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Lines on Hypersurfaces

by

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Introduction

In this thesis we study lines contained in hypersurfaces of the complex projective space. These are the vanishing loci of homogeneous polynomials in $n+1$ variables. The degree of the hypersurface is the degree of the polynomial. We expect that the more lines the hypersurface contains, the more close to a hyperplane should be. In fact, the lines contained in a hyperplane of $\mathbb{P}^n$ are parametrized by the Grassmannian $\mathbb{G}(1, n-1)$, a variety of dimension $2(n-2)$. On the other hand, one can show that the locus of lines contained in an irreducible hypersurface of $\mathbb{P}^n$ has dimension $\leq 2(n-2)$ and the equality holds exactly when the hypersurface is a hyperplane. The above can be rephrased as follows. A hyperplane is an irreducible hypersurface of degree one. So we should expect, as the degree increases the dimension of the locus of lines contained in the hypersurface to decrease and at some point this locus to become empty. This is not quite true. For degree one, we have just one hypersurface, the hyperplane. But for higher degrees this is not the case, and we may find special hypersurfaces containning unexpectedly many lines. But for what we call the general hypersurface of specific degree, we shall see that this is true.

Throughout this thesis we work over the complex numbers. We denote by $\mathbb{P}^n$ the complex projective space of dimension n. We assume some basic familiarity with Algebraic Geometry: projective space, projective varieties, line bundles and vector bundles on manifolds etc. With the term variety we always mean a projective variety, except if otherwise stated. In the first three chapters we recall in short some basic theory: the Chow ring of a smooth variety, the Grassmannian and the theory of Chern classes. In chapter four we study lines on hypersurfaces and in the last chapter we specialize to the case of cubics (hypersurfaces of degree 3), which is a central subject of study on its own.

The main references we use are [**2**] and [**4**].

CHAPTER 1

The Chow Ring

1. The group of Cycles

To the set of the subvarieties of a given projective variety, we correspond the abelian group with elements the formal linear combinations of subvarieties with coefficients in $\mathbb{Z}$. The elements of this group are called *cycles*. When the variety is smooth then modulo an equivalence relation on cycles, which is called rational equivalence of cycles, we can define an intersection product. The theorem which hints the existence of a ring structure given by the intersection product on the group of cycles, modulo the above equivalence relation, is Bezout's theorem (which has various forms):

Theorem 1.1 (A form of Bezout's Theorem)**.**

(1) *Given two plane curves in $\mathbb{P}^2$ of degrees d_1 and d_2 that intersect transversely, then they intersect in $d_1 d_2$ points.*
(2) *More general, given two subvarieties of the projective space of complementary dimension and of degrees d_1 and d_2 that intersect transversely, then they intersect at $d_1 d_2$ points.*
(3) *Even more general, given two subvarieties of the projective space $\mathbb{P}^n$ of codimensions m_1 and m_2, with $m_1 + m_2 \leq n$, and of degrees d_1 and d_2 that intersect transversely then they intersect at a subvariety of codimension $m_1 + m_2$ and of degree $d_1 d_2$.*

In Section 3 we define what it means that two subvarieties intersect transversely. Note that Bezout's Theorem depend on numerical invariants associated to a subvariety and not on the particular choice of the subvariety itself. For example, intersecting either a product of two lines and a curve A or a parabola with the same curve A will give us the same intersection number, given they intersect transversely. This suggests that the number of points of intersection is invariant under some equivalence relation which will have to make clear.

Definition 1.2 (Group of Cycles)**.** Let X be any algebraic variety. The group of *cycles* on X, denoted by $Z(X)$, is the free abelian group generated by the set of subvarieties of X. Its elements are finite formal sums of subvarieties of X with integral coefficients. The group $Z(X)$ is graded by dimension: its k-th component $Z_k(X)$ is the group of formal sums of subvarieties of dimension k with integral coefficients (these are called k-cycles). We then have

$$Z(X) = \bigoplus_{k=0}^{\dim X} Z_k(X).$$

Definition 1.3. A *divisor* is an $(n-1)$-cycle, with $n = \dim X$. A cycle $Z = \sum_i n_i Y_i$, where the Y_i are subvarieties of a variety X, is called *effective* if the coefficients n_i are all non negative.

Notation 1.4. When X is a variety of dimension n, we denote by $Z^k(X)$ the group of cycles of codimension k, that is, $Z^k(X) = Z_{n-k}(X)$.

To any subvariety $Y \subseteq X$ we associate an effective cycle $\sum_{i=1}^{s} r_i Y_i$, where $Y_1, \ldots, Y_s$ are the irreducible components of Y and r_i is the multiplicity of Y along the irreducible component Y_i. It corresponds to the length of the composition series of the local ring $\mathcal{O}_{Y,Y_i}$. For example, if Y is the hypersurface in $\mathbb{P}^n$ given by the homogeneous polynomial F and $F = F_1^{r_1} \cdots F_s^{r_s}$ its factorization into irreducibles, then the associated cycle is the $\sum_{i=1}^{s} r_i Y_i$, with Y_i the irreducible hypersurface defined by the polynomial F_i.

2. Rational equivalence

We now make clear the equivalence between two cycles. Intuitively it is that you can interpolate from one cycle to the other, meaning that there exists a family of subvarieties parametrized by $\mathbb{P}^1$ between the two varieties. This is the algebraic analogous of a continuous deformation of cycles in topology.

Definition 2.1. Let X be a projective variety and let $Rat(X) \subset Z(X)$ be the subgroup generated by the differences of the form

$$\langle \Phi \cap (t_o \times X) \rangle - \langle \Phi \cap (t_1 \times X) \rangle,$$

where t_0, $t_1 \in \mathbb{P}^1$ and Φ is a subvariety of $\mathbb{P}^1 \times X$ which is mapped onto $\mathbb{P}^1$ by the first projection. We say that two cycles are **rationally equivalent** if their difference is in $Rat(X)$. We say that two subvarieties of a given variety are rationally equivalent if their associated cycles are rationally equivalent.

The characteristic example on the level of divisors is

Example 2.2. Let F, G two homogeneous polynomials in $n+1$ variables of the same degree d. We put $X_1 = \mathbb{V}(F)$, $X_2 = \mathbb{V}(G)$ which are subvarieties of $\mathbb{P}^n$. Then X_1, X_2 are rationally equivalent. Indeed, take

$$\Phi = \{([s,t], p) \text{ with } sF(p) + tG(p) = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^n.$$

Let $\pi : \Phi \to \mathbb{P}^1$. This is a family of subvarieties of $\mathbb{P}^n$, parametrized by $\mathbb{P}^1$, with $\pi^{-1}([1,0]) = X_1$, $\pi^{-1}([0,1]) = X_2$ and thus X_1, X_2 are rationally equivalent. In particular, the class of a hypersurface of degree d in $\mathbb{P}^n$ equals the class of the cycle dH, with H any hyperplane.

Example 2.3. Any two lines in $\mathbb{P}^n$ are rational equivalent. Indeed, a line in $\mathbb{P}^n$ is defined by a homogeneous system of $n-1$ equations in $n+1$ variables. Let Σ and Σ' be the systems defining the two lines. Then, as in Example 2.2, we form the family of systems $s\Sigma + t\Sigma'$, parametrized by $\mathbb{P}^1$, which gives the required equivalence.

Given a specific X, it might be hard to determine which subvarieties are rationally equivalent, even in the relative simple case of two points on a surface X: for that, it should exist a rational curve on X joining them. For example, this is always the case when X is the projective space.

Definition 2.4. The *Chow* group of X is the quotient

$$A(X) = Z(X)/Rat(X),$$

the group of *rational equivalence* classes of cycles on X. If $Y \in Z(X)$ is a cycle, we write $[Y] \in A(X)$ for its equivalence class.

Proposition 2.5. *The Chow group $A(X)$ of a variety X is graded by dimension, that is*

$$A(X) = \bigoplus_{k=0}^{\dim X} A_k(X),$$

with $A_k(X)$ the group of the rational equivalence classes of k-cycles.

Notation 2.6. When X is a variety of dimension n, we denote by $A^k(X)$ the group of rationally equivalent classes of cycles of codimension k, that is, $A^k(X) = A_{n-k}(X)$.

Calculating the chow group $A(X)$ can be a really hard task and is possible in few cases, for example that of the projective space. As we shall see at the end of this chapter, see Application 4.3, we have:

Theorem 2.7. *We have that $A_k(\mathbb{P}^n)$ is generated by the class of a k-plane. In other words, any class of a subvariety of $\mathbb{P}^n$ of dimension k equals to $m[\Pi_k]$, with Π_k a fixed k-dimensional plane, and for some (positive) integer m (the degree of the subvariety).*

We close this section by relating cycles classes of the domain and the image of a map. Given a morphism $f : X \to Y$ of smooth projective varieties and a subvariety X' of X we define $f_*(X') = f(X')$ if X' and $f(X')$ have the same dimension and $f_*(X') = 0$ otherwise (note that $\dim f(X') \leq \dim X'$). The above operator is extending naturally to the group of cycles by sending k-cycles to k-cycles. We then have

Proposition 2.8. *We have a well define map of groups*

$$f_* : A(X) \to A(Y) \text{ determined by } f_*([X']) = [f_*(X')] \text{ for any subvariety } X' \text{ of } X.$$

It is called the push forward map.

On the other hand, given $f : X \to Y$ as above and a subvariety Y' of Y it can be proved that we have a well defined map

$$f^* : A(Y) \to A(X)$$

which keeps codimensions, determined by

$$f^*([Y']) = \pi_{1*}([\Gamma_f] \cdot [\pi_2^{-1}(Y')]),$$

with $\Gamma_f \subset X \times Y$ the graph of f and π_1, π_2 the two projections. It is called the pull back map.

The two constructions are related by the projection formula:

Theorem 2.9 (Projection formula)**.** *Given a map $f : X \to Y$ of smooth projective varieties, a class α in $A(Y)$ and a class β in $A(X)$ we have*

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta).$$

3. The Chow ring

Before we introduce the Chow ring we need to define what we mean by saying that two subvarieties intersect transversely. Recall that the dimension of an irreducible subvariety equals the dimension of its tangent space at any smooth point.

Definition 3.1. We say that the subvarieties Y_1, Y_2 of a variety X intersect *transversely* at a point p if p is a smooth point of all Y_1, Y_2 and X and the tangent spaces to Y_1 and Y_2 at p span together the tangent space to X at p, that is,

$$T_pY_1 + T_pY_2 = T_pX.$$

Equivalently,

$$\text{codim}(T_pY_1 \cap T_pY_2) = \text{codim}T_pY_1 + \text{codim}T_pY_2.$$

As a consequence, when Y_1 and Y_2 intersect transversely at p, then p is contained in exactly one irreducible component of Y_1 and one of Y_2, the sum of the dimensions of these components is $\geq \dim X$ and the intersection of those has at p the codimension we expect it to have, i.e. the sum of the codimensions of the above irreducible components. Moreover, the intersection is smooth at p.

We say that the subvarieties Y_1, $Y_2 \subset X$ are *generically transverse*, if every component Z of $Y_1 \cap Y_2$ contains a point p at which Y_1 and Y_2 intersect transversely. Since the property that two subvarieties intersect transversely at a point is an "open property", this is equivalent to saying that every component of $Y_1 \cap Y_2$ contains a Zariski open, at every point of which the subvarieties Y_1 and Y_2 intersect transversely. Two cycles $A = \sum_i n_i Y_i$ and $B = \sum_j m_j Y'_j$ are generically transverse, if each Y_i is generically transverse to each Y'_j. More generally, we will say thata finite family of subvarieties $Y_i \subset X$ intersect transversely at a smooth point $p \in X$, if p is a smooth point on each Y_i and

$$\text{codim}(\bigcap T_pY_i) = \sum_i \text{codim}T_pY_i.$$

There is the analogous definition of when the Y_i's intersect generically transversely.

Under these circumstances and when X is smooth, the following theorem states that the rational equivalence class of the intersection of two subvarieties $Y_1, Y_2 \subset X$ depends only on the corresponding classes of Y_1 and Y_2. This makes possible to define the product structure on the Chow groups, see [2] Theorem-Definition 1.5

Theorem 3.2 (The Chow ring)**.** *If X is a smooth projective variety, then there is a unique product structure on $A(X)$ satisfying the following condition: if two subvarieties Y_1, Y_2 of X are generically transverse, then*

$$[Y_1][Y_2] = [Y_1 \cap Y_2].$$

This structure makes

$$A(X) = \bigoplus_{k=0}^{\dim X} A^k(X)$$

into an associative, commutative ring, graded by codimension, called the Chow ring of X.

The proof of the above Theorem 3.2 is based on the following lemma which ensures that there exists a pair of generically transverse cycles for every two classes in the Chow ring. This is a highly non trivial result, see see [2] Theorem 1.6.

Theorem 3.3 (The Moving Lemma)**.** *Let X be a smooth projective variety.*

(1) For every $\alpha, \beta \in A(X)$ there are generically transverse cycles $A, B \in Z(X)$ with $[A] = \alpha$ and $[B] = \beta$.

(2) The class $[A \cap B]$ is independent from the choice of such cycles A, B.

We finish this section by explaining a relation between the divisors defined by a rational function and the group of rationally equivalent subvarieties.

Definition 3.4. For a given projective variety X, let f be a meromorphic (rational) function on X other than 0. The zeros and poles of X define codimension one subvarieties of X. The irreducible components of these subvarieties defined by f are equipped with multiplicities, positive in the case of zeros (order of vanishing) and negative for the case of poles (order of a pole). We may then define a divisor, that we call it the divisor of f and denoted $Div(f)$, as follows:

$$Div(f) = \sum_{Y \subset X} ord_Y(f)\langle Y\rangle,$$

where Y is irreducible and $ord(f)$ is the order of vanishing or of a pole of f along Y.

Compare now the following fundamental proposition with Example 2.2, by taking into account that F/G is a rational function on $\mathbb{P}^n$.

Proposition 3.5. *Let X be any projective variety, then the group $Rat(X) \subset Z(X)$ is generated by all divisors of rational functions on all subvarieties of X. In particular, if X is of dimension n, then $A_{n-1}(X)$ is equal to the divisor class group of X.*

4. Affine stratifications

As already stated, calculating the Chow groups of a variety X can be very hard. But when X admits a certain *affine stratification*, which is a special kind of decomposition into the union of affine spaces, we may then be able to calculate the generators of the Chow groups. This is how we are going to compute the Chow ring of the projective space and, more general, of the Grassmannian.

Definition 4.1. We say that a projective variety X is *stratified* by a finite collection of locally closed subvarieties U_i (locally closed subvariety= intersection of closed and open subsets in the Zariski topology), if X is a disjoint union of U_i's and, in addition, the closure of any U_i is a union of U_j, meaning that if $\overline{U_i}$ meets U_j, then $U_j \subset \overline{U_i}$. The sets U_i are called the (open) strata of the stratification and their closures closed strata. We call the stratification *affine* if each open stratum is isomorphic to $\mathbb{A}^k$.

Theorem 4.2 ([2] Theorem 1.18 - Totaro (2014))**.** *The classes of the closed strata in an affine stratification of a variety X form a basis of $A(X)$.*

As an application we describe the Chow ring of the projective space, see Theorem 2.7.

Application 4.3.

- $\mathbb{P}^1 = U_1 \sqcup U_0$, with $U_1 = \{[1, a], a \in \mathbb{C}\}$ and $U_0 = \{[0, 1]\}$. We have $U_1 \cong \mathbb{A}^1$ and $U_0 \cong \mathbb{A}^0$ with $\overline{U_1} = U_1 \cup U_0 \cong \mathbb{P}^1$ and $\overline{U_0} = U_0$. We then get an affine stratification of $\mathbb{P}^1$ and by the theorem we have that $A(\mathbb{P}^1)$ is generated by the class of the point $[0, 1]$ and the class of the whole space $\mathbb{P}^1$.
- $\mathbb{P}^2 = U_2 \sqcup U_1 \sqcup U_0$, with $U_2 = \{[1, a, b], a, b \in \mathbb{C}\}$, $U_1 = \{[0, 1, a], a \in \mathbb{C}\}$ and $U_0\{[0, 0, 1]\}$. We have $U_2 \cong \mathbb{A}^2$, $U_1 \cong \mathbb{A}^1$ and $U_0 \cong \mathbb{A}^0$ with $\overline{U_2} = U_2 \cup U_1 \cup U_0 = \mathbb{P}^2$, $\overline{U_1} = U_1 \cup U_0 \cong \mathbb{P}^1$ and $\overline{U_0} = U_0$. We then get an affine stratification of $\mathbb{P}^2$ and by the theorem we have that $A(\mathbb{P}^2)$ is generated by the class of the point $[0, 0, 1]$, the class of the line $\overline{U_{01}} \cong \mathbb{P}^1$ and the class of the whole space $\overline{U_2} \cong \mathbb{P}^2$.
- In the same fashion, we get an affine stratification of $\mathbb{P}^n$ given by $\mathbb{P}^n = \sqcup_{k=0}^n U_k$, with $U_k \cong \mathbb{A}^k$, $\overline{U_k} = \cup_{i=0}^k U_i \cong \mathbb{P}^k$. By the theorem we have that $A(\mathbb{P}^n)$ is generated by the classes of the k-planes $\overline{U_k} \cong \mathbb{P}^k$.

CHAPTER 2

Grassmannians

The Grassmannian is a generalization of the projective space and will be the "ambient" space for the locus of lines in hypersurfaces that we intend to study. We give a brief presentation of its construction and then we find generators for the Chow group along with some formulas which allow us to compute the corresponding class products in the Chow ring.

Definition 0.1 (Grassmannians)**.** Let $n \in \mathbb{N}_{\geq 2}$ and $k \in \mathbb{N}$ with $1 \leq k \leq n$. We denote by $G(k,n)$ the set of all k-dimensional linear subspaces of $\mathbb{C}^n$:

$$G(k,n) := \{V \subset \mathbb{C}^n,\ V \text{ is a vector subspace of } \mathbb{C}^n \text{ of dimension k}\}.$$

It is called the Grassmannian of k-planes in $\mathbb{C}^n$. A k-plane in $\mathbb{C}^n$ is the same thing as a $(k-1)$-plane in the corresponding projective space $\mathbb{P}^{n-1}$, so that we can think of $G(k,n)$ as the set of $(k-1)$-planes in $\mathbb{P}^{n-1}$ and we then use the notation $\mathbb{G}(k-1,n-1)$.

Our main object of interest will be $\mathbb{G}(1,n) = G(2,n+1)$, the Grassmannian of lines in $\mathbb{P}^n$. From now on we will concentrate our study on that (the general case is quite analogous). The first goal is to describe the Grassmannian as a subset of a projective space and then prove that it is indeed a projective subvariety.

1. The Plücker embedding

Let $V = \mathbb{C}^{n+1}$. To any 2-dimensional subspace $L \subset V$ we can associate a multivector as follows: we choose a base v_1, v_2 of L and define

$$\lambda = v_1 \wedge v_2 \in \bigwedge^2 V.$$

Now λ is determined up to a scalar since if we change the basis of W the corresponding vector λ would simply be multiplied by the determinant of the change of basis matrix. We also observe that since we can obtain a different basis by linear operations, these operations change the alternating product of the vectors at most by a scalar. We thus have a well-defined map

$$\psi : G(2,n+1) \ni \operatorname{span}(v_1,v_2) \longrightarrow [v_1 \wedge v_2] \in \mathbb{P}(\bigwedge^2 V) \cong \mathbb{P}^{\binom{n+1}{2}-1}.$$

The map ψ is injective since $v_1 \wedge v_2 \wedge v = 0$ if and only if v_1, v_2, v are linearly dependent. Therefore, given $\psi(L) = [v_1 \wedge v_2]$, we can recover the corresponding subspace L as the space of vectors $v \in V$ such that $v_1 \wedge v_2 \wedge v = 0$. We call the map ψ the **Plücker embedding** of $G(2,n+1)$.

