University of Crete
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UNDERGRADUATE THESIS

Exact Solutions of the Einstein-Maxwell Field Equations: The Electron as a Point-like Particle in General Relativity

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Preface

The following thesis is the result of work conducted during the last one and a half year of my undergraduate studies. I would like to thank my professor, Taxiarchis Papakostas, who was essential in the prosecution of this thesis, offering useful advice on the content of the thesis and general discussions on the theory of general relativity. I would also like to thank my parents for their support and my fellow students from the physics department, who were always willing to discuss matters of this kind with me. Special thanks to George Mouloudakis, for helping me with Mathematica.

Notation

In this text we will use the metric signature convention $(+,-,-,-)$

Greek letters usually denote spacetime indices:

$$\mu, \nu, \kappa, \lambda, ... = 0, 1, 2, 3$$

Latin letters usually denote spatial indices.

$$i, j, k, l, ... = 1, 2, 3$$

Einstein summation convention: every index repeated two times within the same term is summed

$$\sum_{\mu=0}^{4} u^{\mu} u_{\mu} \equiv u^{\mu} u_{\mu}$$

Partial derivative:

$$\partial_{\beta} \equiv \frac{\partial}{\partial x^{\beta}} \equiv \quad_{,\beta}$$

Covariant derivative:

$$\nabla_{\beta} \equiv \quad_{;\beta}$$

Introduction

After publishing his paper titled "On the electrodynamics of moving bodies" in 1905, in which the special theory of relativity is introduced, Albert Einstein wanted to extend his theory, so that it would include gravitational effects. Special relativity had its foundations on the equivalence between observers moving with constant velocity with respect to each other, meaning that these observers experience the same laws of nature. But, who was to say that this doesn't hold for observers that are accelerating or rotating? With that thought in mind, Einstein tried to find a theory that would connect observers that are moving arbitrarily. He succeeded 10 years later and, in 1915, his paper on the general theory of relativity was published. The equations provided by the new theory were non-linear and very difficult to solve, even by Einstein. However, in 1916, astrophysicist Karl Schwarzschild found the first non-trivial exact solution to the Einstein field equations: the Schwarzschild metric. Other exact solutions followed in the next years, with popular examples being the Reissner-Nordström metric, the Kerr metric, the Kerr-Newman metric, the FLRW metric etc. General relativity has been through many experimental tests since its discovery: it calculated the precession of the perihelion of Mercury, predicted that light should be deflected by massive objects like the Sun and explained why light should be redshifted when propagating in a gravitational field. In a more recent event, approximately 100 years after its discovery, the general theory of relativity was once again verified by the long awaited detection of gravitational waves. The theory is being extensively used in attempts at modelling the birth, evolution and collapse of stars, understanding black holes and neutron stars, as well as studying the universe itself. After the detection of gravity waves, it also provided a new way to "listen" the Universe and opened new doors in multi-messenger astronomy. In contrast with how general relativity is most frequently used, in the present work we will not be dealing with any large scale, massive objects, such as stars or black holes. We will attempt to find the effects of gravity on one of the smallest entities of our universe: the electron. The electron, and all other elementary particles, is considered to be of zero size. However, a dimensionless particle with finite mass and charge would have infinite density. This is a known issue in quantum theory, referred to as self-energy or self-interaction problem and it can be sidestepped by employing a number of techniques known as renormalization, so that it is possible to perform relevant calculations. Nevertheless, the theoretical problem remains unsolved.

Before addressing this issue, we are going to present the basic notions of the special and general theory of relativity, as well as the mathematical tools that are needed for our calculations.

1 Special Theory of Relativity

The laws of mechanics, as formulated by Newton, were based on the concepts of absolute space and universal time. Galileo's relativity principle supports these ideas, stating that Newton's laws hold in all reference frames that are in relative motion with respect to each other. Newton's equations of motion are invariant under Galileo's transformation, which connects two observers with relative velocity V (here taken to be along the x-axis):

$$\begin{aligned} t' &= t \\ x' &= x - Vt \\ y' &= y \\ z' &= z \end{aligned}$$

However, Maxwell's equations for electromagnetism are not invariant under the above transformation, but possess another kind of symmetry: they keep their form under Lorentz transformations, which will be presented later. Considering that all laws of nature should be consistent with each other and should possess the same kind of symmetry, the above inconsistency was a strong indication that one of the two theories was wrong.

1.1 Postulates of the New Theory and First Results

The inconsistency stated above was overcome with the generalization of the laws of mechanics by the special theory of relativity, which was developed with contributions from Einstein, Poincare, Lorentz, Michelson, Morley and others and was finally formulated by Einstein in 1905.

The theory is based on two postulates:

Postulate 1: The laws of nature have the same form for all inertial reference frames.

Postulate 2: The speed of light in vacuum has a constant value of $c = 2.99792458 \times 10^8 m/sec$ and is the same for all inertial reference frames.

Using these two statements, it is possible to conduct a series of gedanken experiments, which are essential for the construction of special relativity. The most important results that can be extracted from these thought experiments, predict the relativity of space and time measurements, which should depend on the velocity of the observer conducting the measurement with respect to a motionless observer.

An observer moving with constant velocity V (in the x-direction) with respect to another observer at rest will measure time at different rate, a phenomenon referred to as time dilation:

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{1}$$

Similarly, we can prove that the same observer, when measuring the length of an object , or equivalently, the spatial separation between two events, is going to experience an effect referred to as *length contraction*, meaning that the measured length of an object varies between observers moving with respect to one another. Let us denote as L_0 the length of an object (in the x-direction) (for example a rod) measured in the object's rest frame. An observer moving with velocity V (in the x-direction) with respect to the rod, is going to measure a different length for the object, related to L_0 through the formula:

$$L = L_0\sqrt{1 - \frac{V^2}{c^2}} \tag{2}$$

Generally, observers (or -equivalently- reference frames) that move linearly with constant velocity are referred to as *inertial observers* (or inertial reference frames).

1.2 The Lorentz Transformation

Spacetime measurements made by inertial observers are linked to each other by a specific type of transformation, the Lorentz transformation. Considering two observers O and O' moving with relative velocity V along the x-axis, the coordinates (t, x, y, z) and (t', x', y', z') are linked to each other by the relations

$$t' = \frac{t - \frac{Vx}{c^2}}{1 - V^2/c^2}; \quad x' = \frac{x - Vt}{1 - V^2/c^2}; \quad y' = y; \quad z' = z \tag{3}$$

We can combine these two effects in the following matrix equation:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda(V) \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

The matrix $\Lambda(V)$ is called the Lorentz transformation matrix:

$$\Lambda(V) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

,where

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

is the *Lorentz factor* and

$$\beta \equiv \frac{V}{c}.$$

The Lorentz transformation predicts that events that occur at the same time for one observer do not occur at the same time for another observer moving relative to the first one. This means that the absolute time and space that is used in the Galilean transformations must be abandoned and we should instead describe time and space as a part of the same continuum, known as *spacetime*.

An event is an occurence that is characterized by a definite time and location relative to some reference frame. This means, that an event can be thought as a point in spacetime, fully determined by four elements: (ct, x, y, z). Any four components that transform like $\begin{pmatrix} ct & x & y & z \end{pmatrix}^T$ under Lorentz transformations are said to form the components of a contravariant four-vector. Four vectors are usually noted as

$$x^\mu \ , \quad \mu = 0, 1, 2, 3$$

In Euclidian space, the distance between two points is invariant under rotations in space, something that is expressed mathematically through the Pythagorean theorem:

$$ds^2 = dx^2 + dy^2 + dz^2$$

When we include time as a dimension, the "distance" that remains invariant, in this case under Lorentz transformations, is the spacetime distance between two events:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

and has the same value for all inertial observers. In some sense, Lorentz transformations can be considered as rotations in spacetime.

A relevant quantity is the proper time of an observer, defined as the time as measured by a clock in the observer's rest frame. It is related to ds through the relation:

$$ds = c \cdot d\tau$$

Thus, the proper time between two events is

$$\Delta\tau = \int_P d\tau = \int_P \frac{ds}{c} = \int_P \sqrt{dt^2 - \frac{dx^2}{c^2} - \frac{dy^2}{c^2} - \frac{dz^2}{c^2}} = \int \sqrt{1 - \frac{V^2(t)}{c^2}} dt = \int \frac{dt}{\gamma(t)}$$

where the integral is performed along the trajectory of the observer. The interval ds is invariant under Lorentz transformations, because all inertial observers measure the same value for proper time between two events in their trajectory in spacetime.

We can also write the infinitesimal spacetime interval as:

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$$

,where $\eta_{\alpha\beta}$ is the Minkowski metric:

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Using that notation, we can write the Lorentz transformation in the form:

$$x'^\alpha = \Lambda^\alpha{}_\mu x^\mu$$

,where $\Lambda^\alpha{}_\mu$ are the elements of the Lorentz transfomation matrix.

An equation (e.g a law of physics) that keeps its form under a Lorentz transformation is said to be *Lorentz covariant.*

1.3 Four-vectors

A 4-vector is an object with four components that transforms under Lorentz transformation in a specific way. More specifically, the components of a contravariant 4-vector transform according to:

$$V'^\alpha = \Lambda^\alpha{}_\mu V^\mu$$

and the components of a covariant 4-vector transform as:

$$U'_\alpha = \Lambda_\alpha{}^\mu U_\mu$$

We can also use the metric tensor to lower and raise indices, i.e. to create contravariant vectors from covariant ones and vice-versa:

$$V^\alpha = \eta^{\alpha\beta} V_\beta$$

$$V_\alpha = \eta_{\alpha\beta} V^\beta$$

With the metric tensor we can define an invariant scalar product of 4-vectors:

$$V^2 = \eta^{\alpha\beta} V_\alpha V_\beta = \eta_{\alpha\beta} V^\alpha V^\beta$$

1.4 Relativistic Energy and Momentum

Let $x^\alpha = x^\alpha(\tau) = (ct, \mathbf{x})$ be the trajectory of a particle in spacetime. Then, the 4-vector of the velocity (or simply 4-velocity) of that particle is defined as

$$u^\alpha \equiv \frac{dx^\alpha}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{d\mathbf{x}}{d\tau} \right) = \gamma(c, \mathbf{v})$$

, where τ is the proper-time measured by the particle.

