Quasinormal modes and AdS/CFT applications

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Abstract

The AdS/CFT correspondence has been a remarkable tool in understanding stongly coupled quantum field theories through gravity. One of its most fascinating applications, are considered to be quasinormal modes, which describe damped oscillations of a perturbed gravitational background. Unlike normal modes, quasinormal modes are pure imaginary and they provide us with vital information involving the decay of a field outside of a black hole. In this thesis, we propose a definition of quasinormal modes in asymptotically AdS spacetimes, with the boundary conditions imposed by the AdS/CFT dictionary. We then proceed to compute the corresponding frequencies of a massless scalar field that perturbs a five dimensional Schwarzschild AdS black hole and relate them to the poles of the retarded Green's function of the dual gauge theory at finite temperature. Furthermore, we investigate the behavior of the retarded correlator for high and low frequencies and highlight that our numerical results are verified in both regimes. In particular, we observe that in the low frequency limit the viscosity over entropy ratio takes the universal value of $1/4\pi$.

Acknowledgements

I want to deeply thank my advisor Dr. Christopher Rosen, for his guidance and overall assistance throughout this journey. His incredibly kind and supportive nature made me more determined to learn about this captivating topic. Next, I want thank Prof. Vasileios Niarchos for introducing me to these fascinating concepts. Finally, I want to thank my family and friends for their unconditional support.

Introduction

Characteristic modes of vibration of different objects are discernible all around us. Musical instruments create sounds that are associated with the specific structure and composition of the vibrating object. However, these modes govern not only strings and drums but also black holes.

It is well known that when you perturb a black hole, it will undergo damped oscillations. These oscillations are entirely fixed by the black hole and are independent of the initial perturbation. When we study them, at first glance, they appear similar to normal modes of fixed systems. Nevertheless, when the system is dissipative, the modes decay in time and become complex. These frequencies are called "quasinormal modes" and they have been studied extensively over the past years [1, 2, 3, 4, 5, 6].

The AdS/CFT conjecture [7, 8, 9, 10], stresses that certain string theories on specific asymptotically anti de Sitter spacetimes are dual to quantum field theories in one lower dimension. That is why this conjecture is sometimes referred to as a holographic correspondence [11]. It connects a higher classical gravitational theory with a thermal gauge theory in one lower dimension. A simple yet plausible example of such a duality that we are going to employ in this review is given by the correspondence between the thermal $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills (SYM) at finite temperature and the full type IIB string theory on the $AdS_5 \times S_5$ background. This background, as we shall see, is equivalent to the Schwarzschild-AdS black hole with an infinite horizon radius.

Quasinormal frequencies of AdS black holes have a direct meaning in terms of the dual conformal field theory. According to the AdS/CFT duality, a black hole in AdS space corresponds to a thermal state in CFT at finite temperature. Specifically, small perturbations of the black hole background can be interpreted as small deviations from thermodynamic equilibrium in the dual theory. The timescale to thermodynamic equilibrium of the field theory is given by the lowest quasinormal mode, since this mode is the least damped mode and lives the longest. Furthermore, quasinormal spectra of the gravitational background, give the location of the poles of the retarded correlators of the thermal gauge theory, providing useful information about the transport coefficients of the plasma. There have been an abundance of studies [2, 12, 13, 14, 15] where various quantities of the strongly coupled conformal field theory, such as the shear viscosity and conductivity, have been computed using the gauge/gravity duality.

The outline of the paper is organised as follows. In Chapter 1 we present a brief introduction to the AdS/CFT dictionary and compute various thermodynamic properties of the Schwarzschild-AdS black hole. In the following chapter, we review the definition of quasinormal modes and describe the method we used in order to compute them numerically. In Chapter 3, we formulate a recipe on calculating the retarded Green's function numerically and using our results we determine the viscosity over entropy ratio in the hydrodynamic limit.

Chapter 1

AdS/CFT dictionary

1.1 Preliminaries

In the year 1997 Maldacena [7] proposed a conjecture which equated two separate theories, a gauge theory and a gravitational theory, known as AdS/CFT correspondence where AdS stands for"Anti-de Sitter", which is a spacetime with a constant negative curvature, and CFT "Conformal Field Theory", which is a quantum field theory that is invariant under conformal transformations. In the original example, the full type IIB string theory on the background of $AdS_5 \times S_5$, where S_5 is the five dimensional sphere, is conjectured to be equivalent to the $\mathcal{N} = 4$ $SU(N_c)$ supersymmetric Yang-Mills theory in four dimensions. Let us unravel these two sides of the correspondence in more detail.

First of all, the $\mathcal{N} = 4$ SYM theory is a gauge theory of rank N which is scale invariant both classically and quantum mechanically. Therefore, the physics do not change with the change of the length scale. The $\mathcal{N} = 4$ means that there are four supersymmetries, which are the maximum amount of symmetries in a four dimensional field theory. This particular gauge theory consists of a gauge field, six scalar fields and four Weyl fermions and they transform as a adjoint representation of the $SU(N_c)$ gauge group.

On the string theory side, we have the full IIB string theory with parameters namely the string length l_s and the string coupling g_s. The reason why we study 10 dimensional spacetimes is because superstring theory is quantized consistently in these regimes. In general, we consider the classical gravity approximation of string theory where the curvature of the space is much larger than the string length.

The parameters of the gauge theory are the t' Hooft coupling $\lambda \equiv g_{YM} N_c$ and the number of colours N_c. In the AdS/CFT correspondence, these parameters are mapped into the string theory parameters g_s and L as follows

$$N_c^2 = \frac{\pi L^3}{2G_5} \tag{1.1}$$

and

$$\lambda \sim g_s N_c \sim \frac{L^4}{l_s^4} \tag{1.2}$$

where L is the characteristic radius of AdS_5 spacetime. When we study these separate theories, we essentially choose one theory into being weakly coupled and the corresponding theory becomes strongly coupled. That is the reason why it is often called as a strong/weak duality.