The homogeneous coordinates of $\mathbb{P}^{\binom{n+1}{2}-1}$ are called the **Plücker coordinates**. Explicitly, by choosing the standard base of $V \cong \mathbb{C}^{n+1}$ we can represent the plane L by a $2 \times (n+1)$ matrix M_L whose rows are the vectors v_1, v_2. This matrix is not unique since if we change the basis for L we get some other M'_L . So M_L is determined up to multiplication on the left by an invertible 2×2 matrix A. The Plücker coordinates are then just the 2×2 minors of the matrix M_L. These are well defined, since the 2×2 minors of M'_L are those of M_L multiplied by the fixed non-zero scalar $\det A$.

Example 1.1. The Plücker embedding of $G(1,n+1)$ simply maps a linear subspace $L = \operatorname{span}(a_0e_0 + \cdots + a_ne_n)$ to the point $[a_0,\ldots,a_n] \in \mathbb{P}^n$. Hence $G(1,n+1) = \mathbb{P}^n$ as expected.

Example 1.2. Consider the 2-dimensional subspace $L = \operatorname{span}(e_1+e_2,\ e_1+e_3) \in G(2,3)$. Then

$$(e_1+e_2) \wedge (e_1+e_3) = -e_1 \wedge e_2 + e_1 \wedge e_3 + e_2 \wedge e_3.$$

The coefficients $[-1,1,1]$ of this vector are the Plücker coordinates of L in $\mathbb{P}^{\binom{3}{2}} = \mathbb{P}^2$. Alternatively, these are the maximal minors of the matrix

$$M_L = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The rows of M_L are just the coordinates of the spanning vectors of L.

Now is left to see that the Grassmannian is indeed a closed subset of the projective space. The elements of $\mathbb{P}(\bigwedge^2 V)$ are linear combinations of the *totally decomposable vectors* $w \wedge v$. The image of ψ is exactly the subset of the totally decomposable vectors and have to show that this set is an algebraic subset of $\mathbb{P}(\bigwedge^2 V)$.

Proposition 1.3. *The Grassmannian $G(2, V)$, with $V = \mathbb{C}^{n+1}$, is a closed subset of $\mathbb{P}^{\binom{n+1}{2}-1}$.*

PROOF. We begin with a Lemma:

Lemma 1.4. *Let $v \in V$ a non zero vector and $\omega \in \bigwedge^2 V$. Then $\omega \wedge v = 0$ if and only if $\omega = w \wedge v$ for some $w \in V$.*

PROOF. If $\omega = w \wedge v$ then obviously $\omega \wedge v = 0$. For the other direction: we choose a basis $e_0, e_1, \dots, e_n$ of V with $e_0 = v$ and we write $\omega = \sum_{ij} a_{ij}\, e_i \wedge e_j$. Then $\omega \wedge e_0 = 0$ implies that $\sum_{ij} a_{ij}\, e_i \wedge e_j \wedge e_0 = 0$ in $\bigwedge^3 V$. But if $i, j \neq 0$ then $e_i \wedge e_j \wedge e_0$ is an element of the natural base of $\bigwedge^3 V$ and if i or j equals 0 then $e_i \wedge e_j \wedge e_0 = 0$. Therefore, by linear independence, we should have that $a_{ij} = 0$ when $i, j \neq 0$ and thus any non zero term of ω will be of the form $e_i \wedge e_0$, for some $i \neq 0$. Hence $\omega = w \wedge e_0$. □

Back to the proof of the proposition: by the Lemma, ω is totally decomposable if and only if the kernel of the map

$$f_\omega : V \ni v \longrightarrow v \wedge \omega \in \bigwedge^3 V$$

is $\geq$ 2-dimensional. This is equivalent to saying that ω is totally decomposable if and only if f_ω is of rank $\leq n - 1 = \dim V - 2$. We choose a basis $e_0, \dots, e_n$ for V and the corresponding bases $\{e_i \wedge e_j,\ 0 \leq i < j \leq n\}$ for $\bigwedge^2 V$ and $\{e_i \wedge e_j \wedge e_k,\ 0 \leq i < j < k \leq n\}$ for $\bigwedge^3 V$. We are then looking for the

$$\omega = \sum_{0 \leq i < j \leq n} x_{ij}\, e_i \wedge e_j,$$

for which the map f f_ω is of rank $\leq n - 1 = \dim V - 2$. The matrix of this map with respect to the above bases is determined by the relations

$$f_\omega(e_k) = \sum_{0 \leq i < j \leq n} x_{ij}\, e_k \wedge e_i \wedge e_j,\ k = 0, \dots, n.$$

Hence the entries of the corresponding matrix will be the x_{ij}'s and that this matrix is of rank $\leq n - 1$ corresponds to the vanishing of all $n \times n$ minors. These are polynomials in the Plücker coordinates x_{ij}'s and the vanishing locus of those determine the totally decomposable elements of the $\bigwedge^2 V$ i.e. the image of the $\mathbb{G}(1, n)$ in $\mathbb{P}(\bigwedge^2 V)$ given by the Plücker embedding. □

Example 1.5. Lets obtain the equations determining the variety $G(2, 4)$ as subvariety of $\mathbb{P}^5$. Let e_0, e_1, e_2, e_3 be the standard base if $V \cong \mathbb{C}^4$ and let $\omega = \sum_{0 \leq i < j \leq 3} x_{ij}\, e_i \wedge e_j$. Then

$$\begin{aligned}
f_\omega(e_0) &= \sum_{0 \leq i < j \leq n} x_{ij}\, e_0 \wedge e_i \wedge e_j = x_{12}\, e_0 \wedge e_1 \wedge e_2 + x_{13}\, e_0 \wedge e_1 \wedge e_3 + x_{23}\, e_0 \wedge e_2 \wedge e_3 \\
f_\omega(e_1) &= \sum_{0 \leq i < j \leq n} x_{ij}\, e_1 \wedge e_i \wedge e_j = -x_{02}\, e_0 \wedge e_1 \wedge e_2 - x_{03}\, e_0 \wedge e_1 \wedge e_3 + x_{23}\, e_1 \wedge e_2 \wedge e_3 \\
f_\omega(e_2) &= \sum_{0 \leq i < j \leq n} x_{ij}\, e_2 \wedge e_i \wedge e_j = x_{01}\, e_0 \wedge e_1 \wedge e_2 - x_{03}\, e_0 \wedge e_2 \wedge e_3 - x_{13}\, e_1 \wedge e_2 \wedge e_3 \\
f_\omega(e_3) &= \sum_{0 \leq i < j \leq n} x_{ij}\, e_3 \wedge e_i \wedge e_j = x_{01}\, e_0 \wedge e_1 \wedge e_3 + x_{02}\, e_0 \wedge e_2 \wedge e_3 + x_{12}\, e_1 \wedge e_2 \wedge e_3.
\end{aligned}$$

The corresponding 4×4 matrix is given by

$$A = \begin{pmatrix} x_{12} & -x_{02} & x_{01} & 0 \\ x_{13} & -x_{03} & 0 & x_{01} \\ x_{23} & 0 & -x_{03} & x_{02} \\ 0 & x_{23} & -x_{13} & x_{12} \end{pmatrix}$$

and the equations determining $G(2, 4)$ in the coordinates $x_{01}, x_{02}, x_{03}, x_{12}, x_{13}, x_{23}$ of $\mathbb{P}^5$ are given by the sixteen 3×3 minors of A.

2. Affine cover of the Grassmannian

It is also possible to view the Grassmannian as a smooth complex manifold by looking at certain affine open subsets, in a way analogous to that of the projective space.

We choose the standard base $e_0, \ldots, e_n$ of $V \cong \mathbb{C}^{n+1}$ and for each $0 \le i < j \le n$, let

$$U_{ij} \subset G(2, n+1) \subset \mathbb{P}^{\binom{n+1}{2}-1}$$

be the intersection of $G(2, n+1)$ (embedded by the Plücker embedding in $\mathbb{P}^{\binom{n+1}{2}-1}$) with the corresponding open subset where the $e_i \wedge e_j$-coordinate is non zero, of the standard affine chart of $\mathbb{P}^{\binom{n+1}{2}-1}$. Then a linear subspace $L = \operatorname{span}(v, w) \in G(2, n+1)$ is in U_{ij} if and only if the $2 \times (n+1)$ matrix M_L, i.e. the matrix with rows the coordinates of v and w respectively, has the property that the 2×2 matrix B_{ij} which is formed by the i, j columns is invertible. This in turn is the case if and only if M_L is equivalent by row transformations, i.e a change of basis for L, to a matrix $\tilde{M}_L$ with the property that the 2×2 matrix formed by the i, j-columns is the identity matrix. The rest of the columns form a $2 \times (n-1)$ matrix D_{ij}, i.e. an element of $\mathrm{M}_{2\times(n-1)}(\mathbb{C}) \cong \mathbb{C}^{2(n-1)}$. We write $\tilde{M}_L = (I_{ij}|D_{ij})$. This gives a bijection

$$f : \mathbb{C}^{2(n-1)} = \mathrm{M}_{2\times(n-1)}(\mathbb{C}) \ni D \longrightarrow L \in U_{ij},$$

with L the linear subspace spanned by the rows of $(I_{ij}|D) \in U_{ij}$. The collection of U_{ij}'s, $0 \le i < j \le n$ form an open covering of $G(2, n+1)$. The change of coordinates of the standard affine chart of $\mathbb{P}^{\binom{n+1}{2}-1}$ induces holomorphic changes of coordinates on the open sets U_{ij}, turning $G(2, n+1)$ into a smooth complex manifold of dimension $2(n-1)$.

Example 2.1. In Example 1.5 we showed that $G(2, 4)$ is determined as the zero locus of 16 cubic equations in $\mathbb{P}^5$. We now describe the open set $U_{01} \subset G(2, 4)$, that is, the intersection of $G(2, 4)$ with the corresponding open set in the standard affine chart of $\mathbb{P}^5$ (here we consider $G(2, 4)$ embedded in $\mathbb{P}^5$ by the Plücker embedding). As a consequence we show that $G(2, 4)$ can be actually given as the zero locus of a single quadratic equation!. We keep the notation of Example 1.5.

By the above discussion, the elements of U_{01} are in one to one correspondence with the 2×2 matrices. Given such a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we form the 2×4 matrix

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

and to that we correspond the 2-plane L spanned by the two row vectors, i.e. by the vectors $v = e_0 + ae_2 + be_3$ and $w = e_1 + ce_2 + de_3$. Then the image of L by the Plücker embedding is given by

$$v \wedge w = e_0 \wedge e_1 + c\, e_0 \wedge e_2 + d\, e_0 \wedge e_3 - a\, e_1 \wedge e_2 - b\, e_1 \wedge e_3 + (ad - bc)\, e_2 \wedge e_3$$

Therefore it corresponds to the set of points of $\mathbb{P}^5$:

$$\{[1, c, d, -a, -b, ad - bc],\ a, b, c, d \in \mathbb{C}\} \subset \mathbb{P}^5.$$

As a set in $U_{01} \cong \mathbb{C}^5$ is given by

$$\{(c, d, -a, -b, ad - bc),\ a, b, c, d \in \mathbb{C}\} \subset \mathbb{C}^5.$$

In the coordinates $x_{02}, x_{03}, x_{12}, x_{13}, x_{23}$ of $\mathbb{C}^5$ this is exactly the set determined by the single equation

$$x_{23} = -x_{12}x_{03} + x_{13}x_{02}.$$

By homogenizing with respect to the variable x_{01} we get that $G(2, 4)$ as a subvariety of $\mathbb{P}^5$ is given by the quadratic equation

$$x_{01}x_{23} + x_{12}x_{03} - x_{13}x_{02} = 0.$$

Remark 2.2. The description of the $G(2, 4)$ in terms of one quadratic equation in $\mathbb{P}^5$ is actually due to the fact that $\omega \in \bigwedge^2 V$ is totally decomposable if and only if $\omega \wedge \omega = 0$. When $\dim V = 4$, this translates to the fact that the form $\omega = \sum_{0 \le i < j \le 3} x_{ij} e_i \wedge e_j$ is totally decomposable if and only if

$$\sum_{0 \le i < j \le 3,\, 0 \le k < l \le 3} x_{ij}x_{kl} e_i \wedge e_j \wedge e_k \wedge e_l = 0.$$

The only non-zero term of the above is the one containing (after permutation) the $e_0 \wedge e_1 \wedge e_2 \wedge e_3$ and the corresponding coefficient of that term is the $x_{01}x_{23} + x_{12}x_{03} - x_{13}x_{02}$ which has to be zero. In general, when $\dim V = n+1$ we get $\binom{n+1}{4}$ independent quadratic relations, see [4] Exercise 6.2. Therefore the Grassmannian $G(2,n+1)$ is cut out by quadrics in $\mathbb{P}^{\binom{n+1}{2}-1}$.

3. The Universal and the Quotient bundles

Let $V \cong \mathbb{C}^{n+1}$ and $G = G(2,V)$ the Grassmannian of 2-planes in V. Let $\mathcal{V} := G \times V$ be the trivial vector bundle of rank $n+1$ on G whose fibre at every point of G is the vector space $V \cong \mathbb{C}^{n+1}$.

Definition 3.1. Consider $\mathcal{U}$ to be the rank 2 subbundle of $\mathcal{V}$ whose fiber at a point $[\Lambda] \in G$ is the subspace $\Lambda \subset V$ itself, that is,

$$\begin{tikzcd} \mathcal{U} = \{(\Lambda, v) \in G \times V \mid v \in \Lambda\} \subset G \times V \arrow[d, "\pi"] \\ G. \end{tikzcd}$$

Note that $\mathcal{U}$ is an algebraic subvariety of $G\times V$ since the condition $v \in \Lambda$ translates to $\omega_\lambda \wedge v = 0 \in \bigwedge^3 V$, with $\omega_\lambda \in \bigwedge^2 V$ the image of Λ by the Plücker map. In turn, this translates to the vanishing of some minors which are bilinear functions in the Plücker coordinates and the coordinates of V. The $\mathcal{U}$ is a subbundle of $\mathcal{V}$ called the *universal bundle* (or *tautological bundle*) on G. The quotient $\mathcal{Q} = \mathcal{V}/\mathcal{U}$ is called the *quotient bundle*.

We have an exact sequence

$$0 \to \mathcal{U} \to \mathcal{V} \to \mathcal{Q} \to 0.$$

4. Subvarieties of Grassmannians

We are now going to give a description of certain irreducible subvarieties of the Grassmannian $G(2,V)$, $V \cong \mathbb{C}^{n+1}$, with the property that an open part of them will form an affine stratification of the Grassmannian and therefore, by Theorem 4.2, their classes will serve as generators of the Chow ring. These subvarieties called *Schubert varieties* or *Schubert cycles* and are defined in terms of a chosen *complete flag* $\mathcal{V}$ in V, that is, a nested sequence of subspaces.

$$0 \subset V_1 \subset \cdots \subset V_{n+1} = V$$

with $\dim V_i = i$

For any pair of integers $\alpha = (a_1, a_2)$ with $n-1 \geq a_1 \geq a_2 \geq 0$, we define the *Schubert variety* $\Sigma_\alpha(\mathcal{V}) \subset G(2,V)$ to be the closed subset

$$\begin{aligned} \Sigma_\alpha(\mathcal{V}) =& \{\Lambda \in G(2,V) \mid \dim(V_{n-a_1} \cap \Lambda) \geq 1 \text{ and } \dim(V_{n+1-a_2} \cap \Lambda) \geq 2\} \\ =& \{\Lambda \in G(2,V) \mid V_{n-a_1} \cap \Lambda \neq \{0\} \text{ and } \Lambda \subseteq V_{n+1-a_2}\}. \end{aligned}$$

To simplify the notation, we write $\Sigma_{a_1}(\mathcal{V})$ in place of $\Sigma_{(a_1,0)}(\mathcal{V})$.

For a general topological description of Schubert varieties, see [6] Section 6 or [3] Chapter 1, Section 5.

Example 4.1. We now describe the Schubert varieties of $G(2,4) = \mathbb{G}(1,3)$. First we fix a flag $\mathcal{V}$:

$$0 \subset P_o \subset L_o \subset H_o \subset V = \mathbb{C}^4,$$

where $\dim P_0 = 1$, $\dim L_0 = 2$ and $\dim H_o = 3$. The Schubert varieties are indexed by

$$\alpha = (a_1, a_2), \text{ with } 2 \geq a_1 \geq a_2 \geq 0.$$

So we get the following six cycles:

$$\begin{aligned} \Sigma_0(\mathcal{V}) =& \{\Lambda \in G(2,4) \mid \dim(H_o \cap \Lambda) \geq 1, \dim(V \cap \Lambda) \geq 2\} = G(2,4), \\ \Sigma_1(\mathcal{V}) =& \{\Lambda \in G(2,4) \mid \dim(L_0 \cap \Lambda) \geq 1, \dim(V \cap \Lambda) \geq 2\} = \{\Lambda \in G(2,4) \mid L_o \cap \Lambda \neq \{0\}\}, \\ \Sigma_{1,1}(\mathcal{V}) =& \{\Lambda \in G(2,4) \mid \dim(L_0 \cap \Lambda) \geq 1, \dim(H_o \cap \Lambda) \geq 2\} = \{L_o \cap \Lambda \neq \{0\}, \Lambda \subset H_o\} = \{\Lambda \in G(2,4) \mid \Lambda \subset H_o\}, \\ \Sigma_2(\mathcal{V}) =& \{\Lambda \in G(2,4) \mid \dim(P_o \cap \Lambda) \geq 1, \dim(V \cap \Lambda) \geq 2\} = \{\Lambda \in G(2,4) \mid P_o \subset \Lambda\}, \\ \Sigma_{2,1}(\mathcal{V}) =& \{\Lambda \in G(2,4) \mid \dim(P_o \cap \Lambda) \geq 1, \dim(H_o \cap \Lambda) \geq 2\} = \{\Lambda \in G(2,4) \mid P_o \subset \Lambda \subset H_o\}, \\ \Sigma_{2,2}(\mathcal{V}) =& \{\Lambda \in G(2,4) \mid \dim(P_o \cap \Lambda) \geq 1, \dim(L_o \cap \Lambda) \geq 2\} = \{P_o \subset \Lambda, L_o = \Lambda\} = \{\Lambda \in G(2,4) \mid L_o = \Lambda\}. \end{aligned}$$

Now the corresponding Schubert varieties in $\mathbb{G}(1,3)$ with respect to the corresponding flag

$$p_0 \subset \ell_0 \subset \Pi_0 \subset \mathbb{P}^3,$$

where p_0 is point in $\mathbb{P}^3$, ℓ_0 a line and Π_0 a plane, are just the corresponding projectivations of the previous sets. We get the following varieties

$$\begin{aligned}\Sigma_0(\mathcal{V}) =&\mathbb{G}(1,3),\\ \Sigma_1(\mathcal{V}) =&\{\ell|\ \ell_0\cap\ell\neq\emptyset\},\\ \Sigma_{1,1}(\mathcal{V}) =&\{\ell|\,\ell_0\cap\ell\neq\emptyset,\ \ell\subset\Pi_0\}=\{\ell|\ \ell\subset\Pi_0\},\\ \Sigma_2(\mathcal{V}) =&\{\ell|\ p_0\in\ell\},\\ \Sigma_{2,1}(\mathcal{V}) =&\{\ell|\ p_0\in\ell\subset\Pi_0\},\\ \Sigma_{2,2}(\mathcal{V}) =&\{\ell|\,p_0\in\ell,\ \ell=\ell_0\}=\{\ell|\ \ell=\ell_0\}.\end{aligned}$$

In order to show that all the above varieties are irreducible and determine their dimension, we use the following theorem

Theorem 4.2.

