obtained by measurements based on the standard model of cosmology, ΛCDM. From now on, we will work in units $G_N = c = 1 \Rightarrow \phi_0 = 1$.

By varying (2.4) with respect to (w.r.t.) the metric we obtain[2]

$$\boxed{G_{\mu\nu} + \frac{\Lambda}{\phi}g_{\mu\nu} = \frac{8\pi}{\phi}T^M_{\mu\nu} + \frac{\omega}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla\phi\right)^2\right) + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right)} \tag{2.7}$$

As we can see, the difference from the Einstein equations is that there is also a contribution from the scalar field to the stress-energy tensor of the theory. Analytically it is

$$T_{\mu\nu} = T^M_{\mu\nu} + T^{(\phi)}_{\mu\nu}\,, \tag{2.8}$$

where

$$T^M_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{2.9}$$

is the stress-energy tensor of the matter fields and

$$T^{(\phi)}_{\mu\nu} = \frac{\omega}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla\phi\right)^2\right) + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right)\,. \tag{2.10}$$

Similarly, by varying the action (2.4) w.r.t. the scalar field ϕ we get[2]

$$R - \frac{\omega}{\phi^2}(\nabla\phi)^2 + \frac{2\omega}{\phi}\Box\phi = 0 \tag{2.11}$$

We contract the metric equations (2.7) to find the Ricci scalar

$$R = \frac{4\Lambda - 8\pi T}{\phi} + \frac{\omega}{\phi^2}(\nabla\phi)^2 + \frac{3}{\phi}\Box\phi \tag{2.12}$$

[2]for the analytical derivation see the appendix (6).

where T is the trace of the stress-energy tensor (2.9), and by substituting this into (2.11) we obtain the equation of motion of the Brans Dicke field

$$\boxed{\Box\phi = \frac{8\pi T - 4\Lambda}{2\omega + 3}} \tag{2.13}$$

We want now to solve this set of equations (2.7, 2.13) in order to see the behaviour of the fields $g_{\mu\nu}$ and ϕ. But as we'll see in chapter (4), this is a pretty difficult task. We have to impose an ansatz for the metric, but even in this case it's still impossible to solve the equations analytically. We'll discuss these in details later on.

Chapter 3

Generalization of the no-hair theorem

In this chapter we present the generalization of the no-hair theorem in the context of Brans-Dicke theory with a positive cosmological constant. In particular, we will show that regular, stationary black hole solutions have necessarily constant scalar field ϕ and $\omega = \infty$, and thus coincide with the solutions of GR. Moreover, we prove that if there exists a cosmological horizon, then the scalar field and the Ricci curvature will diverge logarithmically on the horizon.

3.1 Basic tools and definitions

Let us first introduce the basic mathematical tools that we are going to use in this chapter.

Symmetries of a space-time

A space-time is considered to be stationary when it admits a time-like Killing vector field[1], ξ^a. This means that there exists a coordinate system in which the metric components can be time independent, or physically, that in this space-time an observer can see no changes in the gravitational field. If, in addition, this Killing vector field happens to be hyper-surface orthogonal, i.e. is orthogonal to a family of space-like surfaces of constant time, the space-time is called static. A characteristic example of a static space-time is the Schwarzschild solution, which in spherical coordinates is given by

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{3.1}$$

while a characteristic stationary metric is the Kerr metric which is given by

$$\begin{aligned} ds^2 &= \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 \\ &- \left(r^2 + \alpha^2 + \frac{2Mr\alpha^2}{\rho^2}\sin^2\theta\right)\sin^2\theta d\varphi^2 + \frac{4Mr\alpha\sin^2\theta}{\rho^2} dt d\varphi \end{aligned} \tag{3.2}$$

[1]for the definition of a Killing vector field, see (6.3).

where the length scales α, ρ and Δ are given by

$$\alpha = J/M, \ \ \rho^2 = r^2 + \alpha^2 \cos^2\theta, \ \ \Delta = r^2 - 2Mr + \alpha^2 \tag{3.3}$$

with J being the angular momentum of the mass M.

Spherically symmetric is the space-time whose isometry group contains a subgroup which is isomorphic to the rotation group SO(3) and the orbits of this group are 2-spheres. Alternatively we can say that it is a space-time whose metric is invariant under rotations. Similarly, axisymmetric is a space-time if there exists a space-like Killing vector field ψ^a whose orbits are closed curves.

Stationary and axisymmetric metric

If the two Killing vectors ξ^a and ψ^a, that denote stationarity and axisymmetry respectively, commute $[\xi, \psi] = 0$, we say that the space-time is stationary and axisymmetric. This implies that we can always choose coordinates $(x^0 = t, x^1 = \phi, x^2, x^3)$, such that $\xi^a = (\partial/\partial t)^a$ and $\psi^a = (\partial/\partial\phi)^a$ are coordinate vector fields, ending up with a metric that is independent of t and ϕ, $g_{\mu\nu}(x^\lambda) = g_{\mu\nu}(x^2, x^3)$.

A general 2-rank tensor is composed of 16 elements. If it is symmetric, as is the metric, 10 of them are independent. Assuming also that the hypotheses of the theorem (6.3) in the appendix are satisfied (which is happening in a wide range of stationary, axisymmetric space-times of physical interest), we can choose the coordinates x^2 and x^3 in one of the orthogonal 2-surfaces to ξ^a and ψ^a. So the metric has a form

$$g_{\mu\nu} = \begin{bmatrix} -A & B & 0 & 0 \\ B & C & 0 & 0 \\ 0 & 0 & g_{22} & g_{23} \\ 0 & 0 & g_{23} & g_{33} \end{bmatrix} \tag{3.4}$$

where $A = -g_{00} = -\xi^a\xi_a$, $B = g_{01} = \xi^a\psi_a$, $C = g_{11} = \psi^a\psi_a$ and the 2×2 block of zeros expresses the orthogonality of $\partial/\partial x^2$ and $\partial/\partial x^3$ with $\partial/\partial t$ and $\partial/\partial\phi$.

Let us define now the scalar function ρ by $\rho^2 = AC + B^2$ which means that ρ^2 is minus the determinant of the $t - \phi$ part of the metric. If we also assume $\nabla_a\rho \neq 0$ we can identify ρ with one of the coordinates, say x^2, of the 2-surface orthogonal to the Killing vectors. We choose the other coordinate, $z = x^3$, so that $\nabla_a z \perp \nabla_a\rho$. So the metric takes now the form

$$ds^2 = -A(dt - Ed\phi)^2 + A^{-1}\rho^2 d\phi^2 + F^2(d\rho^2 + Ddz^2) \tag{3.5}$$

where $E = B/A$. Thus we have reduced the unknown metric components to four functions, A, E, D, F of two variables ρ, z and eq.(3.5) is the general form of a stationary, axisymmetric space-time satisfying the hypotheses of theorem (6.3).

Energy conditions

As it is known, the distribution of the mass, momentum and stress due to matter or any non-gravitational field is described, in Einstein's equations, by the energy-momentum tensor, $T^{\mu\nu}$. However, the equations alone do not put any restriction on the kind of states of matter or non-gravitational fields. From a point of view, this is not bad, because a good gravity theory should be independent of any assumptions concerning non-gravitational physics, but, on the other hand, it is also a weakness, because without some further criterion, the equations admit solutions with un-physical properties. To eliminate these solutions we impose some energy conditions to the theory. Mathematically speaking, these conditions are essentially restrictions on the eigenvalues and eigenvectors of the stress-energy tensor. They are all summarized in the following table [21].

Table 3.1: Energy conditions

Name	Statement	Conditions
Weak	$T_{ab}u^a u^b \geq 0$	$\rho \geq 0\ ,\ \rho + p_i > 0$
Null	$T_{ab}k^a k^b \geq 0$	$\rho + p_i \geq 0$
Strong	$(T_{ab} - \frac{1}{2}T g_{ab})u^a u^b \geq 0$	$\rho + \sum_i p_i \geq 0\ ,\ \rho + p_i \geq 0$
Dominant	$-T^a{}_b u^b$ future directed	$\rho \geq 0\ ,\ \rho \geq \lvert p_i \rvert$

Congruences of null geodesics

Consider two neighbouring geodesics g_0 and g_1, each described by relations $x^\alpha(t)$, where t is an affine parameter. The geodesics can be either time-like, space-like or null. We introduce an entire family of interpolating geodesics in the space between g_0 and g_1. To each geodesic we assign a label $s \in [0, 1]$, such that g_0 comes with the label $s = 0$ and g_1 with $s = 1$. The vector field $k^\alpha = \partial_t x^\alpha$ is tangent to the geodesics, and it satisfies the equation $k^\alpha \nabla_\alpha k^\beta = 0$. It is easily proven that, there exists a vector field $\xi^\alpha = \partial_s x^\alpha$, which is tangent to the family of curves labelled by t and parametrized by s (which are not geodesics in general). The physical meaning of this vector field is that it represents the deviation vector between g_0 and g_1.

We can see at once that $\partial_s k^\alpha = \partial_t \xi^\alpha$ which can be written covariantly as

$$\pounds_k \xi^a = \pounds_\xi k^a = 0 \Rightarrow k^\beta \nabla_\beta \xi^a = \xi^\beta \nabla_\beta k^a \,. \tag{3.6}$$

It is easy to see that $\xi^\alpha k_\alpha$ is constant along g_0 and thus we can choose $\xi^\alpha k_\alpha = 0$, i.e. ξ^α is orthogonal to k^α. We can see that g_1 moves away from g_0 with acceleration

$$\frac{D^2 \xi^a}{dt^2} = -R^\alpha{}_{\beta\gamma\delta} k^\beta \xi^\gamma k^\delta \tag{3.7}$$

which is called geodesic deviation equation.

We define as a congruence of geodesics, a family of curves such that at each point in an open region O in space-time, there passes one and only one curve from this family. There are congruences of time-like and null geodesics. The case of space-like geodesics does not require a separate treatment, as it is identical to the time-like case and also less interesting from a physical point of view. Here we are going to discuss only the congruence of null geodesics. We want to find the time evolution of a congruence, or alternatively stated, to see how the deviation vector ξ^α behaves in the region between two neighbouring geodesics in the congruence.