The 4-momentum of the particle is defined by

$$p^\alpha \equiv m u^\alpha = m\gamma(c, \mathbf{v}) = (\gamma mc, \mathbf{p})$$

We see that the $0-th$ component of the 4-momentum is equal to the total energy of the particle divided by c:

$$p^\alpha = (E/c, \mathbf{p})$$

We now consider the 4-momentum of a particle as measured by two different observers and denote them as p^α and $p^{\alpha\prime}$ It is possible to construct a scalar with the momentum 4-vector, which -by definition- will have the same value in both reference frames:

$$p^\alpha p_\alpha = p'^\alpha p'_\alpha$$

Now, if the primed frame is chosen to be the rest frame of the particle, then: $p'^\alpha = (mc, 0)$, and the above equation yields the relativistic energy-momentum relation:

$$E^2 = p^2 c^2 + m^2 c^4 \tag{4}$$

1.5 Covariant Form of Maxwell's Equations

Maxwell's equations are able to describe all classical electromagnetic phenomena and dictate the generation and evolution of electric and magnetic fields in a medium. They are usually written in the form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{8}$$

These equations allow us to express the electric field as the gradient of a scalar potential Φ and the magnetic field as the curl of a vector potential $\mathbf{A}$:

$$\mathbf{E} = -\nabla \Phi \quad ; \quad \mathbf{B} = \nabla \times \mathbf{A} \tag{9}$$

It is possible to write Maxwell's equations in a different form, which is going to be useful for relativistic applications, by introducing appropriate 4-vectors. The content will be the same, but the covariance under Lorentz transformation will become more obvious. We define the electromagnetic 4-potential

$$A^\alpha \equiv \left(\frac{\Phi}{c}, \mathbf{A} \right)$$

the 4-current

$$j^\alpha \equiv (c\rho, \mathbf{J})$$

as well as the the electromagnetic field tensor

$$F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1/0 & \end{pmatrix}$$

Now, Maxwell's equations can be written in a more compact form:

$$\partial_\beta F^{\alpha\beta} = -\mu_0 j^\alpha \tag{10}$$

$$\partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0 \tag{11}$$

The quantity that contains information about the energy and momentum stored in an electromagnetic field is the *electromagnetic energy-momentum tensor*:

$$T^{\alpha\beta} = \frac{1}{\mu_0}\left(F^{\alpha\mu}F_{\mu}^{\beta} - \frac{1}{4}\eta^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\right) \tag{12}$$

There is an expression for the energy-momentum tensor corresponding to each different distribution of charges and currents.

2 Elements of Differential Geometry

In the following section, we will present some basic elements of differential geometry, which is the required mathematical framework to formulate the general theory of relativity.

2.1 Manifolds

General relativity deals with curved spacetimes of 3+1 dimensions, which mathematically are dealt with as 4 dimensional Riemannian (or pseudo-Riemannian) manifolds. A manifold is essentially a space, generally with N dimensions, which is locally similar to Euclidian space. A more formal definition is the following:

Definition: a set of points, **M**, is called an N-dimensional manifold if any point of **M** has a neighbourhood that allows one-to-one continuous map onto an open set in $\mathbb{R}^{\mathrm{N}}$ (N-dimensional real space).

To locate a point in a manifold we use a system of N coordinates, but the choice of these coordinates is arbitrary. Therefore we may relabel the points of a manifold by performing a *coordinate transformation*:

$$x^{\mu} \rightarrow x'^{\mu} = x'^{\mu}(x^1, x^2, ..., x^N), \qquad a = 1, 2, ..., N$$

The $N \times N$ transformation matrix $[\frac{\partial x'^a}{\partial x^\beta}]$ is called the Jacobian matrix and its determinant, noted as J, is the Jacobian of the transformation.
If $J \neq 0$ we can (in principle) solve the equations for the old coordinates x^a and obtain the inverse transformation equations:

$$x^a = x^a(x'^1, x'^2, ..., x'^N), \qquad a = 1, 2, ..., N$$

In a similar manner to the above, we define the inverse transformation matrix: $\frac{\partial x^a}{\partial x'^\beta}$ and its Jacobian $J' = \frac{\partial x^a}{\partial x'^\beta}$.

Using the chain rule it is easy to show that the inverse transformation matrix is the inverse of the initial transformation matrix:

$$\sum_{\beta=1}^{N} \frac{\partial x'^a}{\partial x^\beta} \frac{\partial x'^\beta}{\partial x^\gamma} = \delta^a{}_\gamma$$

A subset of points belonging on a manifold define a *curve* or a *surface.* A curve has only one degree of freedom, so it can be parametrized using some parameter u as follows:

$$x^\alpha = x^\alpha(u), \quad \alpha = 1, 2, ..., N$$

On the other hand, an M-dimensional surface has more degrees of freedom, so it can be parametrized as:

$$x^\alpha = x^\alpha(u^1, u^2, ..., u^M), \quad \alpha = 1, 2, ..., N$$

2.1.1 Riemannian Geometry

For developing general relativity, there is need to focus on the study of a certain category of manifolds, in which the distance between two points of the manifold is given by an interval that is quadratic in the coordinate differentials:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{13}$$

,where $g_{\alpha\beta}$ are the components of the metric tensor. In the general case of curved manifolds, the components of the metric tensor are functions of the coordinates, which determinine the local geometry around any point of the manifold.

If $ds^2 > 0$, the manifold is called *Riemannian.* If $ds = 0$ or $ds < 0$, the manifold is called *pseudo-Riemannian.*

2.2 Local Cartesian Coordinates

For a general Riemannian or pseudo-Riemannian manifold, it is not possible to perform a coordinate transformation $x^\alpha \rightarrow x'^\alpha$ that will take the line element $ds^2 = g_{\alpha\beta} dx'^\alpha dx'^\beta$ into the Euclidean form

$$ds^2 = (dx'^1)^2 + (dx'^2)^2 + (dx'^3)^2 = \delta_{\alpha\beta} dx'^\alpha dx'^\beta$$

at every point in the manifold. However, it is possible to make a coordinate transformation such that in the neighbourhood of some specified point P, the line element takes the Euclidean form. In other words, we can always find coordinates x'^α such that, at the point P, the new metric functions $g'_{\alpha\beta}(x')$ satisfy

$$g'_{\alpha\beta}(P) = \delta_{\alpha\beta}$$

$$\left.\frac{\partial g'_{\alpha\beta}}{\partial x'^\sigma}\right|_P = 0$$

Thus, in the neighbourhood of P, we have

$$g_{\alpha\beta}(x') = \delta_{\alpha\beta} + \mathcal{O}[(x' - x'_P)^2]$$

The case that is most interesting in general relativity is the case of four-dimensional spacetime. In this case, we can always find a locally flat coordinate system, i.e. it is always possible to find a coordinate system x'^α for which

$$g_{\alpha\beta}(x') = \eta_{\alpha\beta} + \mathcal{O}[(x' - x'_P)^2]$$

where $\eta_{\alpha\beta}$ is the Minkowski metric.

Let us for simplicity consider a spherical surface as an example of a two-dimensional curved surface. At an arbitrary point P on the surface we can find coordinates x and y, such that in the neighborhood of P we have

$$ds^2 = dx^2 + dy^2$$

It thus follows, that a Euclidean two-dimensional space (a plane) will match the manifold locally at P. This Euclidean space is called the *tangent space* T_P to the manifold at P. This is valid for N-dimensional Riemannian and pseudo-Riemannian spaces: at any point P we can find a coordinate system such that in the neighborhood of P the line element is Euclidean (in the case of general relativity, the line element is that of Minkowski spacetime)

2.3 General Coordinate Systems

In flat spaces, we usually define a basis, which consists of a set of unit vectors orthogonal to each other, and any other vector in that space can be expressed as a linear combination of these vectors. However, in curved spaces, we are obligated to define basis vectors, which may not be mutually perpendicular and may have different length.

Consider a vector $\mathbf{v}$ on a point P of a manifold. At each point P the vector lies in the tangent space T_P of that point. At each point, we can define a set of basis vectors $\boldsymbol{e_\alpha}$ for the tangent space T_P. Thus, we can always express any vector field $\boldsymbol{u}(x)$ at each point in terms of basis vectors:

$$\mathbf{v}(x) = v^\alpha(x)\boldsymbol{e_a}(x) \tag{14}$$

The numbers $v^\alpha(x)$ are the the contravariant components of the vector field $\boldsymbol{v}(x)$ in the basis $\boldsymbol{e_\alpha}$.

