Multivariate Quantile Impulse Response Functions Applied In Macroeconomic Data

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1/12/2020

Summary

In this dissertation a simple multivariate quantile autoregressive model is being used to study heterogeneity in the effects of macroeconomic shocks. In the first half we present a theoretical base as far as quantile regression is concerned. In the second half, we estimate the $VARQ$ and perform $QIRF$ analysis using a three-variable macroeconomic model (with output gap, inflation and Fed Funds rate) for different countries that are part of the E.U (Finland, Germany, Greece and Italy), for a set period of 20 years (1995q_1-2015q_1). We use quantile impulse response functions in order to explore dynamic heterogeneity in the response of endogenous variables to different shocks. The evaluation of the different quantile paths as the dynamic effects for a fixed collection of quantile indexes shows as a result that, some countries have the same response in a given shock, some don't follow the same pattern as others and others can have the exact opposite behavior or their own unique response to the shock. The reason why these differences in the responses of the different counties occur may lay in the unique characteristics of each county's economic structure.

Περίληψη

Σε αυτήν τη διατριβή χρησιμοποιείται ένα απλό πολύ-μεταβλητό ποσοτικό αυτοπαλίνδρομο μοντέλο για τη μελέτη της ετερογένειας στις επιπτώσεις των μακροοικονομικών αιφνίδιων ταραχών. Στο πρώτο μισό παρουσιάζουμε μια θεωρητική βάση όσον αφορά την ποσοτική παλινδρόμηση. Στο δεύτερο μισό, εκτιμούμε το VARQ και πραγματοποιούμε ανάλυση QIRF χρησιμοποιώντας ένα τριών μεταβλητών μακροοικονομικό μοντέλο (παραγωγικό κενό, πληθωρισμό και ποσοστό ομοσπονδιακών κεφαλαίων) για διαφορετικές χώρες που αποτελούν μέρος της Ε.Ε. (Φινλανδία, Γερμανία, Ελλάδα και Ιταλία) , για μια καθορισμένη περίοδο 20 ετών (1995q_1-2015q_1). Χρησιμοποιούμε συναρτήσεις ποσοτικής απόκρισης για να διερευνήσουμε τη δυναμική ετερογένεια στην απόκριση ενδογενών μεταβλητών σε διαφορετικές αιφνίδιες μεταβολές. Η αξιολόγηση των διαφορετικών ποσοτικών διαδρομών, που προκύπτουν από μια σταθερή συλλογή ποσοτικών δεικτών, ως δυναμικά αποτελέσματα μας δείχνουν συμπερασματικά ότι, ορισμένες χώρες έχουν την ίδια απόκριση σε ένα δεδομένο σοκ, μερικές δεν ακολουθούν το ίδιο μοτίβο με άλλες και άλλες μπορούν να έχουν ακριβώς αντίθετη συμπεριφορά ή τη δική τους μοναδική απάντηση στο σοκ. Ο λόγος για τον οποίο εμφανίζονται αυτές οι διαφορές

στις αποκρίσεις των διαφόρων χωρών μπορεί να έγκειται στα μοναδικά χαρακτηριστικά της οικονομικής δομής κάθε χώρας.

Contents

Introduction

Nowadays an important topic of many research studies, as far as economic activity is concerned, is the impact of macroeconomic shocks on the economies, having the effect on lower quantiles that is of utmost importance to policymakers understudied. The tail risk engulfs uncertainty which is transferred to forecasting thus, assessing the uncertainty that surrounds a forecast has the same value as the forecast itself. The benefits of the most common uncertainty measures are that, they are simple to calculate in most of the cases and their interpretation is simple as well. A static specification is a source of them but they mainly derive from recursive model estimates.

The quantile regression methods can provide us with a framework for robust estimation and inference and allow us to explore a variety of forms of conditional heterogeneity under less compelling distributional assumptions. The quantile regression (QR) is a statistical method for estimating models of conditional quantile functions, which offers a systematic strategy for examining how covariates influence the location, scale and shape of the entire response distribution, thus, a variety of heterogeneity in response dynamics can be seen.