The relations above are essential in order for someone to understand the classical gravitational description of this duality. To do so, one needs to take the limit $\lambda \to \infty$, $N_c \to \infty$. That means that, when the size of AdS space is much larger than the string scale $L \gg l_s$, the graviton can be treated as a point particle and string theory can be approximated by classical supergravity. Generally speaking, this duality provides a way of corresponding ten dimensional classical gravity and a gauge theory at strong coupling in four dimensional spacetime.

This equality is mainly understood as an equivalence of quantum partition functions

$$Z_{CFT} = Z_{AdS_5 \times S_5} \tag{1.3}$$

where Z_{CFT} is the generating functional of the gauge theory and $Z_{AdS_5 \times S_5}$ is the generating functional of the gravitational theory. More accurately, one could write the above statement as

$$< e^{-\int \phi_0 \mathcal{O} d^4x} >_{CFT} = e^{-S_{grav}[\phi|_{z=0}=\phi_0]} \tag{1.4}$$

where ϕ acts as a particular field in the gravitational theory and $\mathcal{O}$ is a gauge-invariant operator of the field theory. This equivalence means that the classical gravity action S_{grav} serves as a generating functional for correlation functions of operators $\mathcal{O}$ of the dual theory. On the left hand side of 1.4, ϕ_0 plays the role of the external source of the operator $\mathcal{O}$. On the right hand side, ϕ_0 represents the boundary value of the background field ϕ of AdS space. Namely, AdS/CFT claims that a four dimensional external source can have a five dimensional origin. For example, the field ϕ can be replaced by the metric fluctuation $h_{\mu\nu}$, which in turn corresponds to the boundary operator $T_{\mu\nu}$ of the gauge theory.

An important property of this duality is the existence of D3-branes. In general, D-branes are locations in ten dimensional spacetime where open strings can end and there motion is restricted on the brane. Low energy excitations of a D3-brane can be described by a $U(1)$ gauge theory. If N_c coincident D3-branes are placed on top of one another then one finds that it describes the $\mathcal{N} = 4$ $U(N_c)$ supersymmetric Yang-Mills gauge theory. It splits into a $U(1)$ gauge theory which describes the motion of the stack of the D3-branes and an $SU(N_c)$ part which is interacting and expresses the relative motions of the branes.

1.2 The $SchAdS_5$ black hole metric and thermodynamic quantities

In this section, we introduce the five dimensional Schwarzschild AdS Black Hole. In the holographic correspondence, this metric corresponds to the dual $\mathcal{N} = 4$ SYM gauge theory at finite temperature, so computing thermodynamic quantities of the black hole one can derive thermodynamic properties of the $\mathcal{N} = 4$ SYM theory.

The relevant non extremal metric of the black three-brane background at finite temperature is the following

$$ds^2_{10} = \frac{L^2}{z^2}[-fdt^2 + dx_1^2 + dx_2^2 + dx_3^2] + \frac{L^2}{z^2}f^{-1}dz^2 + L^2 d\Omega_5^2 \tag{1.5}$$

where $f(z) = 1 - z^4/z_H^4$ and $d\Omega_5$ is the metric of the unit S_5 with radius L. The horizon is located at $z = z_H$ where $f = 0$. The $\mathcal{N} = 4$ SYM is defined in flat Minkowski space with coordinates t, x_1, x_2, x_3 and the fifth radial coordinate z represents the energy scale of the gauge theory. More specifically, the boundary $z = 0$ corresponds to the ultraviolet (UV) of the CFT and the horizon corresponds to the infrared (IR). When we say non extremal, we essentially mean a background at non zero temperatures. If one reduces the five sphere part by dimensional reduction, the metric turns into the five dimensional Schwarzschild AdS Black Hole with translationally event horizon

$$ds^2_5 = \frac{L^2}{z^2}[-fdt^2 + dx_1^2 + dx_2^2 + dx_3^2] + \frac{L^2}{z^2}f^{-1}dz^2 \tag{1.6}$$

obeying the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{1.7}$$

with a negative cosmological constant $\Lambda = -6/L^2$. Three flat directions $\mathbf{x}$ can be identified, in contrast with the usual Schwarzschild metric. That is the reason why it is refered as the black three-brane metric. With this metric we can also compute some useful thermodynamic quantities of the dual $\mathcal{N} = 4$ SYM at strong coupling.

We can derive the Hawking temperature in the following simple way. First, we introduce the Euclidean time $t_E = it$ so the metric 1.6 can be extended in the Euclidean space time as

$$ds_E^2 = \frac{L^2}{z^2}(1 - \frac{z^4}{z_H^4})dt_E^2 + \frac{L^2}{z^2}\frac{1}{(1 - \frac{z^4}{z_H^4})}dz^2 + ... \tag{1.8}$$

Near the horizon $z \simeq z_H$, the function $f(z_H)$ becomes zero. Thus, one can approximate the above metric near the horizon by redefining $z = z_H(1 - \rho^2/L^2)$ and at the lowest order in ρ we have

$$ds_E^2 \simeq d\rho^2 + \frac{4\rho^2}{z_H^2}dt_E^2 + \frac{L^2}{z_H^2}(dx_1^2 + dx_2^2 + dx_3^2). \tag{1.9}$$

With this metric transformation our new coordinates take the form of a plane in polar coordinates $ds_{2d}^2 = d\rho^2 + \rho^2 d\phi^2$, if $2t_E/z_H$ has a periodicity of 2π (i.e $\phi = 2t_E/z_H = 2\pi$). The reason we set this constraint is because we require that the Euclidean spacetime, that we just introduced with the metric (1.9), has to be smooth, otherwise it will have a conical singularity at $\rho = 0$. The periodicity of t_E is given by $\Delta t_E = \pi z_H$ and remembering that in a Euclidean thermal field theory the temperature is given by $\Delta t_E = \beta = 1/T$, we get

$$T = \frac{1}{\pi z_H} \tag{1.10}$$

or

$$z_H = \frac{1}{\pi T}. \tag{1.11}$$

We observe that for high temperature T the black hole horizon is close to the boundary of AdS space, while for low temperature the horizon is deep into the bulk.

