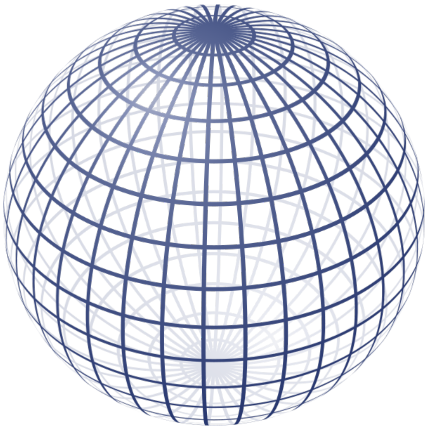

Spherically Symmetric Perfect Fluid

A Thesis Presented in Partial Fulfillment of the Bachelor's Degree

in

Physics

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Abstract

In the present work, we study the case of spherically symmetric distributions of perfect fluids. Within the framework of General Relativity, the Einstein Field Equations are derived and then verified both by recovering elementary solutions and with the assistance of computational algebra software, namely the open-source Maxima. Subsequently, a generalization of the original Ansatz is performed, to further allow for maximal spatial symmetry beyond the spherical one, followed by an investigation of the way in which the familiar FLRW metric emerges as a solution, in the case of a SEM tensor with space-independent proper energy density and pressure. Finally, the time-dependent version of the produced spherically symmetric EFEs is commented upon, as a "toy-model factory" for the treatment of many astrophysically interesting topics, regarding radiative processes and gravitational phenomena alike. The Vaidya metric is showcased as an introductory example of an appropriate time-dependent exterior geometry, and the correspondance between scalar field SEM tensors to that of perfect fluids (pressureless or not) is demonstrated.

To my dear grandmother,

Χρυσή Μπικάκη - Μιχελάκη

The strongest, the kindest, the brightest.

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Chapter 1

Introduction

Gravity, until perhaps proven otherwise, is most accurately described by the General theory of Relativity, as the curvature of spacetime produced by a distribution of matter and radiation. Relativity, as a notion, essentially refers to the physical impossibility of absolute motion detection as imposed by a physical symmetry according to which the expression of Physical laws in equations remains invariant under coordinate transformations. Between only inertial frames, as far as SR is concerned, with GR allowing also for accelerating ones.

In the present work, we begin with a quick overview of the natural language of relativity, that of tensors, manifolds and Riemannian geometry and of the fundamental ideas at the core of both Special and General Relativity and, then, continue with our study of the geometries produced by spherically symmetric distributions of perfect fluid, both static and time-dependant.

1.1 Basic Ingredients

The postulates of SR

- Physical laws have the same form for all inertial observers. No distinction is achievable between two inertial frames of reference.
- The speed of light, c, is an absolute constant with respect to both time and space.

A direct consequence of the second one is that for c to be trully absolute, time must be relative and is therefore merely yet another coordinate, dependent on the particular observer. This conclusion leads to the definitive unification of space and time into a new "entity" we call spacetime. It follows that coordinate transformations constitute the natural language of relativity, since we may relate the physical expressions acquired in different reference frames exactly in means of them. This also allows for verification (by contrasting to experiment) of the first postulate within the framework of the necessary mathematical formalism.

The above inferences provide us with the concrete form of this formalism. The two types of mathematical structures that inherently emerge are manifolds and tensors. The first, as we shall see, in order to provide the means of describing the spacetime, attributing coordinates to it and measuring the appropriate distances that will allow us to perform physics in it and the second in order to describe the physical quantities involved in any phenomenon we wish to study, while ensuring equation form invariance under coordinate transformations, in accordance to the first postulate. Then, it is the job of Riemannnian geometry to account for the proper manipulation of the way in which the physical quantities described in terms of tensors vary accross the spacetime manifold.

1.2 Mathematical Framework

1.2.1 Manifolds

Manifolds are topological spaces with the property that in the neighborood of each point they are homeomorphic to an open subset of Euclidean space with the same number of dimensions as the manifold. An n-manifold therefore is locally homeomorfphic to an open subset of $\mathcal{R}^n$ in the neighborhood of each point. It admits therefore a set of n local coordinates. If the manifold is differentiable it also admits mappings between such sets, called coordinate transformations.

$$x^\mu \to x'^\mu = x'^\mu(x^1, x^2, ..., x^n) \quad , \quad \mu = 1, 2, ..., n$$

The transformation matrix $\frac{\partial x'^\alpha}{\partial x^\beta}$ is the $n \times n$ Jacobian matrix and its determinant, J, is known as the Jacobian of the transformation. The transformation is invertible if $J \neq 0$ and the inverse transformation matrix, $\frac{\partial x^\alpha}{\partial x'^\beta}$, is the inverse of the Jacobian matrix. since

$$x^\mu = x^\mu(x'^1, x'^2, ..., x'^n) \quad , \quad \mu = 1, 2, ..., n$$
$$\frac{\partial x^\alpha}{\partial x'^\beta}\frac{\partial x'^\beta}{\partial x^\gamma} = \delta^\alpha_\gamma$$

where for the second relation we used Einstein's implied summation notation, $\sum_{i=1}^n a_i b^i = a_i b^i$, according to which indices appearing in pairs of upper and lower in mononyms are being summed over, and subsequently applied the chain rule to determine the result.

1.2.2 Tensors

Mathematical objects can be categorised according to the way they transform under a general coordinate transformation, that is under an arbitrary change of reference frame. Scalars transform in the simplest way possible, remaining invariant under any such transformation, while for vectors expressed in contravariant components, we have

$$a'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} a^\mu$$

and when they are expressed in covariant ones

$$a'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} a_\nu$$

More general objects, with an arbitrary number of covariant and contravariant indices, would then transform as

$$T'^{\mu_1\mu_2...\mu_m}_{\nu_1\nu_2...\nu_n} = \prod_{i=1}^{m} \frac{\partial x'^{\mu_i}}{\partial x^{\kappa_i}} \prod_{j=1}^{n} \frac{\partial x^{\lambda_j}}{\partial x'^{\nu_j}} T^{\kappa_1\kappa_2...\kappa_m}_{\lambda_1\lambda_2...\lambda_n}$$

as each index simply trasnforms by use of the Jacobian matrix if it is contravariant and its inverse if it is covariant, independently of all others indices. Objects that transform in this way are what we call tensors and the above relation serves as their definition.

1.2.3 Riemannian Geometry

The additional structure that allows for the definition of distances and angles in a differential manifold is called the metric tensor, g, and for two infinitesimally separated points with coordinates x^μ and $(x^\mu + dx^\mu)$ respectively, the square of their distance, which is a scalar and is the same in all reference frames is provided by the expression

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

where $g_{\mu\nu}$ are the components of the metric tensor. We note that according to the transformation rule of tensors we described in the previous section, the right hand size is also invariant as it should be. Differential manifolds equipped with a metric tensor are called Riemannian if $ds^2 > 0$ and Pseudo-Riemannian if $ds^2 = 0$ or $ds^2 < 0$ are also possible. In the general case, the components of the metric tensor are functions of the coordinates, thus determinining the local geometry around any point of the manifold.

1.2.3.1 Local Coordinates

For a general Riemannian manifold, a global (i.e. for every point of the manifold) coordinate transformation $x^\mu \to x'^\mu$ such that

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \delta_{ab} dx'^a dx'^b$$

is not possible. However, it is possible locally, i.e. in the neighborhood of each point, therefore there always exist coordinates x'^a such that, at the arbitrary point P, we would have

$$g'_{ab}(P) = \delta_{ab}$$
$$\left.\frac{\partial g'_{ab}}{\partial x'^\sigma}\right|_P = 0$$
$$g'_{ab}(\boldsymbol{x}) = \delta_{ab} + \mathcal{O}[(\boldsymbol{x} - \boldsymbol{x}_P)^2]$$

In the more general case of Pseudo-Riemannian manifolds, the same holds with the substituion of δ_{ab} by the diagonal matrix with units and a metric signature corresponding to the particular convention involved. For example, in a manifold with (+,-,-,-) metric signature, the identity matrix would be substituted in the above expression by the diagonal matrix $diag(1, -1, -1, -1)$. Calling such a matrix λ_{ab} for generality, we can define the tangent space T_P of the manifold at P as the one with metric

$$\lambda_{ab} = g'_{ab}(\boldsymbol{x}) - \mathcal{O}[(\boldsymbol{x} - \boldsymbol{x}_P)^2]$$

1.2.3.2 General Coordinate Systems

Since a tangent space can be defined for every point of a manifold, we can define local basis vectors residing in the individual tangent space of each point as

$$\mathbf{e}_a = \lim_{\delta x^a \to 0} \frac{\delta \mathbf{s}}{\delta x^a}$$

where $\delta\mathbf{s}$ is the infinitesimal vector displacement between two points P and Q, along the x^a coordinate, by δx^a. Thus, $\mathbf{e}_a$ is the tangent vector to the x^a coordinate curve at the point P, while the infinitesimal separation of the two points is:

$$d\mathbf{s} = \mathbf{e}_a(x)\, dx^a$$

where we used x in the parenthesis instead of $\boldsymbol{x}$ for simplicity. It follows that

$$ds^2 = d\mathbf{s} \cdot d\mathbf{s} = (dx^a\, \mathbf{e}_a) \cdot (dx^b\, \mathbf{e}_b) = (\mathbf{e}_a \cdot \mathbf{e}_b)\, dx^a dx^b$$

We recovered, therefore, a defining expression for the components of the metric tensor in terms of the local coordinate bases

$$g_{ab}(x) = \mathbf{e}_a(x) \cdot \mathbf{e}_b(x)$$

At the same time, using these local coordinate bases we can define vector fields for the whole manifold as

$$\mathbf{u}(x) = u^a(x)\, \boldsymbol{e_a}(x)$$

The $u^\alpha(x)$ are the contravariant components of the vector field $\mathbf{u}(x)$ in the basis $\boldsymbol{e_a}(x)$. There also exists the dual vector space of each tangent space, where the covectors defined by the expression

$$\boldsymbol{e^a} \cdot \boldsymbol{e_b} = \delta^a_b$$

reside and therefore we also have

$$g^{ab}(x) = \mathbf{e}^a(x) \cdot \mathbf{e}^b(x)$$

and

$$\mathbf{u}(x) = u_a(x)\, \boldsymbol{e^a}(x)$$

Finally, the generalisation of the inner product for manifolds can then be described, again in terms of the local coordinate bases, as

$$\mathbf{v} \cdot \mathbf{w} = (u^a\, \mathbf{e}_a) \cdot (w^b\, \mathbf{e}_b) = g_{ab}\, u^a w^b$$

1.2.3.3 Parallel Transport - Connection

If $\mathbf{v}$ is a vector at a point P of a manifold, we can transport it along a displacement vector $d\mathbf{x}$ into the infinitesimally close point S, in such a way that it remains parallel to itself by

$$\tilde{\mathbf{v}} = \Gamma(P, \mathbf{v}, d\mathbf{x})$$

where $\tilde{\mathbf{v}}$ the resulting vector and Γ the **linear operator of parallel transport**, also called the **connection**. Once this operator is defined at every point of the manifold, can transport any vector on that manifold parallelly to itself. It can be shown, that if the operator is indeed linear the following must hold

$$\tilde{v}^\mu = v^\mu - \Gamma^\mu_{\sigma\lambda} v^\sigma dx^\lambda$$

where $\Gamma^\mu_{\sigma\lambda}$ are the coordinate components of Γ, also known as Christoffel's symbols of the first kind, while the Christoffel symbols of the second kind are defined in torsion free manifolds, being additionally symmetric in the two lower indices, as

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left(\frac{\partial g_{\rho\nu}}{\partial x^\lambda} + \frac{\partial g_{\rho\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\rho} \right)$$

1.2.3.4 Covariant Derivative

Differentiation of a tensor does not necessarily produce new tensors. For a contravariant vector, V^μ, for example, transforming as