(1) *Let $f: X\to Y$ be an epimorphism of projective varieties of dimensions n and m respectively ($n\geq m$). If Y is irreducible and all fibers of f are irreducible, of the same dimension $n-m$, then X is irreducible.*
(2) *$f: X\to Y$ be an epimorphism of projective varieties. If X is irreducible then Y is irreducible.*

We have:

- $\Sigma_0(\mathcal{V})=\mathbb{G}(1,3)$ is irreducible of dimension 4.
- $\Sigma_2(\mathcal{V})=\{\ell|\ p_0\in\ell\}$ is the set of lines in $\mathbb{P}^3$ containing the point p and thus $\cong\mathbb{P}^2$, that is, irreducible of dimension 2. Indeed, if we fix a plane Π in $\mathbb{P}^3$ not containing the point p, then the points of Π are in one to one correspondence with the lines in $\mathbb{P}^3$ going through p: given a point $q\in\Pi$ it determines the line $\overline{qp}$ going through p and, vice versa, a line going through p intersects Π at a unique point (since $p\notin\Pi$, the line is not contained in Π).
- $\Sigma_1(\mathcal{V})=\{\ell|\ \ell_0\cap\ell\neq\emptyset\}$: we consider

$$\Gamma=\{(p,\ell)\in\ell_0\times\mathbb{G}(1,3),\ p\in\ell\}.$$

and let

$$\begin{tikzcd}\Gamma \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathbb{G}(1,3)\\ \ell_0 & \end{tikzcd}$$

the two projections. The image of Γ by the projection pr_2 is exactly the $\Sigma_1(\mathcal{V})$. Now, the fibers of π_1 are all isomorphic to $\mathbb{P}^2$. Indeed, given a point $p\in\ell_0$, the fiber over p is the set of lines in $\mathbb{P}^3$ going through p and thus $\cong\mathbb{P}^2$ (equals the variety $\Sigma_2(\mathcal{V})$ examined before). By the Theorem 4.2 the variety Γ is irreducible of dimension $1+2=3$. Hence its image $\Sigma_1(\mathcal{V})$ by the projection π_2 is irreducible. To find the dimension of $\Sigma_1(\mathcal{V})$, the projection π_2 maps Γ birationally to its image $\Sigma_1(\mathcal{V})$. Indeed, over a point $\ell\in\Sigma_1(\mathcal{V})$, with $\ell\neq\ell_0$, the fiber of π_2 is the pair (p,ℓ) with p the unique point of intersection of ℓ with ℓ_0. Over the point $\ell_0\in\Sigma_1(\mathcal{V})$, the fiber is the set $\{(p,\ell_0),\ p\in\ell_0\}\cong\mathbb{P}^1$. Since irreducible birational varieties have the same dimension, we conclude that the dimension of $\Sigma_1(\mathcal{V})$ is 3.
- $\Sigma_{1,1}(\mathcal{V})=\{\ell|\ \ell\subset\Pi_0\}\cong\mathbb{G}(1,2)\cong\mathbb{P}^2$ and thus irreducible of dimension 2.
- $\Sigma_{2,1}(\mathcal{V})=\{\ell|\ p_0\in\ell\subset\Pi_0\}$ is the set of lines in the plane $\Pi_0\cong\mathbb{P}^2$ containing the point p. With a similar argument as in the case of the cycle $\Sigma_2(\mathcal{V})$, we get that $\Sigma_{2,1}(\mathcal{V})$ is isomorphic to $\mathbb{P}^1$ and thus it is irreducible of dimension 1.
- $\Sigma_{2,2}(\mathcal{V})=\{\ell_0\}$ is just a point, so irreducible of dimension 0.

Remark 4.3. In the above example note that $\mathrm{codim}\Sigma_{a_1,a_2}(\mathcal{V})=a_1+a_2$. This holds in general in $G(2,n+1)$.

Remark 4.4. If we order the pairs (a_1,a_2) by the partial ordering $(b_1,b_2)\succeq(a_1,a_2)$ given by $b_i\geq a_i$, $i=1,2$, then $\Sigma_{b_1,b_2}(\mathcal{V})\subset\Sigma_{a_1,a_2}(\mathcal{V})$ if and only if $(b_1,b_2)\succeq(a_1,a_2)$. For example, in the case of $\mathbb{G}(1,3)$ this partial ordering has two totally ordered chains

$$\Sigma_{2,2}(\mathcal{V})\subset\Sigma_{2,1}(\mathcal{V})\subset\Sigma_{1,1}(\mathcal{V})\subset\Sigma_1(\mathcal{V})\subset\Sigma_0(\mathcal{V})=\mathbb{G}(1,3),$$
$$\Sigma_{2,2}(\mathcal{V})\subset\Sigma_{2,1}(\mathcal{V})\subset\Sigma_2(\mathcal{V})\subset\Sigma_1(\mathcal{V})\subset\Sigma_0(\mathcal{V})=\mathbb{G}(1,3)$$

Now we give an affine stratification of $G(2, n+1)$. For each pair of integers $\alpha = (a_1, a_2)$ with $n-1 \geq a_1 \geq a_2 \geq 0$, we define Σ^o_α to be the complement in Σ_α of the Schubert varieties properly contained in Σ_α. According to Remark 4.4, we have

$$\Sigma^o_\alpha = \Sigma_\alpha \backslash \cup_{\beta \succeq \alpha} \Sigma_\beta.$$

Example 4.5. In the case of $\mathbb{G}(1,3)$ we have:

$\Sigma_0(\mathcal{V})^o = \mathbb{G}(1,3) \backslash \Sigma_1(\mathcal{V}) = \{\ell | \ \ell_0 \cap \ell = \emptyset\}$.
$\Sigma_1(\mathcal{V})^o = \Sigma_1(\mathcal{V}) \backslash (\Sigma_2(\mathcal{V}) \cup \Sigma_{1,1}(\mathcal{V}) = \{\ell | \ \ell_0 \cap \ell \neq \emptyset, \, p_0 \notin \ell \not\subset \Pi_0\}$.
$\Sigma_{1,1}(\mathcal{V})^o = \Sigma_{1,1}(\mathcal{V}) \backslash \Sigma_{2,1}(\mathcal{V}) = \{\ell | \, p_0 \notin \ell \subset \Pi_0\} \mathbb{P}^2 \backslash \mathbb{P}^1 \cong \mathbb{C}^2$.
$\Sigma_2(\mathcal{V})^o = \Sigma_2(\mathcal{V}) \backslash \Sigma_{2,1}(\mathcal{V}) = \{\ell | \ p_0 \in \ell \not\subset \Pi_0\} \cong \mathbb{P}^2 \backslash \mathbb{P}^1 \cong \mathbb{C}^2$.
$\Sigma_{2,1}(\mathcal{V})^o = \Sigma_{2,1}(\mathcal{V}) \backslash \Sigma_{2,2}(\mathcal{V}) = \{\ell | \ p_0 \in \ell \subset \Pi_0, \, \ell \neq \ell_0\} \cong \mathbb{P}^1 \backslash p_0 \cong \mathbb{C}$.
$\Sigma_{2,2}(\mathcal{V})^o = \Sigma_{2,2} \cong \mathbb{C}^0$.

As we have remarked above the last four cases are obviously isomorphic to affine varieties. We now examine the first two cases:

- $\Sigma_0(\mathcal{V})^o = \{\ell | \ \ell_0 \cap \ell = \emptyset\} \cong \mathbb{C}^2 \times \mathbb{C}^2 = \mathbb{C}^4$: indeed, fix two different plane Π_1, Π_2 containing the line ℓ_0. We then have an one to one correspondence between the points of $\Sigma_0(\mathcal{V})^o$ and the product $(\Pi_1 \backslash \ell_o) \times (\Pi_2 \backslash \ell_0) \cong \mathbb{C}^2 \times \mathbb{C}^2$. We have $\Pi_1 \cap \Pi_2 = \ell_0$. A line $\ell \in \Sigma_0(\mathcal{V})^o$ does not belong to either plane, otherwise it will intersect ℓ_0, which is not the case. Therefore it intersect each plane at exactly one point outside ℓ_0 and thus we get a pair of points in $(\Pi_1 \backslash \ell_o) \times (\Pi_2 \backslash \ell_0)$. Conversely, such a pair of points determine a line which does not intersects ℓ_0.
- $\Sigma_1(\mathcal{V})^o = \{\ell | \ \ell_0 \cap \ell \neq \emptyset, \, p_0 \notin \ell \not\subset \Pi_0\} \cong \mathbb{C} \times \mathbb{C}^2 = \mathbb{C}^3$: take a plane $\Pi \neq \Pi_0$ which contains p_0 but $\ell_0 \not\subset \Pi$. Let $\ell_1 = \Pi_0 \cap \Pi$ the line of intersection. The lines ℓ_0 and ℓ_1 intersect at p_0 and they span the plane Π_0. We then have an one to one correspondence between the points of $\Sigma_1(\mathcal{V})^o$ and the product $(\ell_0 \backslash p_0) \times (\Pi \backslash \ell) \cong \mathbb{C} \times \mathbb{C}^2$. Indeed, a line spanned by a point in $\ell_0 \backslash p_0$ and a point of $\Pi \backslash \ell$ intersects ℓ_0 at a point $\neq p_0$ and also since intersects Π outside the line ℓ it does not belong to Π_0. Hence it is a line in $\Sigma_1(\mathcal{V})^o$. And, vice versa, a line in $\Sigma_1(\mathcal{V})^o$ determines a pair of points in $(\ell_0 \backslash p_0) \times (\Pi \backslash \ell) \cong \mathbb{C} \times \mathbb{C}^2$.

Thus the above sets form an affine stratification of $\mathbb{G}(1,3)$ and by Theorem 4.2 the classes of their closures, i.e. the classes determined by the Schubert varieties, generate the Chow group $A(\mathbb{G}(1,3))$. In particular, we have:

$A^0(\mathbb{G}(1,3))$ is generated by the class of $\Sigma_0(\mathcal{V}) = \mathbb{G}(1,3)$.
$A^1(\mathbb{G}(1,3))$ is generated by the class of $\Sigma_1(\mathcal{V})$.
$A^2(\mathbb{G}(1,3))$ is generated by the classes of $\Sigma_2(\mathcal{V})$ and $\Sigma_{1,1}(\mathcal{V})$.
$A^3(\mathbb{G}(1,3))$ is generated by the class of $\Sigma_{2,1}(\mathcal{V})$.
$A^4(\mathbb{G}(1,3))$ is generated by the class of $\Sigma_{2,2}(\mathcal{V})$ = a point in $\mathbb{G}(1,3)$.

For example, the last relation implies that any two point in the Grassmannian are rationally equivalent.

The calculations of the above example generalize to the case of the Grassmannian $G(2, n+1)$ and we have

Proposition 4.6. *The Grassmannian $G(2, n+1)$ has an affine stratification given by*

$$\Sigma^o_\alpha(\mathcal{V}) = \Sigma_\alpha(\mathcal{V}) \backslash \cup_{\beta \succeq \alpha} \Sigma_\beta(\mathcal{V}).$$

The classes of their closures, i.e. the classes determined by the Schubert varieties, generate the Chow group $A(\mathbb{G}(2, n+1))$. In particular, $A^k(\mathbb{G}(2, n+1))$ is generated by the classes of the Schubert varieties $\Sigma_\alpha(\mathcal{V})$ with $|\alpha| = k$. As it turns our these classes do not depend on the choice of the complete flag $\mathcal{V}$. This is due to a theorem by Kleiman, see [2] Theorem 1.7, which in a special form states that

Theorem 4.7 (Kleiman)**.** *Two subvarieties of the Grassmannian $\mathbb{G}(1,n)$ which they differ by a transformation of the linear projective group* $\mathrm{PGL}(n)$ *have the same classes in the Chow group (note that the action of* $\mathrm{PGL}(n)$ *on $\mathbb{P}^n$ induces an action on $\mathbb{G}(1,n)$).*

Two complete flags in $\mathbb{P}^n$ differ by a transformation of the linear projective group $\mathrm{PGL}(n)$ and then one can see that the corresponding Schubert varieties (as subvarieties on $\mathbb{G}(1,n)$) differ by the induced transformation on $\mathbb{G}(1,n)$.

Notation 4.8. We denote by σ_α the class in the Chow group of the Schubert variety $\Sigma_\alpha(\mathcal{V})$. These are called the *Schubert classes.*

5. Intersections in complementary dimension and Pieri's formula

To give a description of the Chow ring of $G(2, n+1)$ we start by evaluating intersections of Schubert varieties $\Sigma_\alpha(\mathcal{V})$, $\Sigma_\beta(\mathcal{W})$ defined in terms of flags $\mathcal{V}$,$\mathcal{W}$ which intersect transversely. This means that

$$V_i \cap W_{n+1-i} = \{0\}, \text{ for each } i = 1, \dots, n+1.$$

We then have

Proposition 5.1. *If $\mathcal{V}$ and $\mathcal{W}$ are transverse flags and $\Sigma_\alpha(\mathcal{V})$, $\Sigma_\beta(\mathcal{W})$ are Schubert cycles with $|\alpha|+|\beta| = 2(n-1)$, then $\Sigma_\alpha(\mathcal{V})$ and $\Sigma_\beta(\mathcal{W})$ intersect transversely in a unique point if $a_1 + b_2 = a_2 + b_1 = n-1$ and are disjoint otherwise. Thus*

$$\deg\ \sigma_\alpha\sigma_\beta = \begin{cases} 1 & \textit{if } a_1 + b_2 = a_2 + b_1 = n-1, \\ 0 & \textit{otherwise.} \end{cases}$$

Remark 5.2. A note on the notation used in the above Proposition. As it is stated before, the Chow group $A^{2(n-1)}(\mathbb{G}(1,n))$ is generated by the class of a (any) point. Hence an element of this is a multiply of this class, which is called the degree (deg) of the class.

We now state Pieri's formula which allows us to give a closed-form expression for the product of any two Schubert classes in terms the generators of the Chow ring of the appropriate codimension.

Proposition 5.3 ([2], Proposition 4.9 or [3] pg 203)**.** *With the above notation, assuming that $a_1 - a_2 \geq b_1 - b_2$ then*

$$\begin{aligned} \sigma_{a_1,a_2}\sigma_{b_1,b_2} &= \sigma_{a_1+b_1,a_2+b_2} + \sigma_{a_1+b_1-1,a_2+b_2+1} + \cdots + \sigma_{a_1+b_2,a_2+b_1} \\ &= \sum_{\substack{|\gamma|=|\alpha|+|\beta| \\ a_1+b_2 \leq c_1 \leq a_1+b_1}} \sigma_{c_1,c_2}. \end{aligned}$$

Remark 5.4. Note that the above sum contains $b_1 - b_2 + 1$ terms. For example:

If $b_1 = b_2 = b$ then $\sigma_{a_1,a_2}\sigma_{b,b} = \sigma_{a_1+b,a_2+b}$.
If $b_2 = 0$ then $\sigma_{a_1,a_2}\sigma_b = \sigma_{a_1+b,a_2} + \sigma_{a_1+b-1,a_2+1} + \cdots + \sigma_{a_1,a_2+b}$.

The relation in the following Corollary, which is a consequence of Pieri's formula has an important interpretation in terms of Chern classes that will explain on the next Chapter.

Corollary 5.5. *In $A(G(2, n+1))$ we have*

$$(1 + \sigma_1 + \ldots + \sigma_{n-1})(1 - \sigma_1 + \sigma_{1,1}) = 1.$$

Proof. By Pieri's formula we have

$$\begin{aligned} \sigma_i\sigma_1 &= \sigma_{i,1} + \sigma_{i+1}, \\ \sigma_i\sigma_{1,1} &= \sigma_{i+1,1} \end{aligned}$$

By plugging in these relations the product on the left hand side of the formula becomes a telescopic sum and the only remaining term is the 1. ☐

In the following proposition we verify (from scratch!) the Pieri's formulas for the products of Schubert classes in the Chow ring of $\mathbb{G}(1,3)$. We have

Proposition 5.6. *In $A(\mathbb{G}(1,3))$, Pieri's formulas give the following relations:*

(1) $\sigma_1^2 = \sigma_{1,1} + \sigma_2$.
(2) $\sigma_1\sigma_{1,1} = \sigma_1\sigma_2 = \sigma_{2,1}$.
(3) $\sigma_1\sigma_{2,1} = \sigma_{1,1}^2 = \sigma_2^2 = \sigma_{2,2} = 1$, $\sigma_{1,1}\sigma_2 = 0$.

Proof. (1) σ_1 is the class of the subvariety of lines intersecting a given line. Since the class does not depend on the choice of the line ℓ_0, σ_1^2 can be realized as follows: we choose two intersecting lines ℓ_0 and ℓ_1 at a point p_0. Then σ_1 is the class of the subvariety of lines intersecting ℓ_0 but also σ_1 is the class of the subvariety of lines intersecting ℓ_1. Then such an ℓ either belongs in the plane spanned by ℓ_0 and ℓ_1 or it is a line of $\mathbb{P}^3$ which passes through the point p_0. The former it is a locus of class $\sigma_{1,1}$ and the latter of class σ_2. Thus $\sigma_1^2 = \sigma_{1,1} + \sigma_2$.
(2) We show first that $\sigma_1\sigma_{1,1} = \sigma_{2,1}$. We choose a 2-plane ℓ_0 and a point $p_0 \notin \ell_0$. Let Π_0 be the plane spanned by the point and the line. We have that σ_1 is the class of the subvariety of lines intersecting a given line ℓ_0 and $\sigma_{1,1}$ is the class of the subvariety of lines belonging in Π_0. Then their intersection is the subvariety of lines belonging in Π_0 and going through the point p_0 and the class of this equals $\sigma_{2,1}$.

We now show that $\sigma_1\sigma_2 = \sigma_{2,1}$. With the previous choices, σ_2 is the class of the subvariety of lines going through p_0. Then their intersection is the same as before.
(3) To show that $\sigma_1\sigma_{2,1} = 1$, choose a plane Π_0, a point $p_0 \in \Pi_0$ and a line $\ell_1 \not\subset \Pi_0$. Let p_1 be the point of intersection of ℓ_1 with Π_0. Then σ_1 is the class of the subvariety of lines intersecting the line ℓ_1 and $\sigma_{1,2}$ is the class of the subvariety of lines belonging in Π_0 and going through p_0. Their intersection is the line $\overline{p_0p_1}$. To show that $\sigma_{1,1}^2 = 1$, take two different planes Π_0 and Π_1 which intersect at a line ℓ_0. Then $\sigma_{1,1}$ is the class of the subvariety of lines which belong in Π_0 but also is the class of the subvariety of lines which belong in Π_1. Their intersection is then the line ℓ_0. To show that $\sigma_2^2 = 1$, we choose two different points p_0 and p_1. Then σ_2 is the class of the subvariety of lines which are going through p_0 but also is the class of the subvariety of lines which are going through p_1. Their intersection is then the single line $\overline{p_0p_1}$. Finally, to show that $\sigma_{1,1}\sigma_2 = 0$, we choose a plane Π_0 and a point p_0 outside the plane Π_0. Then $\sigma_{1,1}$ is the class of the subvariety of lines which belong in Π_0 and σ_2 is the class of the subvariety of lines which are going through p_0. Since $p_0 \notin \Pi_0$ the intersection is empty. $\square$

Remark 5.7. Combining the above relations, we can come up with some well known propositions in projective geometry. For example, by the above we have that $\sigma_1^2\sigma_2 = (\sigma_{1,1} + \sigma_2)\sigma_2 = \sigma_{1,1}\sigma_2 + \sigma_2^2 = 0 + 1 = 1$ which, for example, corresponds to the well known: from a point outside from two skew lines in $\mathbb{P}^3$, there is a unique bisecant to the two lines.