For any set of basis vectors $\boldsymbol{e_\alpha}(x)$, we can define another set of basis vectors, called the *dual basis vectors*, denoted by $\boldsymbol{e^\alpha}(x)$. At any point on the manifold, the dual basis vectors are defined by the relation:

$$\boldsymbol{e}^\alpha(x) \cdot \boldsymbol{e}_\beta(x) = \delta^\alpha_\beta \tag{15}$$

The dual basis vectors at P also lie in the tangent space T_P and form an alternative basis for it. Therefore, we can also express the vector field

$$\mathbf{u}(x) = u_\alpha(x)\boldsymbol{e^a}(x) \tag{16}$$

In any particular coordinate system x^α, we can define at every point P of the manifold a set of *coordinate basis vectors*:

$$\mathbf{e}_\alpha = \lim_{\delta x^\alpha \to 0} \frac{\delta \mathbf{s}}{\delta x^\alpha} \tag{17}$$

where $\delta\mathbf{s}$ is the infinitesimal vector displacement between P and a nearby point Q, whose coordinate separation from P id δx^α along the x^α. Thus, $\mathbf{e}_\alpha$ is the tangent vector to the x^α coordinate curve at the point P. The infinitesimal separation between two nearby points that have coordinates x^α and $x^\alpha + dx^\alpha$ is:

$$d\mathbf{s} = \mathbf{e}_\alpha(x) dx^\alpha \tag{18}$$

We can use this expression to relate the inner product of the coordinate basis vectors at some arbitrary point P to the value of the metric functions $d_{\alpha\beta}$ at that point:

$$ds^2 = d\mathbf{s} \cdot d\mathbf{s} = (dx^\alpha \mathbf{e}_\alpha) \cdot (dx^\beta \mathbf{e}_\beta) = (\mathbf{e}_\alpha \cdot \mathbf{e}_\beta) dx^\alpha x^\beta$$

Comparing with the general expression $ds^2 = g_{\alpha\beta}(x) dx^\alpha dx^\beta$, we conclude that:

$$\mathbf{e}_\alpha(x) \cdot \mathbf{e}_\beta(x) = g_{\alpha\beta}(x) \tag{19}$$

Generally, in a coordinate basis the scalar product of two vectors is given by:

$$\mathbf{v} \cdot \mathbf{w} = (v^\alpha \mathbf{e}_\alpha) \cdot (w^\beta \mathbf{e}_\beta) = g_{\alpha\beta} v^\alpha w^\beta$$

2.4 Tensors

To construct a theory of general relativity, it is essential to find a way to write equations in such a way that they have the same form and physical meaning in any reference frame. To do this, we need to classify mathematical objects according to the way they transform under coordinate transformations between general frames. The simplest transformation rule we can think of is that of scalars, which simply do not change their value under such transformations. Some examples are pure numbers or the spacetime interval ds. The next simplest transformation rule is that of vectors, which can be *contravariant*:

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\mu$$

or *covariant*:

$$U'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} U_\nu$$

A more general object, having an arbitrary number of covariant and contravariant indices, i.e. a *tensor*, transforms as:

$$T'^{\mu_1 \mu_2 \ldots \mu_m}_{\nu_1 \nu_2 \ldots \nu_n} = \prod_{i=1}^{m} \frac{\partial x^{\mu_i}}{\partial x^{\kappa_i}} \prod_{j=1}^{n} \frac{\partial x^{\lambda_j}}{\partial x^{\nu_j}} T^{\kappa_1 \kappa_1 2 \ldots \kappa_1 m}_{\lambda_1 \lambda_2 \ldots \lambda_n}$$

Generally, equations with tensors keep their form under any transformation between general coordinate frames. Therefore, physics equations within the framework of a theory of general relativity should be tensor equations.

2.5 Parallel Transport and Connection on Metric Manifolds

Consider a vector $\mathbf{v}$ at a point P of a metric manifold. We can transport that vector along the displacement vector $d\mathbf{x}$ into the infinitesimally close point S, in such a way that it remains parallel to itself. The resulting vector can be noted as $\check{\mathbf{v}}$. Then, the whole operation can be expressed as:

$$\check{\mathbf{v}} = \Gamma(P, \mathbf{v}, d\mathbf{x}) \tag{20}$$

where Γ is the operator of parallel transport, also called the *connection*. Once this operator is defined at every point of the manifold, we have means of transporting every vector on that manifold parallel to itself. The basic requirements of parallel transport are:

1) If $\mathbf{v} = 0$ then $\check{\mathbf{v}} = 0$

2) If $d\mathbf{x} = 0$ then $\check{\mathbf{v}} = \mathbf{v}$

3) If $\mathbf{v} = \alpha\mathbf{x} + \beta\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are basis vectors, then $\check{\mathbf{v}} = \alpha\check{\mathbf{x}} + \beta\check{\mathbf{y}}$

4) For a displacement $d\mathbf{x}_1$,we have: $v^i - \check{v}^i = dv^i$. Then, for a different displacement $d\mathbf{x}_2 = \kappa d\mathbf{x}_1$,we will have: $v^i - \check{v}^i = \kappa dv^i$

It can be proved, that these requirements are satisfied only if:

$$\check{v}^\mu = v^\mu - \Gamma^\mu_{\sigma\lambda} v^\sigma dx^\lambda \tag{21}$$

where $\Gamma^\mu_{\sigma\lambda}$ are the coordinate components of Γ, also known as *Christoffel's symbols of the first kind.*

2.5.1 Connection on Riemannian Manifolds

In any Riemannian manifold the connection is related to the metric through the relation:

$$\Gamma^{\mu}{}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left(\frac{\partial g_{\rho\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\rho\lambda}}{\partial x^{\nu}} - \frac{\partial g_{\nu\lambda}}{\partial x^{\rho}} \right) \tag{22}$$

$\Gamma^{\mu}{}_{\nu\lambda}$ are often called *Christoffel's symbols of the second kind.*

2.6 Covariant Differentiation

In general, differentiation of a tensor does not yield another tensor. This can be easily seen, considering a contravariant vector V^{μ}, which transforms as:

$$V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}$$

Differentiating with respect to x'^{α} we get:

$$\frac{\partial V'^{\mu}}{\partial x'^{\alpha}} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial V^{\mu}}{\partial x^{\beta}} + \frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\alpha}} V^{\nu}$$

Because of the second term, the derivative does not transform as a tensor.

Using the above relation, there is an obvious way to define a new derivative ∇_{ν}, called the *covariant derivative*, which will transform as a tensor. The covariant derivative of a contravariant vector V^{μ} is defined as:

$$\nabla_{\nu} V^{\mu} \equiv V^{\mu}{}_{;\nu} = \partial_{\nu} V^{\mu} + \Gamma^{\mu}{}_{\nu\lambda} V^{\lambda} \tag{23}$$

Covariant derivatives transform as tensors:

$$\nabla'_{\nu} V'^{\mu} = \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial x'^{\beta}}{\partial x^{\mu}} \nabla'_{\alpha} V'^{\beta}$$

Through this requirement, it is possible to show that the Christoffel symbols transform as:

$$\Gamma'^{\alpha}{}_{\beta\lambda} = \frac{\partial x^{\mu}}{\partial x'^{\beta}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial x'^{\alpha}}{\partial x'^{\nu}} \Gamma^{\nu}{}_{\mu\rho} - \frac{\partial x^{\mu}}{\partial x'^{\beta}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\mu} \partial x^{\rho}}$$

From this transformation law we see that the connection of a manifold is not a tensor.

We can expand the analysis to find the expressions of the covariant derivatives of covariant vectors:

$$\nabla_{\beta} U_{\alpha} \equiv U_{\alpha;\beta} \equiv \partial_{\beta} U_{\alpha} - \Gamma^{\mu}{}_{\alpha\beta} U_{\mu} \tag{24}$$

,as well as higher order tensors:

$$\nabla_\gamma T^{\alpha\beta} = \partial_\gamma T^{\alpha\beta} + \Gamma^\alpha{}_{\mu\gamma} T^{\mu\beta} + \Gamma^\beta{}_{\mu\gamma} T^{\alpha\mu} \tag{25}$$

$$\nabla_\gamma T^\alpha{}_\beta = \partial_\gamma T^\alpha{}_\beta + \Gamma^\alpha{}_{\mu\gamma} T^\mu{}_\beta - \Gamma^\mu{}_{\beta\gamma} T^\alpha{}_\mu \tag{26}$$

$$\nabla_\gamma T_{\alpha\beta} = \partial_\gamma T_{\alpha\beta} - \Gamma^\mu{}_{\alpha\gamma} T_\mu \beta - \Gamma^\mu{}_{\beta\gamma} T_{\alpha\mu} \tag{27}$$

2.7 Geodesics

Consider a differentiable manifold of N-dimensions with metric g and two points A and B on that manifold. Consider also a curve $x^\mu = x^\mu(\lambda)$ connecting these two points. The distance between A and B along this curve is:

$$l_{AB} = \int ds = \int \frac{ds}{d\lambda} d\lambda = \int \sqrt{\left(\frac{ds}{d\lambda}\right)^2} = \int_{\lambda_A}^{\lambda_B} L(x,\dot{x}) d\lambda \tag{28}$$

where the Lagrangian is:

$$L(x,\dot{x}) = [g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu]^{1/2} = \left[g_{\mu\nu}\frac{\partial x^\mu}{\partial\lambda}\frac{\partial x^\nu}{\partial\lambda}\right]^{1/2}$$

The curve for which l_{AB} is minimum is called a *geodesic*. Generally, a geodesic is defined as a curve that extremises distances between all its points.