As before, we consider $k^\alpha = \partial_\lambda x^\alpha$ to be the tangent vector field to the geodesics, with λ an affine parameter. We also assume that k^α is null. Furthermore, we denote with ξ^α the deviation vector which is considered to be orthogonal to k^α, i.e. orthogonal to the geodesics, and also that its Lie derivative in the direction of k^α vanishes. These statements are mathematically translated as

$$k^\alpha k_\alpha = 0 \ , \ k^\beta \nabla_\beta k^\alpha = 0 \ , \ \xi^\beta \nabla_\beta k^\alpha = k^\beta \nabla_\beta \xi^\alpha \ , \ k^\alpha \xi_\alpha = 0 \, . \tag{3.8}$$

Since we'll be interested only in the transverse properties of the congruence, which are determined by the deviation vector ξ^α, we'll isolate the part of the metric, which is transverse to the tangent vector. To do it, we introduce an auxiliary null vector field N_α, such that $k^\alpha N_\alpha = -1$. Thus, the transverse part of the metric is given by

$$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta \, . \tag{3.9}$$

We can easily verify that the relations

$$h_{\alpha\beta} k^\beta = h_{\alpha\beta} N^\beta = 0 \ , \ h^\alpha{}_\alpha = 2 \ , \ h^\alpha{}_\mu h^\mu{}_\beta = h^\alpha{}_\beta \, , \tag{3.10}$$

are satisfied. From these we confirm that $h_{\alpha\beta}$ is purely transverse (orthogonal to both k^α and N^α) and thus two-dimensional. Sadly, $h_{\alpha\beta}$ is not unique, since N_α is not uniquely determined by the null and the normalization conditions.

We wish now to determine the failure of the deviation vector to be parallel transported along the geodesics and for this we introduce the tensor field

$$B_{\alpha\beta} = \nabla_\beta k_\alpha \, . \tag{3.11}$$

such that

$$k^\beta \nabla_\beta \xi^\alpha = B^\alpha{}_\beta \xi^\beta \, . \tag{3.12}$$

From (3.8) we see that $k^\alpha B_{\alpha\beta} = 0 = B_{\alpha\beta} k^\beta$, but $N^\alpha B_{\alpha\beta} \neq 0$, which means that, eq. (3.12) has a non-transverse component that should be removed.

At first, we isolate the transverse part of the deviation vector in order to calculate the transverse components of the relative velocity. Finally, we obtain

$$(\nabla_\beta \tilde{\xi}^\alpha k^\beta)^\sim = \tilde{B}^\alpha{}_\beta \tilde{\xi}^\beta \, , \tag{3.13}$$

where

$$\tilde{B}_{\alpha\beta} = h^\mu{}_\alpha h^\nu{}_\beta B_{\mu\nu} \tag{3.14}$$

and the tilde ($\sim$) denotes that is purely transverse. From eq. (3.9) we can write (3.14) more explicitly

$$\tilde{B}_{\alpha\beta} = (g_\alpha{}^\mu + k_\alpha N^\mu + N_\alpha k^\mu)(g_\beta{}^\nu + k_\beta N^\nu + N_\beta k^\nu) B_{\mu\nu} \tag{3.15}$$

$$= (g_\alpha{}^\mu + k_\alpha N^\mu + N_\alpha k^\mu)(B_{\mu\beta} + k_\beta B_{\mu\nu} N^\nu) \tag{3.16}$$

$$= B_{\alpha\beta} + k_\alpha N^\mu B_{\mu\beta} + k_\beta B_{\alpha\mu} N^\mu + k_\alpha k_\beta B_{\mu\nu} N^\mu N^\nu \, . \tag{3.17}$$

Summarizing, eq. (3.13) describes the purely transverse behaviour of the null congruence and the vector $\tilde{B}^\alpha{}_\beta \tilde{\xi}^\beta$ can be interpreted as the transverse relative velocity between two neighbouring geodesics.

If we decompose now the transverse part of the tensor $B_{\alpha\beta}$, to its irreducible parts, we get

$$\tilde{B}_{\alpha\beta} = \frac{1}{2}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}\,, \tag{3.18}$$

where $\theta = \tilde{B}^{\alpha}{}_{\alpha}$ is the *expansion* scalar, which is the trace part of $\tilde{B}_{\alpha\beta}$ and denotes the fractional rate of change - per unit affine parameter distance - of the congruence's cross-sectional area, $\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2}\theta h_{\alpha\beta}$ is the *shear* tensor, the symmetric and traceless part of $\tilde{B}_{\alpha\beta}$ and $\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$ the *rotation* tensor, the antisymmetric part of $\tilde{B}_{\alpha\beta}$.

If there exists a tangent vector field such that, the rotation tensor vanishes, i.e. $\omega_{\alpha\beta} = 0$, then the congruence is hyper-surface orthogonal.

The equation that describes the evolution of the expansion parameter θ with respect to the affine parameter λ of the geodesics, is called Raychaudhuri's equation and is given by

$$\begin{aligned} \frac{d\theta}{d\lambda} &= -B^{\alpha\beta}B_{\beta\alpha} - R_{\alpha\beta}k^{\alpha}k^{\beta} \\ &= -\frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta}k^{\alpha}k^{\beta} \end{aligned} \tag{3.19}$$

We notice that the second and third terms are zero only when the tensors vanish. Thus, since both the shear and the rotation tensor are purely transverse, it will be $\sigma^{\alpha\beta}\sigma_{\alpha\beta} \geq 0$ and $\omega^{\alpha\beta}\omega_{\alpha\beta} \geq 0$. From this, we conclude that if a congruence of null geodesics is hyper-surface orthogonal, i.e. $\omega_{\alpha\beta} = 0$, the null energy condition holds, i.e. $R_{\alpha\beta}k^{\alpha}k^{\beta} \geq 0$, then the Raychaydhuri equation implies $d\theta/d\lambda \leq 0$, which means that the geodesics are focused during the evolution of the congruence.

3.2 No-hair theorems

In this section we'll describe the "no-hair" theorem and deviations from it, in the context of GR as well as its generalization to Brans-Dicke-like theories with a cosmological constant and an arbitrary potential. In particular, we'll show that

The black hole "no-hair" conjecture states that (see, for example, [22]-[23]) gravitational collapse reaches a stationary final state, characterized by a small number of parameters [24]. Alternatively stated, all the black-hole solutions of the Einstein's equation in vacuum or in the presence of an electromagnetic field, which are stationary and asymptotically flat, can be described only by three parameters, their mass, angular momentum and charge.

Therefore, black-holes are in general simple objects, whose geometry (exterior to the event horizon) is a member of the Kerr-Newman family. The quantities that characterize them are global charges, which can (at least in principle) be measured far from the black hole event horizon. During the process of the formation of a black hole (e.g. gravitational collapse of a dying star), an enormous amount of information about the star which collapsed is being lost. Further progress in observational astronomy and future gravitational wave detectors (e.g. LIGO, VIRGO) may even be able to probe the validity of the "no-hair" conjecture for astrophysical black holes by verifying that the mass, angular momentum and quadrapole moment Q of the black hole satisfy the relation $Q = J^2/M$ which holds for Kerr black holes.

This theorem, as we've already mentioned, has been proved under certain assumptions, stationarity, asymptotic flatness, four-dimensional space-time and vacuum Einstein equations. If one of the aforementioned assumptions does not hold, the theorem is not necessarily valid. For example, if a negative cosmological constant is included, so that the space-time is asymptotically anti-de Sitter (AdS) and not flat, then the event horizon of the black hole is not necessarily spherical, which means that there could exist "topological" black holes (see for details [25, 26, 27]).

Brans-Dicke with a positive Λ

Hawking has proven [28] that stationary and asymptotically flat black holes in Brans-Dicke theory (without a potential) are no different than in GR, i.e. they do not support a non-trivial profile for the scalar field. This was extended [29], to a class of $f(R)$ gravity theories as well as scalar-tensor theories with $\omega = \omega(\phi)$ and generic convex potential $V(\phi)$.

We are going to discuss the case where $\Lambda > 0$ in the framework of the Brans-Dicke theory. Therefore, we assume a black hole surrounded by a stationary cosmological horizon. As we mentioned before, we'll do our calculations in the Jordan frame and we 'll make no precise assumptions for the form of the asymptotic behaviour of the scalar field or the metric. We further assume that the space-time is stationary, axisymmetric, torsion-free and with no naked curvature singularity in the region between the black hole and the cosmological horizon. The trace of the energy-momentum tensor on the right hand side of the equation (2.7) may have several components. Firstly, conformally invariant matter fields like Maxwell, Yang-Mills, conformal scalar or massless fermions, have $T = 0$. Each of the remaining components will be assumed to be a perfect fluid with

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu} \tag{3.20}$$

where $u^a u_a = -1$. We consider the equation of state $P = w\rho$ with P being the pressure and ρ the energy density of the fluid.

We assume that the components of the energy-momentum tensor satisfy the weak energy condition: $\rho \geq 0$ and since they are non-dark energy type, they also satisfy the strong energy condition, which implies $w \geq -1/3$. For non-relativistic matter we have $w \ll 1$ and for radiation, $w = 1/3$, and finally we end up with $-1/3 \leq w \leq 1/3$. Since

$$T = 3P - \rho = -\rho(1 - 3w) \tag{3.21}$$

it will always be $T \leq 0$.

As for the geometric set up we are going to use, its details can be found in [30] and [31] and references therein. It is motivated by the Kerr-Newman-de-Sitter family of space-times.

Any stationary and axisymmetric space-time is endowed, as we mentioned in the previous section, with two Killing vector fields ξ^a (time-like) and ψ^a (space-like), which generate stationarity and axisymmetry, respectively. We assume that these two Killing vector fields commute,

$$[\xi, \psi]^a = \xi^b \partial_b \psi - \psi^b \partial_b \xi = 0 \,. \tag{3.22}$$

We assume that the 2-planes orthogonal to these commuting Killing vector fields foliate the space-time.