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$$

differentiation with respect to x'^α we gives

$$\frac{\partial V'^\mu}{\partial x'^\alpha} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial V^\nu}{\partial x^\beta} + \frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\nu} \frac{\partial x^\sigma}{\partial x'^\alpha} V^\nu$$

which evidently is not a tensor since its second term does not transform in the appropriate way. However, based on the above relation, we can define a derivative that will transform as a tensor, thereby guaranteeing that indeed any differentiation of a tensor by it will also transform as a tensor. We call it the covariant derivative and symbolize it ∇_ν. Its action on a contravariant vector V^μ is defined as

$$\nabla_\nu V^\mu \equiv V^\mu_{;\nu} \equiv \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda$$

while on a covariant one as

$$\nabla_\beta U_\alpha \equiv U_{\alpha;\beta} \equiv \partial_\beta U_\alpha - \Gamma^\mu_{\alpha\beta} U_\mu$$

Subsequently, we can obtain the covariant derivate of a tensor of any higher order straightforwardly, treating each index independently, as usual. For example, for rank-2 contravariant tensors we have

$$\nabla_\gamma T^{\alpha\beta} = \partial_\gamma T^{\alpha\beta} + \Gamma^\alpha{}_{\mu\gamma} T^{\mu\beta} + \Gamma^\beta{}_{\mu\gamma} T^{\alpha\mu}$$

while for scalars the covariant derivative simply reduces to the ordinary one. Furthermore, since the covariant derivative by definition transforms as a tensor

$$\nabla'_\nu V'^\mu = \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x'^\beta}{\partial x^\mu} \nabla'_\alpha V'^\beta$$

we can show that the Christoffel symbols transform as

$$\Gamma'^\alpha_{\beta\lambda} = \frac{\partial x'^\alpha}{\partial x^\nu} \frac{\partial x^\mu}{\partial x'^\beta} \frac{\partial x^\rho}{\partial x'^\lambda} \Gamma^\nu_{\mu\rho} - \frac{\partial x'^\alpha}{\partial x^\nu} \frac{\partial^2 x^\nu}{\partial x'^\beta \partial x'^\lambda}$$

and we can readily see that the connection of a manifold is not a tensor.

1.2.3.5 Geodesics

If A and B are two points in a differentiable n-manifold equipped with a metric, g, and $x^\mu = x^\mu(\lambda)$ is a curve connecting those two points in the manifold, the distance between A and B along this curve is

$$l_{AB} = \int ds = \int \frac{ds}{d\lambda} d\lambda = \int d\lambda \sqrt{\left(\frac{ds}{d\lambda}\right)^2} = \int_{\lambda_A}^{\lambda_B} L(x, \dot{x})\, d\lambda$$

where L is called the Langrangian due to the mathematical equivalence of the above expression for the length of the curve to the one used for the action in Lagrangian Mechanics. It is equal to

$$L(x, \dot{x}) = (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{\frac{1}{2}} = \left(g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda}\right)^{\frac{1}{2}}$$

Extremisation according to the Euler-Lagrange method provides us with the curve of extremum length, l_{AB}, called the geodesic and obtained by the Euler-Lagrange equations for the above Lagrangian

$$\frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\mu} = \frac{\partial L}{\partial x^\mu} \quad , \quad (\mu = 0, 1, ..., n)$$

1.2.3.6 Riemann Curvature Tensor

Parallel transport in a closed curve does not bring you back to the original vector, in general. But we may define a quantity that measures how much not so. We call this quantity the Riemann curvature tensor of the manifold and we produce it starting with the covariant derivative of a covariant vector

$$\nabla_\beta V_\alpha = \partial_\beta V_\alpha - \Gamma^\mu_{\alpha\beta} V_\mu$$

while, acting once again with the covariant derivative we obtain

$$\nabla_\gamma(\nabla_\beta V_\alpha) = \partial_\gamma(\nabla_\beta V_\alpha) - \Gamma^\mu_{\alpha\gamma}(\nabla_\beta V_\mu) - \Gamma^\mu_{\beta\gamma}(\nabla_\mu V_\alpha)$$

which simplifies to the form

$$\nabla_\gamma(\nabla_\beta V_\alpha) - \nabla_\beta(\nabla_\gamma V_\alpha) = V_\mu(\partial_\beta \Gamma^\mu_{\alpha\gamma} - \partial_\gamma \Gamma^\mu_{\alpha\beta} + \Gamma^\nu_{\alpha\gamma}\Gamma^\mu_{\nu\beta} - \Gamma^\nu_{\alpha\beta}\Gamma^\mu_{\nu\gamma})$$

We can therefore define the Riemann curvature tensor as

$$R^\mu{}_{\alpha\beta\gamma} \equiv \partial_\beta \Gamma^\mu_{\alpha\gamma} - \partial_\gamma \Gamma^\mu_{\alpha\beta} + \Gamma^\nu_{\alpha\gamma}\Gamma^\mu_{\nu\beta} - \Gamma^\nu_{\alpha\beta}\Gamma^\mu_{\nu\gamma}$$

It has the following symmetries

$$R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho}$$
$$R_{\mu\nu\lambda\rho} = -R_{\mu\nu\rho\lambda}$$
$$R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}$$
$$R^\mu{}_{\mu\nu\lambda} = 0$$
$$R^\mu{}_{\nu\lambda\rho} + R^\mu_{\lambda\rho\nu} + R^\mu_{\rho\nu\lambda} = 0$$

and it satisfies the following identity (second (differential) Bianchi identity, the first (algebraic) one being the last preceeding relation) with cyclic permutations of the indices

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0$$

By these symmetries the independent components of the Riemann tensor reduce to a total of

$$C_N = \frac{N^2(N^2-1)}{12}$$

with N the dimension of the manifold.

1.2.3.7 Ricci Tensor - Ricci Scalar - Einstein Tensor

The trace of the Riemann curvature tensor, obtained by a contraction, can be defined as

$$R_{\alpha\beta} \equiv R^\mu{}_{\alpha\mu\beta} = \partial_\mu \Gamma^\mu_{\alpha\beta} - \partial_\beta \Gamma^\mu_{\alpha\mu} + \Gamma^\mu_{\nu\mu}\Gamma^\nu_{\alpha\beta} - \Gamma^\mu_{\nu\beta}\Gamma^\nu_{\alpha\mu}$$

It is called the Ricci curvature tensor and using the Bianchi identity of the Riemann tensor, it can be shown that it is symmetric

$$R_{\alpha\beta} = R_{\beta\alpha}$$

Subsequently, the trace of the Ricci tensor is known as the Ricci curvature scalar

$$R \equiv R^\alpha{}_\alpha = g^{\alpha\beta} R_{\alpha\beta}$$

Furthermore, contracting two indices in the Bianchi identity we derive

$$\nabla_\delta R_{\alpha\gamma} + \nabla_\beta R^\beta{}_{\alpha\gamma\delta} - \nabla_\gamma R_{\alpha\delta} = 0$$

and multiplication by $g^{\alpha\gamma}$ yields

$$\nabla_\delta R - 2\nabla_\beta R^\beta_\delta = 0 \quad \Longrightarrow \quad \nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0$$

and we have obtained a symmetric tensor with zero-divergence, the Einstein tensor.

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$

1.3 Special Theory of Relativity - SR

We can now return to the Special theory of Relativity. An event is an occurence characterized by a definite pair of time and location values with respect to some reference frame. Therefore, within the mathematical framework we have established, it essentially is a point in a 4-dimensional manifold, we call the spacetime. The coordinates of the event in the chosen reference frame can then be described in terms, for example, of (ct, x, y, z).

1.3.1 The Lorentz Transformation

In the case of Special Relativity, the spacetime is called Minkowski and in it all observers are inertial. Coordinate transformations between such observers are called Lorentzian transformations and their Jacobian matrix is the Lorentzian transformation matrix. For two inertial observers O and O' with relative velocity V along the x-axis, their coordinates, (ct, x, y, z) and (ct', x', y', z'), are related as

$$t' = \frac{t - \frac{Vx}{c^2}}{1 - V^2/c^2}; \quad x' = \frac{x - Vt}{1 - V^2/c^2}; \quad y' = y; \quad z' = z$$

In this case, the Lorentz transformation matrix has the form

$$\Lambda(V) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \quad \text{and} \quad \beta \equiv \frac{V}{c}$$

with γ known as the Lorentz factor. The complete line element between two events of this spacetime takes the form

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

where we note the spatial distance retains the Euclidean form. According to our established formalism then, the metric of the spacetime in this reference frame takes the simple form

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta_{\alpha\beta}$$

called the Minkowski metric. We therefore write

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

Additionaly, the proper time of an observer is defined as the time measured by a clock in the observer's rest frame, i.e. in a comoving frame where spatial differences vanish

$$ds^2 = c^2 \cdot d\tau^2$$

Therefore

$$\Delta\tau = \int d\tau = \int \frac{ds}{c} = \int\sqrt{dt^2 - \frac{dx^2}{c^2} - \frac{dy^2}{c^2} - \frac{dz^2}{c^2}} = \int\sqrt{1 - \frac{V^2(t)}{c^2}}\,dt = \int \frac{dt}{\gamma(t)}$$

where the integral is performed along the trajectory of the observer.

1.3.2 Four-vectors

Contravariant vectors in this 4-dimensional manifold are called 4-vectors and are defined as the objects transforming as

$$V'^{\mu} = \Lambda^{\mu}_{\nu} V^{\nu}$$

where, as already mentioned, $\Lambda^{\mu}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}$, are the elements of the Lorentz transformation matrix. For covariant 4-vectors we have

$$V'_{\alpha} = \Lambda_{\alpha}{}^{\mu} V_{\mu}$$

And the metric tensor can be used for raising and lowering indices as

$$V^{\alpha} = \eta^{\alpha\beta} V_{\beta}$$
$$V_{\alpha} = \eta_{\alpha\beta} V^{\beta}$$

the invariant scalar product of 4-vectors taking the form

$$V^2 = \eta^{\alpha\beta} V_{\alpha} V_{\beta} = \eta_{\alpha\beta} V^{\alpha} V^{\beta}$$

For conducting physics in this framework, relevant 4-vectors are the 4-velocity, 4-acceleration, 4-momentum and all other commonly useful composites. For example, the 4-velocity is defined as

$$u^{\alpha} \equiv \frac{dx^{\alpha}}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{d\mathbf{x}}{d\tau} \right) = \gamma(c, \mathbf{v})$$

where τ is the proper-time and $\mathbf{v}$ its usual 3D spatial velocity. The scalar product of the 4-velocity is constant, derived as

$$v \cdot v = v^{\mu} v_{\mu} = \eta_{\mu\nu}\, v^{\mu} v^{\nu} = \frac{ds^2}{d\tau^2} = c^2 \tag{1.1}$$

1.4 General Theory of Relativity - GR

The mathematical framework is the same as before, so let's now focus on the physical one. The fundamental idea behind the General theory of Relativity is the Strong Principle of Equivalence, Einstein's "happiest thought", based on which we can generalize the physics of special relativity to also include gravity, while letting the Principle of General Covariance take care of the transformation of the equations describing all other physical phenomena from the inertial referenece frames to accelerating ones as well. According to the Strong Principle of Equivalence (SEP) no distinction on the form of Physical Laws can be made between

- a reference frame **free falling** within a gravitational field and an **inertial** one with **no gravity**.
- a reference frame **accelerating** in a $\vec{g}$ gravitational field and one with **no gravity, but of** $-\vec{g}$ **acceleration**.