Now, let us calculate the entropy S of the black hole, as well as the entropy density s which will be a particularly useful quantity later in our discussion. From the Hawking-Bekenstein area law [16], the entropy is given by

$$S = \frac{A}{4G_5}, \tag{1.12}$$

where A is the horizon area and can be computed via the expression

$$A = \int dx^1 dx^2 dx^3 \sqrt{|g_{3d}|}, \tag{1.13}$$

where $g_{3d} = L^6/z^6$ is the determinant of the part of the metric (1.6) in three dimensions (x_1, x_2, x_3). Thus, the entropy of the five dimensional Schwarzschild AdS Black Hole is written as

$$S = \frac{1}{4G_5} \frac{L^3}{z_H^3} V_3, \tag{1.14}$$

where V_3 is the gauge theory volume. The entropy density is $s = S/V_3$, so one may write

$$s = \frac{1}{4G_5} \frac{L^3}{z_H^3} \tag{1.15}$$

In this particular metric, z runs from $z = 0$ which is the boundary of AdS_5 space to $z = z_H$. According to the AdS/CFT dictionary the four dimensional quantum field theory "lives" at the boundary of the five dimensional AdS space. From this point forward we set $L = 1$ for simplicity.

Chapter 2

Quasinormal modes

2.1 Defining quasinormal modes

Quasinormal modes are characteristic modes that characterize the decay of a perturbation outside of a black hole. In general, they are solutions of a second order differential equation that describe a classical perturbation given a specific gravitational background imposing of course certain boundary conditions. In the case where the geometry is asymptotically flat, the choice of boundary conditions is physically motivated: nothing should emerge from the black hole and no radiation should originate from infinity. However, in an asymptotically AdS spacetime we have to examine the boundary conditions more scrupulously.

In asymptotic AdS_5 space, the horizon is located at z_H, so a condition would be that no outgoing waves should exit the black hole. On the other hand, at the boundary, a vanishing Dirichlet boundary condition would be a natural choice. These boundary conditions, happen only for a discrete set of complex ω called quasinormal frequencies.

Since we are interested in the correlation between classical high-dimensional gravity and the dual four-dimensional gauge theory, a general definition for quasinormal frequencies proposed by [2] is the following: *Quasinormal frequencies of a perturbation in an asymptotically AdS space are defined as the locations in the complex frequency plane of the poles of the* *retarded correlator of the operators dual to that perturbation.* In the AdS/CFT context, the retarded correlator or the so called Green's function of the gauge theory translates into the incoming wave boundary condition at the horizon. In the following section we compute the quasinormal modes numerically for the massless scalar field.

2.2 Quasinormal modes of massless scalar field

The five dimensional action with a general scalar field ϕ can be written as

$$S = \frac{1}{16\pi G_5} \int d^5x\sqrt{-g} \left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \right) \tag{2.1}$$

where $g \equiv det g_{\mu\nu}$ and G_5 is the five dimensional gravitational constant. For simplicity, we set $16\pi G_5 = 1$. Varying the above action with respect to $g_{\mu\nu}$ we get the following expression for the stress-energy tensor

$$T_{\mu\nu} = g_{\mu\nu} \left(-\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - V(\phi) \right) + \partial_\mu\phi\partial_\nu\phi. \tag{2.2}$$

In order for the $SchAdS_5$ black hole to be a solution of $T_{\mu\nu}$, the potential of the scalar field must be $V(\phi = 0) = -6/L^2$. On the other hand, varying the action with respect to ϕ we get

$$\frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu\phi \right) = V'(\phi) \tag{2.3}$$

or

$$\frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu\phi \right) = 0 \tag{2.4}$$

where we have set $V(\phi)$ as a constant. In general, one may solve equation (2.4) using Fourier decomposition which reads,

$$\phi(z, t, \mathbf{x}) = \int \frac{d^4k}{(2\pi)^4} e^{-i\omega t + i\mathbf{k}\mathbf{x}} \phi_k(z) \tag{2.5}$$

where k is the spatial momentum and $\mathbf{x}$ denotes the spatial coordinates of the metric. For simplicity however, we set $k = 0$ and then substitute the "plane-wave" ansatz $\phi(x^\mu) = e^{-i\omega t}\phi(z)$ into (2.6) to obtain the following second order differential equation

$$\frac{z_H^8\omega^2}{(z^4 - z_H^4)^2}\phi(z) + \frac{z^4 + 3z_H^4}{z^5 - zz_H^4}\phi'(z) + \phi''(z) = 0 \tag{2.6}$$

where z_H is the horizon of the $SchAdS_5$ black hole and as we see it is also a regular singularity. One can study this differential equation near the horizon (i.e. $z \to z_H$) and expand it as a series $\phi(z) \approx (z - z_H)^\alpha$. The first few terms have been computed below

$$(z_H - z)^\alpha \left(\frac{\alpha^2 + \frac{z_H^2\omega^2}{16}}{(z - z_H)^2} - \frac{3(8\alpha + z_H^2\omega^2)}{16z_H(z - z_H)} + \mathcal{O}(z - z_H) \right) = 0. \tag{2.7}$$