For our convenience, we shall first construct a family of space-like hyper-surfaces, by defining a vector field $\chi_a = \xi_a - (\xi \cdot \psi/\psi \cdot \psi)\psi_a$, such that $\chi_a \psi^a = 0$ identically. Then, it turns out that (e.g. [32] and references therein):

- χ_a is hyper-surface orthogonal and time-like in our region of interest, $\chi_a \chi^a = -\beta^2 \leq 0$.
- χ_a is Killing and null in any compact β^2 hyper-surface and thus any such hyper-surface is a Killing horizon (has vanishing expansion, shear and rotation for tangent null geodesic congruences).
- for our case we have two $\beta^2 = 0$ hyper-surfaces, the smaller one is the black hole and the larger one is the cosmological event horizon.

The form of the metric in this basis is

$$g_{ab} = -\beta^{-2} \chi_a \chi_\beta + f^{-2} \psi_a \psi_b + \gamma_{ab} \,, \tag{3.23}$$

where f^2 is the norm of ψ^a and γ_{ab} is the metric of the integral sub-manifold orthogonal to both χ^a and ψ^a. Similarly, the Brans-Dicke scalar ϕ is also assumed to be stationary and axisymmetric, i.e. to satisfy $\xi^a \nabla_a \phi = 0 = \psi^a \nabla_a \phi$, which implies that $\chi^a \nabla_a \phi = 0$ as well. Since both ξ^a and ψ^a are Killing vector fields, they have vanishing divergence, so we have $\nabla_a \chi^a = 0$.

In order to show that the scalar field ϕ is generically divergent on any stationary, axisymmetric Killing horizon, we shall solve for it in an infinitesimal neighbourhood of any Killing horizon, $\beta^2 = 0$.

Now, by taking the trace of the eq. (2.7) we obtain

$$R = \frac{4\Lambda - 8\pi T}{\phi} \frac{2\omega}{2\omega + 3} + \frac{\omega}{\phi^2} (\nabla_a \phi)(\nabla^a \phi) \,, \tag{3.24}$$

from which we conclude that if ϕ diverges on the horizon, the R will also diverge and will lead to a naked curvature singularity.

We shall show below that solutions with these properties have necessarily constant ϕ and exist only for $\omega = \infty$. In other words, they reduce to solutions of Einstein's general relativity with a positive cosmological constant.

Indeed, using $\sqrt{-g} = \beta \sqrt{h}$, where h is the determinant of the metric h_{ab} of the spatial hyper-surface orthogonal to χ^a, $\chi^a \nabla_a \phi = 0$ and $\nabla_a \chi^a = 0$, we obtain

$$\Box \phi = \frac{1}{\beta} D_a [\beta D^a \phi] \,, \tag{3.25}$$

where D_a is the spatial derivative associated with h_{ab}. Thus, equation (2.13) becomes

$$D_a (\beta D^a \phi) = \frac{\beta (8\pi T - 4\Lambda)}{2\omega + 3} \,. \tag{3.26}$$

We multiply this with $e^{\epsilon\phi}$ (with $\epsilon = \pm 1$), and integrate over the spatial hyper-surface orthogonal to χ^a (say Σ) between the two horizons. We get

$$\begin{aligned}
\int_\Sigma e^{\epsilon\phi} D_a(\beta D^a\phi) &= \int_\Sigma \beta e^{\epsilon\phi} \frac{8\pi T - 4\Lambda}{2\omega + 3} \\
\int_\Sigma D_a(e^{\epsilon\phi}\beta D^a\phi) &= \int_\Sigma \beta\epsilon e^{\epsilon\phi} D_a\phi D^a\phi + \int_\Sigma \beta e^{\epsilon\phi} \frac{8\pi T - 4\Lambda}{2\omega + 3} \\
\int_{\partial\Sigma} e^{\epsilon\phi}\beta n^a D_a\phi &= \int_\Sigma \beta e^{\epsilon\phi} \left[\epsilon(D^a\phi)(D_a\phi) + \frac{8\pi T - 4\Lambda}{2\omega + 3}\right]
\end{aligned} \tag{3.27}$$

The surface integral is evaluated on the two horizons, where β vanishes, and n^a is the space-like unit normal to the horizons. Then since $\beta = 0$ on $\partial\Sigma$, the left hand side vanishes[2] and we are left with a vanishing integral over Σ.

For $2\omega + 3 > 0$, we take $\epsilon = -1$ in eq. (3.27). Given that the inner product $(D_a\phi)(D^a\phi)$ is space-like and hence always positive definite, $T \leq 0$ and $\Lambda > 0$, the integral is the sum of two negative-definite terms. Thus, the vanishing of the integral over Σ implies $\omega \to \infty$ and ϕ =constant. Similarly for $2\omega + 3 < 0$, with a choice of $\epsilon = +1$. Finally, for finite ω the scalar field and consequently the curvature scalar diverge at the horizons [33].

If at least one of the horizons is not a true horizon, e.g. in the case of a normal star or of a time dependent astrophysical black hole, the left hand side of (3.27) will be non-vanishing and the scalar field cannot be argued to be constant. Let us assume that outside such a stellar object, there exists a surface $\mathcal{S}$, where the Brans-Dicke field has an extremum. Then by (3.27) it becomes obvious that in between $\mathcal{S}$ and the cosmological event horizon, ϕ is constant and $\omega = \infty$. Now, for $\omega = \infty$ the theory reduces to Einstein gravity, so that ϕ is constant inside $\mathcal{S}$ as well. Thus, for finite ω and in the presence of a positive cosmological constant, the field ϕ of a stationary non-black hole space-time must be monotonic.

We have concluded that for finite ω, ϕ necessarily diverges on the horizons ($\beta^2 = 0$). We show next that its behaviour is $\phi \sim \ln\beta$ there. Let us define $Z_a \equiv \nabla_a\beta^2$. It satisfies $\chi^a Z_a = 0 = \psi^a Z_a$. Thus, Z_a is one of the basis vectors spanning γ_{ab} in eq. (3.23). We recall that on any Killing horizon, we have

$$\nabla_a\beta^2 = -2\kappa\chi_a\,, \tag{3.28}$$

where the constant κ, assumed non-zero, is the surface gravity, given by

$$\kappa^2 = \frac{(\nabla_a\beta^2)(\nabla^a\beta^2)}{4\beta^2}|_{\beta^2\to 0} \tag{3.29}$$

Let Z be a parameter along the vector field Z^a: $Z^a\nabla_a Z = 1$. Then using eq. (3.29), it is clear that infinitesimally close to the surface $\beta^2 = 0$, we have

$$(\nabla_a\beta^2)(\nabla^a\beta^2) = Z^a\nabla_a\beta^2 = \frac{d\beta^2}{dZ} = 4\kappa^2\beta^2 \tag{3.30}$$

which can be integrated to give

$$Z = \frac{1}{4\kappa^2}\ln\beta^2\,. \tag{3.31}$$

[2]Clearly the same conclusion holds for more than one black hole horizons.

With this, we shall now solve for the scalar field in an infinitesimal neighbourhood of $\beta^2 = 0$. We note that both χ_a and Z_a become null here as $\sim \beta^2$, eq. (3.28). Then we write $\sqrt{-g} = \beta^2\gamma(x)$, where $\gamma(x)$ is a non-vanishing and well behaved function. Let Θ^a be a basis orthogonal to the near horizon equation of motion. This is essentially the manifestation of the 1+1-dimensional dynamics close to a Killing horizon. Since we are assuming stationarity, our theory becomes one dimensional.

Then using eq. (3.30),(3.31) in (2.13) it is easy to obtain

$$\phi = \int dZ \frac{4\kappa^2}{\gamma(x)} \left(\int^x d\beta^2(x')\Gamma(x') \right) + C_1 Z + C_2 \,, \tag{3.32}$$

with $\Gamma(x') \equiv \frac{8\pi T - 4\Lambda}{2\omega+3}\gamma(x')$ and C_1, C_2 constants.

As $\beta^2 \to 0$, $dZ = d\beta^2/4\kappa^2\beta^2$ is unbounded infinitesimally close to the horizon. Since $\Gamma(x)$ is well behaved, the integral in (3.32) becomes

$$\int \gamma(x) dZ \left(\int^x \Gamma(x') d\beta^2(x') \right) = \frac{Z}{\gamma(x)} \left(\int^x \Gamma(x') d\beta^2(x') \right)_{\beta^2 \to 0} \tag{3.33}$$

from which one concludes $\phi \sim Z \sim \ln \beta^2$.

Generalization to Scalar-Tensor, Brans-Dicke-like theories

If in the action (2.4) we make ω a function of ϕ, $\omega = \omega(\phi)$ and also add an arbitrary potential $V(\phi)$[3], eq. (2.13) becomes

$$(2\omega(\phi) + 3) \Box\phi + \frac{d\omega(\phi)}{d\phi} (\nabla\phi)^2 + \phi \frac{dV}{d\phi} - 2V = 8\pi T - 4\Lambda \tag{3.34}$$

If we now make a conformal transformation as

$$d\psi \equiv \sqrt{2\omega(\phi) + 3} d \ln \phi \,, \; \mathcal{G}_{\alpha\beta} \equiv \phi g_{\alpha\beta} \,, \tag{3.35}$$

we go to Einstein frame, where eq. (3.34) becomes

$$\Box_{\mathcal{G}} \psi = U'(\psi) \equiv \frac{8\pi T - 4\Lambda}{\phi^2 \sqrt{2\omega(\phi) + 3}} + V'(\psi) \,, \tag{3.36}$$

where $\phi = \phi(\psi)$ as a function of ψ.