As a result of this inference, we may actually use a "locally inertial coordinate system", which according to our discussion in (1.2.3.1) always exists, and deduce the nature of Physical law there, with the equations in this locally flat (Minkowski) spacetime not containing the Riemann curvature tensor, Then, the Principle of Equivalence comes to our reassurance, since it demands that physical laws must have the same form in all reference frames. Therefore, an inverse coordinate transformation that would take us back from our "locally inertial coordinate system" to any generalized one, accelerating or not, is bound to retain the form of our equations and since they are produced in terms of tensors, that is automatically guaranteed, the Principle of General Covariance (GCP) constituting a formal statement of exactly that. The passing from the special relativistic (rest-frame) expressions to ones in generalized coordinates, only requires a substitution of partial derivatives with covariant ones and the Minkowski metric, $\eta_{\alpha\beta}$, with the metric tensor of the new frame, $g_{\alpha\beta}(\mathbf{x})$.
The only equations, then, in which the Riemann curvature tensor actually appears are the gravitational field equations, known as the Einstein Field Equations, which retain all the information about the engraving of the presence of matter and energy in the curvature of spacetime.

1.4.1 Einstein Field Equations

The central equations of General Relativity, the Einstein Field Equations, describe gravitational effects by means of spacetime curvature. They can be derived by the variational principle applied for the Einstein-Hilbert action:

$$S = \frac{c^4}{16\pi G}\int R\sqrt{|det(g_{\mu\nu})|}d^4x$$

with respect to the metric $g_{\mu\nu}$. The Euler-Lagrange equations produced in this way, are the **Einstein Field Equations**

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

where in the left hand side we recognise the Einstein tensor that as we have shown is symmetric and has zero-divergence. We rewrite them as

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

$T_{\mu\nu}$ is called the Stress-Energy-Momentum (SEM) Tensor and it describes the flux of energy and momentum in the spacetime. It constitutes the source of the gravitational field according to the General theory of Relativity in the same way mass density does in the framework of Newtonian gravity. For a specific distribution of matter and energy, we can use the corresponding energy-momentum tensor and solve the Einstein Field Equations to obtain the expression of the unknown metric tensor $g_{\mu\nu}$, that will describe the induced geometry of the spacetime. As for the stress-energy tensor, it is a conserved quantity, associated with the invariance of physical laws under spacetime translations and its components specifically describe

- T^{00} : energy density
- $T^{0i} = T^{i0}$: energy flux across the x^i surface or, equivalently, density of the i-th component of linear momentum
- T^{ij} : flux of of i-th component of linear momentum across the x^j surface

The conservation laws inherent in the SEM tensor are summarized in the expression

$$\nabla_\mu T^{\nu\mu} = 0$$

Chapter 2

Spherically Symmetric Perfect Fluid

Our goal is to derive the Einstein Field Equations for the case of a spacetime generated by the mass, momentum, and stress density of a perfect fluid and then explore some of their most interesting properties.

2.1 The Einstein Field Equations - EFEs

We proceed by separately deriving the form of the Einstein tensor, $G_{\mu\nu}$, and the Stress-Energy tensor, $T_{\mu\nu}$, and combining them in the end to obtain our desired EFEs.

2.1.1 The Einstein Tensor - $G_{\mu\nu}$

We begin with our spherically symmetric metric, which allows for radius and time dependance of the g_{tt} and g_{rr} terms

$$ds^2 = c^2 A^2(r,t)\, dt^2 + B^2(r,t)\, dr^2 + r^2\Big(d\theta^2 + sin^2\theta\, d\phi^2\Big) \tag{2.1}$$

We identify our coordinates as $x^1 = ct$, $x^2 = r$, $x^2 = \theta$ and $x^3 = \phi$, so the metric tensor can be represented in matrix form as follows

$$g_{\mu\nu} = \begin{pmatrix} A^2(r,t) & 0 & 0 & 0 \\ 0 & B^2(r,t) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix} \tag{2.2}$$

with inverse

$$g^{\mu\nu} = \begin{pmatrix} \dfrac{1}{A^2(r,t)} & 0 & 0 & 0 \\ 0 & \dfrac{1}{B^2(r,t)} & 0 & 0 \\ 0 & 0 & \dfrac{1}{r^2} & 0 \\ 0 & 0 & 0 & \dfrac{1}{r^2 sin^2\theta} \end{pmatrix} \tag{2.3}$$

The derivatives of $g_{\mu\nu}$ with respect to each of the coordinates are necessary for the evaluation of the Christoffel symbols, so we procced with their calculation. We define $\dot{} \equiv \frac{\partial}{\partial t}$ and $' \equiv \frac{\partial}{\partial r}$. We will work assuming $c\!=\!1$ and, then, restore it in the end. Also, since $g_{\mu\nu}\!=\!0$ for $\mu\!\neq\!\nu$ we have

$$\frac{\partial g_{\mu\nu}}{\partial x^a} = 0 = \frac{\partial g_{\nu\mu}}{\partial x^a} \quad , \quad \text{for } a = 0,1,2,3 \text{ and } \mu\!\neq\!\nu \tag{2.4}$$

and we proceed only with the diagonal terms, $g_{\mu\mu}$. Since no term depends on ϕ we have

$$\frac{\partial g_{\mu\mu}}{\partial x^3} = \frac{\partial g_{\mu\mu}}{\partial \phi} = 0 \tag{2.5}$$

and, as for the θ dependance, only g_{33} has any, thus for $\mu\!\neq\!3$

$$\frac{\partial g_{\mu\mu}}{\partial x^2} = \frac{\partial g_{\mu\mu}}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial g_{33}}{\partial x^2} = \frac{\partial(-r^2 sin^2\theta)}{\partial \theta} = -2r^2 sin\theta\, cos\theta \tag{2.6}$$

Similarly for the time dependance, we obtain

$$\dot{g}_{22} = \dot{g}_{\theta\theta} = 0 = \dot{g}_{33} = \dot{g}_{\phi\phi} \tag{2.7}$$

$$\dot{g}_{00} = \dot{g}_{tt} = 2A\dot{A} \tag{2.8}$$

$$\dot{g}_{11} = \dot{g}_{rr} = -2B\dot{B} \tag{2.9}$$

and, finally, all terms are r dependant and we have

$$g'_{00} = g'_{tt} = 2AA' \tag{2.10}$$

$$g'_{11} = g'_{rr} = -2BB' \tag{2.11}$$

$$g'_{22} = g'_{\theta\theta} = -2r \tag{2.12}$$

$$g'_{33} = g'_{\phi\phi} = -2r sin^2\theta \tag{2.13}$$

Now, the Christoffel symbols are given by

$$\Gamma^{\mu}{}_{\nu\rho} \equiv g^{\mu\sigma}\,[\nu\rho;\sigma] \equiv \frac{1}{2} g^{\mu\sigma}\,[\,\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\,] \tag{2.14}$$

with all of the indices running from 1 to 2. But, since only the diagonal entries of $g^{\mu\nu}$ are non-zero, we have

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\mu}\,[\,\partial_\nu g_{\rho\mu} + \partial_\rho g_{\nu\mu} - \partial_\mu g_{\nu\rho}\,] \tag{2.15}$$

There are 64 Christoffel symbols in total, since each of the three indices has 4 potential values, 0 through 3 and therefore we count 4^3 combinations. Using their symmetry to the two lower indices, we only need to calculate 40 of them. Substituting in Eq.(2.15) the required partial derivatives of the metric from Eqs.(2.7), (2.10), (2.6) and (2.5), crossing out each vanishing term and any mutually cancelling pairs in every expression, we identify the ones not equal to zero as follows

$$\boxed{\begin{aligned} &\Gamma^{t}{}_{tt} = \frac{\dot{A}}{A} \\ &\Gamma^{t}{}_{tr} = \Gamma^{t}{}_{rt} = \frac{A'}{A} \\ &\Gamma^{t}{}_{rr} = \frac{B\dot{B}}{A^2} \end{aligned}} \quad , \quad \boxed{\begin{aligned} &\Gamma^{r}{}_{rr} = \frac{B'}{B} \\ &\Gamma^{r}{}_{tr} = \Gamma^{r}{}_{rt} = \frac{\dot{B}}{B} \\ &\Gamma^{r}{}_{tt} = \frac{AA'}{B^2} \\ &\Gamma^{r}{}_{\theta\theta} = -\frac{r}{B^2} \\ &\Gamma^{r}{}_{\phi\phi} = -\frac{r sin^2\theta}{B^2} \end{aligned}} \tag{2.16}$$

$$\boxed{\begin{aligned}\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{1}{r} \\ \Gamma^{\theta}{}_{\phi\phi} = -sin\theta\, cos\theta\end{aligned}} \quad , \quad \boxed{\begin{aligned}\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{r} \\ \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = cot\theta\end{aligned}} \tag{2.17}$$

We may now procced to the calculation of the components of the Ricci tensor. Using the Riemann tensor formula

$$R^{a}{}_{bcd} = \frac{\partial}{\partial x^{c}}\,\Gamma^{a}{}_{bd} - \frac{\partial}{\partial x^{d}}\,\Gamma^{a}{}_{bc} + \Gamma^{\lambda}{}_{bd}\,\Gamma^{a}{}_{c\lambda} - \Gamma^{\lambda}{}_{bc}\,\Gamma^{a}{}_{d\lambda} \tag{2.18}$$

and the definition of the Ricci tensor

$$R_{\mu\nu} = R^{a}{}_{\mu a\nu} \tag{2.19}$$

we read directly the expression

$$R^{a}{}_{bad} = \frac{\partial}{\partial x^{a}}\,\Gamma^{a}{}_{bd} - \frac{\partial}{\partial x^{d}}\,\Gamma^{a}{}_{ba} + \Gamma^{\lambda}{}_{bd}\,\Gamma^{a}{}_{a\lambda} - \Gamma^{\lambda}{}_{ba}\,\Gamma^{a}{}_{d\lambda} \tag{2.20}$$

We have only 6 non-vanishing elements of the Ricci tensor, namely R_{00}, $R_{01}{=}R_{10}$, R_{11}, R_{22} and $R_{33}{=}sin^2\theta R_{22}$, which explicity are

$$R_{01} = R_{10} = \frac{2}{r}\frac{\dot{B}}{B} \tag{2.21}$$

$$R_{00} = -\frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{AA''}{B^2} - \frac{AA'B'}{B^3} + \frac{2AA'}{rB^2} \tag{2.22}$$

$$R_{11} = -\frac{A''}{A} + \frac{B\ddot{B}}{A^2} - \frac{B\dot{B}\dot{A}}{A^3} + \frac{A'B'}{AB} + \frac{2B'}{rB} \tag{2.23}$$

$$R_{22} = -1 + \frac{1}{B^2} - \frac{rB'}{B^3} + \frac{rA'}{B^2A} \tag{2.24}$$

$$R_{33} = sin^2\theta R_{22} \tag{2.25}$$

Finally, for the Ricci scalar we use Eq.(2.3) again and obtain

$$R = g^{\mu\nu}R_{\mu\nu} \implies \tag{2.26}$$

$$R = \cancelto{0}{g^{01}}R_{01} + \cancelto{0}{g^{10}}R_{10} + g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \implies$$

$$\boxed{R = 2\left[-\frac{\ddot{B}}{BA^2} + \frac{\dot{A}\dot{B}}{BA^3} + \frac{A''}{B^2A} + \frac{2A'}{rB^2A} - \frac{A'B'}{AB^3} - \frac{2B'}{rB^3} - \frac{1}{r^2} + \frac{1}{r^2B^2}\right]} \tag{2.27}$$

where in the second step we only wrote explicitly the terms with non-vanishing Ricci tensor component as a factor, since all others are of course equal to zero. It follows from Eqs.(2.22) - (2.25) and (2.2) that the Einstein tensor has non-vanishing components only for the same index pairs the Ricci tensor does. We obtain them directly by substituting all afformentioned equations, as well as the expression of the Ricci scalar in (2.26), in

