Bergman Spaces

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Preface

In this thesis, we study the functions of the Bergman spaces $\mathcal{A}^p$, that is, the holomorphic functions on the unit disc that are also p-integrable with respect to the Lebesgue area measure. In particular, we explore the properties of these spaces, such as their completeness and duals, obtaining results that are analogous to the properties of L^p spaces. We distinguish the case $p = 1$, for which we introduce the Bloch space as the dual of $\mathcal{A}^1$. Finally, we present a connection between the Bergman spaces and the hyperbolic metric. More precisely, we show that each function of the Bergman space is an infinite sum that depends on specific sequences of the hyperbolic disc.

I would like to thank my supervisor Themis Mitsis, for his guidance, advice, patience and eagerness to help. I am also grateful to N. Frantzikinakis, V. Nestoridis and M. Papadimitrakis, for their remarks and for conversations we had.

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Abstract

In **Chapter 1**, we introduce some elements of hyperbolic geometry in the unit disc, such as the hyperbolic length of curves, the hyperbolic metric and hyperbolic discs. A class of holomorphic functions is widely used throughout this analysis, namely the Möbius group of the unit disc. In the last section of this chapter, we discuss the notion of r-lattices, which are specific sequences in the unit disc, and will play an important role later in our study.

In **Chapter 2**, the theory of Bergman spaces $\mathcal{A}^p$ is presented. In particular, we show that the Bergman spaces are Banach spaces. Next, we move on to the Bergman kernel, which gives an integral representation of the functions of $\mathcal{A}^p$, and is used to show that the dual of the Bergman space $\mathcal{A}^p$ is the Bergman space $\mathcal{A}^q$, where q is the conjugate exponent of $p > 1$. After giving a characterization of the Bergman spaces in terms of derivatives, we finally arrive at the atomic decomposition, which means that we can write each function of $\mathcal{A}^p$ as an infinite sum that uses r-lattices.

Chapter 3 is about the Bloch space, which is a Banach space and the dual of $\mathcal{A}^1$. Lastly, we present a connection between the Bloch space and the hyperbolic metric.

The structure of this thesis follows mainly K. Zhu's book "Operator Theory in Function Spaces" ([15]). Other books we used are [4], [7] and [11].

CHAPTER 1

Hyperbolic Geometry on the Unit Disc

1.1 The Möbius group of $\mathbb{D}$

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and consider the set

$$\text{Aut}(\mathbb{D}) = \{\phi : \mathbb{D} \to \mathbb{D} \,|\, \phi \text{ is 1-1, onto and holomorphic}\}.$$

The functions in Aut($\mathbb{D}$) are called Möbius maps of the unit disc. By [14, Section X.15], the inverse of a Möbius map of $\mathbb{D}$ is also in Aut($\mathbb{D}$). Thus, Aut($\mathbb{D}$) is a group under composition and is called the Möbius group of $\mathbb{D}$.

For the rest of this thesis, we will often use the following theorem from complex analysis (see [12, Theorem 12.6]):

Theorem 1.1.1. *$\phi \in \text{Aut}(\mathbb{D})$ if and only if there exist a real number θ and a point $a \in \mathbb{D}$ such that $\phi = e^{i\theta}\phi_a$ in $\mathbb{D}$, where $\phi_a(z) = \dfrac{a-z}{1-\bar{a}z}$, for all $z \in \mathbb{D}$, is a special Möbius map of $\mathbb{D}$.*

We now give some basic properties of the maps ϕ_a that will be used throughout the thesis.

Proposition 1.1.2. *For any $a \in \mathbb{D}$ and $z \in \mathbb{D}$ we have:*

(i) $\phi_a(0) = a$ *and* $\phi_a(a) = 0$,

(ii) $\phi_a \circ \phi_a(z) = z$,

(iii) $\phi_a'(z) = -\dfrac{1-|a|^2}{(1-\bar{a}z)^2}$, *and*

(iv) $1 - |\phi_a(z)|^2 = \dfrac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}$.

Proof. All properties are derived from straightforward calculations. □

Möbius maps of $\mathbb{D}$ belong, in fact, in a larger class of functions, called Möbius transformations. A Möbius transformation $T : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, where by $\bar{\mathbb{C}}$ we denote the set $\mathbb{C} \cup \{\infty\}$, has the form

$$T(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ (see [1, Section 2.1]). If $c = 0$, then $T(\mathbb{C}) = \mathbb{C}$, and we set $T(\infty) = \infty$. If $c \neq 0$, then $T\big(\mathbb{C} \setminus \{-\frac{d}{c}\}\big) = \mathbb{C} \setminus \{\frac{a}{c}\}$, and we set $T(-\frac{d}{c}) = \infty$ and $T(\infty) = \frac{a}{c}$. In

any case, Möbius transformations are one-to-one functions, which are holomorphic in $\mathbb{C}$ or $\mathbb{C} \setminus \{-\frac{d}{c}\}$ if $c = 0$ or $c \neq 0$, respectively.
An essential property of Möbius transformations is that they preserve the circles of $\bar{\mathbb{C}}$.

Definition 1.1.1. *A circle in $\bar{\mathbb{C}}$ is either a euclidean circle in $\mathbb{C}$ or the union of a euclidean line in $\mathbb{C}$ with $\{\infty\}$.*

Theorem 1.1.3. *Möbius transformations map circles of $\bar{\mathbb{C}}$ onto circles of $\bar{\mathbb{C}}$.*

For more information on the above definition and theorem see, for example, [1, Sections 1.2, 2.1].

1.2 The hyperbolic metric

Definition 1.2.1. *The function $\lambda(z) = \dfrac{1}{1-|z|^2}$, $z \in \mathbb{D}$, is called hyperbolic density.*

Definition 1.2.2. *If $\gamma : [a, b] \to \mathbb{D}$ is a piecewise-C^1 curve, we define the hyperbolic length of γ to be the real number*

$$l_h(\gamma) = \int_\gamma \lambda(z)\,|dz| = \int_a^b \lambda(\gamma(t)) \cdot |\gamma'(t)|\,dt = \int_a^b \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}\,dt.$$

Remark 1.2.1. The hyperbolic length of γ is, indeed, a real number; since $[a, b]$ is a compact set and γ is a curve in $\mathbb{D}$, there exists a point $t_0 \in [a, b]$ such that $\max\{|\gamma(t)| : t \in [a, b]\} = |\gamma(t_0)| < 1$, so

$$l_h(\gamma) = \int_a^b \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}\,dt \leq \int_a^b \frac{|\gamma'(t)|}{1-|\gamma(t_0)|^2}\,dt = \frac{1}{1-|\gamma(t_0)|^2} \cdot L(\gamma) < +\infty,$$

where $L(\gamma)$ denotes the euclidean length of γ.

It follows from Theorem 1.1.1 and Proposition 1.1.2 (iii), (iv), that hyperbolic length is invariant under the action of Möbius maps of $\mathbb{D}$, that is:

Proposition 1.2.1. *If $\gamma : [a, b] \to \mathbb{D}$ is a piecewise-C^1 curve and $\phi \in \mathrm{Aut}(\mathbb{D})$, then $\phi \circ \gamma : [a, b] \to \mathbb{D}$ is a piecewise-C^1 curve and $l_h(\phi \circ \gamma) = l_h(\gamma)$.*

Definition 1.2.3. *If $z, w \in \mathbb{D}$, we define the hyperbolic distance of z, w to be the number*

$$\beta(z, w) = \inf\{l_h(\gamma) : \gamma \textit{ is a piecewise-}C^1 \textit{ curve in } \mathbb{D} \textit{ starting at } z \textit{ and landing at } w\}.$$

Proposition 1.2.2. *The hyperbolic distance is invariant under the action of Möbius maps of $\mathbb{D}$, that is*

$$\beta(\phi(z), \phi(w)) = \beta(z, w), \textit{ for all } z, w \in \mathbb{D} \textit{ and } \phi \in \mathrm{Aut}(\mathbb{D}).$$

Proof. Let $z, w \in \mathbb{D}$ and $\phi \in \mathrm{Aut}(\mathbb{D})$. If γ is a piecewise-C^1 curve in $\mathbb{D}$ starting at z and landing at w, then $\phi \circ \gamma$ is a piecewise-C^1 curve in $\mathbb{D}$ starting at $\phi(z)$ and landing at $\phi(w)$, so the definition of hyperbolic distance and Proposition 1.2.1 give $\beta(\phi(z), \phi(w)) \leq l_h(\phi \circ \gamma) = l_h(\gamma)$; hence

$$\beta(\phi(z), \phi(w)) \leq \beta(z, w). \tag{1.1}$$

The above inequality holds for all $z, w \in \mathbb{D}$ and $\phi \in \mathrm{Aut}(\mathbb{D})$, so we replace z by $\phi(z)$, w by $\phi(w)$, and ϕ by ϕ^{-1}, to get

$$\beta(z, w) \leq \beta(\phi(z), \phi(w)) \tag{1.2}$$

The desired equality follows from (1.1) and (1.2). □

Theorem 1.2.3. *The function $\beta : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ is a metric on $\mathbb{D}$, called the hyperbolic metric of $\mathbb{D}$.*

Proof. Let $z, w \in \mathbb{D}$. Since $l_h(\gamma) \geq 0$ for all piecewise-C^1 curves γ in $\mathbb{D}$ starting at z and landing at w, we have that $\beta(z,w) \geq 0$. Also, if $\gamma : [a,b] \to \mathbb{D}$ is a piecewise-C^1 curve with $\gamma(a) = z$ and $\gamma(b) = w$, then $-\gamma(t) := \gamma(a+b-t),\ t \in [a,b]$, is a piecewise-$C^1$ curve in $\mathbb{D}$ starting at w and landing at z, and

$$l_h(-\gamma) = \int_a^b \frac{|\gamma'(a+b-t)|}{1-|\gamma(a+b-t)|^2}\,dt = \int_a^b \frac{|\gamma'(s)|}{1-|\gamma(s)|^2}\,ds = l_h(\gamma),$$

so $\beta(z,w) = \beta(w,z)$.

Let $z \in \mathbb{D}$. Then, for the curve $\gamma(t) = z$, for all $t \in [0,1]$, we have that $l_h(\gamma) = 0$, hence $\beta(z,z) = 0$. Conversely, let $\beta(z,w) = 0$ for some $z, w \in \mathbb{D}$, and let $\epsilon > 0$. Then there exists a piecewise-C^1 curve $\gamma : [a,b] \to \mathbb{D}$ with $\gamma(a) = z$ and $\gamma(b) = w$, such that $l_h(\gamma) < \epsilon$. If $m = \min\{|\gamma(t)| : t \in [a,b]\} < 1$, then

$$|z-w| \leq L(\gamma) \leq \frac{L(\gamma)}{1-m^2} \leq \int_a^b \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}\,dt = l_h(\gamma) < \epsilon.$$

Since ϵ was arbitrary, we get $|z-w| = 0$; so $z = w$.

Finally, let $u, v, w \in \mathbb{D}$, and $\epsilon > 0$. Then there exist piecewise-C^1 curves $\gamma : [a,b] \to \mathbb{D}$ and $\delta : [b,c] \to \mathbb{D}$, with $\gamma(a) = u$, $\gamma(b) = v$, $\delta(b) = v$ and $\delta(c) = w$, such that

$$l_h(\gamma) < \beta(u,v) + \frac{\epsilon}{2} \text{ and } l_h(\delta) < \beta(v,w) + \frac{\epsilon}{2}.$$

Thus, the curve $\gamma\dot{+}\delta : [a,c] \to \mathbb{D}$, defined by $(\gamma\dot{+}\delta)(t) = \begin{cases} \gamma(t), & \text{if } t \in [a,b], \\ \delta(t), & \text{if } t \in [b,c] \end{cases}$, is a piecewise-$C^1$ curve in $\mathbb{D}$ starting at u and landing at w; hence

$$\beta(u,w) \leq l_h(\gamma\dot{+}\delta) = l_h(\gamma) + l_h(\delta) < \beta(u,v) + \beta(v,w) + \epsilon.$$

Since ϵ was arbitrary, we get $\beta(u,w) \leq \beta(u,v) + \beta(v,w)$, which is the triangle inequality. □

We will now find an explicit formula for the hyperbolic distance, as follows.

Lemma 1.2.4. *Let $-1 < r \leq s < 1$. We have:*

(i)

$$\beta(r,s) = \frac{1}{2}\log\frac{1+\dfrac{s-r}{1-rs}}{1-\dfrac{s-r}{1-rs}}. \tag{1.3}$$

(ii) If $\gamma : [0,1] \to \mathbb{D}$ is a piecewise-C^1 curve with $\gamma(0) = r$ and $\gamma(1) = s$, then

$$l_h(\gamma) = \beta(r,s) \textit{ if and only if } \gamma(t) = u(t), \textit{ for all } t \in [0,1],$$

where $u : [0,1] \to \mathbb{R}$ is an increasing function. In that case, we have $\gamma([0,1]) = [r,s]$.

Proof. (i) Let $\gamma(t) = u(t) + iv(t),\ t \in [a,b]$, be a piecewise-$C^1$ curve in $\mathbb{D}$, with $\gamma(a) = r$ and $\gamma(b) = s$. Since $|\gamma(t)| \geq |u(t)|$, for all $t \in [a,b]$, and $|\gamma'(t)| \geq |u'(t)| \geq u'(t)$, for all $t \in [a,b]$ on which γ is differentiable, we have that

$$l_h(\gamma) = \int_a^b \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}\,dt \geq \int_a^b \frac{u'(t)}{1-u^2(t)}\,dt = \int_r^s \frac{1}{1-x^2}\,dx = \frac{1}{2}\log\frac{1+\dfrac{s-r}{1-rs}}{1-\dfrac{s-r}{1-rs}}, \tag{1.4}$$

so

$$\beta(r,s) \geq \frac{1}{2}\log\frac{1+\dfrac{s-r}{1-rs}}{1-\dfrac{s-r}{1-rs}}. \tag{1.5}$$

Now, consider the parametrization $\gamma(t) = r + t(s-r)$, $t \in [0,1]$, of the line-segment $[r,s]$, for which we have that

$$l_h(\gamma) = \int_0^1 \frac{s-r}{1-|r+t(s-r)|^2}\,dt = \int_r^s \frac{1}{1-x^2}\,dx = \frac{1}{2}\log\frac{1+\dfrac{s-r}{1-rs}}{1-\dfrac{s-r}{1-rs}}. \tag{1.6}$$

Thus, (1.5) and (1.6) give (1.3).

(ii) Let $\gamma = u + iv : [0,1] \to \mathbb{D}$ be a piecewise-C^1 curve, with $\gamma(0) = r$ and $\gamma(1) = s$.

If $l_h(\gamma) = \beta(r,s)$, we have equality in (1.4); so $v(t) = 0$, for all $t \in [0,1]$, which means that $\gamma(t) = u(t)$, for all $t \in [0,1]$. If there exists a point $t_0 \in [0,1]$ such that $u'(t_0) < 0$, then, by the piecewise-continuity of u', there exists an interval $[t_1,t_2] \subset [0,1]$ such that $t_0 \in [t_1,t_2]$ and $u'(t) < 0$, for all $t \in [t_1,t_2]$. Thus,

$$\begin{aligned}
l_h(\gamma) &= \int_0^1 \frac{|u'(t)|}{1-u^2(t)}\,dt \geq \int_0^{t_1} \frac{u'(t)}{1-u^2(t)}\,dt + \int_{t_1}^{t_2} \frac{-u'(t)}{1-u^2(t)}\,dt + \int_{t_2}^1 \frac{u'(t)}{1-u^2(t)}\,dt \\
&> \int_0^{t_1} \frac{u'(t)}{1-u^2(t)}\,dt + \int_{t_1}^{t_2} \frac{u'(t)}{1-u^2(t)}\,dt + \int_{t_2}^1 \frac{u'(t)}{1-u^2(t)}\,dt = \int_0^1 \frac{u'(t)}{1-u^2(t)}\,dt = \beta(r,s),
\end{aligned}$$

which is a contradiction to our hypothesis. Therefore, $u'(t) \geq 0$, for all $t \in [0,1]$ on which u is differentiable. Since u is continuous in $[0,1]$, we get that u is increasing in $[0,1]$, and

$$\gamma([0,1]) = u([0,1]) = [u(0), u(1)] = [r,s].$$

Conversely, let $\gamma(t) = u(t)$, for all $t \in [0,1]$, where $u : [0,1] \to \mathbb{R}$ is an increasing function. Then $u'(t) \geq 0$ for all $t \in [0,1]$ on which u is differentiable; so, using (1.3),

$$l_h(\gamma) = \int_0^1 \frac{|u'(t)|}{1-u^2(t)}\,dt = \int_0^1 \frac{u'(t)}{1-u^2(t)}\,dt = \int_r^s \frac{1}{1-x^2}\,dx = \beta(r,s).$$

□

Theorem 1.2.5. *If $z, w \in \mathbb{D}$, then*

$$\beta(z,w) = \frac{1}{2}\log\frac{1+\left|\dfrac{z-w}{1-\bar{w}z}\right|}{1-\left|\dfrac{z-w}{1-\bar{w}z}\right|}. \tag{1.7}$$

Proof. Let $z, w \in \mathbb{D}$, and consider the Möbius map $\phi_w(\zeta) = \dfrac{w-\zeta}{1-\bar{w}\zeta}$, $\zeta \in \mathbb{D}$. By Proposition 1.2.2,

$$\beta(z,w) = \beta(w,z) = \beta(\phi_w(w), \phi_w(z)) = \beta(0, \phi_w(z)). \tag{1.8}$$

Since rotations are Möbius maps of $\mathbb{D}$, we have that

$$\beta(0,\zeta) = \beta(0,|\zeta|), \text{ for all } \zeta \in \mathbb{D}. \tag{1.9}$$

By (1.8), (1.9) and (1.3),

$$\beta(z,w) = \beta(0, |\phi_w(z)|) = \frac{1}{2}\log\frac{1+|\phi_w(z)|}{1-|\phi_w(z)|} = \frac{1}{2}\log\frac{1+\left|\dfrac{z-w}{1-\bar{w}z}\right|}{1-\left|\dfrac{z-w}{1-\bar{w}z}\right|}.$$

□

The next step is to introduce the concept of hyperbolic geodesics.

Definition 1.2.4. *Let $z, w \in \mathbb{D}$. A C^1 curve $\gamma : [0,1] \to \mathbb{D}$ is called a hyperbolic geodesic that connects z to w if:*

(i) $\gamma(0) = z$, $\gamma(1) = w$,

(ii) $\gamma'(t) \neq 0$, *for all* $t \in [0,1]$, *and*

(iii) $l_h(\gamma) = \beta(z, w)$.

Hyperbolic geodesics are invariant under the action of Möbius maps of $\mathbb{D}$, that is:

Proposition 1.2.6. *Let $z, w \in \mathbb{D}$ and $\phi \in \mathrm{Aut}(\mathbb{D})$. If γ is a hyperbolic geodesic connecting z to w, then $\phi \circ \gamma$ is a hyperbolic geodesic connecting $\phi(z)$ to $\phi(w)$.*

Proof. Since γ is C^1 and ϕ is holomorphic, we have that $\phi \circ \gamma : [0,1] \to \mathbb{D}$ is a C^1 curve, connecting $\phi(z)$ to $\phi(w)$. Also, by Definition 1.2.4 (ii), $\gamma'(t) \neq 0$, for all $t \in [0,1]$, and $\phi'(z) \neq 0$, for all $z \in \mathbb{D}$, because ϕ is a conformal map (see [14, Section X.14]); so

$$(\phi \circ \gamma)'(t) = \phi'(\gamma(t)) \cdot \gamma'(t) \neq 0, \text{ for all } t \in [0,1].$$

Finally, by Propositions 1.2.1 and 1.2.2, and Definition 1.2.4 (iii),

$$l_h(\phi \circ \gamma) = l_h(\gamma) = \beta(z, w) = \beta(\phi(z), \phi(w)).$$

□

Theorem 1.2.7. *Let $z, w \in \mathbb{D}$, with $z \neq w$, and let $\gamma : [0,1] \to \mathbb{D}$ be a C^1 curve connecting z to w, with $\gamma'(t) \neq 0$, for all $t \in [0,1]$.*

(i) Suppose z, w are located on a diameter of $\mathbb{D}$. Then, γ is a hyperbolic geodesic connecting z to w if and only if $\gamma([0,1]) = [z, w]$, where by $[z, w]$ we denote the line-segment with initial point z and ending point w.

(ii) Suppose z, w are not located on a diameter of $\mathbb{D}$. Then, γ is a hyperbolic geodesic connecting z to w if and only if $\gamma([0,1])$ is the arc, with initial point z and ending point w, of the circle C which contains z and w and intersects $\partial\mathbb{D}$ perpendicularly.

Proof. (i) Let z, w be located on a diameter of $\mathbb{D}$. Consider a rotation ϕ such that $\phi(z) = r \in (-1, 1)$, $\phi(w) = s \in (-1, 1)$ and $r < s$.

Let γ be a hyperbolic geodesic connecting z to w. By Proposition 1.2.6, $\phi \circ \gamma$ is a hyperbolic geodesic connecting $\phi(z) = r$ to $\phi(w) = s$, so $l_h(\phi \circ \gamma) = \beta(r, s)$. It follows from Lemma 1.2.4 (ii) that $(\phi \circ \gamma)([0,1]) = [r, s]$, so $\gamma([0,1]) = \phi^{-1}([r, s]) = [z, w]$.

Conversely, let $\gamma([0,1]) = [z, w]$. Then $(\phi \circ \gamma)([0,1]) = \phi([z, w]) = [r, s]$, so $(\phi \circ \gamma)(t) = u(t)$, for all $t \in [0,1]$, where $u : [0,1] \to [r, s]$ is a C^1 function. Note that $u'(t) = \phi'(\gamma(t)) \cdot \gamma'(t) \neq 0$, for all $t \in [0,1]$, because $\phi'(z) \neq 0$, for all $z \in \mathbb{D}$ (see [14, Section X.14]), and since $u(0) = r$ and $u(1) = s > r$, u is a strictly increasing function. It follows from Lemma 1.2.4 (ii) that $l_h(\phi \circ \gamma) = \beta(r, s)$, so Propositions 1.2.1 and 1.2.2 give $l_h(\gamma) = l_h(\phi \circ \gamma) = \beta(r, s) = \beta(\phi^{-1}(r), \phi^{-1}(s)) = \beta(z, w)$. Thus, γ is a hyperbolic geodesic connecting z to w.

(ii) Suppose z, w are not located on a diameter of $\mathbb{D}$. Consider the Möbius map $\phi_w(\zeta) = \dfrac{w - \zeta}{1 - \bar{w}\zeta}$, $\zeta \in \mathbb{D}$. Then $\phi_w(w) = 0$ and $\phi_w(z) \neq 0$, because $z \neq w$. Let ψ be a rotation that maps $\phi_w(z)$ to $|\phi_w(z)| > 0$. Then $\phi := \psi \circ \phi_w$ is a Möbius map of $\mathbb{D}$, with $\phi(w) = 0$ and $\phi(z) = |\phi_w(z)| =: s \in (0, 1)$. It follows from Proposition 1.2.6 and (i) that

$$\gamma \text{ is a hyperbolic geodesic connecting } z \text{ to } w$$

if and only if

$$\phi \circ \gamma \text{ is a hyperbolic geodesic connecting } s \text{ to } 0$$

if and only if

$$(\phi \circ \gamma)([0,1]) = [s, 0]$$

if and only if

$$\gamma([0,1]) = \phi^{-1}([s,0]).$$

Let $C = \phi^{-1}(\mathbb{R} \cup \{\infty\})$. Then, Theorem 1.1.3 implies that C is a circle in $\bar{\mathbb{C}}$. Since ϕ^{-1} is a conformal map, by [14, Section X.14, II.12] we have that ϕ^{-1} preserves the angle between $\mathbb{R}$ and $\partial\mathbb{D}$; hence C intersects $\partial\mathbb{D}$ perpendicularly. Moreover, $z = \phi^{-1}(s) \in C$ and $w = \phi^{-1}(0) \in C$ are not located on a diameter of $\mathbb{D}$, so C is a euclidean circle. Thus,

$$\gamma([0,1]) = \phi^{-1}([s,0])$$

if and only if

$$\gamma([0,1]) \text{ is the arc of } C \cap \mathbb{D} \text{ connecting } z \text{ to } w.$$

□

1.3 The pseudo-hyperbolic metric and hyperbolic discs

Consider the function $\rho(z,w) = \left|\frac{z-w}{1-z\bar{w}}\right|$, for all $z, w \in \mathbb{D}$.

Proposition 1.3.1. *The function ρ is invariant under the action of Möbius maps of $\mathbb{D}$, that is,*

$$\rho(\phi(z), \phi(w)) = \rho(z,w), \textit{ for all } z, w \in \mathbb{D} \textit{ and } \phi \in \mathrm{Aut}(\mathbb{D}).$$

Proof. It follows from Theorem 1.1.1 and straightforward calculations. □

We will show that ρ is a metric on $\mathbb{D}$.

Lemma 1.3.2. $\rho(z,w) \leq |z| + |w|$, *for all $z, w \in \mathbb{D}$.*

Proof. By straightforward calculations, we get that for all $z, w \in \mathbb{D}$,

$$\rho(|z|, -|w|) = \frac{|z| + |w|}{1 + |z||w|} \leq |z| + |w| \tag{1.10}$$

and

$$1 - \rho^2(z,w) = \frac{(1-|z|^2)(1-|w|^2)}{|1-z\bar{w}|^2}. \tag{1.11}$$

Note that (1.11) gives

$$1 - \rho^2(|z|, -|w|) = \frac{(1-|z|^2)(1-|w|^2)}{(1+|z||w|)^2} \tag{1.12}$$

and

$$1 - \rho^2(z,w) \geq \frac{(1-|z|^2)(1-|w|^2)}{(1+|z||w|)^2}, \tag{1.13}$$

for all $z, w \in \mathbb{D}$. Thus, by (1.12), (1.13) and (1.10),

$$1 - \rho^2(|z|, -|w|) \leq 1 - \rho^2(z,w) \;\Rightarrow\; \rho(z,w) \leq \rho(|z|, -|w|) \leq |z| + |w|,$$

for all $z, w \in \mathbb{D}$. □

Proposition 1.3.3. *The function $\rho : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ is a metric on $\mathbb{D}$, called the pseudo-hyperbolic metric of $\mathbb{D}$.*

Proof. For all $z, w \in \mathbb{D}$, we have that $\rho(z,w) \geq 0$, $\rho(z,w) = \rho(w,z)$ and

$$\rho(z,w) = 0 \iff |z-w| = 0 \iff z = w.$$

As for the triangle inequality, by Proposition 1.3.1 and Lemma 1.3.2, we get that for all $z, a, w \in \mathbb{D}$,

$$\begin{aligned}\rho(z,w) &= \rho(\phi_a(z), \phi_a(w)) \leq |\phi_a(z)| + |\phi_a(w)| = \rho(\phi_a(z), 0) + \rho(0, \phi_a(w)) \\ &= \rho(\phi_a(z), \phi_a(a)) + \rho(\phi_a(a), \phi_a(w)) = \rho(z,a) + \rho(a,w).\end{aligned}$$

□

Remark 1.3.1. By formula (1.7) for the hyperbolic metric and the definition of pseudo-hyperbolic metric,

$$\beta(z,w) = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)} = \tanh^{-1}(\rho(z,w)), \tag{1.14}$$

for all $z, w \in \mathbb{D}$.

We will now engage in the concept of pseudo-hyperbolic and hyperbolic discs and explore their connection with euclidean discs.

Definition 1.3.1. *Let $z \in \mathbb{D}$ and $r > 0$.*

(i) The set $D_\rho(z,r) := \{w \in \mathbb{D} : \rho(w,z) < r\}$ is called the pseudo-hyperbolic disc with center z and radius r, and the set $C_\rho(z,r) := \{w \in \mathbb{D} : \rho(w,z) = r\}$ is called the pseudo-hyperbolic circle with center z and radius r.

(ii) The set $D_h(z,r) := \{w \in \mathbb{D} : \beta(w,z) < r\}$ is called the hyperbolic disc with center z and radius r, and the set $C_h(z,r) := \{w \in \mathbb{D} : \beta(w,z) = r\}$ is called the hyperbolic circle with center z and radius r.

Notation. If $z \in \mathbb{C}$ and $r > 0$, we denote by $D(z,r)$ the euclidean disc with center z and radius r, and by $C(z,r)$ the euclidean circle with center z and radius r.

Lemma 1.3.4. *For any $z \in \mathbb{D}$ and $0 < r < 1$, the pseudo-hyperbolic disc $D_\rho(z,r)$ is a euclidean disc with center $C = \dfrac{1-r^2}{1-r^2|z|^2}z$ and radius $R = \dfrac{1-|z|^2}{1-r^2|z|^2}r$.*

Proof. Let $z \in \mathbb{D}$ and $0 < r < 1$. If $w \in \mathbb{C}$, then

$$\begin{aligned}&w \in D_\rho(z,r) \iff w \in \mathbb{D} \text{ and } \rho(w,z) < r \iff w \in \mathbb{D} \text{ and } |z-w|^2 < r^2|1-z\bar{w}|^2 \iff \\ &\iff w \in \mathbb{D} \text{ and } |z|^2 - 2\mathrm{Re}(z\bar{w}) + |w|^2 < r^2(1 - 2\mathrm{Re}(z\bar{w}) + |z|^2|w|^2) \iff \\ &\iff w \in \mathbb{D} \text{ and } |w|^2 - 2\cdot\frac{1-r^2}{1-r^2|z|^2}\cdot\mathrm{Re}(z\bar{w}) < \frac{r^2-|z|^2}{1-r^2|z|^2} \iff \\ &\iff w \in \mathbb{D} \text{ and } |w|^2 - 2\cdot\frac{1-r^2}{1-r^2|z|^2}\cdot\mathrm{Re}(z\bar{w}) + \frac{(1-r^2)^2|z|^2}{(1-r^2|z|^2)^2} < \frac{r^2-|z|^2}{1-r^2|z|^2} + \frac{(1-r^2)^2|z|^2}{(1-r^2|z|^2)^2} \iff \\ &\iff w \in \mathbb{D} \text{ and } |w-C|^2 < R^2 \iff w \in \mathbb{D} \text{ and } w \in D(C,R),\end{aligned}$$

so

$$D_\rho(z,r) = \mathbb{D} \cap D(C,R). \tag{1.15}$$

Note that

$$|C| = \frac{1-r^2}{1-r^2|z|^2}|z| < \frac{1-r^2}{1-r^2|z|^2} < 1,$$

and

$$R < 1 - |C| \iff (1-|z|^2)r < 1 - r^2|z|^2 - (1-r^2)|z| \iff r|z|^2 - (r+1)|z| + 1 > 0.$$

The last inequality holds because $|z| < 1$ and $r \in (0,1)$. Thus, $D(C,R) \subset \mathbb{D}$, and by (1.15), we get $D_\rho(z,r) = D(C,R)$. □

Proposition 1.3.5. *For any $z \in \mathbb{D}$ and $r > 0$, the hyperbolic disc $D_h(z, r)$ is a euclidean disc with center $C = \dfrac{1 - s^2}{1 - s^2|z|^2}\, z$ and radius $R = \dfrac{1 - |z|^2}{1 - s^2|z|^2}\, s$, where $s = \tanh(r) \in (0, 1)$.*

Proof. Let $z \in \mathbb{D}$ and $r > 0$. If $w \in \mathbb{C}$, then it follows from (1.14) and Lemma 1.3.4 that

$$
\begin{aligned}
& w \in D_h(z, r) \;\Leftrightarrow\; w \in \mathbb{D} \text{ and } \beta(w, z) < r \;\Leftrightarrow\; w \in \mathbb{D} \text{ and } \frac{1}{2}\log\frac{1 + \rho(z, w)}{1 - \rho(z, w)} < r \;\Leftrightarrow \\
&\Leftrightarrow\; w \in \mathbb{D} \text{ and } \frac{1 + \rho(z, w)}{1 - \rho(z, w)} < e^{2r} \;\Leftrightarrow\; w \in \mathbb{D} \text{ and } \frac{2\rho(z, w)}{1 - \rho(z, w)} < e^{2r} - 1 \;\Leftrightarrow \\
&\Leftrightarrow\; w \in \mathbb{D} \text{ and } 2\rho(z, w) < (e^{2r} + 1)\tanh(r) \cdot (1 - \rho(z, w)) \;\Leftrightarrow \\
&\Leftrightarrow\; w \in \mathbb{D} \text{ and } \rho(z, w)(e^{2r} + 1) < e^{2r} - 1 \;\Leftrightarrow\; w \in \mathbb{D} \text{ and } \rho(z, w) < s \;\Leftrightarrow \\
&\Leftrightarrow\; w \in D_\rho(z, s) \;\Leftrightarrow\; w \in D(C, R).
\end{aligned}
$$

□

Corollary 1.3.6. *The euclidean metric restricted to $\mathbb{D}$ and the hyperbolic metric produce the same open sets in $\mathbb{D}$ or, equivalently, they are topologically equivalent metrics.*

Proof. Let $z \in \mathbb{D}$ and $r > 0$.

Since $D_h(z, r) \subset \mathbb{D}$ is a euclidean disc, it is an open set in $\mathbb{C}$; hence there exists a real number $\delta_1 > 0$ such than $D(z, \delta_1) \subset D_h(z, r)$.

On the other hand, if $w \in D_h\big(z, \frac{r}{2}\big)$, then by (1.14),

$$|z - w| = |1 - z\bar{w}| \tanh(\beta(z, w)) \le 2\tanh(\beta(z, w)) \le 2\beta(z, w) < r,$$

because $\tanh(x) \le x$, for all $x \ge 0$; hence $D_h\big(z, \frac{r}{2}\big) \subset D(z, r) \cap \mathbb{D}$. □

Remark 1.3.2. Although the euclidean and the hyperbolic metric are topologically equivalent, they are not comparable in $\mathbb{D}$. Indeed, $|z - w| \le 2$, for all $z, w \in \mathbb{D}$, whereas the hyperbolic metric is unbounded in $\mathbb{D}$, as can be clearly seen from (1.14). However, the two metrics are comparable in compact subsets of $\mathbb{D}$.

Proposition 1.3.7. *If $S \subset \mathbb{D}$ is compact, then the euclidean and the hyperbolic metric are comparable in S.*

Proof. Since S is a compact subset of $\mathbb{D}$, there exists a real $R \in (0, 1)$ such that $S \subset \overline{D(0, R)} \subset \mathbb{D}$. Let $z, w \in S$. Then, $|(1 - t)z + tw| \le R$, for all $t \in [0, 1]$, because $\overline{D(0, R)}$ is a convex set. Thus,

$$\beta(z, w) \le l_h([z, w]) = \int_0^1 \frac{|z - w|}{1 - |(1 - t)z + tw|^2}\, dt \le |z - w| \int_0^1 \frac{1}{1 - R^2}\, dt = |z - w| \frac{1}{1 - R^2},$$

so $(1 - R^2)\beta(z, w) \le |z - w|$.

On the other hand, (1.14) gives

$$|z - w| = |1 - z\bar{w}| \tanh(\beta(z, w)) \le 2\beta(z, w), \text{ for all } z, w \in S.$$

Thus, for all $z, w \in S$,

$$(1 - R^2)\beta(z, w) \le |z - w| \le 2\beta(z, w). \tag{1.16}$$

□

Another interesting difference between the euclidean and the hyperbolic metric in $\mathbb{D}$ is that the metric space $(\mathbb{D}, |\cdot|)$ is not complete, in contrast to $(\mathbb{D}, \beta)$. Indeed, the sequence $z_n = 1 - \frac{1}{n+1}$, $n \in \mathbb{N}$, is a Cauchy sequence in $\mathbb{D}$ with respect to the euclidean metric, but it converges to $1 \notin \mathbb{D}$. However, for the hyperbolic metric we have that:

Proposition 1.3.8. *$(\mathbb{D}, \beta)$ is a complete metric space.*

Proof. Let $\{z_k\} \subset \mathbb{D}$ be a Cauchy sequence with respect to the hyperbolic metric, and let $\epsilon > 0$. Then there exists a number $n_0 \in \mathbb{N}$ such that

$$\beta(z_n, z_m) < \epsilon, \text{ for all } n, m \geq n_0. \tag{1.17}$$

Let $n \geq n_0$. By (1.17),

$$\beta(z_n, 0) \leq \beta(z_n, z_{n_0}) + \beta(z_{n_0}, 0) < \epsilon + \beta(z_{n_0}, 0),$$

so $z_n \in D_h(0, r_\epsilon)$, for all $n \geq n_0$, where $r_\epsilon := \epsilon + \beta(z_{n_0}, 0) > 0$. If $s_\epsilon := \tanh(r_\epsilon) \in (0,1)$, then Proposition 1.3.5 yields that $D_h(0, r_\epsilon) = D(0, s_\epsilon)$; so

$$z_n \in \overline{D(0, s_\epsilon)}, \text{ for all } n \geq n_0, \tag{1.18}$$

and $\overline{D(0, s_\epsilon)}$ is a compact subset of $\mathbb{D}$.

Now, by (1.16) and (1.17),

$$|z_n - z_m| \leq 2\beta(z_n, z_m) < \epsilon, \text{ for all } n, m \geq n_0,$$

hence $\{z_k\}$ is a Cauchy sequence with respect to the euclidean metric. Thus, there exists a point $z \in \mathbb{C}$ such that $z_k \to z$, as $k \to +\infty$, and (1.18) gives that $z \in \overline{D(0, s_\epsilon)} \subset \mathbb{D}$. Finally, by (1.16),

$$(1 - s_\epsilon^2)\beta(u, v) \leq |u - v|, \text{ for all } u, v \in \overline{D(0, s_\epsilon)},$$

hence

$$\beta(z_n, z) \leq \frac{1}{1 - s_\epsilon^2}|z_n - z| \to 0, \text{ as } n \to +\infty,$$

that is, $\{z_k\}$ converges to $z \in \mathbb{D}$ with respect to the hyperbolic metric. □

Lemma 1.3.9. *Let $z, a \in \mathbb{D}$ and $r > 0$. Then $\phi_a(D_h(z, r)) = D_h(\phi_a(z), r)$.*

Proof. If $w \in D_h(z, r)$, then $\beta(\phi_a(z), \phi_a(w)) = \beta(z, w) < r$, because of the Möbius invariance of the hyperbolic metric; so $\phi_a(w) \in D_h(\phi_a(z), r)$. For the reverse enclosure, if $w \in D_h(\phi_a(z), r)$, then by Propositions 1.2.2 and 1.1.2 (ii), $\beta(\phi_a(w), z) = \beta(w, \phi_a(z)) < r$, so $w = \phi_a(\phi_a(w)) \in \phi_a(D_h(z, r))$. □

Let μ_2 denote the restriction to $\mathbb{D}$ of the Lebesgue measure of $\mathbb{C}$. Then, by A we denote the normalized area measure of $\mathbb{D}$, that is $A = \frac{1}{\pi}\mu_2$. This way,

$$A(\mathbb{D}) = \int_{\mathbb{D}} dA(z) = \int_0^1 \int_0^{2\pi} \frac{r}{\pi}\, d\theta dr = 1.$$

Also, for all $\alpha \in \mathbb{R}$, define the measure

$$A_\alpha(D) = \int_D dA_\alpha(z) = \int_D c_\alpha (1 - |z|^2)^\alpha \, dA(z),$$

for all Borel-measurable sets $D \subset \mathbb{D}$, where $c_\alpha = \begin{cases} \alpha + 1, & \text{if } \alpha > -1, \\ 1, & \text{if } \alpha \leq -1 \end{cases}$.

Remark 1.3.3. If $\alpha > -1$, the measure A_α is finite. Indeed,

$$\int_{\mathbb{D}} dA_\alpha(z) = \int_{\mathbb{D}} c_\alpha (1 - |z|^2)^\alpha \, dA(z) = \int_0^1 \int_0^{2\pi} (\alpha + 1)(1 - r^2)^\alpha \cdot \frac{r}{\pi}\, d\theta dr = 1 < +\infty.$$

Lemma 1.3.10. *Suppose $z \in \mathbb{D}$ and $r > 0$. Then*

(i) $A(D_h(z,r)) = \dfrac{(1-|z|^2)^2 s^2}{(1-|z|^2 s^2)^2}$,

(ii) $\inf\left\{\dfrac{1}{|1-z\overline{w}|} : w \in D_h(z,r)\right\} = \dfrac{1-s|z|}{1-|z|^2}$, *and*

(iii) $\sup\left\{\dfrac{1}{|1-z\overline{w}|} : w \in D_h(z,r)\right\} = \dfrac{1+s|z|}{1-|z|^2}$,

where $s = \tanh(r) \in (0,1)$.