If we multiply both sides of the first equality of (3.36) with U' and integrate over the spatial hyper-surface, orthogonal to the time-like Killing vector field (say Σ) we obtain

$$\begin{aligned} \int_\Sigma (U' D_\alpha(\beta D^\alpha \psi) &= \int_\Sigma U'^2 \Rightarrow \\ \int_\Sigma (U'' \beta (D_\alpha\psi)(D^\alpha\psi) + U'^2) &= \int_\Sigma D^\alpha (U' \beta D_\alpha \psi) \\ &= \int_{\partial\Sigma} \beta U' n^\mu D_\alpha \psi \,, \end{aligned} \tag{3.37}$$

[3]The scalar field potential $V(\phi)$ constitutes a natural generalization of the cosmological constant and may reduce to a constant, or to a mass term; it is often included in the action when studying early universe or the present-day accelerated universe, as is customary for most scalar fields in particle physics and inflationary theories [34].

where, for the last equality, we have used Stokes theorem and n^μ is the normal vector to the boundary of Σ. The surface integral is evaluated between the horizons where β vanishes. Then since $\beta = 0$ on $\partial\Sigma$ the right hand side is zero, $U'^2 \geq 0$ and $(D_\alpha\psi)(D^\alpha\psi)$ is bounded and space-like, thus non-negative everywhere. Thus, if $U'' \geq 0$ is a local condition we conclude that $\psi = \psi_0 = const.$ and $U'(\psi_0) = 0$. Let us see the different cases:

- For $V = 0$ the no-hair theorem is valid for arbitrary $\omega(\phi)$, $\Lambda \geq 0$ and $T \leq 0$. To verify this, we can write (3.36) in terms of the spatial derivative D, multiply with $e^{-\psi}$ both sides and integrate by parts over the space between the two horizons. Positivity of all terms in the resulting vanishing integral leads to the conclusion that ϕ is constant and $\omega(\phi) \to \infty$.
- For $\Lambda = 0 = T$ the no-hair theorem is always valid as long as $V'' \geq 0$.
- However, once we switch Λ and/or T on, the validity of the theorem requires, as we said, $U'' \geq 0$, which restricts $\omega(\phi)$. Thus the presence of $\Lambda > 0$ can lead to hairy black holes even for $T = 0$ and $V'' \geq 0$.
- Replacing Λ by generic dark energy with $w < -1/3$, does not modify our conclusions provided there exists a cosmological horizon in that case as well. However, although $w < -1/3$ implies repulsive effects, which is necessary for the existence of a cosmological horizon [35], it is not known to be also sufficient.

Chapter 4

Solutions of BD Theory

In the beginning of this chapter, we will consider large ω, which agrees with astrophysical and cosmological constraints [36, 37], and perturb the equations around a Schwarzschild-de Sitter background, keeping linear terms in $1/\omega$ to find an analytical solution, for the dynamical fields. This solution will, of course, contain undetermined integration constants which will be fixed by imposing some boundary conditions in the next chapter. In the end of this chapter we will calculate the maximum turnaround radius in the BD theory, which as we'll see depends on the boundary conditions.

4.1 Analytical Perturbative Solution

Let's consider a static, spherically symmetric metric ansatz[1] for the vacuum solution ($T_{\mu\nu} = 0$)

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\Omega^2 \\ &= -f(r)dt^2 + h(r)dr^2 + r^2 d\Omega^2 \end{aligned} \tag{4.1}$$

We will perturb the fields according to

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{\omega}\delta g_{\mu\nu} \ \ , \ \ \phi\left(r\right) = \phi_0 + \frac{1}{\omega}\delta\phi\left(r\right) \tag{4.2}$$

where $\bar{g}_{\mu\nu} = g^{SdS}_{\mu\nu}$ and $\phi_0 = 1$, while the SdS metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)dt^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{4.3}$$

[1]Birkhoff's theorem in general relativity [38] states that any spherically symmetric solution of the Einstein's equations in the vacuum must be static and asymptotically flat. Alternatively stated, the Schwarzschild solution is the only static, spherically symmetric solution of the vacuum Einstein equations. It can also be generalized to: any spherically symmetric solution of the Einstein-Maxwell equations must be stationary and asymptotically flat (Reissner-Nordström electrovacuum solution). In Brans-Dicke-like scalar-tensor theories, in order for the theorem to hold [39], the scalar field has to be static and then the solution of the field equations can be different from Schwarzschild-de Sitter. If this field is constant the theory reduces to GR and the Birkoff's theorem is valid.

This metric describes a spherical object embedded in a de Sitter space-time and admits two Killing horizons at the vanishing points of the polynomial $1 - 2M/r - \Lambda r^2/3$. In general this polynomial has three real roots

$$\begin{aligned} r_H &= \frac{2}{\sqrt{\Lambda}} \cos\left(\frac{1}{3}\cos^{-1}\left(3M\sqrt{\Lambda}\right) + \frac{\pi}{3}\right) \\ r_C &= \frac{2}{\sqrt{\Lambda}} \cos\left(\frac{1}{3}\cos^{-1}\left(3M\sqrt{\Lambda}\right) - \frac{\pi}{3}\right) \\ r_H &= -\left(r_H + r_C\right) \end{aligned} \tag{4.4}$$

with r_H and r_C to be the only positive solutions and thereby defining the two horizons of the spacetime. For an analytical derivation see [31]. For $3M\sqrt{\Lambda} \ll 1$[2], the horizons are located at $r_H \simeq 2M$ and $r_C \simeq \sqrt{3/\Lambda}$.

The perturbed scalar equation (2.13) is

$$\bar{\Box}\delta\phi = -2\Lambda \tag{4.5}$$

which can be easily solved to give up to an irrelevant additive constant

$$\delta\phi(r) \simeq \frac{C_1}{r_H}\ln\left(1 - \frac{r_H}{r}\right) + \left(1 + \frac{C_1}{2r_C}\right)\ln\left(1 + \frac{r}{r_C}\right) + \left(1 - \frac{C_1}{2r_C}\right)\ln\left(1 - \frac{r}{r_C}\right) \tag{4.6}$$

Assuming $r_H \ll r_c$ and also that $C_1 \ll r_c$, we obtain

$$\delta\phi(r) = \frac{C_1}{2M}\ln(1 - \frac{2M}{r}) + \ln(1 - \frac{\Lambda r^2}{3}) \tag{4.7}$$

which, in our region of interest, i.e. $2M \ll r \ll \sqrt{3/\Lambda}$, can be further simplified to give

$$\boxed{\delta\phi(r) = -\frac{C_1}{r} - \frac{\Lambda r^2}{3}} \tag{4.8}$$

Let us now calculate the backreaction of this solution to the SdS metric. The perturbed metric equations are given by

$$R^{(1)}_{\mu\nu} = \nabla_\mu\delta\phi\nabla_\nu\delta\phi + \nabla_\mu\nabla_\nu\delta\phi + \Lambda\left(\delta g_{\mu\nu} - g^0_{\mu\nu}\delta\phi - g^0_{\mu\nu}\right) \tag{4.9}$$

where $R^{(1)}_{\mu\nu}$ is the first order correction to the Ricci tensor and the solutions of these equations are

$$\delta f(r) = \frac{2M}{r}\left(\frac{C_1}{M} - \frac{C_2}{18M} + 3C_3\right) + \frac{\Lambda r^2}{3}\left(3C_3 + 1\right) - 3C_3 \tag{4.10}$$

$$\delta h(r) = \frac{C_2}{9r} - \frac{2\Lambda r^2}{3} \tag{4.11}$$

The integration constants C_1 and C_2 can be fixed from the boundary conditions, while C_3 can be set to zero by time re-parametrization.

[2]which is true even for $M \sim 10^{17}M_\odot$ since $\Lambda = 10^{-52}m^{-2}$.

Finally, the full solution (including the background) is given by

$$f(r) = 1 - \frac{2M}{r}\left(1 + \frac{1}{\omega}\left(\frac{C_2}{18M} - \frac{C_1}{M}\right)\right) - \frac{\Lambda r^2}{3}\left(1 - \frac{1}{\omega}\right) \tag{4.12}$$

$$\begin{aligned} h(r) &= \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1} + \frac{1}{\omega}\left(\frac{C_2}{9r} - \frac{2\Lambda r^2}{3}\right) \\ &\simeq \left(1 - \frac{2M}{r}\left(1 + \frac{C_2}{18M\omega}\right) - \frac{\Lambda r^2}{3}\left(1 - \frac{2}{\omega}\right)\right)^{-1} \end{aligned} \tag{4.13}$$

Turn around radius

As we discussed in the first chapter the maximum turn around radius of a structure is the distance at which the radial acceleration is zero [10], and corresponds to the point where $f'(r_{ta}) = 0$. From eq.(4.12) we obtain

$$r_{ta} = \left(\frac{3M}{\Lambda}\right)^{1/3}\left(\frac{18C_1 - C_2 - 18M\omega}{-18M\omega + 18M}\right)^{1/3} \tag{4.14}$$

or if we keep only terms of order $1/\omega$

$$r_{ta} = \left(\frac{3M}{\Lambda}\right)^{1/3}\left(1 + \frac{1}{2\omega}\frac{C_2 - 18C_1 + 18M}{27M}\right) . \tag{4.15}$$

Obviously, this maximum bound becomes smaller or bigger, depending on the boundary conditions, i.e. the values of the integration constants.

Chapter 5

Choice of Boundary Conditions

As we mentioned before in order to fix the integration constants we have to impose some boundary conditions to the functions. Also to solve numerically the equations some initial conditions have to be given. In this chapter we derive a solution in the interior and exterior of a star. The exterior solution reduces to the weak field solution, that Brans and Dicke derived in their original paper, for large ω. These solutions are derived in the case where $\Lambda = 0$.

5.1 Exterior and Interior Star Solution

Let us consider a star at the surface of which we're going to set the boundary conditions. At first, we'll find an exterior ($\equiv$ vacuum) solution for a static, spherically symmetric metric and for the Brans-Dicke field. Afterwards, we are going to find an interior solution for the metric and the scalar, considering the inside of the star to be a perfect fluid. Finally, we will match the two solutions at the surface of the star using Israel matching conditions.