The leading term of this expansion is

$$\alpha^2 + \frac{z_H^2\omega^2}{16} = 0 \tag{2.8}$$

which has two solutions $\alpha = i\omega z_H/4$ and $\alpha = -i\omega z_H/4$ so

$$\phi_+(z) \approx (z - z_H)^{\frac{i\omega z_H}{4}} \quad \text{and} \quad \phi_-(z) \approx (z - z_H)^{-\frac{i\omega z_H}{4}}$$

From these two solutions we must obtain the physically acceptable one, i.e. the infalling one, since nothing can escape the black hole. In order to find it, we rewrite them as follows

$$\phi_+(z) \approx e^{(i\omega z_H/4) log(z - z_H)} \quad , \phi_-(z) \approx e^{-(i\omega z_H/4) log(z - z_H)}$$

Observe that $\phi(x^\mu)$ behaves as

$$\phi \approx e^{-i\omega(t \pm \frac{z_H}{4} log(z - z_H))}$$

and if we want a constant phase to the exponential as $t \to \infty$ we must keep the minus sign i.e. for $\alpha = -i\omega z_H/4$.

Having identified the physically acceptable solution, we plug it into our differential equation (2.6) as $\phi(z) = (z - z_H)^{-\frac{i\omega z_H}{4}} F(z)$ so it can help our numerical methods in becoming more stable

$$
\begin{aligned}
-\frac{z_H^2(z^2 + 2zz_H + 3z_H^2)\omega(4(z - iz_H)(-iz + z_H)(z + z_H) + z(z^3 + z^2 z_H + zz_H^2 + 5z_H^3)\omega)}{16z(z - z_H)(z + z_H)^2(z^2 + z_H^2)^2} F(z) \\
+\frac{2z^4 + 6z_H^4 + z(z - iz_H)z_H(-iz + z_H)(z + z_H)\omega}{2(z^5 - zz_H^4)} F'(z) + F''(z) = 0.
\end{aligned}
\tag{2.9}
$$

This corresponds to looking for a Frobenius type solution of the form

$$
F(z) = \sum_{n=0}^{\infty} a_n (z - z_H)^n. \tag{2.10}
$$

Numerically however, it is convenient to use a finite expansion

$$
F(z) = \sum_{n=0}^{N} a_n (z - z_H)^n, \tag{2.11}
$$

where N can be set as high as we wish, but also make our numerics more precise. Substituting (2.10) into (2.9), one may solve the coefficients of the above expansion a_n order by order in terms of for example a_0. Since the equation is linear, we can set $a_0 = 1$ and we still have a solution.

Evidently, our solution depends on ω and z_H, the former being a property of the fluctuation and the latter is related to the temperature of the black hole (i.e. $z_H = 1/\pi T$). These two parameters have the same dimensions and one can rewrite them as a dimensionless ratio $\hat{\omega} = \omega/\pi T$. As we can clearly see from Figure 2.1, our numerical approach provides us with a good approximation of the true solution near the horizon z_H. We can always increase the value of N in the expansion so that our approximation near the horizon fits better with the exact solution.

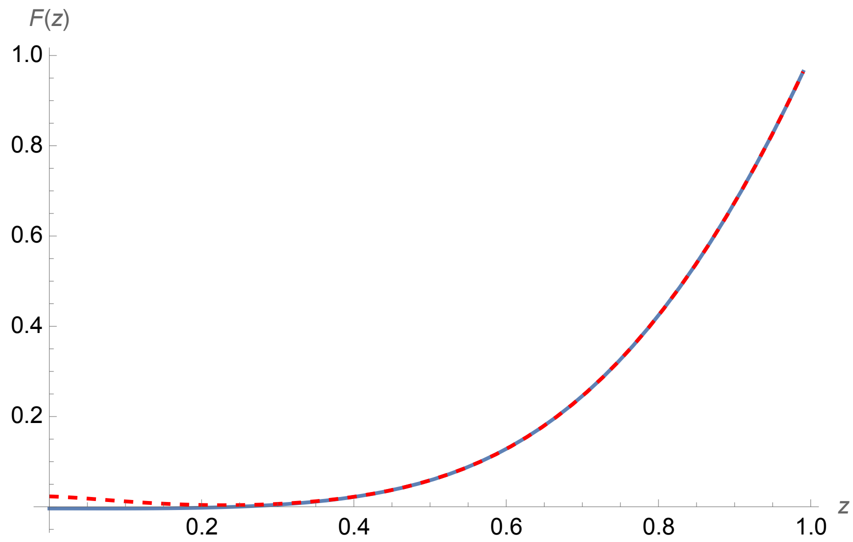


Figure 2.1: The blue line is the numerical solution $F(z)$ where we have set $\omega = 5 - 5i$ and $z_H = 1$. The red dashed line is the near horizon expansion $z \to z_H$ for $N = 10$.

In our coordinates we can set $z = 0$ for the boundary and $z_H = 1$ for the horizon. In order to find the quasinormal modes, one must solve the differential equation (2.9) in the interval $z = [0, 1]$ taking into consideration the appropriate boundary conditions.

The most simple and straightforward method of computing the quasinormal frequencies is the shooting method. The basic idea behind this technique is essentially taking a boundary value problem and turn it into an initial data problem with some educated guess of the initial condition. In our case, we can specify our data at the horizon, considering that we already found an approximation near it. Numerically, it is convenient to specify our data very close to the horizon, since it is a singularity of equation (2.9), so we choose $z_{min} = z_H - \epsilon$ where $z_H = 1$ and $\epsilon = 5 \cdot 10^{-1}$. The value of ϵ is modified as such in order to give us the best possible accuracy. Implementing this technique using the built-in function $NDSolve$ of $Mathematica$ and integrating to the boundary $z_{max} = 10^{-8}$, we get a numerical solution of equation (2.9) for any given ω.