Proof. (i) By Proposition 1.3.5, $D_h(z,r) = D(C,R)$, where $C = \dfrac{1-s^2}{1-s^2|z|^2} z$ and $R = \dfrac{1-|z|^2}{1-s^2|z|^2} s$, hence

$$A(D_h(z,r)) = \int_{D_h(z,r)} dA(z) = \int_0^R \int_0^{2\pi} \frac{\rho}{\pi}\, d\theta d\rho = R^2 = \frac{(1-|z|^2)^2 s^2}{(1-|z|^2 s^2)^2}.$$

(ii) Using Lemma 1.3.9,

$$\begin{aligned}\inf\left\{\frac{1}{|1-z\overline{w}|} : w \in D_h(z,r)\right\} &= \inf\left\{\frac{1}{|1-z\overline{w}|} : w \in \phi_z(D_h(0,r))\right\} \\ &= \inf\left\{\frac{1}{|1-z\overline{\phi_z(w)}|} : w \in D_h(0,r)\right\} \\ &= \inf\left\{\frac{|1-z\overline{w}|}{1-|z|^2} : w \in D_h(0,r)\right\} = \frac{1-s|z|}{1-|z|^2},\end{aligned}$$

where the last equality holds because $D_h(0,r) = D(0,s)$ by Proposition 1.3.5, so for all $w \in D_h(0,r)$, $|1-z\overline{w}| \geq 1-|z||w| \geq 1-|z|s$, and for the sequence $w_n = \left(1-\frac{1}{n}\right)se^{i\mathrm{Arg}z} \in D(0,s)$, $n \in \mathbb{N}$, we have that $|1-z\overline{w}_n| = |1-|z|s\left(1-\frac{1}{n}\right)| \to 1-s|z|$, as $n \to +\infty$.
(iii) It can be proved in exactly the same way as (ii). □

Proposition 1.3.11. *Let $r > 0$ and $\alpha \in \mathbb{R}$. Then*

(i) $|1-z\overline{w}| \sim 1-|z|^2 \sim 1-|w|^2$, *for all $z, w \in \mathbb{D}$ with $\beta(z,w) < r$, and*

(ii) $A_\alpha(D_h(z,r)) \sim (1-|z|^2)^{2+\alpha}$, *for all $z \in \mathbb{D}$.*

Proof. Let $s = \tanh(r) \in (0,1)$.
(i) By Lemma 1.3.10 (ii), (iii), we get that for all $z, w \in \mathbb{D}$ with $\beta(z,w) < r$,

$$\frac{1}{|1-z\overline{w}|} \geq \frac{1-s|z|}{1-|z|^2} \;\Rightarrow\; |1-z\overline{w}| \leq \frac{1-|z|^2}{1-s|z|} \leq \frac{1}{1-s}\cdot(1-|z|^2),$$

and

$$\frac{1}{|1-z\overline{w}|} \leq \frac{1+s|z|}{1-|z|^2} \;\Rightarrow\; |1-z\overline{w}| \geq \frac{1-|z|^2}{1+s|z|} \geq \frac{1}{1+s}\cdot(1-|z|^2),$$

respectively. Thus, $|1-z\overline{w}| \sim 1-|z|^2$. The relation $|1-z\overline{w}| \sim 1-|w|^2$ can be proved in exactly the same way.
(ii) By (i) and Lemma 1.3.10 (i), we have that for all $z \in \mathbb{D}$,

$$\begin{aligned}A_\alpha(D_h(z,r)) &= \int_{D_h(z,r)} dA_\alpha(w) = \int_{D_h(z,r)} c_\alpha(1-|w|^2)^\alpha\, dA(w) \sim \\ &\sim \int_{D_h(z,r)} (1-|z|^2)^\alpha\, dA(w) = (1-|z|^2)^\alpha \cdot \frac{(1-|z|^2)^2 s^2}{(1-|z|^2 s^2)^2} \sim \\ &\sim (1-|z|^2)^{2+\alpha},\end{aligned}$$

because $s^2 \leq \dfrac{s^2}{(1-|z|^2 s^2)^2} \leq \dfrac{s^2}{(1-s^2)^2}$. □

1.4 r-lattices in the hyperbolic metric

Lemma 1.4.1. *There exists a constant $C > 0$ with the following property: If $0 < r < 1$, then every hyperbolic disc $D_h(a, 1)$ can be covered by at most $\frac{C}{r^2}$ hyperbolic discs $D_h(z, r)$.*

Proof. Case 1 (special case). $a = 0$. Then, by Proposition 1.3.5, $D_h(0, 1) = D(0, \tanh(1))$.
We symmetrically place points on $C(0, \tanh(1))$ so that the hyperbolic length of the arc between any two adjacent points be less than or equal to $\frac{r}{4}$. Since

$$l_h(C(0, \tanh(1))) = \int_0^{2\pi} \frac{|\tanh(1) i e^{i\theta}|}{1 - |\tanh(1) e^{i\theta}|^2} \, d\theta = \frac{2\pi \tanh(1)}{1 - \tanh^2(1)} =: t,$$

the number of points we place is equal to $\left[\frac{t}{\frac{r}{4}}\right] + 1 = \left[\frac{C_1}{r}\right] + 1$, where the constant $C_1 > 0$ is independent of r.
Next, we place points on the interval $[0, \tanh(1)]$ so that the hyperbolic distance between any two adjacent points be less than or equal to $\frac{r}{4}$. Since

$$l_h([0, \tanh(1)]) = \int_0^{\tanh(1)} \frac{1}{1 - x^2} \, dx = \frac{1}{2} \log \frac{1 + \tanh(1)}{1 - \tanh(1)} = 1,$$

the number of points we place is equal to $\left[\frac{1}{\frac{r}{4}}\right] + 1 = \left[\frac{C_2}{r}\right] + 1$, where the constant $C_2 > 0$ is independent of r.

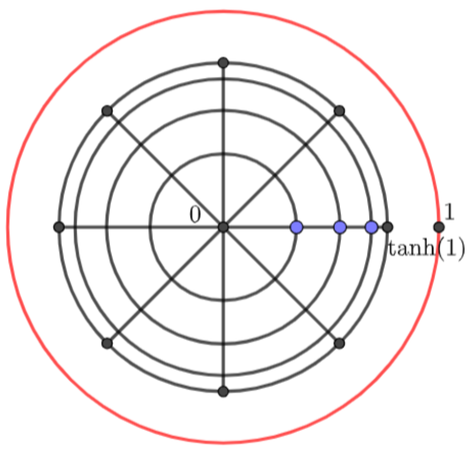

Figure 1.1: The division of the disc $D_h(0, 1)$ into $\frac{C}{r^2}$ pieces

Now, connect the points on $C(0, \tanh(1))$ to the origin and draw circles centered at 0 through the points on $[0, \tanh(1)]$. This way, the hyperbolic disc $D_h(0, 1)$ is divided into

$$\left(\left[\frac{C_1}{r}\right] + 1\right)\left(\left[\frac{C_2}{r}\right] + 1\right) \leq \left(\frac{C_1}{r} + 1\right)\left(\frac{C_2}{r} + 1\right) = \frac{C_1 C_2}{r^2} + \frac{1}{r}(C_1 + C_2) + 1$$
$$\leq \frac{C_1 C_2}{r^2} + \frac{1}{r^2}(C_1 + C_2) + \frac{1}{r^2} = \frac{C}{r^2}$$

pieces, where $C > 0$ is independent of r (see Figure 1.1). Because of its construction, any such piece can be covered by a hyperbolic disc with radius r. Thus, $D_h(0, 1)$ can be covered by at most $\frac{C}{r^2}$ hyperbolic

discs with radius r.
Case 2 (general case). Let $a \in \mathbb{D}$. Then Lemma 1.3.9 gives $\phi_a(D_h(0,1)) = D_h(a,1)$, and, by *Case 1*, there exists a constant $C > 0$, independent of r, such that $D_h(0,1) \subset \bigcup_{j=1}^{n} D_h(z_j, r)$, for some $z_j \in \mathbb{D}$, where $n \le \frac{C}{r^2}$. Thus,

$$D_h(a,1) \subset \phi_a\left(\bigcup_{j=1}^{n} D_h(z_j,r)\right) = \bigcup_{j=1}^{n} \phi_a(D_h(z_j,r)) = \bigcup_{j=1}^{n} D_h(\phi_a(z_j),r).$$

□

Definition 1.4.1. *Let $r > 0$. A sequence $\{a_k\}$ in $\mathbb{D}$ is called an r-lattice in the hyperbolic metric if:*

(i) $\beta(a_i, a_j) \ge \frac{r}{2}$, *for all $i \neq j$, and*

(ii) $\mathbb{D} = \bigcup_{k=1}^{+\infty} D_h(a_k, r)$.

Definition 1.4.2. *A sequence $\{a_k\}$ in $\mathbb{D}$ is said to be separated in the hyperbolic metric if there exists a real $\delta > 0$ such that $\beta(a_i, a_j) \ge \delta$, for all $i \neq j$.*

Remark 1.4.1. Every r-lattice in the hyperbolic metric is a separated sequence in the hyperbolic metric.

Proposition 1.4.2. *There exists a constant $C > 0$ with the following property: If $0 < r < 1$ and $\{a_k\}$ is an r-lattice in the hyperbolic metric, then every point $z \in \mathbb{D}$ belongs to at most $\frac{C}{r^2}$ of the sets $D_h(a_k, 1)$.*

Proof. Let $z \in \mathbb{D}$. Then, there exists a number $n \in \mathbb{N}$ such that $z \in D_h(a_n, r) \subset D_h(a_n, 1)$, by the definition of the r-lattice. Suppose $z \in D_h(a_{k_i}, 1)$ for $i = 1, \dots, N,$ where $N \ge 1$; hence

$$a_{k_i} \in D_h(z,1), \text{ for all } i \in \{1, \dots, N\}. \tag{1.19}$$

By Lemma 1.4.1, there exists a constant $C > 0$, independent of r, and $\lambda_1, \dots, \lambda_K \in \mathbb{D}$ such that

$$D_h(z,1) \subset \bigcup_{n=1}^{K} D_h\Big(\lambda_n, \frac{r}{4}\Big), \tag{1.20}$$

where $K \le \frac{C}{r^2}$. If $N > K$, then it follows from (1.19) and (1.20) that there exist $i, j \in \{1, \dots, N\}, i \neq j$, such that $a_{k_i}, a_{k_j} \in D_h(\lambda_m, \frac{r}{4})$, for some $m \in \{1, \dots, K\}$. But

$$\beta(a_{k_i}, a_{k_j}) \le \beta(a_{k_i}, \lambda_m) + \beta(\lambda_m, a_{k_j}) < \frac{r}{4} + \frac{r}{4} = \frac{r}{2},$$

which is a contradiction to the definition of the r-lattice. Thus, $N \le K \le \frac{C}{r^2}$. □

Theorem 1.4.3. *For any $r > 0$, there exists an r-lattice $\{a_k\}$ in the hyperbolic metric.*

Proof. Let $\mathbb{D} \cap (\mathbb{Q} + i\mathbb{Q}) = \{\lambda_1, \lambda_2, \dots\}$. By the density of $\mathbb{Q} + i\mathbb{Q}$ in $\mathbb{C}$, $\mathbb{D} = \bigcup_{n=1}^{+\infty} D_h(\lambda_n, \frac{r}{2})$. We construct a sequence $\{a_k\}$ in $\mathbb{D}$ as follows:
Set $a_1 = \lambda_1$. If we have chosen a point a_k of the sequence $\{\lambda_n\}$, then we choose a_{k+1} to be the first point after a_k in the sequence $\{\lambda_n\}$ for which the hyperbolic distance from $a_1, \dots, a_k$ is greater than or equal to $\frac{r}{2}$. Inductively, we construct a sequence $\{a_k\}$ in $\mathbb{D}$ which is a subsequence of $\{\lambda_n\}$ and fulfills (i) of Definition 1.4.1.
Let $z \in \mathbb{D}$. Then $z \in D_h(\lambda_n, \frac{r}{2})$, for some $n \in \mathbb{N}$. If $\lambda_n = a_m$ for some $m \in \mathbb{N}$, then

$$z \in D_h\Big(a_m, \frac{r}{2}\Big) \subset D_h(a_m, r).$$

If λ_n is not in $\{a_k\}$, then $n \neq 1$ and we let a_j be the point of $\{a_k\}$ before λ_n with the largest index. Thus, a_{j+1} appears after λ_n. We check the following two possibilities:

(i) $\beta(a_j, \lambda_n) < \frac{r}{2}$. Then $\beta(z, a_j) \le \beta(z, \lambda_n) + \beta(\lambda_n, a_j) < \frac{r}{2} + \frac{r}{2} = r$, so $z \in D_h(a_j, r)$.

(ii) $\beta(a_j, \lambda_n) \ge \frac{r}{2}$. Then, by the choice of a_j and the fact that λ_n does not belong in $\{a_k\}$, there exists an index $k_0 \in \{1, \dots, j-1\}$ such that $\beta(\lambda_n, a_{k_0}) < \frac{r}{2}$. Thus,

$$\beta(z, a_{k_0}) \le \beta(z, \lambda_n) + \beta(\lambda_n, a_{k_0}) < \frac{r}{2} + \frac{r}{2} = r,$$

so $z \in D_h(a_{k_0}, r)$.

In all cases, $z \in D_h(a_k, r)$ for some $k \in \mathbb{N}$. Thus, $\{a_k\}$ fulfills (ii) of Definition 1.4.1 and hence, it is an r-lattice in the hyperbolic metric. □

We will now construct a specific type of r-lattice, called regular r-lattice. First, we need the following two lemmas.

Lemma 1.4.4. *Let $s \in (0,1)$ and $a, b, c \in C(0, s)$ such that $c = e^{i\theta}a$, where $\theta \in (0, \pi] \cup (-\pi, 0)$, and $b = e^{i\tilde{\theta}}a$, where $\tilde{\theta} \in \begin{cases} (0,\theta), & \text{if } \theta \in (0,\pi], \\ (\theta, 0), & \text{if } \theta \in (-\pi, 0) \end{cases}$. Then $\beta(a, c) \ge \beta(a, b)$.*

Proof. Let $a = se^{i\hat{\theta}}$, where $\hat{\theta} \in [0, 2\pi)$, and consider the Möbius maps $\phi_1(z) = e^{-i\hat{\theta}}z$, $z \in \mathbb{D}$, and $\phi = \phi_s \circ \phi_1$ in $\mathbb{D}$. Then, ϕ_1 maps $C(0, s)$ onto itself and ϕ_s maps $[-1, 1]$ onto itself. By Theorem 1.1.3, ϕ_s maps $C(0, s)$ onto a circle in $\mathbb{D}$, which is perpendicular to $\mathbb{R}$, since Möbius maps of $\mathbb{D}$ are conformal, and so they preserve the angles between curves (see [14, Sections X.14, II.12]). Also, ϕ maps a to 0, 0 to s, and if we apply ϕ on $C(0, s)$, the order of the points a, b, c is preserved, by the preservation of angles (see Figure 1.2). Thus, by Proposition 1.2.2 and Theorem 1.2.7 (i),

$$\beta(a, c) = \beta(\phi(a), \phi(c)) = \beta(0, \phi(c)) = l_h([0, \phi(c)]) \ge l_h([0, \phi(b)]) = \beta(0, \phi(b)) = \beta(a, b).$$

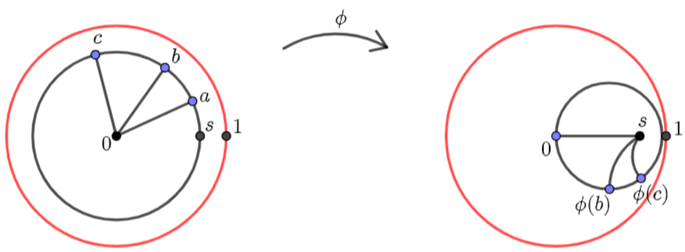

Figure 1.2: The transformation of the circle $C(0, s)$ through the Möbius map ϕ

□

Lemma 1.4.5. *Let $0 < s_1 < s_2 < 1$, $a = s_1 e^{i\theta} \in C(0, s_1)$, where $\theta \in [0, 2\pi)$, $b = s_2 e^{i\theta} \in C(0, s_2)$, and $c \in C(0, s_2)$. Then $\beta(a, c) \ge \beta(a, b)$.*

Proof. Consider the Möbius maps $\phi_1(z) = e^{-i\theta}z$, $z \in \mathbb{D}$, and $\phi = \phi_{s_1} \circ \phi_1$ in $\mathbb{D}$. Then, ϕ_1 maps $C(0, s_1)$ and $C(0, s_2)$ onto themselves and ϕ_{s_1} maps $[-1, 1]$ onto itself. By Theorem 1.1.3, ϕ_{s_1} maps $C(0, s_1)$ and $C(0, s_2)$ onto circles in $\mathbb{D}$, which are perpendicular to $\mathbb{R}$, since Möbius maps of $\mathbb{D}$ are conformal, and so they preserve the angles between curves (see [14, Sections X.14, II.12]). Also, ϕ maps a to 0, 0 to s_1, and b to $\phi(b) = \dfrac{s_1 - s_2}{1 - s_1 s_2} < 0$ (see Figure 1.3). Let $d \in (\phi(C(0, s_2)) \cap \mathbb{R}) \setminus \{\phi(b)\}$. Then, by Proposition 1.2.2 and Theorem 1.2.7 (i),

$$\begin{aligned}
\beta(\phi(b), 0) = \beta(b, a) = l_h([a, b]) &= \tanh^{-1}(s_2) - \tanh^{-1}(s_1) \\
&\le \tanh^{-1}(s_2) = \beta(0, \phi^{-1}(d)) = \beta(s_1, d) \le \beta(s_1, d) + \beta(0, s_1) \\
&= l_h([s_1, d]) + l_h([0, s_1]) = l_h([0, d]) = \beta(0, d),
\end{aligned}$$

so

$$\beta(a,c) = \beta(\phi(a),\phi(c)) = \beta(0,\phi(c)) \geq \beta(0,\phi(b)) = \beta(a,b).$$

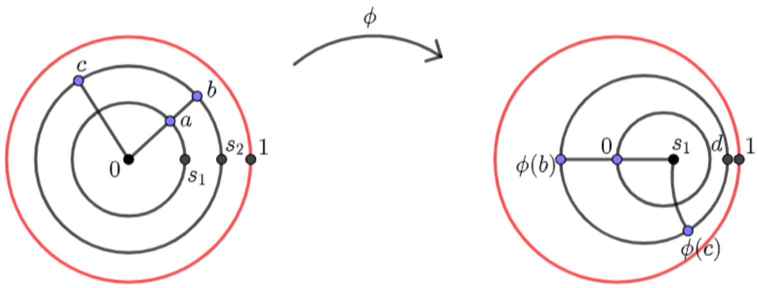

Figure 1.3: The transformation of the circles $C(0,s_1)$ and $C(0,s_2)$ through the Möbius map ϕ

□

Let $r > 0$. We construct a sequence in $\mathbb{D}$ as follows: Set $a_1^0 = 0$. For any $k \in \mathbb{N}\cup\{0\}$, let $S_k = C_h\big(0, \frac{kr}{2}\big)$. Proposition 1.3.5 yields that, for all $k \in \mathbb{N}\cup\{0\}$, $S_k = C(0, \tanh(\frac{kr}{2}))$. For each $k \geq 1$, symmetrically place n_k points $\{a_1^k, \dots, a_{n_k}^k\}$ on S_k such that the hyperbolic distance between any two adjacent points is greater than or equal to $\frac{r}{2}$, but less than $\frac{3r}{4}$.

Proposition 1.4.6. *The sequence $\{0\}\cup\{a_1^1,\dots,a_{n_1}^1\}\cup\cdots\cup\{a_1^k,\dots,a_{n_k}^k\}\cup\dots$ constructed above is an r-lattice in the hyperbolic metric.*

Proof. Let $k \geq 1$ and $i, j \in \{1,\dots,n_k\}, i \neq j$.
If a_i^k, a_j^k are adjacent, then $\beta(a_i^k, a_j^k) \geq \frac{r}{2}$, by the construction of the sequence.
If this is not the case, then $a_j^k = e^{i\theta}a_i^k$, where $\theta \in (0,\pi]\cup(-\pi,0)$, and there exists an index $\lambda \in \{1,\dots,n_k\}$, $\lambda \neq i$, $\lambda \neq j$, such that a_i^k, a_λ^k are adjacent and $a_\lambda^k = e^{i\tilde{\theta}}a_i^k$, where

$$\tilde{\theta} \in \begin{cases} (0,\theta), & \text{if } \theta \in (0,\pi], \\ (\theta,0), & \text{if } \theta \in (-\pi,0) \end{cases}.$$

By Lemma 1.4.4, $\beta(a_i^k, a_j^k) \geq \beta(a_i^k, a_\lambda^k) \geq \frac{r}{2}$.

Let $k, l \geq 1$ with $l > k$, $i \in \{1,\dots,n_k\}$, $j \in \{1,\dots,n_l\}$. If $a_i^k = \tanh(\frac{kr}{2})e^{i\theta}$, where $\theta \in [0,2\pi)$, then by Lemma 1.4.5,

$$\beta(a_i^k, a_j^l) \geq \beta(a_i^k, \tanh\left(\frac{lr}{2}\right)\cdot e^{i\theta}) = \frac{lr}{2} - \frac{kr}{2} \geq \frac{r}{2}.$$

Thus, for all $k, l \in \mathbb{N}\cup\{0\}$ and all $i \in \{1,\dots,n_k\}$, $j \in \{1,\dots,n_l\}$, we have that $\beta(a_i^k, a_j^l) \geq \frac{r}{2}$, so the sequence fulfills (i) of Definition 1.4.1.

Let $z \in \mathbb{D}$. We have one of the following situations:

(i) $z \in S_0 \cup D_h(0, \frac{r}{2})$. Then $z \in D_h(0, r)$.

(ii) $z \in S_k$, for some $k \in \mathbb{N}$. Then there exist indexes $i, j \in \{1,\dots,n_k\}$, $i \neq j$, such that a_i^k, a_j^k are adjacent, $a_j^k = e^{i\hat{\theta}}a_i^k$, where $\hat{\theta} \in (0,\pi]$, and $z = e^{i\theta}a_i^k$, where $\theta \in [0,\hat{\theta}]$. It follows from Lemma 1.4.4 that $\beta(a_i^k, z) \leq \beta(a_i^k, a_j^k) < \frac{3r}{4}$, so $z \in D_h(a_i^k, r)$.

(iii) $z \in D_h\big(0, \frac{(k+1)r}{2}\big) \setminus \overline{D_h\big(0, \frac{kr}{2}\big)}$.
If $\beta(0,z) < \frac{kr}{2} + \frac{r}{4}$, then there exist indexes $i, j \in \{1, \dots, n_k\}$, $i \neq j$, such that a_i^k, a_j^k are adjacent, $a_i^k = \tanh(\frac{kr}{2})e^{i\tilde{\theta}}$ and $a_j^k = e^{i\hat{\theta}}a_i^k$, where $\tilde{\theta} \in [0, 2\pi)$, $\hat{\theta} \in (0, \pi]$, and $z = |z|e^{i\theta}$, where $\theta \in [\tilde{\theta}, \tilde{\theta} + \hat{\theta}]$. Then it follows from Lemma 1.4.4 that

$$\beta(z, a_i^k) \leq \beta\Big(z, \tanh\Big(\frac{kr}{2}\Big) \cdot e^{i\theta}\Big) + \beta\Big(\tanh\Big(\frac{kr}{2}\Big) \cdot e^{i\theta}, a_i^k\Big) \leq \frac{r}{4} + \beta(a_i^k, a_j^k) < \frac{r}{4} + \frac{3r}{4} = r,$$

so $z \in D_h(a_i^k, r)$.
We work for the case $\beta(0,z) \geq \frac{kr}{2} + \frac{r}{4}$ in exactly the same way.

In any case, $z \in D_h(a_i^k, r)$, for some $k \in \mathbb{N} \cup \{0\}$ and $i \in \{1, \dots, n_k\}$, so the sequence fulfills (ii) of Definition 1.4.1 and, hence, it is an r-lattice. □

Proposition 1.4.7. *Let $r > 0$ and $\{a_k\}$ be a regular r-lattice in the hyperbolic metric. For all $R > 0$, there exists a separation of $\{a_k\}$ into finitely many subsequences $\{a_{k1}, \dots, a_{kn}, \dots\}$, $1 \leq k \leq N$ ($N \in \mathbb{N}$ depends on r and R), such that*

$$\beta(a_{ki}, a_{kj}) \geq R, \text{ for all } i \neq j \text{ and } k \in \{1, \dots, N\}. \tag{1.21}$$

Proof. Case 1. $\frac{r}{2} \geq R$. Then, the whole sequence $\{a_k\}$ fulfills (1.21), because it is an r-lattice.
Case 2. $\frac{r}{2} < R$. Then there exists a number $p \in \mathbb{N}$ such that $p\frac{r}{2} \geq R$. From the circles S_k with k small, in which there might not belong many points of the sequence, we choose every point of $\{a_k\}$ to be a subsequence on its own. When we reach a circle S_k where there are enough points, we choose them in groups and then move on to the first circle S_l whose hyperbolic distance from S_k is greater than or equal to $p\frac{r}{2}$. □

Proposition 1.4.8. *Suppose $0 < r \leq 1$ and let $\{a_k\}$ be an r-lattice in the hyperbolic metric. Then, for each $k \in \mathbb{N}$ there exists a Borel-measurable set $D_k \subset \mathbb{D}$ with the following properties:*

(i) $D_h(a_k, \frac{r}{4}) \subset D_k \subset D_h(a_k, r)$, *for all* $k \in \mathbb{N}$,

(ii) $D_i \cap D_j = \emptyset$, *if* $i \neq j$, *and*

(iii) $\mathbb{D} = \bigcup_{k=1}^{+\infty} D_k$.

Proof. Let

$$D_1 = D_h(a_1, r) \setminus \bigcup_{j=2}^{+\infty} D_h\Big(a_j, \frac{r}{4}\Big). \tag{1.22}$$

If $\{D_1, \dots, D_k\}$ have been constructed, we let

$$D_{k+1} = D_h(a_{k+1}, r) \setminus \Bigg[\Bigg(\bigcup_{i=1}^{k} D_i\Bigg) \bigcup \Bigg(\bigcup_{j=k+2}^{+\infty} D_h\Big(a_j, \frac{r}{4}\Big)\Bigg)\Bigg], \tag{1.23}$$

hence, by induction, we arrive at a sequence $\{D_k\}$ of Borel-measurable subsets of $\mathbb{D}$. Then, we get immediately by (1.22) and (1.23) that $D_k \subset D_h(a_k, r)$, for all $k \in \mathbb{N}$, and that $D_i \cap D_j = \emptyset$, if $i \neq j$, because if $z \in D_j$ for some $j > 1$, then $z \notin D_i$, for all $i \in \{1, \dots, j-1\}$.

Let $z \in D_h(a_1, \frac{r}{4})$. If $z \notin D_1$, then by (1.22), $z \in D_h(a_\lambda, \frac{r}{4})$, for some $\lambda \geq 2$; hence

$$\beta(a_1, a_\lambda) \leq \beta(a_1, z) + \beta(z, a_\lambda) < \frac{r}{4} + \frac{r}{4} = \frac{r}{2},$$

which is a contradiction to the definition of the r-lattice. Thus, $D_h(a_1, \frac{r}{4}) \subset D_1$.

Let $k \in \mathbb{N}$. For $i \in \{1, \dots, k\}$, we have $D_i \cap D_h(a_{k+1}, \frac{r}{4}) = \emptyset$. Indeed, $D_1 \cap D_h(a_{k+1}, \frac{r}{4}) = \emptyset$ by (1.22), and if $D_{\lambda+1} \cap D_h(a_{k+1}, \frac{r}{4}) \neq \emptyset$ for some $\lambda \in \{1, \dots, k-1\}$, then there exists a point

$z \in D_{\lambda+1} \cap D_h(a_{k+1}, \frac{r}{4})$, so by (1.23), $z \notin D_h(a_j, \frac{r}{4})$, for all $j \geq \lambda + 2,$ and $z \in D_h(a_{k+1}, \frac{r}{4})$, which is a contradiction, since $k + 1 \geq \lambda + 2$. Thus,

$$D_h\left(a_{k+1}, \frac{r}{4}\right) \bigcap \left(\bigcup_{i=1}^{k} D_i\right) = \emptyset. \tag{1.24}$$

On the other hand, if there exists a point $z \in D_h(a_{k+1}, \frac{r}{4}) \cap D_h(a_\lambda, \frac{r}{4})$, for some $\lambda \geq k + 2$, then $\beta(a_{k+1}, a_\lambda) \leq \beta(a_{k+1}, z) + \beta(z, a_\lambda) < \frac{r}{4} + \frac{r}{4} = \frac{r}{2}$, which is a contradiction to the definition of the r-lattice. Thus,

$$D_h\left(a_{k+1}, \frac{r}{4}\right) \bigcap \left(\bigcup_{j=k+2}^{+\infty} D_h\left(a_j, \frac{r}{4}\right)\right) = \emptyset. \tag{1.25}$$

By (1.23), (1.24) and (1.25), $D_h(a_{k+1}, \frac{r}{4}) \subset D_{k+1}$, for all $k \in \mathbb{N}$.

Finally, let $z \in \mathbb{D}$. By (ii) of Definition 1.4.1, $z \in D_h(a_\lambda, r)$, for some $\lambda \in \mathbb{N}$. If $\lambda = 1$, then, by (1.22), either $z \in D_1$ or $z \in D_h(a_j, \frac{r}{4}) \subset D_j$, for some $j \geq 2$. If $\lambda = k + 1$, for some $k \in \mathbb{N}$, then, by (1.23), either $z \in D_{k+1}$, or $z \in D_i$, for some $i \in \{1, \ldots, k\}$, or $z \in D_h(a_j, \frac{r}{4}) \subset D_j$, for some $j \geq k + 2$. In any case, $z \in \bigcup_{k=1}^{+\infty} D_k$. □

CHAPTER 2

Bergman Spaces

Definition 2.0.1. *For $p > 0$ and $\alpha > -1$, we define $\mathcal{A}^p(dA_\alpha) = H(\mathbb{D}) \cap L^p(\mathbb{D}, dA_\alpha)$. These spaces are called Bergman spaces with standard weights.*

Remark 2.0.1. The Bergman spaces are vector spaces over $\mathbb{C}$, as $H(\mathbb{D})$ and $L^p(\mathbb{D}, dA_\alpha)$ are vector spaces over $\mathbb{C}$.

Remark 2.0.2. If $p \geq 1$, then the normed space $(L^p(\mathbb{D}, dA_\alpha), \|\cdot\|_{p,\alpha})$, where

$$\|f\|_{p,\alpha} = \Big(\int_{\mathbb{D}} |f|^p \, dA_\alpha \Big)^{1/p}, \text{ for all } f \in L^p(\mathbb{D}, dA_\alpha),$$

is a Banach space.
If $0 < p < 1$, then the metric space $(L^p(\mathbb{D}, dA_\alpha), d_{p,\alpha})$, where

$$d_{p,\alpha}(f, g) = \int_{\mathbb{D}} |f - g|^p \, dA_\alpha, \text{ for all } f, g \in L^p(\mathbb{D}, dA_\alpha),$$

is complete.

Notation. We will denote by $\|f\|_{p,\alpha}$ the number $\Big(\int_{\mathbb{D}} |f|^p \, dA_\alpha \Big)^{1/p}$, even if $0 < p < 1$.

2.1 Completeness and other properties

Proposition 2.1.1. *Suppose $p > 0$ and $0 < r < 1$. Then*

$$|f(0)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta, \textit{ for all } f \in H(\mathbb{D}). \tag{2.1}$$

Proof. If $f \equiv 0$, then (2.1) obviously holds. Thus, we assume that $f \not\equiv 0$. Then, by the Identity Theorem (see Sarason [14, Sections VII.13, VII.14]), the set $f^{-1}(\{0\})$ is at most countable.

Fix $r \in (0, 1)$ and suppose that f has no zero on $|z| = r$. We also assume that $f(0) \neq 0$ (otherwise (2.1) holds obviously). We consider the function

$$g : D\Big(0, \frac{1}{r}\Big) \to \mathbb{C}, \text{ with } g(z) = f(rz), \text{ for all } z \in D\Big(0, \frac{1}{r}\Big),$$

which is holomorphic in $D\big(0, \frac{1}{r}\big) \supset \mathbb{D}$ and nonvanishing on $|z| = 1$. Let $\{a_1, a_2, \ldots, a_n\}$ be the zeros of g in $\mathbb{D}$, repeated according to multiplicity. Then, by [14, Section VII.13],

$$g(z) = (z - a_1) \cdot \ldots \cdot (z - a_n)\, G(z), \text{ for all } z \in D\Big(0, \frac{1}{r}\Big), \tag{2.2}$$

where $G : D\big(0, \frac{1}{r}\big) \to \mathbb{C}$ is holomorphic and $G(z) \neq 0$, for all $z \in \overline{\mathbb{D}}$. Since $f(0) \neq 0$, we have that $a_k \neq 0$, for all $k \in \{1, \ldots, n\}$, so we can define the Blaschke product

$$B(z) = \prod_{k=1}^{n} \frac{|a_k|}{a_k} \cdot \frac{a_k - z}{1 - \bar{a}_k z}, \text{ for all } |z| < \min\Big\{\Big|\frac{1}{\bar{a}_k}\Big| : k = 1, \ldots, n\Big\} =: \rho \in (1, +\infty).$$

Note that

$$|B(0)| = |a_1| \cdot \ldots \cdot |a_n| \in (0, 1) \tag{2.3}$$

and

$$|B(z)| = \prod_{k=1}^{n} \frac{|a_k - z|}{|1 - \bar{a}_k z|} = |\phi_{a_1}(z)| \cdot \ldots \cdot |\phi_{a_n}(z)| = 1, \text{ for all } z \in \partial\mathbb{D}. \tag{2.4}$$

Now, set $R := \min\big\{\frac{1}{r}, \rho\big\} > 1$, and consider the function $h : D(0, R) \to \mathbb{C}$, with

$$h(z) = \frac{g(z)}{B(z)} \overset{(2.2)}{=} (-1)^n G(z) \prod_{k=1}^{n} \frac{a_k}{|a_k|}(1 - \bar{a}_k z), \text{ for all } z \in D(0, R).$$

Then $h \in H(D(0, R))$ and $h(z) \neq 0$, for all $z \in \overline{\mathbb{D}}$, hence there exists a real number $R' \in (1, R)$ such that $h(z) \neq 0$, for all $z \in D(0, R')$. Since $D(0, R')$ is a simply connected domain, by [14, Section X.5] there exists a branch of $\log h$ in $D(0, R')$, so we can define a holomorphic function h^p in $D(0, R')$. Then, the Mean Value Property ([14, Section VII.6]), (2.3) and (2.4) give

$$\begin{aligned} h^p(0) = \frac{1}{2\pi} \int_0^{2\pi} h^p(e^{i\theta})\, d\theta \;\Rightarrow\; |h(0)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |h(e^{i\theta})|^p\, d\theta \;\Rightarrow \\ \Rightarrow \frac{|g(0)|^p}{|B(0)|^p} \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|g(e^{i\theta})|^p}{|B(e^{i\theta})|^p}\, d\theta \;\Rightarrow \\ \Rightarrow |f(0)|^p \leq \frac{1}{2\pi} |B(0)|^p \int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta. \end{aligned}$$

Suppose now that there exists a point $z \in C(0, r)$ such that $f(z_0) = 0$. Since $f^{-1}(\{0\})$ is at most countable, we can find a sequence $\{r_n\}$ such that $\lim_{n\to+\infty} r_n = r$, and, for all $n \in \mathbb{N}$, $r_n \in (r, 1)$, $r_{n+1} < r_n$, and f do not vanish on $C(0, r_n)$. Thus, by the previous case,

$$|f(0)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(r_n e^{i\theta})|^p\, d\theta, \text{ for all } n \in \mathbb{N}. \tag{2.5}$$

If $f_n : [0, 2\pi] \to \mathbb{R}$, with $f_n(\theta) = |f(r_n e^{i\theta})|^p$, for all $\theta \in [0, 2\pi]$ and $n \in \mathbb{N}$, then

$$\lim_{n\to+\infty} f_n(\theta) = |f(re^{i\theta})|^p, \text{ for all } \theta \in [0, 2\pi],$$

and

$$|f_n(\theta)| = |f(r_n e^{i\theta})|^p \leq (\max\{|f(z)| : z \in \overline{D(0, r_1)}\})^p, \text{ for all } \theta \in [0, 2\pi] \text{ and } n \in \mathbb{N},$$

since f is continuous on the compact set $\overline{D(0, r_1)}$. Applying [6, 2.24 The Dominated Convergence Theorem] to (2.5), we take (2.1). □

Corollary 2.1.2. *Let $p > 0$. If f is holomorphic in an open set $\Omega \subset \mathbb{C}$ and $a \in \Omega$, then*

$$|f(a)|^p \leq \frac{1}{2\pi}\int_0^{2\pi} |f(a + re^{i\theta})|^p \, d\theta,$$

for all $r > 0$, provided that $\overline{D(a,r)} \subset \Omega$.

Proof. Since $\overline{D(a,r)} \subset \Omega$ is compact and Ω is open, there exists a real number $R > r$ such that $\overline{D(a,r)} \subset D(a,R) \subset \Omega$. Let $\phi(z) = a + Rz$, for all $z \in \mathbb{C}$. Then $\phi(\mathbb{D}) = D(a,R)$ and $\phi \in H(\mathbb{D})$, so Proposition 2.1.1 implies that

$$|f(a)|^p = |f(\phi(0))|^p \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(\phi\left(\frac{r}{R}e^{i\theta}\right)\right)\right|^p d\theta = \frac{1}{2\pi}\int_0^{2\pi} |f(a + re^{i\theta})|^p \, d\theta.$$

□

Corollary 2.1.3. *Let $p > 0$ and $\alpha > -1$. Then*

$$|f(0)|^p \leq \int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z), \text{ for all } f \in H(\mathbb{D}).$$

Proof. Using polar coordinates and Proposition 2.1.1,

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) &= \int_{\mathbb{D}} |f(z)|^p \cdot c_\alpha (1-|z|^2)^\alpha \, dA(z) = \int_0^1 c_\alpha (1-r^2)^\alpha \cdot \frac{r}{\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta dr \\ &\geq \int_0^1 c_\alpha (1-r^2)^\alpha \cdot 2r|f(0)|^p \, dr = |f(0)|^p \int_0^1 (\alpha+1)(1-r^2)^\alpha \cdot 2r \, dr = |f(0)|^p.\end{aligned}$$

□

Theorem 2.1.4. *Let $p > 0$, $\alpha > -1$, and $f \in \mathcal{A}^p(dA_\alpha)$. Then*

$$|f(z)| \leq \frac{\|f\|_{p,\alpha}}{(1-|z|^2)^{(2+\alpha)/p}}, \text{ for all } z \in \mathbb{D}.$$

Proof. Let $z \in \mathbb{D}$. Then the function

$$F(w) = (f \circ \phi_z)(w) \cdot \frac{(1-|z|^2)^{(2+\alpha)/p}}{(1-w\bar{z})^{2(2+\alpha)/p}}, \text{ for all } w \in \mathbb{D},$$

is holomorphic in $\mathbb{D}$, and

$$\begin{aligned}\int_{\mathbb{D}} |F(w)|^p \, dA_\alpha(w) &= \int_{\mathbb{D}} |f \circ \phi_z(w)|^p \cdot \frac{(1-|z|^2)^{2+\alpha}}{|1-w\bar{z}|^{2(2+\alpha)}} \cdot c_\alpha (1-|w|^2)^\alpha \, dA(w) \\ &= \int_{\mathbb{D}} |f(w)|^p \cdot \frac{(1-|z|^2)^{2+\alpha}}{|1-\phi_z(w)\bar{z}|^{2(2+\alpha)}} \cdot c_\alpha (1-|\phi_z(w)|^2)^\alpha \cdot |\phi_z'(w)|^2 \, dA(w) \\ &= \int_{\mathbb{D}} |f(w)|^p \cdot \frac{(1-|z|^2)^{2+\alpha}}{|1-\phi_z(w)\bar{z}|^{2(2+\alpha)}} \cdot \frac{(1-|z|^2)^{2+\alpha}}{|1-\bar{z}w|^{2(2+\alpha)}} \, dA_\alpha(w) \\ &= \int_{\mathbb{D}} |f(w)|^p \, dA_\alpha(w) = \|f\|_{p,\alpha}^p < +\infty,\end{aligned}$$

where we used the Change of Variables Theorem ([6, Theorem 2.47a]) for the second equality, and Proposition 1.1.2 (iii), (iv) for the third equality. Therefore, $F \in \mathcal{A}^p(dA_\alpha)$, with $\|F\|_{p,\alpha} = \|f\|_{p,\alpha}$. Using Corollary 2.1.3,

$$|f(z)|(1-|z|^2)^{(2+\alpha)/p} = |F(0)| \leq \|F\|_{p,\alpha} = \|f\|_{p,\alpha},$$

and the desired inequality follows. □

Corollary 2.1.5. *Let* $p \geq 1$, $\alpha > -1$ *and* $z \in \mathbb{D}$. *The function* $T_z : \mathcal{A}^p(dA_\alpha) \to \mathbb{C}$, *with* $T_z(f) = f(z)$, *for all* $f \in \mathcal{A}^p(dA_\alpha)$, *is a bounded linear functional on* $\mathcal{A}^p(dA_\alpha)$.

Proof. The linearity of T_z can be proved easily. Regarding the boundedness, if $f \in \mathcal{A}^p(dA_\alpha)$, then by Theorem 2.1.4,

$$|T_z(f)| = |f(z)| \leq \frac{1}{(1-|z|^2)^{(2+\alpha)/p}} \|f\|_{p,\alpha}.$$

□

Corollary 2.1.6. *Let* $p > 0$, $\alpha > -1$ *and* $S \subset \mathbb{D}$ *be a compact set. Then there exists a real constant* $C = C(p, \alpha, S) > 0$ *such that*

$$\sup\{|f(z)| : f \in \mathcal{A}^p(dA_\alpha),\ \|f\|_{p,\alpha} \leq 1,\ z \in S\} \leq C < +\infty. \tag{2.6}$$

Proof. Since S is a compact subset of $\mathbb{D}$, there exists a real $R \in (0,1)$ such that $S \subset \overline{D(0,R)} \subset \mathbb{D}$. By Theorem 2.1.4,

$$|f(z)| \leq \frac{\|f\|_{p,\alpha}}{(1-|z|^2)^{(2+\alpha)/p}} \leq \frac{1}{(1-|z|^2)^{(2+\alpha)/p}} \leq \frac{1}{(1-R^2)^{(2+\alpha)/p}},$$

for all $f \in \mathcal{A}^p(dA_\alpha)$ with $\|f\|_{p,\alpha} \leq 1,$ and $z \in S$. Setting $C = \dfrac{1}{(1-R^2)^{(2+\alpha)/p}} > 0$, we get (2.6). □

Theorem 2.1.7. *For any* $p > 0$ *and* $\alpha > -1$, *the Bergman space* $\mathcal{A}^p(dA_\alpha)$ *is closed in* $L^p(\mathbb{D}, dA_\alpha)$.

Proof. Suppose $\{f_k\}$ is a sequence in $\mathcal{A}^p(dA_\alpha)$ which converges in $L^p(\mathbb{D}, dA_\alpha)$ to $g \in L^p(\mathbb{D}, dA_\alpha)$; so $\{f_k\}$ is a Cauchy sequence in $L^p(\mathbb{D}, dA_\alpha)$.

Let S be a compact set in $\mathbb{D}$. By Corollary 2.1.6, there exists a positive constant C (which depends on p, α and S) such that $|f(z)| \leq C$, for all $f \in \mathcal{A}^p(dA_\alpha)$ with $\|f\|_{p,\alpha} \leq 1,$ and $z \in S$. Let $\epsilon > 0$. Since $\{f_k\}$ is a Cauchy sequence, there exists a number $k_0 \in \mathbb{N}$ such that

$$\left(\int_{\mathbb{D}} |f_n - f_m|^p \, dA_\alpha\right)^{1/p} < \frac{\epsilon}{C}, \text{ for all } n, m \geq k_0 \ \Rightarrow \ \left\|\frac{C}{\epsilon} f_n - \frac{C}{\epsilon} f_m\right\|_{p,\alpha} < 1, \text{ for all } n, m \geq k_0,$$

hence Corollary 2.1.6 gives that, for all $n, m \geq k_0$ and $z \in S$,

$$\left|\frac{C}{\epsilon} f_n(z) - \frac{C}{\epsilon} f_m(z)\right| \leq C, \text{ or equivalently, } |f_n(z) - f_m(z)| \leq \epsilon.$$

Thus, $\{f_k\}$ converges uniformly on S to a function f. Since S was an arbitrary compact subset of $\mathbb{D}$, by the Weierstrass Convergence Theorem ([12, Theorem 10.28]), $f \in H(\mathbb{D})$.