CHAPTER 3

Chern Classes

As we have seen in the previous chapter, the basic classes in the Chow group of the Grassmannian were given by the Schubert varieties which where defined using geometric properties of the projective space. But in many occasions, given a projective variety, we can construct some interesting classes in the Chow group which arise in a different way, namely, they come from vector bundles on the variety. The most interesting of those are the so called degeneracy loci: roughly, given a map $\phi : \mathcal{E} \to \mathcal{F}$ of vector bundles on a variety X (we can think of that as family of linear maps parametrized by the points of X) the degeneracy locus is the locus of points of X where the corresponding linear map fails to be of maximal rank. A special case of this is the zero locus of a holomorphic section of a vector bundle. These loci arise very often in problems and the question is whether f we can calculate their classes in the Chow ring of X in terms of data related to the vector bundles $\mathcal{E}$ and $\mathcal{F}$. These data are the Chern classes of a vector bundle.

1. The first Chern class of a line bundle

We start by defining the first Chern class of a line bundle and then build to a general vector bundle and stating various computing formulas. Given a line bundle $\mathcal{L}$ on a smooth projective variety X, a non-trivial theorem asserts that it possesses a non-zero meromorphic section σ. This is given by a collection of meromorphic functions $\sigma = \{s_a\}$ on an open covering U_a of X, where $\mathcal{L}$ can be trivialized. That is, on each U_a we have isomorphisms $f_a : \mathcal{L}|_{U_a} \to U_a \times \mathbb{C}$ with $g_{ab} = f_a \circ f_b^{-1} : U_a \cap U_b \to \mathbb{C}^*$ the holomorphic transition functions. Here we view $g_{ab}(z)$ as the linear isomorphism $\mathbb{C}_z \to \mathbb{C}_z$ (and thus a non zero constant) induced by the restriction of $f_a \circ f_b^{-1}$ on the fiber $\mathbb{C}_z$.

By definition, the s_a's satisfy the relation $s_a = g_{ab} s_b$ on $U_a \cap U_b$. Since the g_{ab}'s are holomorphic and nowhere vanishing, this implies that s_a and s_b have the same zeros and poles on $U_a \cap U_b$. Therefore to a meromorphic section $\sigma = \{s_a\}$ we can assign the divisor (σ) of its zeros and poles defined as the union of zeros and poles on U_a's of the meromorphic functions s_a. It will be

$$(\sigma) = \sum_i \mathrm{ord}_{V_i} V_i,$$

where V_i are the codimension one irreducible subvarieties of V where s vanishes or has a pole and ord_{V_i} is the multiplicity of vanishing or of a pole of s on V_i (a positive integer for zeros and a negative for poles).

We now observe that if we choose any other non-zero meromorphic section σ', then (σ) and (σ') are rationally equivalent, i.e. they define the same class in the Chow group of X. Indeed, the quotients s_a/s'_a on the U_a's are glued together to give a meromorphic function on X which makes (σ) and (σ') rationally equivalent, see Proposition 3.5. Indeed, since $s_a = g_{ab} s_b$ and $s'_a = g_{ab} s_b$ then $s_a/s'_a = s_b/s'_b$ on $U_a \cap U_b$.

Definition 1.1. Given a line bundle $\mathcal{L}$ on a smooth projective variety X, we define the first Chern class

$$c_1(\mathcal{L}) \in A^1(X)$$

to be the rational equivalence class of the divisor (σ) of any non-zero rational section σ.

Since the trivial line bundle $\mathcal{O}_X := \mathbb{C} \times X$ has a non-zero constant section, we have

Corollary 1.2. *The trivial line bundle $\mathcal{O}_X$ on X has $c_1(\mathcal{O}_X) = 0$.*

As it turns out, a line bundle is determined up to isomorphism by its transition functions. The dual bundle $\mathcal{L}^\vee$ of a line bundle $\mathcal{L}$ with transition functions g_{ab} is the line bundle given by transition functions g_{ab}^{-1}. Given a meromorphic section $\sigma = \{s_a\}$ of the line bundle $\mathcal{L}$, it induces a meromorphic section $\sigma^\vee$ on the dual bundle $\mathcal{L}^\vee$, with $\sigma^\vee = \{s_a^{-1}\}$. We then have $(\sigma^\vee) = -(\sigma)$ and therefore,

Corollary 1.3. *The first Chern class of the dual $\mathcal{L}^\vee$ of a line bundle $\mathcal{L}$ is given by*

$$c_1(\mathcal{L}^\vee) = -c_1(\mathcal{L}).$$

The tensor product $\mathcal{L} \otimes \mathcal{L}'$ of two line bundles $\mathcal{L}$ and $\mathcal{L}'$ with transition functions g_{ab} and g'_{ab} respectively, is the line bundle given by the transition functions $g_{ab}\, g'_{ab}$. If $\sigma = \{s_a\}$ and $\sigma' = \{s'_a\}$ are meromorphic sections of $\mathcal{L}$ and $\mathcal{L}'$ respectively, then the $\sigma\,\sigma' = \{s_a s'_a\}$ is a meromorphic section of $\mathcal{L} \otimes \mathcal{L}'$. Since $(\sigma\,\sigma') = (\sigma) + (\sigma')$, we have:

Corollary 1.4. *The tensor product $\mathcal{L} \otimes \mathcal{L}'$ of two line bundles $\mathcal{L}$ and $\mathcal{L}'$ has first Chern class given by*

$$c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}').$$

2. Characterizing Chern classes

We now extend the definition of $c_1(\mathcal{L})$ to introduce Chern classes of higher rank vector bundles, which will give us higher codimension subvarieties.

Theorem 2.1 (Theorem 5.3 in [2]). *There is a unique way of assigning to each vector bundle $\mathcal{E}$ of rank r on a smooth projective variety X a class $c(\mathcal{E})$, called the Chern class of $\mathcal{E}$, which has the form*

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \cdots \in A(X),$$

with $c_i(\mathcal{E}) \in A^i(X)$, in such a way that:

(1) *(Line bundles) If $\mathcal{L}$ is a line bundle on X then the Chern class of $\mathcal{L}$ is*

$$c(\mathcal{L}) = 1 + c_1(\mathcal{L}),$$

where $c_1(\mathcal{L}) \in A^1(X)$ is the first Chern class of $\mathcal{L}$ defined in 1.1.

(2) *(Bundles with enough sections) If $\tau_0, \ldots \tau_{r-i}$ are holomorphic sections of $\mathcal{E}$ and the degeneracy locus D where they are dependent has codimension i in X, then*

$$c_i(\mathcal{E}) = [D] \in A^i(X).$$

(3) *(Whitney's formula) If*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

is a short exact sequence of vector bundles on X then

$$c(\mathcal{F}) = c(\mathcal{E})\, c(\mathcal{G}) \in A(X).$$

(4) *(Functoriality) If $\phi : Y \to X$ is a morphism of smooth varieties, then*

$$\phi^*(c(\mathcal{E})) = c(\phi^*(\mathcal{E})).$$

Corollary 2.2. *When $\mathcal{F} = \mathcal{E} \oplus \mathcal{G}$, Whitney's formula implies that*

$$c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G}).$$

In particular, when $\mathcal{E}$ is the trivial rank r bundle, i.e. $\mathcal{E} = \mathcal{O}_X^{\oplus r}$, then by the Corollary we get that $c(\mathcal{E}) = 1$. In other words, $c_i(\mathcal{O}_X^{\oplus r}) = 0$, for all $i \geq 1$.

Corollary 2.3. *Let $\mathcal{E}$ be a rank r bundle on X and τ a holomorphic section. If the zero locus $Z(\tau)$ of τ has codimension r in X then its class is given by*

$$[Z(\tau)] = c_r(\mathcal{E}).$$

Note when $\mathcal{E} = \mathcal{L}$ a line bundle and τ a holomorphic section, this reduces to Definition 1.1.

Given a rank r vector bundle $\mathcal{E}$ on X we can construct the so called projectivized bundle $\mathbb{P}(\mathcal{E})$. Roughly, the fiber over $x \in X$ is the projectivisation of the fiber of $\mathcal{E}$ over x. It turns out that $\mathbb{P}(\mathcal{E})$ is a projective variety and we have a natural map

$$\pi : \mathbb{P}(\mathcal{E}) \to X.$$

A point $p \in \mathbb{P}(\mathcal{E})$ in the fiber over $x \in X$ represents a line ℓ_p in the fiber $\mathcal{E}_x$ of $\mathcal{E}$ over x. As in the case of the Grasmmannian, we then have on $\mathbb{P}(\mathcal{E})$ a universal (tautological) line bundle $\mathcal{U}$ with fiber over p the line ℓ_p. It fits in a short exact sequence

$$0 \to \mathcal{U}_\mathcal{E} \to \pi^*\mathcal{E} \to \mathcal{Q}_\mathcal{E} \to 0,$$

with $\pi^*\mathcal{E}$ the pull back of the vector bundle $\mathcal{E}$ to $\mathbb{P}(\mathcal{E})$ by the map π and $\mathcal{Q}_\mathcal{E}$is the quotient bundle. We define

$$\zeta = c_1(\mathcal{U}_\mathcal{E}^\vee) \in A^1(\mathbb{P}(\mathcal{E})).$$

Remark 2.4. When $\mathcal{E} = \mathbb{C}^{r=n+1} \times X = \mathcal{O}^{\oplus(n+1)}$ the trivial bundle, then $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}^n$ and it turns out $\zeta = [H]$is the class of a hyperplane.

Theorem 2.5 (Theorem 5.9 in [2])**.** *Let $\mathcal{E}$ be a vector bundle of rank r on a smooth projective variety X and let $\pi : \mathbb{P}(\mathcal{E}) \to X$ be the projectivized vector bundle. Let ζ be the first Chern class of $\mathcal{U}_{\mathcal{E}}^{\vee}$ as above. Then*

(1) The pull-back map $\pi^* : A(X) \to A(\mathbb{P}(\mathcal{E}))$ is injective.
(2) The element $\zeta \in A(\mathbb{P}(\mathcal{E}))$ satisfies a unique monic polynomial relation of degree r with coefficients in $\pi^*(A(X))$, namely

$$\zeta^r + \pi^* c_1(\mathcal{E})\, \zeta^{r-1} + \cdots + \pi^* c_r(\mathcal{E}) = 0.$$

Since the above polynomial relation is unique, it gives an alternative and more geometric way to define the chern classes $c_i(\mathcal{E})$, $i = 1, \ldots r$ (as will shortly see, the chern classes $c_i(\mathcal{E})$, for $i > r$, vanish).

The following theorem is extremely useful for computing chern classes, see [2] Section 5.4 for a detailed discussion.

Theorem 2.6 (Splitting principle)**.** *Any identity among Chern classes of bundles that is true for bundles that are direct sums of line bundles is true in general.*

3. Computing Examples

We now present some examples where we use Whitney's formula in combination with the splitting principle in order to calculate Chern classes.

Example 3.1. If $\mathcal{E}$ is a vector bundle of rank r, then $c_i(\mathcal{E}) = 0$ for $i > r$: indeed, say $\mathcal{E}$ split as

$$\mathcal{E} = \bigoplus_{i=1}^{r} \mathcal{L}_i,$$

for some line bundles $\mathcal{L}_i$. Then , since $c(\mathcal{L}_i) = 1 + c_1(\mathcal{L}_i)$, Whitney's formula would imply that

$$c(\mathcal{E}) = \prod_{i=1}^{r} c(\mathcal{L}_i) = \prod_{i=1}^{r} (1 + c_1(\mathcal{L}_i)),$$

which has no terms of degree $> r$.

Example 3.2. If $\mathcal{E} = \bigoplus_{i=1}^{r} \mathcal{L}_i$ as above, then

$$c(\mathcal{E}^{\vee}) = \prod_{i=1}^{r} (1 + c_1(\mathcal{L}_i^{\vee})) = \prod_{i=1}^{r} (1 - c_1(\mathcal{L}_i)),$$

with $\mathcal{L}^{\vee}$ the dual bundle, see Corollary 1.3. Whitney's formula gives us then the basic identity

$$c_i(\mathcal{E}^{\vee}) = (-1)^i c_i(\mathcal{E})$$

By the splitting principle, this identity holds for any bundle.

Example 3.3. By the *determinant* $\det\mathcal{E}$ of a bundle $\mathcal{E}$ we mean the line bundle that is the highest exterior power of the bundle $\mathcal{E}$, namely $\det\mathcal{E} := \bigwedge^{rank\mathcal{E}} \mathcal{E}$. If $\mathcal{E}$ has transition matrices G_{ab} then $\det\mathcal{E}$ is given by the transition functions $g_{ab} = \det G_{ab}$. If we assume $\mathcal{E} = \bigoplus_{i=1}^{r} \mathcal{L}_i$, then $\det\mathcal{E} = \bigotimes_{i=1}^{r} \mathcal{L}_i$ and hence

$$c_1(\det\mathcal{E}) = \sum c_1(\mathcal{L}) = c_1(\mathcal{E}).$$

The splitting principle tells us this identity holds in general.

Example 3.4. Suppose that $\mathcal{E}$ is a bundle of rank 2. If $\mathcal{E}$ splits as a direct $\mathcal{E} = \mathcal{L} \bigoplus \mathcal{M}$ of line bundles $\mathcal{L}$ and $\mathcal{M}$ with Chern classes $c_1(\mathcal{L}) = \alpha$ and $c_1(\mathcal{M}) = \beta$ then, by Whitney's formula

$$c(\mathcal{E}) = (1 + \alpha)(1 + \beta)$$

and therefore

$$c_1(\mathcal{E}) = \alpha + \beta \text{ and } c_2(\mathcal{E}) = \alpha\beta.$$

Further, we would have

$$\mathrm{Sym}^2(\mathcal{E}) = \mathcal{L}^{\otimes 2} \oplus (\mathcal{L} \otimes \mathcal{M}) \oplus \mathcal{M}^{\otimes 2},$$

from which we would deduce, see Corollary 1.4,

$$\begin{aligned} c(\mathrm{Sym}^2\mathcal{E}) =& (1+2\alpha)(1+\alpha+\beta)(1+2\beta) \\ =& 1+2(\alpha+\beta)+(2\alpha^2+8\alpha\beta+2\beta^2)+4\alpha\beta(\alpha+\beta) \\ =& 1+2c_1(\mathcal{E})+(2c_1(\mathcal{E})^2+4c_2(\mathcal{E}))+4c_1(\mathcal{E})c_2(\mathcal{E}). \end{aligned}$$

By the splitting principle, this is a valid expression for $c(\mathrm{Sym}^2\mathcal{E})$, whether or not $\mathcal{E}$ actually splits. Therefore, with $\mathcal{E}$ a vector bundle of rank 2 we have:

$$\begin{aligned} c_1(\mathrm{Sym}^2\mathcal{E}) =& 2c_1(\mathcal{E}), \\ c_2(\mathrm{Sym}^2\mathcal{E}) =& 2c_1(\mathcal{E})^2+4c_2(\mathcal{E}), \\ c_3(\mathrm{Sym}^2\mathcal{E}) =& 4c_1(\mathcal{E})c_2(\mathcal{E}). \end{aligned}$$

Similarly,

$$\mathrm{Sym}^3\mathcal{E} = \mathcal{L}^{\otimes 3} \oplus (\mathcal{L}^2 \otimes \mathcal{M}) \oplus (\mathcal{L} \otimes \mathcal{M}^2) \oplus \mathcal{M}^{\otimes 3},$$

from which we would deduce

$$c(\mathrm{Sym}^3\mathcal{E}) = (1+3\alpha)(1+2\alpha+\beta)(1+\alpha+3\beta)(1+3\beta).$$

From this we can calculate as before all the Chern classes of $\mathrm{Sym}^3\mathcal{E}$ (a bundle of rank 4). For example, the top Chern class is given by

$$c_4(\mathrm{Sym}^3\mathcal{E}) = 9\alpha\beta(2(\alpha+\beta)^2+\alpha\beta). \tag{1}$$

Example 3.5. If $\mathcal{E}$ is a rank r vector bundle that we assume it splits as $\mathcal{E} = \bigoplus_{i=1}^r \mathcal{L}_i$, with $\mathcal{L}_i$ line bundles, and $\mathcal{L}$ a line bundle, then

$$\mathcal{E} \otimes \mathcal{L} = \oplus_{i=1}^r (\mathcal{L}_i \otimes \mathcal{L}).$$

Let $c_1(\mathcal{L}_i) = \alpha_i$ and $c_1(\mathcal{L}) = \alpha$. Then by Whitney's formula we have

$$c(\mathcal{E} \otimes \mathcal{L}) = (1+\alpha_1+\alpha)\cdots(1+\alpha_r+\alpha).$$

Therefore

$$c_1(\mathcal{E} \otimes \mathcal{L}) = a_1+\ldots+a_r+r\alpha = c_1(\mathcal{E})+rc_1(\mathcal{L}).$$

In general, the above implies that

$$c_k(\mathcal{E} \otimes \mathcal{L}) = \sum_{l=0}^{k} \binom{r-l}{k-l} c_1(\mathcal{L})^{k-l} c_l(\mathcal{E}) = \sum_{i=0}^{k} \binom{r-k+i}{i} c_1(\mathcal{L})^i c_{k-i}(\mathcal{E}).$$

Similarly, we can prove for example, that if $\mathcal{E}$, $\mathcal{F}$ are vector bundles of rank e and f respectively then

$$c_1(\mathcal{E} \otimes \mathcal{F}) = f\, c_1(\mathcal{E}) + e\, c_1(\mathcal{F}).$$

4. Chern classes of the Universal (tautological) and Quotient bundles

The projective space $\mathbb{P}^n$ can be identified with the Grassmannian $G(1, n+1)$ and as such it carries a universal (tautological) bundle $\mathcal{U}$ and a quotient bundle $\mathcal{Q}$ fitting in a short exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathbb{C}^{n+1} \times \mathbb{P}^n \longrightarrow \mathcal{Q} \longrightarrow 0.$$

As we have seen, $c_1(\mathcal{U}) = \zeta := -[H]$, with H a hyperplane. Also, by Corollary 2.2, we have that $c_1(\mathbb{C}^{n+1} \times \mathbb{P}^n) = 1$ and therefore, by Whitney's formula we have

$$c(\mathcal{Q}) = \frac{1}{1-\zeta} = 1+\zeta+\zeta^2+\cdots+\zeta^n.$$

We now examine the case of the Grassmannian $G(2, n+1)$. We have again a short exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathbb{C}^{n+1} \times G(2, n+1) \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0,$$

with $\mathcal{U}$ the universal bundle and $\mathcal{Q}$ the quotient bundle. We first compute the chern classes of the quotient bundle. Recall that for any integer a, with $0 \le a \le n-1$, and a complete flag $\mathcal{V} = \{V_i\}_{i=1,\ldots,n+1}$ with $\dim V_i = i$, we have defined the Schubert variety

$$\Sigma_a(\mathcal{V}) = \{\Lambda \in G(2, n+1) |\, V_{n-a} \cap \Lambda \neq \{0\}\}.$$

This is a subvariety of codimension a with σ_a its Schubert class in the Chow group.