To search for quasinormal modes, one needs to create a function that provides the value of the above solution at the boundary z_{max} for multiple ω. The zeros of this function tell us the quasinormal modes. In order to identify approximately where the locations of the modes are, we simply use the "graphical inspection" method and subsequently apply the *FindRoot* built-in function to gives us a precise value of the frequencies. A simple example of such an inspection can be seen in Figure 2.2.

The first 10 quasinormal frequencies $\hat{\omega}_n$ that have been obtained are listed in Table 2.1 and ω_n is written as

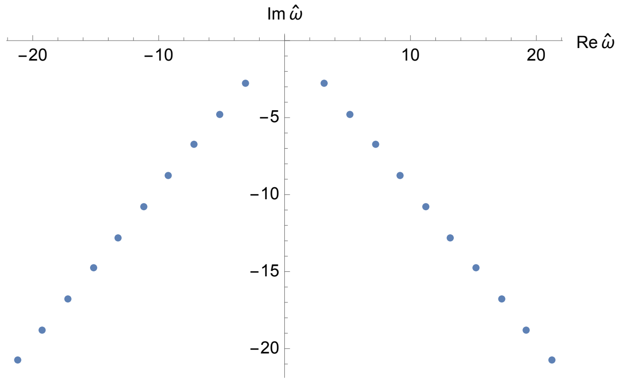
$$\omega_n = \omega_{n,R} - i\omega_{n,I} \tag{2.12}$$

so that $\omega_{n,I}$ is positive for all n. According to AdS/CFT correspondence, the "minus" sign represents the inverse damping time of the field theory side and the approach to thermal equilibrium. We also observe that the frequencies appear to be symmetric about the

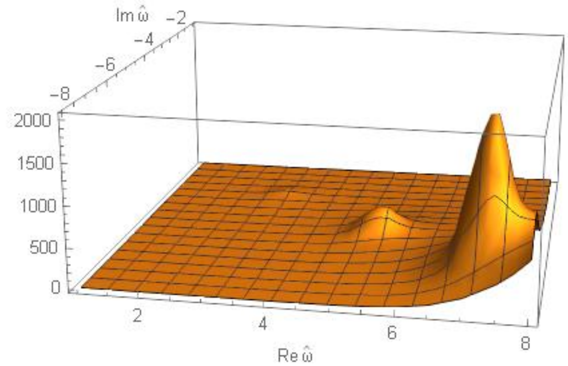
n	Re$\hat{\omega}_n$	Im$\hat{\omega}_n$
1	±3.119452	-2.746676
2	±5.169521	-4.763570
3	±7.187931	-6.769565
4	±9.197199	-8.772481
5	±11.202676	-10.774162
6	±13.206247	-12.775239
7	±15.208736	-14.775979
8	±17.210558	-16.776515
9	±19.211943	-18.776919
10	±21.213032	-20.777237

Table 2.1: The 10 lowest quasinormal frequencies $\hat{\omega}_n$.

imaginary $\hat{\omega}$ axis, a property that does not seem to be trivial from (2.9). They seem to be pairs of complex-conjucate solutions of (2.9). For the lowest ones our approach works well



(a) The first 10 quasinormal frequencies.



(b) A 3D plot of the first 3 lowest quasinormal frequencies in the complex $\hat{\omega}$ plane.

Figure 2.2: Two graphs that depict the first few quasinormal modes.

and they are comparable with the work of Starinets [2] as well as the result of Horowitz and Hubeny [1] for large 5d Schwarzschild AdS black holes. However, as we go to higher n terms the results do not match well, since we need to go higher than $N = 10$ from 2.11 to achieve better accuracy.

Another way of computing the quasinormal modes of the Schwarzschild AdS black hole is to approximate the series near the horizon by a finite sum, as explained above, and solve the equation below

$$F_n(0) \approx \sum_{n=0}^{N} a_n(\omega) = 0 \tag{2.13}$$

numerically for $z_H = 1$. This method works for a large number of N and so it is somewhat difficult for a computer to analyze it. Also, one may observe that some of the roots of (2.13) do not match with the frequencies of the exact solution. Those particular roots are called "false frequencies" and may be neglected for our particular purpose. However, knowing these frequencies gives us confidence about the accuracy of the values of the quasinormal modes that we have computed above.

Chapter 3

Prescription for calculating the Green's function and the viscosity/entropy ratio

In this chapter, we will examine the retarded Green's function of the strongly coupled gauge theory and provide a recipe on how to compute it numerically. With our numerical results, we calculate the viscosity over entropy ratio in the hydrodynamic limit.

Let us take a closer look at what happens at the boundary $z = 0$. If we expand equation (2.6) as a series near the boundary $\phi(z) \approx z^{\Delta}$, we will get

$$z^{\Delta}\left(\frac{\Delta(-4+\Delta)}{z^2} + \omega^2 - \frac{4\Delta z^2}{z_H^4} + \mathcal{O}(z^4)\right) = 0. \tag{3.1}$$

The leading term of the expansion is

$$\Delta(-4+\Delta) = 0 \tag{3.2}$$

so the two solutions are $\Delta_- = 0$ and $\Delta_+ = 4$. The local solution that satisfies the near the horizon boundary condition (2.10) can be expressed as a linear combination of the other two local solutions at the boundary

$$\phi(z) = \mathcal{A}F_I(z) + \mathcal{B}F_{II}(z), \tag{3.3}$$

where

$$F_I(z) = z^{\Delta_-}(1 + ...) \tag{3.4}$$

$$F_{II}(z) = z^{\Delta_+}(1 + ...) \tag{3.5}$$

and the ellipses denote higher powers of z. The coefficients $\mathcal{A}$ and $\mathcal{B}$ typically depend on the parameters of the ODE (i.e. ω and z_H).