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{2.28}$$

They are

$$
\boxed{
\begin{aligned}
&G_{01} = G_{10} = \frac{2}{r}\frac{\dot{B}}{B} && (2.29)\\
&G_{00} = A^2\left[\frac{2B'}{rB^3} + \frac{1}{r^2} - \frac{1}{r^2B^2}\right] && (2.30)\\
&G_{11} = B^2\left[\frac{2A'}{rB^2A} - \frac{1}{r^2} + \frac{1}{r^2B^2}\right] && (2.31)\\
&G_{22} = r^2\left(-\frac{\ddot{B}}{BA^2} + \frac{\dot{A}\dot{B}}{BA^3} + \frac{A''}{B^2A} - \frac{A'B'}{AB^3}\right) - r\left(\frac{B'}{B^3} - \frac{A'}{B^2A}\right) && (2.32)\\
&G_{33} = sin^2\theta\, G_{22} && (2.33)
\end{aligned}
}
$$

2.1.2 The Stress-Energy-Momentum (SEM) Tensor - $T_{\mu\nu}$

In the case of a perfect fluid the covariant form of the SEM Tensor, for a (+,-,-,-) metric signature as the one we use, is provided by the expression

$$T_{\mu\nu} = \left(\rho_m + \frac{p}{c^2}\right) v_\mu v_\nu - p\, g_{\mu\nu} \tag{2.34}$$

where ρ is the proper (rest frame) mass density, p the rest frame pressure, which is isotropic for perfect fluids, and v_a the covariant form of the macroscopic 4-velocity vector field of the fluid. Calling $e = \rho_m c^2$ the proper energy density, we rewrite the above expression as

$$T_{\mu\nu} = \frac{e+p}{c^2} v_\mu v_\nu - p\, g_{\mu\nu} \tag{2.35}$$

which, by setting $q = \frac{e+p}{c^2}$, takes the simpler form

$$T_{\mu\nu} = q v_\mu v_\nu - p\, g_{\mu\nu} \tag{2.36}$$

Now, we need the explicit expression for the 4-velocity. By definition it is a tangent vector to the worldline and its contravariant form is

$$v^\mu = \frac{dx^i}{d\tau} \tag{2.37}$$

where τ the proper time. Since our coordinates are $x^1 = ct$, $x^2 = r$, $x^2 = \theta$ and $x^3 = \phi$, we may write

$$v^\mu = \left(\frac{cdt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\phi}{d\tau}\right) = \frac{dt}{d\tau}\left(c, \frac{dr}{dt}, \frac{d\theta}{dt}, \frac{d\phi}{dt}\right) \tag{2.38}$$

Since we used $c = 1$ in obtaining the Einstein tensor we will continue doing so here, for consistency. Recalling also that $\dot{} = \frac{d}{dt}$, we may write

$$v^\mu = h\Big(1, \dot{r}, \dot{\theta}, \dot{\phi}\Big) \tag{2.39}$$

where $h = h(r,t) = \frac{dt}{d\tau}$, for which given the form of our metric in Eq.(2.3) we have

$$h^{-2} = A^2 - (B^2 - 1)\,\dot{r}^2 - \Big[\dot{r}^2 + (r\dot{\theta})^2 + (r\, sin\theta\dot{\phi})^2\Big] \tag{2.40}$$

and since the sum in the brackets is simply the squared norm of the 3-d velocity vector in spherical coordinates, setting $v = |\vec{v}|$, we can write

$$h^{-2} = A^2 - B^2\dot{r}^2 + \dot{r}^2 - v^2 \tag{2.41}$$

Since we need the covariant form of our 4-velocity, using

$$v_\mu = g_{\mu\nu} v^\nu \tag{2.42}$$

we obtain

$$v_\mu = h\left(A^2, -B^2\dot{r}, -r^2\dot{\theta}, -r^2 sin^2\theta\dot{\phi}\right) \tag{2.43}$$

Therefore the components of $T_{\mu\nu}$ are

$$\boxed{\begin{aligned} &T_{01} = T_{10} = -qh^2A^2B^2\dot{r} && (2.44)\\ &T_{00} = A^2\left[qh^2A^2 - p\right] && (2.45)\\ &T_{11} = B^2\left[qh^2B^2\dot{r}^2 + p\right] && (2.46)\\ &T_{22} = r^2\left[qh^2r^2\dot{\theta}^2 + p\right] && (2.47)\\ &T_{33} = r^2sin^2\theta\left[qh^2r^2sin^2\theta\dot{\phi}^2 + p\right] && (2.48)\end{aligned}}$$

and

$$\boxed{\begin{aligned} &T_{02} = T_{20} = -qh^2A^2r^2\dot{\theta} && (2.49)\\ &T_{03} = T_{30} = -qh^2A^2r^2sin^2\theta\dot{\phi} && (2.50)\\ &T_{12} = T_{21} = -qh^2B^2r^2\dot{r}\dot{\theta} && (2.51)\\ &T_{13} = T_{31} = -qh^2B^2r^2sin^2\theta\dot{r}\dot{\phi} && (2.52)\\ &T_{23} = T_{32} = -qh^2r^4sin^2\theta\dot{\theta}\dot{\phi} && (2.53)\end{aligned}}$$

We separated them in this manner, because we will equate them with the corresponding components of the Einstein tensor, therefore the second group will be set equal to zero and our set of field equations will be restricted to the ones corresponding to the first group, a total of 5.

2.1.3 The EFEs

Combining the components of the two tensors, according to the Einstein Field Equations formula

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.54}$$

we explicitly derive

- For the (02) component:

$$\cancelto{0}{G_{02}} = \kappa T_{02} \implies -qh^2A^2r^2\dot{\theta} = 0 \implies \boxed{\dot{\theta} = 0} \tag{2.55}$$

- For the (03) component:

$$\cancelto{0}{G_{02}} = \kappa T_{02} \implies -qh^2A^2r^2sin^2\theta\dot{\phi} = 0 \implies \boxed{\dot{\phi} = 0} \tag{2.56}$$

- For the (12),(13),(23) and their symmetric components no new information arises, since the same terms appear.

We conclude that, since $\dot{\phi} = 0 = \dot{\theta}$, the velocity is simply $\vec{v} = \dot{r}\hat{r}$ and motion is allowed only in the radial direction for our spherically symmetric distribution of perfect fluid, as was intuitively expected. Then, Eq.(2.41) takes the simpler form

$$h^{-2} = A^2 - B^2\dot{r}^2 \tag{2.57}$$

Moreover, T_{22} and T_{33} simplify into $-pg_{22}$ and $-pg_{33}$ respectively. Since $g_{33}=sin^2\theta g_{22}$, we have

$$G_{33}=sin^2\theta G_{22} \quad \text{and} \quad T_{33}=sin^2\theta T_{22} \tag{2.58}$$

and therefore the two field equations they produce, are essentially the same. We proceed keeping only the (22) one.

Summarising our work up to now, we have began with 8 unknown functions of (r,t), namely the components of the metric A^2, B^2, the pressure and energy density of the perfect fluid, p, e and the components of the 4-velocity in terms of h, $\dot{r}$, $\dot{\theta}$ and $\dot{\phi}$. We discovered the last two are equal to zero if the demand for spherical symmetry is to be satisfied and, thus we are left with 6 unknowns. One may notice, then, that Eq.(2.57) yields the h function solely in terms of A^2, B^2 and $\dot{r}$. Therefore, in fact our unknowns are 5 and by also adopting a particular Equation of State (EoS) for our fluid, we can further relate the p and e functions, finally limiting the number of unknowns to 4. This is in perfect accordance with the number of independent field equations we have obtained, namely the ones corresponding to the (01), (00), (11) and (22) components of our tensors. Their explicit form is

$$\text{(01):}\quad 2\dot{B} = -\kappa q h^2 r A^2 B^3 \dot{r} \tag{2.59}$$

$$\text{(00):}\quad A^2\left[\left(\frac{2B'}{rB^3} + \frac{1}{r^2} - \frac{1}{r^2B^2} + \kappa p\right)h^{-2} - \kappa q A^2\right] = 0 \tag{2.60}$$

$$\text{(11):}\quad B^2\left[\left(\frac{2A'}{rB^2A} - \frac{1}{r^2} + \frac{1}{r^2B^2} - \kappa p\right)h^{-2} - \kappa q B^2\dot{r}^2\right] = 0 \tag{2.61}$$

$$\text{(22):}\quad r^2\left(-\frac{\ddot{B}}{BA^2} + \frac{\dot{A}\dot{B}}{BA^3} + \frac{A''}{B^2A} - \frac{A'B'}{AB^3} - \kappa p\right) - r\left(\frac{B'}{B^3} - \frac{A'}{B^2A}\right) = 0 \tag{2.62}$$

and we can readily see that we have a closed system for our A^2, B^2, $\dot{r}$ and, say, e (r,t)-dependent unknowns. Combination of the (00) and (11) equations and explicit substitution of q gives

$$\kappa(p - e) + \frac{2}{r}\left[\frac{A'}{B^2A} - \frac{B'}{B^3} + \frac{1}{rB^2} - \frac{1}{r}\right] = 0 \tag{2.63}$$

Another straightforwrad observation upon simple visual inspection of our equations is that certain (combinations of) terms are simply derivates of others and a variety of simplifications and further combinations of these EFEs are possible. Nevertheless, we choose to not follow through with an extensive analysis of a wide range of potential function choices for the relation between the energy density and the pressure of our fluid (EoS) that could be imported in the equations and lead to specific solutions (or solution families). Since our fluid is by definition perfect, i.e. without internal heat conduction, shear stresses or viscocity, not many of them are expected to be of true physical interest and it is true that if a special EoS ever, following some potential reasoning on a microscopic basis, became more instructive than most, one could always return to the set of equations provided above, plug it in and obtain the corresponding solution. Therefore, although such a survey and comparison of the behaviour of diverse solutions could be mathematically interesting, we choose to instead shed light on two different notable results of this present study of perfect fluids with spherical symmetry.

But before continuing with their more detailed discussion, we proceed to the conduction of some elementary checks in order to verify that the form of our derived Einstein Field Equations is correct, by examinig whether they can reproduce some known results, namely the exterior and interior Schwarzschild solutions, both static, that should be included in the broad collection of solutions of our EFEs, non-static and static alike.

2.2 Elementary checks of the EFEs

Since the cases we shall examine here are static, we begin by simplifying our EFEs by when all derivatives with respect to time vanish. Noting that in this case we can always use the rest frame of the fluid, we also have for the 4-velocity

$$h^{-2} = A^2 - B^2\cancelto{0}{\dot{r}^2} \quad \Longrightarrow \quad h^{-2} = A^2 \tag{2.64}$$

and

$$v^\mu = \left(A^{-1}, 0, 0, 0\right) \tag{2.65}$$

Then for (01) we obtain

$$\dot{B} = 0 \tag{2.66}$$

In (00) the last two terms inside the brackets give

$$A^2\left[\kappa p - \kappa(e+p)\right] = -\kappa e A^2 \tag{2.67}$$

and therefore our system reduces to

$$\boxed{\begin{aligned}
&(00){:} \quad \frac{2B'}{rB^3} + \frac{1}{r^2} - \frac{1}{r^2 B^2} = \kappa e && (2.68)\\
&(11){:} \quad \frac{2A'}{rB^2 A} - \frac{1}{r^2} + \frac{1}{r^2 B^2} = \kappa p && (2.69)\\
&(22){:} \quad \frac{A''}{A} - \frac{A'B'}{AB} - \kappa p B^2 = \frac{1}{r}\left(\frac{B'}{B} - \frac{A'}{A}\right) && (2.70)
\end{aligned}}$$

with a combination of (11) and (22) also providing that involves directly only the unknown components of the metric and not the ones of the stress energy tensor.

$$\frac{A''}{A} - \frac{A'B'}{AB} - \frac{1}{r}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{B^2}{r^2}\left(1 - \frac{1}{B^2}\right) = 0 \tag{2.71}$$