Since $f_k \to g$, as $k \to +\infty$, in $L^p(\mathbb{D}, dA_\alpha)$, there exists a subsequence $\{f_{k_n}\}$ such that $f_{k_n} \to g$, as $n \to +\infty$, almost everywhere in $\mathbb{D}$. Thus, $f = g$ almost everywhere in $\mathbb{D}$, and hence $f \in \mathcal{A}^p(dA_\alpha)$. □

Corollary 2.1.8. *For any* $p > 0$ *and* $\alpha > -1$, *the space* $(\mathcal{A}^p(dA_\alpha), \|\cdot\|_{p,\alpha})$ *or* $(\mathcal{A}^p(dA_\alpha), d_{p,\alpha})$ *is a complete metric space.*

Proof. By Remark 2.0.2, the spaces $(L^p(\mathbb{D}, dA_\alpha), \|\cdot\|_{p,\alpha})$ or $(L^p(\mathbb{D}, dA_\alpha), d_{p,\alpha})$ are complete metric spaces, so Theorem 2.1.7 gives that Bergman spaces are complete with respect to the corresponding metrics. □

Proposition 2.1.9. *Suppose* $0 < p \leq 1$ *and* $\alpha > -1$. *Then*

$$\int_{\mathbb{D}} |f(z)| \, dA_\gamma(z) \leq \frac{\gamma+1}{\alpha+1}\left(\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z)\right)^{1/p}, \text{ for all } f \in \mathcal{A}^p(dA_\alpha),$$

where $\gamma = \frac{2+\alpha}{p} - 2 > -1$.

Proof. Let $f \in \mathcal{A}^p(dA_\alpha)$. Then, Theorem 2.1.4 gives that for all $z \in \mathbb{D}$,

$$|f(z)| = |f(z)|^p \cdot |f(z)|^{1-p} \leq |f(z)|^p \cdot \left(\frac{\|f\|_{p,\alpha}}{(1-|z|^2)^{(2+\alpha)/p}}\right)^{1-p},$$

so

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|\, dA_\gamma(z) &\leq \int_{\mathbb{D}} |f(z)|^p \cdot \left(\frac{\|f\|_{p,\alpha}}{(1-|z|^2)^{(2+\alpha)/p}}\right)^{1-p} dA_\gamma(z)\\ &= \|f\|_{p,\alpha}^{1-p} \int_{\mathbb{D}} |f(z)|^p \cdot \frac{(\gamma+1)(1-|z|^2)^\gamma}{(1-|z|^2)^{(2+\alpha)(1-p)/p}}\, dA(z)\\ &= (\gamma+1)\cdot \|f\|_{p,\alpha}^{1-p} \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha\, dA(z)\\ &= \frac{\gamma+1}{\alpha+1}\cdot \|f\|_{p,\alpha}.\end{aligned}$$

□

Remark 2.1.1. The above Proposition means that if $0 < p \leq 1$, $\alpha > -1$ and $\gamma = \frac{2+\alpha}{p} - 2 > -1$, then $\mathcal{A}^p(dA_\alpha) \subset \mathcal{A}^1(dA_\gamma)$.

Proposition 2.1.10. *Let $p > 0$, $\alpha \in \mathbb{R}$ and $r > 0$. Then there exists a constant $C = C(\alpha, r) > 0$ such that*

$$|f(z)|^p \leq \frac{C}{(1-|z|^2)^{2+\alpha}} \int_{D_h(z,r)} |f(w)|^p\, dA_\alpha(w),$$

for all $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$.

Proof. Proposition 1.3.5 gives that $D_h(0,r) = D(0,s)$, where $s = \tanh(r) \in (0,1)$, so, using Proposition 2.1.1, we get that for all $f \in H(\mathbb{D})$,

$$\begin{aligned}\int_{D_h(0,r)} |f(w)|^p\, dA_\alpha(w) &= \int_{D(0,s)} |f(w)|^p \cdot c_\alpha (1-|w|^2)^\alpha\, dA(w)\\ &= \int_0^s c_\alpha(1-\rho^2)^\alpha \cdot \frac{\rho}{\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^p\, d\theta d\rho\\ &\geq \int_0^s c_\alpha (1-\rho^2)^\alpha \cdot 2\rho |f(0)|^p\, d\rho\\ &= |f(0)|^p A_\alpha(D_h(0,r)),\end{aligned}$$

hence

$$|f(0)|^p \leq \frac{1}{A_\alpha(D_h(0,r))} \int_{D_h(0,r)} |f(w)|^p\, dA_\alpha(w), \text{ for all } f \in H(\mathbb{D}). \tag{2.7}$$

Let $z \in \mathbb{D}$ and replace f by $f \circ \phi_z$ in (2.7) to get

$$|f(z)|^p = |f\circ\phi_z(0)|^p \leq \frac{1}{A_\alpha(D_h(0,r))} \int_{D_h(0,r)} |f\circ\phi_z(w)|^p\, dA_\alpha(w), \text{ for all } f \in H(\mathbb{D}). \tag{2.8}$$

By the Change of Variables Theorem ([6, Theorem 2.47a]), Lemma 1.3.9 and Proposition 1.1.2 (iii), (iv),

$$\begin{aligned}\int_{D_h(0,r)} |f\circ\phi_z(w)|^p\, dA_\alpha(w) &= \int_{D_h(0,r)} |f\circ\phi_z(w)|^p \cdot c_\alpha(1-|w|^2)^\alpha\, dA(w)\\ &= \int_{D_h(z,r)} |f(w)|^p \cdot c_\alpha(1-|\phi_z(w)|^2)^\alpha \cdot |\phi_z'(w)|^2\, dA(w)\\ &= \int_{D_h(z,r)} |f(w)|^p \cdot \frac{(1-|z|^2)^{2+\alpha}}{|1-\bar{z}w|^{2(2+\alpha)}}\, dA_\alpha(w).\end{aligned}$$

Thus, (2.8) becomes

$$|f(z)|^p \leq \frac{1}{A_\alpha(D_h(0,r))} \int_{D_h(z,r)} |f(w)|^p \cdot \frac{(1-|z|^2)^{2+\alpha}}{|1-\bar{z}w|^{2(2+\alpha)}}\, dA_\alpha(w), \tag{2.9}$$

for all $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$.
Let $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. If $w \in D_h(z,r)$, then, by Proposition 1.3.11 (i), $|1 - z\bar{w}| \sim 1 - |z|^2$. Therefore, there exists a constant $\tilde{C} = \tilde{C}(\alpha, r) > 0$ such that for all $w \in D_h(z,r)$,

$$\frac{(1-|z|^2)^{2+\alpha}}{|1-\bar{z}w|^{2(2+\alpha)}} \leq \frac{\tilde{C}}{(1-|z|^2)^{2+\alpha}}. \tag{2.10}$$

By (2.9) and (2.10),

$$\begin{aligned} |f(z)|^p &\leq \frac{1}{A_\alpha(D_h(0,r))} \int_{D_h(z,r)} |f(w)|^p \cdot \frac{\tilde{C}}{(1-|z|^2)^{2+\alpha}}\, dA_\alpha(w) \\ &= \frac{C}{(1-|z|^2)^{2+\alpha}} \int_{D_h(z,r)} |f(w)|^p\, dA_\alpha(w), \end{aligned}$$

where $C = C(\alpha, r) := \dfrac{\tilde{C}}{A_\alpha(D_h(0,r))} > 0$. □

Remark 2.1.2. The denominator $(1-|z|^2)^{2+\alpha}$ in the above Proposition can be replaced by $A_\alpha(D_h(z,r))$ because of Proposition 1.3.11 (ii).

Proposition 2.1.11. *Let $p > 0$, $\alpha > -1$ and $\{a_k\} \subset \mathbb{D}$ be a separated sequence in the hyperbolic metric. Then there exists a constant $C > 0$ such that*

$$\sum_{k=1}^{+\infty} (1-|a_k|^2)^{2+\alpha} |f(a_k)|^p \leq C \int_{\mathbb{D}} |f(z)|^p\, dA_\alpha(z), \text{ for all } f \in \mathcal{A}^p(dA_\alpha). \tag{2.11}$$

Proof. Since $\{a_k\}$ is separated in the hyperbolic metric, there exists a real number $\delta > 0$ such that $\beta(a_i, a_j) \geq \delta$, for all $i \neq j$. Let $r = \frac{\delta}{2} > 0$ and note that $D_h(a_i, r) \cap D_h(a_j, r) = \emptyset$, for all $i \neq j$, because of the definition of the separated sequence. Thus, if $f \in \mathcal{A}^p(dA_\alpha)$, then

$$\sum_{k=1}^{n} \int_{D_h(a_k,r)} |f(z)|^p\, dA_\alpha(z) = \int_{\bigcup_{k=1}^n D_h(a_k,r)} |f(z)|^p\, dA_\alpha(z) \leq \int_{\mathbb{D}} |f(z)|^p\, dA_\alpha(z), \text{ for all } n \in \mathbb{N},$$

so

$$\sum_{k=1}^{+\infty} \int_{D_h(a_k,r)} |f(z)|^p\, dA_\alpha(z) \leq \int_{\mathbb{D}} |f(z)|^p\, dA_\alpha(z). \tag{2.12}$$

Also, by Proposition 2.1.10, there exists a constant $C = C(\alpha, \delta) > 0$ such that

$$|f(a_k)|^p \leq \frac{C}{(1-|a_k|^2)^{2+\alpha}} \int_{D_h(a_k,r)} |f(z)|^p\, dA_\alpha(z), \text{ for all } k \in \mathbb{N},$$

hence

$$\sum_{k=1}^{+\infty} (1-|a_k|^2)^{2+\alpha} |f(a_k)|^p \leq C \sum_{k=1}^{+\infty} \int_{D_h(a_k,r)} |f(z)|^p\, dA_\alpha(z). \tag{2.13}$$

By (2.12) and (2.13), we get (2.11). □

Finally, we discuss a few things about H^∞, which is the space of all bounded analytic functions in $\mathbb{D}$. Note that H^∞ is a linear subspace of $B(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \,|\, f \text{ is bounded in } \mathbb{D}\}$. It is known from elementary Functional Analysis that $B(\mathbb{D})$, when equipped with the norm $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}$, becomes a Banach space (see [10, Section 1.10.1]).

Proposition 2.1.12. *$(H^\infty, \|\cdot\|_\infty)$ is a Banach space.*

Proof. Since $(B(\mathbb{D}), \|\cdot\|_\infty)$ is a Banach space, it suffices to show that H^∞ is a closed subspace of $B(\mathbb{D})$. Thus, suppose $\{f_k\}$ is a sequence in H^∞ that converges in $B(\mathbb{D})$ to a function $f \in B(\mathbb{D})$. Then

$$\lim_{k\to+\infty} \sup\{|f_k(z) - f(z)| : z \in \mathbb{D}\} = \lim_{k\to+\infty} \|f_k - f\|_\infty = 0,$$

hence $f_k \to f$ uniformly in $\mathbb{D}$, as $k \to +\infty$. By the Weierstrass Convergence Theorem ([12, Theorem 10.28]), $f \in H(\mathbb{D})$, so $f \in H^\infty$. This means that H^∞ is closed in $B(\mathbb{D})$. □

Proposition 2.1.13. *H^∞ is a Banach algebra.*

Proof. For the definition of a Banach algebra see, for example, [2, Chapter VII, Definition 1.1]. Now, let $f, g \in H^\infty$. Then, $f \leq \|f\|_\infty$ and $g \leq \|g\|_\infty$ in $\mathbb{D}$, so

$$|(fg)(z)| = |f(z)g(z)| \leq \|f\|_\infty \|g\|_\infty, \text{ for all } z \in \mathbb{D},$$

which implies that $fg \in H^\infty$ and $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$. □

2.2 The reproducing kernel for $\mathcal{A}^2(dA_\alpha)$

It is known from Functional Analysis that the Banach space $L^2(\mathbb{D}, dA_\alpha)$, where $\alpha > -1$, can be supplied with the inner product

$$\langle f, g\rangle_{2,\alpha} = \int_{\mathbb{D}} f(w)\overline{g(w)}\, dA_\alpha(w), \text{ for all } f, g \in L^2(\mathbb{D}, dA_\alpha),$$

and that $\|\cdot\|_{2,\alpha}$ is induced by this inner product. This fact makes $(L^2(\mathbb{D}, dA_\alpha), \langle\cdot,\cdot\rangle_{2,\alpha})$ a Hilbert space, so $(\mathcal{A}^2(dA_\alpha), \langle\cdot,\cdot\rangle_{2,\alpha})$ is also a Hilbert space.
Let $z \in \mathbb{D}$. By Corollary 2.1.5, $T_z \in (\mathcal{A}^2(dA_\alpha))^*$, so by the Riesz representation theorem ([6, Theorem 5.25]), there exists a unique function $h_{z,\alpha} \in \mathcal{A}^2(dA_\alpha)$ such that for all $f \in \mathcal{A}^2(dA_\alpha)$,

$$T_z(f) = \langle f, h_{z,\alpha}\rangle_{2,\alpha} \quad \Leftrightarrow \quad f(z) = \int_{\mathbb{D}} f(w)\overline{h_{z,\alpha}(w)}\, dA_\alpha(w).$$

Let $K_\alpha : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ with $K_\alpha(z, w) = \overline{h_{z,\alpha}(w)}$, for all $z, w \in \mathbb{D}$. Then K_α is called the reproducing kernel of $\mathcal{A}^2(dA_\alpha)$, because of the formula

$$f(z) = \int_{\mathbb{D}} f(w)K_\alpha(z, w)\, dA_\alpha(w), \text{ for all } z \in \mathbb{D} \text{ and } f \in \mathcal{A}^2(dA_\alpha). \tag{2.14}$$

When $\alpha = 0$, we write K instead of K_0 and we call K the Bergman kernel of $\mathbb{D}$.

Remark 2.2.1. Recall that every Hilbert space has an orthonormal basis ([6, Proposition 5.28]).

Theorem 2.2.1. *Suppose $\{e_n\}$ is an orthonormal basis of $\mathcal{A}^2(dA_\alpha)$. Then*

$$K_\alpha(z, w) = \sum_{n=1}^{+\infty} e_n(z)\overline{e_n(w)}, \textit{ for all } z, w \in \mathbb{D}, \tag{2.15}$$

and the series (2.15) converges uniformly on compact subsets of $\mathbb{D} \times \mathbb{D}$.

Proof. We will prove the theorem in steps.

Step 1.

$$\sum_{n=1}^{+\infty} |e_n(z)|^2 < +\infty, \text{ for all } z \in \mathbb{D}. \tag{2.16}$$

Let $z \in \mathbb{D}$. Using Parseval's Identity ([6, Theorem 5.27b]) and (2.14),

$$\|h_{z,\alpha}(\cdot)\|_{2,\alpha}^2 = \|\overline{K_\alpha(z,\cdot)}\|_{2,\alpha}^2 = \sum_{n=1}^{+\infty} |\langle \overline{K_\alpha(z,\cdot)}, e_n\rangle_{2,\alpha}|^2 = \sum_{n=1}^{+\infty} |\langle e_n, K_\alpha(z,\cdot)\rangle_{2,\alpha}|^2$$
$$= \sum_{n=1}^{+\infty} \left| \int_{\mathbb{D}} e_n(w) K_\alpha(z,w)\, dA_\alpha(w) \right|^2 = \sum_{n=1}^{+\infty} |e_n(z)|^2,$$

and (2.16) follows.

Step 2. The series (2.15) converges absolutely for all $z, w \in \mathbb{D}$.

Indeed, by the Cauchy-Schwarz inequality and (2.16),

$$\sum_{n=1}^{+\infty} |e_n(z)\overline{e_n(w)}| \le \left(\sum_{n=1}^{+\infty} |e_n(z)|^2\right)^{1/2} \cdot \left(\sum_{n=1}^{+\infty} |e_n(w)|^2\right)^{1/2} < +\infty, \text{ for all } z, w \in \mathbb{D}.$$

Step 3. Let S be a compact subset of $\mathbb{D}$. Then

$$\sup\left\{\left(\sum_{n=1}^{+\infty} |e_n(z)|^2\right)^{1/2} : z \in S\right\} = \sup\left\{\left|\sum_{n=1}^{+\infty} a_n e_n(z)\right| : z \in S, \sum_{n=1}^{+\infty} |a_n|^2 = 1\right\}. \tag{2.17}$$

Let $z \in S$ and set $t := \left(\sum_{n=1}^{+\infty} |e_n(z)|^2\right)^{1/2} \in [0, +\infty)$, because of (2.16). If $t = 0$, then $e_n(z) = 0$, for all $n \in \mathbb{N}$, so $t = |\sum_{n=1}^{+\infty} a_n e_n(z)|$, where $a_n = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \ge 2 \end{cases}$. If $t > 0$, then

$$t = \frac{1}{t}\sum_{n=1}^{+\infty} |e_n(z)|^2 = \sum_{n=1}^{+\infty} \frac{1}{t} \cdot \overline{e_n(z)} e_n(z) = \left|\sum_{n=1}^{+\infty} \frac{1}{t} \cdot \overline{e_n(z)} e_n(z)\right|$$

and

$$\sum_{n=1}^{+\infty} \left|\frac{1}{t} \cdot \overline{e_n(z)}\right|^2 = \frac{1}{t^2}\sum_{n=1}^{+\infty} |e_n(z)|^2 = 1.$$

Thus,

$$\left\{\left(\sum_{n=1}^{+\infty} |e_n(z)|^2\right)^{1/2} : z \in S\right\} \subset \left\{\left|\sum_{n=1}^{+\infty} a_n e_n(z)\right| : z \in S, \sum_{n=1}^{+\infty} |a_n|^2 = 1\right\}. \tag{2.18}$$

Let $z \in S$ and $\{a_n\} \subset \mathbb{C}$ such that $\sum_{n=1}^{+\infty} |a_n|^2 = 1$. Then, by the Cauchy-Schwarz inequality,

$$\left|\sum_{n=1}^{+\infty} a_n e_n(z)\right| \le \left(\sum_{n=1}^{+\infty} |a_n|^2\right)^{1/2} \cdot \left(\sum_{n=1}^{+\infty} |e_n(z)|^2\right)^{1/2} = \left(\sum_{n=1}^{+\infty} |e_n(z)|^2\right)^{1/2}. \tag{2.19}$$

(2.18) and (2.19) give (2.17).

Step 4. Let $f \in \mathcal{A}^2(dA_\alpha)$. Then

$$f(z) = \sum_{n=1}^{+\infty} \langle f, e_n\rangle_{2,\alpha} e_n(z), \text{ for all } z \in \mathbb{D}. \tag{2.20}$$

By [6, Theorem 5.27c],

$$f = \sum_{n=1}^{+\infty} \langle f, e_n\rangle_{2,\alpha} e_n \text{ in } \mathcal{A}^2(dA_\alpha) \;\Leftrightarrow\; \Big\| \sum_{n=1}^{N} \langle f, e_n\rangle_{2,\alpha} e_n - f \Big\|_{2,\alpha} \longrightarrow 0, \text{ as } N \to +\infty. \tag{2.21}$$

Also, by Theorem 2.1.4, we have that for all $z \in \mathbb{D}$ and $N \in \mathbb{N}$,

$$\Big| \sum_{n=1}^{N} \langle f, e_n\rangle_{2,\alpha} e_n(z) - f(z) \Big| \le \frac{\| \sum_{n=1}^{N} \langle f, e_n\rangle_{2,\alpha} e_n - f \|_{2,\alpha}}{(1-|z|^2)^{(2+\alpha)/2}},$$

so, using (2.21),

$$\lim_{N\to+\infty} \Big| \sum_{n=1}^{N} \langle f, e_n\rangle_{2,\alpha} e_n(z) - f(z) \Big| = 0, \text{ for all } z \in \mathbb{D},$$

and (2.20) follows.

Step 5. Let S be a compact subset of $\mathbb{D}$. Then

$$\sup \Big\{ \Big| \sum_{n=1}^{+\infty} a_n e_n(z) \Big| : z \in S, \sum_{n=1}^{+\infty} |a_n|^2 = 1 \Big\} = \sup\{|f(z)| : z \in S, \|f\|_{2,\alpha} = 1\}. \tag{2.22}$$

Let $f \in \mathcal{A}^2(dA_\alpha)$ with $\|f\|_{2,\alpha} = 1$, and $z \in S$. Then $|f(z)| = \big| \sum_{n=1}^{+\infty} \langle f, e_n\rangle_{2,\alpha} e_n(z) \big|$, because of (2.20), and Parseval's Identity ([6, Theorem 5.27b]) gives $\sum_{n=1}^{+\infty} |\langle f, e_n\rangle_{2,\alpha}|^2 = \|f\|_{2,\alpha}^2 = 1$; hence

$$\{|f(z)| : z \in S, \|f\|_{2,\alpha} = 1\} \subset \Big\{ \Big| \sum_{n=1}^{+\infty} a_n e_n(z) \Big| : z \in S, \sum_{n=1}^{+\infty} |a_n|^2 = 1 \Big\}. \tag{2.23}$$

Let $z \in S$ and $\{a_n\} \subset \mathbb{C}$ with $\sum_{n=1}^{+\infty} |a_n|^2 = 1 < +\infty$. By the Riesz-Fischer Theorem ([10, Section 2.10]), the series $\sum_{n=1}^{+\infty} a_n e_n$ converges in $\mathcal{A}^2(dA_\alpha)$ to a function $f \in \mathcal{A}^2(dA_\alpha)$, $\langle f, e_n\rangle_{2,\alpha} = a_n$, for all $n \in \mathbb{N}$, and $\|f\|_{2,\alpha}^2 = \sum_{n=1}^{+\infty} |a_n|^2 = 1$. Thus, it follows from (2.20) that $|f(z)| = |\sum_{n=1}^{+\infty} a_n e_n(z)|$, and so

$$\Big\{ \Big| \sum_{n=1}^{+\infty} a_n e_n(z) \Big| : z \in S, \sum_{n=1}^{+\infty} |a_n|^2 = 1 \Big\} \subset \{|f(z)| : z \in S, \|f\|_{2,\alpha} = 1\}. \tag{2.24}$$

(2.23) and (2.24) give (2.22).

Step 6. If S, T are compact sets in $\mathbb{D}$, then there exist constants $C_1 = C_1(\alpha, S)$, $C_2 = C_2(\alpha, T) > 0$ such that

$$\sum_{n=1}^{+\infty} |e_n(z)\overline{e_n(w)}| \le C_1 C_2, \text{ for all } z \in S \text{ and } w \in T. \tag{2.25}$$

By Corollary 2.1.6, there exist constants $C_1 = C_1(\alpha, S)$, $C_2 = C_2(\alpha, T) > 0$ such that

$$\sup\{|f(z)| : z \in S, \|f\|_{2,\alpha} = 1\} \le C_1 \text{ and } \sup\{|f(w)| : w \in T, \|f\|_{2,\alpha} = 1\} \le C_2,$$

so (2.17) and (2.22) give that

$$\Big(\sum_{n=1}^{+\infty} |e_n(z)|^2 \Big)^{1/2} \le C_1, \text{ for all } z \in S, \text{ and } \Big(\sum_{n=1}^{+\infty} |e_n(w)|^2 \Big)^{1/2} \le C_2, \text{ for all } w \in T. \tag{2.26}$$

Thus, (2.25) follows from the Cauchy-Schwarz inequality and (2.26).

Step 7. The series (2.15) converges uniformly if z, w stay in compact sets of $\mathbb{D}$.

Let $f_n(u,v) = \sum_{k=1}^{n} e_k(u)\overline{e_k(v)}$, for all $u, v \in \mathbb{D}$ and $n \in \mathbb{N}$. By *Step 2*,

$$\lim_{n\to+\infty} f_n(u,v) = \sum_{k=1}^{+\infty} e_k(u)\overline{e_k(v)} =: f(u,v), \text{ for all } u, v \in \mathbb{D}. \tag{2.27}$$

Let $g_n(u,v) = f_n(u,\bar{v}) = \sum_{k=1}^{n} e_k(u)\overline{e_k(\bar{v})}$, for all $u, v \in \mathbb{D}$ and $n \in \mathbb{N}$. Then $g_n(u,\cdot)$, $g_n(\cdot,v)$ are holomorphic in $\mathbb{D}$, for all $n \in \mathbb{N}$ and $u, v \in \mathbb{D}$ (see [14, Exercise II.8.2]).
Let S, T be compact subsets of $\mathbb{D}$, and $z \in S$, $w \in T$. Then, there exists a real $r \in (0,1)$ such that $S, T \subset D(0,r)$. By Cauchy's formula for a circle ([14, Section VII.5]),

$$\begin{aligned} g_n(z,w) &= \frac{1}{2\pi i}\int_{|u|=r} \frac{g_n(u,w)}{u-z}\,du = \frac{1}{2\pi i}\int_{|u|=r} \frac{1}{u-z}\cdot\frac{1}{2\pi i}\int_{|v|=r}\frac{g_n(u,v)}{v-w}\,dvdu \\ &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{g_n(re^{i\theta}, re^{it})}{(re^{i\theta}-z)(re^{it}-w)} r^2 e^{i\theta}e^{it}\,dtd\theta, \end{aligned}$$

for all $n \in \mathbb{N}$. Thus, if $n, m \in \mathbb{N}$,

$$\begin{aligned} |g_n(z,w) - g_m(z,w)| &\le \frac{r^2}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{|g_n(re^{i\theta}, re^{it}) - g_m(re^{i\theta}, re^{it})|}{|re^{i\theta}-z||re^{it}-w|}\,dtd\theta \\ &\le C\int_0^{2\pi}\int_0^{2\pi} |g_n(re^{i\theta}, re^{it}) - g_m(re^{i\theta}, re^{it})|\,dtd\theta, \end{aligned} \tag{2.28}$$

where $C = C(r,S,T) := \dfrac{r^2}{4\pi^2}\cdot\dfrac{1}{\operatorname{dist}(C(0,r),S)\cdot\operatorname{dist}(C(0,r),T)} > 0$.
Note that by *Step 6*, there exists a constant $\tilde{C} = \tilde{C}(\alpha, C(0,r)) > 0$ such that

$$\sum_{k=1}^{+\infty} |e_k(u)\overline{e_k(v)}| \le \tilde{C}^2, \text{ for all } u, v \in C(0,r),$$

so

$$\begin{aligned} |g_n(re^{i\theta}, re^{it}) - g_m(re^{i\theta}, re^{it})| &\le |g_n(re^{i\theta}, re^{it})| + |g_m(re^{i\theta}, re^{it})| \\ &\le \sum_{k=1}^{n} |e_k(re^{i\theta})\overline{e_k(re^{-it})}| + \sum_{k=1}^{m} |e_k(re^{i\theta})\overline{e_k(re^{-it})}| \\ &\le 2\sum_{k=1}^{+\infty} |e_k(re^{i\theta})\overline{e_k(re^{-it})}| \le 2\tilde{C}^2, \end{aligned}$$

for all $n, m \in \mathbb{N}$ and $\theta, t \in [0, 2\pi]$. Also, by (2.27),

$$\lim_{m\to+\infty} g_m(re^{i\theta}, re^{it}) = \lim_{m\to+\infty} f_m(re^{i\theta}, re^{-it}) = f(re^{i\theta}, re^{-it}), \text{ for all } \theta, t \in [0, 2\pi].$$

Thus, by applying ([6, 2.24 The Dominated Convergence Theorem]) to (2.28), we get

$$|g_n(z,w) - f(z,\bar{w})| \le C\int_0^{2\pi}\int_0^{2\pi} |g_n(re^{i\theta}, re^{it}) - f(re^{i\theta}, re^{-it})|\,dtd\theta, \text{ for all } n \in \mathbb{N},$$

hence

$$\sup_{z\in S,\, w\in T} |g_n(z,w) - f(z,\bar{w})| \le C\int_0^{2\pi}\int_0^{2\pi} |g_n(re^{i\theta}, re^{it}) - f(re^{i\theta}, re^{-it})|\,dtd\theta, \text{ for all } n \in \mathbb{N}.$$

It follows from [6, 2.24 The Dominated Convergence Theorem] again that

$$\lim_{n\to+\infty} C\int_0^{2\pi}\int_0^{2\pi} |g_n(re^{i\theta}, re^{it}) - f(re^{i\theta}, re^{-it})|\, dtd\theta = 0,$$

so

$$\lim_{n\to+\infty}\sup_{z\in S,\, w\in T} |g_n(z,w) - f(z,\bar{w})| = 0, \tag{2.29}$$

which means that $g_n \to g$ uniformly on $S\times T$, where $g(u,v) := f(u,\bar{v})$, for all $u, v \in \mathbb{D}$.
Now, let $U := \{\bar{w} : w \in T\}$. Since (2.29) was proven for arbitrary compact sets in $\mathbb{D}$, it holds also for S, U. Therefore,

$$\begin{aligned}\lim_{n\to+\infty}\sup_{z\in S,\, w\in T} |f_n(z,w) - f(z,w)| &= \lim_{n\to+\infty}\sup_{z\in S,\, w\in T} |g_n(z,\bar{w}) - g(z,\bar{w})|\\ &= \lim_{n\to+\infty}\sup_{z\in S,\, a\in U} |g_n(z,a) - g(z,a)| = 0,\end{aligned}$$

which means that $f_n \to f$ uniformly on $S\times T$.

Step 8. By (2.16) and the F.Riesz-Fischer Theorem ([10, Section 2.10]) we have that, for all $z\in\mathbb{D}$, the series $\sum_{n=1}^{+\infty}\overline{e_n(z)}e_n$ converges in $\mathcal{A}^2(dA_\alpha)$ to a function $g_{z,\alpha}\in\mathcal{A}^2(dA_\alpha)$, and $\langle g_{z,\alpha}, e_n\rangle_{2,\alpha} = \overline{e_n(z)}$, for all $n\in\mathbb{N}$. Let $f\in\mathcal{A}^2(dA_\alpha)$. Then (2.20) and [2, Chapter 1, Theorem 4.13] give that for all $z\in\mathbb{D}$,

$$\begin{aligned}f(z) &= \sum_{n=1}^{+\infty}\langle f, e_n\rangle_{2,\alpha} e_n(z) = \sum_{n=1}^{+\infty}\langle f, e_n\rangle_{2,\alpha}\overline{\langle g_{z,\alpha}, e_n\rangle}_{2,\alpha}\\ &= \langle f, g_{z,\alpha}\rangle_{2,\alpha} = \int_{\mathbb{D}} f(w)\sum_{n=1}^{+\infty} e_n(z)\overline{e_n(w)}\, dA_\alpha(w).\end{aligned}$$

By (2.14) and the uniqueness of the Riesz representation theorem ([6, Theorem 5.25]), we get (2.15).

□

The above Theorem will be used to find an explicit formula for the reproducing kernel. First, we need to find an appropriate orthonormal basis for $\mathcal{A}^2(dA_\alpha)$. We use ideas in [7, Proposition 1.4].

Lemma 2.2.2. *Let* $e_n(z) = \sqrt{\dfrac{\Gamma(n+\alpha+2)}{n!\Gamma(\alpha+2)}}\cdot z^n$, *for all* $z\in\mathbb{D}$ *and* $n\in\mathbb{N}\cup\{0\}$. *Then* $\{e_n\}\subset H(\mathbb{D})$ *forms an orthonormal basis for* $\mathcal{A}^2(dA_\alpha)$.

Proof. We will prove this lemma in steps.

Step 1. $\{e_n\}$ is an orthonormal subset of $\mathcal{A}^2(dA_\alpha)$.

Indeed, note that

$$\begin{aligned}\int_{\mathbb{D}} |e_n(z)|^2\, dA_\alpha(z) &= \frac{\Gamma(n+\alpha+2)}{n!\Gamma(\alpha+2)}\int_{\mathbb{D}} |z|^{2n}\, dA_\alpha(z)\\ &= (\alpha+1)\frac{\Gamma(n+\alpha+2)}{n!\Gamma(\alpha+2)}\int_0^1\int_0^{2\pi}\rho^{2n}(1-\rho^2)^\alpha\cdot\frac{\rho}{\pi}\, d\theta d\rho\\ &= (\alpha+1)\frac{\Gamma(n+\alpha+2)}{n!\Gamma(\alpha+2)}\int_0^1 2\rho^{2n+1}(1-\rho^2)^\alpha\, d\rho\\ &= (\alpha+1)\frac{\Gamma(n+\alpha+2)}{n!\Gamma(\alpha+2)}\int_0^1 r^n(1-r)^\alpha\, dr\\ &= (\alpha+1)\frac{\Gamma(n+\alpha+2)}{n!\Gamma(\alpha+2)}\cdot B(n+1,\alpha+1)\\ &= (\alpha+1)\frac{\Gamma(n+\alpha+2)}{n!\Gamma(\alpha+2)}\cdot\frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)} = 1,\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$, so $e_n \in \mathcal{A}^2(dA_\alpha)$ and $\|e_n\|_{2,\alpha} = 1$, for all $n \in \mathbb{N} \cup \{0\}$. Also, if $n > m \geq 0$, then

$$\begin{aligned}\langle e_n, e_m \rangle_{2,\alpha} &= \sqrt{\frac{\Gamma(n+\alpha+2)}{n!\Gamma(\alpha+2)}}\sqrt{\frac{\Gamma(m+\alpha+2)}{m!\Gamma(\alpha+2)}} \int_{\mathbb{D}} (\alpha+1) z^n \bar{z}^m (1-|z|^2)^\alpha \, dA(z) \\ &= C(n,m,\alpha) \int_0^1 \int_0^{2\pi} r^{n+m} e^{i(n-m)\theta} (1-r^2)^\alpha \cdot \frac{r}{\pi} \, d\theta dr = 0,\end{aligned}$$

because $\displaystyle\int_0^{2\pi} e^{i(n-m)\theta} \, d\theta = \begin{cases} 2\pi, & \text{if } n = m, \\ 0, & \text{if } n \neq m \end{cases}.$

Step 2. Let $f \in \mathcal{A}^2(dA_\alpha)$. Then $f(z) = \sum_{n=0}^{+\infty} a_n z^n = \sum_{n=0}^{+\infty} a_n b_{n,\alpha} e_n(z)$, for all $z \in \mathbb{D}$, where $a_n = \dfrac{f^{(n)}(0)}{n!}$ and $b_{n,\alpha} = \sqrt{\dfrac{n!\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)}}$, for all $n \in \mathbb{N} \cup \{0\}$. We will show that

$$\|f\|_{2,\alpha}^2 = \sum_{n=0}^{+\infty} |a_n|^2 b_{n,\alpha}^2. \tag{2.30}$$

Let $S_N(z) = \sum_{n=0}^{N} a_n z^n$, for all $z \in \mathbb{D}$ and $N \in \mathbb{N} \cup \{0\}$, and $\rho \in (0,1)$. Then, for all $N \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}\int_{D(0,\rho)} |S_N(z)|^2 \, dA_\alpha(z) &= \int_{D(0,\rho)} S_N(z)\overline{S_N(z)} \, dA_\alpha(z) = \sum_{n=0}^{N} \left(a_n \sum_{m=0}^{N} \bar{a}_m \int_{D(0,\rho)} z^n \bar{z}^m \, dA_\alpha(z) \right) \\ &= \sum_{n=0}^{N} \left(a_n \sum_{m=0}^{N} \bar{a}_m (\alpha+1) \int_0^\rho \int_0^{2\pi} r^{n+m} e^{i(n-m)\theta} (1-r^2)^\alpha \cdot \frac{r}{\pi} \, d\theta dr \right) \\ &= \sum_{n=0}^{N} |a_n|^2 (\alpha+1) \int_0^\rho 2r^{2n+1} (1-r^2)^\alpha \, dr.\end{aligned}$$

Since $S_N(z) \to f(z)$, as $N \to +\infty$, for all $z \in D(0,\rho)$, and

$$|S_N(z)| \leq \sum_{n=0}^{N} |a_n||z|^n \leq \sum_{n=0}^{+\infty} |a_n| \rho^n < +\infty, \text{ for all } z \in D(0,\rho) \text{ and } N \in \mathbb{N} \cup \{0\}$$

(see [12, 10.5 Power Series]), it follows from [6, 2.24 The Dominated Convergence Theorem] that

$$\int_{D(0,\rho)} |f(z)|^2 \, dA_\alpha(z) = \lim_{N \to +\infty} \int_{D(0,\rho)} |S_N(z)|^2 \, dA_\alpha(z) = \sum_{n=0}^{+\infty} |a_n|^2 (\alpha+1) \int_0^\rho 2r^{2n+1} (1-r^2)^\alpha \, dr.$$

Let $\{\rho_k\} \subset (0,1)$, with $\rho_k \leq \rho_{k+1}$, for all $k \in \mathbb{N}$, and $\lim_{k \to +\infty} \rho_k = 1$. For all $k \in \mathbb{N}$, consider the functions $f_k(z) = |f(z)|^2 \chi_{D(0,\rho_k)}(z)$, for all $z \in \mathbb{D}$, and $g_k(r) = 2r^{2n+1}(1-r^2)^\alpha \chi_{(0,\rho_k)}(r)$, for all $r \in (0,1)$, where $n \in \mathbb{N} \cup \{0\}$ is fixed. Then, by applying [6, 2.24 The Dominated Convergence Theorem] to the sequences $\{f_k\}$ and $\{g_k\}$, we have that

$$\lim_{k \to +\infty} \int_{D(0,\rho_k)} |f(z)|^2 \, dA_\alpha(z) = \int_{\mathbb{D}} \lim_{k \to +\infty} f_k(z) \, dA_\alpha(z) = \int_{\mathbb{D}} |f(z)|^2 \, dA_\alpha(z) = \|f\|_{2,\alpha}^2,$$

and

$$\lim_{k \to +\infty} \int_0^{\rho_k} 2r^{2n+1} (1-r^2)^\alpha \, dr = \int_0^1 \lim_{k \to +\infty} g_k(r) \, dr = \int_0^1 2r^{2n+1} (1-r^2)^\alpha \, dr,$$

so

$$\begin{aligned}
\|f\|_{2,\alpha}^2 &= \lim_{\rho\to 1^-}\int_{D(0,\rho)} |f(z)|^2\,dA_\alpha(z) = \lim_{\rho\to 1^-}\sum_{n=0}^{+\infty}|a_n|^2(\alpha+1)\int_0^\rho 2r^{2n+1}(1-r^2)^\alpha\,dr\\
&= \sup_{\rho\in(0,1)}\sum_{n=0}^{+\infty}|a_n|^2(\alpha+1)\int_0^\rho 2r^{2n+1}(1-r^2)^\alpha\,dr\\
&= \sup_{\rho\in(0,1)}\sup_{N\in\mathbb{N}\cup\{0\}}\sum_{n=0}^{N}|a_n|^2(\alpha+1)\int_0^\rho 2r^{2n+1}(1-r^2)^\alpha\,dr\\
&= \sup_{N\in\mathbb{N}\cup\{0\}}\sup_{\rho\in(0,1)}\sum_{n=0}^{N}|a_n|^2(\alpha+1)\int_0^\rho 2r^{2n+1}(1-r^2)^\alpha\,dr\\
&= \sup_{N\in\mathbb{N}\cup\{0\}}\lim_{\rho\to 1^-}\sum_{n=0}^{N}|a_n|^2(\alpha+1)\int_0^\rho 2r^{2n+1}(1-r^2)^\alpha\,dr\\
&= \sup_{N\in\mathbb{N}\cup\{0\}}\sum_{n=0}^{N}|a_n|^2(\alpha+1)\int_0^1 2r^{2n+1}(1-r^2)^\alpha\,dr\\
&= \sum_{n=0}^{+\infty}|a_n|^2(\alpha+1)\int_0^1 2r^{2n+1}(1-r^2)^\alpha\,dr = \sum_{n=0}^{+\infty}|a_n|^2 b_{n,\alpha}^2.
\end{aligned}$$

Step 3. For all $n\in\mathbb{N}\cup\{0\}$,

$$\langle f, e_n\rangle_{2,\alpha} = a_n b_{n,\alpha}. \tag{2.31}$$

Let $r\in(0,1)$ and $n\in\mathbb{N}\cup\{0\}$. Then, the series $\sum_{k=0}^{+\infty} a_k r^k e^{i(k-n)\theta}$ converges absolutely, because the series $\sum_{k=0}^{+\infty} a_k z^k$ converges absolutely in $\mathbb{D}$, and

$$\left|\sum_{k=0}^{N} a_k r^k e^{i(k-n)\theta} - \sum_{k=0}^{+\infty} a_k r^k e^{i(k-n)\theta}\right| \le \sum_{k=N+1}^{+\infty}|a_k| r^k, \text{ for all } \theta\in[0,2\pi] \text{ and } N\in\mathbb{N}\cup\{0\}.$$

Since $\sum_{k=N+1}^{+\infty}|a_k|r^k \to 0$, as $N\to+\infty$, we get that $\sum_{k=0}^{N} a_k r^k e^{i(k-n)\theta} \to \sum_{k=0}^{+\infty} a_k r^k e^{i(k-n)\theta}$ uniformly in $[0,2\pi]$, as $N\to+\infty$. Thus,

$$\begin{aligned}
\langle f, e_n\rangle_{2,\alpha} &= \int_{\mathbb{D}} f(z)\frac{\bar{z}^n}{b_{n,\alpha}}(\alpha+1)(1-|z|^2)^\alpha\,dA(z)\\
&= \frac{(\alpha+1)}{b_{n,\alpha}}\int_0^1\int_0^{2\pi} f(re^{i\theta})r^n e^{-in\theta}(1-r^2)^\alpha\cdot\frac{r}{\pi}\,d\theta dr\\
&= \frac{(\alpha+1)}{b_{n,\alpha}}\int_0^1 r^{n+1}(1-r^2)^\alpha\cdot\frac{1}{\pi}\int_0^{2\pi}\sum_{k=0}^{+\infty} a_k r^k e^{i(k-n)\theta}\,d\theta dr\\
&= \frac{(\alpha+1)}{b_{n,\alpha}}\int_0^1 r^{n+1}(1-r^2)^\alpha\cdot\frac{1}{\pi}\sum_{k=0}^{+\infty}\int_0^{2\pi} a_k r^k e^{i(k-n)\theta}\,d\theta dr\\
&= \frac{a_n}{b_{n,\alpha}}(\alpha+1)\int_0^1 2r^{2n+1}(1-r^2)^\alpha\,dr = a_n b_{n,\alpha}.
\end{aligned}$$

Step 4. By (2.30) and (2.31),

$$\|f\|_{2,\alpha}^2 = \sum_{n=0}^{+\infty}|\langle f, e_n\rangle_{2,\alpha}|^2, \text{ for all } f\in\mathcal{A}^2(dA_\alpha),$$

hence it follows from [2, Chapter I, Theorem 4.13] that $\{e_n\}$ is a basis for $\mathcal{A}^2(dA_\alpha)$. □

Proposition 2.2.3. *The reproducing kernel of $\mathcal{A}^2(dA_\alpha)$ is given by*

$$K_\alpha(z,w) = \frac{1}{(1-z\bar{w})^{2+\alpha}}, \text{ for all } z,w \in \mathbb{D}.$$

Proof. Let $\{e_n\}$ be the orthonormal basis of $\mathcal{A}^2(dA_\alpha)$ that was defined in Lemma 2.2.2. By Theorem 2.2.1 and a generalization of the Binomial Theorem ([13, Exercise 5.2.4]),

$$\begin{aligned}
K_\alpha(z,w) &= \sum_{n=0}^{+\infty} e_n(z)\overline{e_n(w)} = \sum_{n=0}^{+\infty} \frac{\Gamma(n+\alpha+2)}{n!\Gamma(\alpha+2)}(z\bar{w})^n \\
&= 1 + (\alpha+2)z\bar{w} + \frac{(\alpha+2)(\alpha+3)}{2!}(z\bar{w})^2 + \dots \\
&= 1 + (-\alpha-2)(-z\bar{w}) + \frac{(-\alpha-2)(-\alpha-2-1)}{2!}(-z\bar{w})^2 + \dots \\
&= (1+(-z\bar{w}))^{-\alpha-2} = \frac{1}{(1-z\bar{w})^{\alpha+2}},
\end{aligned}$$

for all $z,w \in \mathbb{D}$. □

An interesting property of the Bergman kernel is its Möbius invariance.