The use of a VAR model (vector autoregressive) is a very important and good way to present the dynamics of macroeconomic data. This method provides us with an efficient way to forecast, to describe our data and to study the dynamics in a multivariate time series model though the structural inference of this approach. The main drawback is the track of average outcomes of the VAR models that have standard impulse response functions in their linear structure. A model with a constant coefficient used in time series is not good enough because, the effects of a succession of small and varied shocks on the structure of dynamic economic models can be ignored, especially in a case with highly aggregated data series. In addition, these models are unable to take into account the asymmetric and heterogeneous dynamic responses that are present in various cases.

The uncertainty measurements are based on past forecasting errors that are liked to root mean squared forecasting errors ($RMSFE$), or mean absolute errors (MAE). The forecasters mainly use these kinds of measures for their forecasts. Although their simple calculation and interpretation mechanisms, these models are limited due to the normality assumption, the lack of being up to date with the data and the most recent developments, and the proneness to large outliers.

Engle and Manganelli (2004) propose a quantile autoregressive framework to model value-at-risk where the quantiles follow an autoregressive process. Gourieroux and Jasiak (2008) study dynamic additive quantile model. Xiao (2009) proposes quantile regression (QR) with co-integrated time series. Galvao *et al.* (2013) interpret the quantile regression (QR) time-series framework as modeling the business cycle, where high conditional realizations of a distributed lag model correspond to high quantiles and low conditional realizations of a distributed lag model correspond to low quantiles. Montes-Rojas (2019) develops a reduced form vector directional quantile estimator based on the multivariate directional quantiles framework of Hallin *et al.* (2010).

In our dissertation, we will apply the Montes-Rojas (2019) of the vector autoregressive quantile ($VARQ$) model to contemporary macroeconomics EU data. This approach generalizes the quantile autoregressive framework proposed by Koenker and Xiao (2006) and Galvao *et al.* (2013) to the multivariate case. A collection of directional quantile models for a fixed orthonormal basis; in which each component represents a directional quantile that corresponds to a particular endogenous variable, can have a solution by this model. A map from the space of the σ-field that is generated by the available information at a specific time and a unit ball, whose dimension is given by the number of endogenous variables to the space of endogenous variables, is described by this model. The heterogeneity in time series can by explored by the $VARQ$ model, by the estimation of conditional models of each endogenous variable conditional an all other contemporaneous endogenous variable and set of information available at the time. These conditional models are used to construct a simultaneous system of directional quantile regression (QR) models, whose solution is a reduced-form multivariate quantile model.

In the first half we will present a theoretical base as far as quantile regression is concerned. In the second half, we will estimate the $VARQ$ and perform $QIRF$ analysis using a three-variable macroeconomic model for different countries, with output gap, inflation and Fed Funds rate, for a set period. We then will evaluate the effect of a standard deviation shock in the government bonds, that is, the fiscal shock, after our estimation, using the Cholesky decomposition of Christiano *et al.* (1996), and explore dynamic heterogeneity applying the $QIRFs$. This new analysis reveals important asymmetries and heterogeneity in the response to fiscal shocks in terms of different quantile paths of high or low conditional output and inflation.