2.2.1 Exterior Schwarzschild solution

We start with the exterior Schwarzschild case, that corresponds to the choice $T_{\mu\nu}=0$ or, equivalently, $e=0$, $p=0$ and has the Minkowski metric, $\eta_{\mu\nu}$, as a limit for $r \to \infty$. The above equations now further reduce to the form

$$(00){:} \quad \frac{2B'}{rB^3} + \frac{1}{r^2} - \frac{1}{r^2 B^2} = 0 \tag{2.72}$$

$$(11){:} \quad \frac{2A'}{rB^2 A} - \frac{1}{r^2} + \frac{1}{r^2 B^2} = 0 \tag{2.73}$$

The first of the two involves only one of our two remaining functions, therefore we start with it and we obtain

$$-\frac{2rB'}{B^3}+\frac{1}{B^2}-1=0 \implies \left(\frac{r}{B^2}-r\right)'=0 \tag{2.74}$$

$$\frac{r}{B^2}-r=c \implies B^2(r+c)=r \overset{r\neq -c}{\implies} \boxed{B^2=\frac{1}{1+\frac{c}{r}}} \tag{2.75}$$

while by summing Eqs. (00) and (11) we obtain

$$\frac{B'}{B}+\frac{A'}{A}=0 \implies AB'+A'B=0 \implies (AB)'=0 \implies AB=c_2$$

and by imposing the demand that our metric should go to $\eta_{\mu\nu}$ for $r\to\infty$ we have

$$A^2(r\to\infty)=1 \quad \text{and} \quad B^2(r\to\infty)=1 \tag{2.76}$$

thus $c_2=1$ since it is a constant and has the same value for all r, therefore

$$\boxed{A^2=1+\frac{c}{r}} \tag{2.77}$$

And the metric takes the form

$$\boxed{ds^2=\left(1+\frac{c}{r}\right)dt^2-\left(1+\frac{c}{r}\right)^{-1}dr^2-r^2d\Omega^2} \tag{2.78}$$

which is indeed immediately recognised as the exterior Schwarzschild solution, where $c=-r_s$ the Schwarzschild radius that can be derived through the weak field approximation, from which for large distances we get $g_{00}\to 1+2\Phi$, with $\Phi=-\frac{GM}{r}$.

2.2.2 Interior Schwarzschild solution

In this case the assumption for the proper energy density and pressure is that the first is a constant, e, and the second a function of r, regular at $r=0$ and with $p(r=r_0)=0$, where r_0 is the assumed boundary of our perfect fluid distribution. From Eq.(2.68) we then have

$$-\frac{2B'}{rB^3}+\frac{1}{r^2B^2}-\frac{1}{r^2}=\kappa e \tag{2.79}$$

$$-\frac{2rB'}{B^3}+\frac{1}{B^2}-1-\kappa e r^2=0 \implies \left(\frac{r}{B^2}-r-\kappa e\frac{r^3}{3}\right)'=0 \tag{2.80}$$

$$\frac{r}{B^2}-r-\kappa e\frac{r^3}{3}=c \implies B^2(c_1+r+\kappa e\frac{r^3}{3})=r \tag{2.81}$$

$$\overset{\kappa e\frac{r^3}{3}+r\neq -c_1}{\implies} \boxed{B^2=\frac{1}{1+\frac{r^2}{R^2}+\frac{c_1}{r}}} \tag{2.82}$$

where we set $R^{-2}=\frac{\kappa e}{3}$. Now, in order to obtain A we notice that in Eq.(2.69) although we know B, we still have two unknown functions, but a relationship between them can be acquired with use of the conserving property of the stress-energy tensor, $\nabla_\mu T^{\nu\mu}=0$, specifically for the $\nu=1$ equation, from which we obtain

$$\frac{dp}{dr}+\frac{e+p}{2}\frac{A'}{A}=0 \tag{2.83}$$

where we treated e as a constant, as assumed. Equating the $\frac{A'}{A}$ terms of the above equation and Eq.(2.69), we get an expression involving only the pressure and the known function B. Substituting the latter and integrating, we obtain the explicit form for $p(r)$. It then follows directly by Eq.(2.83) that

$$\boxed{A^2 = \left[a - b\left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}} + \frac{c_1}{r}\right]^2} \tag{2.84}$$

with a and b integration constants. Demanding regularity of A at $r=0$ we get $c_1=0$, which we also substitute in our expression for B^2 in Eq.(2.82) and the solution finally takes the form

$$\boxed{ds^2 = \left[a - b\left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}}\right]^2 dt^2 - \frac{dr^2}{1 + \frac{r^2}{R^2}} - r^2 d\Omega^2} \tag{2.85}$$

with the shouldering with an exterior Schwarzschild solution at the boundary, $r=r_0$, also providing the values of the integration constants. According to the literature, this is indeed the correct expression for the interior Schwarzschild solution, therefore we proceed with the confidence that at least the reduced for the static case EFEs must be of the correct form.

2.2.3 Computer assisted confirmation

Further static spherically symmetric perfect fluid solutions, also known as ssspfs, have been produced since the beginning of General Relativity, and they could also be used as more checks for the static version of our obtained EFEs. Again, though, instead of applying a case-by-case extensive approach, now for the final confirmation that our EFEs are of the correct form, we resorted to the help of symbolic computation software, specifically an open-source Computer Algebra System (CAS), Maxima (**maxima**), that provides a really useful, ready-to-use tool for direct calculation of a wide range of relevant for General Relativity tensors, *ctensor*, a component tensor manipulation package. With the help of this tool, we were able to directly reproduce all the intermediate results involved in our analysis as well as the final form of the Einstein tensor, in the general time-dependent case. The results were identical with our own, as one might verify by examining them in the Code Appendix at Lst.((ii)).

Before moving on, though, we would like to refer the reader to an article by Boonserm et al., 2005, a work concerned with static spherically symmetric perfect fluid solutions, where an algorithm for generating new ones is analysed and insightful remarks regarding the intrinsic relations between the known solutions are provided.

2.3 Notable Results

Now that we have established the validity of our produced Einstein Field Equations for the case of a spherically symmetric perfect fluid, we care to expand our analysis in two directions. Firstly, we are interested in examining whether a slight generalisation of our original metric Ansatz, that could allow not just for spherical symmetry but for maximal symmetry in the 3D spatial part of the spacetime manifold, combined with the assumption of another special case for the proper energy density and pressure of our perfect fluid, namely that $\partial_r e = 0 = \partial p$, could lead us to a direct reproduction of the FLRW metric, the foundation of the current standard cosmological model. And, secondly, we are interested in further studying the possible implications of the time-dependance the metric Ansatz in our original analysis exhibits.

2.3.1 Maximally symmetric perfect fluid - The FLRW model

Taking a step back in our original spherically symmetric Ansatz and also allowing time-dependence for all the components of our metric, we obtain

$$ds^2 = c^2 A^2(r,t)\, dt^2 + B^2(r,t)\, dr^2 + Q^2(r,t)\Big(d\theta^2 + sin^2\theta\, d\phi^2\Big) \tag{2.86}$$

Spherical symmetry of our perfect fluid of course still holds and we proceed to computing the Einstein tensor for this metric as well. Since to this end we followed the exact same steps as the ones described in our original analysis, we will not detail them here, but we have listed the results provided by Maxima in Lst.(iii) at the relevant Appendix. We may also simplify the metric as

$$ds^2 = c^2 A^2(r,t)\, dt^2 + B^2(r,t)\left(dr^2 + f^2(r,t) d\Omega^2\right) \tag{2.87}$$

bringing it closer to our familiar form of FLRW, where $Q^2(r,t) = B^2(r,t) f^2(r,t)$. Now, our goal is to examine whether we can "discover" the FLRW metric by directly imposing the assumptions of homogeneity and isotropy on our equations in the form of a special choice for our perfect fluid's stress-energy tensor, namely $e = e(t)$ and $p = p(t)$. It is, in principle expected that we should. Although the most common approach of obtaining the FLRW metric is by simply stating our demands and providing convincing arguments for its resultant form that should just comply with the requested geometric properties and use of the Einstein Field Equations is only reserved for explicitly calculating the scale factor, we would like to see where in the equations one can see the effect of the assumption and how they are directly expressed in mathematical language. We should emphasize of course that the following will not constitute a proof, only a check that everything works as expected and an investigation of where in the equations the assumptions have the biggest effect.

If we consider homogeneity and isotropy, they essentially mean that the origin point with respect to which we have stated the metric is arbitrary and the same metric would in fact be the Ansatz with respect to any point of our fluid. Therefore we can consider the metric to be that of a comoving with the fluid observer, it would then follow that $\dot{r} = 0$, we are in the rest frame and the stress-energy tensor is diagonal. Then the (01) equation would simple become

$$\frac{2A'B\dot{Q} - 2AB\dot{Q}' + 2A\dot{B}Q'}{ABQ} = 0 \tag{2.88}$$

It then becomes evident that if $Q(r,t) = B(t) f(r)$ the last two terms are identical and they cancel out and $A' = 0$ which means that $A^2 = A^2(t)$. This essentially means that the rest frame of one observer is the rest frame of every other observer as well. Time is **everywhere** orthogonal to the spatial hypersurfaces and we can define a new time coordinate without altering the rest of the coordinates at all, as $d\tilde{t} = A(t) dt$. Our metric then reduces structurally to the familiar form of the FLRW metric

$$ds^2 = dt^2 + a^2(t)\left(dr^2 + f^2(r) d\Omega^2\right) \tag{2.89}$$

It remains to be seen if the $f(r)$ function inherently admits the well known form of $S_k(r)$, being r for $k = 0$, $\frac{sin(r\sqrt{k})}{\sqrt{k}}$ for $k > 0$ and $\frac{\sinh(r\sqrt{|k|})}{\sqrt{|k|}}$ for $k < 0$. Before continuing though, we should note that in the straightforward approach of solving the equations, rather than commenting upon them, the $A' = 0$, would have already been derived in means of the conservation laws of the stress-energy tensor and the Killing vectors of this spacetime, since using them one can show that the rest frame timelines are indeed geodesics for **every** part of the fluid. Therefore, Eq.(2.88) would then directly yield the $Q(r,t) = B(t) f(r)$ relation. We proceed with the search for maximal symmetry in the solution of our EFEs. In the Appendix in Lst.((iv)) we have listed the components of the Einstein tensor for this metric also, as well as the Ricci curvature. They read

$$\boxed{\begin{aligned} G_{00} &= -\frac{1}{a^2}\left[\frac{2f''}{f}+\left(\frac{f'}{f}\right)^2-\frac{1}{f^2}\right]+3\left(\frac{\dot{a}}{a}\right)^2 && (2.90)\\ G_{11} &= \frac{(f')^2-1}{f^2}-2a\ddot{a}-\dot{a}^2 && (2.91)\\ G_{22} &= f^2\frac{f''}{f}+f^2\left(-2a\ddot{a}-\dot{a}^2\right) && (2.92)\\ G_{33} &= sin^2\theta\left[f^2\frac{f''}{f}+f^2\left(-2a\ddot{a}-\dot{a}^2\right)\right] && (2.93)\end{aligned}}$$

and the Ricci curvature

$$R=\frac{4ff''+2(f')^2-2+\left(-6a\ddot{a}-6\dot{a}^2\right)f^2}{a^2f^2} \tag{2.94}$$

The stress-energy tensor is as we demonstrated that of the comoving frame of our fluid, therefore it reads

$$T_{\mu\nu}=\mathrm{diag}\left(e,pa^2,pa^2f^2,pa^2f^2\sin(\theta)^2\right) \tag{2.95}$$

Since we are interested only in the spatial 3D part of the metric for all times, i.e. regardless of the particular value the scale factor acquires at each time, we may set it everywhere $a=1$. One may check their correspondence to the ones of the ones directly acquired by a direct calculation for just the spatial part of the metric, listed in the Appendix, in Lst.((v)). We then have for the Ricci scalar

$$R^{(3D)}=\frac{4ff''+2(f')^2-2}{f^2} \tag{2.96}$$

$$\left.\begin{aligned} \frac{(f')^2-1}{f^2} &= \kappa p\\ \cancel{f^2}\frac{f''}{f} &= \kappa p\cancel{f^2}\\ \cancel{\sin^2\theta f^2}\frac{f''}{f} &= \kappa p\cancel{f^2\sin^2\theta}\end{aligned}\right\}\Rightarrow \boxed{\frac{2ff''+(f')^2-1}{f^2}=\frac{R^{(3D)}}{2}=3\kappa p} \tag{2.97}$$

But p is a constant (since we chose a particular time when we fixed the scale factor) therefore the curvature of the 3D hypersurfaces is fixed, allowed to only vary with time following the evolution of the scale factor and the left hand side, a function only of r in this expression, is also a constant. Essentially, not only did we derive the constant (uniform) nature of the Ricci curvature accross all space but we also obtained the ordinary differential equation that fixes the $f(r)$ function. From here on it is straightforward to solve the equation and verify that indeed $f(r)=S_k(r)$, therefore the maximally symmetric, constant curvature known solutions are directly acquired.