Proposition 2.2.4. *If $\phi \in \mathrm{Aut}(\mathbb{D})$, then*

$$K(z,w) = \phi'(z)K(\phi(z),\phi(w))\overline{\phi'(w)}, \text{ for all } z,w \in \mathbb{D},$$

or equivalently,

$$\frac{1}{(1-z\bar{w})^2} = \frac{\phi'(z)\overline{\phi'(w)}}{(1-\phi(z)\overline{\phi(w)})^2}, \text{ for all } z,w \in \mathbb{D}.$$

Proof. Let $\{e_n\}$ be an orthonormal basis of $\mathcal{A}^2(dA)$, and set $\sigma_n(z) = e_n(\phi(z))\phi'(z)$, for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$. Then:

(i) Using the Change of Variables Theorem ([6, Theorem 2.47a]),

$$\int_{\mathbb{D}} |\sigma_n(z)|^2\, dA(z) = \int_{\mathbb{D}} |e_n(\phi(z))|^2|\phi'(z)|^2\, dA(z) = \int_{\mathbb{D}} |e_n(z)|^2\, dA(z) < +\infty, \text{ for all } n \in \mathbb{N},$$

so $\sigma_n \in \mathcal{A}^2(dA)$ and $\|\sigma_n\|_2 = \|e_n\|_2 = 1$, for all $n \in \mathbb{N}$.

(ii) Let $n > m \geq 1$. By the Cauchy-Schwarz inequality, $e_n\bar{e}_m \in L^1(\mathbb{D}, dA)$, so using again [6, Theorem 2.47a],

$$\begin{aligned}
\langle \sigma_n, \sigma_m \rangle_2 &= \int_{\mathbb{D}} \sigma_n(z)\overline{\sigma_m(z)}\, dA(z) = \int_{\mathbb{D}} e_n(\phi(z))\phi'(z)\overline{e_m(\phi(z))} \cdot \overline{\phi'(z)}\, dA(z) \\
&= \int_{\mathbb{D}} e_n(z)\overline{e_m(z)}\, dA(z) = \langle e_n, e_m \rangle_2 = 0.
\end{aligned}$$

(iii) Let $f \in \mathcal{A}^2(dA)$. Then $\dfrac{f\circ\phi^{-1}}{\phi'\circ\phi^{-1}} \in H(\mathbb{D})$, and by [6, Theorem 2.47a],

$$\int_{\mathbb{D}} \left| \frac{(f\circ\phi^{-1})(z)}{(\phi'\circ\phi^{-1})(z)} \right|^2 dA(z) = \int_{\mathbb{D}} |f(\phi^{-1}(z))|^2 \cdot |(\phi^{-1})'(z)|^2\, dA(z) = \int_{\mathbb{D}} |f(z)|^2\, dA(z) < +\infty,$$

so $\dfrac{f\circ\phi^{-1}}{\phi'\circ\phi^{-1}} \in \mathcal{A}^2(dA)$. Thus, [6, Theorem 5.27c] yields that

$$\frac{f\circ\phi^{-1}}{\phi'\circ\phi^{-1}} = \sum_{n=1}^{+\infty} a_n e_n \text{ in } \mathcal{A}^2(dA), \tag{2.32}$$

where $a_n = \left\langle \frac{f\circ\phi^{-1}}{\phi'\circ\phi^{-1}}, e_n \right\rangle_2$, for all $n \in \mathbb{N}$. Note that for all $N \in \mathbb{N}$,

$$\begin{aligned}
\left\| \frac{f\circ\phi^{-1}}{\phi'\circ\phi^{-1}} - \sum_{n=1}^{N} a_n e_n \right\|_2^2 &= \int_{\mathbb{D}} \left| f(\phi^{-1}(z))(\phi^{-1})'(z) - \sum_{n=1}^{N} a_n e_n(z) \right|^2 dA(z) \\
&= \int_{\mathbb{D}} \left| f(z)\cdot(\phi^{-1})'(\phi(z)) - \sum_{n=1}^{N} a_n e_n(\phi(z)) \right|^2 \cdot |\phi'(z)|^2 \, dA(z) \\
&= \int_{\mathbb{D}} \left| \frac{f(z)}{\phi'(z)} - \sum_{n=1}^{N} a_n e_n(\phi(z)) \right|^2 \cdot |\phi'(z)|^2 \, dA(z) \\
&= \int_{\mathbb{D}} \left| f(z) - \sum_{n=1}^{N} a_n e_n(\phi(z))\phi'(z) \right|^2 dA(z) = \left\| f - \sum_{n=1}^{N} a_n \sigma_n \right\|_2^2,
\end{aligned}$$

where we used [6, Theorem 2.47a] for the second equality. Therefore,

$$\lim_{N\to+\infty} \left\| f - \sum_{n=1}^{N} a_n \sigma_n \right\|_2 = \lim_{N\to+\infty} \left\| \frac{f\circ\phi^{-1}}{\phi'\circ\phi^{-1}} - \sum_{n=1}^{N} a_n e_n \right\|_2 = 0,$$

because of (2.32).

Thus, $\{\sigma_n\}$ forms an orthonormal basis for $\mathcal{A}^2(dA)$. By Theorem 2.2.1, for all $z, w \in \mathbb{D}$,

$$\begin{aligned}
K(z,w) = \sum_{n=1}^{+\infty} \sigma_n(z)\overline{\sigma_n(w)} &= \sum_{n=1}^{+\infty} e_n(\phi(z))\phi'(z)\overline{e_n(\phi(w))}\cdot\overline{\phi'(w)} \\
&= \phi'(z)K(\phi(z),\phi(w))\overline{\phi'(w)}.
\end{aligned}$$

□

2.3 The reproducing kernel for $\mathcal{A}^p(dA_\alpha)$, where $p \geq 1$

Our next goal is to show that (2.14) holds for all $f \in \mathcal{A}^p(dA_\alpha)$ and $z \in \mathbb{D}$, where $p \geq 1$. For this, we need the following results.

Proposition 2.3.1. *Let $\alpha > -1$. Then*

$$f(0) = \int_{\mathbb{D}} f(z)\, dA_\alpha(z), \textit{ for all } f \in H^\infty. \tag{2.33}$$

Proof. Let $f \in H^\infty$ and $\{r_n\} \subset (0,1)$, with $r_n \leq r_{n+1}$, for all $n \in \mathbb{N}$, and $\lim_{n\to+\infty} r_n = 1$. Using the Mean Value Property ([14, Section VII.6]), we have that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\int_{D(0,r_n)} f(z)\, dA_\alpha(z) &= \int_0^{r_n} (\alpha+1)(1-\rho^2)^\alpha \cdot \frac{\rho}{\pi} \int_0^{2\pi} f(\rho e^{i\theta})\, d\theta d\rho \\
&= \int_0^{r_n} (\alpha+1)(1-\rho^2)^\alpha \cdot 2\rho f(0)\, d\rho = f(0)(1-(1-r_n^2)^{\alpha+1}),
\end{aligned}$$

so

$$\lim_{n\to+\infty} \int_{D(0,r_n)} f(z)\, dA_\alpha(z) = f(0). \tag{2.34}$$

On the other hand, $|f(z)\chi_{D(0,r_n)}(z)| \leq |f(z)| \leq \|f\|_\infty$, for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$, hence, by [6, 2.24 The Dominated Convergence Theorem],

$$\lim_{n\to+\infty} \int_{D(0,r_n)} f(z)\, dA_\alpha(z) = \int_{\mathbb{D}} f(z)\, dA_\alpha(z). \tag{2.35}$$

(2.34) and (2.35) give (2.33). □

Corollary 2.3.2. *Let $\alpha > -1$. If $f \in H^\infty$ and $a \in \mathbb{D}$, then*

$$f(a) = (1-|a|^2)^{2+\alpha} \int_{\mathbb{D}} \frac{f(z)}{|1-\bar{a}z|^{2\alpha+4}}\, dA_\alpha(z). \tag{2.36}$$

Proof. Indeed, if $g = f \circ \phi_a$ in $\mathbb{D}$, then $g \in H^\infty$, hence by the Change of Variables Theorem ([6, Theorem 2.47a]) and Proposition 1.1.2 (iii), (iv),

$$\begin{aligned} f(a) = g(0) &\overset{(2.33)}{=} \int_{\mathbb{D}} g(z)\, dA_\alpha(z) = \int_{\mathbb{D}} (f\circ\phi_a)(z)\cdot(\alpha+1)(1-|z|^2)^\alpha\, dA(z) \\ &= \int_{\mathbb{D}} f(z)\cdot(\alpha+1)(1-|\phi_a(z)|^2)^\alpha \cdot |\phi_a'(z)|^2\, dA(z) = \int_{\mathbb{D}} f(z)\cdot \frac{(1-|a|^2)^{\alpha+2}}{|1-\bar{a}z|^{2\alpha+4}}\, dA_\alpha(z). \end{aligned}$$

□

Corollary 2.3.3. *Let $\alpha > -1$. Then*

$$f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-z\bar{w})^{2+\alpha}}\, dA_\alpha(w), \text{ for all } f \in H^\infty \text{ and } z \in \mathbb{D}. \tag{2.37}$$

Proof. Let $f \in H^\infty$ and $z \in \mathbb{D}$. By [14, Section X.5], the function $g(w) = f(w)(1-\bar{z}w)^{\alpha+2}$, $w \in \mathbb{D}$, is well-defined and holomorphic in $\mathbb{D}$. Moreover, $g \in H^\infty$, because

$$|g(w)| = |f(w)||1-\bar{z}w|^{\alpha+2} \le \|f\|_\infty \cdot 2^{\alpha+2}, \text{ for all } w \in \mathbb{D}.$$

Thus, by (2.36),

$$\begin{aligned} & g(z) = (1-|z|^2)^{2+\alpha} \int_{\mathbb{D}} \frac{g(w)}{|1-\bar{z}w|^{2\alpha+4}}\, dA_\alpha(w) \;\Rightarrow \\ \Rightarrow\; & f(z)(1-|z|^2)^{2+\alpha} = (1-|z|^2)^{2+\alpha} \int_{\mathbb{D}} \frac{f(w)(1-\bar{z}w)^{\alpha+2}}{(1-\bar{z}w)^{\alpha+2}(1-z\bar{w})^{\alpha+2}}\, dA_\alpha(w) \;\Rightarrow \\ \Rightarrow\; & f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-z\bar{w})^{\alpha+2}}\, dA_\alpha(w). \end{aligned}$$

□

Lemma 2.3.4. *Let $p > 0$, $\alpha > -1$, and consider the function $M(r, f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta$, for all $f \in H(\mathbb{D})$ and $r \in [0, 1)$. Then, $M(\cdot, f)$ is increasing, for all $f \in H(\mathbb{D})$.*

Proof. Let $f \in H(\mathbb{D})$ and $0 \le R_1 < R_2 < 1$. By Poisson's Theorem ([3, Chapter X, Corollary 2.10]), there exists a function $u : \overline{D(0, R_2)} \to \mathbb{R}$, which is harmonic in $D(0, R_2)$, continuous in $\overline{D(0, R_2)}$ and $u(z) = |f(z)|^p$, for all $z \in \partial D(0, R_2)$. Since u is harmonic in $D(0, R_2)$, it satisfies the Mean Value Property ([3, Chapter X, 1.4 Mean Value Theorem]), so

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta, \text{ for all } r \in [0, R_2),$$

and by the continuity of u in $\overline{D(0, R_2)}$, the above equality holds also for $r = R_2$.

Note that $|f|^p$ is a subharmonic function in $\mathbb{D}$, due to Corollary 2.1.2, so it follows from [3, Chapter X, Corollary 3.5] that $|f(z)|^p \le u(z)$, for all $z \in D(0, R_2)$. Thus,

$$\begin{aligned} M(R_1, f) &= \frac{1}{2\pi}\int_0^{2\pi} |f(R_1e^{i\theta})|^p\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} u(R_1e^{i\theta})\, d\theta = u(0) \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(R_2e^{i\theta})\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} |f(R_2e^{i\theta})|^p\, d\theta = M(R_2, f). \end{aligned}$$

□

The proof of the following Proposition is based on [4, §2.3, Theorem 3].

Proposition 2.3.5. *Let $p > 0$ and $\alpha > -1$. If $e_n(z) = z^n$, for all $z \in \mathbb{D}$ and $n \in \mathbb{N} \cup \{0\}$, then the set* $\operatorname{span}\{e_n : n \in \mathbb{N} \cup \{0\}\}$ *(which is the set of all polynomials in $\mathbb{C}$) is dense in $\mathcal{A}^p(dA_\alpha)$.*

Proof. Let $f \in \mathcal{A}^p(dA_\alpha)$ and $\{r_n\} \subset (0,1)$, with $r_n < r_{n+1}$, for all $n \in \mathbb{N}$, and $\lim_{n\to+\infty} r_n = 1$. Consider the functions $f_n(z) = f(r_n z)$, for all $z \in D\big(0, \frac{1}{r_n}\big)$ and $n \in \mathbb{N}$, and denote by S_N^n the Taylor polynomial of degree N of the function f_n; that is, $S_N^n(z) = \sum_{k=0}^N a_k r_n^k z^k$, for all $z \in D\big(0, \frac{1}{r_n}\big)$ and $N \in \mathbb{N} \cup \{0\}$, where $a_k = \frac{f^{(k)}(0)}{k!}$, for all $k \in \mathbb{N} \cup \{0\}$. By [12, 10.5 Power Series], $S_N^n \to f_n$ uniformly on $\mathbb{D}$, as $N \to +\infty$, for all $n \in \mathbb{N}$. Since

$$|f_n(z) - S_N^n(z)| \leq |f_n(z)| + \sum_{k=0}^{N} |a_k| r_n^k |z|^k \leq \max_{z\in\overline{\mathbb{D}}} |f_n(z)| + \sum_{k=0}^{+\infty} |a_k| r_n^k < +\infty,$$

for all $z \in \mathbb{D}$ and $N \in \mathbb{N} \cup \{0\}$, it follows from the Bounded Convergence Theorem ([9, Exercise 3.1.10]) that

$$\|f_n - S_N^n\|_{p,\alpha}^p = \int_{\mathbb{D}} |f_n(z) - S_N^n(z)|^p \, dA_\alpha(z) \underset{N\to+\infty}{\longrightarrow} 0, \text{ for all } n \in \mathbb{N}. \tag{2.38}$$

Now, let S be a compact subset of $\mathbb{D}$. Then there exists a real $R \in (0,1)$ such that $S \subset \overline{D(0,R)} \subset \mathbb{D}$. Let $\epsilon > 0$. Since f is uniformly continuous in $\overline{D(0,R)}$, there exists a real number $\delta > 0$ such that if $z, w \in \overline{D(0,R)}$ and $|z - w| < \delta$, then $|f(z) - f(w)| < \epsilon$. Also, by the convergence of $\{r_n\}$, there exists a number $n_0 \in \mathbb{N}$ such that $1 - r_n < \delta$, for all $n \geq n_0$. Let $z \in S$ and $n \geq n_0$. Then $|r_n z| \leq |z| \leq R$ and $|r_n z - z| = |z|(1 - r_n) < \delta$, hence

$$|f_n(z) - f(z)| = |f(r_n z) - f(z)| < \epsilon.$$

Thus, we have proved that $f_n \to f$ uniformly on compact sets of $\mathbb{D}$, as $n \to +\infty$. By [6, 2.24 The Dominated Convergence Theorem],

$$\lim_{n\to+\infty} M(\rho, f_n - f) = \lim_{n\to+\infty} \frac{1}{2\pi} \int_0^{2\pi} |f_n(\rho e^{i\theta}) - f(\rho e^{i\theta})|^p \, d\theta = 0, \text{ for all } \rho \in [0,1), \tag{2.39}$$

where M is the function defined in Lemma 2.3.4. Moreover,

$$\begin{aligned} M(\rho, f_n - f) &\leq \frac{1}{2\pi} \int_0^{2\pi} 2^p (|f_n(\rho e^{i\theta})|^p + |f(\rho e^{i\theta})|^p) \, d\theta = 2^p (M(\rho, f_n) + M(\rho, f)) \\ &= 2^p (M(r_n\rho, f) + M(\rho, f)) \leq 2^{p+1} M(\rho, f), \text{ for all } \rho \in [0,1) \text{ and } n \in \mathbb{N}, \end{aligned} \tag{2.40}$$

where we used Lemma 2.3.4 for the last inequality, and note that

$$\begin{aligned} \int_{[0,1)} M(\rho, f) \cdot (\alpha+1)(1-\rho^2)^\alpha \rho \, d\rho &= \frac{1}{2} \int_0^1 \int_0^{2\pi} |f(\rho e^{i\theta})|^p \cdot (\alpha+1)(1-\rho^2)^\alpha \cdot \frac{\rho}{\pi} \, d\theta d\rho \\ &= \frac{1}{2} \int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) < +\infty. \end{aligned} \tag{2.41}$$

(2.39), (2.40), (2.41) and [6, 2.24 The Dominated Convergence Theorem] give

$$\begin{aligned} \|f_n - f\|_{p,\alpha}^p &= \int_{\mathbb{D}} |f_n(z) - f(z)|^p \, dA_\alpha(z) \\ &= \int_0^1 \int_0^{2\pi} |f_n(\rho e^{i\theta}) - f(\rho e^{i\theta})|^p \cdot (\alpha+1)(1-\rho^2)^\alpha \cdot \frac{\rho}{\pi} \, d\theta d\rho \\ &= 2 \int_0^1 M(\rho, f_n - f) \cdot (\alpha+1)(1-\rho^2)^\alpha \rho \, d\rho \underset{n\to+\infty}{\longrightarrow} 0. \end{aligned} \tag{2.42}$$

Let $\epsilon > 0$. By (2.42), there exists a number $n_0 \in \mathbb{N}$ such that

$$\|f - f_{n_0}\|_{p,\alpha} < \frac{\epsilon}{2}. \tag{2.43}$$

By (2.38), there exists a number $N_0 \in \mathbb{N}$ such that

$$\|f_{n_0} - S_{N_0}^{n_0}\|_{p,\alpha} < \frac{\epsilon}{2}. \tag{2.44}$$

(2.43) and (2.44) give

$$\|f - S_{N_0}^{n_0}\|_{p,\alpha} \le \|f - f_{n_0}\|_{p,\alpha} + \|f_{n_0} - S_{N_0}^{n_0}\|_{p,\alpha} < \epsilon,$$

which means that polynomials are dense in $\mathcal{A}^p(dA_\alpha)$. □

In the proof of the following Theorem we use an idea of [11, Theorem 8.2.1].

Theorem 2.3.6. *Let $p \ge 1$ and $\alpha > -1$. Then*

$$f(z) = \int_{\mathbb{D}} f(w) K_\alpha(z,w)\, dA_\alpha(w), \text{ for all } z \in \mathbb{D} \text{ and } f \in \mathcal{A}^p(dA_\alpha).$$

Proof. Let $z \in \mathbb{D}$. Since

$$\int_{\mathbb{D}} \frac{|g(w)|}{|1 - z\bar{w}|^{2+\alpha}}\, dA_\alpha(w) \le \frac{1}{(1-|z|)^{2+\alpha}} \int_{\mathbb{D}} |g(w)|\, dA_\alpha(w) < +\infty, \text{ for all } g \in \mathcal{A}^1(dA_\alpha),$$

the functional Λ_z given by

$$\Lambda_z(g) = \int_{\mathbb{D}} \frac{g(w)}{(1 - z\bar{w})^{2+\alpha}}\, dA_\alpha(w), \text{ for all } g \in \mathcal{A}^1(dA_\alpha),$$

is well-defined, linear and bounded on $\mathcal{A}^1(dA_\alpha)$. Consider, also, the functional $T_z \in (\mathcal{A}^1(dA_\alpha))^*$, defined in Corollary 2.1.5. By [2, Chapter III, Proposition 2.1], Λ_z and T_z are continuous with respect to $\|\cdot\|_{1,\alpha}$.

Now, let $f \in \mathcal{A}^p(dA_\alpha)$. Since A_α is a finite measure (see Remark 1.3.3), it follows from [9, Exercise 7.3.3] that $f \in \mathcal{A}^1(dA_\alpha)$. By Proposition 2.3.5, there exists a sequence of polynomials $\{p_n\}$ such that

$$\lim_{n\to+\infty} \|p_n - f\|_{1,\alpha} = 0.$$

Since $p_n \in H^\infty$ for all $n \in \mathbb{N}$,

$$\begin{aligned} f(z) = T_z(f) = T_z\Big(\lim_{n\to+\infty} p_n\Big) &= \lim_{n\to+\infty} T_z(p_n) = \lim_{n\to+\infty} p_n(z) \\ &\overset{(2.37)}{=} \lim_{n\to+\infty} \int_{\mathbb{D}} \frac{p_n(w)}{(1 - z\bar{w})^{2+\alpha}}\, dA_\alpha(w) = \lim_{n\to+\infty} \Lambda_z(p_n) \\ &= \Lambda_z\Big(\lim_{n\to+\infty} p_n\Big) = \Lambda_z(f) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^{2+\alpha}}\, dA_\alpha(w). \end{aligned}$$

□

2.4 Projections and the dual of $\mathcal{A}^p(dA_\alpha)$

Since $\mathcal{A}^2(dA_\alpha)$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA_\alpha)$, there exists a bounded linear operator P_α from $L^2(\mathbb{D}, dA_\alpha)$ onto $\mathcal{A}^2(dA_\alpha)$, called the orthogonal projection of $L^2(\mathbb{D}, dA_\alpha)$ onto $\mathcal{A}^2(dA_\alpha)$ (see [10, Sections 2.8, 2.11] for the definition and properties of the orthogonal projection). If $\alpha = 0$, we write P instead of P_0 and we call P the Bergman projection on $\mathbb{D}$.

Proposition 2.4.1. *Let $\alpha > -1$ and $f \in L^2(\mathbb{D}, dA_\alpha)$. Then*

$$P_\alpha f(z) = \int_{\mathbb{D}} f(w) K_\alpha(z,w)\, dA_\alpha(w), \text{ for all } z \in \mathbb{D}.$$

Proof. Let $z \in \mathbb{D}$. Using the functional T_z defined in Corollary 2.1.5, and (2.14),

$$P_\alpha f(z) = T_z(P_\alpha f) = \langle P_\alpha f, \overline{K_\alpha(z,\cdot)}\rangle_{2,\alpha} = \langle f, P_\alpha(\overline{K_\alpha(z,\cdot)})\rangle_{2,\alpha} = \langle f, \overline{K_\alpha(z,\cdot)}\rangle_{2,\alpha},$$

where the third equality holds because of [10, Proposition 2.15 (iv)], and the last equality holds because $P_\alpha g = g$, for all $g \in \mathcal{A}^2(dA_\alpha)$. □

Our next goal is to find projections from $L^p(\mathbb{D}, dA_\alpha)$ onto $\mathcal{A}^p(dA_\alpha)$, for $p \geq 1$. First we need the following results.

Lemma 2.4.2. *Let $r \in (0,1)$ and $b \in \mathbb{R}$. Then there exists a constant $C = C(r,b) > 0$ such that*

$$|(1-z)^b - (1-w)^b| \leq C|z-w|, \text{ for all } z, w \in \overline{D(0,r)}.$$

Proof. Let $z, w \in \overline{D(0,r)}$. By a generalization of the Binomial Theorem ([13, Exercise 5.2.4]),

$$\begin{aligned}
|(1-z)^b - (1-w)^b| &= \Big|\sum_{k=0}^{+\infty}(-1)^k\binom{b}{k}z^k - \sum_{k=0}^{+\infty}(-1)^k\binom{b}{k}w^k\Big| \leq \sum_{k=0}^{+\infty}\Big|\binom{b}{k}\Big||z^k - w^k| \\
&= \sum_{k=1}^{+\infty}\Big|\binom{b}{k}\Big| \cdot |z-w| \cdot |z^{k-1} + z^{k-2}w + \ldots + zw^{k-2} + w^{k-1}| \\
&\leq \sum_{k=1}^{+\infty}\Big|\binom{b}{k}\Big| \cdot |z-w| \cdot kr^{k-1} = |z-w| \sum_{k=1}^{+\infty} \frac{|b(b-1)\cdot \ldots \cdot (b-k+1)|}{(k-1)!} \cdot r^{k-1} \\
&= |z-w| \sum_{k=0}^{+\infty} \frac{|b(b-1)\cdot \ldots \cdot (b-k)|}{k!} \cdot r^k,
\end{aligned}$$

and the series $\sum_{k=0}^{+\infty} \frac{|b(b-1)\cdot \ldots \cdot (b-k)|}{k!} \cdot r^k$ converges (we can easily verify this by doing a ratio test). Thus, for $C > \sum_{k=0}^{+\infty} \frac{|b(b-1)\cdot \ldots \cdot (b-k)|}{k!} \cdot r^k \geq 0$, we have that $|(1-z)^b - (1-w)^b| \leq C|z-w|$. □

Proposition 2.4.3. *Let $\alpha > -1$ and $F : \mathbb{D} \to \mathbb{C}$ be a Borel-measurable function on $\mathbb{D}$ such that*

$$\int_{\mathbb{D}} |F(w) K_\alpha(z,w)|\, dA_\alpha(w) < +\infty, \text{ for all } z \in \mathbb{D}. \tag{2.45}$$

Then, the function $G : \mathbb{D} \to \mathbb{C}$ given by

$$G(z) = \int_{\mathbb{D}} F(w) K_\alpha(z,w)\, dA_\alpha(w), \text{ for all } z \in \mathbb{D},$$

is holomorphic in $\mathbb{D}$, with

$$G'(z) = \int_{\mathbb{D}} F(w) \cdot \frac{\partial K_\alpha}{\partial z}(z,w)\, dA_\alpha(w), \text{ for all } z \in \mathbb{D}.$$

Proof. Let $z \in \mathbb{D}$. Then there exists a real $R > 0$ such that $\overline{D(z,R)} \subset \mathbb{D}$. Since

$$\frac{\partial K_\alpha}{\partial z}(z,w) = \frac{(2+\alpha)\bar{w}}{(1-z\bar{w})^{3+\alpha}}, \text{ for all } w \in \mathbb{D},$$

we get that for every sequence $\{h_n\}$ in $\overline{D(0,R)} \setminus \{0\}$ with $\lim_{n\to+\infty} h_n = 0$,

$$\lim_{n\to+\infty} \frac{K_\alpha(z+h_n,w) - K_\alpha(z,w)}{h_n} = \frac{(2+\alpha)\bar{w}}{(1-z\bar{w})^{3+\alpha}}, \text{ for all } w \in \mathbb{D}. \tag{2.46}$$

Set $r := |z| + R \in (0,1)$. Then $(z+h_n)\bar{w}, z\bar{w} \in \overline{D(0,r)}$, for all $n \in \mathbb{N}$ and $w \in \mathbb{D}$, hence by Lemma 2.4.2, there exists a constant $C_1 = C_1(r,\alpha) > 0$ such that

$$|(1-z\bar{w})^{2+\alpha} - (1-(z+h_n)\bar{w})^{2+\alpha}| \le C_1|z\bar{w} - (z+h_n)\bar{w}| = C_1|h_n w|, \text{ for all } n \in \mathbb{N} \text{ and } w \in \mathbb{D}.$$

Thus, for all $n \in \mathbb{N}$ and $w \in \mathbb{D}$,

$$\begin{aligned}\left|\frac{K_\alpha(z+h_n,w) - K_\alpha(z,w)}{h_n}\right| &= \frac{1}{|h_n|}\left|\frac{1}{(1-(z+h_n)\bar{w})^{2+\alpha}} - \frac{1}{(1-z\bar{w})^{2+\alpha}}\right| \\ &= \frac{1}{|h_n|}\frac{|(1-z\bar{w})^{2+\alpha} - (1-(z+h_n)\bar{w})^{2+\alpha}|}{|1-(z+h_n)\bar{w}|^{2+\alpha}|1-z\bar{w}|^{2+\alpha}} \\ &\le \frac{C_1|w|}{|1-(z+h_n)\bar{w}|^{2+\alpha}|1-z\bar{w}|^{2+\alpha}} \le \frac{C_1}{(1-r)^{2(2+\alpha)}} =: C.\end{aligned}$$

It follows that

$$|F(w)|\left|\frac{K_\alpha(z+h_n,w) - K_\alpha(z,w)}{h_n}\right| \le C|F(w)|, \text{ for all } w \in \mathbb{D} \text{ and } n \in \mathbb{N}, \tag{2.47}$$

and note that

$$\int_{\mathbb{D}} |F(w)|\, dA_\alpha(w) = 2^{2+\alpha}\int_{\mathbb{D}} \frac{|F(w)|}{2^{2+\alpha}}\, dA_\alpha(w) \le 2^{2+\alpha}\int_{\mathbb{D}} \frac{|F(w)|}{|1-z\bar{w}|^{2+\alpha}}\, dA_\alpha(w) < +\infty, \tag{2.48}$$

because of (2.45). By (2.46), (2.47), (2.48) and [6, 2.24 The Dominated Convergence Theorem],

$$\begin{aligned}\lim_{n\to+\infty} \frac{G(z+h_n) - G(z)}{h_n} &= \lim_{n\to+\infty} \int_{\mathbb{D}} F(w)\frac{K_\alpha(z+h_n,w) - K_\alpha(z,w)}{h_n}\, dA_\alpha(w) \\ &= \int_{\mathbb{D}} F(w)\frac{(2+\alpha)\bar{w}}{(1-z\bar{w})^{3+\alpha}}\, dA_\alpha(w),\end{aligned}$$

so

$$G'(z) = \lim_{h\to 0}\frac{G(z+h) - G(z)}{h} = \int_{\mathbb{D}} F(w)\cdot\frac{\partial K_\alpha}{\partial z}(z,w)\, dA_\alpha(w).$$

□

Proposition 2.4.4. *Let $p \ge 1$, $\alpha > -1$ and $\gamma > -1$ such that $p(\gamma+1) > \alpha+1$. The operator P_γ given by*

$$P_\gamma f(z) = \int_{\mathbb{D}} f(w)K_\gamma(z,w)\, dA_\gamma(w) = (\gamma+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\gamma}{(1-z\bar{w})^{2+\gamma}} f(w)\, dA(w), \tag{2.49}$$

is a bounded projection from $L^p(\mathbb{D}, dA_\alpha)$ onto $\mathcal{A}^p(dA_\alpha)$.

Proof. By Theorem A.0.2, the operator P_γ is bounded on $L^p(\mathbb{D}, dA_\alpha)$ and

$$\int_{\mathbb{D}} |K_\gamma(z,w)||f(w)|\, dA_\gamma(w) = (\gamma+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\gamma}{|1-z\bar{w}|^{2+\gamma}}|f(w)|\, dA(w) < +\infty,$$

for all $z \in \mathbb{D}$ and $f \in L^p(\mathbb{D}, dA_\alpha)$. Then, Lemma 2.4.3 yields that $P_\gamma f$ is holomorphic in $\mathbb{D}$, for all $f \in L^p(\mathbb{D}, dA_\alpha)$, so $P_\gamma(L^p(\mathbb{D}, dA_\alpha)) \subset \mathcal{A}^p(dA_\alpha)$.

It remains to show that P_γ is onto $\mathcal{A}^p(dA_\alpha)$. Let $z \in \mathbb{D}$ and consider the functional

$$\Lambda_z(f) = P_\gamma f(z) = T_z(P_\gamma f),$$

for all $f \in L^p(\mathbb{D}, dA_\alpha)$, where T_z is the bounded linear functional defined in Corollary 2.1.5. Then

$$|\Lambda_z(f)| = |T_z(P_\gamma f)| \leq \|T_z\| \|P_\gamma f\|_{p,\alpha} \leq \|T_z\| \|P_\gamma\| \|f\|_{p,\alpha}, \text{ for all } f \in L^p(\mathbb{D}, dA_\alpha).$$

Thus, Λ_z is a bounded linear functional, for all $z \in \mathbb{D}$.
Now, let $f \in \mathcal{A}^p(dA_\alpha)$. By Proposition 2.3.5, there exists a sequence $\{p_n\}$ of polynomials such that

$$\lim_{n\to+\infty} \|p_n - f\|_{p,\alpha} = 0.$$

Let $z \in \mathbb{D}$. Since $p_n \in H^\infty$ for all $n \in \mathbb{N}$,

$$\begin{aligned} f(z) = T_z(f) = T_z\Big(\lim_{n\to+\infty} p_n\Big) &= \lim_{n\to+\infty} T_z(p_n) = \lim_{n\to+\infty} p_n(z) \\ &\overset{(2.37)}{=} \lim_{n\to+\infty} \int_{\mathbb{D}} p_n(w) K_\gamma(z,w)\, dA_\gamma(w) = \lim_{n\to+\infty} \Lambda_z(p_n) \\ &= \Lambda_z\Big(\lim_{n\to+\infty} p_n\Big) = \Lambda_z(f) = P_\gamma f(z), \end{aligned}$$

hence $f = P_\gamma f$ in $\mathbb{D}$. □

Theorem 2.4.5. *Let $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > -1$. Then $(\mathcal{A}^p(dA_\alpha))^*$ is isomorphic to $\mathcal{A}^q(dA_\alpha)$, and the two spaces have equivalent norms.*

Proof. Consider the operator $J : \mathcal{A}^q(dA_\alpha) \to (\mathcal{A}^p(dA_\alpha))^*$, given by $Jh = l_h$, for all $h \in \mathcal{A}^q(dA_\alpha)$, where

$$l_h(f) = \int_{\mathbb{D}} f(z)\overline{h(z)}\, dA_\alpha(z), \text{ for all } f \in \mathcal{A}^p(dA_\alpha) \text{ and } h \in \mathcal{A}^q(dA_\alpha).$$

(i) J is well-defined. Indeed, [6, 6.2 Hölder's Inequality] gives

$$\int_{\mathbb{D}} |f(z)||\overline{h(z)}|\, dA_\alpha(z) \leq \|f\|_{p,\alpha} \|h\|_{q,\alpha} < +\infty, \text{ for all } f \in \mathcal{A}^p(dA_\alpha) \text{ and } h \in \mathcal{A}^q(dA_\alpha),$$

so $Jh \in (\mathcal{A}^p(dA_\alpha))^*$ with

$$\|Jh\| \leq \|h\|_{q,\alpha}, \tag{2.50}$$

for all $h \in \mathcal{A}^q(dA_\alpha)$.

(ii) $J(h_1 + h_2) = Jh_1 + Jh_2$ and $J(\lambda h) = \overline{\lambda} Jh$, for all $h_1, h_2, h \in \mathcal{A}^q(dA_\alpha)$ and $\lambda \in \mathbb{C}$.

(iii) Let $l \in (\mathcal{A}^p(dA_\alpha))^*$. By the Hahn-Banach Theorem ([8, Theorem 3.2]), there exists a bounded linear functional $\Lambda : L^p(\mathbb{D}, dA_\alpha) \to \mathbb{C}$, such that $\Lambda(f) = l(f)$, for all $f \in \mathcal{A}^p(dA_\alpha)$, and $\|\Lambda\| = \|l\|$. The classical duality between L^p spaces ([9, Theorem 7.15]) yields that there exists a unique $g \in L^q(\mathbb{D}, dA_\alpha)$ such that $\Lambda = \Lambda_g$ in $L^p(\mathbb{D}, dA_\alpha)$, where

$$\Lambda_g(f) = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA_\alpha(z), \text{ for all } f \in L^p(\mathbb{D}, dA_\alpha).$$

Also, Theorem A.0.2 implies that the operator

$$L_\alpha f(z) = \int_{\mathbb{D}} |f(w)||K_\alpha(z,w)|\, dA_\alpha(w),$$

is well-defined and bounded on $L^p(\mathbb{D}, dA_\alpha)$, hence, if $f \in \mathcal{A}^p(dA_\alpha)$, then [6, 6.2 Hölder's Inequality] implies that

$$\int_{\mathbb{D}}\int_{\mathbb{D}} |f(w)||K_\alpha(z,w)||g(z)|\, dA_\alpha(w)dA_\alpha(z) = \int_{\mathbb{D}} |g(z)|L_\alpha f(z)\, dA_\alpha(z) \\ \leq \|g\|_{q,\alpha} \cdot \|L_\alpha f\|_{p,\alpha} < +\infty.$$

By Proposition 2.4.4 and [6, 2.37 The Fubini-Tonelli Theorem], we have that for all $f \in \mathcal{A}^p(dA_\alpha)$,

$$\begin{aligned} l(f) = \Lambda_g(f) &= \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA_\alpha(z) = \int_{\mathbb{D}} P_\alpha f(z)\overline{g(z)}\, dA_\alpha(z) \\ &= \int_{\mathbb{D}}\int_{\mathbb{D}} f(w)K_\alpha(z,w)\overline{g(z)}\, dA_\alpha(w)dA_\alpha(z) \\ &= \int_{\mathbb{D}}\int_{\mathbb{D}} f(w)K_\alpha(z,w)\overline{g(z)}\, dA_\alpha(z)dA_\alpha(w) \\ &= \int_{\mathbb{D}} f(w)\overline{\int_{\mathbb{D}} g(z)K_\alpha(w,z)\, dA_\alpha(z)}\, dA_\alpha(w) \\ &= \int_{\mathbb{D}} f(w)\overline{P_\alpha g(w)}\, dA_\alpha(w) = l_{P_\alpha g}(f) = [J(P_\alpha g)](f), \end{aligned}$$

so J is onto $(\mathcal{A}^p(dA_\alpha))^*$.

(iv) Suppose that $Jh_1 = Jh_2 = l \in (\mathcal{A}^p(dA_\alpha))^*$, for some $h_1, h_2 \in \mathcal{A}^q(dA_\alpha)$. Then

$$h_1(z) = \sum_{k=0}^{+\infty} a_k z^k \text{ and } h_2(z) = \sum_{k=0}^{+\infty} b_k z^k, \text{ for all } z \in \mathbb{D},$$

where $a_k = \dfrac{h_1^{(k)}(0)}{k!}$ and $b_k = \dfrac{h_2^{(k)}(0)}{k!}$, for all $k \in \mathbb{N} \cup \{0\}$.
Let $n \in \mathbb{N} \cup \{0\}$ and set $e_n(z) = z^n$, for all $z \in \mathbb{D}$. Note that the series $\sum_{k=0}^{+\infty} \bar{a}_k z^n \bar{z}^k$ converges absolutely in $\mathbb{D}$, so for a fixed $r \in (0,1)$, the series $\sum_{k=0}^{+\infty} \bar{a}_k r^{n+k} e^{i(n-k)\theta}$ converges uniformly on $[0, 2\pi]$. Also, consider a sequence $\{r_m\} \subset (0,1)$ with $\lim_{m\to+\infty} r_m = 1$ and $r_m < r_{m+1}$, for all $m \in \mathbb{N}$. Then [6, 2.24 The Dominated Convergence Theorem] yields that

$$\lim_{m\to+\infty} \int_{D(0,r_m)} z^n\overline{h_1(z)}\, dA_\alpha(z) = \int_{\mathbb{D}} z^n\overline{h_1(z)}\, dA_\alpha(z) \tag{2.51}$$

and

$$\lim_{m\to+\infty} \int_0^{r_m} \bar{a}_n(\alpha+1)r^{2n}(1-r^2)^\alpha \cdot 2r\, dr = \int_0^1 \bar{a}_n(\alpha+1)r^{2n}(1-r^2)^\alpha \cdot 2r\, dr. \tag{2.52}$$

Thus,

$$\begin{aligned} Jh_1(e_n) &= \int_{\mathbb{D}} z^n\overline{h_1(z)}\, dA_\alpha(z) \overset{(2.51)}{=} \lim_{m\to+\infty} \int_{D(0,r_m)} \sum_{k=0}^{+\infty} \bar{a}_k z^n \bar{z}^k\, dA_\alpha(z) \\ &= \lim_{m\to+\infty} \int_0^{r_m} (\alpha+1)(1-r^2)^\alpha \cdot \frac{r}{\pi} \int_0^{2\pi} \sum_{k=0}^{+\infty} \bar{a}_k r^{n+k} e^{i(n-k)\theta}\, d\theta dr \\ &= \lim_{m\to+\infty} \int_0^{r_m} (\alpha+1)(1-r^2)^\alpha \cdot \frac{r}{\pi} \sum_{k=0}^{+\infty} \bar{a}_k r^{n+k} \int_0^{2\pi} e^{i(n-k)\theta}\, d\theta dr \\ &= \lim_{m\to+\infty} \int_0^{r_m} (\alpha+1)(1-r^2)^\alpha \cdot \frac{r}{\pi} \bar{a}_n r^{2n} \cdot 2\pi\, dr \\ &\overset{(2.52)}{=} \int_0^1 \bar{a}_n(\alpha+1)r^{2n}(1-r^2)^\alpha \cdot 2r\, dr = \bar{a}_n(\alpha+1)B(n+1,\alpha+1). \end{aligned} \tag{2.53}$$

Similarly,

$$Jh_2(e_n) = \bar{b}_n(\alpha+1)B(n+1,\alpha+1). \tag{2.54}$$

Since $Jh_1(e_n) = Jh_2(e_n)$, for all $n \in \mathbb{N}\cup\{0\}$, (2.53) and (2.54) give

$$\bar{a}_n(\alpha+1)B(n+1,\alpha+1) = \bar{b}_n(\alpha+1)B(n+1,\alpha+1), \text{ for all } n \in \mathbb{N}\cup\{0\},$$

that is, $a_n = b_n$, for all $n \in \mathbb{N}\cup\{0\}$, and hence, $h_1 = h_2$ in $\mathbb{D}$. This means that J is one-to-one.