Applying the gauge-gravity duality recipe for Minkowski correlators [17], one finds that the retarded Green's function of the operator $\mathcal{O}$ of the field theory that is dual to the perturbation $\phi(z)$ is given by

$$G^R(\omega) \approx 2\nu\frac{\mathcal{B}(\omega)}{\mathcal{A}(\omega)} + \quad \text{contact terms.} \tag{3.6}$$

The factor in front of the above two point function reads $2\nu = 2\Delta - d = 4$, where Δ is the conformal dimension of the massless scalar and $d = 4$ is the dimension of the gauge theory. Zeros of the coefficient $\mathcal{A}(\omega)$ correspond to poles of the two-point function $G^R(\omega)$. On the contrary, from the general relativity side, $\mathcal{A} = 0$ is precisely the vanishing Dirichlet condition that we imposed for our ODE (2.6) at $z = 0$.

One may compute the coefficients of $\mathcal{A}$ and $\mathcal{B}$ simply by substituting the expression (3.3) into (2.6) and then expanding it as a series in terms of z. However, in order to calculate the coefficient $\mathcal{B}$ numerically, we need to add logarithms into our $F_I(z)$ series. The series that we obtained can be written as

$$-\frac{3\mathcal{A}_1}{z} + (\omega^2\mathcal{A}_0 - 4\mathcal{A}_2) + (\omega^2\mathcal{A}_1 - 3\mathcal{A}_3)z + (\omega^2\mathcal{A}_2 + 4\mathcal{A}_4)z^2 + ... = 0 \tag{3.7}$$

where ... denotes higher powers of z. After that, we solve the $\mathcal{A}$ terms order by order in terms of $\mathcal{A}_0$. Note that every coefficient in our series is set to zero, since our series is linear.

The equation that we derived looks something along the lines of the following expression

$$\phi(z) = \mathcal{A}_0 + \mathcal{A}_0 \frac{\omega^2}{4} z^2 - \mathcal{A}_0 \frac{\omega^4}{16} z^4 \log(z) + ... + \mathcal{B}_0 z^4 + ... \tag{3.8}$$

Note that odd terms of z vanish; only even terms survive. The $\mathcal{A}_0$ coefficient can be found in the same way that we computed the quasinormal modes, by identifying the numerical solution of $\phi(z)$ at the boundary. For every given ω there is a $\mathcal{A}_0$, which is indeed complex.

The procedure is not the same for the $\mathcal{B}_0$ case. One way to find it is to subtract the $\mathcal{A}_0$ terms from ϕ and then divide it by z^4 which is the first $\mathcal{B}_0$ term from the expansion of (3.8)

$$\frac{\phi(z) - (\mathcal{A}_0 + \mathcal{A}_0 \frac{\omega^2}{4} z^2 - \mathcal{A}_0 \frac{\omega^4}{16} z^4 \log(z) + ...)}{z^4} = \mathcal{B}_0. \tag{3.9}$$

We can call the numerator of the above expression as ϕ_{UVA}, which now depends on $\omega, \mathcal{A}_0$ and z. Plotting this new function with specific ω and $\mathcal{A}_0$ gives us the coefficient that we are looking for. Two key features can be observed from Figure 3.1. The first one is that as

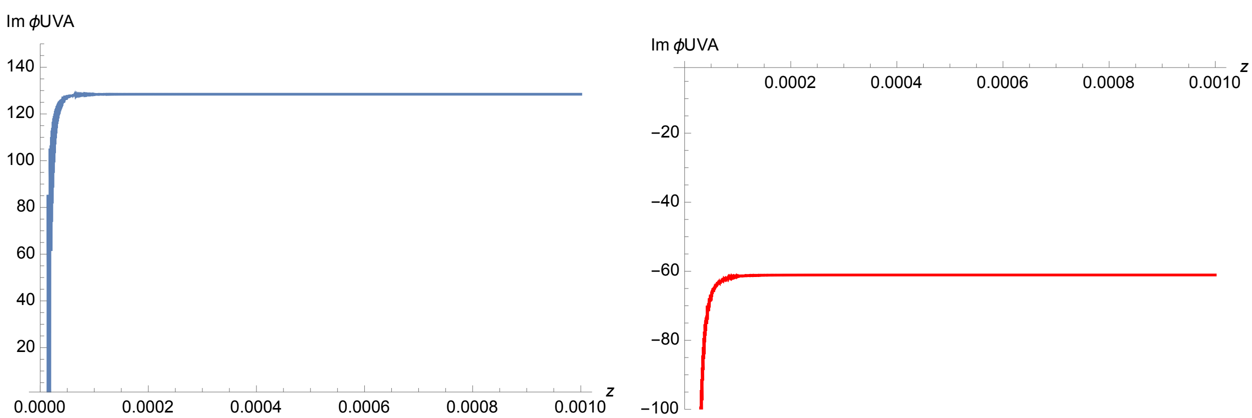


Figure 3.1: Example of plots where we take the imaginary part of the new function that we mentioned above ϕ_{UVA} over the radial coordinate z with fixed ω and $\mathcal{A}_0$.

we approach closer and closer to the boundary $z = 0$ the curve diverges. The second and most important feature is that both graphs show a constant line which stops at a point and proceeds to diverge. That particular point gives us a rough estimate of the $\mathcal{B}_0$ coefficient.

However, while this "graphical inspection" method can give us an insight on the quantity that we seek, it is prone to numerical error and rather time consuming.

Another similar but more precise method can be applied in this case. First, we create a list of points from the numerical solution of ϕ for multiple z, points that are close to the boundary, and then use the built-in Mathematica function $FindFit$ to search for the best possible value of $\mathcal{B}_0$ that fit the data that we specified above in equation (3.8).

Now, we have the necessary tools to obtain the Green's function $G(\omega)$ for any given ω. One may compute the Green's function for the quasinormal modes that we found in Table 2.1 and notice that the correlator blows up to infinity due to the fact that $\mathcal{A}_0$ becomes approximately zero.