Consequently, geodesics are the resulting solutions of the Euler-Lagrange equations for the above Lagrangian:

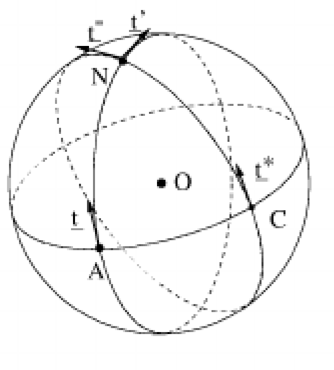
$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\mu} = \frac{\partial L}{\partial x^\mu} \quad ; \quad (\mu = 0, 1, ..., N)$$

Substituting the Lagrangian $L(x,\dot{x}) = [g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu]^{1/2}$, we obtain the so called geodesic equation:

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\kappa}\frac{dx^\nu}{d\lambda}\frac{dx^\kappa}{d\lambda} = 0 \tag{29}$$

2.8 Riemann Curvature Tensor

Parallel transport on Riemannian manifolds has a number of properties not seen in Euclidean space. This can be demonstrated by considering the simple example of parallel transport of vectors on a spherical surface. Generally, any vector tangent to a geodesic remains tangent during parallel transport along this geodesic. Moreover, since the angle between two parallel transported vector is constant, so must be the angle between a vector parallel transported along a geodesic and the geodesic itself.



It is obvious from the image, that the result of parallel transport depends, not only on the initial and final points, but also on the path along which the transport is carried out. In addition, we observe that parallel transport along a closed curve does not resut in the original vector. For example, parallel transport of the vector **t'** along the path $NACN$ yields the vector **t''**.

These properties stem from the fact that the sphere is a curved surface. Curvature of surfaces and, in general, manifolds is described by the *Riemann curvature tensor*. Consider the covariant derivative of a covariant vector:

$$\nabla_\beta V_\alpha = \partial_\beta V_\alpha - \Gamma^\mu_{\alpha\beta} V_\mu$$

Since $\nabla_\beta V_\alpha$ is a second order covariant tensor, a second differentiation yields:

$$\nabla_\gamma \left(\nabla_\beta V_\alpha\right) = \partial_\gamma \left(\nabla_\beta V_\alpha\right) - \Gamma^\mu_{\alpha\gamma} \left(\nabla_\beta V_\mu\right) - \Gamma^\mu_{\beta\gamma} \left(\nabla_\mu V_\alpha\right)$$

Conducting the differentiations, one can show that:

$$\nabla_\gamma(\nabla_\beta V_\alpha) - \nabla_\beta(\nabla_\gamma V_\alpha) = V_\mu \left(\partial_\beta \Gamma^\mu_{\alpha\gamma} - \partial_\gamma \Gamma^\mu_{\alpha\beta} + \Gamma^\nu_{\alpha\gamma}\Gamma^\mu_{\nu\beta} - \Gamma^\nu_{\alpha\beta}\Gamma^\mu_{\nu\gamma}\right) \tag{30}$$

The above relation allows us to define the Riemann curvature tensor as:

$$R^\mu{}_{\alpha\beta\gamma} \equiv \partial_\beta \Gamma^\mu_{\alpha\gamma} - \partial_\gamma \Gamma^\mu_{\alpha\beta} + \Gamma^\nu_{\alpha\gamma}\Gamma^\mu_{\nu\beta} - \Gamma^\nu_{\alpha\beta}\Gamma^\mu_{\nu\gamma} \tag{31}$$

The Riemann tensor has the following properties:

$$R_{\mu\nu\lambda\rho} = -R_{\mu\nu\rho\lambda}$$

$$R_{\mu\nu\lambda\rho} = -R_{\nu\mu\rho\lambda}$$

$$R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}$$

$$R^{\mu}{}_{\mu\nu\lambda} = 0$$

$$R^{\mu}{}_{\nu\lambda\rho} + R^{\mu}{}_{\lambda\rho\nu} + R^{\mu}{}_{\rho\nu\lambda} = 0$$

The Riemann tensor also satisfies the *Bianchi Identity*:

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0$$

These properties reduce the independent components of the Riemann tensor to:

$$C_N = \frac{1}{12} N^2 (N^2 - 1)$$

where N is the dimension of the manifold.

2.9 Ricci Curvature Tensor and Ricci Scalar

Another important tensor related to curved spaces is the Ricci tensor. Consider an N-dimensional manifold and geodesics of that manifold close to one another. The Ricci tensor describes the small volume enclosed by these geodesics.

It is obtained simply by contraction of the Riemann curvature tensor:

$$R_{\alpha\beta} \equiv R^{\mu}{}_{\alpha\mu\beta} = \partial_\mu \Gamma^{\mu}_{\alpha\beta} - \partial_\beta \Gamma^{\mu}_{\alpha\mu} + \Gamma^{\mu}_{\nu\mu}\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\mu}_{\nu\beta}\Gamma^{\nu}_{\alpha\mu} \quad (32)$$

From the cyclic identity of the Riemann tensor, it can be shown that the Ricci tensor is symmetric:

$$R_{\alpha\beta} = R_{\beta\alpha} \quad (33)$$

The trace of the Ricci tensor is known as the Ricci scalar:

$$R \equiv R^{\alpha}{}_{\alpha} = g^{\alpha\beta} R_{\alpha\beta} \quad (34)$$

Contraction of two indices in the Bianchi identity yields:

$$\nabla_\delta R_{\alpha\gamma} + \nabla_\beta R^{\beta}{}_{\alpha\gamma\delta} - \nabla_\gamma R_{\alpha\delta} = 0$$

Multiplying by $g^{\alpha\gamma}$ we get:

$$\nabla_\delta R - 2\nabla_\beta R^\beta_\delta = 0 \ \Rightarrow \ \nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0 \tag{35}$$

This leads us to the definition of the *Einstein tensor*:

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \tag{36}$$

which is symmetric and has zero-divergence. The Einstein tensor has great importance in general relativity, as it it shows up in the Einstein equation and is related to curvature.

3 General Theory of Relativity

With the basics of the mathematical foundation of general relativity described in the previous chapter, we now use it to discuss the physics of the theory. We start with the equivalence principle, which generalizes the physics of special relativity to include gravity, and then introduce the principle of general covariance, which will let us generalize electrodynamics to curved spacetime.

3.1 The Principle of Equivalence

The principle of equivalence tells us how an arbitrary physical system responds to an external gravitational field.

The Principle of Equivalence: At every point in spacetime in an arbitrary gravitational field, it is possible to choose a "locally inertial coordinate system" such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in a non-accelerated Cartesian coordinate system in the absence of gravitation. In the framework of general relativity, there are no global inertial frames. The curvature of spacetime, inflicted by gravitational interactions, does not allow us to introduce a global Cartesian coordinate system. However, one can introduce local Cartesian coordinates around some region of spacetime. These correspond to *locally inertial frames*, which experience the laws of special relativity.

In a locally inertial Cartesian coordinate system with coordiantes ξ^α the metric is given by

$$ds^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta$$

An arbitrary coordinate transformation to a general coordinate system x^μ is expressed as:

$$d\xi^\alpha = \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu$$

For any such change of coordinates we have

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

where we have defined

$$g_{\alpha\beta} \equiv \eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta}$$

Locally, in the coordinate system of ξ^α, the equations of motion of a free particle are:

$$\frac{d^2 \xi^\alpha}{d\lambda^2} = 0 \tag{37}$$

where λ is a parameter (for massive particles, it can be taken to be the proper time). This coordinate system corresponds to an observer who, within a confined region of space and time, experiences the laws of special relativity. Changing to the coordinates x^μ and using the chain rule, we can write the equations of motion in the new coordinate system as:

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{d^2x^\mu}{d\lambda^2}\frac{d^2x^\nu}{d\lambda^2} = 0 \tag{38}$$

where we defined

$$\Gamma^\alpha_{\mu\nu} = \frac{\partial x^\alpha}{\partial \xi^\sigma}\frac{\partial^2 \xi^\sigma}{\partial x^\mu \partial x^\nu} \tag{39}$$

This is a slightly different derivation of the geodesic equation.

The previous statement is also related to another central idea of the theory, called the General Principle of Relativity. According to that principle, it is not possible to detect the effects of gravitational interactions via local measurements. In other words, an observer carrying out experiments within a sufficiently small free-falling laboratory, lasting for small time intervals, cannot detect the presence of nearby gravitating bodies or a possible acceleration of the laboratory. This statement implies, that all local physical laws must have exactly the same form as in the flat spacetime of Special Relativity, and, therefore, the Riemann curvature tensor can not appear in them. The only exception, as we will see later, are the equations of the gravitational field, which show exactly how the curvature is imposed on spacetime by the presence of mass and energy.

3.2 The Principle of General Covariance

According to the principle of equivalence, we can always find a locally inertial frame at any point in a gravitational field. Therefore, physical laws must have the same form, no matter what generalized coordinates are used to label the events in spacetime. Formally, we state that the physics equations must satisfy the ***principle of general covariance***:

1) Physics equations must preserve their form under general coordinate transformations, which leave the infinitesimal spacetime interval ds^2 invariant. This is charecteristic of tensor equations.
2) Physics equations should reduce to the correct special relativistic form in the local inertial frames, where gravity is absent. Additionally, gravitational equations reduce to Newtonian equations in the limit of low velocity particles in a weak and static field.

By the equivalence principle one can write down an equation that holds in a locally inertial coordinate system and then make a general coordinate transformation to find the corresponding equation in that coordinate system. The principle of equivalence also tells us that there exists a locally inertial frame at every point in spacetime, in which the effects of gravity are absent. With the principle of general covariance, finding the equations that hold in a general coordinate system is much simpler. If we assume an equation that holds in special relativity, i.e. in locally inertial frames, it must hold in all coordinate systems. The method to find equations that are valid in a general gravitational field is to simply take the equations and definitions that are valid in special relativity and replace partial derivatives with covariant ones and the Minkowski metric $\eta_{\alpha\beta}$ with the general metric tensor $g_{\alpha\beta}$

3.3 Einstein's Field Equation

The central equation of General Relativity, as proposed by Einstein, can efficiently describe gravitational effects by means of spacetime curvature. It is derived by applying variations on the Einstein-Hilbert action:

$$S = \frac{c^4}{16\pi G} \int R\sqrt{|\ det(g_{\mu\nu})\ |} d^4x \tag{40}$$

with respect to $g_{\mu\nu}$.