We consider a static, spherically symmetric metric of the form

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\Omega^2 \\ &= -e^{2\Phi(r)}dt^2 + e^{2\Psi(r)}dr^2 + r^2 d\Omega^2 \end{aligned} \tag{5.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the 2-sphere. We also denote the scalar field as

$$\phi = e^{\chi} \tag{5.2}$$

where (as in the previous chapter) ϕ and consequently χ are only functions of the radial coordinate and do not depend on any other coordinate. The equations of motion (2.7) and (2.13) become respectively for this metric ansatz,

$$G_{\mu\nu} = 8\pi e^{-\chi}\left(T_{\mu\nu} - \frac{Tg_{\mu\nu}}{2\omega+3}\right) + (\omega+1)\nabla_\mu\chi\nabla_\nu\chi + \nabla_\mu\nabla_\nu\chi - \frac{\omega}{2}g_{\mu\nu}(\nabla\chi)^2 \tag{5.3}$$

$$\Box\chi + (\nabla\chi)^2 = \frac{8\pi T}{2\omega+3}e^{-\chi} \tag{5.4}$$

The energy momentum tensor of a perfect fluid is given by

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu} \tag{5.5}$$

where the four-velocity is normalized so that $u_\mu u^\mu = -1$, ρ is the energy density and P is the pressure. Finally, for the metric ansatz (5.1) the tt, rr and $\theta\theta$ components of the metric equations (5.3) become

$$\frac{2}{r}\Psi' + \frac{e^{2\Psi}-1}{r^2} = 8\pi\rho e^{2\Psi-\chi} + \chi'' + \chi'\left[\frac{2}{r} - \Psi' + (1+\frac{\omega}{2}\chi')\right] \quad (5.6)$$

$$\frac{2}{r}\Phi' - \frac{e^{2\Psi}-1}{r^2} = 8\pi P e^{2\Psi-\chi} - \chi'\left[\frac{2}{r} + \Phi' - \frac{\omega}{2}\chi'\right] \quad (5.7)$$

$$\Phi'' + (\Phi'-\Psi')(\Phi'+\frac{1}{r}) = 8\pi P e^{2\Psi-\chi} - \chi'' - \chi'\left[\frac{1}{r} + \Phi' - \Psi' + (1+\frac{\omega}{2})\chi'\right] \quad (5.8)$$

while the scalar equation (5.4) becomes

$$\chi'' + \chi'\left[\frac{2}{r} + \Phi' - \Psi' + \chi'\right] = \frac{8\pi}{2\omega+3}(3P-\rho)e^{2\Psi-\chi}\,. \quad (5.9)$$

Exterior solution

Let us now find the exterior vacuum solution. We keep only linear terms of the dynamical fields in equations (5.6-5.9) and we obtain respectively

$$\frac{2}{r}\Psi' + \frac{2\Psi}{r^2} = \chi'' + \frac{2}{r}\chi' \quad (5.10)$$

$$\frac{2}{r}\Phi' - \frac{2\Psi}{r^2} = -\frac{2}{r}\chi' \quad (5.11)$$

$$\Phi'' + \frac{\Phi'-\Psi'}{r} = -\chi'' - \frac{1}{r}\chi' \quad (5.12)$$

$$\chi'' + \frac{2}{r}\chi' = 0\,. \quad (5.13)$$

This is a system of differential equations and admits an exact solution given by

$$\Phi(r) = \frac{c_1}{r} + c_2 \quad , \quad \Psi(r) = -\frac{c_1+c_3}{r} \quad , \quad \chi(r) = \frac{c_3}{r} + c_4 \quad (5.14)$$

where c_2 can be set to zero by time re-parametrization. Hence, the exterior metric (5.1) is approximately

$$ds^2 = -(1+\frac{2c_1}{r})dt^2 + (1-\frac{2(c_1+c_3)}{r})dr^2 + r^2 d\Omega^2 \quad (5.15)$$

If we transform to isotropic coordinates, we can identify $c_1 = -M$ and $c_1 + c_3 = \gamma M$, where γ is the PPN parameter[1], which in GR is equal to one, and get

$$ds^2 = -(1-\frac{2M}{r})dt^2 + (1+\frac{2\gamma M}{r})(dr^2 + r^2 d\Omega^2) \quad (5.16)$$

Finally, the solution takes the form

$$\Phi_{ext} = -\frac{M}{r} \quad , \quad \Psi_{ext} = \frac{\gamma M}{r} \quad , \quad \chi_{ext} = \chi_0^{ext} + \frac{(1-\gamma)M}{r} \quad (5.17)$$

[1]"The formalism of Newtonian theory plus post-Newtonian corrections is called the "post-Newtonian approximation". It contains a set of parameters (called "PPN parameters") that can be specified arbitrarily", C.W.Misner-K.S.Thorne-J.A.Wheeler *Gravitation*. (see chapter 6)

Interior solution

In order to find the interior solution we define a scale L (to be determined), such that $\epsilon = r/L \ll 1$ within the range of interest, i.e. within the star. We expand the equations (5.6-5.9) in ϵ, which means that $\epsilon \ll 1 \Rightarrow r \ll L$ inside the star.

$$\begin{aligned}
\frac{e^{2\Psi_0}-1}{\epsilon^2} + \frac{2\left(\left(e^{2\Psi_0}+1\right)\Psi_1 - \chi_1\right)}{\epsilon} - 3\chi_2 + \chi_1\Psi_1 &- \frac{\omega}{2}\chi_1^2 \\
-\chi_1^2 + 2\Psi_2 + e^{2\Psi_0}\left(\Psi_2 + 2\Psi_1^2\right) - 8\pi\rho_0 e^{2\Psi_0-\chi_0} &= 0 \qquad (5.18) \\
\frac{1-e^{2\Psi_0}}{\epsilon^2} + \frac{2\left(\Phi_1 + \chi_1 - e^{2\Psi_0}\Psi_1\right)}{\epsilon} + 2\Phi_2 + \Phi_1\chi_1 &+ 2\chi_2 \\
-\frac{\omega}{2}\chi_1^2 - e^{2\Psi_0}\left(\Psi_2 + 2\Psi_1^2\right) - 8\pi P_0 e^{2\Psi_0-\chi_0} &= 0 \qquad (5.19) \\
\frac{\Phi_1 + \chi_1 - \Psi_1}{\epsilon} + 2\Phi_2 + \Phi_1\chi_1 - \Phi_1\Psi_1 + \Phi_1^2 &+ 2\chi_2 \\
-\chi_1\Psi_1 + \frac{\omega}{2}\chi_1^2 + \chi + 1^2 - \Psi_2 - 8\pi P_0 e^{2\Psi_0-\chi_0} &= 0 \qquad (5.20) \\
\frac{2\chi_1}{\epsilon} + \Phi_1\chi_1 + 3\chi_2 - \chi_1\Psi_1 &+ \chi_1^2 \\
-8\pi e^{2\Psi_0-\chi_0}\frac{(3P_0-\rho_0)}{2\omega+3} &= 0 \qquad (5.21)
\end{aligned}$$

where $\Psi_0 = \Psi(\epsilon), \Psi_1 = \Psi'(\epsilon), \Psi_2 = \Psi''(\epsilon), \Phi_0 = \Phi(\epsilon), \Phi_1 = \Phi'(\epsilon), \Phi_2 = \Phi''(\epsilon), \chi_0 = \chi(\epsilon), \chi_1 = \chi'(\epsilon), \chi_2 = \chi''(\epsilon)$. These equations have to be satisfied for any ϵ and thus the coefficients of all the terms in the expansion have to be zero, including the zero order terms. We, thus, obtain a system of ten algebraic equations, but only seven of them are independent. There are eleven parameters to be determined

$$\{\Psi_0, \Psi_1, \Psi_2, \chi_0, \chi_1, \chi_2, \Phi_0, \Phi_1, \Phi_2, P_0, \rho_0\} \qquad (5.22)$$

so that we will be left with four undetermined quantities, while the other seven will be expressed in terms of these four by solving the system.

After choosing to leave $P_0, \rho_0, \chi_0^{int}, \Phi_0$ free we get

$$\begin{aligned}
\Phi_{int} &= \Phi_0 + \frac{2\pi}{3}e^{-\chi_0^{int}}(3P_0+\rho_0)\left(1 - \frac{1}{2\omega+3}\right)r^2 \qquad (5.23) \\
\Psi_{int} &= \frac{4\pi}{3}e^{-\chi_0^{int}}\left(\frac{3P_0-\rho_0}{2\omega+3} + \rho_0\right)r^2 \qquad (5.24) \\
\chi_{int} &= \chi_0^{int} + \frac{4\pi}{3}e^{-\chi_0^{int}}\frac{3P_0-\rho_0}{2\omega+3}r^2 \qquad (5.25)
\end{aligned}$$

where we have substituted back $r = \epsilon$.

Matching the two solutions

As we have seen in the previous sections we have four free parameters in the interior solution, $\{P_0, \rho_0, \chi_0^{int}, \Phi_0\}$, and three in the exterior solution, $\{\chi_0^{ext}, M, \gamma\}$. We can determine some of these parameters by matching the two solutions at the surface of the star, at $r = R_0$.

Let us first, review how this matching can be done. Once a given four-dimensional space-time is endowed with a metric tensor $g_{\alpha\beta}$, we can define an induced metric on a specified hyper-surface, which is three-dimensional. While this is pretty straightforward in the case of a time-like or a space-like hyper-surface, the null case requires special care. It turns out that, we can further define a vectorial surface element that allows vector fields to be integrated over the hyper-surface. The above properties have to do with the intrinsic geometry of a hyper-surface, while the way that this is embedded in the space-time manifold has to do with its extrinsic geometry. In general, the space-time curvature tensor can be decomposed into a purely intrinsic part, the curvature tensor of the hyper-surface, and an extrinsic part that measures the bending of the hyper-surface in space-time; this bending is described by a three dimensional tensor $K_{\alpha\beta}$ known as the extrinsic curvature.