A useful quantity that is related to what we have discussed so far in this section is the imaginary part of the Green's function, since it provides us with important information about the transport coefficients of the $\mathcal{N} = 4$ SYM theory, such as the shear viscosity η [14, 13].

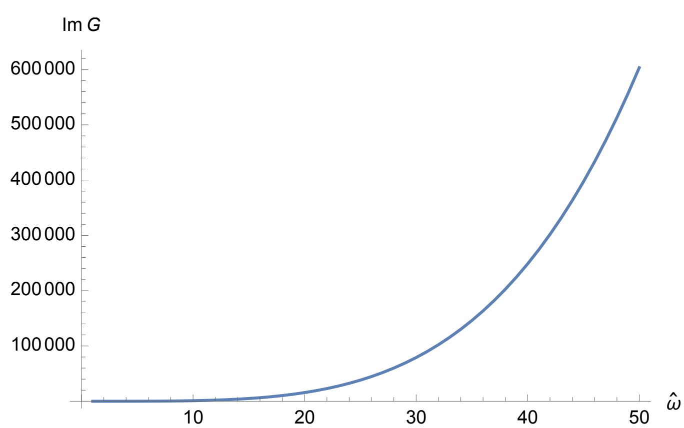


Figure 3.2: The imaginary part of the two-point function $Im(G(\omega))$ over the frequency of the perturbation $\hat{\omega} = \omega/\pi T$.

In order to find this relationship between them, we need to take a closer look at Figure 3.2. The curve that is starting off near zero and blows up to infinity appears to be similar to a power law behavior. Plotting our data into a logarithmic plot we can see this point more clearly.

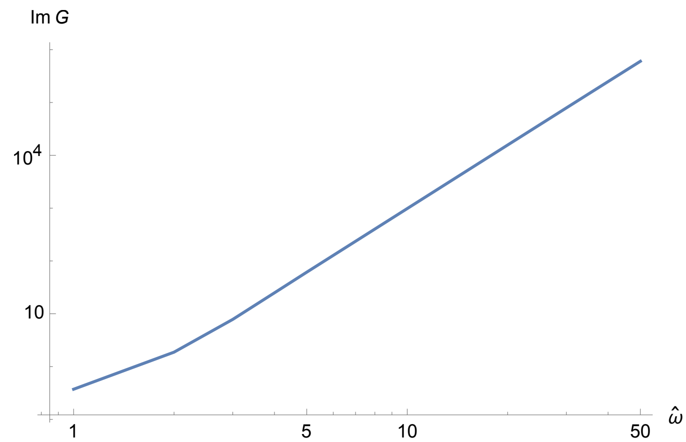


Figure 3.3: Log-Log plot of the imaginary part of the Green's function $Im(G(\omega))$ over the frequency $\hat{\omega} = \omega/\pi T$.

In the following graph 3.3, there are two regions where the function looks to be linear, one for low $\hat{\omega}$ and one for high $\hat{\omega}$. This clue suggests that these two regions have a separate power law behavior.

In order to identify these power law behaviors, one can zoom into these two distinctive areas. The power law equation that governs these areas could be written as

$$ImG = q\omega^a, \tag{3.10}$$

where q and a are both numerical constants. To determine these constants, one has to rewrite equation (3.10) in logarithmic terms, so it becomes linear

$$\log(ImG(\omega)) = a\log(\omega) + \log q. \tag{3.11}$$

In order to extract the a and q values, we can perform a linear model fit for different ω. Below, in Fig 3.4 we present logarithmic figures for both high and low frequencies. The numerical values of the constants evaluated are $a \sim 1.00003, q \sim 1.00018$ for small $\hat{\omega}$ and

$a \sim 3.97885, q \sim 0.43336$ for high $\hat{\omega}$. We observe from the values above that for very small frequencies, the power law equation (3.10) becomes linear.

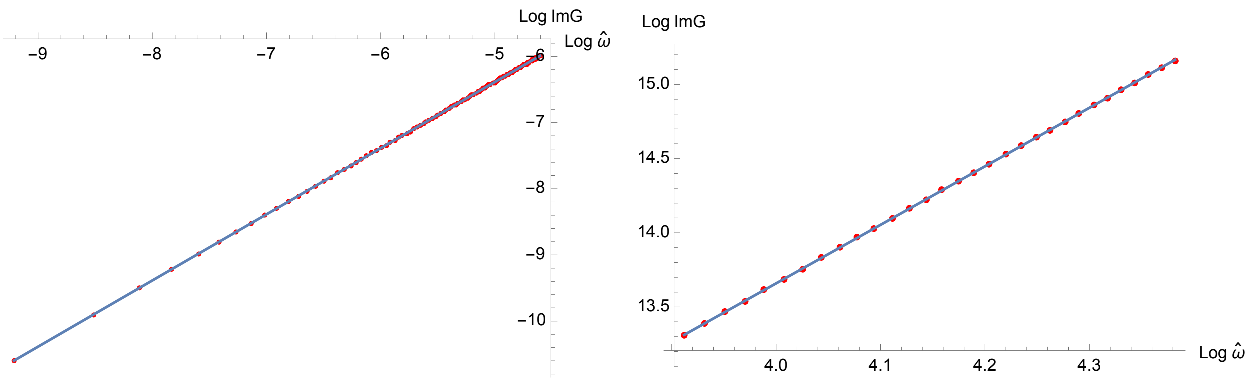


Figure 3.4: The left graph is a logarithmic graph of the imaginary part of the Green's function over the frequency for small $\hat{\omega}$, while the right plot is for high $\hat{\omega}$; the red dots represent the numerical data computed and the blue line the linear fit.