The main conclusions of this analysis are, firstly, that of course the FLRW metric is included in the family of solutions of perfect fluids with spherical symmetry, once they are also allowed to have time-dependence in the g_{22} and g_{33} components of the metric to account for the necessary transition to maximal symmetry and that direct solution of the corresponding EFEs coupled with the identically in the rest frame stress-energy tensor of the fluid, in the special case that $e=e(t)$ and $p=p(t)$, exactly reproduces the metric with all of its geometric properties inherently satisfied.

2.3.2 Time-dependent exterior - Further Remarks

When time-dependance is allowed for in our spherically symmetric metric for a perfect fluid distribution, the main interesting outcome is the general $\dot{r} \neq 0$ property of the fluid's 4-velocity, which results in a freedom for time-dependant radial motion, essentially flow of energy or matter in the radial direction, as detailed by the conservation laws of its stress energy tensor, summarized in $\nabla_\mu T^{\nu\mu} = 0$, specifically for a choice of $\nu = 0$ (or 1). This property can be translated not only as freedom for the boundary of the fluid to vary over time but fundamentally as freedom for radial energy flow potentially even to the exterior region. Such a communication may not be allowed when we a priori choose a vacuum exterior described necessarily by the Schwarzschild solution due to its spherical symmetry, the demand for a Minkowski limit as $r \to \infty$ and the limitation imposed thereafter by the Birkhoff theorem that it should also be static, but we need not make such an assumption (vacuum) for the exterior region in the first place. Since, in the present case, the metric for our perfect fluid distribution allows, in principle, for such a communication, we can also use for the exterior another metric, derived by the Schwarzschild one, but such that it also allows for this exact type of communication with the interior. The simplest such choice is the Vaidya metric (Chunilal Vaidya, 1943; Vaidya, 1951) or the Vaidya-Bonner metric, an extension of the Reissner-Nordstroem metric if one also wants to account for electrically non-neutral matter exchange between the interior and exterior regions. Development for cases with further more desired properties of the assumed matter distribution and its evolution can, then, follow directly by building upon this basic idea in a standard and straightforward manner.

It should be stated that, although one should also take into account the physicality and microscopic reasoning behind each choice of a particular realisation, both in terms of a Lagrangian, SEM Tensor and EoS that could describe each region (interior and exterior) individually, as well as the underlying mechanisms through which potential transformations between the possibly distinct types residing in the two regions could take place, in principle, a macroscopic description is always possible in the above mentioned time-dependant framework. The outlook is that a variety of fundamental physics driven choices could be explored, initially as idealised toy-models but with the potential to gradually assist us with advancing towards all the more realistic ones.

A specific application of interest is the case of a time-dependant stellar evolution toy-model. Predictions of such descriptions of a stellar model should be contrasted against the currently used spherically symmetric quasi-static one, to determine whether any potentially new insights with implications that could help expand our understanding of the real world scenario could be drawn, regarding both radiative processes and gravitational collapse.

2.3.2.1 Vaidya Metric

Now, coming back to the simple starting case of the Vaidya metric as a realisation of an exterior geometry we should at first present its line element

$$ds^2 = \left(1 - \frac{2M(u)}{r}\right) du^2 \pm 2dudr - r^2 \left(d\theta^2 + sin^2\theta \, d\phi^2\right) \tag{2.98}$$

obtained by the Schwarzschild metric by changing to the null coordinate u, defined as

$$u = t \pm \left[r + 2m \ln\left(\frac{r}{2M} - 1\right)\right] \tag{2.99}$$

with the plus sign corresponding to the retarded (outgoing) case and the minus to the advanced (ingoing) one. We notice an interesting property of our metric in the new coordinates, namely that now both the mass, $M(u)$, and the Schwarzschild radius, r_s, have become time-dependant, directly reflecting the freedom of radial motion and following the flow of energy of the system. Furthermore, we note that this particular metric can be produced by a pressureless perfect fluid corresponding to some form of radiation (null-dust), with stress-energy tensor thus given by

$$T_{\mu\nu} = e\, l_\mu l_\nu \quad , \quad l^\mu l_\mu = 0 \tag{2.100}$$

with l_a the null co-vectors of the directions in which the radiation may be moving. Here the relevant ones are the null radial geodesics. Shouldering with our interior solution, even if for it we used a general perfect fluid with pressure, can be achieved as usually by letting p go to zero at the boundary. Concerns regarding the underlying phenomenology of a transformation between a potentially massive fluid in the interior and a massless one (radiation) required for the exterior should be addressed, as previously noted, nevertheless this description is viable from the simple macroscopic perspective.

2.3.2.2 Scalar Field - Perfect Fluid Correspondance

We will now proceed with presenting one particular choice of realisation that could give rise to the SEM tensor of Eq.(2.100), as required by the Vaidya metric, detailed also in (Faraoni & Côté, 2019; Papacostas, 1991b). It is the case of a massless and pressureless spherically symmetric time-dependant scalar field, ϕ, which we will demonstrate that can indeed be identified as the required null-dust. We begin with the general expression for the stress-energy tensor of a massive scalar field that satisfies the Klein-Gordon equation and can therefore potentially correspond to axions, neutral mesons or any other neutral spin-0 particle for the present study.

$$T_{\mu\nu} = \nabla_\mu \phi\, \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \left[\nabla_a \phi\, \nabla^a \phi - m^2 \phi^2 \right] - V(\phi)\, g_{\mu\nu} \tag{2.101}$$

where ∇_μ is the covariant derivative and $V(\phi)$ the scalar field potential. The velocity of this scalar field then is simply

$$v_\mu = \frac{\nabla_\mu \phi}{\sqrt{\nabla^\alpha \phi \nabla_\alpha \phi}} \tag{2.102}$$

Comparing the scalar field stress-energy tensor in Eq.(2.101) and the perfect fluid one in (2.35) with use of the above expression for the covariant 4-velocity of the scalar field, we first obtain its version describing only dust, by only allowing for the term with the product of 4-velocities, thus setting $V(\phi) = 0$ and $m = 0$, which was a desired condition either way, since we are interested in a description of radiation. We have

$$T_{\mu\nu} = \nabla_\mu \phi\, \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_a \phi\, \nabla^a \phi \tag{2.103}$$

Subsequently, we derive the null-dust case when the fluid's 4-velocity is of the null type, namely when its gradient is

$$\nabla^\mu \phi = 0 \tag{2.104}$$

where $\nabla^\mu = g^{\mu\nu} \nabla_\nu$. Then the stress-energy tensor further is further simplified to the form

$$T_{\mu\nu} = \nabla_\mu \phi\, \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \cancelto{0}{\nabla_a} \phi\, \nabla^a \phi \tag{2.105}$$

$$\Longrightarrow\; T_{\mu\nu} = e\, l_\mu l_\nu \quad , \quad \text{where } l_a = \frac{\nabla_a \phi}{\sqrt{e}} \tag{2.106}$$

and the identification of this stress-energy tensor as one of null-dust becomes evident. Therefore we have successfully acquired a viable although simplistic physical realisation for the radiation type content of a possible exterior Vaidya metric. If we would then additionally like to perform the same for the interior of the spherically symmetric perfect fluid in our original analysis, we may proceed directly with use of Eq.(2.101) with the terms we omitted for the dust case now necessary to represent the proper pressure dependant part of our perfect fluid's stress energy tensor, denoting that the presence of the scalar field potential is necessary for the elimination of viscous (cross) terms that would otherwise arise, therefore guaranteeing that the tensor is indeed one of perfect fluid, with the mass being reinstated in its expression also guaranteeing that general matter and energy distributions may all be included in an effective description of scalar field perfect fluids.

Chapter 3

Epilogue

Within the framework of General relativity, we studied the case of spherically symmetric distributions of perfect fluids. We begun with the derivation of the Einstein Field Equations and after performing some necessary checks for the validity of our results, both through reproduction of elementary known solutions and the use of computational algebra software for direct confirmation we directed our attention to the interpretation of some notable inferences. Initially, we generalized our original Ansatz, to a form that beyond spherical symmetry could also allow for maximal spatial symmetry and we examined how the familiar FLRW metric emerged as a solution, when we focused on the special case of a SEM tensor with space-independent proper energy density and pressure. Thereafter, we commented upon our main outcome, concerning the time-dependent version of our produced EFEs and their corresponding solutions. We showcased how, allowing for time-dependent exterior solutions, toy-models of many astrophysically interesting and highly relevant processess could be constructed. We presented the Vaidya metric as an introductory example of such an exterior geometry, while we demonstrated how physically motivated choices of the SEM tensor could be identified as perfect fluids. Concluding, we seek to once again emphasize the relevance of spherically symmetric time-dependent Einstein Field Equations, coupled even with idealized perfect fluids, for approximating and studying a wide range of celestial objects and processes, in search of deeper insights. From protostars to stellar remnants, to exotic stars and even the universe as a whole, they all pose as excellent candidates for potentially fruitful new results. For radiative processes and gravitational phenomena alike, the need of paradigm shift in the description of stellar evolution, gravitational collapse and even gravitational waves (from spherically symmetric sources, when not limited by the Birkhoff theorem of static treatments) is a given and focusing on the time-dependent EFEs, even of highly symmetric, idealized perfect fluids as a starting point, seems like the best way to go.

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Abbreviations

Acronym	Definition
GR	General Relativity
SR	Special Relativity
EFE	Einstein Field Equations
EoS	Equation of State
WEP	Weak Equivalence Principle
SEP	Strong Equivalence Principle
GCP	General Covariance Principle
FLRW (metric/model)	Friedmanni-Lemaitre-Robertson-Walker (metric/model)
ΛCDM (cosmology/model)	Lambda Cold Darm Matter (cosmology/model)

Listings

Calculating the Einstein Tensor with Maxima

All following results were obtained with use of the *ctensor* (component tensor manipulation) package of **maxima**.