When a given hyper-surface "splits" the space-time into two regions and for each region we have a metric tensor, in order for the union of the two metrics to form a valid solution to the Einstein field equations, some conditions have to be satisfied. These conditions are that, the induced metric and the extrinsic curvature must be the same on both sides of the hyper-surface. In our case, it also must be satisfied that the scalar field and its first derivative are continuous on the surface.

$$g_{ij}^{(int)} = g_{ij}^{(ext)} \quad , \quad K_{ij}^{(int)} = K_{ij}^{(ext)} \tag{5.26}$$

$$\chi^{(int)}(R_0) = \chi^{(ext)}(R_0) \quad , \quad \frac{d\chi^{(int)}}{dr}(R_0) = \frac{d\chi^{(ext)}}{dr}(R_0) \tag{5.27}$$

where $i, j = 0, 2, 3$ and K_{ij} is the extrinsic curvature of the space orthogonal to ∂_r. Since $g_{1i} = 0$ we find that $K_{ij} = -\frac{1}{2}e^{-\Psi}g'_{ij}$. So the matching conditions are

$$\Phi^{(int)}(R_0) = \Phi^{(ext)}(R_0) \ , \ \Psi^{(int)}(R_0) = \Psi^{(ext)}(R_0) \ , \ \frac{d\Phi^{(int)}}{dr}(R_0) = \frac{d\Phi^{(ext)}}{dr}(R_0)$$

$$\chi^{(int)}(R_0) = \chi^{(ext)}(R_0) \quad , \quad \frac{d\chi^{(int)}}{dr}(R_0) = \frac{d\chi^{(ext)}}{dr}(R_0)$$

We can now solve the above system of five equations leaving ρ_0, χ_0^{int} free to get

$$P_0 = 0 \tag{5.28}$$

$$\gamma = \frac{1+\omega}{2+\omega} \tag{5.29}$$

$$\Phi_0 = -\frac{3M}{2R_0} \tag{5.30}$$

$$M = \frac{4\pi\rho_0}{3}\frac{4+2\omega}{3+2\omega}e^{-\chi_0^{int}}R_0^3 \tag{5.31}$$

$$\chi_0^{ext} = \chi_0^{int} - \frac{3M}{2R_0(2+\omega)} \tag{5.32}$$

As we can see from (5.31), in this definition M is the dynamical mass rather than the actual mass obtained by summing all the masses of the fluid particles (i.e. it inherits a contribution from the BD parameter, ω). We can see from (5.31) that, if we had considered $G \neq 1$ it would be

$$GM = \frac{4\pi\rho_0}{3}\frac{4+2\omega}{3+2\omega}e^{-\chi_0^{int}}R_0^3 \tag{5.33}$$

where we can identify from GR $M = \frac{4\pi\rho_0}{3}R_0^3$ at $r = R_0$ and thus

$$G_{eff} = \frac{4+2\omega}{3+2\omega}e^{-\chi_0^{int}} = \frac{4+2\omega}{3+2\omega}\frac{1}{\phi_0} = \frac{4+2\omega}{3+2\omega}G_N\,. \tag{5.34}$$

It is clear that in the GR limit $\omega \to \infty$ we recover $G_{eff} = G_N$.

Hence the full interior solution, is

$$ds^2 = -e^{-\frac{3M}{R_0}}(1+\frac{M}{R_0^3}r^2)dt^2 + (1+\frac{2\gamma M}{R_0^3}r^2)dr^2 + r^2 d\Omega^2 \tag{5.35}$$

$$\chi = \chi_0^{int} + \frac{(\gamma-1)M}{2R_0^3}r^2 \tag{5.36}$$

It is convenient to identify the scale $L = \sqrt{R_0^3/M}$. Now by rewriting the interior solution, we obtain

$$ds^2 = -e^{-\frac{3M}{R_0}}(1+\frac{r^2}{L^2})dt^2 + (1+\frac{2\gamma r^2}{L^2})dr^2 + r^2 d\Omega^2 \tag{5.37}$$

$$\chi = \chi_0^{int} + (\gamma-1)\frac{r^2}{2L^2} \tag{5.38}$$

$$\phi = e^{\chi_0^{int}}\left(1+(\gamma-1)\frac{r^2}{2L^2}\right) \tag{5.39}$$

We can further fix the parameter χ_0^{int} if we consider the large ω limit. In that limit it is $\gamma \sim 1$ and consequently $\phi \sim e^{\chi_0^{int}}$. Thus χ_0^{int} has to be set to zero, since from GR we know that $\phi_0 = 1$. In this limit also, the metric reduces to the GR solution.

Furthermore, the exterior solution is given by

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2\gamma M}{r}\right)dr^2 + r^2 d\Omega^2 \tag{5.40}$$

$$\chi = \chi_0^{int} - \frac{3M}{2R_0(2+\omega)} + \frac{(1-\gamma)M}{r} \tag{5.41}$$

where, as we mentioned before, the mass M is given by (5.31). Thus, if we consider that ω is large, and keep only up to $1/\omega$ terms we get for the metric

$$ds^2 = -\left(1-\frac{2M}{r}\left(1+\frac{1}{2\omega}\right)\right)dt^2 + \left(1-\frac{2M}{r}\left(1-\frac{1}{2\omega}\right)\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{5.42}$$

Finally, in the calculations we will have to give a value to R_0 in order to solve the equations. For the expansion in ϵ to be valid in the inside region of the star, we need $R_0/L \ll 1$, which implies that $M/R_0 \ll 1$. This is a very natural assumption since the Schwarzschild radius is well within the surface of any reasonable astrophysical object of our present interest. So we have obtained a lower bound to the value of R_0. For an upper bound, the value of R_0 is arbitrary, since we can consider an interior structure of any size. However, for our purposes it should definitely be smaller than the maximum turnaround radius. These two assumptions give a range $M \ll R_0 < r_{ta}$ within which the interior solution can be expanded and the matching procedure can be used to determine the parameters of the expansion.

5.2 Weak Field Limit

In the paper at which Brans and Dikce introduced their theory [17], they derived also an analytical solution considering the weak field approximation. They perturbed the metric around flat space-time, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and kept only linear terms in $h_{\mu\nu}$. This solution is of first order in mass densities and is equally important as the one derived in general relativity. In order for our analytical, perturbative solution (4.12,4.13) to reduce to this one at the $\Lambda \to 0$ limit, we have to equate these two and fix the integration constants. Their solution is

$$f_{wf}(r) = 1 - \frac{2M}{r}\left(1 + \frac{1}{2\omega+3}\right) \sim 1 - \frac{2M}{r}\left(1 + \frac{1}{2\omega}\right) \quad (5.43)$$

$$h_{wf}(r) = 1 + \frac{2M}{r}\left(1 - \frac{1}{2\omega+3}\right) \sim 1 + \frac{2M}{r}\left(1 - \frac{1}{2\omega}\right) \quad (5.44)$$

$$\phi_{wf}(r) = 1 + \frac{2M}{(2\omega+3)r} \sim 1 + \frac{M}{\omega r} \quad (5.45)$$

where in the right hand side we kept only $1/\omega$ terms. As we can see, for large ω this solution is exactly the same with the exterior star solution (5.42). By equating this with (4.8,4.12,4.13) we obtain

$$C_1 = -M \ , \ C_2 = -9M \, . \quad (5.46)$$

so our solution (with $\Lambda \neq 0$) becomes

$$f(r) = 1 - \frac{2M}{r}\left(1 + \frac{1}{2\omega}\right) - \frac{\Lambda r^2}{3}\left(1 - \frac{1}{\omega}\right) \quad (5.47)$$

$$h(r) = \left(1 - \frac{2M}{r}\left(1 - \frac{1}{2\omega}\right) - \frac{\Lambda r^2}{3}\left(1 - \frac{2}{\omega}\right)\right)^{-1} \quad (5.48)$$

$$\phi(r) = 1 + \frac{1}{\omega}\left(\frac{M}{r} - \frac{\Lambda r^2}{3}\right) \quad (5.49)$$

5.3 Other possible solutions

As we saw in chapter 3 and verified by the eq. (4.6), the scalar field diverges logarithmically on both horizons. However, since in realistic astrophysical systems either one or both horizon can be "hidden", special cases of (4.6) are of physical interest.

(i) If we assume that the cosmological horizon is absent, in the sense that it is not stationary or Killing due to breaking of de Sitter symmetry by some matter field, we see that (4.6) can be used only up to a certain $r \ll r_c$. Thus, to avoid the divergence at r_H we set $C_1 = 0$, ending up with,

$$\delta\phi = \ln\left(1 - \frac{r^2}{r_c^2}\right) \, , \quad (5.50)$$

which is finite, reliable for any $r_H \leq r \ll r_c$ and non-trivial at r_H. Thus, black-holes can support $\phi-$hair if the stationary cosmological horizon is somehow "hidden".

(ii) If we now assume that the horizon at r_H is "hidden", e.g. in the interior of a "star", we choose $C_1 = 2r_c$ in (4.6), so that

$$\delta\phi = \frac{2r_c}{r_H} \ln\left(1 - \frac{r_H}{r}\right) + 2\ln\left(1 + \frac{r}{r_c}\right) , \tag{5.51}$$

is regular, monotonacally increasing, unique and certainly reliable in the region where $r_c/\omega \ll r \leq r_c$.

We know that the interior solution of a star is decreasing at the surface. This can be seen by integrating (3.26) from $r = 0$ to the surface $r = R$ and is also verified by (5.35).

$$\int_0^R D_\alpha(\beta D^\alpha \phi) = \int_0^R \beta \frac{8\pi T - 4\Lambda}{2\omega + 3} \tag{5.52}$$

which for $T < 0$ and also $2\omega + 3 > 0$ gives

$$\int_\Sigma D_\alpha(\beta D^\alpha \phi) < 0 \tag{5.53}$$

and for spherical configurations regular at $r = 0$, this implies

$$\left.\frac{d\phi}{dr}\right|_{r=R} < 0\,. \tag{5.54}$$

Also let us assume for the exterior that asymptotically the space-time is not de Sitter, but there is a cosmological horizon, on which the scalar field is regular. Let P be a point inside the cosmological horizon and let us integrate the scalar equation (3.26) from this point P to the horizon (H)

$$\int_P^H D_\alpha(\beta D^\alpha \phi) = \int_P^H \beta \frac{8\pi T - 4\Lambda}{2\omega + 3}$$
$$\int_H (\beta n_\alpha D^\alpha \phi) - \int_P (\beta \epsilon_\alpha D^\alpha \phi) < 0$$

where n_α and ϵ_α are the normal vectors on the horizon and on P respectively. On the horizon β vanishes and inside it has to be positive so

$$\int_P (\beta \epsilon_\alpha D^\alpha \phi) > 0$$
$$D^\alpha \phi|_{r=P} > 0\,. \tag{5.55}$$

Any smooth exterior solution with positive slope on the horizon and negative slope on the surface of a star has necessarily an extremum in between. This contradicts the non-perturbative monotonicity relust for ϕ. Concluding, smooth solution regular at both $r = 0$ and $r = r_c$ can exist only for $\omega = \infty$ and ϕ =constant everywhere, while the exterior is just the SdS space-time.

We saw that in order to describe a star in the context of BD theory with a positive cosmological constant with finite ω, one has to search for solutions with a "hidden" cosmological horizon.

(iii) This is the case we described at section (5.2).