We now fix a base $e_0, \ldots, e_n$ of $\mathbb{C}^{n+1}$ and let $\mathcal{V}$ be the complete flag formed by the $V_i = \mathrm{span}(e_0, \ldots, e_{i-1})$, $i = 1, \ldots, n+1$. The e_i's correspond to constant sections $\{e_i\} \times G(2, n+1)$, $i = 0, \ldots n$, of the trivial bundle

$\mathbb{C}^{n+1} \times G(2, n+1)$. Their images by π give rise to sections τ_i, $i = 0, \ldots, n$, of the quotient bundle $\mathcal{Q}$. Note that $\mathcal{Q}$ is a bundle of rank n. We now apply Theorem 2.1(2). We fix an i with $0 \leq i \leq n$. Then $\tau_0, \ldots, \tau_{n-i}$ are linearly dependent on the fiber $\mathbb{C}^{n+1}/\Lambda$ of $\mathcal{Q}$ over a point $\Lambda \in G(2, n+1)$ if and only if there is a non trivial linear combination of the e_i's, $i = 0, \ldots, n-i$, which belongs in Λ. This is equivalent to saying that $V_i \cap \Lambda \neq \{0\}$. Therefore the degeneracy locus is exactly the variety $\Sigma_a(\mathcal{V})$ and hence by the above mentioned Theorem we have

$$c_i(\mathcal{Q}) = \sigma_i \text{ and so, } c(\mathcal{Q}) = 1 + \sigma_1 + \cdots + \sigma_{n-1}.$$

To calculate now the Chern class of the universal bundle $\mathcal{U}$ we apply Whitney's relation in the above exact sequence and we have $c(\mathcal{Q})\, c(\mathcal{U}) = 1$. Recall now the formula given in Corollary 5.5

$$(1 \ + \ \sigma_1 \ + \ldots \ + \ \sigma_{n-1})(1 \ - \ \sigma_1 \ + \sigma_{1,1}) = 1.$$

We then get

$$c(\mathcal{U}) = 1 \ - \ \sigma_1 \ + \sigma_{1,1}.$$

CHAPTER 4

Lines on Hypersurfaces

1. Hypersurfaces in $\mathbb{C}^n$ and in $\mathbb{P}^n$

Let $X \subset \mathbb{C}^n$ be a hypersurface in $\mathbb{C}^n$ defined by the degree d polynomial $f(x_1, \ldots, x_n)$. Two hypersurfaces are considered to be the same if and only if their defining polynomials differ by a multiplicative constant. We denote by $\mathbb{V}(f)$ the zero locus of f. A point $p \in X$ is a smooth point of X if $\frac{\partial f}{\partial x_i}(p) \neq 0$ for some i. The hypersurface X is smooth if it is smooth at every point $p \in X$. The singular locus of X is the set of non-smooth points of X. Since it is defined by the vanishing of f and all the first order partial derivatives, it is a closed subvariety of X. The smooth locus of X is the complement of its singular locus and thus is a Zariski open of X. When $p = (a_1, \ldots, a_n)$ is a smooth point of a hypersurface X, the tangent space T_pX of X at p is given by

$$T_pX = \mathbb{V}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - a_i)\right) \subset \mathbb{C}^n.$$

The Taylor expansion of f at $p = (a_1, \ldots, a_n)$ is given by

$$f = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - a_i) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i x_j}(p)(x_i - a_i)(x_j - a_j) + \cdots$$

Then the point $p \in X$ is smooth if and only if the first non-zero term of the Taylor expansion of f at p is the linear term.

In homogeneous terms we have the following: Let X be a hypersurface in $\mathbb{P}^n$ defined by the degree d homogeneous polynomial $F(x_0, \ldots, x_n)$. Then X is called a hypersurface of degree d. The property for a point p to be a smooth point of X is local and is then defined by the standard decomposition of $\mathbb{P}^n = \cup_{i=0}^n U_i$ into affine sets. For example, if $p \in U_0$ then we may write $p = [1, a_1, \ldots, a_n]$ and $X \cap U_0$ is given by the equation $f = 0$ with $f(x_1, \ldots, x_n) = F(1, x_1, \ldots, x_n)$. Then p is a smooth point of X if and only if $\frac{\partial f}{\partial x_i}(a_1, \ldots, a_n) \neq 0$ for some $i = 1, \ldots, n$. This is a well defined notion, i.e. independent from the choice of U_i. Note that this is equivalent to saying that $\frac{\partial F}{\partial x_i}(1, a_1, \ldots, a_n) \neq 0$ for some $i = 0, \ldots, n$. Indeed, if $i \geq 1$ then $\frac{\partial F}{\partial x_i}(1, a_1, \ldots, a_n) = \frac{\partial f}{\partial x_i}(a_1, \ldots, a_n)$. If $i = 0$ then by the Euler's relation $dF(x_0, \ldots, x_n) = \sum_{i=0}^n x_i \frac{\partial F}{\partial x_i}(x_0, \ldots, x_n)$, we have

$$\frac{\partial F}{\partial x_0}(1, a_1, \ldots, a_n) + \sum_{i=1}^{n} a_i \frac{\partial F}{\partial x_i}(1, a_1, \ldots, a_n) = 0.$$

Hence if $\frac{\partial F}{\partial x_0}(1, a_1, \ldots, a_n) \neq 0$ then there is an $i = 1, \ldots, n$ with $\frac{\partial F}{\partial x_i}(1, a_1, \ldots, a_n) \neq 0$. Moreover, when p is a smooth point of X, the tangent hyperplane at a point $p = [a_0, \ldots, a_n]$ is given by

$$T_p(X) = \mathbb{V}(\sum_{i=0}^{n} \frac{\partial F}{\partial x_i}(p)\, x_i).$$

Indeed, if $p \in U_0$ we may write $p = [1, a_1, \ldots, a_n]$ and then in $U_0 \cong \mathbb{C}^n$ we have $T_pX = \mathbb{V}\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)(x_i - a_i)\right)$. By the Euler's relation we have that $\sum_{i=0}^n a_i \frac{\partial F}{\partial x_i}(p) = 0$, with $a_0 = 1$, hence $-\sum_{i=1}^n a_i \frac{\partial F}{\partial x_i}(p) = \frac{\partial F}{\partial x_0}(p)$. This give the result.

By the above discussion we also have the following: p is a singular point of X if and only if $\frac{\partial F}{\partial x_i}(p) = 0$, for every $i = 0, \ldots n$. Note that we don't have to include the relation $F(p) = 0$ since this is implied by the Euler's relation.

2. Lines through a point contained in a hypersurface

Suppose now that a hypersurface $X \subset \mathbb{C}^n$ defined by the polynomial $f(x_1, \ldots, x_n)$ of degree d contains a line ℓ which is not contained in the singular locus of X. Then the points of ℓ where X is singular, as a proper closed subset of the line, is a finite set. Let p be a smooth point of ℓ. We may assume, after performing a translation, that p is the origin. Then, in parametric form, the line ℓ is given by

$$\ell = \{t(a_1, \ldots, a_n),\ t \in \mathbb{C}\},$$

for some direction vector $(a_1, \ldots, a_n)$. We now write the Taylor expansion of f at the origin as

$$f = f_1 + f_2 + \ldots + f_d,$$

with f_i homogeneous of degree i. By restricting on the points of ℓ, we have $f(ta_0, \ldots, ta_n) = 0$, for every $t \in \mathbb{C}$, which gives that

$$tf_1(a_1, \ldots, a_n) + t^2 f_2(a_1, \ldots, a_n) + \ldots + t^d f_d(a_1, \ldots, a_n) = 0,\ \forall t \in \mathbb{C}.$$

Therefore, a line ℓ with direction vector $(a_1, \ldots, a_n)$ is contained in X if and only if

$$f_i(a_1, \ldots, a_n) = 0,\ \text{for all } i = 1, \ldots, d.$$

Since two lines in $\mathbb{C}^n$ are the same if and only if their direction vectors define the same point in $\mathbb{P}^{n-1}$ we have that the locus of lines contained in X and pass through the origin is given by the subvariety of $\mathbb{P}^{n-1}$ defined by the homogeneous equations $f_i = 0$, $i = 1, \ldots, d$. In particular, since $f_1 = 0$ gives the equation of the tangent hyperplane to X at the origin and since $f_1(ta_1, \ldots, ta_n) = 0$, for all $t \in \mathbb{C}$, we get that

Lemma 2.1. *Suppose ℓ is a line contained in a hypersurface X. Then the tangent hyperplane T_pX to X at a smooth point $p \in \ell$ contains the line ℓ.*

The above can also be seen by interpreting the tangent space of X at p as the space of tangent vectors at p of all smooth arcs going through that point. Since a line which is contained in X and pass through p defines such an arc then its tangent vector, which is supported on the line belongs to T_pX and thus the line itself is contained in T_pX.

By performing a rotation, we may further assume that the tangent hyperplane to X at the origin is given by the equation $x_n = 0$. Then the locus of lines in X passing through the origin is a subvariety of $\mathbb{V}(x_n) \cong \mathbb{P}^{n-2}$ defined by the homogeneous equations

$$f_i(x_1, \ldots, x_{n-1}, 0) = 0,\ \text{for all } i = 2, \ldots, d.$$

In other words, the above locus corresponds to the intersection of $d-1$ hypersurfaces in $\mathbb{P}^{n-2}$.

We recall the following Theorem, based on Krull's principal ideal theorem, see [2], Theorem 0.1:

Theorem 2.2. *Given hypersurfaces X_i in $\mathbb{P}^m$, $i = 1, \ldots, r$ with $r \leq m$, defined by the homogeneous polynomials F_i then their intersection has codimension $\leq r$. In particular their intersection is non empty. Moreover, if the F_i's are "independent" (the ideal $\langle F_1, \ldots, F_r \rangle$ is a complete intersection ideal) then the above codimension equals to r.*

Remark 2.3. Indeed, we start with the hypersurface X_1 which has codimension 1 and then we intersect with the hypersurface X_2. Since the zeros of the polynomial F_2 on the components of X_1 are of codimension at most 1 (either the F_2 vanishes on the component or its zero locus is of codimension 1), we get that $X_1 \cap X_2$ is of codimension at most 2 and so on. That the polynomials are general means that any time we intersect the codimension increases exactly by one.

The above discussion, combined with Theorem 2.2, implies

Proposition 2.4. *Let X be a hypersurface of degree $d \leq n-1$ in $\mathbb{P}^n$. Then through every smooth point of X it passes a line which is contained in X and, in fact, the lines through that point which are contained in X are parametrized by a projective variety of dimension $\geq n-1-d$ (=codimension $d-1$ in $\mathbb{P}^{n-2}$).*

When $d \leq n-1$, then for a "general" hypersurface X of degree d in $\mathbb{P}^n$, we expect the locus of lines which pass through a "general point" of X and contained in X to be of dimension $n-d-1$.

Remark 2.5. Suppose $\mathcal{T}$ is an irreducible algebraic variety the points of which parametrize a family of algebraic objects. We say that a property is satisfied by the general member of the family, if there exists a Zariski open (hence dense) subset U of $\mathcal{T}$ such that every member of the family represented by a point in U satisfies this property. For example, the Grassmannian $\mathbb{G}(1,n)$ parametrizes the family of lines in $\mathbb{P}^n$. Given a plane Π_0 of codimension 2 in $\mathbb{P}^n$, the general line in $\mathbb{P}^n$ does not intersect the plane Π_0 (indeed, in this case the complement of U is a closed Schubert cycle of codimension one).

3. The Fano variety of lines of a hypersurface

From now on and in order to avoid confusion, given a line ℓ in $\mathbb{P}^n$ we denote by $[\ell] \in \mathbb{G}(1,n)$ the corresponding point in the Grassmannian of lines.

Definition 3.1. Let X be a hypersurface in $\mathbb{P}^n$. We define

$$\mathcal{F}_X = \{[\ell],\ \ell \subset X\} \subset \mathbb{G}(1,n),$$

the space which parametrizes lines contained in the hypersurface X.

We shall now prove that $\mathcal{F}_X$ is a projective variety. For this we use, see [3], pg 167:

Theorem 3.2 (Chow's Theorem)**.** *Any analytic subvariety of the projective space is projective. In particular, any analytic subvariety of a projective variety is projective.*

To start with, we recall the following:

Remark 3.3. The homogeneous polynomials of degree d in $n+1$ variables are parametrized by their coefficients, the number of which is $N+1 := \binom{n+d}{d}$. Therefore they correspond to the points of a linear space $V_{n+1,d} \cong \mathrm{Sym}^d\mathbb{C}^{n+1}$ of dimension $N+1$. Since two hypersurfaces defined by the homogeneous polynomials F and G are considered to be the same if and only if $F = cG$, $c \in \mathbb{C}^*$, we get that the space $\mathbb{P}^N = \mathbb{P}(V_{n+1,d})$ parametrizes hypersurfaces of degree d in $\mathbb{P}^n$. Singular hypersurfaces form a proper closed subvariety of codimension one in $\mathbb{P}^N$, see Application 4.3 for a proof, and thus the smooth hypersurfaces form a non empty Zariski open U_{sm} of $\mathbb{P}^N$. Note that a smooth hypersurface must be defined by irreducible polynomials, because reducible polynomials always define singular hypersurfaces: the non -empty (by Bezout's Theorem) intersection locus of any pair of their irreducible components lies in the singular locus. Inside the locus of singular hypersurfaces lies the locus of hypersurfaces defined by polynomials F which decompose as $F = G^2H$. These hypersurfaces are characterized by the fact that they have components, the ones defined by $G = 0$, which are everywhere singular. A dimension count shows that the locus of such hypersurfaces in $\mathbb{P}^N$ is of big codimension and we will exclude them from our study. The singularities of the hypersurfaces that we deal with will form, on each one of their irreducible components, a closed proper subset.

Theorem 3.4. *Let X be a hypersurface of degree d in $\mathbb{P}^n$. Then, if non-empty, the locus $\mathcal{F}_X$ is a projective subvariety of $\mathbb{G}(1,n)$. It is called the* **Fano variety** *of lines of X.*

Proof. The proof uses the following basic fact: let E be a vector bundle on a projective variety X and σ a holomorphic section of E. The variety X is embedded in the total space of E by the zero section of E. Then the zero locus $Z(\sigma) = \{p \in X,\ \sigma(p) = 0\}$ of σ (i.e. the intersection of the image of σ with that of the zero section) is a projective subvariety of X. Indeed, by Chow's Theorem 3.2 it suffices to show that $Z(\sigma)$ is an analytic subvariety of X. Locally on a neighbourhood U of X, the section σ is given by an analytic map $F : U \to \mathbb{C}^n$, with $n = \mathrm{rank}E$. Let $f_i = \pi_i \circ F : U \to \mathbb{C}$, with π_i the i-projection. Then $Z(\sigma)|_U = \mathbb{V}(f_1, \ldots, f_n)$ an analytic subvariety of U.

Let $\mathbb{G} = \mathbb{G}(1,n)$ be the Grassmannian of lines in $\mathbb{P}^n$. Let $\mathcal{U}$ be the universal bundle over $\mathbb{G}$. The inclusion $\mathcal{U} \hookrightarrow \mathbb{C}^{n+1} \times \mathbb{G}$ induces a dual map

$$(\mathbb{C}^{n+1})^\vee \times \mathbb{G} \to \mathcal{U}^\vee \to 0.$$

Over a point $[\ell] \in \mathbb{G}$ the above is the restriction map of a functional in $(\mathbb{C}^{n+1})^\vee$ on the two dimensional subspace ℓ of $\mathbb{C}^{n+1}$. The above induces a map $\phi : \mathrm{Sym}^d(\mathbb{C}^{n+1})^\vee \times \mathbb{G} \to \mathrm{Sym}^d\mathcal{U}^\vee$. The space $\mathrm{Sym}^d(\mathbb{C}^{n+1})^\vee$ is identified with the space $V_{n+1,d}$: a polynomial of degree d in $n+1$ variables induces a *linear* map $\mathrm{Sym}^d(\mathbb{C}^{n+1}) \to \mathbb{C}$ and vice versa. Therefore an element of $V_{n+1,d}$ can be viewed as a constant section of the trivial bundle $\mathrm{Sym}^d(\mathbb{C}^{n+1})^\vee \times \mathbb{G}$. We then get an induced map from $V_{n+1,d}$ to the space of holomorphic sections of the bundle $\mathrm{Sym}^d\mathcal{U}^\vee$, given by the restriction on ℓ. Let now $F \in V_{n+1,d}$ be the defining polynomial of X and let σ_F be the induced section of $\mathrm{Sym}^d\mathcal{U}^\vee$. We claim that the Fano variety

$\mathcal{F}_X$ is the zero locus of the section σ_F: indeed, a point $[\ell] \in \mathbb{G}$ is in the zero locus of σ_F if and only if the restriction of F on ℓ vanishes i.e. $\ell \subset X$. As the zero locus of a section of a vector bundle, $\mathcal{F}$ is a projective subvariety of $\mathbb{G}(1, n)$. □

Remark 3.5. Let σ be a section of a rank r vector bundle E on a projective variety X of dimension n and let $Z(\sigma)$ be its zero locus. We may have that $Z(\sigma) = \emptyset$, for example, take $E = \mathbb{C}^r \times X$ the trivial bundle and σ to be the section $p \times X$, with $p \neq (0, \ldots, 0)$ a constant non-zero section. But in case $Z(\sigma) \neq \emptyset$ then $\text{codim}Z(\sigma) \leq r$: we can see this locally; as we stated before $Z(\sigma)$ is defined as the zero locus of r analytic functions, i.e. the intersection of the zero locus of these functions. Every time we intersect, the codimension increases at most by one as in Remark 2.3. Moreover, we expect that the general situation will be that, at each step the codimension increases by one, and thus the codimension of $Z(\sigma)$ to be exactly r. By the above proof, $\mathcal{F}_X$ is the zero section of the bundle $\text{Sym}^d\mathcal{U}^\vee$, a bundle of rank $d+1$. According to what was said before, we expect the general situation to be that the codimension of $\mathcal{F}_X$ in $\mathbb{G}(1, n)$ equals $d + 1$ and thus $\dim(\mathcal{F}_X) = 2(n-1) - (d+1) = 2n - d - 3$. In other words, we expect in general $\mathcal{F}_X = \emptyset$ when $d > 2n - 3$ or otherwise to be a variety of dimension $2n - d - 3$. Will discuss this claim in what follows.

By Corollary 2.3 we then have:

Corollary 3.6. *When $\mathcal{F}_X$ has the expected dimension $2n-d-3$ then its Chern class in the Chow group of the Grassmannian $\mathbb{G}(1, n)$ equals*

$$[\mathcal{F}_X] = \mathrm{c}_{d+1}(\text{Sym}^d\mathcal{U}^\vee) \in A^{d+1}(\mathbb{G}(1, n)),$$

with $\mathcal{U}$ the universal bundle on $\mathbb{G}(1, n)$.