The principle of least action yields the Euler-Lagrange equation for that system, namely, ***Einstein's Field Equation***:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{41}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor.

We can now write:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{42}$$

The unknown quantity of the Einstein equation is the metric tensor $g_{\mu\nu}$, which contains all the information needed to describe the geometry of spacetime.

$T_{\mu\nu}$ is called the *Energy-Momentum Tensor* and it describes the flux of energy and momentum in spacetime. The Energy-Momentum tensor, which is symmetric, is the source of the gravitational field in the Einstein field equations of general relativity, just as mass density is the source of such a field in Newtonian gravity. If we consider a specific distribution of matter and energy, we can find the corresponding energy-momentum tensor and, by solving the Einstein equation, determine how that mass and energy affects the spacetime around it. A better understanding of the tensor can be achieved by taking a look at its components separately:

$\cdot\, T^{00}$: energy density $\cdot\, c^{-2}$

$\cdot\, T^{0i} = T^{i0}$: flux of relativistic mass across the x^i surface or, equivalently, density of the i-th component of linear momentum

$\cdot\, T^{ij}$: flux of of i-th component of linear momentum across the x^j surface

,in particular :

T^{ii}(no sum) : pressure (independent of direction)

The energy momentum tensor is a conserved quantity, associated with invariance of physics laws under spacetime translations:

$$\nabla_\nu T^{\mu\nu} = 0 \tag{43}$$

3.4 Gravitational Time Dilation and Redshift

In Special Relativity, time and space measurements are different between observers and they are related to each other by Lorentz transformations, that depend only on the relative velocity between the observers. A similar phenomenon occurs in General Relativity, but here the time and space measurements are modified, not due to the relative velocity between two inertial frames, which is a very special case, but due to the non-flat geometry of spacetime. A non-flat geometry can originate at a source of gravitation, such as mass and energy, but also can be caused by lienear acceleration or rotation.

Let's consider an observer at rest on a point A inside an arbitrary gravitational field, that emits light waves towards another observer at rest, placed on a different point B in the field. The time taken for a wave crest to travel from A to B is a constant. Therefore, the time between the departure from point A of successive crests will equal the time between their arrival at point B. Using the general form of the spacetime interval, with all spatial differentials equal to zero, we write:

$$d\tau_A^2 = g_{00}(x_A)dt^2 \quad ; \quad d\tau_B^2 = g_{00}(x_B)dt^2 \tag{44}$$

,which can be translated as a difference in frequency between the emitted and the detected light:

$$\frac{d\tau_A}{d\tau_B} = \frac{\nu_B}{\nu_A} = \sqrt{\frac{g_{00}(x_A)}{g_{00}(x_A)}} \tag{45}$$

3.5 Electrodynamics in General Relativity

In a previous section, we presented the covariant form of Maxwell's equations, written in terms of 4-vectors and the electromagnetic tensor. Obviously, when one is not studying electromagnetic fields just in inertial frames, but in more general ones, those expressions do not hold. In this section, we present how Maxwell's equations are modified, so that they hold in such general frames, i.e. so that they keep their form under GCTs.

Fortunately, as we mentioned before, all we have to do is substitute partial derivatives in the Lorentz covariant formulation with covariant derivatives. Then, Maxwell's equations read:

$$\nabla_\mu F^{\mu\nu} = \mu_0 j^\nu$$

or, in potential form:

$$g^{\mu\nu}(\nabla_\mu \nabla_\sigma A_\lambda - \nabla_\mu \nabla_\lambda A_\sigma) = \mu_0 j^\lambda$$

In vacuum, the electromagnetic field tensor satisfies the equation:

$$\nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\sigma\mu} + \nabla_\mu F_{\nu\sigma} = 0$$

The electromagnetic energy-momentum tensor is generalized to the form:

$$T_{\mu\nu} = \frac{1}{\mu_0}[F_{\mu\rho}F^\rho_\nu - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}]$$

These expressions reduce to the ones presented in the context of special relativity, if we move to a locally inertial frame.

4 Exact Solutions of Einstein's Field Equations

Einstein's equations are in general difficult to solve, because, for a generic energy-momentum tensor, they compose a system of ten, non-linear, partial differential equations. However, we can find exact solutions to these equations by specifying the symmetry properties of the system. The exact solutions that we are going to present in this section are static and possess spherical symmetry.

4.1 The General Static and Isotropic Metric

A static and spherically symmetric spacetime can be described by a spacetime interval (or metric) with the general form:

$$ds^2 = A^2(r)(cdt)^2 - B^2(r)dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{46}$$

,where A^2 and B^2 are unknown functions of r.

For that form of the metric, we can calculate all quantities that relate to the Einstein equation in terms of A^2 and B^2.

The Christoffel symbols arising from that metric are:

$$\Gamma^t_{tr} = \frac{A'}{A}$$

$$\Gamma^r_{rr} = \frac{B'}{B} \;\; ; \;\; \Gamma^r_{tt} = \frac{AA'}{B^2} \;\; ; \;\; \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{B^2} \;\; ; \;\; \Gamma^r_{\theta\theta} = -\frac{r}{B^2}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \;\; ; \;\; \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{r\phi} = \frac{1}{r} \;\; ; \;\; \Gamma^\psi_{\theta\phi} = \cot\theta$$

The components of the Riemann tensor are:

$$R_{rtrt} = -R_{rttr} = R_{trtr} = -AA' + \frac{AA'B'}{B}$$

$$R_{r\theta r\theta} = R_{\theta r\theta r} = -\frac{rB'}{B}$$

$$R_{r\phi r\phi} = R_{\phi r\phi r} = -r\sin^2\theta\frac{B'}{B}$$

$$R_{\theta\phi\theta\phi} = R_{\phi\theta\phi\theta} = -r^2\sin^2\theta\left(1 - \frac{1}{B^2}\right)$$

$$R_{\theta t\theta t} = R_{t\theta t\theta} = -\frac{rAA'}{B^2}$$

$$R_{\phi t\phi t} = R_{t\phi t\phi} = -r\sin^2\theta\frac{AA'}{B^2}$$

.

The components of the Ricci tensor are:

$$R_{tt} = -\frac{AA''}{B^2} + \frac{AA'B'}{B^3} - \frac{2AA'}{rB^2}$$

$$R_{rr} = \frac{A''}{A} - \frac{A'B'}{AB} - \frac{2B'}{rB}$$

$$R_{\theta\theta} = \frac{1}{B^2} - \frac{rB'}{B^3} + \frac{rA'}{AB^2} - 1$$

$$R_{\phi\phi} = \sin^2\theta\left(\frac{1}{B^2} - \frac{rB'}{B^3} + \frac{rA'}{AB^2} - 1\right)$$

Finally, the Ricci scalar is:

$$R = -\frac{2A''}{AB^2} + \frac{2A'B'}{AB^3} - \frac{4A'}{rAB^2} + \frac{4B'}{rB^3} - \frac{2}{r^2B^2} + \frac{2}{r^2}$$

4.2 The Schwarzschild Spacetime

The simplest, non-trivial exact solution to Einstein's equations is the **Schwarzschild solution** and was derived by Karl Schwarzschild in 1916. It describes the spacetime outside a spherical mass, while $T_{\mu\nu} = 0$, which means that the spacetime around the source-mass is empty.

The Einstein equation reads:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0.$$

By taking the trace of that equation, we see that $R = 0$, which means that the equation we have to solve becomes:

$$R_{\mu\nu} = 0.$$

We can find a non-trivial solution to that equation:

$$\frac{R_{tt}}{A^2} + \frac{R_{rr}}{B^2} = 0 \Rightarrow -\frac{2}{rB^2}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \Rightarrow \frac{A'}{A} = -\frac{B'}{B} \Rightarrow \frac{dA}{A} = -\frac{dB}{B} \Rightarrow A = \frac{const.}{B}.$$

Considering that the spacetime has to be asymptotically flat, it is obvious that the constant is unity :

$$A = \frac{1}{B}.$$

Also:

$$R_{\theta\theta} = 0 \Rightarrow \frac{1}{B^2} - \frac{rB'}{B^3} + \frac{rA'}{AB^2} - 1 = 0 \Rightarrow A^2 + rAA' + rAA' - 1 = 0 \Rightarrow A^2 + 2rAA' - 1 = 0 \Rightarrow$$

$$\Rightarrow \frac{AdA}{1 - A^2} = \frac{dr}{2r} \Rightarrow A^2(r) = 1 - \frac{const.}{r}$$

To find the exact value of the constant, we need to take the non-relativistic or Newtonian limit, i.e. the regime of low intensity, stationary gravitational fields and small velocities. Therefore, the velocity of a test particle satisfies the condition:

$$v \ll c \Rightarrow \frac{d\boldsymbol{x}}{dt} \ll c \Rightarrow \frac{d\boldsymbol{x}}{ds} \ll \frac{cdt}{ds} \tag{47}$$

The geodesic equation(eqn name) becomes:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{00}\left(\frac{dt}{dt}\right)^2 = 0 \tag{48}$$

Since the field is stationary, all time derivatives of the metric are zero, so that:

$$\Gamma^\mu_{00} = -\frac{1}{2}g^{\mu\nu}\frac{\partial g_{00}}{\partial x^\nu} \tag{49}$$

We can consider a weak field, by assuming that the spacetime is similar to the Minkowski spacetime, with some small deformations:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad ; \quad \mid h_{\mu\nu} \mid \ll 1 \tag{50}$$

Then, to first order in h, we have:

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\nu}\frac{\partial h_{00}}{\partial x^\nu}$$

Substitution in equation (48) yields the relation:

$$\frac{d^2\boldsymbol{x}}{ds^2} = -\frac{1}{2}\left(\frac{dt}{ds}\right)^2 \nabla h_{00} \tag{51}$$

Fron Newton's theory, we recall the result:

$$\frac{d^2\boldsymbol{x}}{ds^2} = -\nabla\Phi \tag{52}$$

The gravitational potential at a distance r from a spherical body with mass M is:

$$\Phi = -\frac{GM}{r}$$

, so by comparison of equations (51) and (52), we find:

$$h_{00} = 2\Phi + constant \tag{53}$$

The constant is easily set to zero, by taking the limit of a flat metric when the distance from the object is very large. Hence, h_{00} and

$$g_{00} = 1 - \frac{2GM}{c^2 r} \tag{54}$$

Then :

$$A^2(r) = 1 - \frac{2GM}{c^2 r}$$

Finally, we can write down the Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)(cdt)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{55}$$

or:

$$ds^2 = \left(1 - \frac{r_s}{r}\right)(cdt)^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{56}$$

The quantity

$$r_s \equiv \frac{2GM}{c^2}$$

is called the Schwarzschild radius.