5.4 Numerical Solutions

We can use now the analytical, perturbative solution (5.47,5.48,5.49) as initial conditions at $r = R_0 = 10^3 M$ to integrate the equations numerically. The tt and rr components of eq. (2.7) and also eq. (2.13) become for the metric (4.1)

$$\frac{f'\phi'}{2h\phi} + f\left(\frac{1}{r^2} + \frac{h'}{rh^2} - \frac{1}{h}\left(\frac{1}{r^2} + \frac{\omega}{2}\frac{\phi'}{\phi^2}\right)\right) = \frac{2\omega - 1}{2\omega + 3}\frac{\Lambda f}{\phi} \tag{5.56}$$

$$\frac{rf' - fh + f}{r^2 f} - \frac{\phi''}{\phi} + \frac{h'\phi'}{2\phi h} - \frac{\omega}{2}\frac{\phi'^2}{\phi^2} = -\frac{2\omega - 1}{2\omega + 3}\frac{\Lambda h}{\phi} \tag{5.57}$$

$$\frac{f'\phi'}{2fh} - \frac{h'\phi'}{2h^2} + \frac{\phi''}{h} + \frac{2\phi'}{rh} = -\frac{4\Lambda}{2\omega + 3} \tag{5.58}$$

The solutions are plotted in the fig. (5.4).

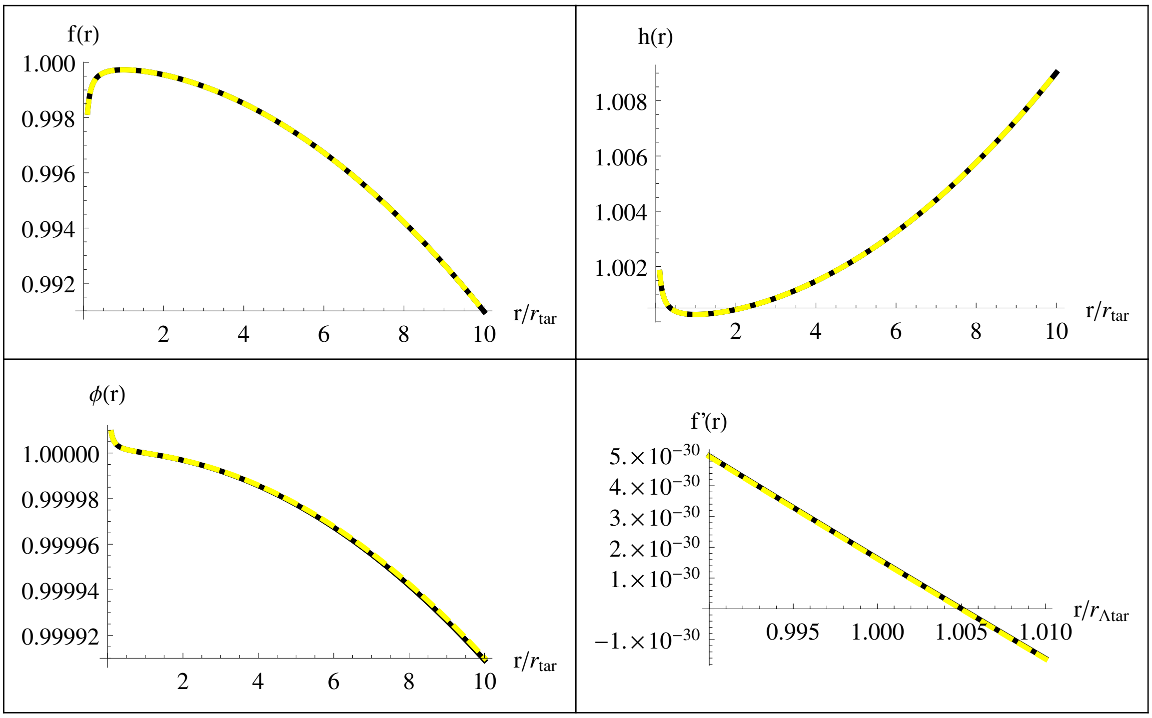

Figure 5.1: Analytical (continuous black) and numerical (yellow dashed) solutions, are plotted, in units of the maximum turnaround radius of the structure with $M = 10^{17} M_\odot, R_0 = 10^3 M, \omega = 100$. Although the analytical solution is perturbative and the numerical is not, the two solutions are in very good agreement in the region of interest. In the bottom right plot the derivative of $f(r)$ is plotted in units of the ΛCDM turnaround radius. As it can be seen the turnaround radius, corresponding to the vanishing point of $f'(r)$, is larger than in the ΛCDM case.

Mass	Maximum Turnaround Radius (m)				
($M_\odot$)	ΛCDM	an.sol. $\omega_{BD} = 10^2$	an.sol. $\omega_{BD} = 10^4$	num. sol. $\omega_{BD} = 10^2$	num. sol. $\omega_{BD} = 10^4$
10^{11}	1.64359×10^{22}	1.65181×10^{22}	1.64368×10^{22}	1.65176×10^{22}	1.64368×10^{22}
10^{13}	7.62888×10^{22}	7.66703×10^{22}	7.62927×10^{22}	7.66681×10^{22}	7.62926×10^{22}
10^{15}	3.54101×10^{23}	3.55872×10^{23}	3.54119×10^{23}	3.55862×10^{23}	3.54119×10^{23}
10^{17}	1.64359×10^{24}	1.65181×10^{24}	1.64368×10^{24}	1.65176×10^{24}	1.64368×10^{24}

Table 5.1: The maximum turnaround radius is given, for different values of the mass and of ω. In the second column are the values of the ΛCDM model. The other columns are for the BD theory, calculated with the analytical and the numerical solutions respectively.

Chapter 6

Appendix

6.1 Equivalence Principle

Einstein's motivation for apprehending GR was the equivalence principle (EP), that is the state of motion of an observer should not affect the laws of physics. In simpler words, there measurement an observer makes in an inertial frame at rest, should be the same with the one that another observer makes in moving inertial frame relative to the first. If we exclude gravity, this immediately leads to special relativity. When including gravity, we in fact have several choices as to how strictly we enforce the EP [40]. In order of increasing restriction we have,

- *Weak equivalence principle*: all uncharged test-particles[1] move ("free-fall") on the same space-time trajectory, if initial position and velocity is given, independently of composition.

- *Einstein's equivalence principle*: the WEP holds, and in addition the laws of physics are the same for any "freely-falling" observer, following such a test body.

- *Strong equivalence principle*: the WEP is valid also for small bodies, meaning that any finite-size correction falls off as r^{-3} or faster, where r is the distance to the body. This means that all freely-falling observers cannot see any change in the laws of physics.

6.2 PPN formalism

After the discovery of general relativity from Einstein, a wide range of candidate gravity theories was proposed over the years. In order to compare these theories with astrophysical observations (such as solar systems tests), it was useful to have a framework on which the predictions of different theories are parametrized in a systematic way. This was provided in the form of parametrized-Post-Newtonian (PPN) formalism by K. Nordtvedt, K. Thorne and C. Will.

[1]By test-particle we mean that its mass does not affect the gravitational field.

Let us assume a theory of gravity which is similar to GR but differs in details. In particular, let say that the geometry of the space-time outside an isolated star would be an excellent approximation to Schwarzschild solution but would differ from it in details. These differences could be summarized in a few PPN parameters.

The general static, spherically symmetric metric is given by

$$ds^2 = -f(r)(cdt)^2 + h(r)dr^2 + r^2 d\Omega^2 \,. \tag{6.1}$$

We could have another arbitrary function multiplying the angular part of the metric, but we can always define a new radial coordinate such that the metric takes the above form. The Schwarzschild metric has this form with

$$f(r) = 1 - \frac{2GM}{c^2 r} \,, \quad h(r) = f^{-1}(r) \,. \tag{6.2}$$

The mass M is the only parameter that determines the spherical geometry outside the star, and thus let us expand (6.1) in powers of GM/c^2r, which is the only dimensionless combination of G, M, c and r, in order to get the Newtonian limit plus some post-Newtonian corrections.

In the non-relativistic limit we should recover the Newtonian theory, and the predictions for orbits in that limit are determined but the first relativistic correction to the flat space, ending up with

$$f(r) = 1 - \frac{2GM}{c^2 r} + ..., \quad h(r) = 1 + ... \,. \tag{6.3}$$

To get the first post-Newtonian corrections, we keep the next terms in both f and h

$$f(r) = 1 - \frac{2GM}{c^2 r} + 2(\beta - \gamma)\left(\frac{2GM}{c^2 r}\right)^2 + ... \tag{6.4}$$

$$h(r) = 1 + 2\gamma\frac{2GM}{c^2 r} + ... \,. \tag{6.5}$$

The coefficients in front of the post-Newtonian terms are related to the PPN parameters β and γ according to standard usage. These parameters may be different in different theories of gravity. For GR the values are those of the Schwarzschild metric $\beta = 1 = \gamma$.

The bending of light by the sun, the precession of perihelion of a planet, and the time delay of light can all be worked out for the PPN metric obtained by inserting (6.4,6.5) into (6.1). The results to leading order in $1/c^2$ are as follows

- For the deflection angle $\delta\phi_{def}$ of a light ray passing by a mass M at an impact parameter b

$$\delta\phi_{def} = \left(\frac{1+\gamma}{2}\right)\left(\frac{4GM}{c^2 b}\right) \,. \tag{6.6}$$

- For the precession $\delta\phi_{prec}$ of the perihelion of a planet per orbit

$$\delta\phi_{prec} = \frac{1}{3}(2 + 2\gamma - \beta)\frac{6\pi GM}{c^2 a(1-\epsilon^2)} \tag{6.7}$$

  where M is the mass of the orbited star, a is the orbit's semi-major axis, and ϵ eccentricity.

- For the "excess" time delay of light, Δt_{exc}, in the approximation that the radii $r_\oplus$ of the emitter at the earth and responder r_R are much greater than the distance r_1 of closest to the gravitating body

$$\Delta t_{exc} = \left(\frac{\gamma+1}{2}\right)\frac{4GM}{c^3}\left[\log\left(\frac{4r_\oplus r_R}{r_1^2}\right)+1\right] \tag{6.8}$$

We can use these tests to compare the values of the PPN parameters in different gravity theories.