Listing (i): Our original spherically symmetric metric Ansatz - HTML output

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<!--          Created with wxMaxima version 16.12.0          -->
<!-- ***************************************************** -->
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      \newcommand{\ensuremath}[1]{\mbox{$#1$}}
\)

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backslash ctensor.mac\mbox{}\\Enter the dimension of the coordinate system:
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list containing the names of the coordinates in order [t,r,theta,phi];\mbox
{}\\Do you want to
1. Enter a new metric?
2. Enter a metric from a file?
3. Approximate a metric with a Taylor series? 1;\mbox{}\\\mbox{Is the matrix
1. Diagonal  2. Symmetric  3. Antisymmetric  4. General}Answer 1, 2, 3 or 4
 : 1;Row 1 Column 1: A\^{}2;Row 2 Column 2: -B\^{}2;Row 3 Column 3: -r
\^{}2;Row 4 Column 4: -r\^{}2*sin(theta)\^{}2;\mbox{}\\\mbox{Matrix entered
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\[\tag{\% t67} {{\mathit{lein}}_{2,1}}=\frac{2 \left( {B_t}\right) }{B r}\]
\[\tag{\% t68} {{\mathit{lein}}_{2,2}}=\frac{2 \left( {A_r}\right)  r-A\, {{B}^{2}}+A}{A\, {{r}^{2}}}\]
\[\tag{\% t69} {{\mathit{lein}}_{3,3}}=-\frac{r\, \left( \left( A\, {{B}^{2}}\, \left( {B_{tt}}\right) -\left( {A_t}\right) \, {{B}^{2}}\, \left( {B_t}\right) +{{A}^{2}}\, \left( {A_r}\right) \, \left( {B_r}\right) -{{A}^{2}}\, \left( {A_{rr}}\right)  B\right)  r+{{A}^{3}}\, \left( {B_r}\right) -{{A}^{2}}\, \left( {A_r}\right)  B\right) }{{{A}^{3}}\, {{B}^{3}}}\]
```

```
\[\tag{\% t70} {{\mathit{lein}}_{4,4}}=-\]
\[\tag{\% o70} \mathit{done}\]

 <HR>
 <SMALL> Created with <A HREF="http://wxmaxima.sourceforge.net/">wxMaxima</A
   >.</SMALL>
 </BODY>
</HTML>
```

Listing (ii): Our original spherically symmetric metric Ansatz

```
<!-- ************************************************************** -->
<!--           Created with wxMaxima version 16.12.0            -->
<!-- ************************************************************** -->
```

```
-->   csetup();
Type KILL(ALL); and then TENSORKILL:TRUE;
          before you enter a new metric.
#0: csetup() (C:\maxima-5.39.0\share\maxima\5.39.0_2_g5a49f11_dirty\share\tensor\ctensor.mac line 208)
 -- an error. To debug this try: debugmode(true);
```

```
-->   kill(all);
(%o0) done
```

```
-->   tensorkill:true;
(tensorkill) true
```

```
-->   load(ctensor);
      csetup();
(%o3) C:\maxima-5.39.0\share\maxima\5.39.0_2_g5a49f11_dirty\share\tensor\ctensor.mac
Enter the dimension of the coordinate system: 4;
Do you wish to change the coordinate names? y;
Enter a list containing the names of the coordinates in order [t,r,theta,phi];
Do you want to 1. Enter a new metric? 2. Enter a metric from a file? 3. Approximate a metric with a Taylor series? 1;
Is the matrix  1. Diagonal  2. Symmetric  3. Antisymmetric  4. General
Answer 1, 2, 3 or 4 : 1;
Row 1 Column 1: A^2;
Row 2 Column 2: -B^2;
Row 3 Column 3: -r^2;
Row 4 Column 4: -r^2*sin(theta)^2;
Matrix entered.
Enter functional dependencies with DEPENDS or 'N' if none depends([A,B],[t,r]);
Do you wish to see the metric? y;
```

$$\begin{bmatrix} A^2 & 0 & 0 & 0 \\ 0 & -B^2 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin(\theta)^2 \end{bmatrix}$$

```
(%o4) done
```

(%i4) `christof(mcs);`

(%o3) done

(%t4) $mcs_{1,1,1}=\frac{A_t}{A}$

(%t5) $mcs_{1,1,2}=\frac{A\left(A_r\right)}{B^2}$

(%t6) $mcs_{1,2,1}=\frac{A_r}{A}$

(%t7) $mcs_{1,2,2}=\frac{B_t}{B}$

(%t8) $mcs_{2,2,1}=\frac{B\left(B_t\right)}{A^2}$

(%t9) $mcs_{2,2,2}=\frac{B_r}{B}$

(%t10) $mcs_{2,3,3}=\frac{1}{r}$

(%t11) $mcs_{2,4,4}=\frac{1}{r}$

(%t12) $mcs_{3,3,2}=-\frac{r}{B^2}$

(%t13) $mcs_{3,4,4}=\frac{\cos(\theta)}{\sin(\theta)}$

(%t14) $mcs_{4,4,2}=-\frac{r\sin(\theta)^2}{B^2}$

(%i16) `ricci(dis);`

(%t15) $mcs_{4,4,3}=-\cos(\theta)\sin(\theta)$

(%o15) done

(%t16) $ric_{1,1}=\frac{2A\left(A_r\right)}{B^2 r}-\frac{B_{tt}}{B}+\frac{\left(A_t\right)\left(B_t\right)}{AB}-\frac{A\left(A_r\right)\left(B_r\right)}{B^3}+\frac{A\left(A_{rr}\right)}{B^2}$

(%t17) $ric_{1,2}=\frac{2\left(B_t\right)}{Br}$

(%t18) $ric_{2,2}=\frac{2\left(B_r\right)}{Br}+\frac{B\left(B_{tt}\right)}{A^2}-\frac{\left(A_t\right)B\left(B_t\right)}{A^3}+\frac{\left(A_r\right)\left(B_r\right)}{AB}-\frac{A_{rr}}{A}$

(%t19) $ric_{3,3}=\frac{\left(B_r\right)r}{B^3}-\frac{\left(A_r\right)r}{AB^2}-\frac{1}{B^2}+1$

(%t20) $ric_{4,4}=\frac{\left(B_r\right)r\sin(\theta)^2}{B^3}-\frac{\left(A_r\right)r\sin(\theta)^2}{AB^2}-\frac{\sin(\theta)^2}{B^2}+\sin(\theta)^2$

(%i21) `scurvature();`

(%o20) done

(%o21) $-\frac{\left(2AB^2\left(B_{tt}\right)-2\left(A_t\right)B^2\left(B_t\right)+2A^2\left(A_r\right)\left(B_r\right)-2A^2\left(A_{rr}\right)B\right)r^2+\left(4A^3\left(B_r\right)-4A^2\left(A_r\right)B\right)r+2A^3B^3-2A^3B}{A^3B^3r^2}$

```
leinstein(dis);
```

(%t66) $lein_{1,1}=\frac{A^2\left(2\left(B_r\right)r+B^3-B\right)}{B^3\,r^2}$

(%t67) $lein_{2,1}=\frac{2\left(B_t\right)}{B\,r}$

(%t68) $lein_{2,2}=\frac{2\left(A_r\right)r-A\,B^2+A}{A\,r^2}$

(%t69) $lein_{3,3}=-\frac{r\left(\left(A\,B^2\left(B_{t\,t}\right)-\left(A_t\right)B^2\left(B_t\right)+A^2\left(A_r\right)\left(B_r\right)-A^2\left(A_{r\,r}\right)B\right)r+A^3\left(B_r\right)-A^2\left(A_r\right)B\right)}{A^3\,B^3}$

(%t70) $lein_{4,4}=-\frac{r\left(\left(A\,B^2\left(B_{t\,t}\right)-\left(A_t\right)B^2\left(B_t\right)+A^2\left(A_r\right)\left(B_r\right)-A^2\left(A_{r\,r}\right)B\right)r+A^3\left(B_r\right)-A^2\left(A_r\right)B\right)\sin(\theta)^2}{A^3\,B^3}$

(%o70) done

Listing (iii): Towards FLRW - Original Ansatz

```
<!-- ***************************************************** -->
<!--           Created with wxMaxima version 16.12.0         -->
<!-- ***************************************************** -->
```

```
-->  kill(all);
(%o0) done

-->  tensorkill:true;
(tensorkill) true

-->  load(ctensor);
     csetup();
(%o2) C:\maxima-5.39.0\share\maxima\5.39.0_2_g5a49f11_dirty\share\tensor\ctensor.mac
Enter the dimension of the coordinate system: 4;
Do you wish to change the coordinate names? y;
Enter a list containing the names of the coordinates in order [t,r,theta,phi];
Do you want to 1. Enter a new metric? 2. Enter a metric from a file? 3. Approximate a metric with a Taylor series? 1
;
Is the matrix  1. Diagonal  2. Symmetric  3. Antisymmetric  4. General
Answer 1, 2, 3 or 4 : 1;
Row 1 Column 1: A^2;
Row 2 Column 2: -B^2;
Row 3 Column 3: -Q^2;
Row 4 Column 4: -Q^2*sin(theta)^2;
Matrix entered.
Enter functional dependencies with DEPENDS or 'N' if none depends([A,B,Q],[t,r]);
Do you wish to see the metric? y;
```

$$\begin{bmatrix} A^2 & 0 & 0 & 0 \\ 0 & -B^2 & 0 & 0 \\ 0 & 0 & -Q^2 & 0 \\ 0 & 0 & 0 & -Q^2\sin(\theta)^2 \end{bmatrix}$$

(%i4) christof(mcs);

(%o3) *done*

(%t4) $mcs_{1,1,1}=\frac{A_t}{A}$

(%t5) $mcs_{1,1,2}=\frac{A\left(A_r\right)}{B^2}$

(%t6) $mcs_{1,2,1}=\frac{A_r}{A}$

(%t7) $mcs_{1,2,2}=\frac{B_t}{B}$

(%t8) $mcs_{1,3,3}=\frac{\varrho_t}{\varrho}$

(%t9) $mcs_{1,4,4}=\frac{\varrho_t}{\varrho}$

(%t10) $mcs_{2,2,1}=\frac{B\left(B_t\right)}{A^2}$

(%t11) $mcs_{2,2,2}=\frac{B_r}{B}$

(%t12) $mcs_{2,3,3}=\frac{\varrho_r}{\varrho}$

(%t13) $mcs_{2,4,4}=\frac{\varrho_r}{\varrho}$

(%t14) $mcs_{3,3,1}=\frac{\varrho\left(\varrho_t\right)}{A^2}$

(%t15) $mcs_{3,3,2}=-\frac{\varrho\left(\varrho_r\right)}{B^2}$

(%t16) $mcs_{3,4,4}=\frac{\cos(\theta)}{\sin(\theta)}$

(%t17) $mcs_{4,4,1}=\frac{\varrho\left(\varrho_t\right)\sin(\theta)^2}{A^2}$

(%t18) $mcs_{4,4,2}=-\frac{\varrho\left(\varrho_r\right)\sin(\theta)^2}{B^2}$

(%i20) ricci(dis);

(%t19) $mcs_{4,4,3}=-\cos(\theta)\sin(\theta)$

(%o19) done

(%t20) $$ric_{1,1}=-\frac{2\left(Q_{tt}\right)}{Q}+\frac{2\left(A_t\right)\left(Q_t\right)}{A\,Q}+\frac{2\,A\left(A_r\right)\left(Q_r\right)}{B^2\,Q}-\frac{B_{tt}}{B}+\frac{\left(A_t\right)\left(B_t\right)}{A\,B}-\frac{A\left(A_r\right)\left(B_r\right)}{B^3}+\frac{A\left(A_{rr}\right)}{B^2}$$

(%t21) $$ric_{1,2}=\frac{2\left(A_r\right)\left(Q_t\right)}{A\,Q}-\frac{2\left(Q_{rt}\right)}{Q}+\frac{2\left(B_t\right)\left(Q_r\right)}{B\,Q}$$

(%t22) $$ric_{2,2}=\frac{2\,B\left(B_t\right)\left(Q_t\right)}{A^2\,Q}-\frac{2\left(Q_{rr}\right)}{Q}+\frac{2\left(B_r\right)\left(Q_r\right)}{B\,Q}+\frac{B\left(B_{tt}\right)}{A^2}-\frac{\left(A_t\right)B\left(B_t\right)}{A^3}+\frac{\left(A_r\right)\left(B_r\right)}{A\,B}-\frac{A_{rr}}{A}$$

(%t23) $$ric_{3,3}=\frac{Q\left(Q_{tt}\right)}{A^2}+\frac{\left(Q_t\right)^2}{A^2}+\frac{\left(B_t\right)Q\left(Q_t\right)}{A^2\,B}-\frac{\left(A_t\right)Q\left(Q_t\right)}{A^3}-\frac{Q\left(Q_{rr}\right)}{B^2}-\frac{\left(Q_r\right)^2}{B^2}+\frac{\left(B_r\right)Q\left(Q_r\right)}{B^3}-\frac{\left(A_r\right)Q\left(Q_r\right)}{A\,B^2}+1$$

(%t24) $$ric_{4,4}=\frac{Q\left(Q_{tt}\right)\sin(\theta)^2}{A^2}+\frac{\left(Q_t\right)^2\sin(\theta)^2}{A^2}+\frac{\left(B_t\right)Q\left(Q_t\right)\sin(\theta)^2}{A^2\,B}-\frac{\left(A_t\right)Q\left(Q_t\right)\sin(\theta)^2}{A^3}-\frac{Q\left(Q_{rr}\right)\sin(\theta)^2}{B^2}-\frac{\left(Q_r\right)^2\sin(\theta)^2}{B^2}+\frac{\left(B_r\right)Q\left(Q_r\right)\sin(\theta)^2}{B^3}-\frac{\left(A_r\right)Q\left(Q_r\right)\sin(\theta)^2}{A\,B^2}+\sin(\theta)^2$$

(%i25) scurvature();

(%o24) done

(%o25) $$-\frac{4\,A\,B^3\,Q\left(Q_{tt}\right)+2\,A\,B^3\left(Q_t\right)^2+\left(4\,A\,B^2\left(B_t\right)-4\left(A_t\right)B^3\right)Q\left(Q_t\right)-4\,A^3\,B\,Q\left(Q_{rr}\right)-2\,A^3\,B\left(Q_r\right)^2+\left(4\,A^3\left(B_r\right)-4\,A^2\left(A_r\right)B\right)Q\left(Q_r\right)+\left(2\,A\,B^2\left(B_{tt}\right)-2\left(A_t\right)B^2\left(B_t\right)+2\,A^2\left(A_r\right)\left(B_r\right)-2\,A^2\left(A_{rr}\right)B\right)Q^2+2\,A^3\,B^3}{A^3\,B^3\,Q^2}$$

(%i26) leinstein(dis);

(%t26) $$lein_{1,1}=\frac{B^3\left(Q_t\right)^2+2\,B^2\left(B_t\right)Q\left(Q_t\right)-2\,A^2\,B\,Q\left(Q_{rr}\right)-A^2\,B\left(Q_r\right)^2+2\,A^2\left(B_r\right)Q\left(Q_r\right)+A^2\,B^3}{B^3\,Q^2}$$

(%t27) $$lein_{2,1}=\frac{2\left(A_r\right)B\left(Q_t\right)-2\,A\,B\left(Q_{rt}\right)+2\,A\left(B_t\right)\left(Q_r\right)}{A\,B\,Q}$$

(%t28) $$lein_{2,2}=-\frac{2\,A\,B^2\,Q\left(Q_{tt}\right)+A\,B^2\left(Q_t\right)^2-2\left(A_t\right)B^2\,Q\left(Q_t\right)-A^3\left(Q_r\right)^2-2\,A^2\left(A_r\right)Q\left(Q_r\right)+A^3\,B^2}{A^3\,Q^2}$$

(%t29) $$lein_{3,3}=-\frac{Q\left(A\,B^3\left(Q_{tt}\right)+\left(A\,B^2\left(B_t\right)-\left(A_t\right)B^3\right)\left(Q_t\right)-A^3\,B\left(Q_{rr}\right)+\left(A^3\left(B_r\right)-A^2\left(A_r\right)B\right)\left(Q_r\right)+\left(A\,B^2\left(B_{tt}\right)-\left(A_t\right)B^2\left(B_t\right)+A^2\left(A_r\right)\left(B_r\right)-A^2\left(A_{rr}\right)B\right)Q\right)}{A^3\,B^3}$$

(%t30) $$lein_{4,4}=-\frac{Q\left(A\,B^3\left(Q_{tt}\right)+\left(A\,B^2\left(B_t\right)-\left(A_t\right)B^3\right)\left(Q_t\right)-A^3\,B\left(Q_{rr}\right)+\left(A^3\left(B_r\right)-A^2\left(A_r\right)B\right)\left(Q_r\right)+\left(A\,B^2\left(B_{tt}\right)-\left(A_t\right)B^2\left(B_t\right)+A^2\left(A_r\right)\left(B_r\right)-A^2\left(A_{rr}\right)B\right)Q\right)\sin(\theta)^2}{A^3\,B^3}$$

(%o30) done

Listing (iv): Towards FLRW - Final Metric