4. The main tools

In studing the dimension of the Fano variety will use repeatedly the following construction. Let $I = \{(p, [\ell]), p \in \ell\} \subset X \times \mathcal{F}_X$. I is called the incidence variety and fits in a diagram

$$\begin{tikzcd} I \arrow[r, "q"] \arrow[d, "p"] & \mathcal{F}_X \\ X & \end{tikzcd} \tag{2}$$

where p and q are the projections. Then:

For $[\ell] \in \mathbb{G}(1, n)$ we have $q^{-1}([\ell]) = \ell \subset X$. Thus the map q is a $\mathbb{P}^1$-fibration.
For $x \in X$ we have $p^{-1}(x) = \{[\ell] \in \mathcal{F}_X, x \in \ell\}$.

We now state the following fundamental Theorem which will play a key role in our study, see [4], Theorem 11.12:

Theorem 4.1 (Dimension of the fibers)**.** *Let $\phi : Y \to X$ be a surjective morphism of irreducible projective varieties with $\dim Y = m \geq \dim X = n$. Then for every $x \in X$ we have $\dim\phi^{-1}(x) \geq m - n$, in fact any irreducible component of the fiber $\phi^{-1}(x)$ has dimension $\geq m - n$. Moreover, there exists a Zariski open set $U \subset X$ such that for any $x \in U$ we have $\dim\phi^{-1}(x) = m - n$. The latter translates to: the dimension of the general fiber of the map ϕ is the minimum one $(= m - n)$ and, actually, the general fiber is of pure dimension $m - n$, i.e., every irreducible component of the fiber has dimension $m - n$. Moreover, if X and Y are smooth, the general fiber of ϕ is smooth (but may be reducible, with all components of the same dimension $m - n$).*

Example 4.2. A characteristic example is the blow up $\tilde{\mathbb{P}^2}$ of the projective plane $\mathbb{P}^2$ at a point $p_0 \in \mathbb{P}^2$. The blowing up map $\pi : \tilde{\mathbb{P}^2} \to \mathbb{P}^2$ has the following property:

Over a point p which belongs in the open set $U = \mathbb{P}^2 \backslash \{p_0\}$ the fiber is just one point and thus of dimension zero.

Over the point p_0 the fiber is the exceptional curve E and thus of dimension one.

Application 4.3. As an application of the above theorem we show that in the parameter space $\mathbb{P}^N$ of hypersurfaces in $\mathbb{P}^n$ of degree d, see Remark 3.3, the locus of singular hypersurfaces $\mathcal{S}$ is of codimension one. Indeed, we let

$$\mathcal{X} = \{([X], x) \subset \mathbb{P}^N \times \mathbb{P}^n, \, x \in X \text{ a singular point}\}$$

and we consider the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "q"] \arrow[d, "p"] & \mathbb{P}^n \\ \mathbb{P}^N, & \end{tikzcd}$$

with p, q the projections. Note that $\mathcal{S} = \mathrm{Im}p$. We now claim that each fiber of the map q is isomorphic to $\mathbb{P}^{N-(n+1)}$. Indeed, the fiber over a point $x \in \mathbb{P}^n$ is the locus of hypersurfaces in $\mathbb{P}^n$ of degree d which contain x as a singular point. Since this are properties invariant by a projective transformation of $\mathbb{P}^n$, all fibers of the map p are isomorphic. We may then assume that $x = [1, 0, \dots, 0]$ and, by dehomogenizing, we are asking for the locus of degree d polynomials in $\mathbb{C}[x_1, \dots, x_n]$ which have a singularity at the origin. By what we have explained in Section 2, the Taylor expansion of such a polynomial at the origin will be of the form

$$f_2 + \cdots + f_d,$$

with f_i homogeneous of degree i. In other words, the above expansion will have $f_0 = f_1 = 0$ which translates to the vanishing of $1 + n$ coefficients of f. This proves the claim. By the above Theorem 4.1, will then get that

$$\dim\mathcal{X} = n + N - 1 = N - 1.$$

and, also, by Theorem 4.2, we get get that it is irreducible. On the other hand, we claim that there is a hypersurface in $\mathbb{P}^n$ of degree d with a single singular point, for example the one defined by

$$x_1^d + \cdots + x_n^d = 0$$

has the point $[1, 0, \dots, 0]$ as the only singular point (geometrically, this is a cone over a smooth subvariety of $\mathbb{P}^{n-1}$). The same theorem then implies that the general fiber of the map p is of dimension 0 and, also, that

$$\dim\mathcal{S} = \dim(\mathrm{Im}p) = \dim\mathcal{X} = N - 1.$$

We fare closing this section with the following theorem, see [4], Theorem 3.13.

Theorem 4.4. *Let $\phi : Y \to X$ a morphism of projective varieties. Then $\phi(Y)$ is a projective (closed) subvariety of X.*

5. An upper bound

In this section we demonstrate the following claim stated in the Introduction, see Proposition 5.2 below. For a more general result, see [7]. Before we state our proposition, we mention the following basic result, see [2] Theorem 0.2.

Lemma 5.1. *Let A and B be subvarieties of a smooth projective variety X of dimension n. Then every (non-empty) component C of the intersection $A \cap B$ satisfies*

$$\mathrm{codim}C \leq \mathrm{codim}A + \mathrm{codim}B, \ \text{i.e. } \dim C \geq \dim A + \dim B - n.$$

Proposition 5.2. *Let X be an irreducible hypersurface of $\mathbb{P}^n$. Then $\dim\mathcal{F}_X \leq 2(n-2)$ and the equality holds exactly when the hypersurface is a hyperplane.*

Proof. When X is a hyperplane then $X \cong \mathbb{P}^{n-1}$ and $\mathcal{F}_X = \mathbb{G}(1, n-1)$. Therefore $\dim\mathcal{F}_X = \dim\mathbb{G}(1, \mathrm{n}-1) = 2(\mathrm{n}-2)$. To show that $\dim\mathcal{F}_X \leq 2(n-2)$, we work with induction on n. For $n = 2$, an irreducible hypersurface is an irreducible curve. If the curve is a line then $\dim\mathcal{F}_X = 0$ which achieves the bound. Otherwise $\mathcal{F}_X = \emptyset$, i.e. $\dim\mathcal{F}_X = -1$. Suppose the result holds for irreducible hypersurfaces in $\mathbb{P}^{n-1}$ then we prove it for irreducible hypersurfaces in $\mathbb{P}^n$.

Let X be an irreducible hypersurface in $\mathbb{P}^n$. If X does not contain any line, then the bound of the dimension of $\mathcal{F}_X$ is satisfied trivially. We may then assume that X contains a line ℓ. Let H be a hyperplane containing the line and not containing the hypersurface. Then every component C_i of the intersection $X \cap H$ is of codimension 1 in $H \cong \mathbb{P}^{n-1}$ and we have

$$\mathcal{F}_{X\cap H} = \mathcal{F}_X \cap \mathcal{F}_H (\neq \emptyset).$$

By Lemma 5.1 we get

$$\begin{aligned} \dim(\mathcal{F}_{X\cap H}) \geq& \dim\mathcal{F}_X + \dim\mathcal{F}_H - \dim G(1, n) \\ =& \dim\mathcal{F}_X + \dim G(1, n-1) - \dim G(1, n) = \dim\mathcal{F}_X - 2, \ \text{i.e.} \\ \dim\mathcal{F}_X \leq& \dim(\mathcal{F}_{X\cap H}) + 2. \end{aligned}$$

We now have

$$\mathcal{F}_{X\cap H} = \cup_i \mathcal{F}_{C_i}$$

and then by induction we have

$$\dim \mathcal{F}_{C_i} \leq 2(n-3)$$

which proves the result.

We show now that we have equality exactly when X is a hyperplane. Suppose then that X is an irreducible hypersurface in $\mathbb{P}^n$ with $\dim \mathcal{F}_X = 2(n-2)$. We first claim that the map p of Diagram 2 is onto. Note that the dimension of a fiber $p^{-1}(x) = \{[\ell] \in \mathcal{F}_X, x \in \ell\}$ is $\leq n-2$. Indeed, this is a subvariety of the locus of lines in $\mathbb{P}^n$ which pass through x. This is a variety $\cong \mathbb{P}^{n-1}$. Since $p^{-1}(x)$ is a proper subvariety of this (since X is a hypersurface in $\mathbb{P}^n$), its dimension is $\leq n-2$. By Theorem 4.1, we get that

$$\dim I = 1 + \dim \mathcal{F}_X = 1 + 2(n-2) = 2n-3,$$

and then, by the same theorem we get

$$\dim \mathrm{Im} p \geq \dim I - (n-2) = 2n-3-(n-2) = n-1$$

Since $\mathrm{Im}p$ is a subvariety of X, see Theorem 4.4, of dimension $\geq \dim X$, we get $\mathrm{Im}p = X$, i.e., the map p is onto. By Theorem 4.1 we conclude that the generic fiber of the map p has dimension equal to $\dim I - \dim X = 2n-3-(n-1) = n-2$. Tis means that the locus of lines which pass throug the generic point of $x \in X$ and are contained in X has dimension $n-2$ and thus the lines through x spann the whole X. This implies, that for the generic $x \in X$, the hypersurface X is a cone with vertex x. But this means that X is a hyperplane: if not, the points of X spann the whole $\mathbb{P}^n$ and the same happens for the points of a (non empty) Zariski open of X. We may then choose $n+1$ generic points of X which spann the whole $\mathbb{P}^n$. For each such point, X is a cone with vertex the point, which implies that the linear span of these points (ie the $\mathbb{P}^n$) is contained in X; a contradiction. ☐

6. Dimension estimates

We start this section with the following proposition.

Proposition 6.1. *Suppose $d \leq n-1$. Then for every hypersurface X of degree d in $\mathbb{P}^n$ we have $\dim \mathcal{F}_X \geq 2n-d-3$.*

Proof. The singular locus of a hypersurface is always a closed set of codimension ≥ 1 and thus the general point of the hypersurface is smooth. With $d \leq n-1$, we have by Proposition 2.4 that for any smooth point of X, the lines which are contained in X and pass through that point are parametrized by a projective variety of dimension $\geq n-d-1$. Then the general fiber of the map p in Diagram 2 is of dimension $\geq n-1-d$ and hence $\dim I \geq \mathrm{dim X} + n - 1 - d = 2n-d-2$. But then

$$\dim \mathcal{F}_X = \dim I - 1 \geq 2n-d-3.$$

☐

We now study the, so called, universal Fano variety of lines of hypersurfaces of degree d in $\mathbb{P}^n$. Recall by Remark 3.3 the definition of $\mathbb{P}^N = \mathbb{P}(V_{n+1,d})$. We define

$$\Phi(n,d) = \{([X],[\ell]), \ell \subset X\} \subset \mathbb{P}^N \times \mathbb{G}(1,n).$$

This fits in the diagram

$$\begin{tikzcd} \Phi(n,d) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathbb{G}(1,n) \\ \mathbb{P}^N. \end{tikzcd} \tag{3}$$

The fiber over a point $[X] \in \mathbb{P}^N$ under the first projection $\pi_1 : \Phi(n,d) \to \mathbb{P}^N$ is the Fano variety $\mathcal{F}_X$ of X. For that reason, $\Phi(n,d)$ is called the universal Fano variety. Note that the fiber over $[\ell] \in \mathbb{G}(1,n)$ under the second projection π_2 is the locus of hypersurfaces which contain ℓ.

Proposition 6.2. *The universal Fano variety $\Phi(n,d)$ is a smooth irreducible variety of dimension*

$$\dim \Phi(n,d) = N + 2n - d - 3.$$

Proof. As we sated above, the fiber over a point $[\ell] \in \mathbb{G}(1,n)$ is the set of hypersurfaces which contain the line ℓ. This is a linear subspace of $\mathbb{P}^N$. Indeed, we have a restriction linear map $V_{n+1,d} \to \mathrm{Sym}^d(\ell)^\vee$, where we now view the line ℓ as a 2-dimensional subspace of $\mathbb{C}^{n+1}$ (see also the proof of Theorem 3.4) . The Kernel K_ℓ of this map is the linear subspace of $V_{n+1,d}$ of homogeneous polynomials vanishing on ℓ. Hence the points of $\mathbb{P}^N = \mathbb{P}(V_{n+1,d})$ which correspond to hypersurfaces containing ℓ is the linear subspace $\mathbb{P}(K_\ell)$ of $\mathbb{P}^N$. Since $\dim V_{n+1,d} = N+1$ and $\dim \mathrm{Sym}^d(\ell)^\vee = d+1$, this is a space of projective dimension $N-(d+1)$. Therefore the fibration $\pi_2 : \Phi(n,d) \to \mathbb{G}(1,n)$ is a fibration of smooth irreducible fibers of dimension $N-(d+1)$ over the irreducible smooth variety $\mathbb{G}(1,n)$. By Theorem 4.2, we conclude that $\Phi(n,d)$ is a smooth irreducible variety of dimension $N-(d+1)+\dim\mathbb{G}(1,n) = N-(d+1)+2(n-1) = N+2n-d-3$. ☐

By the above Theorem 4.1 we have:

Corollary 6.3.

(1) *When $d > 2n-3$ the general hypersurface of degree d in $\mathbb{P}^n$ contains no lines.*
(2) *When $d \leq 2n-3$, a hypersurface X of degree d in $\mathbb{P}^n$ which contains at least one line has $\dim\mathcal{F}_X \geq 2n-d-3 \geq 0$.*
(3) *When $d \leq n-1$, the general hypersurface X of degree d in $\mathbb{P}^n$ has $\dim\mathcal{F}_X = 2n-d-3$. Moreover, through the general point of X the lines which pass through this point and are contained in X are parametrized by a projective variety of dimension $n-d-1 \geq 0$.*

Proof. (1) Suppose that the general hpersurface of degree d in $\mathbb{P}^n$ contains a line. Then the image of $\pi_1 : \Phi(n,d) \to \mathbb{P}^N$ will contain a dense set and, since it is closed, the map will be surjective and the general fiber will have dimension $\dim\Phi(n,d) - \dim\mathbb{P}^N = 2n-d-3 < 0$, a contradiction.
(2) If a hypersurface X contains a line then $[X] \in \mathbb{P}^n$ lies in the image of π_1 and by the Theorem 4.1 we have

$$\dim\mathcal{F}_X = \dim\pi_1^{-1}([X]) \geq \dim\Phi(n,d) - \dim\mathrm{Im}\pi_1 \geq (N+2n-d-3) - N = 2n-d-3 \geq 0.$$

(3) For the first claim: by Proposition 6.1, the map π_1 is onto. Then by Theorem 4.1 the general fiber of the map has dimension equal to $(N+2d-n-3)-N = 2d-n-3$. For the second claim: by Proposition 2.4, the map p of the Diagram 2 is onto. Then, by Theorem 4.1, for the general $x \in X$ we have

$$\dim p^{-1}(x) = \dim I - \dim X = (\dim\mathcal{F}_X + 1) - (n-1) = (2n-d-3+1) - (n-1) = n-d-1.$$

☐

7. Tangent spaces to the Grassmannian $\mathbb{G}(1,n) = G(2,n+1)$

We describe now in an informal, but useful way, the tangent space $T_{[\ell]}G(2,n+1)$ of the Grassmannian $\mathbb{G}(1,n) = G(2,n+1)$ at a given point $[\ell]$. We follow the exposition given in [4], Lecture 16. A tangent vector in $T_{[\ell]}G(2,n+1)$ corresponds to the velocity vector at 0 of a local holomorphic arc $\gamma(t) \subset G(2,n+1)$ with $\gamma(0) = [\ell]$. Let $\gamma(t) = [\ell(t)]$.

We will interpret how the plane $\ell \subset \mathbb{C}^{n+1}$ moves when the point $[\ell] \in G(2,n+1)$ moves along the arc γ. For that, given a $v \in \ell$ we shall see that γ induces a path in $\mathbb{C}^{n+1}$ with origin at $v \in \ell$ and thus a movements of v. But this path is well defined modulo motions of v inside ℓ. Nevertheless, this allows us to measure the movement of v away from ℓ. By choosing a frame of ℓ we may then measure how this frame moves away form ℓ and thus how ℓ itself moves.

Let $I_G = \{(p,[\ell]),\ p \in \ell\} \subset \mathbb{C}^{n+1} \times G(2,n+1)$ be the incidence variety. The second projection $I_G \to G(2,n+1)$ is a $\mathbb{C}^2$-fibration. Given a $v \in \ell$, by appropriately lifting this arc to the incidence variety I_G and then projecting to $\mathbb{C}^{n+1}$ we get a local holomorphic path $\Gamma_v(t)$ in $\mathbb{C}^{n+1}$ with $\Gamma_v(t) \in \ell(t)$ for all t, $\Gamma_v(0) = v$ and such that $\Gamma(t)$ intersects the space $\ell(t)$ transversely. The above procedure defines a homomorphism

$$\phi_\gamma : \ell \to \mathbb{C}^{n+1}/\ell \text{ with } \phi_\gamma(v) = \frac{d\Gamma_v(t)}{dt}|_{t=0}.$$

The above is well defined, i.e. it does not depend on the choice of $\Gamma_v(t)$. Indeed, if $\Gamma_v^1(t)$ is a different choice, then since, $\Gamma_v(0) = v = \Gamma_v^1(0)$ we have $\Gamma_v^1(t) = \Gamma_v(t) + tE(t)$ for some arc $E(t)$ with $E(t) \in \ell(t)$, for all t. Then

$$\frac{d\Gamma_v^1(t)}{dt}|_{t=0} = \frac{d\Gamma_v(t)}{dt}|_{t=0} + E(0), \text{ with } E(0) \in \ell$$

and thus ϕ_γ is well defined. Therefore the local holomorphic arc $\gamma(t) \subset G(2, n+1)$ with $\gamma(0) = [\ell]$ induces naturally the homomorphism ϕ_γ which indicates how $\ell \subset X$ moves. We then get the description of the tangent space to the Grasmmannian at a point by

$$T_{[\ell]}G(2, n+1) \cong \mathrm{Hom}(\ell, \mathbb{C}^n/\ell).$$

8. Tangent spaces to the Fano Variety of lines of a hypersurface

Let X be a hypersurface in $\mathbb{P}^n$ defined by a homogeneous polynomial F of degree d. Let $\ell \subset X$ be a line which we assume for simplicity that is contained in the smooth locus of X. The tangent space $T_{[\ell]}\mathcal{F}_X$ of the Fano variety of lines at the point $[\ell]$ will satisfy $T_{[\ell]}\mathcal{F}_X \subset T_{[\ell]}G(2, n+1) = \mathrm{Hom}(\ell, \mathbb{C}^{n+1}/\ell)$. Following the notation of the preceding section, we now choose $\gamma(t)$ a holomorphic arc in $\mathcal{F}_X$ and then, for each $v \in \ell$, will have $\Gamma_v(t) \subset X$. But since $\Gamma_v(t) \subset X$ we have $\phi_\gamma(v) = \Gamma'_v(0) \perp \nabla F(\Gamma_v(0) = v)$. In other words, by viewing v as a point $p \in \ell \subset X$, will have $\phi_\gamma(p) \in T_pX$ and therefore

$$T_{[\ell]}\mathcal{F}_X \subseteq \mathcal{H} := \{\phi : \ell \to \mathbb{C}^{n+1}/\ell, \text{ with } \phi(p) \in T_pX/\ell \text{ for all } p \in \ell\}.$$

We now interpret $\mathcal{H}$ as follows: we recall the notation $V_{2,d}$ for the space of homogeneous polynomials of degree d in two variables. Let

$$G : \mathrm{Hom}(\ell, \mathbb{C}^{n+1}/\ell) \to V_{2,d} \text{ with } \phi \mapsto G_\phi$$

where G_ϕ is the polynomial of degree d defined by the Euclidean inner product

$$G_\phi(x_0, x_1) = \nabla F(x_0, x_1) \cdot \phi(x_0, x_1).$$

Although $\phi(x_0, x_1)$ is defined modulo ℓ the above map is well defined by Lemma 2.1 which says that $\nabla F(x_0, x_1) \cdot v = 0$, for any $v \in \ell$. By the above discussion, we have $\mathcal{H} = \mathrm{Ker}G$. Moreover, if the map G is onto, we get

$$\mathrm{dim}\mathcal{H} = \mathrm{dimHom}(\ell, \mathbb{C}^{n+1}/\ell) - \mathrm{dim}V_{2,d} = 2(n-1) - (d+1) = 2n - d - 3$$

and thus

$$\mathrm{dim}T_{[\ell]}\mathcal{F}_X \leq \mathrm{dim}\mathcal{H} = 2n - d - 3.$$

9. The main Theorem

We now prove the main Theorem:

Theorem 9.1. *When $d \leq 2n-3$ then the Fano variety of lines of a general smooth hypersurface of degree d in $\mathbb{P}^n$ is a smooth projective variety of dimension $2n-d-3$.*

Proof. That is smooth projective follows from Theorem 4.1.