(v) Let $l \in (\mathcal{A}^p(dA_\alpha))^*$. Using the notation in (iii), there exists a function $g \in L^q(\mathbb{D}, dA_\alpha)$ such that $J(P_\alpha g) = l$. Since J is one-to-one, $J^{-1}l = P_\alpha g$, and by Proposition 2.4.4 and the fact that $L^q(\mathbb{D}, dA_\alpha)$ and $(L^p(\mathbb{D}, dA_\alpha))^*$ are isometrically isomorphic ([9, Theorem 7.15]),

$$\|J^{-1}l\|_{q,\alpha} = \|P_\alpha g\|_{q,\alpha} \le \|P_\alpha\|\|g\|_{q,\alpha} = \|P_\alpha\|\|\Lambda_g\| = \|P_\alpha\|\|l\|. \tag{2.55}$$

By (i), (ii), (iii), (iv) and (v), J is an isomorphism, and it follows from (2.50) and (2.55) that $(\mathcal{A}^p(dA_\alpha))^*$ and $\mathcal{A}^q(dA_\alpha)$ have equivalent norms. □

Proposition 2.4.6. *Let $p \ge 1$, $\alpha > -1$ and $\gamma > -1$ such that $p(\gamma+1) > \alpha+1$. Then the spaces $\mathcal{A}^p(dA_\alpha)$ and $L^p(\mathbb{D}, dA_\alpha)/\mathrm{Ker}P_\gamma$ are isomorphic.*

Proof. Since P_γ is onto, the above result is known from elementary functional analysis (see [8, Section 3.4]). □

2.5 A characterization of $\mathcal{A}^p(dA_\alpha)$ in terms of derivatives

Lemma 2.5.1. *Let $p > 0$, $\alpha > -1$ and $n \in \mathbb{N}$. Then, there exists a constant $C = C(p,\alpha,n) > 0$ such that*

$$\int_{\mathbb{D}} |f(z)|^p\, dA_\alpha(z) \le C \int_{\mathbb{D}} |z^n f(z)|^p\, dA_\alpha(z), \text{ for all } f \in H(\mathbb{D}).$$

Proof. Let $f \in H(\mathbb{D})$ and $z \in D\big(0, \frac{1}{4}\big)$. By Corollary 2.1.3,

$$\begin{aligned} |f(z)|^p = |f\circ\phi_z(0)|^p &\le \int_{\mathbb{D}} |f\circ\phi_z(w)|^p\, dA_\alpha(w) \\ &= \int_{|w|<\frac{1}{2}} |f\circ\phi_z(w)|^p\, dA_\alpha(w) + \int_{\frac{1}{2}<|w|<1} |f\circ\phi_z(w)|^p\, dA_\alpha(w). \end{aligned} \tag{2.56}$$

Set $I_z := \int_{|w|<\frac{1}{2}} |f\circ\phi_z(w)|^p\, dA_\alpha(w)$. Since $f\circ\phi_z \in H\big(D(0,\frac{3}{4})\big) \cap C\big(\overline{D(0,\frac{3}{4})}\big)$, by the Maximum Modulus Principle ([14, Section VII.16]) there exists a real $\theta_0 \in [0, 2\pi)$ such that

$$|f\circ\phi_z(w)| \le \Big|f\circ\phi_z\Big(\frac{3}{4}e^{i\theta_0}\Big)\Big|, \text{ for all } w \in D\Big(0, \frac{3}{4}\Big),$$

so

$$I_z \le \int_{|w|<\frac{1}{2}} \Big|f\circ\phi_z\Big(\frac{3}{4}e^{i\theta_0}\Big)\Big|^p dA_\alpha(w) = B_1\Big|f\circ\phi_z\Big(\frac{3}{4}e^{i\theta_0}\Big)\Big|^p, \tag{2.57}$$

where $B_1 = B_1(\alpha) = \int_{|w|<\frac{1}{2}} dA_\alpha(w) > 0$. Also, by Proposition 2.1.10 and the fact that we can find a hyperbolic disc inside any euclidean disc (Corollary 1.3.6), there exists a constant $B_2 = B_2(\alpha) > 0$ such that

$$\Big|f\circ\phi_z\Big(\frac{3}{4}e^{i\theta_0}\Big)\Big|^p \le \frac{B_2}{\big(1-\big(\frac{3}{4}\big)^2\big)^{2+\alpha}} \int_{D\left(\frac{3}{4}e^{i\theta_0}, \frac{1}{8}\right)} |f\circ\phi_z(w)|^p\, dA_\alpha(w). \tag{2.58}$$

It follows from (2.57) and (2.58) that

$$I_z \le B \int_{D\left(\frac{3}{4}e^{i\theta_0}, \frac{1}{8}\right)} |f\circ\phi_z(w)|^p\, dA_\alpha(w) \le B \int_{\frac{1}{2}<|w|<1} |f\circ\phi_z(w)|^p\, dA_\alpha(w),$$

where $B = B(\alpha) = B_1 \cdot \dfrac{B_2}{\left(1-\left(\frac{3}{4}\right)^2\right)^{2+\alpha}} > 0$, hence by (2.56),

$$|f(z)|^p \leq C_1 \int_{\frac{1}{2}<|w|<1} |f\circ\phi_z(w)|^p \, dA_\alpha(w), \tag{2.59}$$

where $C_1 = C_1(\alpha) = B + 1 > 0$.

Set $\Omega_z = \phi_z^{-1}\big(\{w \in \mathbb{D} : \frac{1}{2} < |w| < 1\}\big) = \Big\{w \in \mathbb{D} : \dfrac{1}{2} < \Big|\dfrac{z-w}{1-\bar{z}w}\Big| < 1\Big\}$. By the Change of Variables Theorem ([6, Theorem 2.47a]) and Proposition 1.1.2 (iii), (iv),

$$\begin{aligned}\int_{\frac{1}{2}<|w|<1} |f\circ\phi_z(w)|^p \, dA_\alpha(w) &= \int_{\Omega_z} |f(w)|^p(\alpha+1)(1-|\phi_z(w)|^2)^\alpha |\phi_z'(w)|^2 \, dA(w)\\ &= \int_{\Omega_z} |f(w)|^p \cdot \frac{(1-|z|^2)^{2+\alpha}}{|1-\bar{z}w|^{2(2+\alpha)}} \, dA_\alpha(w),\end{aligned}$$

so (2.59) becomes

$$|f(z)|^p \leq C_1 \int_{\Omega_z} |f(w)|^p \cdot \frac{(1-|z|^2)^{2+\alpha}}{|1-\bar{z}w|^{2(2+\alpha)}} \, dA_\alpha(w). \tag{2.60}$$

Note that if $w \in \Omega_z$, then

$$\frac{1}{2} < \left|\frac{z-w}{1-\bar{z}w}\right| \leq \frac{|z|+|w|}{1-|z|},$$

so

$$|w| \geq \frac{1}{2} - \frac{3}{2}|z| > \frac{1}{2} - \frac{3}{2}\cdot\frac{1}{4} = \frac{1}{8}.$$

Also,

$$\frac{(1-|z|^2)^{2+\alpha}}{|1-\bar{z}w|^{2(2+\alpha)}} \leq \frac{1}{(1-|z|)^{2(2+\alpha)}} \leq \frac{1}{\left(1-\frac{1}{4}\right)^{2(2+\alpha)}}, \text{ for all } w \in \mathbb{D},$$

hence (2.60) becomes

$$|f(z)|^p \leq C_2 \int_{\frac{1}{8}<|w|<1} |f(w)|^p \, dA_\alpha(w), \tag{2.61}$$

where $C_2 = C_2(\alpha) = \dfrac{C_1}{\left(1-\frac{1}{4}\right)^{2(2+\alpha)}} > 0$. Set $C_3 = C_3(p,\alpha,n) = 8^{np}\cdot C_2 > 0$. Then (2.61) gives

$$\begin{aligned}|f(z)|^p &\leq \frac{C_3}{8^{np}} \int_{\frac{1}{8}<|w|<1} |f(w)|^p \, dA_\alpha(w) \leq C_3 \int_{\frac{1}{8}<|w|<1} |w|^{np}|f(w)|^p \, dA_\alpha(w)\\ &\leq C_3 \int_{\mathbb{D}} |w^n f(w)|^p \, dA_\alpha(w).\end{aligned} \tag{2.62}$$

Since (2.62) holds for an arbitrary $z \in D\left(0, \frac{1}{4}\right)$,

$$\begin{aligned}\int_{|z|<\frac{1}{4}} |f(z)|^p \, dA_\alpha(z) &\leq C_3 \int_{|z|<\frac{1}{4}} \int_{\mathbb{D}} |w^n f(w)|^p \, dA_\alpha(w) dA_\alpha(z)\\ &= C_4 \int_{\mathbb{D}} |w^n f(w)|^p \, dA_\alpha(w),\end{aligned} \tag{2.63}$$

where $C_4 = C_4(p,\alpha,n) = C_3 \displaystyle\int_{|z|<\frac{1}{4}} dA_\alpha(z) > 0$. On the other hand,

$$\begin{aligned}\int_{\frac{1}{4}<|z|<1} |f(z)|^p \, dA_\alpha(z) &\leq 4^{np} \int_{\frac{1}{4}<|z|<1} |z|^{np}|f(z)|^p \, dA_\alpha(z)\\ &\leq 4^{np} \int_{\mathbb{D}} |z^n f(z)|^p \, dA_\alpha(z).\end{aligned} \tag{2.64}$$

Set $\tilde{C} = \tilde{C}(p, \alpha, n) = \max\{C_4, 4^{np}\} > 0$. Then (2.63) and (2.64) imply that

$$\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) = \int_{|z|<\frac{1}{4}} |f(z)|^p \, dA_\alpha(z) + \int_{\frac{1}{4}<|z|<1} |f(z)|^p \, dA_\alpha(z)$$
$$\leq 2\tilde{C} \int_{\mathbb{D}} |z^n f(z)|^p \, dA_\alpha(z) = C \int_{\mathbb{D}} |z^n f(z)|^p \, dA_\alpha(z),$$

where $C = C(p, \alpha, n) = 2\tilde{C} > 0$. □

Proposition 2.5.2. *Let $\alpha > -1$, $n \in \mathbb{N}$ and $f \in H(\mathbb{D})$ such that*

$$f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0. \tag{2.65}$$

Then, for all $z \in \mathbb{D}$,

$$f(z) = \frac{1}{(\alpha+1) \cdot \ldots \cdot (\alpha+n)} \int_{\mathbb{D}} \frac{(1-|w|^2)^n f^{(n)}(w)}{\bar{w}^n (1-z\bar{w})^{2+\alpha}} \, dA_\alpha(w), \tag{2.66}$$

provided that the integral in the right side of (2.66) *converges.*

Proof. Assume that the integral in the right side of (2.66) converges for all $z \in \mathbb{D}$, which means that

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^n |f^{(n)}(w)|}{|\bar{w}|^n |1-z\bar{w}|^{2+\alpha}} \, dA_\alpha(w) < +\infty, \text{ for all } z \in \mathbb{D}.$$

By Proposition 2.4.3, the function g given by

$$g(z) = \frac{1}{(\alpha+1) \cdot \ldots \cdot (\alpha+n)} \int_{\mathbb{D}} \frac{(1-|w|^2)^n f^{(n)}(w)}{\bar{w}^n (1-z\bar{w})^{2+\alpha}} \, dA_\alpha(w), \text{ for all } z \in \mathbb{D},$$

is holomorphic in $\mathbb{D}$ and, working as in the proof of Proposition 2.4.3, we can differentiate under the integral sign. Thus, if $0 \leq k \leq n-1$,

$$g^{(k)}(z) = \frac{1}{(\alpha+1)(\alpha+k+2) \cdot \ldots \cdot (\alpha+n)} \int_{\mathbb{D}} \frac{(1-|w|^2)^n f^{(n)}(w)}{\bar{w}^{n-k} (1-z\bar{w})^{\alpha+k+2}} \, dA_\alpha(w), \text{ for all } z \in \mathbb{D},$$

while

$$g^{(n)}(z) = \frac{\alpha+n+1}{\alpha+1} \int_{\mathbb{D}} \frac{(1-|w|^2)^n f^{(n)}(w)}{(1-z\bar{w})^{\alpha+n+2}} \, dA_\alpha(w), \text{ for all } z \in \mathbb{D}. \tag{2.67}$$

By the Taylor series expansion of f in $\mathbb{D}$,

$$f(w) = \sum_{m=0}^{+\infty} \frac{f^{(m)}(0)}{m!} w^m, \text{ for all } w \in \mathbb{D},$$

so [12, Theorem 10.6] gives

$$f^{(n)}(w) = \sum_{m=n}^{+\infty} m(m-1) \cdot \ldots \cdot (m-n+1) \frac{f^{(m)}(0)}{m!} w^{m-n} = \sum_{m=0}^{+\infty} \frac{f^{(m+n)}(0)}{m!} w^m, \text{ for all } w \in \mathbb{D},$$

and the series converges absolutely in $\mathbb{D}$. Thus, if we fix $0 \le k \le n-1$ and $r \in [0,1)$, then the series $\sum_{m=0}^{+\infty} \frac{f^{(m+n)}(0)}{m!} r^m e^{i(m+n-k)\theta}$ converges uniformly in $[0, 2\pi]$, so

$$\begin{aligned} g^{(k)}(0) &= \frac{1}{(\alpha+1)(\alpha+k+2)\cdot \ldots \cdot (\alpha+n)} \int_{\mathbb{D}} \frac{(1-|w|^2)^n f^{(n)}(w)}{\bar{w}^{n-k}} \, dA_\alpha(w) \\ &= \frac{1}{(\alpha+k+2)\cdot \ldots \cdot (\alpha+n)} \int_0^1 \int_0^{2\pi} \frac{(1-r^2)^n f^{(n)}(re^{i\theta})}{r^{n-k} e^{-i(n-k)\theta}} (1-r^2)^\alpha \cdot \frac{r}{\pi} \, d\theta dr \\ &= \frac{1}{(\alpha+k+2)\cdot \ldots \cdot (\alpha+n)} \int_0^1 \frac{1}{\pi} \cdot \frac{(1-r^2)^{n+\alpha}}{r^{n-k-1}} \int_0^{2\pi} \sum_{m=0}^{+\infty} \frac{f^{(m+n)}(0)}{m!} r^m e^{i(m+n-k)\theta} \, d\theta dr \\ &= \frac{1}{(\alpha+k+2)\cdot \ldots \cdot (\alpha+n)} \int_0^1 \frac{1}{\pi} \cdot \frac{(1-r^2)^{n+\alpha}}{r^{n-k-1}} \sum_{m=0}^{+\infty} \frac{f^{(m+n)}(0)}{m!} r^m \int_0^{2\pi} e^{i(m+n-k)\theta} \, d\theta dr \\ &= 0, \end{aligned}$$

because $m+n-k > 0$ for all $m \in \mathbb{N} \cup \{0\}$. Therefore, by (2.65),

$$f^{(k)}(0) = 0 = g^{(k)}(0), \text{ for all } k \in \{0, \ldots, n-1\}. \tag{2.68}$$

Note, also, that

$$\begin{aligned} \int_{\mathbb{D}} |f^{(n)}(w)| \, dA_{\alpha+n}(w) &= \frac{\alpha+n+1}{\alpha+1} \int_{\mathbb{D}} (1-|w|^2)^n |f^{(n)}(w)| \, dA_\alpha(w) \\ &\le \frac{\alpha+n+1}{\alpha+1} \int_{\mathbb{D}} \frac{(1-|w|^2)^n |f^{(n)}(w)|}{|\bar{w}|^n} \, dA_\alpha(w) \\ &\le \frac{2^{2+\alpha}(\alpha+n+1)}{\alpha+1} \int_{\mathbb{D}} \frac{(1-|w|^2)^n |f^{(n)}(w)|}{|\bar{w}|^n |1-z\bar{w}|^{2+\alpha}} \, dA_\alpha(w) < +\infty \end{aligned}$$

by our hypothesis, hence $f^{(n)} \in \mathcal{A}^1(dA_{\alpha+n})$. Then, Theorem 2.3.6 gives that for all $z \in \mathbb{D}$,

$$\begin{aligned} f^{(n)}(z) &= \int_{\mathbb{D}} \frac{f^{(n)}(w)}{(1-z\bar{w})^{2+\alpha+n}} \, dA_{\alpha+n}(w) \\ &= \frac{\alpha+n+1}{\alpha+1} \int_{\mathbb{D}} (\alpha+1) \frac{f^{(n)}(w)(1-|w|^2)^n (1-|w|^2)^\alpha}{(1-z\bar{w})^{2+\alpha+n}} \, dA(w) \overset{(2.67)}{=} g^{(n)}(z), \end{aligned}$$

so $f^{(n-1)}(z) = g^{(n-1)}(z) + c$, for all $z \in \mathbb{D}$, where c is a constant. In particular, by (2.68), $c = 0$, hence $f^{(n-1)} = g^{(n-1)}$ in $\mathbb{D}$. Using (2.68) repeatedly, we get that $f = g$ in $\mathbb{D}$. □

For the following Theorem we used [7, Section 2.3].

Theorem 2.5.3. *Let $p > 0$, $n \in \mathbb{N}$, $\alpha > -1$ and $f \in H(\mathbb{D})$. Then, $f \in \mathcal{A}^p(dA_\alpha)$ if and only if the function $g(z) = (1-|z|^2)^n f^{(n)}(z)$, $z \in \mathbb{D}$, is in $L^p(\mathbb{D}, dA_\alpha)$.*

Proof. First assume that $f \in \mathcal{A}^p(dA_\alpha)$. If $\gamma > \frac{2+\alpha}{p} > 0$, then using Theorem 2.1.4 and Remark 1.3.3,

$$\begin{aligned} \int_{\mathbb{D}} |f(z)| \, dA_\gamma(z) &\le \int_{\mathbb{D}} \frac{\|f\|_{p,\alpha}}{(1-|z|^2)^{(2+\alpha)/p}} \, dA_\gamma(z) = \|f\|_{p,\alpha}(\gamma+1) \int_{\mathbb{D}} (1-|z|^2)^{\gamma - \frac{2+\alpha}{p}} \, dA(z) \\ &= \frac{(\gamma+1)\|f\|_{p,\alpha}}{\gamma - \frac{2+\alpha}{p} + 1} < +\infty, \end{aligned}$$

hence $f \in \mathcal{A}^1(dA_\gamma)$, and so Theorem 2.3.6 yields that

$$f(z) = (\gamma+1) \int_{\mathbb{D}} \frac{(1-|w|^2)^\gamma}{(1-z\bar{w})^{2+\gamma}} f(w) \, dA(w), \text{ for all } z \in \mathbb{D}. \tag{2.69}$$

Working as in the proof of Proposition 2.4.3, we can integrate under the integral sign in (2.69) to get

$$f^{(n)}(z) = (\gamma+1)\cdot \ldots \cdot(\gamma+n+1)\int_{\mathbb{D}} \frac{\bar{w}^n(1-|w|^2)^\gamma}{(1-z\bar{w})^{n+2+\gamma}} f(w)\,dA(w),\ \text{for all } z\in\mathbb{D},$$

so

$$g(z) = C_1(1-|z|^2)^n \int_{\mathbb{D}} \frac{(1-|w|^2)^\gamma}{(1-z\bar{w})^{n+2+\gamma}} \bar{w}^n f(w)\,dA(w),\ \text{for all } z\in\mathbb{D}, \tag{2.70}$$

where $C_1 = C_1(\gamma, n) = (\gamma+1)\cdot \ldots \cdot (\gamma+n+1) > 0$. We now distinguish two different cases with respect to the values of p.

Case 1. $p \geq 1$. Since

$$\int_{\mathbb{D}} |\bar{w}^n f(w)|^p\,dA_\alpha(w) \leq \int_{\mathbb{D}} |f(w)|^p\,dA_\alpha(w) < +\infty$$

and $-pn < \alpha+1 < p(\gamma+1)$ by the choice of γ, Theorem A.0.2 for $a = n$ and $b = \gamma$ implies that $g \in L^p(\mathbb{D}, dA_\alpha)$.

Case 2. $0 < p < 1$. Set $\lambda = p(\gamma+2)-2$ and note that $\lambda > \alpha > -1$ by the choice of γ. Also, for any $z \in \mathbb{D}$, consider the function $F_z(w) = \dfrac{f(w)}{(1-\bar{z}w)^{n+2+\gamma}}$, $w\in\mathbb{D}$. Then, for all $z\in\mathbb{D}$, $F_z \in H(\mathbb{D})$ and

$$\begin{aligned}
\int_{\mathbb{D}} |F_z(w)|^p\,dA_\lambda(w) &= \int_{\mathbb{D}} \frac{|f(w)|^p}{|1-\bar{z}w|^{p(n+2+\gamma)}}(\lambda+1)(1-|w|^2)^\lambda\,dA(w)\\
&\leq \int_{\mathbb{D}} \frac{|f(w)|^p}{(1-|z|)^{p(n+2+\gamma)}}(\lambda+1)(1-|w|^2)^\alpha\,dA(w)\\
&= \frac{1}{(1-|z|)^{p(n+2+\gamma)}}\frac{\lambda+1}{\alpha+1}\int_{\mathbb{D}} |f(w)|^p\,dA_\alpha(w) < +\infty,
\end{aligned}$$

hence $F_z \in \mathcal{A}^p(dA_\lambda)$. By (2.70) and Proposition 2.1.9, we have that for all $z\in\mathbb{D}$,

$$\begin{aligned}
|g(z)| &\leq C_1(1-|z|^2)^n \int_{\mathbb{D}} \frac{(1-|w|^2)^\gamma}{|1-z\bar{w}|^{n+2+\gamma}}|\bar{w}|^n|f(w)|\,dA(w)\\
&\leq \frac{C_1}{\gamma+1}(1-|z|^2)^n\int_{\mathbb{D}} |F_z(w)|\,dA_\gamma(w)\\
&\leq \frac{C_1}{\gamma+1}(1-|z|^2)^n\cdot\frac{\gamma+1}{\lambda+1}\left(\int_{\mathbb{D}}|F_z(w)|^p\,dA_\lambda(w)\right)^{1/p},
\end{aligned}$$

so

$$|g(z)|^p \leq C_2(1-|z|^2)^{np}\int_{\mathbb{D}}\frac{|f(w)|^p}{|1-\bar{z}w|^{p(n+2+\gamma)}}(1-|w|^2)^\lambda\,dA(w),\ \text{for all } z\in\mathbb{D},$$

where $C_2 = C_2(n,p,\gamma) = C_1^p(\lambda+1)^{1-p} > 0$. Using [6, 2.37a The Fubini-Tonelli Theorem],

$$\begin{aligned}
\int_{\mathbb{D}}|g(z)|^p\,dA_\alpha(z) &\leq C_2\int_{\mathbb{D}}\int_{\mathbb{D}}(1-|z|^2)^{np}\frac{|f(w)|^p}{|1-\bar{z}w|^{p(n+2+\gamma)}}(1-|w|^2)^\lambda\,dA(w)dA_\alpha(z)\\
&= C_2(\alpha+1)\int_{\mathbb{D}}\int_{\mathbb{D}}(1-|z|^2)^{np+\alpha}\frac{|f(w)|^p}{|1-\bar{z}w|^{p(n+2+\gamma)}}(1-|w|^2)^\lambda\,dA(w)dA(z)\\
&= C_2(\alpha+1)\int_{\mathbb{D}}|f(w)|^p(1-|w|^2)^\lambda\int_{\mathbb{D}}\frac{(1-|z|^2)^{np+\alpha}}{|1-\bar{z}w|^{p(n+2+\gamma)}}\,dA(z)dA(w).
\end{aligned}$$

Let $w\in\mathbb{D}$ and set $I(w) = \displaystyle\int_{\mathbb{D}}\frac{(1-|z|^2)^{np+\alpha}}{|1-\bar{z}w|^{p(n+2+\gamma)}}\,dA(z)$. If $w\in\mathbb{D}\setminus D(0,\frac{1}{2})$, then Lemma A.0.1 for $t = np+\alpha > -1$ and $c = p(2+\gamma)-\alpha-2 = \lambda-\alpha > 0$ implies that there exists a constant $C_3 = C_3(p,\gamma,\alpha,n) > 0$ such that

$$I(w) \leq C_3\cdot\frac{1}{(1-|w|^2)^{\lambda-\alpha}}. \tag{2.71}$$

If $w \in D(0, \frac{1}{2})$, then

$$I(w) \leq \int_{\mathbb{D}} \frac{(1-|z|^2)^{np+\alpha}}{(1-\frac{1}{2})^{p(n+2+\gamma)}} \, dA(z) = \frac{2^{p(n+2+\gamma)}}{np+\alpha+1} \leq \frac{2^{p(n+2+\gamma)}}{np+\alpha+1} \cdot \frac{1}{(1-|w|^2)^{\lambda-\alpha}}. \tag{2.72}$$

Thus, if $C_4 = C_4(p, \gamma, \alpha, n) = \max \left\{ C_3, \frac{2^{p(n+2+\gamma)}}{np+\alpha+1} \right\} > 0$, we have that

$$\begin{aligned} \int_{\mathbb{D}} |g(z)|^p \, dA_\alpha(z) &\leq C_2(\alpha+1) \int_{\mathbb{D}} |f(w)|^p (1-|w|^2)^\lambda I(w) \, dA(w) \\ &\leq C_2 C_4 (\alpha+1) \int_{\mathbb{D}} |f(w)|^p (1-|w|^2)^\alpha \, dA(w) \\ &= C_2 C_4 \int_{\mathbb{D}} |f(w)|^p \, dA_\alpha(w) < +\infty, \end{aligned}$$

hence $g \in L^p(\mathbb{D}, dA_\alpha)$.

Conversely, assume that $g \in L^p(\mathbb{D}, dA_\alpha)$. We distinguish two cases for f :

Case 1 (special case). We assume that

$$f(0) = f'(0) = \ldots = f^{(2n)}(0) = 0. \tag{2.73}$$

Then by the Taylor series expansion of f in $\mathbb{D}$ we have that

$$f(w) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{k!} w^k, \text{ for all } w \in \mathbb{D},$$

so [12, Theorem 10.6] and (2.73) give

$$\begin{aligned} f^{(n)}(w) &= \sum_{k=n}^{+\infty} k(k-1) \cdot \ldots \cdot (k-n+1) \frac{f^{(k)}(0)}{k!} w^{k-n} = \sum_{k=2n}^{+\infty} \frac{f^{(k)}(0)}{(k-n)!} w^{k-n} \\ &= w^n \sum_{k=2n}^{+\infty} \frac{f^{(k)}(0)}{(k-n)!} w^{k-2n} = w^n \sum_{k=0}^{+\infty} \frac{f^{(k+2n)}(0)}{(k+n)!} w^k, \text{ for all } w \in \mathbb{D}. \end{aligned}$$

This means that the function $G(w) = \begin{cases} \frac{f^{(n)}(w)}{w^n}, & \text{if } w \in \mathbb{D} \setminus \{0\} \\ 0, & \text{if } w = 0 \end{cases}$, is holomorphic in $\mathbb{D}$, so there exists a real $M > 0$ such that $|G(w)| \leq M$, for all $w \in \overline{D(0, \frac{1}{2})}$. Consider the function

$$h(w) = \frac{(1-|w|^2)^n f^{(n)}(w)}{\bar{w}^n}, \text{ for all } w \in \mathbb{D} \setminus \{0\}.$$

Let us consider the case where $p \geq 1$. First note that if $w \in \overline{D(0, \frac{1}{2})} \setminus \{0\}$, then

$$|h(w)| = (1-|w|^2)^n |G(w)| \leq M, \tag{2.74}$$

and if $w \in \mathbb{D} \setminus \overline{D(0, \frac{1}{2})}$, then

$$|h(w)| = \frac{(1-|w|^2)^n |f^{(n)}(w)|}{|w|^n} \leq 2^n |g(w)|. \tag{2.75}$$

Thus, (2.74), (2.75) and our hypothesis that $g \in L^p(\mathbb{D}, dA_\alpha)$ give

$$\begin{aligned}\int_{\mathbb{D}} |h(w)|^p \, dA_\alpha(w) &= \int_{\overline{D(0,\frac{1}{2})}} |h(w)|^p \, dA_\alpha(w) + \int_{\mathbb{D}\setminus \overline{D(0,\frac{1}{2})}} |h(w)|^p \, dA_\alpha(w) \\ &\leq \int_{\overline{D(0,\frac{1}{2})}} M^p \, dA_\alpha(w) + 2^{np} \int_{\mathbb{D}\setminus \overline{D(0,\frac{1}{2})}} |g(w)|^p \, dA_\alpha(w) < +\infty,\end{aligned}$$

hence $h \in L^p(\mathbb{D}, dA_\alpha) \subset L^1(\mathbb{D}, dA_\alpha)$. If $\gamma > \alpha > -1$, then

$$\begin{aligned}\int_{\mathbb{D}} \frac{|h(w)|}{|1 - z\bar{w}|^{2+\gamma}} \, dA_\gamma(w) &\leq \frac{\gamma+1}{(1-|z|)^{2+\gamma}} \int_{\mathbb{D}} |h(w)|(1-|w|^2)^\gamma \, dA(w) \\ &\leq \frac{\gamma+1}{(\alpha+1)(1-|z|)^{2+\gamma}} \int_{\mathbb{D}} |h(w)|(\alpha+1)(1-|w|^2)^\alpha \, dA(w) \\ &= \frac{\gamma+1}{(\alpha+1)(1-|z|)^{2+\gamma}} \|h\|_{1,\alpha} < +\infty, \text{ for all } z \in \mathbb{D},\end{aligned}$$

so Proposition 2.5.2 yields that

$$f(z) = \frac{1}{(\gamma+2)\cdot \ldots \cdot (\gamma+n)} \int_{\mathbb{D}} \frac{(1-|w|^2)^\gamma}{(1-z\bar{w})^{2+\gamma}} h(w) \, dA(w), \text{ for all } z \in \mathbb{D}.$$

By the choice of γ and the fact that $h \in L^p(\mathbb{D}, dA_\alpha)$, Theorem A.0.2 gives that $f \in L^p(\mathbb{D}, dA_\alpha)$, and so $f \in \mathcal{A}^p(dA_\alpha)$.

Let $0 < p < 1$. Suppose that $\gamma > -1$ such that $p(\gamma+2) > \alpha + 2$. Also, set $\lambda = p(n+\gamma+2) - 2 > pn + \alpha$, and $J(w) = \displaystyle\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{|1-\bar{z}w|^{p(\gamma+2)}} \, dA(z)$, for all $w \in \mathbb{D}$. Using Lemma A.0.1 and the fact that J is bounded on $D\big(0, \frac{1}{2}\big)$, we derive that there exists a constant $C_5 = C_5(p, \alpha, \gamma) > 0$ such that

$$J(w) \leq C_5 \cdot \frac{1}{(1-|w|^2)^{\lambda - pn - \alpha}}, \text{ for all } w \in \mathbb{D}. \tag{2.76}$$

By [6, 2.37a The Fubini-Tonelli Theorem] and (2.76),

$$\begin{aligned}\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f^{(n)}(w)|^p (1-|w|^2)^\lambda}{|1-z\bar{w}|^{p(2+\gamma)}} \, dA(w) dA_\alpha(z) &= \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f^{(n)}(w)|^p (1-|w|^2)^\lambda}{|1-z\bar{w}|^{p(2+\gamma)}} \, dA_\alpha(z) dA(w) \\ &= \int_{\mathbb{D}} (\alpha+1)|f^{(n)}(w)|^p (1-|w|^2)^\lambda J(w) \, dA(w) \\ &\leq C_5 \int_{\mathbb{D}} |f^{(n)}(w)|^p (1-|w|^2)^{pn} \, dA_\alpha(w) \\ &= C_5 \int_{\mathbb{D}} |g(w)|^p \, dA_\alpha(w) < +\infty,\end{aligned} \tag{2.77}$$

which means that

$$\int_{\mathbb{D}} \frac{|f^{(n)}(w)|^p (1-|w|^2)^\lambda}{|1-z\bar{w}|^{p(2+\gamma)}} \, dA(w) < +\infty, \text{ for almost every } z \in \mathbb{D}. \tag{2.78}$$

Using the same arguments as in the proof of (A.15), we get that (2.78) holds for all $z \in \mathbb{D}$. By Lemma 2.5.1, there exists a constant $C_6 = C_6(\gamma, n) > 0$ such that, for all $z \in \mathbb{D}$,

$$\begin{aligned}\int_{\mathbb{D}} \frac{|h(w)|}{|1-z\bar{w}|^{2+\gamma}} \, dA_\gamma(w) &= \frac{1}{\gamma+n+1} \int_{\mathbb{D}} \left| \frac{G(w)}{(1-\bar{z}w)^{2+\gamma}} \right| dA_{\gamma+n}(w) \\ &\leq C_6 \int_{\mathbb{D}} \left| w^n \cdot \frac{G(w)}{(1-\bar{z}w)^{2+\gamma}} \right| dA_{\gamma+n}(w) \\ &= C_6 \int_{\mathbb{D}} \left| \frac{f^{(n)}(w)}{(1-\bar{z}w)^{2+\gamma}} \right| dA_{\gamma+n}(w),\end{aligned} \tag{2.79}$$

and Proposition 2.1.9 and (2.78) give that, for all $z \in \mathbb{D}$,

$$\int_{\mathbb{D}} \left| \frac{f^{(n)}(w)}{(1-\bar{z}w)^{2+\gamma}} \right| dA_{\gamma+n}(w) \leq \frac{\gamma+n+1}{\lambda+1} \left(\int_{\mathbb{D}} \frac{|f^{(n)}(w)|^p}{|1-z\bar{w}|^{p(2+\gamma)}} \, dA_\lambda(w) \right)^{1/p} < +\infty. \tag{2.80}$$

Thus, (2.79), (2.80) and Proposition 2.5.2 imply that

$$f(z) = \frac{1}{(\gamma+1) \cdot \ldots \cdot (\gamma+n)} \int_{\mathbb{D}} \frac{1}{(1-z\bar{w})^{2+\gamma}} h(w) \, dA_\gamma(w), \text{ for all } z \in \mathbb{D}. \tag{2.81}$$

By (2.79), (2.80) and (2.81),

$$|f(z)|^p \leq C_7 \int_{\mathbb{D}} \frac{|f^{(n)}(w)|^p (1-|w|^2)^\lambda}{|1-z\bar{w}|^{p(2+\gamma)}} \, dA(w), \text{ for all } z \in \mathbb{D}, \tag{2.82}$$

where $C_7 = C_7(p,\gamma,n) = \dfrac{C_6^p (\gamma+n+1)^p (\lambda+1)^{1-p}}{(\gamma+1)^p \cdot \ldots \cdot (\gamma+n)^p} > 0$. By (2.82) and (2.77),

$$\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) \leq C_7 \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f^{(n)}(w)|^p (1-|w|^2)^\lambda}{|1-z\bar{w}|^{p(2+\gamma)}} \, dA(w) dA_\alpha(z) < +\infty,$$

which means that $f \in L^p(\mathbb{D}, dA_\alpha)$, and so $f \in \mathcal{A}^p(dA_\alpha)$.

Case 2 (general case). In this case we assume only that $f \in H(\mathbb{D})$. Then, by the Taylor series expansion of f in $\mathbb{D}$,

$$f(z) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{k!} z^k, \text{ for all } z \in \mathbb{D}.$$

Consider the polynomial $p_{2n}(z) = \sum_{k=0}^{2n} \frac{f^{(k)}(0)}{k!} z^k$ and the function $F(z) = f(z) - p_{2n}(z)$, for all $z \in \mathbb{D}$. Note that for all $0 \leq m \leq 2n$,

$$F^{(m)}(z) = f^{(m)}(z) - \sum_{k=m}^{2n} \frac{f^{(k)}(0)}{(k-m)!} z^{k-m}, \text{ for all } z \in \mathbb{D}, \tag{2.83}$$

so

$$F(0) = F'(0) = \ldots = F^{(2n)}(0) = 0.$$

Also, consider the function

$$G(z) = (1-|z|^2)^n F^{(n)}(z) \overset{(2.83)}{=} g(z) - (1-|z|^2)^n \sum_{k=n}^{2n} \frac{f^{(k)}(0)}{(k-n)!} z^{k-n}, \text{ for all } z \in \mathbb{D}.$$

Then

$$\int_{\mathbb{D}} (1-|z|^2)^{np} \left| \sum_{k=n}^{2n} \frac{f^{(k)}(0)}{(k-n)!} z^{k-n} \right|^p dA_\alpha(z) \leq \frac{\alpha+1}{\alpha+np+1} \int_{\mathbb{D}} \left(\sum_{k=n}^{2n} \frac{|f^{(k)}(0)|}{(k-n)!} \right)^p dA_{\alpha+np}(z) < +\infty,$$

so $G \in L^p(\mathbb{D}, dA_\alpha)$. By the special case, $F \in \mathcal{A}^p(dA_\alpha)$, and since $p_{2n} \in H^\infty \subset \mathcal{A}^p(dA_\alpha)$, we finally get that $f \in \mathcal{A}^p(dA_\alpha)$.

□

2.6 Preparation for the Atomic Decomposition

Let $p \geq 1$ and $b > \frac{2}{p}$. For each $a \in \mathbb{D}$, consider the function

$$f_a(z) = \frac{(1-|a|^2)^{(pb-2)/p}}{(1-\bar{a}z)^b}, \text{ for all } z \in \mathbb{D}, \tag{2.84}$$

which is well-defined in $\mathbb{D}$ (see [14, Section X.5]), and holomorphic in $\mathbb{D}$. Note, also, that for all $a \in \mathbb{D}$,

$$\int_{\mathbb{D}} |f_a(z)|^p \, dA(z) = \int_{\mathbb{D}} \frac{(1-|a|^2)^{pb-2}}{|1-\bar{a}z|^{pb}} \, dA(z) \leq \int_{\mathbb{D}} \frac{(1-|a|^2)^{pb-2}}{(1-|a|)^{pb}} \, dA(z) < +\infty,$$

so $f_a \in \mathcal{A}^p(dA)$.

Proposition 2.6.1. *There exist constants* $c = c(p,b)$, $C = C(p,b) > 0$ *such that*

$$c \leq \|f_a\|_p \leq C, \textit{ for all } a \in \mathbb{D}.$$

Proof. Let $a \in \mathbb{D}$.
Case 1. $a \in \mathbb{D} \setminus D\big(0, \frac{1}{2}\big)$. Since $pb - 2 > 0$, by Lemma A.0.1 there exist constants $\tilde{c} = \tilde{c}(p,b)$, $\tilde{C} = \tilde{C}(p,b) > 0$ such that

$$\frac{\tilde{c}}{(1-|a|^2)^{pb-2}} \leq \int_{\mathbb{D}} \frac{1}{|1-\bar{a}z|^{pb}} \, dA(z) \leq \frac{\tilde{C}}{(1-|a|^2)^{pb-2}},$$

hence

$$\tilde{c}^{1/p} \leq \|f_a\|_p \leq \tilde{C}^{1/p}. \tag{2.85}$$

Case 2. $a \in D\big(0, \frac{1}{2}\big)$. Then

$$|1-\bar{a}z| \leq 1 + |a||z| \leq 1 + \frac{1}{2}, \text{ for all } z \in \mathbb{D},$$

so

$$\|f_a\|_p^p \geq \int_{\mathbb{D}} \frac{(1-|a|^2)^{pb-2}}{\big(\frac{3}{2}\big)^{pb}} \, dA(z) \geq \left(\frac{2}{3}\right)^{pb} \left(\frac{3}{4}\right)^{pb-2}, \tag{2.86}$$

and

$$|1-\bar{a}z| \geq 1 - |a||z| \geq 1 - \frac{1}{2}, \text{ for all } z \in \mathbb{D},$$

so

$$\|f_a\|_p^p \leq \int_{\mathbb{D}} \frac{(1-|a|^2)^{pb-2}}{\big(\frac{1}{2}\big)^{pb}} \, dA(z) \leq 2^{pb}. \tag{2.87}$$

In any case, if $c = c(p,b) := \min\left\{\tilde{c}^{1/p}, \big(\frac{2}{3}\big)^b \big(\frac{3}{4}\big)^{b-\frac{2}{p}}\right\} > 0$ and $C = C(p,b) := \max\left\{\tilde{C}^{1/p}, 2^b\right\} > 0$, then (2.85), (2.86) and (2.87) give

$$c \leq \|f_a\|_p \leq C.$$

□

Lemma 2.6.2. *Suppose* $R > 0$ *and* $b \in \mathbb{R}$. *Then there exists a constant* $C = C(R,b) > 0$ *such that*

$$\left|\frac{(1-z\bar{u})^b}{(1-z\bar{v})^b} - 1\right| \leq C\beta(u,v), \textit{ for all } z,u,v \in \mathbb{D}, \textit{ with } \beta(u,v) \leq R. \tag{2.88}$$

In particular, there exists a constant $\tilde{C} = \tilde{C}(R,b) > 0$ *such that*

$$\frac{1}{\tilde{C}} \leq \frac{|1-z\bar{u}|^b}{|1-z\bar{v}|^b} \leq \tilde{C}, \textit{ for all } z,u,v \in \mathbb{D}, \textit{ with } \beta(u,v) \leq R. \tag{2.89}$$

Proof. Set $r := \tanh(R) \in (0,1)$, and let $z, u, v \in \mathbb{D}$ with $\beta(u,v) \le R$. Then there exist $w, \lambda \in \mathbb{D}$ such that $v = \phi_u(w)$ and $z = \phi_u(\lambda)$. By (1.14) and Proposition 1.2.2,

$$\tanh^{-1}(|w|) = \beta(0,w) = \beta(\phi_u(0), \phi_u(w)) = \beta(u,v) \le R,$$

so $|w| \le r$, and hence $|u\bar{w}| \le r$ and $|\lambda\bar{w}| \le r$. Lemma 2.4.2 now gives that there exists a constant $C_1 = C_1(r,b) > 0$ such that

$$|(1-u\bar{w})^b - (1-\lambda\bar{w})^b| \le C_1|u\bar{w} - \lambda\bar{w}| \le 2C_1|w|. \tag{2.90}$$

Also, it follows from a straightforward calculation that

$$\frac{(1-z\bar{u})^b}{(1-z\bar{v})^b} - 1 = \frac{(1-\phi_u(\lambda)\bar{u})^b}{(1-\phi_u(\lambda)\overline{\phi_u(w)})^b} - 1 = \frac{(1-u\bar{w})^b - (1-\lambda\bar{w})^b}{(1-\lambda\bar{w})^b},$$

and, since $1 - r \le 1 - |\lambda||w| \le |1-\lambda\bar{w}| \le 2$,

$$\left|\frac{(1-z\bar{u})^b}{(1-z\bar{v})^b} - 1\right| = \frac{|(1-u\bar{w})^b - (1-\lambda\bar{w})^b|}{|1-\lambda\bar{w}|^b} \overset{(2.90)}{\le} \frac{2C_1|w|}{|1-\lambda\bar{w}|^b} \le C_2|w|, \tag{2.91}$$

where $C_2 = C_2(R,b) > 0$. Finally, (1.16) yields that $|w| \le 2\beta(0,w) = 2\beta(u,v)$, so we get (2.88) for $C = C(R,b) = 2C_2 > 0$.

Regarding (2.89), note that

$$\left|\frac{(1-z\bar{u})^b}{(1-z\bar{v})^b}\right| - 1 \le \left|\frac{(1-z\bar{u})^b}{(1-z\bar{v})^b} - 1\right| \overset{(2.88)}{\le} C\beta(u,v) \le CR,$$

hence

$$\frac{|1-z\bar{u}|^b}{|1-z\bar{v}|^b} \le \tilde{C}, \tag{2.92}$$

where $\tilde{C} = \tilde{C}(R,b) = CR + 1 > 0$. Since (2.92) holds for all $u, v \in \mathbb{D}$ that satisfy $\beta(u,v) \le R$, we can replace u by v, and v by u to get

$$\frac{|1-z\bar{v}|^b}{|1-z\bar{u}|^b} \le \tilde{C}. \tag{2.93}$$

(2.89) follows from (2.92) and (2.93). □

Lemma 2.6.3. *Let $p \ge 1$. Then there exists a constant $C = C(p) > 0$ such that*

$$\int_{D_h(0,r)} |f(z) - f(0)|\, dA(z) \le Cr^3 \Big(\int_{D_h(0,1)} |f(z)|^p\, dA(z) \Big)^{1/p} \tag{2.94}$$

and

$$\int_{D_h(0,r)} |f(z)|\, dA(z) \le Cr^2 \Big(\int_{D_h(0,1)} |f(z)|^p\, dA(z) \Big)^{1/p}, \tag{2.95}$$

for all $r \in (0,1)$ and $f \in H(\mathbb{D})$.