In the hydrodynamic limit $\omega \to 0$ (i.e. for very small frequencies) we can compute the ratio of $Im(G^R(\omega))/\omega = q$, since we found it is a constant. Then from Kubo's formula, the shear viscosity of the $\mathcal{N} = 4$ SYM gauge theory is given by

$$\eta \sim \lim_{\omega \to 0} \frac{1}{\omega} Im G^R_{xy,xy}(\omega, k = 0) \tag{3.12}$$

where $G^R_{xy,xy}$ is the Green's function of the stress-energy tensor T_{xy}, which acts as a response to the metric perturbation h_{xy}[1] . But, the equation of motion of h_{xy} is identical to the equation of motion we derived for the scalar field (2.6). One may now compute the ratio of the shear viscosity over the entropy density η/s. Combining equation (1.15), with $L = 1$ and $z_H = 1$, and recalling that $G_5 = 1/16\pi$, we find the famous ratio

$$\frac{\eta}{s} = \frac{1}{4\pi} \tag{3.13}$$

in the limit of infinite 't Hooft coupling λ and infinite number of colours N_c.

[1] The metric perturbation may take the form of $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, where we have taken the five dimensional metric 1.6

By applying the AdS/CFT dictionary from equation (1.1) and the temperature of the black hole (1.11), we obtain the entropy density in terms of the temperature

$$s = \frac{\pi^2}{2} N_c^2 T^3. \tag{3.14}$$

Thus, the value of η itself from equation (3.13) is given by

$$\eta = \frac{\pi}{8} N_c^2 T^3. \tag{3.15}$$

We have seen what happens at low frequencies, but what about high frequencies $\omega \gg T$? If we take the scalar field fluctuation (2.6) and make a change of variables $\zeta = z\omega$ and $\zeta_H = \omega z_H$ the equation transforms to

$$\frac{\zeta_H^8 \phi(\zeta)}{(\zeta^4 - \zeta_H^4)^2} + \frac{(\zeta^4 + 3\zeta_H^4)\phi'(\zeta)}{(\zeta^5 - \zeta\zeta_H^4)} + \phi''(\zeta) = 0. \tag{3.16}$$

One can study this equation for very large $\zeta_H \to \infty$ and observe that it can be reduced to

$$\zeta^2 \phi(\zeta) - 3\zeta\phi'(\zeta) + \zeta^2 \phi''(\zeta) = 0, \tag{3.17}$$

which is exactly the equation of motion of a scalar field in pure AdS space (without the black hole). Making a change of variables $\phi(\zeta) = \zeta^2 g(\zeta)$ in equation (3.17), we get

$$\zeta^2 g''(\zeta) + \zeta g'(\zeta) + (\zeta^2 - 4) g(\zeta) = 0. \tag{3.18}$$

The solutions of the above equation are just the Bessel functions and the equation has a regular singularity at $\zeta = 0$ and an irregular singularity at $\zeta = \infty$. It is convenient to use the Hankel functions $H_2^{(1,2)}$ since they can take the plane-wave form for large ζ. By reinstating the original variables $\zeta = \omega z$ the Hankel function asymptotically behaves as

$$H_2^{(1,2)}(\omega z) \sim \sqrt{\frac{2}{\pi \omega z}} e^{\pm i\omega z \mp 5\pi i/4}. \tag{3.19}$$

We want to impose the "incoming wave" boundary condition there, so we pick out the solution of interest $g \propto H_2^{(1)}(\omega z)$ and the full solution takes the form

$$\phi(z) \sim z^2 \omega^2 H_2^{(1)}(\omega z). \tag{3.20}$$

Furthermore, we can expand the Hankel function close to the boundary $z \to 0$ of AdS space and the first few terms are written below

$$\phi(z) \sim -\frac{4i}{\pi} - \frac{iz^2\omega^2}{\pi} + \frac{z^4\omega^4}{8} - \frac{3iz^4\omega^4}{16\pi} + \frac{i\gamma z^4\omega^4}{4\pi} + \frac{iz^4\omega^4 \log(\frac{z\omega}{2})}{4\pi} + ..., \tag{3.21}$$

where γ is Euler's constant and ... represents higher orders of z. From this expansion and equation (3.8) we can extract the $\mathcal{A}_0$ and $\mathcal{B}_0$ terms. If we want to examine the imaginary part of the Green's function, one would get

$$Im(G(\omega)) \simeq 2\nu \frac{\mathcal{B}_0}{\mathcal{A}_0} = \frac{\pi\omega^4}{8} \tag{3.22}$$

which gives us the correct power law behavior for high frequencies. One may also extract the real part of the Green's function, simply by taking the leading term of the Hankel expansion from 3.21 (i.e. $i\omega^4 \log \omega$) and divide by the constant term $\mathcal{A}_0 = -4i/\pi$.

Conclusions

In this thesis, we gave a definition and numerically computed the quasinormal modes of the five dimensional Schwarzschild- AdS black hole. These modes, govern the late time decay of a massless scalar field and have a direct interpretation as the poles of the retarded thermal correlator in the holographically dual gauge theory. Moreover, we presented a general approach on computing the retarded Green's function of the strongly coupled $\mathcal{N} = 4$ SYM gauge theory numerically applying the AdS/CFT framework and validated our results by calculating the viscosity over entropy ratio η/s in the hydrodynamic limit.

The viscosity over entropy ratio has been computed in other strongly coupled quantum field theories that have pure Einstein gravity duals, including Dp branes [18]. In these examples the ratio is given by the same number $1/4\pi$, which implies that this result is universal and is independent of the metric or the dimensionality of the spacetime. Moreover, there is an increasing amount of experimental data [19] that suggest that the strongly coupled quark-gluon plasma, or better known as QGP, approaches this constant ratio. This fact, in some sense, proves the power of the gauge/gravity duality.

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