```
<!-- ************************************************** -->
<!--           Created with wxMaxima version 16.12.0           -->
<!-- ************************************************** -->
```

```
(%i1) kill(all);
(%o0) done

(%i1) tensorkill:true;
(tensorkill) true

(%i3) load(ctensor);
      csetup();
(%o2) C:\maxima-5.39.0\share\maxima\5.39.0_2_g5a49f11_dirty\share\tensor\ctensor.mac
Enter the dimension of the coordinate system: 4;
Do you wish to change the coordinate names? y;
Enter a list containing the names of the coordinates in order [t,r,theta,phi];
Do you want to1. Enter a new metric?2. Enter a metric from a file?3. Approximate a metric with a Taylor series? 1;
Is the matrix  1. Diagonal  2. Symmetric  3. Antisymmetric  4. General
Answer 1, 2, 3 or 4 : 1;
Row 1 Column 1: 1;
Row 2 Column 2: -a^2;
Row 3 Column 3: -a^2*f^2;
Row 4 Column 4: -a^2*f^2*sin(theta)^2;
Matrix entered.
Enter functional dependencies with DEPENDS or 'N' if none depends([a],[t],[f],[r]);
Do you wish to see the metric? y;
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 \\ 0 & 0 & -a^2 f^2 & 0 \\ 0 & 0 & 0 & -a^2 f^2 \sin(\theta)^2 \end{bmatrix}$$

```
(%i4) christof(mcs);
(%o3) done
```

(%t4) $mcs_{1,2,2} = \frac{a_t}{a}$

(%t5) $mcs_{1,3,3} = \frac{a_t}{a}$

(%t6) $mcs_{1,4,4} = \frac{a_t}{a}$

(%t7) $mcs_{2,2,1} = a\left(a_t\right)$

(%t8) $mcs_{2,3,3} = \frac{f_r}{f}$

(%t9) $mcs_{2,4,4} = \frac{f_r}{f}$

(%t10) $mcs_{3,3,1} = a\left(a_t\right) f^2$

(%t11) $mcs_{3,3,2} = -f\left(f_r\right)$

(%t12) $mcs_{3,4,4} = \frac{\cos(\theta)}{\sin(\theta)}$

(%t13) $mcs_{4,4,1} = a\left(a_t\right) f^2 \sin(\theta)^2$

(%t14) $mcs_{4,4,2} = -f\left(f_r\right) \sin(\theta)^2$

(%i16) ricci(dis);

(%t15) $mcs_{4,4,3}=-\cos(\theta)\sin(\theta)$

(%o15) done

(%t16) $ric_{1,1}=-\frac{3\left(a_{t\,t}\right)}{a}$

(%t17) $ric_{2,2}=-\frac{2\left(f_{r\,r}\right)}{f}+a\left(a_{t\,t}\right)+2\left(a_{t}\right)^{2}$

(%t18) $ric_{3,3}=-f\left(f_{r\,r}\right)-\left(f_{r}\right)^{2}+a\left(a_{t\,t}\right)f^{2}+2\left(a_{t}\right)^{2}f^{2}+1$

(%t19) $ric_{4,4}=-f\left(f_{r\,r}\right)\sin(\theta)^{2}-\left(f_{r}\right)^{2}\sin(\theta)^{2}+a\left(a_{t\,t}\right)f^{2}\sin(\theta)^{2}+2\left(a_{t}\right)^{2}f^{2}\sin(\theta)^{2}+\sin(\theta)^{2}$

(%i20) scurvature();

(%o19) done

(%o20) $\frac{4f\left(f_{r\,r}\right)+2\left(f_{r}\right)^{2}+\left(-6a\left(a_{t\,t}\right)-6\left(a_{t}\right)^{2}\right)f^{2}-2}{a^{2}f^{2}}$

(%i21) leinstein(dis);

(%t21) $lein_{1,1}=-\frac{2f\left(f_{r\,r}\right)+\left(f_{r}\right)^{2}-3\left(a_{t}\right)^{2}f^{2}-1}{a^{2}f^{2}}$

(%t22) $lein_{2,2}=\frac{\left(f_{r}\right)^{2}+\left(-2a\left(a_{t\,t}\right)-\left(a_{t}\right)^{2}\right)f^{2}-1}{f^{2}}$

(%t23) $lein_{3,3}=f\left(f_{r\,r}+\left(-2a\left(a_{t\,t}\right)-\left(a_{t}\right)^{2}\right)f\right)$

(%t24) $lein_{4,4}=f\left(f_{r\,r}+\left(-2a\left(a_{t\,t}\right)-\left(a_{t}\right)^{2}\right)f\right)\sin(\theta)^{2}$

(%o24) done

Listing (v): Towards Maximal Symmetry - 3D Spatial Metric

```
<!-- ************************************************** -->
<!--           Created with wxMaxima version 16.12.0      -->
<!-- ************************************************** -->
```

```
(%i1)   kill(all);
(%o0)   done

(%i1)   tensorkill:true;
(tensorkill)   true

(%i3)   load(ctensor);
        csetup();
(%o2)   C:\maxima-5.39.0\share\maxima\5.39.0_2_g5a49f11_dirty\share\tensor\ctensor.mac
Enter the dimension of the coordinate system:  3;
Do you wish to change the coordinate names?  y;
Enter a list containing the names of the coordinates in order  [w,theta,phi];
Do you want to1. Enter a new metric?2. Enter a metric from a file?3. Approximate a metric with a Taylor series?    1;
Is the matrix  1. Diagonal  2. Symmetric  3. Antisymmetric  4. General
Answer 1, 2, 3 or 4 :  1;
Row 1 Column 1:  1;
Row 2 Column 2:  X^2;
Row 3 Column 3:  X^2*sin(theta)^2;
Matrix entered.
Enter functional dependencies with DEPENDS or 'N' if none  depends([X],[w]);
Do you wish to see the metric?  y;
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & X^2 & 0 \\ 0 & 0 & X^2 \sin(\theta)^2 \end{bmatrix}$$

```
(%i4)   christof(mcs);
(%o3)   done
```

(%t4) $mcs_{1,2,2} = \frac{X_w}{X}$

(%t5) $mcs_{1,3,3} = \frac{X_w}{X}$

(%t6) $mcs_{2,2,1} = -X \left(X_w\right)$

(%t7) $mcs_{2,3,3} = \frac{\cos(\theta)}{\sin(\theta)}$

(%t8) $mcs_{3,3,1} = -X \left(X_w\right) \sin(\theta)^2$

(%i10) ricci(dis);

(%t9) $mcs_{3,3,2}=-\cos(\theta)\sin(\theta)$

(%o9) done

(%t10) $ric_{1,1}=-\frac{2\left(x_{w\,w}\right)}{x}$

(%t11) $ric_{2,2}=-x\left(x_{w\,w}\right)-\left(x_{w}\right)^2+1$

(%t12) $ric_{3,3}=-x\left(x_{w\,w}\right)\sin(\theta)^2-\left(x_{w}\right)^2\sin(\theta)^2+\sin(\theta)^2$

(%o12) done

(%i13) scurvature();

(%o13) $-\frac{4\,x\left(x_{w\,w}\right)+2\left(x_{w}\right)^2-2}{x^2}$

(%i14) leinstein(dis);

(%t14) $lein_{1,1}=\frac{\left(x_{w}\right)^2-1}{x^2}$

(%t15) $lein_{2,2}=x\left(x_{w\,w}\right)$

(%t16) $lein_{3,3}=x\left(x_{w\,w}\right)\sin(\theta)^2$

(%o16) done