This metric describes the spacetime around a spherical body of mass M. If the body has radius r_*, then the metric is valid only for $r > r_*$, because for $r < r_*$ we are no longer in vacuum, i.e. $T_{\mu\nu} \neq 0$. Meanwhile, the detailed distribution of matter in the interior of the body does not influence the external gravitational field, a feature also present in Newton's theory in the case of a spherically symmetric body. For any spherically symmetric object with radius larger that r_s, the metric experiences no singularities. Consider, now, an object that has a radius smaller than r_s. Then, the Schwarzschild solution holds at $r = r_s$ and there is a singularity at that distance, as well as for $r = 0$. An object with a radius $r < r_s$ is called a *black hole*. This name is justified by the fact, that no massive particle can escape if it approaches the object at a distance smaller than r_s. This "boundary" in spacetime is called an *event horizon* and it marks the surface for which events inside of it cannot affect the outside. An interpretation of this fact is that a clock falling towards a black hole experiences infinite time dilation as it approaches the event horizon. Hence, it would require an infinite time for the clock to reach the horizon, as seen from an outside observer. Also, the redshift of an electromagnetic travelling radially outwards the black hole goes to infinity, as the point of emission approaches the event horizon.

4.3 The Reissner-Nordström Spacetime

The Reissner-Nordström solution is an exact solution of the coupled Einstein-Maxwell equations and it describes the spacetime outside a uniformly charged spherical body at rest. It was discovered by Hans Reissner, Hermann Weyl, Gunnar Nordström and George Barker Jeffery. Due to the stationarity and spherical symmetry of the problem, we will use the metric of section (5.1), or , in other words, the left side of the Einstein equation is exactly the same. Although this is also a vacuum solution, in the sense that matter density is zero, there is energy and momentum flow, due to the electromagnetic field.

A uniformly charged spherical body would create a radial electric field around it, while the magnetic field would be zero, due to stationarity. The electromagnetic field tensor reads:

$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{E_r}{c} & 0 & 0 \\ -\frac{E_r}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{57}$$

The non-zero components of the electromagnetic energy-momentum tensor are:

$$T^{00}_{em} = \frac{\epsilon_0 B^2 E_r^2}{2}, \quad T^{11}_{em} = -\frac{\epsilon_0 A^2 E_r^2}{2}, \quad T^{22}_{em} = \frac{\epsilon_0 A^2 B^2 E_r^2}{2r^2}, \quad T^{33}_{em} = sin^2\theta \frac{\epsilon_0 A^2 B^2 E_r^2}{2r^2} \tag{58}$$

Maxwell's equations in empty space give:

$$\nabla_\mu F^{\mu\nu} = 0 \Rightarrow \frac{1}{\sqrt{|\, g \,|}}(\partial_\mu \sqrt{|\, g \,|} F^{\mu\nu}) = 0 \Rightarrow \frac{1}{ABr^2 sin^2\theta} \partial_\mu (ABr^2 sin^2\theta F^{\mu\nu}) = 0 \tag{59}$$

The only non-zero equation is for $\mu = 1$:

$$\frac{\partial}{\partial r}(ABr^2 F^{10}) = 0 \Rightarrow \frac{\partial}{\partial r}\Big(ABr^2 \frac{E_r}{cA^2B^2}\Big) = 0 \Rightarrow \frac{\partial}{\partial r}\Big(\frac{r^2 E_r}{AB}\Big) = 0 \Rightarrow \tag{60}$$

$$E_r(r) = \frac{c_1 A(r) B(r)}{r^2} \tag{61}$$

We can find the constant c_1 considering, that the spacetime we are interested in is asymptotically flat, i.e. it approaches the Minkowski spacetime for large r: A Also, we use the fact that for large values of r, the electric field will approach that of a point charge, as calculated from Gauss' Law:

$$E_r(r) \to \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2} \tag{62}$$

, which means that

$$c_1 = \frac{Q}{4\pi\epsilon_0}.$$

The Reissner-Nordström metric is:

$$ds^2 = \Big(1-\frac{2GM}{c^2r}+\frac{Q^2G}{4\pi\epsilon_0c^4r^2}\Big)(cdt)^2-\Big(1-\frac{2GM}{c^2r}+\frac{Q^2G}{4\pi\epsilon_0c^4r^2}\Big)^{-1}dr^2-r^2(d\theta^2+sin^2\theta d\phi^2) \tag{63}$$

or

$$ds^2 = \left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right)(cdt)^2 - \left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right)^{-1}dr^2 - r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{64}$$

Possible event horizons can occur when

$$r^2 - r_s r + r_Q^2 \Rightarrow r_\pm = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4r_Q^2}\right)$$

From this equation, we see that, depending on the relative values of r_s and r_Q, there could be two, one or zero real-valued solutions.

5 A Model for the Electron

In 2014, Prof. Hans Dekker (arXiv:1408.4796v3) had the idea to model the electron and the other charged leptons as a relativistic mass and charge distribution of finite size. By finding an exact solution to the Einstein-Maxwell equations, he used the experimental values for the mass and charge of the electron to calculate the characteristic size of the distribution. The following section is devoted to solving the Einstein equations for a perfect charged fluid, which consists a simple source for the Reissner-Nordström spacetime.

5.1 The Basic Concept

Modern theoretical physics is essentially based on the existence of a finite set of elementary point particles and their electromagnetic, gravitational, and weak or strong interactions. Apart from the neutrinos, all fundamental particles carry electric charge. However, the very concept of a stable point charge -such as the electron- is an old and as yet basically unsolved problem. Namely, a finite amount of electric charge concentrated in an infinitely small volume would be unstable and would explode, if there wasn't an internal force to act against the repulsive self-interaction.

Historically, the problems with a point charge were already recognized in classical physics. For instance, in Poincaré's 'electron model', presented in his paper "Sur la dynamique de l'électron", the electric force on the charged sphere was counteracted by an elastic force of unspecified, non-electromagnetic and non-gravitational nature. So, what if the electron has finite size? And if that is true, how small does it have to be, so that it's gravitational self-attraction makes it stable? These are the central questions that we are going to deal with and attempt to answer. Recently, based on state-of-the-art precision measurements of the electron's gyromagnetic g-factor, 1989 Nobel laureate H.G. Dehmelt has pointed out that "Today everybody 'knows' the electron is a Dirac point particle with radius $r = 0$ and $g = 2$. But is it? The value $r \approx 10^{22}m$ given here constitutes an important new upper limit. [..] Thus, the electron may have size."

In any case, in quantum theory the electron can be treated successfully as a structureless point particle, as it is possible to neglect the problem of its infinite self-energy by mass renormalization, which yields results in terms of the particle's experimentally observed mass m. Here, we will attempt to solve that problem, by considering that such particles can -in principle- be stable due to gravity. Specifically, from the point of view of Einstein's theory, the enormous amount of electrostatic energy compressed into an infinitely small volume must give rise to significant local gravitational effects.

We will ignore weak interactions, in order to construct a simple model, and we will not a priori assume that the mass-charge density is rigorously zero outside some radius r_m. Instead, we will only demand that our solution becomes asymptotically equal to the Schwarzschild metric for large r. Therefore, we will study the gravitational field equations for the Ricci tensor $R_{\mu\nu}$ corresponding to a classical charged mass distribution, i.e. with the full energy-momentum tensor for the material mass and the electromagnetic fields as the source of gravitational energy.

5.2 The Solution to the Einstein-Maxwell Equations

We will attempt to find an exact solution to the Einstein-Maxwell equations that correspond to a spherically symmetric mass-charge distribution with zero internal pressure.

The energy-momentum tensor will have two terms: one term for the perfect, pressureless fluid (or dust) and one term for the radial electric field, which are, respectively:

$$T^{(m)}_{\mu\nu} = \rho_m u_\mu u_\nu \quad \text{and} \quad T^{(em)}_{\mu\nu} = \frac{1}{\mu_0}[F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}] \tag{65}$$

where ρ_μ is the matter density of the fluid and u_μ is the 4-velocity field of the dust particles. In addition, the 4-velocities of the fluid satisfy the normalization condition

$$u_\mu u^\mu = 1$$

The general expression for the density of a system of n point-particles is:

$$\rho_m = g^{-1/2}\sum_n m_n \int \delta^4[x - x_n(\tau)]d\tau \tag{66}$$

and their charge density is:

$$\rho_Q = g^{-1/2}\sum_n q_n \int \delta^4[x - x_n(\tau)]d\tau$$

where g is the trace of the metric tensor.