6.3 Symmetries

There are cases where a space-time is equipped with specific symmetries, the existence of which simplifies its study a lot. Specifically, as we will see below, geometrical symmetries of a space-time turn out to be physical symmetries, since gravity is interpreted as geometry.

Lie derivative

Let P be a point of a $n-$dimensional Riemannian space, R^n, with coordinates (x^a), at the $\{x^a\}$ frame of reference. Suppose that P is related a second point P' of the same space, which has coordinates (x'^a), at the same frame of reference. This is called a representation of R^n on itself. Let us consider, the relation

$$x'^a = x^a + \epsilon\xi^a \tag{6.9}$$

where ϵ is an arbitrarily small (scalar) quantity and $\xi^a = \xi^a(x^b)$ is a vector field. Eq. (6.9) gives the relation between P and P'. Consequently, eq.(6.9) defines an infinitesimal representation of R^n on itself, which is fully described by ξ^a.

Consider a scalar field $\phi = \phi(x^a)$ defined on R^n and P, P' two points on this space, which are related through (6.9). Furthermore, $\phi = \phi_P = \phi(x^a)$ and $\phi' = \phi_{P'} = \phi(x'^a)$ are the values of the scalar field at these points. If we take the Taylor expansion of ϕ' around P we end up with

$$\phi' \simeq \phi + \epsilon\xi^a\partial_a\phi\,. \tag{6.10}$$

Except from ϕ' at P', we can define a second scalar quantity, $\tilde{\phi}$, from (6.9). We denote the limit

$$(\pounds_\xi\phi)_P = \lim_{\epsilon\to 0}\left(\frac{\phi'-\tilde{\phi}}{\epsilon}\right) \tag{6.11}$$

as the *Lie derivative* of the scalar field ϕ at P in the direction ξ^a.

Assuming that eq. (6.9) preserves the scalar quantities, we have $\tilde{\phi} = \phi$. Thus, eq. (6.11) becomes from (6.10)

$$(\pounds_\xi\phi)_P = [\xi^a(\partial_a\phi)]_P \tag{6.12}$$

Finally, since P was chosen to be an arbitrary point the above relation leads to the general definition

$$\pounds_\xi\phi = \xi^a\partial_a\phi \tag{6.13}$$

of the Lie derivative of a scalar field ϕ in the direction of ξ^a.

Similarly, the Lie derivative of a (contravariant and covariant respectively) vector field is given by

$$\begin{aligned} \pounds_\xi u^a &= \xi^b \partial_b u^a - u^b \partial_b \xi^a \,, &(6.14)\\ \pounds_\xi \upsilon_a &= \xi^b \partial_b \upsilon_a + \upsilon_b \partial_a \xi^b \,. &(6.15)\end{aligned}$$

For a 2-rank covariant tensor we'll have

$$\pounds_\xi T_{ab} = \xi^c \partial_c T_{ab} + T_{cb} \partial_a \xi^c + T_{ac} \partial_b \xi^c \qquad (6.16)$$

which in the case of a metric tensor in a Riemannian space becomes

$$\pounds_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a \,. \qquad (6.17)$$

As a consequence, in contrast with the covariant derivative, the Lie derivative of a metric tensor is not always vanishing.

Killing vectors and Isometries

Eq. (6.9) is called *isometry* when it preserves the line element of this space. In this case, the vector field ξ^a, which causes the isometry, is called *Killing vector*.

Let $g_{\mu\nu} dx^\mu dx^\nu$ be the line element of the space at a point P and $g'_{\mu\nu} dx'^\mu dx'^\nu$ at P' which is related to P through (6.9). Assuming that $g'_{\mu\nu} = g_{\mu\nu}(x'^\rho)$, if we Taylor expand we obtain

$$g'_{\mu\nu} = g_{\mu\nu} + \epsilon \xi^\rho \partial_\rho g_{\mu\nu} \,, \qquad (6.18)$$

with the right hand side calculated at P. At the same time, we get from (6.9) that

$$dx'^a = dx^a + \epsilon dx^b \partial_b \xi^a \qquad (6.19)$$

Keeping only first order terms in ϵ, we conclude that (6.9) changes the line element like

$$g'_{\mu\nu} dx'^\mu dx'^\nu - g_{\mu\nu} dx^\mu dx^\nu = \epsilon \left(\xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho \right) dx^\mu dx^\nu \,. \qquad (6.20)$$

Thus, (6.9) will be an isometry if and only if the condition

$$\xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho = 0 \,, \qquad (6.21)$$

is satisfied. This can easily be written as

$$\pounds_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \,. \qquad (6.22)$$

This equation is known as *Killing equation* and is a necessary and sufficient condition in order for (6.9) to be an isometry. Every vector (ξ^a) that satisfies (6.22) is called a Killing vector. Hence, from (6.17) and (6.22), we conclude that the Lie derivative of the metric tensor in the direction of a Killing vector will vanish.

Theorem: *Let ξ^a and ψ^a be two commuting Killing vector fields such that*

- *$\xi_{[a} \psi_b \nabla_c \xi_{d]}$ and $\xi_{[a} \psi_b \nabla_c \psi_{d]}$ each vanishes at at least one point of the space-time (which, in particular, will be true if either ξ^a or ψ^a vanishes at one point) and*
- *$\xi^a R_a{}^{[b} \xi^c \psi^{d]} = \psi^a R_a{}^{[b} \xi^c \psi^{d]} = 0$.*

Then the 2-planes orthogonal to ξ^a and ψ^a are integrable.

6.4 Variations

We present and prove here some of the basic variations that have, extensively, been used throughout the derivations. Let us start by the definition of the inverse metric tensor

$$g_{\mu\nu}g^{\nu\rho} = \delta_\mu{}^\rho \tag{6.23}$$

and vary the last by noting that $\delta_\mu{}^\rho$ is constant, to get

$$(\delta g_{\mu\nu})g^{\nu\rho} + g_{\mu\nu}(\delta g^{\nu\rho}) = 0 \tag{6.24}$$

Now, contracting with $g_{\rho\sigma}$ we obtain

$$(\delta g_{\mu\nu})\delta^\nu{}_\sigma + g_{\mu\nu}g_{\rho\sigma}(\delta g^{\nu\rho}) = 0 \tag{6.25}$$

$$\Rightarrow \qquad \delta g_{\mu\sigma} = -g_{\mu\nu}g_{\rho\sigma}(\delta g^{\nu\rho}) \tag{6.26}$$

Let us now compute the variation of the square root of the determinant of the metric tensor. We do the calculation for n-dimensional Riemannian spaces and then apply it for pseudo-Riemannian 4-dimensional space of GR. Firstly, we write the determinant simply as

$$g = \det(g_{\mu\nu}) \tag{6.27}$$

$$\Rightarrow \qquad \delta(\sqrt{g}) = \frac{1}{2\sqrt{g}}\delta g \tag{6.28}$$

It is known that for any square $n \times n$ matrix $\mathcal{A}$ it holds that

$$\det(\mathcal{A}) = e^{Tr(\mathcal{A})} \tag{6.29}$$

and substituting $\mathcal{A} \to g_{\mu\nu}$ in the above, we get

$$g = \det(g_{\mu\nu}) = e^{Tr(g_{\mu\nu})} \tag{6.30}$$

which under the variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ gives

$$\det(g_{\mu\nu} + \delta g_{\mu\nu}) = e^{Tr(g_{\mu\nu}+\delta g_{\mu\nu})} = e^{Tr(g_{\mu\nu})}e^{Tr(\delta g_{\mu\nu})} = ge^{Tr(\delta g_{\mu\nu})} \tag{6.31}$$

where we have used the linearity of the trace. Since the variations δg are small, in the expansion of $e^{Tr(\delta g_{\mu\nu})}$ we can neglect second and higher order terms and thus simplify it as

$$e^{Tr(\delta g_{\mu\nu})} \simeq g\left(1 + Tr\left(\delta g_{\mu\nu}\right)\right) \tag{6.32}$$

but from the definition of the trace, we know that $Tr(\delta g_{\mu\nu}) = g^{\mu\nu}\delta g_{\mu\nu}$ and thus

$$\det(g_{\mu\nu} + \delta g_{\mu\nu}) \simeq g\left(1 + g^{\mu\nu}\delta g_{\mu\nu}\right) \tag{6.33}$$

It has been proven before that $g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu}$ and finally

$$\delta g = -gg_{\mu\nu}\delta g^{\mu\nu} \tag{6.34}$$

$$\Rightarrow \qquad \delta(\sqrt{g}) = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu} \tag{6.35}$$

Now, by replacing $g \to -g$ we end up with

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{6.36}$$

After some non-trivial calculations, it turns out that for the metric variation

$$g_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}, \;\; g^{\mu\nu} = g^{\mu\nu} + \delta g^{\mu\nu} \tag{6.37}$$

the Christoffel symbols will become

$$\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\alpha\beta} + \delta\Gamma^{\mu}{}_{\alpha\beta} \tag{6.38}$$

where

$$\begin{aligned}
\Gamma^{\mu}{}_{\alpha\beta} &= \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right), && (6.39)\\
\delta\Gamma^{\mu}{}_{\alpha\beta} &= \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha \delta g_{\beta\nu} + \partial_\beta \delta g_{\nu\alpha} - \partial_\nu \delta g_{\alpha\beta}\right) + \Gamma^{\rho}{}_{\alpha\beta}\delta g^{\mu}_{\rho} && (6.40)\\
&= \frac{1}{2}g^{\mu\nu}\left(\nabla_\alpha \delta g_{\beta\nu} + \nabla_\beta \delta g_{\nu\alpha} - \nabla_\nu \delta g_{\alpha\beta}\right) && (6.41)
\end{aligned}$$

where ∇_α is the covariant derivative with respect to the background metric. After some manipulations we can obtain

$$\begin{aligned}
\delta R_{\mu\nu} &= \nabla^\alpha \nabla_{(\mu}\delta g_{\nu)\alpha} - \frac{1}{2}\left(\Box \delta g_{\mu\nu} + \nabla_\mu \nabla_\nu \delta g\right) && (6.42)\\
\delta R &= \nabla^\mu \nabla^\nu \delta g_{\mu\nu} - \Box \delta g - R^{\mu\nu}\delta g_{\mu\nu} && (6.43)
\end{aligned}$$

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