Proof. Let $f \in H(\mathbb{D})$ and suppose first that $0 < r < \frac{1}{3}$. By Proposition 1.3.5, $D_h(0,r) = D(0,\tanh(r))$ and $D_h(0,1) = D(0,\tanh(1))$. If we set $t = \tanh(\frac{1}{2}) \in (\tanh(\frac{1}{3}), \tanh(1))$, then Cauchy's formula for a circle ([14, Section VII.5]) implies that for all $z \in D_h(0,r)$,

$$\begin{aligned} |f(z)| &= \left| \frac{1}{2\pi i} \int_{|\zeta|=t} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| = \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f(te^{i\theta})}{te^{i\theta} - z} \cdot tie^{i\theta}\, d\theta \right| \\ &\le \frac{t}{2\pi} \int_0^{2\pi} \frac{|f(te^{i\theta})|}{|te^{i\theta} - z|}\, d\theta \le C_1 \int_0^{2\pi} |f(te^{i\theta})|\, d\theta, \end{aligned} \tag{2.96}$$

where $C_1 = \dfrac{t}{2\pi \cdot \text{dist}\big(C(0, \tanh\left(\frac{1}{3}\right)), C(0,t)\big)} > 0.$
Now, consider the function $F(z) = f(\tanh(1)z)$, for all $z \in \mathbb{D}$. Then $F \in H(\mathbb{D})$ and by the Change of Variables Theorem ([6, Theorem 2.47a]),

$$\begin{aligned}\int_{\mathbb{D}} |F(z)|^p \, dA(z) &= \int_{\mathbb{D}} |f(\tanh(1)z)|^p \, dA(z) = \int_{D(0,\tanh(1))} |f(z)|^p \cdot \frac{1}{|\tanh(1)|^2} \, dA(z) \\ &\leq \frac{1}{\tanh^2(1)} \int_{D_h(0,1)} \Big(\max_{D_h(0,1)} |f| \Big)^p \, dA(z) < +\infty,\end{aligned}$$

so $F \in \mathcal{A}^p(dA)$ and $\|F\|_p = \dfrac{1}{(\tanh(1))^{2/p}} \Big(\int_{D_h(0,1)} |f(z)|^p \, dA(z) \Big)^{1/p}$. By Theorem 2.1.4,

$$|f(\tanh(1)z)| \leq \frac{1}{(\tanh(1))^{2/p}} \Big(\int_{D_h(0,1)} |f(w)|^p \, dA(w) \Big)^{1/p} \frac{1}{(1-|z|^2)^{2/p}}, \text{ for all } z \in \mathbb{D}. \tag{2.97}$$

By setting $z = \frac{\zeta}{\tanh(1)}$ in (2.97), where $\zeta \in C(0,t)$, we get

$$|f(\zeta)| \leq C_2 \Big(\int_{D_h(0,1)} |f(w)|^p \, dA(w) \Big)^{1/p}, \text{ for all } \zeta \in C(0,t), \tag{2.98}$$

where $C_2 = C_2(p) = \left(\frac{\tanh(1)}{\tanh^2(1) - t^2} \right)^{2/p} > 0$. Using (2.98), (2.96) becomes

$$|f(z)| \leq 2\pi C_1 C_2 \Big(\int_{D_h(0,1)} |f(w)|^p \, dA(w) \Big)^{1/p}, \text{ for all } z \in D_h(0,r), \tag{2.99}$$

and by integrating each side of (2.99), we have

$$\begin{aligned}\int_{D_h(0,r)} |f(z)| \, dA(z) &\leq 2\pi C_1 C_2 \Big(\int_{D_h(0,1)} |f(w)|^p \, dA(w) \Big)^{1/p} \int_{D_h(0,r)} dA(z) \\ &= 2\pi C_1 C_2 \Big(\int_{D_h(0,1)} |f(w)|^p \, dA(w) \Big)^{1/p} \int_0^{\tanh(r)} \int_0^{2\pi} \frac{\rho}{\pi} \, d\theta d\rho \\ &= 2\pi C_1 C_2 \tanh^2(r) \Big(\int_{D_h(0,1)} |f(w)|^p \, dA(w) \Big)^{1/p} \\ &\leq 2\pi C_1 C_2 r^2 \Big(\int_{D_h(0,1)} |f(w)|^p \, dA(w) \Big)^{1/p},\end{aligned} \tag{2.100}$$

because $\tanh(x) \leq x$, for all $x \geq 0$.
Working similarly, we can derive that there exists a constant $\tilde{C}_1 = \tilde{C}_1(p) > 0$ such that

$$\int_{D_h(0,r)} |f(z) - f(0)| \, dA(z) \leq \tilde{C}_1 r^3 \Big(\int_{D_h(0,1)} |f(w)|^p \, dA(w) \Big)^{1/p}. \tag{2.101}$$

Suppose now that $\frac{1}{3} \leq r < 1$. Then, if q is the conjugate exponent of p, there exists a constant $C_3 = C_3(q) > 0$ such that

$$C_3 \Big(\frac{1}{3} \Big)^2 \geq (\tanh(1))^{2/q} \geq (\tanh(r))^{2/q}.$$

Thus, by [9, Hölder's Inequalities, p. 213],

$$\begin{aligned}\int_{D_h(0,r)} |f(z)|\,dA(z) &\le \Big(\int_{D_h(0,r)} |f(z)|^p\,dA(z)\Big)^{1/p}\Big(\int_{D_h(0,r)} dA(z)\Big)^{1/q}\\ &\le \Big(\int_{D_h(0,1)} |f(z)|^p\,dA(z)\Big)^{1/p}(\tanh(r))^{2/q}\\ &\le C_3\Big(\frac{1}{3}\Big)^2\Big(\int_{D_h(0,1)} |f(z)|^p\,dA(z)\Big)^{1/p}\\ &\le C_3r^2\Big(\int_{D_h(0,1)} |f(z)|^p\,dA(z)\Big)^{1/p}.\end{aligned} \tag{2.102}$$

Finally, there exists a constant $\tilde{C}_2 > 0$ such that

$$\tilde{C}_2\Big(\frac{1}{3}\Big)^3 \ge 1 \ge r^2,$$

hence, using (2.95) and (2.97),

$$\begin{aligned}\int_{D_h(0,r)} |f(z)-f(0)|\,dA(z) &\le \int_{D_h(0,r)} |f(z)|\,dA(z) + \int_{D_h(0,r)} |f(0)|\,dA(z)\\ &\le Cr^2\Big(\int_{D_h(0,1)} |f(z)|^p\,dA(z)\Big)^{1/p} + |f(0)|\tanh^2(r)\\ &\le \Big(C+\frac{1}{(\tanh(1))^{2/p}}\Big)r^2\Big(\int_{D_h(0,1)} |f(z)|^p\,dA(z)\Big)^{1/p}\\ &\le \Big(C+\frac{1}{(\tanh(1))^{2/p}}\Big)\tilde{C}_2\Big(\frac{1}{3}\Big)^3\Big(\int_{D_h(0,1)} |f(z)|^p\,dA(z)\Big)^{1/p}\\ &\le \Big(C+\frac{1}{(\tanh(1))^{2/p}}\Big)\tilde{C}_2r^3\Big(\int_{D_h(0,1)} |f(z)|^p\,dA(z)\Big)^{1/p}.\end{aligned} \tag{2.103}$$

Thus, (2.100) and (2.102) give (2.95), and (2.101) and (2.103) give (2.94). □

Corollary 2.6.4. *Let $p \ge 1$. Then there exists a constant $C = C(p) > 0$ such that*

$$\int_{D_h(a,r)} |f(z)-f(a)|\,dA(z) \le Cr^3(1-|a|^2)^{2\left(1-\frac{1}{p}\right)}\Big(\int_{D_h(a,1)} |f(z)|^p\,dA(z)\Big)^{1/p} \tag{2.104}$$

and

$$\int_{D_h(a,r)} |f(z)|\,dA(z) \le Cr^2(1-|a|^2)^{2\left(1-\frac{1}{p}\right)}\Big(\int_{D_h(a,1)} |f(z)|^p\,dA(z)\Big)^{1/p}, \tag{2.105}$$

for all $a \in \mathbb{D}$, $r \in (0,1)$ and $f \in H(\mathbb{D})$.

Proof. Let $a \in \mathbb{D}$, $r \in (0,1)$ and $f \in H(\mathbb{D})$. Replace f by $f \circ \phi_a$ in (2.94) to get

$$\int_{D_h(0,r)} |f(\phi_a(z)) - f(\phi_a(0))|\,dA(z) \le Cr^3\Big(\int_{D_h(0,1)} |f(\phi_a(z))|^p\,dA(z)\Big)^{1/p}, \tag{2.106}$$

for some constant $C = C(p) > 0$. Using the Change of Variables Theorem ([6, Theorem 2.47a]), Lemma 1.3.9 and Proposition 1.1.2 (iii),

$$\begin{aligned}\int_{D_h(0,r)} |f(\phi_a(z)) - f(\phi_a(0))|\,dA(z) &= \int_{D_h(a,r)} |f(z)-f(a)|\cdot|\phi_a'(z)|^2\,dA(z)\\ &= \int_{D_h(a,r)} |f(z)-f(a)|\cdot\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}\,dA(z)\end{aligned} \tag{2.107}$$

and

$$\begin{aligned} \int_{D_h(0,1)} |f(\phi_a(z))|^p \, dA(z) &= \int_{D_h(a,1)} |f(z)|^p \cdot |\phi_a'(z)|^2 \, dA(z) \\ &= \int_{D_h(a,1)} |f(z)|^p \cdot \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \, dA(z). \end{aligned} \tag{2.108}$$

By Proposition 1.3.11 (i), $|1-\bar{a}z| \sim 1-|a|^2$, for all $z \in D_h(a,1)$, so there exist constants C_1, $C_2 > 0$, independent of a, z, r, f, p, such that

$$C_1 \frac{1}{(1-|a|^2)^2} \le \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \le C_2 \frac{1}{(1-|a|^2)^2}, \text{ for all } z \in D_h(a,1). \tag{2.109}$$

By (2.106), (2.107), (2.108) and (2.109),

$$\begin{aligned} C_1 \int_{D_h(a,r)} \frac{|f(z)-f(a)|}{(1-|a|^2)^2} \, dA(z) &\le \int_{D_h(a,r)} |f(z)-f(a)| \cdot \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \, dA(z) \\ &\le Cr^3 \Big(\int_{D_h(a,1)} |f(z)|^p \cdot \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \, dA(z) \Big)^{1/p} \\ &\le Cr^3 \Big(C_2 \int_{D_h(a,1)} \frac{|f(z)|^p}{(1-|a|^2)^2} \, dA(z) \Big)^{1/p}, \end{aligned}$$

and (2.104) follows.

Working similarly, we can derive (2.105). □

For any $r \in (0,1]$, fix an r-lattice $\{a_n\} \subset \mathbb{D}$ in the hyperbolic metric, and a disjoint decomposition $\{D_n\}$ of $\mathbb{D}$ satisfying the conditions of Proposition 1.4.8. For any parameter $b > 1$, we define an operator $S : \mathcal{A}^1(dA_{b-2}) \to H(\mathbb{D})$ given by

$$Sf(z) = \sum_{n=1}^{+\infty} \frac{A_{b-2}(D_n) f(a_n)}{(1-z\bar{a}_n)^b}, \text{ for all } z \in \mathbb{D} \text{ and } f \in \mathcal{A}^1(dA_{b-2}). \tag{2.110}$$

Proposition 2.6.5. *The operator S defined in (2.110) is well-defined.*

Proof. Let $z \in \mathbb{D}$ and $f \in \mathcal{A}^1(dA_{b-2})$. By Proposition 1.4.8, $D_n \subset D_h(a_n, r) \subset D_h(a_n, 1)$, for all $n \in \mathbb{N}$, so $A_{b-2}(D_n) \le A_{b-2}(D_h(a_n,1))$, for all $n \in \mathbb{N}$. Now, Proposition 1.3.11 (ii) implies that $A_{b-2}(D_h(a_n,1)) \sim (1-|a_n|^2)^b$, for all $n \in \mathbb{N}$, which means that there exists a constant $C_1 = C_1(b) > 0$ such that

$$A_{b-2}(D_n) \le A_{b-2}(D_h(a_n,1)) \le C_1(1-|a_n|^2)^b, \text{ for all } n \in \mathbb{N}. \tag{2.111}$$

Also, Proposition 2.1.11 yields that there exists a constant $C_2 = C_2(b,r) > 0$ such that

$$\sum_{n=1}^{+\infty} (1-|a_n|^2)^b \cdot |f(a_n)| \le C_2 \int_{\mathbb{D}} |f(w)| \, dA_{b-2}(w) < +\infty. \tag{2.112}$$

By (2.111) and (2.112),

$$\sum_{n=1}^{+\infty} \frac{A_{b-2}(D_n)|f(a_n)|}{|1-z\bar{a}_n|^b} \le \frac{C_1 C_2}{(1-|z|)^b} \int_{\mathbb{D}} |f(w)| \, dA_{b-2}(w) < +\infty.$$

Thus, the series in (2.110) converges for all $z \in \mathbb{D}$.

Consider the functions $g_n(z) = \dfrac{A_{b-2}(D_n) f(a_n)}{(1 - z\bar{a}_n)^b}$, for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$, which are holomorphic in $\mathbb{D}$, and let K be a compact subset of $\mathbb{D}$. Then there exists a real $R \in (0,1)$ such that $K \subset D(0,R)$, hence

$$|g_n(z)| = \frac{A_{b-2}(D_n)|f(a_n)|}{|1 - z\bar{a}_n|^b} \le \frac{A_{b-2}(D_n)|f(a_n)|}{(1-R)^b} \overset{(2.111)}{\le} \frac{C_1}{(1-R)^b}(1-|a_n|^2)^b \cdot |f(a_n)|,$$

for all $z \in K$ and $n \in \mathbb{N}$. Since

$$\sum_{n=1}^{+\infty} \frac{C_1}{(1-R)^b}(1-|a_n|^2)^b \cdot |f(a_n)| \overset{(2.112)}{\le} \frac{C_1 C_2}{(1-R)^b} \int_{\mathbb{D}} |f(w)|\, dA_{b-2}(w) < +\infty,$$

by the Weierstrass criterion for the uniform convergence of series, $\sum_{n=1}^{+\infty} g_n$ converges uniformly on K. Thus, Weierstrass Convergence Theorem ([12, Theorem 10.28]) implies that $\sum_{n=1}^{+\infty} g_n \in H(\mathbb{D})$. □

Lemma 2.6.6. *Let $p \ge 1$. For any $r \in (0,1)$ and $b > 1$ consider the operator S given in (2.110). Then there exists a constant $C = C(p,b) > 0$ such that*

$$|f(z) - Sf(z)| \le C r^3 \sum_{k=1}^{+\infty} \frac{(1-|a_k|^2)^{(pb-2)/p}}{|1 - \bar{a}_k z|^b} \left(\int_{D_h(a_k,1)} |f(w)|^p \, dA(w) \right)^{1/p}, \tag{2.113}$$

for all $z \in \mathbb{D}$ and $f \in \mathcal{A}^1(dA_{b-2})$.

Proof. Let $z \in \mathbb{D}$ and $f \in \mathcal{A}^1(dA_{b-2})$. By Theorem 2.3.6,

$$f(z) = \int_{\mathbb{D}} f(w) K_{b-2}(z,w)\, dA_{b-2}(w) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^b}\, dA_{b-2}(w),$$

and since $\{D_k\}$ is a partition of $\mathbb{D}$,

$$\begin{aligned} f(z) - Sf(z) &= \int_{\bigcup_{k=1}^{+\infty} D_k} \frac{f(w)}{(1-z\bar{w})^b}\, dA_{b-2}(w) - \sum_{k=1}^{+\infty} \frac{A_{b-2}(D_k) f(a_k)}{(1 - z\bar{a}_k)^b} \\ &= \sum_{k=1}^{+\infty} \int_{D_k} \frac{f(w)}{(1-z\bar{w})^b}\, dA_{b-2}(w) - \sum_{k=1}^{+\infty} \int_{D_k} \frac{f(a_k)}{(1-z\bar{a}_k)^b}\, dA_{b-2}(w) \\ &= \sum_{k=1}^{+\infty} \int_{D_k} \left(\frac{f(w)}{(1-z\bar{w})^b} - \frac{f(a_k)}{(1-z\bar{a}_k)^b} \right) dA_{b-2}(w). \end{aligned}$$

Thus,

$$\begin{aligned} |f(z) - Sf(z)| &\le \sum_{k=1}^{+\infty} \int_{D_k} \left| \frac{f(w)}{(1-z\bar{w})^b} - \frac{f(a_k)}{(1-z\bar{a}_k)^b} \right| dA_{b-2}(w) \\ &\le \sum_{k=1}^{+\infty} \int_{D_k} \left(\left| \frac{f(w)}{(1-z\bar{w})^b} - \frac{f(w)}{(1-z\bar{a}_k)^b} \right| + \left| \frac{f(w) - f(a_k)}{(1-z\bar{a}_k)^b} \right| \right) dA_{b-2}(w) \\ &= I(z) + J(z), \end{aligned} \tag{2.114}$$

where

$$I(z) = \sum_{k=1}^{+\infty} \frac{1}{|1 - z\bar{a}_k|^b} \int_{D_k} \left| \frac{(1-z\bar{a}_k)^b}{(1-z\bar{w})^b} - 1 \right| \cdot |f(w)|\, dA_{b-2}(w) \tag{2.115}$$

and

$$J(z) = \sum_{k=1}^{+\infty} \frac{1}{|1 - z\bar{a}_k|^b} \int_{D_k} |f(w) - f(a_k)|\, dA_{b-2}(w). \tag{2.116}$$

For each $k \in \mathbb{N}$, let

$$I_k = \int_{D_k} \left| \frac{(1 - z\bar{a}_k)^b}{(1 - z\bar{w})^b} - 1 \right| \cdot |f(w)| \, dA_{b-2}(w)$$

and

$$J_k = \int_{D_k} |f(w) - f(a_k)| \, dA_{b-2}(w).$$

By Proposition 1.4.8,

$$I_k \leq \int_{D_h(a_k,r)} \left| \frac{(1 - z\bar{a}_k)^b}{(1 - z\bar{w})^b} - 1 \right| \cdot |f(w)|(b-1)(1 - |w|^2)^{b-2} \, dA(w) \tag{2.117}$$

and

$$J_k \leq \int_{D_h(a_k,r)} |f(w) - f(a_k)|(b-1)(1 - |w|^2)^{b-2} \, dA(w). \tag{2.118}$$

Now, if $w \in D_h(a_k, r)$, then $w \in D_h(a_k, 1)$, so Proposition 1.3.11 (i) implies that there exists a constant $C_1 = C_1(b) > 0$, independent of w, a_k, r, such that

$$(1 - |w|^2)^{b-2} \leq C_1(1 - |a_k|^2)^{b-2}, \tag{2.119}$$

for all $w \in D_h(a_k, 1)$ and $k \in \mathbb{N}$. Also, by Lemma 2.6.2, there exists a constant $C_2 = C_2(b) > 0$ such that

$$\left| \frac{(1 - z\bar{a}_k)^b}{(1 - z\bar{w})^b} - 1 \right| \leq C_2 \beta(a_k, w), \text{ for all } w \in D_h(a_k, 1) \text{ and } k \in \mathbb{N}. \tag{2.120}$$

Thus, by (2.119) and (2.120), (2.117) and (2.118) become

$$\begin{aligned} I_k &\leq \int_{D_h(a_k,r)} C_2 \beta(a_k, w)|f(w)|(b-1)C_1(1 - |a_k|^2)^{b-2} \, dA(w) \\ &\leq C_1 C_2 r(b-1)(1 - |a_k|^2)^{b-2} \int_{D_h(a_k,r)} |f(w)| \, dA(w) \end{aligned} \tag{2.121}$$

and

$$J_k \leq C_1(b-1)(1 - |a_k|^2)^{b-2} \int_{D_h(a_k,r)} |f(w) - f(a_k)| \, dA(w), \tag{2.122}$$

respectively, for all $k \in \mathbb{N}$. By Corollary 2.6.4, there exists a constant $C_3 = C_3(p) > 0$ such that

$$\int_{D_h(a_k,r)} |f(w)| \, dA(w) \leq C_3 r^2 (1 - |a_k|^2)^{2\left(1-\frac{1}{p}\right)} \left(\int_{D_h(a_k,1)} |f(w)|^p \, dA(w) \right)^{1/p} \tag{2.123}$$

and

$$\int_{D_h(a_k,r)} |f(w) - f(a_k)| \, dA(w) \leq C_3 r^3 (1 - |a_k|^2)^{2\left(1-\frac{1}{p}\right)} \left(\int_{D_h(a_k,1)} |f(w)|^p \, dA(w) \right)^{1/p}, \tag{2.124}$$

for all $k \in \mathbb{N}$. Therefore, for all $k \in \mathbb{N}$,

$$\begin{aligned} I_k &\overset{(2.121)}{\leq} C_1 C_2 r(b-1)(1 - |a_k|^2)^{b-2} \int_{D_h(a_k,r)} |f(w)| \, dA(w) \\ &\overset{(2.123)}{\leq} C_1 C_2 C_3 r^3 (b-1)(1 - |a_k|^2)^{b-\frac{2}{p}} \left(\int_{D_h(a_k,1)} |f(w)|^p \, dA(w) \right)^{1/p} \\ &= \tilde{C} r^3 (1 - |a_k|^2)^{b-\frac{2}{p}} \left(\int_{D_h(a_k,1)} |f(w)|^p \, dA(w) \right)^{1/p}, \end{aligned} \tag{2.125}$$

where $\tilde{C} = \tilde{C}(p, b) = C_1C_2C_3(b-1) > 0$, and, similarly, (2.122) and (2.124) give

$$J_k \le \hat{C}r^3(1-|a_k|^2)^{b-\frac{2}{p}}\left(\int_{D_h(a_k,1)} |f(w)|^p\,dA(w)\right)^{1/p}, \tag{2.126}$$

where $\hat{C} = \hat{C}(p, b) = C_1C_3(b-1) > 0$. Using (2.125) and (2.126), (2.115) and (2.116) become

$$I(z) \le \tilde{C}r^3 \sum_{k=1}^{+\infty} \frac{(1-|a_k|^2)^{(pb-2)/p}}{|1-z\bar{a}_k|^b}\left(\int_{D_h(a_k,1)} |f(w)|^p\,dA(w)\right)^{1/p} \tag{2.127}$$

and

$$J(z) \le \hat{C}r^3 \sum_{k=1}^{+\infty} \frac{(1-|a_k|^2)^{(pb-2)/p}}{|1-z\bar{a}_k|^b}\left(\int_{D_h(a_k,1)} |f(w)|^p\,dA(w)\right)^{1/p}. \tag{2.128}$$

If $C = C(p, b) := 2\max\{\tilde{C}, \hat{C}\} > 0$, then (2.113) follows from (2.114), (2.127) and (2.128). □

2.7 Atomic Decomposition on Bergman Spaces

Theorem 2.7.1. *Let $p \ge 1$ and $b > 2 \ge 1 + \frac{1}{p}$. Then there exists a constant $\sigma = \sigma(p, b) > 0$ such that for any r-lattice $\{a_k\}$ in the hyperbolic metric, where $0 < r < \sigma$, the space $\mathcal{A}^p(dA)$ consists exactly of functions of the form*

$$f(z) = \sum_{k=1}^{+\infty} c_k \frac{(1-|a_k|^2)^{(pb-2)/p}}{(1-z\bar{a}_k)^b}, \text{ for all } z \in \mathbb{D}, \tag{2.129}$$

where $\{c_k\} \in l^p$, and the series (2.129) converges in norm in $\mathcal{A}^p(dA)$.

Proof. We will prove the Theorem in steps.

Step 1. The series in (2.129) converges for all $z \in \mathbb{D}$ and defines a function $f \in \mathcal{A}^p(dA)$. Also, the series in (2.129) converges in norm to f in $\mathcal{A}^p(dA)$.

Indeed, let $\{c_k\} \in l^p$ and $\{a_k\}$ be any r-lattice in the hyperbolic metric, where $0 < r < 1$. Also, consider the functions

$$f_k(z) = \frac{(1-|a_k|^2)^{(pb-2)/p}}{(1-z\bar{a}_k)^b}, \text{ for all } z \in \mathbb{D} \text{ and } k \in \mathbb{N}.$$

The functions f_k are of the form (2.84), so $f_k \in \mathcal{A}^p(dA)$, for all $k \in \mathbb{N}$, and Proposition 2.6.1 yields that there exists a constant $C = C(p, b) > 0$ such that

$$\|f_k\|_p^p = \int_{\mathbb{D}} |f_k(z)|^p\,dA(z) \le C, \text{ for all } k \in \mathbb{N}. \tag{2.130}$$

Now, choose a decomposition $\{D_k\}$ of $\mathbb{D}$ according to Proposition 1.4.8 and let χ_k be the characteristic function of D_k. If $z \in \mathbb{D}$, then there exists a unique $m \in \mathbb{N}$ such that $z \in D_m$, so

$$\sum_{k=1}^{+\infty} |c_k| A(D_k)^{-1/p}\chi_k(z) = |c_m| A(D_m)^{-1/p} < +\infty.$$

Thus, we can define a function $F : \mathbb{D} \to [0, +\infty)$ by $F(z) = \sum_{k=1}^{+\infty} |c_k| A(D_k)^{-1/p}\chi_k(z)$, for all $z \in \mathbb{D}$. Then

$$\begin{aligned}\int_{\mathbb{D}} |F(z)|^p\,dA(z) &= \int_{\bigcup_{k=1}^{+\infty} D_k} |F(z)|^p\,dA(z) = \sum_{k=1}^{+\infty}\int_{D_k} |F(z)|^p\,dA(z)\\ &= \sum_{k=1}^{+\infty}\int_{D_k} |c_k|^p A(D_k)^{-1}\,dA(z) = \sum_{k=1}^{+\infty} |c_k|^p < +\infty,\end{aligned}$$

so $F \in L^p(\mathbb{D}, dA)$ and $\|F\|_p^p = \sum_{k=1}^{+\infty} |c_k|^p$. Next, note that the choice of b implies that $p(b-1) > 1$, so Theorem A.0.2 yields that the integral operator $T : L^p(\mathbb{D}, dA) \to L^p(\mathbb{D}, dA)$ defined by

$$Th(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^{b-2}}{|1-z\bar{w}|^b} h(w)\, dA(w), \text{ for all } z \in \mathbb{D} \text{ and } h \in L^p(\mathbb{D}, dA),$$

is bounded on $L^p(\mathbb{D}, dA)$. Using [6, Theorem 2.15], we have that for all $z \in \mathbb{D}$,

$$\begin{aligned} TF(z) &= \int_{\mathbb{D}} \frac{(1-|w|^2)^{b-2}}{|1-z\bar{w}|^b} \cdot \sum_{k=1}^{+\infty} |c_k| A(D_k)^{-1/p} \chi_k(w)\, dA(w) \\ &= \sum_{k=1}^{+\infty} |c_k| A(D_k)^{-1/p} \int_{D_k} \frac{(1-|w|^2)^{b-2}}{|1-z\bar{w}|^b}\, dA(w). \end{aligned} \tag{2.131}$$

Let $k \in \mathbb{N}$ and $z \in \mathbb{D}$. By Proposition 1.4.8, $D_k \subset D_h(a_k, r)$, so, using Lemma 1.3.10 (i),

$$A(D_k) \leq A(D_h(a_k, r)) = \frac{\tanh^2(r)}{(1-|a_k|^2 \tanh^2(r))^2}(1-|a_k|^2)^2 \leq C_1(1-|a_k|^2)^2,$$

where $C_1 = C_1(r) = \dfrac{\tanh^2(r)}{(1-\tanh^2(r))^2} > 0$. Thus,

$$A(D_k)^{-1/p} \geq C_1^{-1/p}(1-|a_k|^2)^{-2/p}. \tag{2.132}$$

Also, $D_k \subset D_h(a_k, r) \subset D_h(a_k, 1)$, hence Proposition 1.3.11 (i) yields that there exists a constant $C_2 = C_2(b) > 0$ such that

$$(1-|w|^2)^{b-2} \geq C_2(1-|a_k|^2)^{b-2}, \text{ for all } w \in D_k, \tag{2.133}$$

and by Lemma 2.6.2, there exists a constant $C_3 = C_3(b) > 0$ such that

$$\frac{1}{|1-z\bar{w}|^b} \geq \frac{C_3}{|1-z\bar{a}_k|^b}, \text{ for all } w \in D_k. \tag{2.134}$$

On the other hand, Proposition 1.4.8 yields that $D_h\left(a_k, \frac{r}{4}\right) \subset D_k$, so using Lemma 1.3.10 (i),

$$A(D_k) \geq A\left(D_h\left(a_k, \frac{r}{4}\right)\right) = \frac{\tanh^2\left(\frac{r}{4}\right)}{\left(1-|a_k|^2 \tanh^2\left(\frac{r}{4}\right)\right)^2}(1-|a_k|^2)^2 \geq C_4(1-|a_k|^2)^2, \tag{2.135}$$

where $C_4 = C_4(r) = \tanh^2\left(\frac{r}{4}\right) > 0$. It follows from (2.133), (2.134) and (2.135) that

$$\begin{aligned} \int_{D_k} \frac{(1-|w|^2)^{b-2}}{|1-z\bar{w}|^b}\, dA(w) &\geq C_2 C_3 \int_{D_k} \frac{(1-|a_k|^2)^{b-2}}{|1-z\bar{a}_k|^b}\, dA(w) \\ &= C_2 C_3 \frac{(1-|a_k|^2)^{b-2}}{|1-z\bar{a}_k|^b} A(D_k) \\ &\geq C_2 C_3 C_4 \frac{(1-|a_k|^2)^b}{|1-z\bar{a}_k|^b}. \end{aligned} \tag{2.136}$$

By (2.131), (2.132) and (2.136),

$$TF(z) \geq \sum_{k=1}^{+\infty} |c_k| C_1^{-1/p} C_2 C_3 C_4 \frac{(1-|a_k|^2)^{(pb-2)/p}}{|1-z\bar{a}_k|^b} = \delta \sum_{k=1}^{+\infty} |c_k| |f_k(z)|, \text{ for all } z \in \mathbb{D},$$

where $\delta = \delta(p, b, r) = C_1^{-1/p} C_2 C_3 C_4 > 0$. Thus,

$$\sum_{k=1}^{+\infty} |c_k||f_k(z)| \leq \frac{1}{\delta} TF(z) < +\infty, \text{ for all } z \in \mathbb{D}, \tag{2.137}$$

because of (A.15), so we can define the functions $f(z) = \sum_{k=1}^{+\infty} c_k f_k(z)$ and $g(z) = \sum_{k=1}^{+\infty} |c_k||f_k(z)|$, for all $z \in \mathbb{D}$. By (2.137) and the fact that T is bounded,

$$\begin{aligned} \int_{\mathbb{D}} |f(z)|^p \, dA(z) &\leq \int_{\mathbb{D}} \left(\sum_{k=1}^{+\infty} |c_k||f_k(z)| \right)^p dA(z) \leq \int_{\mathbb{D}} \frac{1}{\delta^p} (TF(z))^p \, dA(z) \\ &= \frac{1}{\delta^p} \|TF\|_p^p \leq \frac{1}{\delta^p} \|T\|^p \|F\|_p^p < +\infty, \end{aligned} \tag{2.138}$$

so $f \in L^p(\mathbb{D}, dA)$. Similarly, we derive that $g \in L^p(\mathbb{D}, dA)$. It remains to show that the series in (2.129) converges to f in $\mathcal{A}^p(dA)$. Note that

$$\lim_{N \to +\infty} \left| \sum_{k=1}^{N} c_k f_k(z) - f(z) \right|^p = 0,$$

and

$$\left| \sum_{k=1}^{N} c_k f_k(z) - f(z) \right|^p \leq 2^p \left(\sum_{k=1}^{N} |c_k||f_k(z)| \right)^p + 2^p |f(z)|^p \leq 2^p g^p(z) + 2^p |f(z)|^p,$$

for all $z \in \mathbb{D}$ and $N \in \mathbb{N}$. Since $f, g \in L^p(\mathbb{D}, dA)$, by [6, 2.24 The Dominated Convergence Theorem],

$$\lim_{N \to +\infty} \left\| \sum_{k=1}^{N} c_k f_k - f \right\|_p^p = \lim_{N \to +\infty} \int_{\mathbb{D}} \left| \sum_{k=1}^{N} c_k f_k(z) - f(z) \right|^p dA(z) = 0,$$

which means that $f = \sum_{k=1}^{+\infty} c_k f_k$ in $L^p(\mathbb{D}, dA)$. By the fact that $f_k \in \mathcal{A}^p(dA)$, for all $k \in \mathbb{N}$, and Theorem 2.1.7, $f \in \mathcal{A}^p(dA)$.

We have now completed *Step 1* of the proof. Note that, so far, the proof worked for any r-lattice in the hyperbolic metric, with $0 < r < 1$.

Step 2. Every function in $\mathcal{A}^p(dA)$ admits a representation (2.129).

Indeed, let $f \in \mathcal{A}^p(dA)$ and fix an r-lattice $\{a_k\}$ in the hyperbolic metric, where r is sufficiently small and to be specified later. Since $b > 2$,

$$\int_{\mathbb{D}} |f(z)|^p \, dA_{b-2}(z) = \int_{\mathbb{D}} |f(z)|^p (b-1)(1-|z|^2)^{b-2} \, dA(z) \leq (b-1) \int_{\mathbb{D}} |f(z)|^p \, dA(z) < +\infty,$$

hence $f \in \mathcal{A}^p(dA_{b-2})$. By Lemma 2.6.6, there exists a constant $\tilde{C}_1 = \tilde{C}_1(p, b) > 0$ such that

$$|f(z) - Sf(z)| \leq \tilde{C}_1 r^3 \sum_{k=1}^{+\infty} |f_k(z)| \left(\int_{D_h(a_k, 1)} |f(w)|^p \, dA(w) \right)^{1/p}, \text{ for all } z \in \mathbb{D}. \tag{2.139}$$

Set $c_k := \left(\int_{D_h(a_k,1)} |f(w)|^p \, dA(w) \right)^{1/p}$, for all $k \in \mathbb{N}$. By Proposition 1.4.2, there exists a constant $\tilde{C}_2 > 0$, independent of r, such that every point $z \in \mathbb{D}$ belongs to at most $\frac{\tilde{C}_2}{r^2}$ of the sets $D_h(a_k, 1)$; so

$$\begin{aligned} \sum_{k=1}^{+\infty} |c_k|^p &= \sum_{k=1}^{+\infty} \int_{D_h(a_k,1)} |f(w)|^p \, dA(w) = \sum_{k=1}^{+\infty} \int_{\mathbb{D}} |f(w)|^p \chi_{D_h(a_k,1)}(w) \, dA(w) \\ &= \int_{\mathbb{D}} |f(w)|^p \sum_{k=1}^{+\infty} \chi_{D_h(a_k,1)}(w) \, dA(w) \leq \frac{\tilde{C}_2}{r^2} \int_{\mathbb{D}} |f(w)|^p \, dA(w) < +\infty, \end{aligned} \tag{2.140}$$

that is, $\{c_k\} \in l^p$. By *Step 1*, $\sum_{k=1}^{+\infty} |c_k||f_k(z)| < +\infty$, for all $z \in \mathbb{D}$, and (2.138) implies that

$$\int_{\mathbb{D}} \left(\sum_{k=1}^{+\infty} |c_k||f_k(z)| \right)^p dA(z) \leq \frac{1}{\delta^p} \|T\|^p \|F\|_p^p = \frac{1}{\delta^p} \|T\|^p \sum_{k=1}^{+\infty} |c_k|^p < +\infty. \tag{2.141}$$

By (2.139) (2.140) and (2.141),

$$\begin{aligned} \int_{\mathbb{D}} |f(z) - Sf(z)|^p \, dA(z) &\leq \tilde{C}_1^p r^{3p} \int_{\mathbb{D}} \left(\sum_{k=1}^{+\infty} |c_k||f_k(z)| \right)^p dA(z) \\ &\leq \tilde{C}_1^p r^{3p} \cdot \frac{1}{\delta^p} \|T\|^p \sum_{k=1}^{+\infty} |c_k|^p \\ &\leq \tilde{C}_1^p r^{3p-2} \cdot \frac{1}{\delta^p} \|T\|^p \tilde{C}_2 \int_{\mathbb{D}} |f(w)|^p \, dA(w) \\ &= \tilde{C}_1^p \tilde{C}_2 \|T\|^p \frac{C_1}{C_2^p C_3^p C_4^p} r^{3p-2} \int_{\mathbb{D}} |f(w)|^p \, dA(w) \\ &= \tilde{C} \cdot \frac{1}{(1 - \tanh^2(r))^2} \cdot \frac{r^{3p} \tanh^2(r)}{r^2 (\tanh\left(\frac{r}{4}\right))^{2p}} \int_{\mathbb{D}} |f(w)|^p \, dA(w), \end{aligned} \tag{2.142}$$

where $\tilde{C} = \tilde{C}(p,b) = \frac{\tilde{C}_1^p \tilde{C}_2 \|T\|^p}{C_2^p C_3^p} > 0$. Thus, $I - S$ is a bounded linear operator on $\mathcal{A}^p(dA)$, and so S is a bounded linear operator on $\mathcal{A}^p(dA)$. Since

$$\lim_{r \to 0^+} \frac{1}{(1 - \tanh^2(r))^2} \cdot \frac{r^{3p} \tanh^2(r)}{r^2 (\tanh\left(\frac{r}{4}\right))^{2p}} = 0,$$

there exists a real $\sigma = \sigma(p, b) \in (0, 1)$ such that

$$\tilde{C} \cdot \frac{1}{(1 - \tanh^2(r))^2} \cdot \frac{r^{3p} \tanh^2(r)}{r^2 (\tanh\left(\frac{r}{4}\right))^{2p}} < 1, \text{ for all } r \in (0, \sigma). \tag{2.143}$$

By (2.142) and (2.143), $\|I - S\| < 1$, so Proposition A.0.3 yields that S is invertible, that is there exists a bounded linear operator S^{-1} on $\mathcal{A}^p(dA)$ such that $SS^{-1} = I = S^{-1}S$. Set $g = S^{-1}f \in \mathcal{A}^p(dA)$. Then

$$f(z) = Sg(z) = \sum_{k=1}^{+\infty} \frac{A_{b-2}(D_k) g(a_k)}{(1 - z\bar{a}_k)^b} = \sum_{k=1}^{+\infty} \frac{A_{b-2}(D_k) g(a_k)}{(1 - |a_k|^2)^{(pb-2)/p}} \cdot \frac{(1 - |a_k|^2)^{(pb-2)/p}}{(1 - z\bar{a}_k)^b}, \tag{2.144}$$

for all $z \in \mathbb{D}$. Set $d_k = \dfrac{A_{b-2}(D_k) g(a_k)}{(1 - |a_k|^2)^{(pb-2)/p}}$, for all $k \in \mathbb{N}$, and note that, by Proposition 1.3.11 (ii), there exists a constant $C_5 = C_5(b) > 0$ such that

$$A_{b-2}(D_k) \leq A_{b-2}(D_h(a_k, 1)) \leq C_5 (1 - |a_k|^2)^b, \text{ for all } k \in \mathbb{N}. \tag{2.145}$$

Using (2.145),

$$\begin{aligned} \sum_{k=1}^{+\infty} |d_k|^p &= \sum_{k=1}^{+\infty} \frac{(A_{b-2}(D_k))^p |g(a_k)|^p}{(1 - |a_k|^2)^{pb-2}} \\ &\leq C_5^p \sum_{k=1}^{+\infty} (1 - |a_k|^2)^2 |g(a_k)|^p \\ &\leq C_5^p C_6 \int_{\mathbb{D}} |g(z)|^p \, dA(z) < +\infty, \end{aligned} \tag{2.146}$$

where $C_6 = C_6(r) > 0$ is the constant appearing in Proposition 2.1.11. It follows from (2.146) and (2.144) that f can be written in the form (2.129).

□

CHAPTER 3

The Bloch Space

Let $f \in H(\mathbb{D})$ and set $\|f\|_{\mathcal{B}} = \sup\{(1-|z|^2)|f'(z)| : z \in \mathbb{D}\}$. We define the Bloch space $\mathcal{B}$ of $\mathbb{D}$ to be the space of all analytic functions f on $\mathbb{D}$ such that $\|f\|_{\mathcal{B}} < +\infty$.

Remark 3.0.1. The Bloch space is a vector space over $\mathbb{C}$.

Proof. Let $f, g \in \mathcal{B}$ and $\lambda \in \mathbb{C}$. Then

$$(1-|z|^2)|(f+g)'(z)| \leq (1-|z|^2)|f'(z)| + (1-|z|^2)|g'(z)| \leq \|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}} < +\infty,$$

for all $z \in \mathbb{D}$, hence $f + g \in \mathcal{B}$, with $\|f+g\|_{\mathcal{B}} \leq \|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}}$. Also,

$$\sup\{(1-|z|^2)|\lambda f'(z)| : z \in \mathbb{D}\} = |\lambda| \sup\{(1-|z|^2)|f'(z)| : z \in \mathbb{D}\} < +\infty,$$

hence $\lambda f \in \mathcal{B}$, with $\|\lambda f\|_{\mathcal{B}} = |\lambda| \|f\|_{\mathcal{B}}$. □

3.1 Completeness

Our goal is to show that the Bloch space, equipped with a specific norm, can become a Banach space. The completeness is achieved due to the completeness of the semi-norm $\|\cdot\|_{\mathcal{B}}$.

Lemma 3.1.1. *$\|\cdot\|_{\mathcal{B}}$ is a complete semi-norm on $\mathcal{B}$.*

Proof. It follows from Remark 3.0.1 that $\|\cdot\|_{\mathcal{B}}$ is a semi-norm on $\mathcal{B}$. To show the completeness, let $\{f_k\} \subset \mathcal{B}$ be a Cauchy sequence with respect to the semi-norm, and let $\epsilon > 0$. Then there exists a number $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\|_{\mathcal{B}} < \epsilon, \text{ for all } n, m \geq n_0,$$

so

$$(1-|z|^2)|f_n'(z) - f_m'(z)| < \epsilon, \text{ for all } z \in \mathbb{D} \text{ and } n, m \geq n_0, \tag{3.1}$$

hence $\{f_k'(z)\}$ is a Cauchy sequence for all $z \in \mathbb{D}$. Thus, there exists a function $g : \mathbb{D} \to \mathbb{C}$ such that $g(z) = \lim_{k \to +\infty} f_k'(z)$, for all $z \in \mathbb{D}$.