We can perform the integration in equations (66) and (67) and obtain the total mass and total charge of the distribution:

$$M = \sum_n m_n = \int \gamma^{1/2}\rho_m d^3x$$

and

$$Q = \sum_n q_n = \int \gamma^{1/2}\rho_Q d^3x$$

where $\gamma = -g/g_{00}$ is the determinant of the three-dimensional metric tensor $\gamma_{ij} = (g_{ij} - g_{0i}g_{0j}/g_{00})$. The Einstein equation reads:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}\left(\rho_m u_\mu u_\nu + \frac{1}{\mu_0}[F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}]\right) \tag{67}$$

First, let us calculate the components of $T^{(m)}_{\mu\nu} = \rho_m u_\mu u_\nu$. Setting $\mu = 0$ and $\nu = i = 1,2,3$ in the Einstein equation and taking into account the normalization condition and the fact that $R_{0i} = g_{0i} = 0$, we get :

$$u_0 u_i = 0 \Rightarrow u_i = 0 \tag{68}$$

meaning that the spacelike components of the fluid velocities are zero, which is a result originating only from the spherical symmetry of the system. Again, combining with the normalization condition we find that:

$$u^\mu = \frac{1}{A(r)}(1,0,0,0) \qquad \text{and} \qquad u_\mu = A(r)(1,0,0,0) \tag{69}$$

Finally, we have:

$$T^{(m)}_{00} = \rho_m A^2(r) \quad \text{and} \quad T^{(m)}_{11} = T^{(m)}_{22} = T^{(m)}_{33} = 0 \tag{70}$$

Einstein's field equations for the static isotropic mass-charge system may be written as:

$$R_{00} = \frac{8\pi G}{c^4}\rho_m\left(u_0u_0 - \frac{1}{2}g_{00}c^2\right) + \frac{8\pi G}{c^4}T^{(em)}_{00} = \frac{4\pi G}{c^2}\rho_m A^2 + \frac{4\pi G}{c^4}\epsilon_0 A^4 B^2 E_r^2 \tag{71}$$

$$R_{11} = \frac{8\pi G}{c^4}\rho_m\left(-\frac{1}{2}g_{11}c^2\right) + \frac{8\pi G}{c^4}T^{(em)}_{11} = \frac{4\pi G}{c^2}\rho_m B^2 - \frac{4\pi G}{c^4}\epsilon_0 A^2 B^4 E_r^2 \tag{72}$$

$$R_{22} = \frac{8\pi G}{c^4}\rho_m\left(\frac{1}{2}r^2c^2\right) + \frac{8\pi G}{c^4}T^{(em)}_{22} \Rightarrow \frac{1}{B^2} + \frac{2rA'}{AB^2} - 1 = \frac{4\pi G\epsilon_0}{c^4}A^2B^2r^2E_r^2 \tag{73}$$

$$R_{33} = sin^2\theta \cdot R_{22} \tag{74}$$

Adding the first two Einstein equations we can obtain an expression for the density of the fluid:

$$\frac{R_{00}}{A^2} + \frac{R_{11}}{B^2} = \frac{8\pi G}{c^2}\rho_m \Rightarrow \rho_m = \frac{c^2}{4\pi G r B^2}\left(\frac{A'}{A} + \frac{B'}{B}\right) \tag{75}$$

We can also subtract the first to Einstein equations by parts:

$$\frac{R_{00}}{A^2} - \frac{R_{11}}{B^2} = -\frac{8\pi G}{c^4}\epsilon_0 A^2 B^2 E_r^2 \tag{76}$$

Differentiation of equation (73) with respect to r yields:

$$-\frac{B'}{B^3}+\frac{A'}{AB^2}\frac{rA''}{AB^2}-\frac{r(A')^2}{A^2B^2}-\frac{2rA'B'}{AB^3}=\frac{2\pi G\epsilon_0}{c^4}\frac{d}{dr}\left(A^2B^2r^2E_r^2\right) \tag{77}$$

Combination of equations (76) and (77) results in a relation with important physical meaning considering the fluid distribution and the consequent spacetime deformation:

$$\frac{A'}{A}=\frac{AB^2E_r\rho_Q}{\rho_m c^2} \tag{78}$$

This is an equilibrium condition, that reflects the balance between gravitational self-attraction and electrostatic self-repulsion.

For a structureless charged mass the intrinsic charge-to-mass ratio e_n/m_n should be an n-independent constant, let's call it "k" , This is equivalent to considering the simple equation of state for a uniformly charged fluid :

$$\rho_m = k\rho_Q \tag{79}$$

Without loss of generality, we can imply that $M = kQ$, where M and Q are the total mass and charge of the fluid. We can try to simplify some of the expressions, by defining the parameter:

$$\mu=\frac{Mc^2}{Q} \tag{80}$$

Now we can rewrite the equilibrium condition in a simpler form:

$$B^2E_r=\mu\frac{A'}{A^2} \tag{81}$$

By taking both the Scwarzschild limit

$$A^2(r)=1-\frac{2Gm}{rc^2} \tag{82}$$

and the Poisson limit:

$$B^2E_r=\frac{Q}{4\pi\epsilon_0 r^2} \tag{83}$$

at $r\to\infty$,we can obtain an expression for the experimentally observed mass m of the object:

$$m=\frac{Q^2}{4\pi\epsilon_0 GM} \tag{84}$$

The observed mass m is practically defined by equation (84).

The equilibrium equation is essential, because it reduces our problem to finding the temporal metric $A^2(r)$. If we do that, we can also find $B^2(r)$ from the equation:

$$B^2(r) = 1 + \frac{2rA'}{A} + \frac{4\pi G\epsilon_0}{c^4}k^2r^2\big(\frac{A'}{A}\big) \tag{85}$$

or:

$$B^2(r) = 1 + 2y + 4\lambda y^2 \tag{86}$$

,where we have defined:

$$y(r) \equiv \frac{rA'}{A} \qquad and \qquad \lambda \equiv \frac{\pi G\epsilon_0 k^2}{c^4}$$

The value of the parameter λ depends on the equation of state of the distribution. Specifically for the electron: $\lambda \approx 10^{42}$.

Going a few steps back, equation (76) can be written as:

$$A'' = \frac{A'B'}{B} - \frac{A'}{r} + \frac{AB'}{rB} + \frac{4\pi G\epsilon_0}{c^4}A^3B^4E_r^2 \tag{87}$$

which, together with

$$E_r = \frac{A'}{A^2B^2}\mu$$

results in :

$$\frac{rA''}{A} = \frac{rA'B'}{AB}\frac{A'}{A} + \frac{B'}{B} + 4\lambda r\left(\frac{A'}{A}\right)^2 \tag{88}$$

Next, we differentiate equation (86) :

$$BB' = y' + 4\lambda yy' \tag{89}$$

Combining with (88) we obtain :

$$(4\lambda - 1)\frac{y^2}{r}(1 + 2y + 4\lambda y^2) = (1 - 4\lambda)yy' \tag{90}$$

This is the equation that contains the full solution to the problem. Let us, first, deal with the case, where $y \neq 0$ and $\lambda \neq 1/4$. We are lead to solve the non-linear differential equation:

$$\frac{r}{2}(A^2)'' + (1 + 2\lambda y^2)(A^2)' = 0 \tag{91}$$

We should modify the form of this equation, introducing an auxiliary function

$$\zeta(r) \equiv \frac{1}{y(r)} = \frac{A}{rA'} \tag{92}$$

as the differential equation (91), in terms of $\zeta(r)$, takes the simple form:

$$r\zeta\zeta' = \zeta^2 + 2\zeta + 4\lambda \tag{93}$$

The above equation has an exact solution:

$$\frac{1}{2}\ln(\zeta^2 + 2\zeta + 4\lambda) + \frac{1}{\sqrt{4\lambda - 1}} \operatorname{arccot}\left(\frac{\zeta + 1}{\sqrt{4\lambda - 1}}\right) = lnr + C \tag{94}$$

We now require that the solution should approach the Schwarzschild solution at large distances. By applying this constraint, we can calculate the constant:

$$\text{When } r \to \infty : A^2 \to 1 - \frac{2Gm}{c^2 r} = 1 - \frac{r_s}{r} \quad and \quad \zeta \approx \frac{2r}{r_s} \tag{95}$$

,which leads to the particular solution of the equation:

$$\frac{1}{2}\ln(\zeta^2 + 2\zeta + 4\lambda) + \frac{1}{\sqrt{4\lambda - 1}} \operatorname{arccot}\left(\frac{\zeta + 1}{\sqrt{4\lambda - 1}}\right) = ln\Big(\frac{2r}{r_s}\Big) \tag{96}$$

This is a relation that matches a value of the somewhat abstract auxiliary function ζ to the more familiar radial distance r.

Now, it is easy to write the metric functions in terms of ζ. For $A(r)$ we have:

$$\frac{dA}{d\zeta} = \frac{dA}{dr}\frac{dr}{d\zeta} = \frac{A'}{\zeta'}$$

or

$$\frac{dA}{d\zeta} = \frac{A}{r\zeta\zeta'} \Rightarrow A^2(r) = \exp\left(-\frac{1}{\sqrt{\lambda - \frac{1}{4}}} \operatorname{arccot}\left(\frac{\zeta + 1}{\sqrt{4\lambda - 1}}\right)\right)$$

or, equivalently :

$$A^2(r) = \left(\frac{r_s}{2r}\right)^2 (\zeta^2 + 2\zeta + 4\lambda)$$

Also, from equation (86):

$$B^2(r) = \frac{\zeta^2 + 2\zeta + 4\lambda}{\zeta^2}$$

For finite values of r/r_c and $\lambda \to \infty$:

$$\frac{1}{B^2} = 1 - \left(\frac{r_c}{r}\right)$$

We notice, that this solution is valid only for non-negative values of ζ. Especially, for $\zeta = 0$ this equation (96) leads us to a specific value of radial distance

$$r_c = r_s\sqrt{\lambda}\exp\left[\frac{1}{\sqrt{4\lambda - 1}}\operatorname{arccot}\left(\frac{1}{\sqrt{4\lambda - 1}}\right)\right].$$

This is a quantity of great importance for our problem: we will show that it is the characteristic size of the system.