To prove that its dimension is $2n-d-3$, we need to show that the map $\pi_1 : \Phi(n, d) \to \mathbb{P}^N$ is surjective. Then by Theorem 4.1 and Proposition 6.2 the general fiber will have the required dimension $(N + 2n - d - 3) - N = 2n - d - 3$. In fact, it suffices to show that there exists a hypersurface X of degree d in $\mathbb{P}^n$ such that its Fano variety $\mathcal{F}_X$ contains an irreducible component W of dimension $\mathrm{dim}W \leq 2n-d-3$: indeed, if this is the case, let $m = \mathrm{dimIm}\pi_1 \leq N$. Then, by Theorem 4.1, we have: $2n - d - 3 \geq \mathrm{dim}W \geq (N + 2n - d - 3) - m$ hence $m \geq N$. But, also, $m \leq N$ hence $m = N$ which implies that $\mathrm{Im}\pi_1 = \mathbb{P}^N$.

By general theory, the dimension of an irreducible variety is the dimension of its tangent space at a smooth point. We use the description of the tangent space of the Fano variety given in the previous Section 8 in order to find a hypersurface X of degree d and a line ℓ in the smooth locus of X such that the tangent space of the Fano variety $\mathcal{F}_X$ at the point $[\ell]$ has dimension $2n-d-3$. Then, the irreducible component of the $\mathcal{F}_X$ where $[\ell]$ belongs will have the required dimension.

We find now a hypersurface X defined by a degree d homogeneous polynomial F in $n+1$ variables which contains the line $\ell = \mathbb{V}(x_2, \ldots, x_n)$ in its smooth locus and such that the map G defined in Section 8 is onto. We may write

$$F = \sum_{i=2}^{n} x_i g_i(x_0, x_1) + h, \text{ with } h \in \langle x_2, \ldots, x_n \rangle^2 \text{ (= the square of the ideal generated by } x_2, \ldots, x_n),$$

with g_i of degree $d-1$. Then

$$\nabla F(x_0, x_1) = (0, 0, g_2(x_0, x_1), \ldots, g_n(x_0, x_1)).$$

A $\phi \in \mathrm{Hom}(\ell, \mathbb{C}^{n+1}/\ell)$ corresponds to a vector

$$\phi = (y_{00}x_0 + y_{01}x_1, y_{10}x_0 + y_{11}x_1, \ldots, y_{i0}x_0 + y_{i1}x_1, \ldots, y_{d0}x_0 + y_{d1}x_1).$$

We are then looking for an F as above such that: given any $T(x_0, x_n) = \sum_{k=0}^{d} B_k x_0^j x_1^{d-k} \in V_{2,d}$, there is a choice of a ϕ as above with

$$(0, 0, g_2(x_0, x_1), \ldots, g_n(x_0, x_1)) \cdot (y_{00}x_0 + y_{01}x_1, y_{10}x_0 + y_{11}x_1, \ldots, y_{i0}x_0 + y_{i1}x_1, \ldots, y_{d0}x_0 + y_{d1}x_1)$$
$$= \sum_{k=0}^{d} B_k x_0^k x_1^{d-k}.$$

In other words, when $d \leq 2n-3$, we want

$$V_{2,d} \subseteq \langle g_2(x_0, x_1), \ldots, g_n(x_0, x_1) \rangle .$$

We do the case $d = 2n-3$, the others are easier. We take $g_i(x_0, x_1) = x_0^{2(n-i)} x_1^{2(i-2)}$, $i = 2, \ldots, n$. In other words,

$$(g_1(x_0, x_1), g_2(x_0, x_1), \ldots, g_n(x_0, x_1)) = (x_0^{2(n-2)} x_1^0,\ x_0^{2(n-3)} x_1^2,\ \ldots, x_0^0 x_1^{2(n-2)}).$$

Then, for the base $x_0^{2n-3-k} x_1^k$, $k = 0, \ldots, 2n-3$, of $V_{2,2n-3}$ we have: when $k = 2i+1$, $i = 0, \ldots, n-2$, then $x_0^{2n-3-k} x_1^k = x_0^{2(n-2-i)} x_1^{2i+1} = x_1 g_{i+2}(x_0, x_1)$ and when $k = 2i$, $i = 0, \ldots, n-2$ then $x_0^{2n-3-k} x_1^k = x_0^{2n-2i-3} x_1^{2i} = x_0 g_{i+2}(x_0, x_1)$. Moreover such an F is smooth at the line ℓ. This proves the main Theorem. □

In the above proof we showed that the map $\pi_1 : \Phi(n, d) \to \mathbb{P}^N$ is surjective. We therefore have

Corollary 9.2. *When $d \leq 2n-3$ i.e. $n \geq (d+3)/2$, any hypersurface X of degree d in $\mathbb{P}^n$ contains a line.*

The above Theorem 9 combined with Corollary 3.6 shows

Corollary 9.3. *When $d \leq 2n-3$ then the Chern class in the Chow group of the Grassmannian $\mathbb{G}(1, n)$ of the Fano variety of lines of a general smooth hypersurface X of degree d in $\mathbb{P}^n$ is given by*

$$[\mathcal{F}_X] = \mathrm{c}_{d+1}(\mathrm{Sym}^d \mathcal{U}^\vee) \in A^{d+1}(\mathbb{G}(1, n)).$$

As we have seen in Corollary 6.3, when $d \leq n-1$, the general hypersurface X of degree d in $\mathbb{P}^n$ has $\dim \mathcal{F}_X = 2n-d-3$. Moreover, through the general point of X the lines which pass through this point and are contained in X are parametrized by a projective variety of dimension $n-d-1$. We now have

Corollary 9.4. *When $n \leq d \leq 2n-3$, the general hypersurface X of degree d in $\mathbb{P}^n$ contains a line, but through a general point of X it passes no line contained in X.*

Proof. By the Theorem, X contains lines. If we assume that through the general point it passes a line, then the map $p : I \to X$ of diagram 2 is onto. But then $\dim I \geq \dim X = n-1$. On the other hand, the map $q : I \to \mathcal{F}_X$ gives: $\dim I = 1 + \dim \mathcal{F}_X = 1 + 2n - d - 3 = 2n - d - 2 \geq n-1$, i.e. $d \leq n-1$; a contradiction. □

Example 9.5. For the case of quadric hypersurfaces ($d = 2$): The Fano variety of any irreducible quadric hypersurface X in $\mathbb{P}^n$, $n \geq 3$, has dimension $\dim \mathcal{F}_X = 2n-5$. Indeed, by Corollary 6.3 (3), we get $\dim \mathcal{F}_X = 2n-5 = 2(n-2)-1$, which is one less than the upper bound of Proposition 5.2. But since the upper bound is obtained exactly in the case of a hyperplane, we get that $\dim \mathcal{F}_X = 2n-5$. By the same Corollary, we get that form any point of X it passes a line contained in X.

Example 9.6. For the case of cubic hypersurfaces ($d = 3$):

(1) Any cubic hypersurface in $\mathbb{P}^n$, $n \geq 3$, contains a line.
(2) A general cubic surface X in $\mathbb{P}^3$ ($n = 3$) contains finitely many lines, because $\dim \mathcal{F}_X = 0$. In fact, we will see in the next Chapter that the number of lines is 27.
(3) Through a general point of a general cubic threefold in $\mathbb{P}^4$ ($n = 4$) they pass a finite number of lines which are contained in X. In fact, we will see in the next Chapter that they pass six lines.
(4) The family of lines which pass through a general point of a general cubic fourfold in $\mathbb{P}^5$ ($n = 5$) is parametrized by a curve. In fact, we will see that it is a genus four curve.
(5) and so on....

Example 9.7. For the case of quartic hypersurfaces ($d = 4$):

(1) Any quartic hypersurface in $\mathbb{P}^n$, $n \geq 4$, contains a line.
(2) A general quartic surface X in $\mathbb{P}^3$ ($n = 3$) contains no lines.
(3) A general quartic threefold X in $\mathbb{P}^4$ ($n = 4$) contains lines, but through a general point of X they pass no lines contained in X.
(4) Through a general point of a general quartic fourfold in $\mathbb{P}^5$ ($n = 5$) they pass a finite number of lines which are contained in X.
(5) The family of lines which pass through a general point of a general quartic fivefold in $\mathbb{P}^6$ ($n = 6$) is parametrized by a curve.
(6) and so on....

CHAPTER 5

Cubic Hypersurfaces

As it is stated in [5]: *Algebraic geometry starts with cubic polynomial equations. Everything of smaller degree, like linear maps or quadratic forms, belongs to the realm of linear algebra.*

1. Cubic hypersurfaces and the rationality problem

The theory of cubic hypersurfaces, although it is one of the most studied areas which lies in the origins of algebraic geometry, it still poses some of the most fascinating, and to the great extend, open questions of the subject:

When is a cubic hypersurface rational?
For what $n \in \mathbb{N}$ is the general cubic hypersurface in $\mathbb{P}^n$ rational?

Before we explain what we mean with the term "rational", we should point out that in several cases where a hypersurface turns out to be rational, the proof is based on the existence of lines (or, more general, of linear subspaces) contained in the hypersurface.

We say that an irreducible variety X of dimension n is rational if it is birational to a projective space. This means that there is a map defined on a non empty Zariski open subset U of X which induces an isomorphism from U to a Zariski open subset of $\mathbb{P}^n$. A characteristic example is the blowing up map of Example 4.2. This has an algebraic interpretation: the function field $k(X)$ of X is isomorphic to the polynomial function filed $k(x_1, \ldots, x_n)$. It corresponds to the fact that X can be parametrized almost everywhere, i.e. on a Zariski open set.

Hypersurfaces of degree one in $\mathbb{P}^n$ are hyperplanes $\cong \mathbb{P}^{n-1}$ and thus are rational. Irreducible hypersurfaces of degree two in $\mathbb{P}^n$ are rational since the projection from a smooth point gives a birational map to $\mathbb{P}^{n-1}$. And then we come to cubic hypersurfaces:

- Any singular irreducible cubic hypersurface X in $\mathbb{P}^n$ with a nodal singularity x is rational. Nodal singularities are the simplest ones. With $x = [1, 0, \ldots, 0]$, such a hypersurface has the form
$$x_0 Q(x_1, \ldots, x_n) + C(x_1, \ldots, x_n),$$
with Q a smooth (irreducible) quadric and C a smooth (irreducible) cubic in $\mathbb{P}^{n-1}$. The lines going through x are parametrized by $\mathbb{V}(Q, C) \subset \mathbb{P}^{n-1}$, a codimension 2 subvariety. Hence the locus $S = \cup_{x \in \ell \subset X} \ell \subset X$ is of codimension 1 in X. The birational map to $\mathbb{P}^{n-1}$ is given by the projection from the singular point x. It is defined on $X \backslash S$. By Bezout's theorem, any line which passes through the singular point and is not contained in S will intersect the hypersurface at one additional point which makes the projection a birational map (note that the degree of the intersection of such a line with X at the point x is ≥ 2).
- Smooth cubic (elliptic) curves in $\mathbb{P}^2$ are never rational since their genus is one.
- All smooth cubic surfaces in $\mathbb{P}^3$ are rational; as it turns out any smooth cubic surface can be realized as the blowing up of $\mathbb{P}^2$ at six points.
- The general cubic threefold in $\mathbb{P}^4$ is not rational. This is one of the most famous and influential works in algebraic geometry, done by H. Clemens and Ph. Griffiths on 1972, [1]. In the course of the proof they developed the theory of intermediate Jacobians.
- Any known irreducible cubic fourfold in $\mathbb{P}^5$ is rational but is a major open problem in algebraic geometry if the general cubic fourfold is rational (the belief is that it is not).
- When $n \geq 6$ we know very little about the rationality problem of cubic hypersurfaces in $\mathbb{P}^n$, apart from some special cases. For example, smooth cubic hypersurfaces in $\mathbb{P}^{2m+1}$ which contain two disjointed linear subspaces of dimension m are rational. We shall see shortly that this is always the case for smooth cubic surfaces. But there is no known example of a rational smooth cubic hypersurface in the even dimensional projective space $\mathbb{P}^{2m}$.

Apart from the notion of rationality there are two weaker notions: that of unirationality and that of rationally connected. A variety X of dimension m is called unirational if there exist a dominant rational map $\phi : \mathbb{P}^m \to X$, i.e. a map ϕ defined on a Zariski open of $\mathbb{P}^m$ with dense image in X. This has an algebraic interpretation: the function field $k(X)$ of X is embedded in the polynomial function field $k(x_1, \ldots, x_m)$. It corresponds to the fact that there is a Zariski open set of X that can be multi-parametrized. A variety X is called rationally connected if through any two general points of X it passes a rational curve.

- The property that a variety is rationally connected is a geometric property with no algebraic analogue, nevertheless it is a better understood property!.
- Rational and unirational varieties are rationally connected: this is due to Lüroth's Theorem, or in algebrogeometric terms to the fact that the image of a rational curve under a morphism is a rational curve. Since any two points of the projective space are joined by a line, then its image under the rational or the unirational map is a rational curve which joins the images of the points.
- On the other hand, there is no known example of a rationally connected variety which is not unirational (the belief is that it should exist such a variety).
- For curves and surfaces (over the complex numbers): rational = unirational = rationally connected. For the case of curves, this is the Lüroth's Theorem. The equality for the case of surfaces is based on a characterization of rationality in terms of arithmetic invariants given by Castelnuovo (Castelnuovo's rationality criterion).
- Rationally connected varieties seems to be the right higher dimensional analogue of rational curves and surfaces.
- Mori has proposed an arithmetic criterion (analogous to Castelnuovo's criterion for surfaces) for a smooth complex variety to be rationally connected.

Proposition 1.1. *Any smooth cubic hypersurface in $\mathbb{P}^n$, $n \geq 3$, is unirational.*

Proof. Let X be a smooth cubic hypersurface. As we have seen in Example 9.6, it contains a line ℓ_0. For any $x \in \ell_0$ the tangent hyperplane T_xX contains ℓ_0. We take the variety

$$\mathcal{X} = \{(x, [\ell]) \in \ell_0 \times \mathbb{G}(1, n),\ x \in \ell \subset T_xX\}$$

and we consider the projection

$$\pi : \mathcal{X} \to \ell_0.$$

Given a point $x \in \ell_0$ the fiber $\pi^{-1}(x)$ is the set of lines in $T_xX \cong \mathbb{P}^{n-1}$ containing the point x. These lines are parametrized by the points of $\mathbb{G}(1, n-1)$ which correspond to lines going through the point $x \in T_x$. This locus is isomorphic to $\mathbb{P}^{n-2}$: indeed, fix a hyperplane $H \cong \mathbb{P}^{n-2}$ which does not contain x. Then any line through x intersects H at a unique point and, vice versa, any point of H defines a unique line going through x. Hence, $\pi^{-1}(x) \cong \mathbb{P}^{n-2}$. Therefore $\mathcal{X}$ is a $\mathbb{P}^{n-2}$-fibration over $\ell_0 \cong \mathbb{P}^1$. As such it is birational to $\mathbb{P}^{n-1}$. We now have a dominant rational $\psi : \mathcal{X} \to X$ defined as follows: suppose $(x, \ell) \in \mathcal{X}$; since the line ℓ lies in T_xX it has intersection multiplicity ≥ 2 with X at x. We then assign to $(x, [\ell])$ the third point of intersection of ℓ with the cubic X. Note that the above map is not well defined on the locus of points $(x, \ell) \in \mathcal{X}$ with $\ell \subset X$. This is a proper closed subvariety of $\mathcal{X}$: as we have seen in Section 2, the locus lines through x contained in X is isomorphic to a closed subvariety of $\mathbb{P}^{n-2}$ defined as the intersection locus of the vanishing loci of a quadric and a cubic polynomials. If both polynomials are the zero polynomials then this is $\cong \mathbb{P}^{n-2}$ otherwise it is a closed subvariety of $\mathbb{P}^{n-2}$. By assuming that $x = [1, 0, \ldots, 0]$ and $T_xX = \mathbb{V}(x_n)$ the former happens exactly when X is given by the polynomial $x_0^2 x_n$, which defines a singular hypersurface (which is not our case). Therefore, for any point $x \in \ell_0$ we have that the above locus is a proper closed locus in $\mathbb{P}^{n-2}$ and then the same happens when x moves in ℓ_0.