Let S be a compact subset of $\mathbb{D}$. Then there exists a real $R \in (0,1)$ such that $S \subset \overline{D(0,R)} \subset \mathbb{D}$. Let $\epsilon > 0$. Since $\{f_k\}$ is Cauchy with respect to the semi-norm, there exists a number $n_1 \in \mathbb{N}$ (which depends on ϵ and R) such that

$$|f_n'(z) - f_m'(z)| < \frac{\epsilon(1-R^2)}{(1-|z|^2)} \leq \epsilon, \text{ for all } z \in S \text{ and } n, m \geq n_1,$$

so $\{f'_k\}$ converges to g uniformly on S. By the Weierstrass Convergence Theorem ([12, Theorem 10.28]), $g \in H(\mathbb{D})$.

Since $\mathbb{D}$ is a simply connected domain, there exists a function $f \in H(\mathbb{D})$ such that $f' = g$ in $\mathbb{D}$ ([14, Section X.4]). Letting $n \to +\infty$ in (3.1), we have

$$(1 - |z|^2)|f'(z) - f'_m(z)| \le \epsilon, \text{ for all } z \in \mathbb{D} \text{ and } m \ge n_0, \tag{3.2}$$

which means that $f - f_{n_0} \in \mathcal{B}$. Since $f_{n_0} \in \mathcal{B}$, it follows that $f \in \mathcal{B}$, and (3.2) gives

$$\|f - f_m\|_{\mathcal{B}} \le \epsilon, \text{ for all } m \ge n_0,$$

so $\|f - f_k\|_{\mathcal{B}} \to 0$, as $k \to +\infty$. □

Remark 3.1.1. In the above proof we can choose f such that $f(0) = a_0$ for some given $a_0 \in \mathbb{C}$. Indeed, if h is a primitive of g, then we consider the function $f(z) = h(z) + a_0 - h(0)$, which is also a primitive of g, with $f(0) = a_0$.

We now introduce the norm $\|f\| = |f(0)| + \|f\|_{\mathcal{B}}$ in $\mathcal{B}$.

Proposition 3.1.2. *$(\mathcal{B}, \|\cdot\|)$ is a Banach space.*

Proof. Let $\{f_k\} \subset \mathcal{B}$ be a Cauchy sequence with respect to $\|\cdot\|$, and let $\epsilon > 0$. Then there exists a number $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\| < \epsilon, \text{ for all } n, m \ge n_0,$$

so

$$|f_n(0) - f_m(0)| < \epsilon \text{ and } \|f_n - f_m\|_{\mathcal{B}} < \epsilon, \text{ for all } n, m \ge n_0.$$

This means that $\{f_k(0)\}$ is a Cauchy sequence, so let $a_0 = \lim_{k\to+\infty} f_k(0) \in \mathbb{C}$, and $\{f_k\}$ is a Cauchy sequence with respect to the semi-norm. By Lemma 3.1.1 and Remark 3.1.1, there exists a function $f \in \mathcal{B}$ such that $\|f - f_k\|_{\mathcal{B}} \to 0$, as $k \to +\infty$, and $f(0) = a_0$. Thus,

$$\|f - f_k\| = |f(0) - f_k(0)| + \|f - f_k\|_{\mathcal{B}} \to 0, \text{ as } k \to +\infty.$$

□

3.2 The dual of $\mathcal{A}^1(dA)$

Proposition 3.2.1. *$H^\infty \subset \mathcal{B}$ and $\|f\|_{\mathcal{B}} \le \|f\|_\infty$, for all $f \in H^\infty$.*

Proof. Let $f \in H^\infty$. Then $f \in H(\mathbb{D})$ and $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\} < +\infty$.
Case 1. $\|f\|_\infty \le 1$. Then $|f(z)| \le 1$, for all $z \in \mathbb{D}$. If there exists a point $z_0 \in \mathbb{D}$ such that $|f(z_0)| = 1$, then by the Maximum Modulus Principle ([14, Section VII.16]), f will be constant, so $|f'(0)| = 0 \le 1$. If $|f(z)| < 1$, for all $z \in \mathbb{D}$, then $|f'(0)| \le 1 - |f(0)|^2 \le 1$, by the Schwarz-Pick Lemma ([14, Exercise VII.17.3]). Thus, in any case we have that $|f'(0)| \le 1$.
Case 2 (general case). If $\|f\|_\infty = 0$, then $f = 0$ in $\mathbb{D}$, so $|f'(0)| = 0 \le \|f\|_\infty$. If $\|f\|_\infty \ne 0$, then the function $g = \dfrac{f}{\|f\|_\infty}$ is holomorphic and bounded. In particular,

$$|g(z)| = \frac{|f(z)|}{\|f\|_\infty} \le 1, \text{ for all } z \in \mathbb{D},$$

so $\|g\|_\infty \le 1$. By *Case 1*, $|g'(0)| \le 1$, so $|f'(0)| \le \|f\|_\infty$. Thus, in any case we have that

$$|f'(0)| \le \|f\|_\infty. \tag{3.3}$$

Now, let $f \in H^\infty$ and $z \in \mathbb{D}$. Then $f \circ \phi_z \in H^\infty$, with $\|f \circ \phi_z\|_\infty = \|f\|_\infty$, so

$$|f'(z)|(1 - |z|^2) = |f'(\phi_z(0))| \cdot |\phi'_z(0)| = |(f \circ \phi_z)'(0)| \le \|f \circ \phi_z\|_\infty = \|f\|_\infty < +\infty,$$

where the first equality holds because of Proposition 1.1.2 (iii), and the inequality holds because of (3.3). Thus, $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \le \|f\|_\infty$. □

Remark 3.2.1. The containment $H^\infty \subset \mathcal{B}$ is proper. Indeed, since $\mathbb{D}$ is a simply connected domain and $g(z) = 1 - z$ is a nowhere vanishing holomorphic function in $\mathbb{D}$, there exists a branch of $\log g$ in $\mathbb{D}$ ([14, Section X.5]). Let f be such a branch. Then $f \in H(\mathbb{D})$ and

$$f'(z) = \frac{g'(z)}{g(z)} = -\frac{1}{1-z}, \text{ for all } z \in \mathbb{D},$$

so

$$(1-|z|^2)|f'(z)| = \frac{1-|z|^2}{|1-z|} \leq \frac{(1-|z|)(1+|z|)}{1-|z|} < 2, \text{ for all } z \in \mathbb{D}.$$

Thus, $f \in \mathcal{B}$. However, f is not bounded, because

$$\lim_{\substack{z\to 1\\ |z|<1}} \mathrm{Re} f(z) = \lim_{\substack{z\to 1\\ |z|<1}} \log|1-z| = -\infty.$$

Proposition 3.2.2. *The operator P given by*

$$P\phi(z) = \int_{\mathbb{D}} \frac{\phi(w)}{(1-z\bar{w})^2}\, dA(w), \tag{3.4}$$

is a bounded linear operator from $L^\infty(\mathbb{D}, dA)$ onto $\mathcal{B}$.

Proof. Since $L^\infty(\mathbb{D}, dA) \subset L^p(\mathbb{D}, dA)$, for all $p \in (1, +\infty)$, (A.15) implies that the integral in (3.4) is defined for all $\phi \in L^\infty(\mathbb{D}, dA)$ and $z \in \mathbb{D}$, and $P\phi \in L^p(\mathbb{D}, dA)$, for all $\phi \in L^\infty(\mathbb{D}, dA)$ and $p > 1$.

Let $\phi \in L^\infty(\mathbb{D}, dA)$ and set $f = P\phi$, hence

$$f(z) = \int_{\mathbb{D}} \frac{\phi(w)}{(1-z\bar{w})^2}\, dA(w), \text{ for all } z \in \mathbb{D}. \tag{3.5}$$

By Proposition 2.4.3, $f \in H(\mathbb{D})$ and

$$f'(z) = \int_{\mathbb{D}} \phi(w) \frac{2\bar{w}}{(1-z\bar{w})^3}\, dA(w), \text{ for all } z \in \mathbb{D},$$

so

$$(1-|z|^2)|f'(z)| \leq 2(1-|z|^2) \int_{\mathbb{D}} \frac{|\bar{w}||\phi(w)|}{|1-z\bar{w}|^3}\, dA(w) \leq 2\|\phi\|_\infty (1-|z|^2) I(z), \tag{3.6}$$

for all $z \in \mathbb{D}$, where $I(z) = \displaystyle\int_{\mathbb{D}} \frac{1}{|1-z\bar{w}|^3}\, dA(w)$. By Lemma A.0.1 and the fact that I is bounded on $D\big(0, \frac{1}{2}\big)$, there exists a constant $C_1 > 0$ such that

$$I(z) \leq C_1 \cdot \frac{1}{1-|z|^2}, \text{ for all } z \in \mathbb{D}. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$(1-|z|^2)|f'(z)| \leq C_2 \|\phi\|_\infty < +\infty, \text{ for all } z \in \mathbb{D}, \tag{3.8}$$

where $C_2 = 2C_1 > 0$, which means that $f \in \mathcal{B}$ and

$$\|P\phi\|_{\mathcal{B}} = \|f\|_{\mathcal{B}} \leq C_2 \|\phi\|_\infty. \tag{3.9}$$

Also, by (3.5),

$$|f(0)| \leq \int_{\mathbb{D}} |\phi(w)|\, dA(w) \leq \|\phi\|_\infty, \tag{3.10}$$

so (3.9) and (3.10) imply that

$$\|P\phi\| = |f(0)| + \|f\|_{\mathcal{B}} \leq (1 + C_2)\|\phi\|_\infty,$$

that is, P is a bounded linear operator from $L^\infty(\mathbb{D}, dA)$ into $\mathcal{B}$.

It remains to show that P is onto. Let $f \in \mathcal{B}$. By the Taylor series expansion of f in $\mathbb{D}$, we can write

$$f(z) = f(0) + f'(0)z + f_1(z), \text{ for all } z \in \mathbb{D}, \tag{3.11}$$

where $f_1(z) = \sum_{n=2}^{+\infty} \frac{f^{(n)}(0)}{n!} z^n$, for all $z \in \mathbb{D}$. Then

$$f'(z) = f'(0) + f_1'(z), \text{ for all } z \in \mathbb{D}, \tag{3.12}$$

so

$$\begin{aligned}(1-|z|^2)|f_1'(z)| &= (1-|z|^2)|f'(z) - f'(0)| \leq (1-|z|^2)|f'(z)| + |f'(0)| \\ &\leq \|f\|_{\mathcal{B}} + |f'(0)| < +\infty,\end{aligned} \tag{3.13}$$

for all $z \in \mathbb{D}$, which means that $f_1 \in \mathcal{B}$. Also, (3.11) and (3.12) yield that $f_1(0) = f_1'(0) = 0$. Consider the function

$$\phi(z) = f(0) + f'(0)z + \frac{(1-|z|^2)f_1'(z)}{\bar{z}}, \text{ for all } z \in \mathbb{D} \setminus \{0\}.$$

Since

$$\lim_{z\to 0} \left|\frac{f_1'(z)}{\bar{z}}\right| = \lim_{z\to 0} \left|\frac{f'(z) - f'(0)}{z}\right| = |f''(0)|,$$

there exists a real $\delta \in (0,1)$ such that $\left|\frac{f_1'(z)}{\bar{z}}\right| < 1 + |f''(0)|$, for all $z \in D(0,\delta) \setminus \{0\}$, hence

$$|\phi(z)| \leq |f(0)| + |f'(0)||z| + \frac{(1-|z|^2)|f_1'(z)|}{|\bar{z}|} \leq |f(0)| + |f'(0)| + 1 + |f''(0)|, \tag{3.14}$$

for all $z \in D(0,\delta) \setminus \{0\}$. If $z \in \mathbb{D} \setminus D(0,\delta)$, then

$$|\phi(z)| \leq |f(0)| + |f'(0)||z| + \frac{(1-|z|^2)|f_1'(z)|}{|\bar{z}|} \overset{(3.13)}{\leq} |f(0)| + |f'(0)| + \frac{\|f\|_{\mathcal{B}} + |f'(0)|}{\delta}. \tag{3.15}$$

Thus, (3.14) and (3.15) imply that $\phi \in L^\infty(\mathbb{D}, dA)$. Finally, by Theorem 2.3.6,

$$\int_{\mathbb{D}} \frac{f(0)}{(1-z\bar{w})^2}\, dA(w) = f(0) \text{ and } \int_{\mathbb{D}} \frac{f'(0)w}{(1-z\bar{w})^2}\, dA(w) = f'(0)z, \text{ for all } z \in \mathbb{D},$$

and by Proposition 2.5.2,

$$f_1(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)f_1'(w)}{\bar{w}(1-z\bar{w})^2}\, dA(w), \text{ for all } z \in \mathbb{D},$$

so

$$P\phi(z) = \int_{\mathbb{D}} \frac{\phi(w)}{(1-z\bar{w})^2}\, dA(w) = f(0) + f'(0)z + f_1(z) \overset{(3.11)}{=} f(z), \text{ for all } z \in \mathbb{D}.$$

□

Proposition 3.2.3. *The spaces $\mathcal{B}$ and $L^\infty(\mathbb{D}, dA)/\mathrm{Ker}P$ are isomorphic.*

Proof. Since P is onto, the above result is known from elementary functional analysis (see [8, Section 3.4]). □

The following are based on [4, §2.6, Lemma 7, Theorem 8].

Consider the function $V : H(\mathbb{D}) \to H(\mathbb{D})$, defined by

$$Vf(z) = \begin{cases} \frac{f(z)-f(0)}{z}, & \text{if } z \in \mathbb{D} \setminus \{0\} \\ f'(0), & \text{if } z = 0 \end{cases}, \text{ for all } f \in H(\mathbb{D}).$$

Proposition 3.2.4. *Let $f, g \in H(\mathbb{D})$ and $t \in (0,1)$. Then*

$$\int_{D(0,t)} f(z)\overline{g(z)}\,dA(z) = \int_{D(0,t)} Vf(z)\cdot\overline{g'(z)}(t^2-|z|^2)\,dA(z) + t^2 f(0)\overline{g(0)}. \tag{3.16}$$

Proof. By the Taylor series expansion of f and g in $\mathbb{D}$, we have that, for all $z \in \mathbb{D}$,

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n,\ (Vf)(z) = \sum_{n=0}^{+\infty} a_{n+1} z^n,\ g(z) = \sum_{n=0}^{+\infty} b_n z^n \text{ and } g'(z) = \sum_{n=0}^{+\infty} b_{n+1}(n+1)z^n,$$

where $a_n = \frac{f^{(n)}(0)}{n!}$ and $b_n = \frac{g^{(n)}(0)}{n!}$, for all $n \in \mathbb{N}\cup\{0\}$. We make the following observations:
(a) Fix $r \in (0,t)$ and let

$$S_N(f)(\theta) = \sum_{n=0}^{N} a_n r^n e^{in\theta},\ S_N(g)(\theta) = \sum_{n=0}^{N} b_n r^n e^{in\theta},$$
$$S_N(Vf)(\theta) = \sum_{n=0}^{N} a_{n+1} r^n e^{in\theta} \text{ and } S_N(g')(\theta) = \sum_{n=0}^{N} b_{n+1}(n+1) r^n e^{in\theta},$$

for all $\theta \in [0,2\pi]$ and $N \in \mathbb{N}\cup\{0\}$. Since the series $\sum_{n=0}^{+\infty} a_n z^n$ converges absolutely in $\mathbb{D}$, we get that $S_N(f)$ is uniformly bounded in $[0,2\pi]$ and converges uniformly in $[0,2\pi]$. The same goes for $S_N(g), S_N(Vf)$ and $S_N(g')$; so $S_N(f)\overline{S_N(g)}$ and $S_N(Vf)\cdot\overline{S_N(g')}$ converge uniformly in $[0,2\pi]$.
(b) For all $r \in [0,t^2]$ and $n \in \mathbb{N}\cup\{0\}$,

$$|a_n\bar{b}_n|r^n \le |a_n||b_n|t^{2n} \text{ and } |a_{n+1}\overline{b_{n+1}}(n+1)|r^n \le |a_{n+1}||b_{n+1}|(n+1)t^{2n}.$$

Since

$$\limsup(\sqrt[n]{|a_n||b_n|t^{2n}}) \le t^2 \limsup(\sqrt[n]{|a_n|})\limsup(\sqrt[n]{|b_n|}) \le t^2 < 1$$

and

$$\limsup(\sqrt[n]{|a_{n+1}||b_{n+1}|(n+1)t^{2n}}) \le t^2 \limsup(\sqrt[n]{|a_{n+1}|})\limsup(\sqrt[n]{|b_{n+1}|(n+1)}) \le t^2 < 1,$$

we get that $\sum_{n=0}^{+\infty}|a_n||b_n|t^{2n} < +\infty$ and $\sum_{n=0}^{+\infty}|a_{n+1}||b_{n+1}|(n+1)t^{2n} < +\infty$, hence the Weierstrass criterion for the uniform convergence of series yields that the series

$$\sum_{n=0}^{+\infty} a_n\bar{b}_n r^n \text{ and } \sum_{n=0}^{+\infty} a_{n+1}\overline{b_{n+1}}(n+1)r^n$$

converge uniformly in $[0,t^2]$.
Using (a) and (b), we have

$$\begin{aligned}
\int_{D(0,t)} f(z)\overline{g(z)}\,dA(z) &= \int_0^t\int_0^{2\pi} f(re^{i\theta})\overline{g(re^{i\theta})}\cdot\frac{r}{\pi}\,d\theta dr\\
&= \int_0^t \frac{r}{\pi}\int_0^{2\pi}\lim_{N\to+\infty}(S_N(f)(\theta)\cdot\overline{S_N(g)(\theta)})\,d\theta dr\\
&= \int_0^t \frac{r}{\pi}\lim_{N\to+\infty}\sum_{n=0}^{N}\sum_{m=0}^{N} a_n\bar{b}_m r^{n+m}\int_0^{2\pi} e^{i(n-m)\theta}\,d\theta dr\\
&= \int_0^t 2r\lim_{N\to+\infty}\sum_{n=0}^{N} a_n\bar{b}_n r^{2n}\,dr = \int_0^{t^2}\lim_{N\to+\infty}\sum_{n=0}^{N} a_n\bar{b}_n r^n\,dr\\
&= \sum_{n=0}^{+\infty} a_n\bar{b}_n\int_0^{t^2} r^n\,dr = t^2 f(0)\overline{g(0)} + \sum_{n=1}^{+\infty} a_n\bar{b}_n\frac{t^{2(n+1)}}{n+1}
\end{aligned} \tag{3.17}$$

and, similarly,

$$\begin{aligned}\int_{D(0,t)}(Vf)(z)\cdot\overline{g'(z)}(t^2-|z|^2)\,dA(z) &= \int_0^t\int_0^{2\pi}\lim_{N\to+\infty}(S_N(Vf)(\theta)\cdot\overline{S_N(g')(\theta)})\frac{r(t^2-r^2)}{\pi}\,d\theta dr\\ &= \int_0^t 2r(t^2-r^2)\lim_{N\to+\infty}\sum_{n=0}^{N}a_{n+1}\overline{b_{n+1}}(n+1)r^{2n}\,dr\\ &= \int_0^{t^2}(t^2-r)\lim_{N\to+\infty}\sum_{n=0}^{N}a_{n+1}\overline{b_{n+1}}(n+1)r^{n}\,dr\\ &= \sum_{n=0}^{+\infty}a_{n+1}\overline{b_{n+1}}(n+1)\int_0^{t^2}r^n(t^2-r)\,dr\\ &= \sum_{n=0}^{+\infty}a_{n+1}\overline{b_{n+1}}\frac{t^{2(n+2)}}{n+2},\end{aligned}$$

that is,

$$\int_{D(0,t)}(Vf)(z)\cdot\overline{g'(z)}(t^2-|z|^2)\,dA(z)=\sum_{n=1}^{+\infty}a_n\bar{b}_n\frac{t^{2(n+1)}}{n+1}. \tag{3.18}$$

(3.16) follows from (3.17) and (3.18). □

Proposition 3.2.5. *V is a bounded linear operator on $\mathcal{A}^1(dA)$.*

Proof. First we will show that $Vf\in\mathcal{A}^1(dA)$, for all $f\in\mathcal{A}^1(dA)$. Let $f\in\mathcal{A}^1(dA)$. Since Vf is continuous in $\mathbb{D}$, there exists a constant $M>0$ such that $|Vf(z)|\le M$, for all $z\in\overline{D(0,\frac{1}{2})}$. Thus,

$$\begin{aligned}\int_{\mathbb{D}}|Vf(z)|\,dA(z) &= \int_{D\left(0,\frac{1}{2}\right)}|Vf(z)|\,dA(z)+\int_{\mathbb{D}\setminus D\left(0,\frac{1}{2}\right)}\frac{|f(z)-f(0)|}{|z|}\,dA(z)\\ &\le M\int_{D\left(0,\frac{1}{2}\right)}dA(z)+2\int_{\mathbb{D}\setminus D\left(0,\frac{1}{2}\right)}|f(z)-f(0)|\,dA(z)\\ &\le M+2\int_{\mathbb{D}\setminus D\left(0,\frac{1}{2}\right)}|f(z)|\,dA(z)+2\int_{\mathbb{D}\setminus D\left(0,\frac{1}{2}\right)}|f(0)|\,dA(z)\\ &\le M+2\|f\|_1+2|f(0)|<+\infty,\end{aligned}$$

hence $Vf\in\mathcal{A}^1(dA)$.

It remains to show that V is a bounded operator. Let $f_n,f,g\in\mathcal{A}^1(dA)$, for all $n\in\mathbb{N}$, such that $\lim_{n\to+\infty}\|f_n-f\|_1=0$ and $\lim_{n\to+\infty}\|Vf_n-g\|_1=0$. By Theorem 2.1.4,

$$|f_n(z)-f(z)|\le\frac{\|f_n-f\|_1}{(1-|z|^2)^2}\ \text{and}\ |Vf_n(z)-g(z)|\le\frac{\|Vf_n-g\|_1}{(1-|z|^2)^2},$$

for all $z\in\mathbb{D}$ and $n\in\mathbb{N}$, so

$$\lim_{n\to+\infty}f_n(z)=f(z), \tag{3.19}$$

$$\lim_{n\to+\infty}Vf_n(z)=g(z), \tag{3.20}$$

for all $z\in\mathbb{D}$. By (3.19),

$$\lim_{n\to+\infty}Vf_n(z)=\lim_{n\to+\infty}\frac{f_n(z)-f_n(0)}{z}=\frac{f(z)-f(0)}{z}=Vf(z),\ \text{if}\ z\in\mathbb{D}\setminus\{0\},$$

so (3.20) implies that $Vf(z) = g(z)$, for all $z \in \mathbb{D}\setminus\{0\}$. Since Vf, g are continuous in $\mathbb{D}$, we also have that

$$Vf(0) = \lim_{z\to 0} Vf(z) = \lim_{z\to 0} g(z) = g(0).$$

Thus, $Vf = g$ in $\mathbb{D}$, so the Closed Graph Theorem ([8, Theorem 3.21]) yields that V is a bounded operator on $\mathcal{A}^1(dA)$. □

Corollary 3.2.6. *Let $f \in \mathcal{A}^1(dA)$ and $g \in \mathcal{B}$. Then*

$$\lim_{t\to 1^-} \int_{D(0,t)} f(z)\overline{g(z)}\,dA(z) = \int_{\mathbb{D}} Vf(z)\cdot\overline{g'(z)}(1-|z|^2)\,dA(z) + f(0)\overline{g(0)}. \tag{3.21}$$

Proof. Let $\{t_n\} \subset (0,1)$, with $\lim_{n\to+\infty} t_n = 1$ and $t_n < t_{n+1}$, for all $n \in \mathbb{N}$. Then

$$\lim_{n\to+\infty} Vf(z)\cdot\overline{g'(z)}(t_n^2-|z|^2)\chi_{D(0,t_n)}(z) = Vf(z)\cdot\overline{g'(z)}(1-|z|^2),\ \text{for all } z\in\mathbb{D}, \tag{3.22}$$

and

$$|Vf(z)\cdot\overline{g'(z)}(t_n^2-|z|^2)||\chi_{D(0,t_n)}(z)| \le |Vf(z)|\cdot|g'(z)|(1-|z|^2) \le |Vf(z)|\|g\|_{\mathcal{B}}, \tag{3.23}$$

for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$. By Proposition 3.2.5, $Vf \in \mathcal{A}^1(dA)$, so

$$\int_{\mathbb{D}} |Vf(z)|\|g\|_{\mathcal{B}}\,dA(z) < +\infty. \tag{3.24}$$

Using (3.22) (3.23), (3.24) and [6, 2.24 The Dominated Convergence Theorem],

$$\lim_{n\to+\infty} \int_{D(0,t_n)} Vf(z)\cdot\overline{g'(z)}(t_n^2-|z|^2)\,dA(z) = \int_{\mathbb{D}} Vf(z)\cdot\overline{g'(z)}(1-|z|^2)\,dA(z) \in \mathbb{C}. \tag{3.25}$$

Since (3.25) holds for an arbitrary sequence $\{t_n\}$, we get that

$$\lim_{t\to 1^-} \int_{D(0,t)} Vf(z)\cdot\overline{g'(z)}(t^2-|z|^2)\,dA(z) = \int_{\mathbb{D}} Vf(z)\cdot\overline{g'(z)}(1-|z|^2)\,dA(z). \tag{3.26}$$

(3.26) and Proposition 3.2.4 give (3.21). □

Theorem 3.2.7. $(\mathcal{A}^1(dA))^*$ *is isomorphic to $\mathcal{B}$, and the two spaces have equivalent norms.*

Proof. Consider the operator $J : \mathcal{B} \to (\mathcal{A}^1(dA))^*$, given by $Jg = l_g$, for all $g \in \mathcal{B}$, where

$$l_g(f) = \lim_{t\to 1^-} \int_{D(0,t)} f(z)\overline{g(z)}\,dA(z),\ \text{for all } f \in \mathcal{A}^1(dA) \text{ and } g \in \mathcal{B}.$$

(i) J is well-defined. Indeed, by Corollary 3.2.6,

$$l_g(f) = \int_{\mathbb{D}} Vf(z)\cdot\overline{g'(z)}(1-|z|^2)\,dA(z) + f(0)\overline{g(0)},\ \text{for all } f \in \mathcal{A}^1(dA) \text{ and } g \in \mathcal{B},$$

and, using Corollary 2.1.3,

$$\begin{aligned}|l_g(f)| &\le \int_{\mathbb{D}} |Vf(z)|\cdot|g'(z)|(1-|z|^2)\,dA(z) + |f(0)||g(0)| \le \|Vf\|_1\|g\|_{\mathcal{B}} + |f(0)||g(0)|\\ &\le \|V\|\|f\|_1\|g\|_{\mathcal{B}} + \|f\|_1|g(0)| \le C\|f\|_1(\|g\|_{\mathcal{B}} + |g(0)|) = C\|g\|\|f\|_1,\end{aligned}$$

for all $f \in \mathcal{A}^1(dA)$ and $g \in \mathcal{B}$, where $C = \max\{\|V\|, 1\} > 0$. Thus, $l_g \in (\mathcal{A}^1(dA))^*$, for all $g \in \mathcal{B}$, and

$$\|Jg\| = \|l_g\| \le C\|g\|,\ \text{for all } g \in \mathcal{B}. \tag{3.27}$$

(ii) $J(g_1+g_2) = Jg_1 + Jg_2$ and $J(\lambda g) = \bar{\lambda} Jg$, for all $g_1, g_2, g \in \mathcal{B}$ and $\lambda \in \mathbb{C}$.

(iii) Let $l \in (\mathcal{A}^1(dA))^*$. By the Hahn-Banach Theorem ([8, Theorem 3.2]), there exists a bounded linear functional $\Lambda : L^1(\mathbb{D}, dA) \to \mathbb{C}$, such that $\Lambda(f) = l(f)$, for all $f \in \mathcal{A}^1(dA)$, and $\|\Lambda\| = \|l\|$. The classical duality between L^p spaces ([9, Theorem 7.15]) yields that there exists a unique $\phi \in L^\infty(\mathbb{D}, dA)$ such that $\Lambda = \Lambda_\phi$ in $L^1(\mathbb{D}, dA)$, where

$$\Lambda_\phi(f) = \int_{\mathbb{D}} f(z)\overline{\phi(z)}\, dA(z), \text{ for all } f \in L^1(\mathbb{D}, dA).$$

By Proposition 3.2.2, $g = P\phi \in \mathcal{B}$. Also, Theorem A.0.2 implies that the operator

$$Lf(z) = \int_{\mathbb{D}} |f(w)||K(z,w)|\, dA(w),$$

is well-defined and bounded on $L^p(\mathbb{D}, dA)$, for all $p > 1$.
Let Q be a polynomial. Then

$$\int_{\mathbb{D}}\int_{\mathbb{D}} |Q(w)||K(z,w)||\phi(z)|\, dA(w)dA(z) = \int_{\mathbb{D}} |\phi(z)|LQ(z)\, dA(z) \le \|\phi\|_\infty \cdot \|LQ\|_1 < +\infty.$$

By Theorem 2.3.6, [6, 2.37 The Fubini-Tonelli Theorem] and Proposition 3.2.2,

$$\begin{aligned} l(Q) = \Lambda_\phi(Q) &= \int_{\mathbb{D}} Q(z)\overline{\phi(z)}\, dA(z) = \int_{\mathbb{D}}\int_{\mathbb{D}} Q(w)K(z,w)\overline{\phi(z)}\, dA(w)dA(z) \\ &= \int_{\mathbb{D}}\int_{\mathbb{D}} Q(w)K(z,w)\overline{\phi(z)}\, dA(z)dA(w) = \int_{\mathbb{D}} Q(w)\overline{\int_{\mathbb{D}} \phi(z)K(w,z)\, dA(z)}\, dA(w) \\ &= \int_{\mathbb{D}} Q(w)\overline{P\phi(w)}\, dA(w) = \int_{\mathbb{D}} Q(w)\overline{g(w)}\, dA(w), \end{aligned} \tag{3.28}$$

which means that $Q\bar{g}$ is integrable. By [6, 2.24 The Dominated Convergence Theorem] and (3.28),

$$l_g(Q) = \lim_{t\to 1^-} \int_{D(0,t)} Q(z)\overline{g(z)}\, dA(z) = \int_{\mathbb{D}} Q(z)\overline{g(z)}\, dA(z) = l(Q). \tag{3.29}$$

Now, let $f \in \mathcal{A}^1(dA)$. Since the polynomials are dense in $\mathcal{A}^1(dA)$ (Proposition 2.3.5), there exists a sequence $\{p_n\}$ of polynomials such that $\lim_{n\to+\infty}\|p_n - f\|_1 = 0$. Thus, (3.29) gives

$$l(f) = l\Big(\lim_{n\to+\infty} p_n\Big) = \lim_{n\to+\infty} l(p_n) = \lim_{n\to+\infty} l_g(p_n) = l_g\Big(\lim_{n\to+\infty} p_n\Big) = l_g(f) = (Jg)(f),$$

so J is onto $(\mathcal{A}^1(dA))^*$.

(iv) Suppose that $Jg_1 = Jg_2 = l \in (\mathcal{A}^1(dA))^*$, for some $g_1,\ g_2 \in \mathcal{B}$. Then

$$g_1(z) = \sum_{k=0}^{+\infty} a_k z^k \text{ and } g_2(z) = \sum_{k=0}^{+\infty} b_k z^k, \text{ for all } z \in \mathbb{D},$$

where $a_k = \dfrac{g_1^{(k)}(0)}{k!}$ and $b_k = \dfrac{g_2^{(k)}(0)}{k!}$, for all $k \in \mathbb{N}\cup\{0\}$.
Let $n \in \mathbb{N}\cup\{0\}$ and set $e_n(z) = z^n$, for all $z \in \mathbb{D}$. Note that the series $\sum_{k=0}^{+\infty} \bar{a}_k z^n \bar{z}^k$ converges absolutely in $\mathbb{D}$, so for a fixed $r \in (0,1)$, the series $\sum_{k=0}^{+\infty} \bar{a}_k r^{n+k} e^{i(n-k)\theta}$ converges uniformly on $[0, 2\pi]$. Also, consider a sequence $\{r_m\} \subset (0,1)$ with $\lim_{m\to+\infty} r_m = 1$ and $r_m < r_{m+1}$, for all $m \in \mathbb{N}$. Then [6, 2.24 The Dominated Convergence Theorem] yields that

$$\lim_{m\to+\infty} \int_0^{r_m} \bar{a}_n r^{2n} \cdot 2r\, dr = \int_0^1 \bar{a}_n r^{2n} \cdot 2r\, dr. \tag{3.30}$$

Thus,

$$\begin{aligned} Jg_1(e_n) &= \lim_{m\to+\infty}\int_{D(0,r_m)} z^n\overline{g_1(z)}\,dA(z) = \lim_{m\to+\infty}\int_{D(0,r_m)}\sum_{k=0}^{+\infty}\bar{a}_k z^n\bar{z}^k\,dA(z) \\ &= \lim_{m\to+\infty}\int_0^{r_m}\frac{r}{\pi}\int_0^{2\pi}\sum_{k=0}^{+\infty}\bar{a}_k r^{n+k}e^{i(n-k)\theta}\,d\theta dr \\ &= \lim_{m\to+\infty}\int_0^{r_m}\frac{r}{\pi}\sum_{k=0}^{+\infty}\bar{a}_k r^{n+k}\int_0^{2\pi}e^{i(n-k)\theta}\,d\theta dr \\ &= \lim_{m\to+\infty}\int_0^{r_m}\frac{r}{\pi}\bar{a}_n r^{2n}\cdot 2\pi\,dr \overset{(3.30)}{=} \int_0^1 \bar{a}_n r^{2n}\cdot 2r\,dr = \frac{\bar{a}_n}{n+1}. \end{aligned} \tag{3.31}$$

Similarly,

$$Jg_2(e_n) = \frac{\bar{b}_n}{n+1}. \tag{3.32}$$

Since $Jg_1(e_n) = Jg_2(e_n)$, for all $n\in\mathbb{N}\cup\{0\}$, (3.31) and (3.32) give $a_n = b_n$, for all $n\in\mathbb{N}\cup\{0\}$, and hence, $g_1 = g_2$ in $\mathbb{D}$. This means that J is one-to-one.

(v) Let $l\in(\mathcal{A}^1(dA))^*$. Using the notation in (iii), there exists a function $\phi\in L^\infty(\mathbb{D},dA)$ such that $J(P\phi) = l$. Since J is one-to-one, $J^{-1}l = P\phi$, and by Proposition 3.2.2 and the fact that $L^\infty(\mathbb{D},dA)$ and $(L^1(\mathbb{D},dA))^*$ are isometrically isomorphic ([9, Theorem 7.15]),

$$\|J^{-1}l\| = \|P\phi\| \le \|P\|\|\phi\|_\infty = \|P\|\|\Lambda_\phi\| = \|P\|\|l\|. \tag{3.33}$$

By (i), (ii), (iii), (iv) and (v), J is an isomorphism, and it follows from (3.27) and (3.33) that $(\mathcal{A}^1(dA))^*$ and $\mathcal{B}$ have equivalent norms. □

Remark 3.2.2. The integral

$$\int_{\mathbb{D}}|f(z)\overline{g(z)}|\,dA(z)$$

is not always finite for $f\in\mathcal{A}^1(dA)$ and $g\in\mathcal{B}$.

Indeed, let $g\in\mathcal{B}\setminus H^\infty$ and suppose that $fg\in\mathcal{A}^1(dA)$, for all $f\in\mathcal{A}^1(dA)$. Consider the linear operator $T_g:\mathcal{A}^1(dA)\to\mathcal{A}^1(dA)$, given by $T_gf = fg$, for all $f\in\mathcal{A}^1(dA)$, and let $f_n, f, h\in\mathcal{A}^1(dA)$, for all $n\in\mathbb{N}$, such that $\lim_{n\to+\infty}\|f_n - f\|_1 = 0$ and $\lim_{n\to+\infty}\|T_gf_n - h\|_1 = 0$. By Theorem 2.1.4,

$$|f_n(z) - f(z)| \le \frac{\|f_n - f\|_1}{(1-|z|^2)^2} \text{ and } |T_gf_n(z) - h(z)| \le \frac{\|T_gf_n - h\|_1}{(1-|z|^2)^2},$$

for all $z\in\mathbb{D}$ and $n\in\mathbb{N}$, so

$$\lim_{n\to+\infty} f_n(z) = f(z), \tag{3.34}$$

$$\lim_{n\to+\infty} T_gf_n(z) = h(z), \tag{3.35}$$

for all $z\in\mathbb{D}$. By (3.34),

$$\lim_{n\to+\infty} T_gf_n(z) = \lim_{n\to+\infty} f_n(z)g(z) = f(z)g(z) = T_gf(z), \text{ for all } z\in\mathbb{D},$$

so (3.35) implies that $T_gf(z) = h(z)$, for all $z\in\mathbb{D}$. Thus, the Closed Graph Theorem ([8, Theorem 3.21]) yields that T_g is a bounded operator on $\mathcal{A}^1(dA)$.
Let $z\in\mathbb{D}$. We will use the functional $T_z\in(\mathcal{A}^1(dA))^*$, which was introduced in Corollary 2.1.5. For all $f\in\mathcal{A}^1(dA)$, with $\|f\|_1 = 1$, we have that

$$|g(z)||T_z(f)| = |g(z)f(z)| = |T_z(fg)| = |T_z(T_gf)| \le \|T_z\|\|T_gf\|_1 \le \|T_z\|\|T_g\|. \tag{3.36}$$

Taking the supremum for all $f \in \mathcal{A}^1(dA)$, with $\|f\|_1 = 1$, in (3.36), we get $|g(z)|\|T_z\| \le \|T_z\|\|T_g\|$, and so

$$|g(z)| \le \|T_g\|. \tag{3.37}$$

Since (3.37) holds for an arbitrary $z \in \mathbb{D}$, we conclude that $g \in H^\infty$, which is a contradiction.

3.3 The Bloch space and the hyperbolic metric

A useful property of $\|\cdot\|_{\mathcal{B}}$ is its Möbius invariance.

Proposition 3.3.1. *For all $f \in \mathcal{B}$ and $\phi \in \mathrm{Aut}(\mathbb{D})$, $f \circ \phi \in \mathcal{B}$ and $\|f \circ \phi\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$.*

Proof. Let $f \in \mathcal{B}$ and $\phi \in \mathrm{Aut}(\mathbb{D})$. Then, by Theorem 1.1.1, $\phi = e^{i\theta}\phi_a$ in $\mathbb{D}$, for some real number θ and some point $a \in \mathbb{D}$. By Proposition 1.1.2 (iii), (iv), and the fact that

$$(1 - |\phi(z)|^2) \cdot |f'(\phi(z))| \le \|f\|_{\mathcal{B}}, \text{ for all } z \in \mathbb{D},$$

we have

$$\begin{aligned}(1 - |z|^2) \cdot |(f \circ \phi)'(z)| &= (1 - |z|^2) \cdot |f'(\phi(z))| \cdot |\phi'(z)| \\ &\le (1 - |z|^2) \cdot \frac{\|f\|_{\mathcal{B}}}{1 - |\phi(z)|^2} \cdot |\phi_a'(z)| = \|f\|_{\mathcal{B}} < +\infty,\end{aligned}$$

for all $z \in \mathbb{D}$, so

$$f \circ \phi \in \mathcal{B} \text{ and } \|f \circ \phi\|_{\mathcal{B}} \le \|f\|_{\mathcal{B}}. \tag{3.38}$$

Since (3.38) holds for all $f \in \mathcal{B}$ and $\phi \in \mathrm{Aut}(\mathbb{D})$, we replace f by $f \circ \phi$, and ϕ by ϕ^{-1} in (3.38) to get

$$\|f\|_{\mathcal{B}} \le \|f \circ \phi\|_{\mathcal{B}}. \tag{3.39}$$

(3.38) and (3.39) give the desired equality. □

The following results show that there exists a relation between the Bloch space and the hyperbolic metric. First, we need to calculate a specific limit.

Lemma 3.3.2. *Let $z \in \mathbb{D}$. Then*

$$\lim_{w \to z} \frac{|w - z|}{\beta(z, w)} = 1 - |z|^2.$$

Proof. Let $w \in \mathbb{D}$. Then, using (1.14) and (1.11),

$$\begin{aligned}&\beta(z, w) = \tanh^{-1}(\rho(z, w)) \;\Rightarrow\; \rho(z, w) = \tanh(\beta(z, w)) \;\Rightarrow \\ &\Rightarrow\; \sinh(\beta(z, w)) = \rho(z, w)\cosh(\beta(z, w)) \;\Rightarrow \\ &\Rightarrow\; \sinh^2(\beta(z, w)) = \rho^2(z, w)(1 + \sinh^2(\beta(z, w))) \;\Rightarrow \\ &\Rightarrow\; (1 - \rho^2(z, w))\sinh^2(\beta(z, w)) = \rho^2(z, w) \;\Rightarrow \\ &\Rightarrow\; \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z\bar{w}|^2}\sinh^2(\beta(z, w)) = \frac{|z - w|^2}{|1 - z\bar{w}|^2} \;\Rightarrow \\ &\Rightarrow\; \sqrt{(1 - |z|^2)(1 - |w|^2)}\,\sinh(\beta(z, w)) = |z - w|.\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{w \to z} \frac{|z - w|}{\beta(z, w)} &= \lim_{w \to z} \frac{\sqrt{(1 - |z|^2)(1 - |w|^2)}\,\sinh(\beta(z, w))}{\beta(z, w)} \\ &= (1 - |z|^2) \cdot \lim_{w \to z} \frac{\sinh(\beta(z, w))}{\beta(z, w)} = (1 - |z|^2) \cdot \lim_{x \to 0+} \frac{\sinh x}{x} \\ &= (1 - |z|^2) \cdot \lim_{x \to 0+} \frac{\cosh x}{1} = 1 - |z|^2.\end{aligned}$$

□

Theorem 3.3.3. *If $f \in \mathcal{B}$, then* $\|f\|_{\mathcal{B}} = \sup\left\{\dfrac{|f(z)-f(w)|}{\beta(z,w)} : z,w \in \mathbb{D},\ z \neq w\right\}$.