But first, let's consider the case where $\lambda = 1/4$. In this case. equation (90) becomes a trivial zero identity for all metric functions $B(r)$. Hence, the problem can be solved by any mass-charge distribution, which is an irrelevant case for our cause. Thus, we can safely consider that $\lambda \neq 1/4$. We should also note, that the parameter λ takes a very large value for the electron. We notice that for very large values of λ, the observable classical mass is $m = |Q|/\sqrt{4\pi\epsilon_0 G}$, so that the Schwarzschild radius is:

$$r_s = r_c = |Q|/\sqrt{G/4\pi\epsilon_0 c^4} = l_P\sqrt{\alpha_e}$$

However, in the case of an electron, i.e. for $Q = e$, we get $m = 10^{-9}$, which is many orders of magnitude larger than the mass of the electron $m_e \sim 10^{-30}$ kg. Therefore, in order to achieve equilibrium due to self-gravitation, λ has to be a lot larger than 1/4.

By now we have obtained only one part of the solution, which corresponds to spatial areas satisfying $r > r_c$, because this is the range in which the function ζ is defined. Looking back at equation (90), we are left with only one option for the spacetime inside r_c: y has to be equal to zero, which means that both A and B are constant for $r < r_c$. More specifically:

$$y = \frac{rA'}{A} = 0 \Rightarrow A' = 0.$$

Then equation (86) yields :

$$B^2(r < r_c) = 1.$$

Also, the "exterior" solution for $\zeta = 0$ gives us the value of the temporal metric on the boundary $r = r_c$, as well as inside the radial distance r_c :

$$A^2(r < r_c) = \lambda\left(\frac{r_s}{r_c}\right)^2$$

The electric field of the entire distribution can be extracted from the equilibrium equation (98):

$$E_r(r) = \left(\frac{\mu}{2rA}\right)\frac{\zeta}{4\lambda + 2\zeta + \zeta^2}\Theta(r - r_c)$$

where Θ is the *Heaviside* or *step function*, which is added "by hand", to make sure that the electric field is zero for $r < r_c$.

One further gets:

$$r^2ABE_r = \frac{\mu}{2}\frac{r}{\sqrt{\lambda + \zeta + \zeta^2}}\Theta(r - r_c) \tag{97}$$

Finally, through the Poisson equation we obtain the charge density distribution

$$r^2B\rho_Q = \frac{\mu\epsilon_0 r}{2\sqrt{\zeta^2 + 2\zeta + 4\lambda}}\delta(r - r_c) - \frac{\mu\epsilon_0}{2}\frac{1}{\zeta\sqrt{\zeta^2 + 2\zeta + 4\lambda}}\Theta(r - r_c) \tag{98}$$

where $\delta(r)$ is the Dirac distribution.

It is obvious that the density has two "parts", a singular one and a continuous one. For finite values of r/r_c and $\lambda \to \infty$ the continuous part of the density becomes

$$\rho_Q^{cont} = -\frac{Q}{8\pi r_c^3\sqrt{4\lambda}}\left(\frac{r_c}{r}\right)^4$$

Clearly, for $\lambda \approx 10^{42}$, $\rho_Q^{cont} \approx 0$, so charge and mass almost completely accumulate at radius r_c. Therefore, we could say that the distribution resembles a spherical shell of radius r_c. Specifically, for the electron:

$$r_c^{electron} = 1.38063 \times 10^{-36}m$$

A different method, which will allow us to better visualize the behaviour of the resulting spacetime, is to solve equations numerically, keeping λ as a free parameter. The numerical solutions for the temporal function, as well as the radial electric field are presented graphically below, as a function of the non-dimensional radial variable r/r_c for three different values of λ.

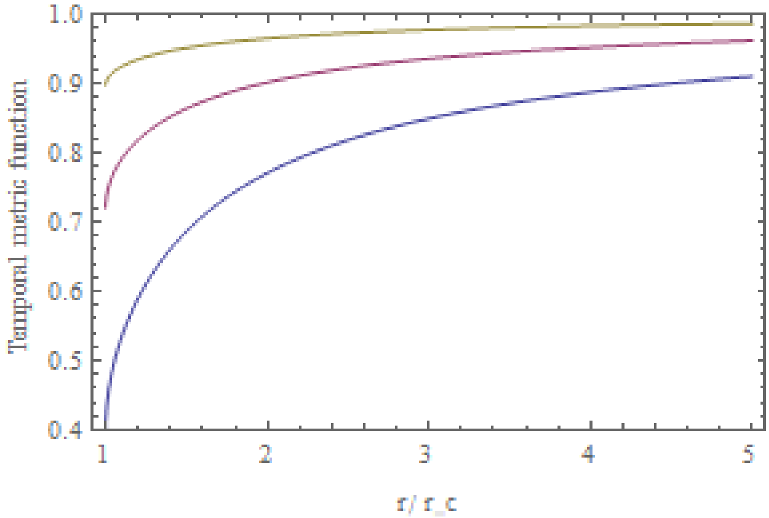


Figure 1: Temporal Metric Function $A^2(r)$ as a function of r/r_c

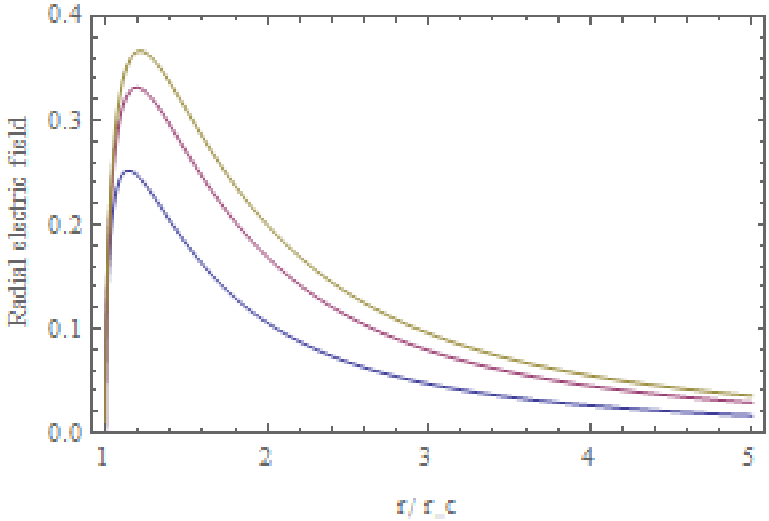


Figure 2: Radial electric field E_r/E_c, where $E_c = \mu/4r_c\sqrt{\lambda}$ as a function of r/r_c

5.3 Discussion

The solution of the Einstein-Maxwell equations for a collection of dust particles has yielded a density distribution which does not resemble the interior fluid solutions usually encountered in General Relativity, like the Schwarzschild interior solution. The distribution is virtually consisting of two parts, conjoined at radial distance r_c.

The charge and matter densities are rigorously zero for $r < r_c$, which originates from the fact, that the components of the metric are constant in that range. It can be easily shown, that any metric that is not a function of the coordinates, i.e. it is constant can be transformed to the Minkowski metric. In other words, the spacetime for $r < r_c$ is flat. This means that the solution resembles a spherical shell, with infinitely small thickness. Almost all of the mass and charge almost completely accumulate at radial distance r_c. More specifically, the "tail" of the density distribution, i.e. the part that extends at $r > r_c$ contains only $10^{-21}Q$.

The application we presented included using the solution we found to construct a model for the electron. It is known, that the electron is a quantum particle and any analysis without the inclusion of quantum mechanics should be incomplete. The result for the characteristic length of the electron is approximately $10^{-36}m$, which is one order of magnitude smaller than the Planck length ℓ_P. At that scale, both gravitational and quantum effects are believed to be significant. Specifically, quantum zero oscillations of the gravitational field distort Euclidean geometry, while the smaller the scale under study, the greater the deviations from Euclidean geometry. As noted in Regge (1958) "for the space-time region with dimensions ℓ, the uncertainty of the Christoffel symbols $\Delta\Gamma$ is of the order of ℓ_P^3/ℓ^3, and the uncertainty of the metric tensor Δg is of the order of l_P^2/l^2 . If ℓ is a macroscopic length, the quantum constraints are fantastically small and can be neglected even on atomic scales. If the value ℓ is comparable to ℓ_P, then the maintenance of the former (usual) concept of space becomes more and more difficult and the influence of micro curvature becomes obvious". In our case: $\ell_P/\ell \sim 10$, so $\Delta g \sim 100$. It is obvious, that if we accoount for quantum fluctuations of spacetime, there is a large uncertainty in our results, especially for A^2, as it takes values very close to unity, even in the vicinity of the mass-charge distribution. In any case, the description of such quantum gravitational effects is not complete whatsoever, let alone dealt with here.

Further work that could be possibly be done on this model, would be to solve the Einstein equations for a rotating mass-charge distribution, taking the large r limits of the Kerr or the Kerr-Newmann spacetime. This way, it would be possible to calculate the characteristic size of a rotating body, which would probably be more physically relevant.

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