One actually can see that $\psi : \mathcal{X} \to X$ is a finite map of degree two. Given a point $x \in X$ in the image of ψ (and thus outside ℓ_0), we take the plane Π_x spanned by x and ℓ_0. Then Π_x intersects X at the line ℓ_0 and a conic Q. Let $Q \cap \ell_0 = \{x_1, x_2\}$ (in general $x_1 \neq x_2$). Then $\psi^{-1}(x) = \{x_1, x_2\}$. Indeed, it suffices to show that the lines $\overline{xx_i}$, $i = 1, 2$ belong in the tangent planes $T_{x_i}X$, $i = 1, 2$, respectively: But since ℓ_0 is contained in X it belongs in the tangent planes $T_{x_i}X$, $i = 1, 2$ and Π_x is spanned by ℓ_0 and the tangent to the conic Q at x_i (the latter belongs in the $T_{x_i}X$). Hence $T_{x_i}X = \Pi_x$, $i = 1, 2$. $\square$

2. Cubic surfaces in $\mathbb{P}^3$

We first show that any smooth cubic surface X in $\mathbb{P}^3$ is rational. We follow the approach in [8], Lecture 15. We start with the following remark: by Bezout's theorem, the intersection of X with a 2-plane is a plane cubic. The generic situation is that this cubic is irreducible; but it also may break into a line and an irreducible conic, or to a triple of lines, that, a priori, two of them may coincide. We start with some lemmas:

Lemma 2.1. *Suppose ℓ is a line contained in X. Then $l^2 = -1$ (the intersection takes place in the Chow ring of X).*

Proof. Take $x \in X$ and a 2-plane Π containing ℓ, different from T_xX. Then Π intersects X generically transversely along ℓ and hence, by Bezout, $\ell \cdot X = \ell + Q$, with Q a quadric curve not containing ℓ. But then (in the Chow ring)

$$\begin{aligned} 1 =& \ell \cdot \Pi \text{ (in } \mathbb{P}^3) \\ =& \ell \cdot \ell + Q \text{ (in } X) \\ =& \ell^2 + \ell \cdot Q = \ell^2 + 2, \end{aligned}$$

hence the result. □

Lemma 2.2. *When a 2 plane Π intersects X at lines, then the intersection consists of three distinct lines.*

Proof. Suppose not, so suppose that the intersection of Π with X is of the form $2\ell + \ell'$, with ℓ, ℓ' lines (not necessarily different). We then have in the Chow ring

$$\begin{aligned} 1 =& \ell \cdot \Pi \text{ (the intersection takes place in } \mathbb{P}^3) \\ =& \ell \cdot (2\ell + \ell') \text{ (the intersection takes place in } X) \\ =& 2\ell^2 + 1 \text{ in case } \ell \neq \ell', \text{ or } 3\ell^2 \text{ in case } \ell = \ell'. \end{aligned}$$

The former case contradicts $\ell^2 = -1$, see Lemma 2.1, and the latter that the intersection number is an integer. □

Corollary 2.3. *If Π is a 2-plane containing a line $\ell \subset X$, then $\Pi \cap X$ is either a conic and a line, or three distinct lines.*

Proposition 2.4. *The cubic surface X contains at most 3 lines through any given point $p \in X$. If X contains two or three lines passing through $p \in X$, then these lie in the tangent plane $\Pi = T_pX$ and $\Pi \cap S$ has one of the following possible configurations: Three distinct lines in a plane as a triangle where two of them meet at p or an asterisk meeting in p .*

Proof. By Lemma 2.1, any line which passes through p and is contained in X belongs to $\Pi = T_pX$. Since $X \cap \Pi$ is of degree 3 this intersection contains at most three lines. This proves the first two claims. The last claim is then a consequence of Corollary 2.3. □

For the proof of the next proposition we will use the following Lemma

Lemma 2.5. *A plane conic given by*

$$Q = Ax_0^2 + 2Bx_0x_1 + Cx_1^2 + 2Dx_0x_2 + 2Ex_1x_2 + Fx_2^2$$

is singular if and only if

$$\det \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix} = 0$$

and in that case is a union of two lines.

Proof. The last claim is a consequence of Bezout's Theorem. If the conic Q is singular at p it cannot be irreducible, otherwise the line joining p and another point $q \in Q$ will not be contained in Q and intersects Q at three points; a contradiction. Hence Q is reducible and thus a union of two lines.

By what we have discussed on Section 1, Q has a singular point p if and only if $\frac{\partial Q}{\partial x_i}(p) = 0$, for each $i = 0, 1, 2$ which is equivalent to saying that p is a solution of the homogeneous linear system defined by the partial derivatives. The condition that this system has a non zero solution is given by the vanishing of the above determinant. □

Proposition 2.6. *Let X be a smooth cubic surface and $\ell \in X$ a line. Then there are exactly 10 lines which intersect ℓ. These lines are given by five pairs of lines (ℓ_i, ℓ_i'), $1 \le i \le 5$, which have the following properties:*

(1) *$\ell \cup \ell_i \cup \ell_i'$ lie in a plane. In particular, the lines ℓ_i and ℓ_i' intersect the line ℓ for $i = 1, ...5$*
(2) *$(\ell_i \cup \ell_i') \cap (\ell_j \cup \ell_j') = \emptyset$ for $i \ne j$. (Note that it is also possible that ℓ, ℓ_i, ℓ_i' meet at a point.)*

Proof. Let Π be a plane containing ℓ then as we already stated $\Pi \cap S = \ell + Q$ where Q is a conic. Now Q is either irreducible or $\Pi \cap X$ has one of the configurations of the above Proposition 2.4. So we must show there are exactly 5 planes containing ℓ where the conic Q decomposes into lines. Each of these planes gives a pair of lines and the previous proposition implies they have the given configuration. We can assume by fixing coordinates that ℓ is given by $\ell = \{x_2 = x_3 = 0\}$.

We now consider the pencil of planes $\Pi_{\lambda,\mu} : \{\mu x_2 - \lambda x_3 = 0\}$ containing the line ℓ. We want to determine for which values $[\lambda, \mu] \in \mathbb{P}^1$, the intersection $\Pi_{\lambda,\mu} \cap X$ consists of three lines. Since X contains ℓ it is given by a homogeneous cubic polynomial f with no term involving only x_0, x_1. Hence F has the form

$$f = Ax_0^2 + Bx_0x_1 + Cx_1^2 + Dx_0 + Ex_1 + F = 0, \tag{4}$$

where $A, B, C, D, E, F \in k[x_2, x_3]$ are homogeneous polynomials, with A, B, C linear, D, E quadratic and F cubic.

Observe now that the points of $\Pi_{\lambda,\mu}$ have the form $[x_0, x_1, \lambda t, \mu t]$, $x_0, x_1, t \in \mathbb{C}$. In the parametric coordinates x_0, x_1, t of the plane $\Pi_{\lambda,\mu}$, the line ℓ is given by the equation $t = 0$. Then h restricted to the plane that plane takes the form

$$f|_{H_{\lambda,\mu}} = tQ_{\lambda,\mu}(x_0, x_1, t),$$

where $Q_{\lambda,\mu}$ is the conic:

$$Q_{\lambda,\mu}(x_0, x_1, t) = A(\lambda, \mu)x_0^2 + B(\lambda, \mu)x_0x_1 + C(\lambda, \mu)x_1^2 + D(\lambda, \mu)x_0t + E(\lambda, \mu)x_1t + F(\lambda, \mu)t^2.$$

By the Lemma 2.5, the conic $Q_{\lambda,\mu}$ is singular (and hence a union of two distinct lines) for those $[\lambda, \mu]$ for which

$$\Delta(\lambda, \mu) := \det \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix} = ACF + 2DBE - AE^2 - B^2F - D^2C = 0.$$

This is a degree 5 homogeneous polynomial in λ, μ and we need to show that it has 5 distinct roots, i.e has no double roots. By a coordinate transformation we can take $[0, 1]$ to be a zero of $\Delta(\lambda, \mu)$. This means that λ divides $\Delta(\lambda, \mu)$ and we want to show that λ^2 does not divide $\Delta(\lambda, \mu)$.

By Proposition 2.6, the possible configurations of the intersection $\Pi_{0,1} \cap X$ in the coordinate x_0, x_1, t of the plane $\Pi_{0,1}$ (after performing a projective transformation on the plane) are

(1) $\ell = \{t = 0\}$, $\ell_1 = \{x_0 = 0\}$, $\ell_1' = \{x_1 = 0\}$
(2) $\ell = \{t = 0\}$, $\ell_1 = \{x_0 = 0\}$, $\ell_1' = \{x_0 - t = 0\}$ (all lines are going through the point $[0, 1, 0] \in \Pi_{0,1}$).

In the first case the lines ℓ_1 and ℓ_1' as a lines of $\mathbb{P}^3$ are given by $\ell_1 = \{x_0 = x_2 = 0\}$ $\ell_1' = \{x_2 = x_0 - x_3 = 0\}$. Since these lines are contained in X, the expression of f given in (4) its easy to see that it should satisfy $x_2 \mid A, C, D, E, F$. Then F takes the form

$$f = x_0x_1x_3 + x_2g(x_0, x_1, x_2, x_3),$$

with g of degree 2. But then

$$\Delta \equiv B^2F = -x_3^2F \mod \lambda^2,$$

In addition, the point $p = [0 : 0 : 0 : 1] \in X$, p is a smooth point means that F must contain the term $x_2x_3^2$ with non-zero coefficient. In particular, x_2^2 does not divide F. Therefore $(\lambda = 0)$ is a simple root of Δ.

In the second case, similar considerations show that

$$f = x_0x_3(x_0 - x_3) + x_2g,$$

with g is a quadratic again. We then get $D(x_2, x_3) = -x_3^2 + x_2\beta(x_2, x_3)$ and $x_2|B, C, E, F$. To conclude, we have

$$\Delta \equiv -CD^2 \mod \lambda^2$$

and since C is linear $C(x_2, x_3) = cx_2$ and X is smooth at $p = [0 : 1 : 0 : 0]$, we have $c \ne 0$ and we conclude as before. $\square$

Corollary 2.7. *A smooth cubic surface X contains a pair of skew lines.*

Proof. Take two of the above 5 planes, say Π_1 and Π_2 and choose a line $\neq \ell$ on each one of them, say ℓ_1, ℓ_2 respectively. If these lines meet, they have to meet at a point of the line ℓ. If the two lines $\neq \ell$ of Π_1 pass through the same point p of ℓ then ℓ_2 does not pass through this (recall: at most three lines contained in X can pass through a point) and hence ℓ_1 and ℓ_2 are skew. If the above two lines of Π_1 meet ℓ at two different points, then ℓ_2 and one of them are skew lines. $\square$

Lemma 2.8. *Let ℓ, μ two skew lines in $\mathbb{P}^3$. Then from every point $x \in \mathbb{P}^3$ outside $x \notin \ell \cup \mu$, there is a unique bisecant to ℓ and μ.*

Proof. Take the 2-plane Π spanned by the point x and the line ℓ. Then Π intersects μ at a unique point x': indeed, μ does not belong to Π otherwise is coplanar with ℓ (and then they intersect). The line $\overline{xx'}$ is the unique bisecant through x. $\square$

Theorem 2.9. *A smooth cubic surface X in $\mathbb{P}^3$ is rational (birational to $\mathbb{P}^2$).*

Proof. Given two skew lines ℓ and $\mu \in \mathbb{P}^3$ we can construct a rational map

$$\pi_{\ell,\mu} : \mathbb{P}^3 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1,$$

as follows. From a point $x \notin \ell \cup \mu$ there is a unique line L through x that intersects ℓ, say at the point Q, and μ , say at T. Thus we define the map

$$\begin{aligned} \pi_{\ell,\mu} : \mathbb{P}^3 \backslash \{\ell \cup \mu\} &\longrightarrow \quad \ell \times \mu \cong \mathbb{P}^1 \times \mathbb{P}^1 \\ x &\longrightarrow \quad (L \cap \ell = Q, L \cap \mu = T) \end{aligned}$$

This defines a morphism whose fiber over $(Q, T) \in \ell \times \mu$ is the line $L = \overline{PQ}$ of $\mathbb{P}^3$. By transformation of coordinates we can take ℓ to be $\mathbb{V}(x_2, x_3)$ and $\mu = \mathbb{V}(x_0, x_1)$. Then $\pi_{\ell,\mu}$ is given by

$$\begin{aligned} \pi_{\ell,\mu} : \mathbb{P}^3 \backslash \{\ell \cup \mu\} &\longrightarrow \quad \ell \times \mu \\ (x_0 : x_1 : x_2 : x_3) &\longrightarrow \quad ((x_0, x_1), (x_2, x_3)) \end{aligned}$$

which shows it is a rational map. We will show that its restriction to X

$$\phi = \pi_{\ell,\mu}|_X : X \longrightarrow \ell \times \mu$$

has a rational inverse

$$\psi : \ell \times \mu \longrightarrow X.$$

For $(Q, T) \in \ell \times \mu$, let L be the line $L = \overline{QT}$ in $\mathbb{P}^3$. By Proposition 2.6 there are only finitely many lines contained in X meeting ℓ, meaning that only for a finite number of points (Q, T) the bisecant is entirely in X . And so, for a general point in $\ell \times \mu$ the bisecant which is the fiber L meets, by Bezout, X at three points Q, R, T, of which Q and T are the given points on ℓ and μ. We thus define the rational map

$$\psi =: \ell \times \mu \longrightarrow X \text{ with } (Q, T) \mapsto R.$$

Now $\mathbb{P}^1 \times \mathbb{P}^1$ is birational to $\mathbb{P}^2$ and hence that X is birational to $\mathbb{P}^2$. $\square$

3. 27 lines on a cubic

Theorem 3.1. *A general cubic surface X in $\mathbb{P}^3$ contains 27 (different) lines. In fact, it can be shown that every smooth cubic contains 27 (different) lines.*

Proof. We know that $\dim \mathcal{F}_X = 0$ and hence the universal Fano variety $\Phi(3, 3)$ introduced in Section 6 is smooth and maps generically finitely to one on the parameter space of cubic surfaces $\mathbb{P}^{N=19}$. This implies that the general fiber, that is, the Fano variety of the general cubic surface is a finite number m of points (without multiplicities as a cycle in the Chow group).

The Chern class of this locus is a zero cycle and as such its class is a m times the generator $\sigma_{2,2}$ of $A^4(\mathbb{G}(1, 3))$ and by By Corollary 9.3 it is given by the class $c_4(\mathrm{Sym}^3 \mathcal{U}^\vee)$, with $\mathcal{U}$ the universal bundle on $\mathbb{G}(1, 3)$. In Section 4 we saw that

$$c(\mathcal{U}) = 1 - \sigma_1 + \sigma_{1,1} \text{ and hence } c(\mathcal{U}^\vee) = 1 + \sigma_1 + \sigma_{1,1}$$

and then, Equation (1) gives

$$c_4(\mathrm{Sym}^3 \mathcal{U}^\vee) = 9\alpha\beta(2(\alpha + \beta)^2 + \alpha\beta),$$

with $\alpha + \beta = c_1(\mathcal{U}^\vee) = \sigma_1$ and $\alpha\,\beta = c_2(\mathcal{U}^\vee) = \sigma_{1,1}$. We conclude,

$$c_4(\mathrm{Sym}^3 \mathcal{U}^\vee) = 9\sigma_{1,1}(2\sigma_1^2 + \sigma_{1,1}) = 27\sigma_{2,2},$$

by Pieri's formula 5.6. ☐

Example 3.2. An explicit example is the Fermat cubic surface given by the polynomial

$$F = x_0^3 + x_1^3 + x_2^3 + x_3^2$$

In this case we can determine the 27 lines. They are given by the system of equation

$$x_0 + \omega x_i = x_j + \omega' x_k = 0, \text{ with } \{i, j, k\} = \{1, 2, 3\}, \ j < k,$$

where ω, ω' are some of the three roots of unity, possibly the same.

Remark 3.3. To show that every smooth cubic surfaces contains 27 lines, one shows that the restriction of the map $\pi : \Phi(3, 3) \to \mathbb{P}^{19}$ over the open locus $U_{sm} \subset \mathbb{P}^{19}$ of smooth cubic surfaces is a topological covering. In other words, the exceptional locus of the generically finite map π lies inside the divisor of singular cubics.

4. Cubic threefolds, fourfolds, etc

Proposition 4.1. *Through a general point of a general cubic threefold in $\mathbb{P}^4$ there pass six (different) lines.*

Proof. We take $Q(x_1, x_2, x_3), C(x_1, x_2, x_3)$ a general quadratic and a general cubic homogeneous polynomial which define an irreducible quadric Q and an irreducible cubic C in $\mathbb{P}^2$, respectively. General here means that the quadric and the cubic are smooth and they and intersect transversely. Then by Bezout's theorem Q and C intersect at six different points. We then take the cubic threefold X_0 defined by the degree 3 homogeneous polynomial

$$F_0 = x_0^2 x_4 + x_0 Q(x_1, x_2, x_3) + C(x_1, x_2, x_3).$$

X_0 is smooth at the origin $[1, 0, 0, 0, 0]$ with tangent plane given by $x_4 = 0$. By the discussion in Section 2, the lines through the origin which are contained in X_0 correspond to the intersection points of Q with C in $\mathbb{P}^2$ and thus there are exactly 6 lines going through the origin and contained in X_0. We now consider the incidence variety

$$\mathcal{J} = \{(p; [\ell], [X]), p \in \ell \subset X\} \subset \mathbb{P}^4 \times \mathbb{G}(1, 4) \times \mathbb{P}^{34},$$

with $\mathbb{P}^{34}$ the space which parametrizes cubic threefolds in $\mathbb{P}^4$. The image of the projection $\pi : \mathcal{J} \to \mathbb{P}^4 \times \mathbb{P}^{19}$ is the locus $\{(p, [X]), p \in X\} \subset \mathbb{P}^4 \times \mathbb{P}^{34}$ which is easily seen to be of dimension $34 + 3 = 37$. The fiber of π over a point $(p, [X])$, $p \in X$, is exactly the lines going through p and contained in X. The other projection $q : \mathcal{J} \to \mathbb{P}^4$ is a $\mathbb{P}^{33}$-fibration and thus $\mathcal{J}$ is a smooth irreducible space: indeed the fiber over a point $p \in \mathbb{P}^4$ is the locus in $\mathbb{P}^{34}$ of the cubic hypersurfaces in $\mathbb{P}^4$ containing the point p. This is a linear condition on the coefficients of the cubic equation and thus define a hyperplane in $\mathbb{P}^{34}$, i.e. space $\cong \mathbb{P}^{33}$. A dimension count then shows that $\dim\mathcal{J} = 33 + 4 = 37 = \dim\mathrm{Im}\pi$. Hence the general fiber of the map $\pi : \mathcal{J} \to \mathrm{Im}\pi$ is zero dimensional i.e. a finite number of points and π is a generically finite map, i.e. there is an open $U \subset \mathrm{Im}\pi$ such that the restriction map $\pi : \mathcal{J}|_U \to U$ is a finite map. Note that since $\mathcal{J}$ is irreducible then also its restriction $\mathcal{J}|_U$ on the Zariski open $\pi^{-1}U$ is irreducible. This implies that through a general couple $(p, [X])$, $p \in X$ there is a finite number of lines going through p and are contained in X. Since the latter corresponds to the intersection of a quadric and a cubic in $\mathbb{P}^2$, this means that the quadric and the cubic have no common components and thus they intersect at ≤ 6 distinct points. We therefore have that the finite irreducible cover $\mathcal{J}|_U \to U$ is of degree at most 6. But since over the point $([1, 0, 0, 0, 0], X_0)$ is of degree exactly 6 then, by general theory, it is a cover of degree 6, which means over the general point of U the fiber consists of 6 distinct points. This completes the proof. ☐

Remark 4.2. Actually, more is true: Any smooth cubic threefold contains at most a finite number of points through which they pass an infinite number of lines contained in the threefold. These points are called Eckardt points. Moreover, the general cubic threefold X contains no Eckardt points and the locus of points from which they pass ≤ 5 distinct lines is an irreducible surface in X.

Proposition 4.3. *The locus of lines which pass through a general point of a general cubic fourfold in $\mathbb{P}^5$ is parametrized by a smooth curve of genus g.*

PROOF. We work as in the proof of Proposition 4.1. By the discussion in Section 2, the lines through the origin which are contained in X_0 correspond to the intersection points of a quadric and a cubic in $\mathbb{P}^3$. We choose a general quadric Q and a cubic C. These are smooth irreducible and intersect transversely. This is a $(2,3)$ intersection in $\mathbb{P}^3$ which, by well known theorey, is a smooth curve of genus g (canonically embedded). SInce the condition of "smooth irreducible and intersect transversely" is an open condition then this is true for a Zariski open subset of points of the fourfold. $\square$

Remark 4.4. In general, the locus of lines which pass through a general point of a general cubic in $\mathbb{P}^n$ is parametrized by a smooth variety of dimension $n-4$, which is a $(2,3)$ intersection in $\mathbb{P}^{n-2}$.

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