Proof. Suppose $f \in \mathcal{B}$ and $a \in \mathbb{D}\setminus\{0\}$. Then $(1-|at|^2)|f'(at)| \leq \|f\|_{\mathcal{B}}$, for all $t \in [0,1]$, so

$$\begin{aligned} |f(a)-f(0)| &= \left|\int_{[0,a]} f'(\zeta)\,d\zeta\right| = \left|\int_0^1 f'(at)\cdot a\,dt\right| \leq |a|\cdot\int_0^1 |f'(at)|\,dt \\ &\leq |a|\cdot\int_0^1 \frac{\|f\|_{\mathcal{B}}}{1-|at|^2}\,dt = |a|\cdot\|f\|_{\mathcal{B}}\cdot\int_0^{|a|}\frac{1}{1-s^2}\cdot\frac{1}{|a|}\,ds \\ &= \frac{1}{2}\|f\|_{\mathcal{B}}\log\frac{1+|a|}{1-|a|} \overset{(1.7)}{=} \|f\|_{\mathcal{B}}\cdot\beta(a,0). \end{aligned}$$

Since the above inequality holds also for $a=0$, we have

$$|f(a)-f(0)| \leq \|f\|_{\mathcal{B}}\cdot\beta(a,0), \text{ for all } a \in \mathbb{D}. \tag{3.40}$$

Let $z,w \in \mathbb{D}$ and $f \in \mathcal{B}$. By Proposition 3.3.1, the Möbius invariance of the hyperbolic metric and (3.40),

$$|f(w)-f(z)| = |(f\circ\phi_z)(\phi_z(w)) - (f\circ\phi_z)(0)| \leq \|f\circ\phi_z\|_{\mathcal{B}}\cdot\beta(\phi_z(w),0) = \|f\|_{\mathcal{B}}\cdot\beta(z,w),$$

so

$$M := \sup\left\{\frac{|f(z)-f(w)|}{\beta(z,w)} : z,w\in\mathbb{D},\ z\neq w\right\} \leq \|f\|_{\mathcal{B}} < +\infty. \tag{3.41}$$

For the reverse inequality, Lemma 3.3.2 gives

$$\lim_{w\to z}\frac{|f(z)-f(w)|}{\beta(z,w)} = \lim_{w\to z}\frac{|w-z|}{\beta(z,w)}\cdot\frac{|f(w)-f(z)|}{|w-z|} = (1-|z|^2)|f'(z)|, \text{ for all } z\in\mathbb{D},$$

so $(1-|z|^2)|f'(z)| \leq M$, for all $z \in \mathbb{D}$, and hence

$$\|f\|_{\mathcal{B}} \leq M. \tag{3.42}$$

Inequalities (3.41) and (3.42) give the desired equality. □

Corollary 3.3.4. *If $f \in H(\mathbb{D})$, then $f \in \mathcal{B}$ if and only if there exists a constant $C > 0$ such that*

$$|f(z)-f(w)| \leq C\beta(z,w), \textit{ for all } z,w\in\mathbb{D}. \tag{3.43}$$

Proof. Let $f \in \mathcal{B}$. Then (3.43) holds for $C > \|f\|_{\mathcal{B}} \geq 0$, because of Theorem 3.3.3.

Conversely, let $|f(z)-f(w)| \leq C\beta(z,w)$, for all $z,w \in \mathbb{D}$, for some constant $C > 0$. Then, by Lemma 3.3.2,

$$(1-|z|^2)|f'(z)| = \lim_{w\to z}\frac{|f(z)-f(w)|}{\beta(z,w)} \leq C < +\infty, \text{ for all } z\in\mathbb{D},$$

so $f \in \mathcal{B}$. □

Remark 3.3.1. Let $f \in \mathcal{B}$. For $w = 0$, the above corollary gives

$$|f(z)-f(0)| \leq C\cdot\beta(z,0) = \frac{C}{2}\log\frac{1+|z|}{1-|z|} \leq \frac{C}{2}\log 2 - \frac{C}{2}\log(1-|z|), \text{ for all } z\in\mathbb{D},$$

for some constant $C > 0$. This means that a Bloch function can grow at most as fast as $-\log(1-|z|)$. Note that by Remark 3.2.1, the function $f(z) = \log(1-z)$, $z \in \mathbb{D}$, is in the Bloch space, and as far as radial growth is concerned, f is the worst function in $\mathcal{B}$.

Theorem 3.3.5. *We have* $\beta(z,w) = \sup\{|f(z) - f(w)| : f \in \mathcal{B},\ \|f\|_{\mathcal{B}} \le 1\}$, *for all* $z, w \in \mathbb{D}$.

Proof. Let $z \in \mathbb{D}$. By Theorem 3.3.3,

$$|f(z) - f(0)| \le \|f\|_{\mathcal{B}} \cdot \beta(z,0) \le \beta(z,0), \text{ for all } f \in \mathcal{B}, \text{ with } \|f\|_{\mathcal{B}} \le 1,$$

so

$$\sup\{|f(z) - f(0)| : f \in \mathcal{B},\ \|f\|_{\mathcal{B}} \le 1\} \le \beta(z,0).$$

If $z \in \mathbb{D} \setminus \{0\}$, then $z = |z|e^{-i\theta}$, for some $\theta \in \mathbb{R}$. Consider the function

$$h(w) = \frac{1 + we^{i\theta}}{1 - we^{i\theta}}, \text{ for all } w \in \mathbb{D},$$

which is holomorphic and nonvanishing in $\mathbb{D}$. Since $\mathbb{D}$ is a simply connected domain, there exists a branch of $\log h$ in $\mathbb{D}$ ([14, Section X.5]). We choose l to be a branch of $\log h$ in $\mathbb{D}$ such that $l(0) = 0$. Then, for the function $g = \frac{1}{2}l$ we have:

(i) $g \in H(\mathbb{D})$, with $g'(w) = \dfrac{1}{2} \cdot l'(w) = \dfrac{1}{2}\dfrac{h'(w)}{h(w)} = \dfrac{e^{i\theta}}{1 - e^{2i\theta} \cdot w^2}$, for all $w \in \mathbb{D}$,

(ii) $(1 - |w|^2)|g'(w)| \le \dfrac{1 - |w|^2}{1 - |e^{2i\theta}| \cdot |w|^2} = 1$, for all $w \in \mathbb{D}$, so $g \in \mathcal{B}$ with $\|g\|_{\mathcal{B}} \le 1$, and

(iii) $|g(z) - g(0)| = \left|\dfrac{1}{2} \cdot l(z) - \dfrac{1}{2} \cdot l(0)\right| = \dfrac{1}{2} \log \dfrac{1 + |z|}{1 - |z|} = \beta(z,0)$.

Thus,

$$\sup\{|f(z) - f(0)| : f \in \mathcal{B},\ \|f\|_{\mathcal{B}} \le 1\} = \beta(z,0), \tag{3.44}$$

and note that (3.44) holds also for $z = 0$.

Now, let $z, w \in \mathbb{D}$. By Theorem 3.3.3,

$$|f(z) - f(w)| \le \|f\|_{\mathcal{B}} \cdot \beta(z,w) \le \beta(z,w), \text{ for all } f \in \mathcal{B}, \text{ with } \|f\|_{\mathcal{B}} \le 1,$$

so

$$\sup\{|f(z) - f(w)| : f \in \mathcal{B},\ \|f\|_{\mathcal{B}} \le 1\} \le \beta(z,w). \tag{3.45}$$

Moreover, by Proposition 3.3.1, the Möbius invariance of the hyperbolic metric and (3.44),

$$\begin{aligned} \beta(z,w) &= \beta(\phi_w(z), 0) = \sup\{|f(\phi_w(z)) - f(0)| : f \in \mathcal{B},\ \|f\|_{\mathcal{B}} \le 1\} \\ &= \sup\{|(f \circ \phi_w)(z) - (f \circ \phi_w)(w)| : f \in \mathcal{B},\ \|f \circ \phi_w\|_{\mathcal{B}} \le 1\} \\ &\le \sup\{|f(z) - f(w)| : f \in \mathcal{B},\ \|f\|_{\mathcal{B}} \le 1\}. \end{aligned} \tag{3.46}$$

The desired equality follows from (3.45) and (3.46). □

Theorem 3.3.6. *Suppose* $f \in H(\mathbb{D})$ *and* $s, t \in (0, +\infty)$ *with* $s + t = 1$. *Then* $f \in \mathcal{B}$ *if and only if there exists a constant* $C > 0$ *such that*

$$(1 - |z|^2)^s (1 - |w|^2)^t \cdot \frac{|f(z) - f(w)|}{|z - w|} \le C, \textit{ for all } z, w \in \mathbb{D}, \textit{ with } z \ne w. \tag{3.47}$$

Proof. First assume that (3.47) holds for some positive constant C, and let $z \in \mathbb{D}$. By letting $w \to z$ in (3.47), we get $(1 - |z|^2)|f'(z)| \le C < +\infty$, so $f \in \mathcal{B}$.

Conversely, assume that $f \in \mathcal{B}$. Then, by Theorem 3.3.3,

$$\frac{|f(z) - f(w)|}{\beta(z,w)} \le \|f\|_{\mathcal{B}}, \text{ for all } z, w \in \mathbb{D}, \text{ with } z \ne w. \tag{3.48}$$

Now consider the function $F(z,w) = (1-|z|^2)^s(1-|w|^2)^t \cdot \dfrac{\beta(z,w)}{|z-w|}$, for all $z, w \in \mathbb{D}$, with $z \neq w$. If there exists a constant $M > 0$ such that

$$F(z,w) \leq M, \text{ for all } z, w \in \mathbb{D}, \text{ with } z \neq w, \tag{3.49}$$

then, by multiplying (3.48) and (3.49) by parts, we get (3.47) for $C = \|f\|_{\mathcal{B}} M + 1 > 0$. Thus, it suffices to show that F is bounded.
Since s, t are positive numbers with $s+t=1$, we may assume without loss of generality that $0 < t \leq \frac{1}{2}$. Let $z, w \in \mathbb{D}$, with $z \neq w$. Then there exists a point $u \in \mathbb{D} \setminus \{0\}$ such that $w = \phi_z(u)$, so, using the Möbius invariance of the hyperbolic metric and Proposition 1.1.2 (iv),

$$\begin{aligned} F(z,w) &= (1-|z|^2)^s(1-|\phi_z(u)|^2)^t \cdot \frac{\beta(z,\phi_z(u))}{|z-\phi_z(u)|} \\ &= (1-|z|^2) \cdot \frac{(1-|u|^2)^t}{|1-\bar{z}u|^{2t}} \cdot \frac{\beta(0,u)|1-\bar{z}u|}{|z(1-\bar{z}u)-(z-u)|} \\ &= |1-\bar{z}u|^{1-2t} \cdot \frac{(1-|u|^2)^t}{2|u|} \cdot \log\frac{1+|u|}{1-|u|} = |1-\bar{z}u|^{1-2t} \cdot h(|u|), \end{aligned}$$

where $h(x) = \dfrac{(1-x^2)^t}{2x} \cdot \log\dfrac{1+x}{1-x}$, for all $x \in (0,1)$. We have:

(i)
$$\lim_{x\to 0^+} h(x) = \frac{1}{2}\lim_{x\to 0^+}\frac{1}{x}\cdot\log\frac{1+x}{1-x} = \frac{1}{2}\lim_{x\to 0^+}\frac{1-x}{1+x}\cdot\frac{2}{(1-x)^2} = 1,$$

so there exists some $\delta \in (0, \frac{1}{3})$ such that $h(x) < 2$, for all $x \in (0,\delta)$,

(ii)

$$\begin{aligned} \lim_{x\to 1^-} h(x) &= \frac{1}{2}\lim_{x\to 1^-}(1+x)^t(1-x)^t(\log(1+x)-\log(1-x)) \\ &= -\frac{2^t}{2}\lim_{x\to 1^-}(1-x)^t\log(1-x) \\ &= -2^{t-1}\cdot\lim_{y\to 0^+} y^t\log y = 0, \end{aligned}$$

so there exists some $\tilde{\delta} \in (\frac{2}{3}, 1)$ such that $h(x) < 1$, for all $x \in (\tilde{\delta}, 1)$, and

(iii) h is continuous in $[\delta, \tilde{\delta}]$, so there exists a constant $L > 0$ such that $h(x) \leq L$, for all $x \in [\delta, \tilde{\delta}]$.

Thus, for $\tilde{L} = \max\{2, 1, L\} > 0$ we have that

$$h(x) \leq \tilde{L}, \text{ for all } x \in (0,1). \tag{3.50}$$

Moreover, since $1-2t \geq 0$, we have

$$|1-\bar{z}u|^{1-2t} \leq (1+|\bar{z}||u|)^{1-2t} \leq 2^{1-2t}. \tag{3.51}$$

By (3.50) and (3.51),

$$F(z,w) = |1-\bar{z}u|^{1-2t}\cdot h(|u|) \leq 2^{1-2t}\cdot\tilde{L} =: M,$$

and M is independent of z, w, hence F is bounded. This completes the proof of the theorem. □

APPENDIX A

Some functional analytic results

Lemma A.0.1. *Let $z \in \mathbb{D}$, $c \in \mathbb{R}$, $t > -1$ and*

$$I_{c,t}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-z\bar{w}|^{2+t+c}}\, dA(w). \tag{A.1}$$

(i) If $c < 0$, then $I_{c,t}$ is bounded as a function of z.

(ii) If $c = 0$ and $|z| \geq \frac{1}{2}$, then there exist constants $C_1 = C_1(t)$, $C_2 = C_2(t) > 0$ such that

$$C_1 \log \frac{1}{1-|z|^2} \leq I_{0,t}(z) \leq C_2 \log \frac{1}{1-|z|^2}.$$

(iii) If $c > 0$ and $|z| \geq \frac{1}{2}$, then there exist constants $C_1 = C_1(c,t)$, $C_2 = C_2(c,t) > 0$ such that

$$\frac{C_1}{(1-|z|^2)^c} \leq I_{c,t}(z) \leq \frac{C_2}{(1-|z|^2)^c}.$$

Proof. First, note that the integral in (A.1) is defined for all $z \in \mathbb{D}$. Indeed, let $z \in \mathbb{D}$ and distinguish the following cases:
Case 1. $2 + t + c \geq 0$. Then

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-z\bar{w}|^{2+t+c}}\, dA(w) \leq \int_{\mathbb{D}} \frac{(1-|w|^2)^t}{(1-|z|)^{2+t+c}}\, dA(w) = \frac{1}{(1-|z|)^{2+t+c}} \cdot \frac{1}{t+1} < +\infty. \tag{A.2}$$

Case 2. $2 + t + c < 0$. Then

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-z\bar{w}|^{2+t+c}}\, dA(w) \leq \int_{\mathbb{D}} \frac{(1-|w|^2)^t}{2^{2+t+c}}\, dA(w) = \frac{1}{2^{2+t+c}} \cdot \frac{1}{t+1} < +\infty. \tag{A.3}$$

Next, set $\lambda = \frac{1}{2}(2+t+c)$. If $c \geq 0$, then $\lambda > \frac{1}{2}$, so if $\lambda = 0$ or $\lambda = -n$ for some $n \in \mathbb{N}$, then $c < 0$, and (A.2) and (A.3) imply that $I_{c,t}$ is bounded in $\mathbb{D}$.

If $\lambda \neq 0$ and $\lambda \neq -n$ for all $n \in \mathbb{N}$, then $\Gamma(\lambda)$ and $\Gamma(n+\lambda)$ are defined for all $n \in \mathbb{N}$. Let $z \in \mathbb{D}$. Then for all $w \in \mathbb{D}$, a generalization of the Binomial Theorem ([13, Exercise 5.2.4]) yields that

$$(1-z\bar{w})^{-\lambda} = \sum_{n=0}^{+\infty} \frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)} z^n \bar{w}^n \text{ and } (1-\bar{z}w)^{-\lambda} = \sum_{n=0}^{+\infty} \frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)} \bar{z}^n w^n,$$

hence

$$\frac{1}{|1-z\bar{w}|^{2+t+c}} = \frac{1}{|1-z\bar{w}|^{2\lambda}} = \frac{1}{(1-z\bar{w})^{\lambda}(1-\bar{z}w)^{\lambda}} = \lim_{N\to+\infty}\sum_{n=0}^{N}\sum_{m=0}^{N} a_n a_m z^n \bar{w}^n \bar{z}^m w^m, \quad \text{(A.4)}$$

where $a_k = \dfrac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)}$, for all $k \in \mathbb{N}\cup\{0\}$.

Now, consider a sequence $\{r_k\} \subset (0,1)$ such that $\lim_{k\to+\infty} r_k = 1$ and $r_k < r_{k+1}$, for all $k \in \mathbb{N}$. We make the following observations in order to interchange the integral and sum signs:

(a) By [6, 2.24 The Dominated Convergence Theorem],

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-z\bar{w}|^{2\lambda}}\, dA(w) = \lim_{k\to+\infty}\int_{D(0,r_k)} \frac{(1-|w|^2)^t}{|1-z\bar{w}|^{2\lambda}}\, dA(w).$$

(b) Let $r \in (0,1)$ and consider the functions

$$s_N(\theta) = \sum_{n=0}^{N} a_n \bar{z}^n r^n e^{in\theta}, \text{ for all } \theta \in [0,2\pi] \text{ and } N \in \mathbb{N}\cup\{0\}.$$

Since the power series $\sum_{n=0}^{+\infty} a_n \bar{z}^n w^n$ converges absolutely in $\mathbb{D}$, we get that s_N is uniformly bounded in $[0,2\pi]$ and converges uniformly in $[0,2\pi]$, hence $s_N\bar{s}_N$ converges uniformly in $[0,2\pi]$.

(c) Let $R \in (0,1)$. Then the series $\sum_{n=0}^{+\infty} a_n^2|z|^{2n}r^{2n}$ converges uniformly in $[0,R]$. Indeed,

$$a_n^2|z|^{2n}r^{2n} \leq a_n^2|z|^{2n}R^{2n}, \text{ for all } r \in [0,R] \text{ and } n \in \mathbb{N}\cup\{0\},$$

and we can verify that the series $\sum_{n=0}^{+\infty} a_n^2|z|^{2n}R^{2n}$ converges by doing a root test. The desired result follows by the Weierstrass criterion for the uniform convergence of series.

(d) Since

$$\int_0^1 (1-r^2)^t r^{2n} 2r\, dr = \int_0^1 (1-r)^t r^n\, dr = B(n+1,t+1) = \frac{\Gamma(n+1)\Gamma(t+1)}{\Gamma(n+t+2)} < +\infty,$$

by [6, 2.24 The Dominated Convergence Theorem] we get

$$\lim_{k\to+\infty}\int_0^{r_k} (1-r^2)^t r^{2n} 2r\, dr = \int_0^1 (1-r^2)^t r^{2n} 2r\, dr.$$

Using (A.4), (a), (b), (c) and (d),

$$\begin{aligned}
I_{c,t}(z) &= \lim_{k\to+\infty}\int_{D(0,r_k)} (1-|w|^2)^t \lim_{N\to+\infty}\sum_{n=0}^{N}\sum_{m=0}^{N} a_n a_m z^n \bar{w}^n \bar{z}^m w^m\, dA(w) \\
&= \lim_{k\to+\infty}\int_0^{r_k}\int_0^{2\pi} (1-r^2)^t \cdot \frac{r}{\pi} \lim_{N\to+\infty} (\bar{s}_N(\theta)s_N(\theta))\, d\theta dr \\
&= \lim_{k\to+\infty}\int_0^{r_k} (1-r^2)^t \cdot \frac{r}{\pi} \lim_{N\to+\infty}\int_0^{2\pi}\sum_{n=0}^{N}\sum_{m=0}^{N} a_n a_m z^n \bar{z}^m r^{n+m} e^{i(m-n)\theta}\, d\theta dr \\
&= \lim_{k\to+\infty}\int_0^{r_k} (1-r^2)^t \cdot \frac{r}{\pi} \lim_{N\to+\infty}\sum_{n=0}^{N}\sum_{m=0}^{N} a_n a_m z^n \bar{z}^m r^{n+m} \int_0^{2\pi} e^{i(m-n)\theta}\, d\theta dr \\
&= \lim_{k\to+\infty}\int_0^{r_k} (1-r^2)^t \cdot 2r \lim_{N\to+\infty}\sum_{n=0}^{N} a_n^2|z|^{2n}r^{2n}\, dr \\
&= \lim_{k\to+\infty}\lim_{N\to+\infty}\sum_{n=0}^{N} a_n^2|z|^{2n}\int_0^{r_k} (1-r^2)^t r^{2n} 2r\, dr
\end{aligned}$$

$$= \sup_{k\in\mathbb{N}} \sup_{N\in\mathbb{N}\cup\{0\}} \sum_{n=0}^{N} a_n^2 |z|^{2n} \int_0^{r_k} (1-r^2)^t r^{2n} 2r\, dr$$
$$= \sup_{N\in\mathbb{N}\cup\{0\}} \sup_{k\in\mathbb{N}} \sum_{n=0}^{N} a_n^2 |z|^{2n} \int_0^{r_k} (1-r^2)^t r^{2n} 2r\, dr$$
$$= \lim_{N\to+\infty} \lim_{k\to+\infty} \sum_{n=0}^{N} a_n^2 |z|^{2n} \int_0^{r_k} (1-r^2)^t r^{2n} 2r\, dr = \sum_{n=0}^{+\infty} a_n^2 |z|^{2n} \int_0^{1} (1-r^2)^t r^{2n} 2r\, dr$$
$$= \sum_{n=0}^{+\infty} \frac{\Gamma^2(n+\lambda)}{(n!)^2\Gamma^2(\lambda)} \frac{\Gamma(n+1)\Gamma(t+1)}{\Gamma(n+t+2)} |z|^{2n} = \frac{\Gamma(t+1)}{\Gamma^2(\lambda)} \sum_{n=0}^{+\infty} \frac{\Gamma^2(n+\lambda)}{n!\Gamma(n+t+2)} |z|^{2n},$$

so

$$I_{c,t}(z) = \frac{1}{t+1} + \frac{\Gamma(t+1)}{\Gamma^2(\lambda)} \sum_{n=1}^{+\infty} \frac{\Gamma^2(n+\lambda)}{n!\Gamma(n+t+2)} |z|^{2n}. \tag{A.5}$$

At this point, we use a version of Stirling's formula, namely that for any $A > 0$,

$$\lim_{x\to+\infty} \sup_{0\le a\le A} \left| \frac{x^a \Gamma(x)}{\Gamma(x+a)} - 1 \right| = 0 \tag{A.6}$$

(see [5, Lemma 7.61]). By (A.6) we derive that

$$\lim_{n\to+\infty} \frac{\Gamma(n+\lambda)}{\Gamma(n+2\lambda-c) n^{c-\lambda}} = 1$$

and

$$\lim_{n\to+\infty} \frac{\Gamma(n+\lambda)}{n!\cdot n^{\lambda-1}} = 1,$$

from which it follows that

$$\lim_{n\to+\infty} \frac{\Gamma^2(n+\lambda)}{n!\Gamma(n+t+2) n^{c-1}} = 1.$$

Thus, there exists a number $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\Gamma^2(n+\lambda)}{n!\Gamma(n+t+2) n^{c-1}} - 1 \right| < \frac{1}{2}, \text{ for all } n \ge n_0,$$

or

$$\frac{1}{2} n^{c-1} < \frac{\Gamma^2(n+\lambda)}{n!\Gamma(n+t+2)} < \frac{3}{2} n^{c-1}, \text{ for all } n \ge n_0.$$

Consequently, we can find constants $\tilde{C}_1 = \tilde{C}_1(c,t),\ \tilde{C}_2 = \tilde{C}_2(c,t) > 0$ such that

$$\tilde{C}_1 n^{c-1} \le \frac{\Gamma^2(n+\lambda)}{n!\Gamma(n+t+2)} \le \tilde{C}_2 n^{c-1}, \text{ for all } n \in \mathbb{N}. \tag{A.7}$$

By (A.5) and (A.7),

$$\frac{\Gamma(t+1)}{\Gamma^2(\lambda)} \tilde{C}_1 \sum_{n=1}^{+\infty} n^{c-1} |z|^{2n} \le I_{c,t}(z) \le \frac{1}{t+1} + \frac{\Gamma(t+1)}{\Gamma^2(\lambda)} \tilde{C}_2 \sum_{n=1}^{+\infty} n^{c-1} |z|^{2n}. \tag{A.8}$$

If $c < 0$, then (A.8) gives

$$I_{c,t}(z) \le \frac{1}{t+1} + \frac{\Gamma(t+1)}{\Gamma^2(\lambda)} \tilde{C}_2 \sum_{n=1}^{+\infty} n^{c-1} |z|^{2n} \le \frac{1}{t+1} + \frac{\Gamma(t+1)}{\Gamma^2(\lambda)} \tilde{C}_2 \sum_{n=1}^{+\infty} \frac{1}{n^{1-c}} < +\infty,$$

that is, $I_{c,t}$ is bounded in $\mathbb{D}$.
If $c = 0$ and $|z| \geq \frac{1}{2}$, then

$$\sum_{n=1}^{+\infty} n^{-1}|z|^{2n} = \log \frac{1}{1-|z|^2} \tag{A.9}$$

and there exists a constant $\hat{C} = \hat{C}(t) > 0$ such that

$$\hat{C} \log \frac{1}{1-\left(\frac{1}{2}\right)^2} \geq \frac{1}{t+1}. \tag{A.10}$$

By (A.8),

$$I_{0,t}(z) \geq \frac{\Gamma(t+1)}{\Gamma^2(\lambda)} \tilde{C}_1 \sum_{n=1}^{+\infty} n^{-1}|z|^{2n} = C_1 \log \frac{1}{1-|z|^2},$$

where $C_1 = C_1(t) = \dfrac{\Gamma(t+1)}{\Gamma^2(\lambda)} \tilde{C}_1 > 0$, and by (A.8), (A.9) and (A.10),

$$\begin{aligned} I_{0,t}(z) &\leq \frac{1}{t+1} + \frac{\Gamma(t+1)}{\Gamma^2(\lambda)} \tilde{C}_2 \sum_{n=1}^{+\infty} n^{-1}|z|^{2n} \leq \hat{C} \log \frac{1}{1-\left(\frac{1}{2}\right)^2} + \frac{\Gamma(t+1)}{\Gamma^2(\lambda)} \tilde{C}_2 \log \frac{1}{1-|z|^2} \\ &\leq \hat{C} \log \frac{1}{1-|z|^2} + \frac{\Gamma(t+1)}{\Gamma^2(\lambda)} \tilde{C}_2 \log \frac{1}{1-|z|^2} = C_2 \log \frac{1}{1-|z|^2}, \end{aligned}$$

where $C_2 = C_2(t) = \hat{C} + \dfrac{\Gamma(t+1)}{\Gamma^2(\lambda)} \tilde{C}_2 > 0$.
Finally, if $c > 0$ and $|z| \geq \frac{1}{2}$, then (A.6) gives that

$$\lim_{n \to +\infty} \frac{\Gamma(n+c)}{n! \cdot n^{c-1}} = 1,$$

so we can find constants $\tilde{C}_3 = \tilde{C}_3(c,t)$, $\tilde{C}_4 = \tilde{C}_4(c,t) > 0$ such that

$$\tilde{C}_3 \frac{\Gamma(n+c)}{n!} \leq n^{c-1} \leq \tilde{C}_4 \frac{\Gamma(n+c)}{n!}, \text{ for all } n \in \mathbb{N}. \tag{A.11}$$

Also, note that

$$\sum_{n=1}^{+\infty} \frac{\Gamma(n+c)}{n!\Gamma(c)} |z|^{2n} = \frac{1}{(1-|z|^2)^c}, \tag{A.12}$$

and there exists a constant $C = C(c,t) > 0$ such that

$$C \cdot \frac{1}{\left(1-\left(\frac{1}{2}\right)^2\right)^c} \geq \frac{1}{t+1}. \tag{A.13}$$

By (A.8), (A.11) and (A.12),

$$I_{c,t}(z) \geq \frac{\Gamma(t+1)}{\Gamma^2(\lambda)} \tilde{C}_1 \sum_{n=1}^{+\infty} n^{c-1}|z|^{2n} \geq \frac{\Gamma(t+1)\Gamma(c)}{\Gamma^2(\lambda)} \tilde{C}_1 \tilde{C}_3 \sum_{n=1}^{+\infty} \frac{\Gamma(n+c)}{n!\Gamma(c)} |z|^{2n} = C_1 \frac{1}{(1-|z|^2)^c},$$

where $C_1 = C_1(c,t) = \dfrac{\Gamma(t+1)\Gamma(c)}{\Gamma^2(\lambda)} \tilde{C}_1 \tilde{C}_3 > 0$, and by (A.8), (A.11), (A.12) and (A.13),

$$\begin{aligned} I_{c,t}(z) &\leq \frac{1}{t+1} + \frac{\Gamma(t+1)}{\Gamma^2(\lambda)} \tilde{C}_2 \sum_{n=1}^{+\infty} n^{c-1}|z|^{2n} \\ &\leq C \cdot \frac{1}{\left(1-\left(\frac{1}{2}\right)^2\right)^c} + \frac{\Gamma(t+1)\Gamma(c)}{\Gamma^2(\lambda)} \tilde{C}_2 \tilde{C}_4 \sum_{n=1}^{+\infty} \frac{\Gamma(n+c)}{n!\Gamma(c)} |z|^{2n} \\ &\leq C \cdot \frac{1}{(1-|z|^2)^c} + \frac{\Gamma(t+1)\Gamma(c)}{\Gamma^2(\lambda)} \tilde{C}_2 \tilde{C}_4 \frac{1}{(1-|z|^2)^c} = C_2 \frac{1}{(1-|z|^2)^c}, \end{aligned}$$

where $C_2 = C_2(c,t) = C + \dfrac{\Gamma(t+1)\Gamma(c)}{\Gamma^2(\lambda)}\tilde{C}_2\tilde{C}_4 > 0.$ $\square$

Theorem A.0.2. *Let $p \geq 1$, $\alpha > -1$ and $a, b \in \mathbb{R}$ such that*

$$-pa < \alpha + 1 < p(b+1). \tag{A.14}$$

Then

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^b}{|1-z\bar{w}|^{2+a+b}}|f(w)|\,dA(w) < +\infty, \text{ for all } z \in \mathbb{D} \text{ and } f \in L^p(\mathbb{D}, dA_\alpha), \tag{A.15}$$

and the operators $L, S, T : L^p(\mathbb{D}, dA_\alpha) \to L^p(\mathbb{D}, dA_\alpha)$ given by

$$Lf(z) = (1-|z|^2)^a \int_{\mathbb{D}} \frac{(1-|w|^2)^b}{|1-z\bar{w}|^{2+a+b}}|f(w)|\,dA(w),$$

$$Sf(z) = (1-|z|^2)^a \int_{\mathbb{D}} \frac{(1-|w|^2)^b}{(1-z\bar{w})^{2+a+b}}f(w)\,dA(w),$$

$$Tf(z) = (1-|z|^2)^a \int_{\mathbb{D}} \frac{(1-|w|^2)^b}{|1-z\bar{w}|^{2+a+b}}f(w)\,dA(w),$$

are well-defined and bounded on $L^p(\mathbb{D}, dA_\alpha)$.

Proof. We distinguish two cases for p :

Case 1. $p > 1$. Let $q \in (1, +\infty)$ be the conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. By (A.14) we can derive that

$$-\frac{b+1}{q} < \frac{a}{q} \text{ and } -\frac{a+\alpha+1}{p} < \frac{b-\alpha}{p},$$

because $-pa < p(b+1)$, as well as

$$p(b+1) > \alpha+1 \;\Leftrightarrow\; b+1 > \frac{\alpha}{p} + \frac{1}{p} \;\Leftrightarrow\; b\Big(\frac{1}{p}+\frac{1}{q}\Big) - \frac{\alpha}{p} > -\frac{1}{q} \;\Leftrightarrow\; \frac{b-\alpha}{p} > -\frac{b+1}{q},$$

and

$$-pa < \alpha+1 \;\Leftrightarrow\; a\Big(\frac{1}{p}+\frac{1}{q}\Big) > -\frac{\alpha+1}{p} \;\Leftrightarrow\; \frac{a}{q} > -\frac{a+\alpha+1}{p}.$$

Thus, we can choose a real σ such that

$$\sigma \in \Big(-\frac{b+1}{q}, \frac{a}{q}\Big) \bigcap \Big(-\frac{a+\alpha+1}{p}, \frac{b-\alpha}{p}\Big) \neq \emptyset.$$

Now, consider the functions

$$h(z) = (1-|z|^2)^\sigma, \text{ for all } z \in \mathbb{D}, \text{ and } H(z,w) = \frac{1}{\alpha+1} \cdot \frac{(1-|z|^2)^a(1-|w|^2)^{b-\alpha}}{|1-z\bar{w}|^{2+a+b}}, \text{ for all } z, w \in \mathbb{D}.$$

We have

$$\int_{\mathbb{D}} H(z,w)h^q(w)\,dA_\alpha(w) = (1-|z|^2)^a \int_{\mathbb{D}} \frac{(1-|w|^2)^{\sigma q+b}}{|1-z\bar{w}|^{2+a+b}}\,dA(w), \text{ for all } z \in \mathbb{D}. \tag{A.16}$$

If $|z| \geq \frac{1}{2}$, then Lemma A.0.1 (iii) gives that there exists a constant $C_1 = C_1(q, a, b, \sigma) > 0$ such that

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^{\sigma q+b}}{|1-z\bar{w}|^{2+a+b}}\,dA(w) \leq C_1 \cdot \frac{1}{(1-|z|^2)^{a-\sigma q}} = C_1 \cdot \frac{h^q(z)}{(1-|z|^2)^a}, \tag{A.17}$$

so (A.16) and (A.17) give

$$\int_{\mathbb{D}} H(z,w)h^q(w)\,dA_\alpha(w) \le C_1 h^q(z). \tag{A.18}$$

If $|z| < \frac{1}{2}$, then $2+a+b = 2+\sigma q + b + a - \sigma q > 1$, by the choice of σ, hence

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^{\sigma q+b}}{|1-z\bar{w}|^{2+a+b}}\,dA(w) \le 2^{2+a+b}\int_{\mathbb{D}}(1-|w|^2)^{\sigma q+b}\,dA(w) = \frac{2^{2+a+b}}{\sigma q+b+1}, \tag{A.19}$$

so (A.16) and (A.19) give

$$\int_{\mathbb{D}} H(z,w)h^q(w)\,dA_\alpha(w) \le \frac{2^{2+a+b}(1-|z|^2)^a}{\sigma q+b+1} \le \frac{2^{2+a+b}(1-|z|^2)^{\sigma q}}{\sigma q+b+1} = \frac{2^{2+a+b}}{\sigma q+b+1}h^q(z). \tag{A.20}$$

Thus, if we set $C = C(q,a,b,\sigma) = \max\left\{C_1, \dfrac{2^{2+a+b}}{\sigma q+b+1}\right\} > 0$, then (A.18) and (A.20) give

$$\int_{\mathbb{D}} H(z,w)h^q(w)\,dA_\alpha(w) \le Ch^q(z), \text{ for all } z \in \mathbb{D}. \tag{A.21}$$

Working similarly, we can derive that there exists a constant $\tilde{C} = \tilde{C}(p,\alpha,a,b,\sigma) > 0$ such that

$$\int_{\mathbb{D}} H(z,w)h^p(z)\,dA_\alpha(z) \le \tilde{C}h^p(w), \text{ for all } w \in \mathbb{D}. \tag{A.22}$$

By (A.21), (A.22) and the proof of Schur's Theorem ([15, Theorem 3.6]), $Lf \in L^p(\mathbb{D}, dA_\alpha)$, for all $f \in L^p(\mathbb{D}, dA_\alpha)$, and the operator L is bounded on $L^p(\mathbb{D}, dA_\alpha)$.

Case 2. $p = 1$. Let $f \in L^1(\mathbb{D}, dA_\alpha)$. Then, by [6, 2.37a The Fubini-Tonelli Theorem],

$$\begin{aligned}\int_{\mathbb{D}} |Lf(z)|\,dA_\alpha(z) &= \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|z|^2)^a(1-|w|^2)^b}{|1-z\bar{w}|^{2+a+b}}|f(w)|\,dA(w)dA_\alpha(z)\\ &= \int_{\mathbb{D}}(1-|w|^2)^b\cdot|f(w)|\int_{\mathbb{D}}\frac{(1-|z|^2)^a}{|1-z\bar{w}|^{2+a+b}}\,dA_\alpha(z)dA(w)\\ &= \int_{\mathbb{D}}(1-|w|^2)^b\cdot|f(w)|(\alpha+1)\int_{\mathbb{D}}\frac{(1-|z|^2)^{a+\alpha}}{|1-\bar{z}w|^{2+a+b}}\,dA(z)dA(w).\end{aligned} \tag{A.23}$$

If $|w| \ge \frac{1}{2}$, then Lemma A.0.1 (iii) gives that there exists a constant $C_1 = C_1(\alpha,a,b) > 0$ such that

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^{a+\alpha}}{|1-\bar{z}w|^{2+a+b}}\,dA(z) \le C_1\cdot\frac{1}{(1-|w|^2)^{b-\alpha}}. \tag{A.24}$$

If $|w| < \frac{1}{2}$, then $b-\alpha > 0$ and $2+a+b > 1$, because of (A.14), hence

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^{a+\alpha}}{|1-\bar{z}w|^{2+a+b}}\,dA(z) \le 2^{2+a+b}\int_{\mathbb{D}}(1-|z|^2)^{a+\alpha}\,dA(z) \le \frac{2^{2+a+b}}{a+\alpha+1}\cdot\frac{1}{(1-|w|^2)^{b-\alpha}}. \tag{A.25}$$

Thus, if we set $C = C(\alpha,a,b) = \max\left\{C_1, \dfrac{2^{2+a+b}}{a+\alpha+1}\right\} > 0$, then (A.24) and (A.25) give that

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^{a+\alpha}}{|1-\bar{z}w|^{2+a+b}}\,dA(z) \le C\cdot\frac{1}{(1-|w|^2)^{b-\alpha}}, \text{ for all } w \in \mathbb{D}. \tag{A.26}$$

By (A.23) and (A.26),

$$\begin{aligned}\int_{\mathbb{D}} |Lf(z)|\,dA_\alpha(z) &\le \int_{\mathbb{D}}(1-|w|^2)^b\cdot|f(w)|(\alpha+1)C\cdot\frac{1}{(1-|w|^2)^{b-\alpha}}\,dA(w)\\ &= C\int_{\mathbb{D}}|f(w)|\,dA_\alpha(w) < +\infty,\end{aligned}$$

so $Lf \in L^1(\mathbb{D}, dA_\alpha)$, and $\|Lf\|_{1,\alpha} \le C\|f\|_{1,\alpha}$, that is, L is bounded on $L^1(\mathbb{D}, dA_\alpha)$.

In any case, we have proved that L is a well-defined bounded operator on $L^p(\mathbb{D}, dA_\alpha)$. Then, it easy to see that S, T are well-defined bounded operators on $L^p(\mathbb{D}, dA_\alpha)$.

It remains to show (A.15). Let $f \in L^p(\mathbb{D}, dA_\alpha)$. Since $Lf \in L^p(\mathbb{D}, dA_\alpha)$, there exists a set $E_f \subset \mathbb{D}$ such that $A_\alpha(E_f) = 0$ and

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^b}{|1-z\bar{w}|^{2+a+b}} |f(w)| \, dA(w) < +\infty, \text{ for all } z \in \mathbb{D} \setminus E_f. \tag{A.27}$$

Let $z \in E_f$. Since $A_\alpha(E_f) = 0$, there exists a sequence $\{z_n\} \subset \mathbb{D} \setminus E_f$ with $\lim_{n \to +\infty} z_n = z$. Thus, there exists a real $R \in (0,1)$ such that $|z_n| \leq R$, for all $n \in \mathbb{N}$. Note that

$$\lim_{n \to +\infty} \frac{(1-|w|^2)^b}{|1-z_n\bar{w}|^{2+a+b}} |f(w)| = \frac{(1-|w|^2)^b}{|1-z\bar{w}|^{2+a+b}} |f(w)|, \text{ for all } w \in \mathbb{D}, \tag{A.28}$$

and

$$\frac{(1-|w|^2)^b}{|1-z_n\bar{w}|^{2+a+b}} |f(w)| \leq \frac{(1-|w|^2)^b}{(1-R)^{2+a+b}} |f(w)|, \text{ for all } w \in \mathbb{D} \text{ and } n \in \mathbb{N}. \tag{A.29}$$

Also, since $|1 - z_1\bar{w}| \leq 2$, for all $w \in \mathbb{D}$, there exists a constant $\tilde{C} > 0$ such that

$$\frac{1}{(1-R)^{2+a+b}} \leq \tilde{C} \cdot \frac{1}{2^{2+a+b}} \leq \tilde{C} \cdot \frac{1}{|1-z_1\bar{w}|^{2+a+b}}, \text{ for all } w \in \mathbb{D}, \tag{A.30}$$

and by (A.27),

$$\int_{\mathbb{D}} \tilde{C} \cdot \frac{(1-|w|^2)^b}{|1-z_1\bar{w}|^{2+a+b}} |f(w)| \, dA(w) < +\infty. \tag{A.31}$$

By (A.28), (A.29), (A.30) and (A.31), [6, 2.24 The Dominated Convergence Theorem] yields that

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^b}{|1-z\bar{w}|^{2+a+b}} |f(w)| \, dA(w) < +\infty.$$

Thus, we have shown that the integral in (A.27) is finite for all $z \in \mathbb{D}$. □

Let $(X, \|\cdot\|_X)$ be a Banach space. By $L(X)$ we denote the space of all bounded linear operators $T : X \to X$, and by $\|T\|$ the norm of T. Then, it is known from functional analysis that $(L(X), \|\cdot\|)$ is a Banach space. Also, if $T_1, T_2 \in L(X)$, then $T_1T_2 := T_1 \circ T_2 \in L(X)$ and $\|T_1T_2\| \leq \|T_1\|\|T_2\|$. For more information on these topics see, for example, [6, Section 5.1].

Proposition A.0.3. *Let $(X, \|\cdot\|_X)$ be a Banach space and $S \in L(X)$. If $\|I - S\| < 1$, where I is the identity operator, then S is invertible, that is, there exists an operator $T \in L(X)$ such that*

$$ST = I = TS.$$

Proof. Since $\|I - S\| < 1$,

$$\sum_{k=0}^{+\infty} \|(I-S)^k\| \leq \sum_{k=0}^{+\infty} \|I-S\|^k = \frac{1}{1-\|I-S\|} < +\infty.$$

Thus, $\sum_{k=0}^{+\infty}(I-S)^k$ converges in $L(X)$, because $L(X)$ is a Banach space, and

$$\lim_{k \to +\infty} (I-S)^k = O \text{ in } L(X), \tag{A.32}$$

where O is the null operator. Let $T = \sum_{k=0}^{+\infty}(I-S)^k$ and $T_n = \sum_{k=0}^{n}(I-S)^k$, for all $n \in \mathbb{N} \cup \{0\}$. Then

$$ST_n = (I-(I-S)) \sum_{k=0}^{n} (I-S)^k = \sum_{k=0}^{n} (I-S)^k - \sum_{k=1}^{n+1} (I-S)^k = I - (I-S)^{n+1}, \tag{A.33}$$

and

$$\|ST_n - ST\| = \|S(T_n - T)\| \leq \|S\|\|T_n - T\|, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{A.34}$$

Since $\lim_{n\to+\infty} T_n = T$ in $L(X)$, (A.34) implies that $\lim_{n\to+\infty} ST_n = ST$ in $L(X)$. Thus, by taking limits as $n \to +\infty$ in (A.33) and using (A.32), we get that $ST = I$. By analogous arguments, $TS = I$; hence S is invertible. □

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