1.1. Mean Regression

Over the years Classic Econometrics has been the answer to many rising questions, not only from an economic aspect but from a social aspect as a whole. The results though classical regression models have to do with the study of the average in a conditional distribution. The main focus of the mean regression is the expectation of a variable Y, which is conditional on the values of a set of variables $\boldsymbol{X}$**,** common known as the regression function $E(Y|\boldsymbol{X})$ (Weisberg, 2005). A function like this restricts on a specific location of the Y conditional distribution. So, the main point of interest is the changes in $Y = y_1, y_2, \ldots, y_n$ observations, with y_i to represent the $i-th$ observation, as$X = x_1, x_2, \ldots, x_m$ variate, with x_i to represent the $i-th$ variable that change. We have the mean function and the variance function respectfully, that consist the simplest form of a linear regression model:

$$y_i = E(Y|\boldsymbol{X} = x_i) = \beta_0 + \beta_1 x \qquad (1.1.1)$$

where β_0 is the value of $E(Y|\boldsymbol{X} = x_i)$ when $x = 0$ and β_1 is the rate of change in $E(Y|\boldsymbol{X} = x_i)$ for a given unit of change in $\boldsymbol{X}$ and gives us the slope.

$$Var(Y_i) = Var(Y|\boldsymbol{X} = x_i) = \sigma^2 \qquad (1.1.2)$$

Eq.1.1.2 is assumed to be constant with $\sigma^2 > 0$. Because of the positive value of the variance; $\sigma^2 > 0$, a difference between the observed value of the $i-th$ response y_i and the expected value $E\ (Y|\boldsymbol{X} = x_i)$ is observed. That difference is called a statistical error e_i for case i defined implicitly by $y_i = E(Y|\boldsymbol{X} = x_i) + e_i$ or explicitly by $e_i = y_i - E(Y|\boldsymbol{X} = x_i)$. The parameters that the errors e_i are depended upon are commonly unknown in the mean function and there are also not observable. They are random variables and correspond to the vertical distance between the point Y_i and the mean function $E(Y|\boldsymbol{X} = x_i)$ (Shewhart & Wilks, 2005). As far as errors are concerned, there are two very important assumptions that must be made. Firstly, we assume that $E(e_i|x_i) = 0$ and secondly, we assume that all errors are independent with each other, causing the value of the error for one case not to give any information about the value of the error for a different case. Also, a general assumption that errors follow the normal distribution is needed, though it may not always be true.

After the definition of our model that suits our problem comes the estimation of the parameters of that model. There are many methods that can be applied to achieve that purpose and the most common and simple is that of the ordinary least squares (OLS), in which we look for the parameters estimates that can minimize our residual sum of squares:

$$\hat{y}_i = \hat{E}(Y|X = x_i) = \hat{\beta}_0 + \hat{\beta}_1 x_i \qquad (1.1.3)$$

Where $\hat{y}_i$ is the fitted value for case i given by $\hat{E}(Y|X = x_i)$, $\hat{\beta}_1$is the estimator of β_1 and $\hat{\beta}_0$ is the estimator of β_0.

The equation for the statistical errors is:

$$e_i = y_i - (\beta_0 + \beta_1 x_i), i = 1, \dots, n \qquad (1.1.4)$$

The least squares for simple regression depend on averages, sums of squares and sums of cross-products. (Weisberg, 2005). We can use the Least Square Criterion, which is based on the residuals in order to obtain estimators. Here an inherent asymmetry in the response and the predictor in regression problems can be seen through the residuals.The values of β_0 and β_1 that minimize the following function are called OLS estimators:

$$RSS(\beta_0, \beta_1) = \sum_{i=1}^{n}[y_i - (\beta_0 + \beta_1 x_i)]^2 \qquad (1.1.5)$$

where, when the evaluation of (β_0, β_1) is made at $(\hat{\beta}_0, \hat{\beta}_1)$, the $RSS(\hat{\beta}_0, \hat{\beta}_1)$ is called the residual sum of squares.

There are many different ways to find the least squares estimates, one expression of them is:

$$\hat{\beta}_1 = \frac{SXY}{SXX} = r_{xy}\frac{SD_y}{SD_x} = r_{xy}(\frac{SYY}{SXX})^2$$

(1.1.6)

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

where:

- $\bar{x}$ stands for the sample average of x, derived from $\sum\frac{x_i}{n}$
- $\bar{y}$ stands for the sample average of y, derived from $\sum\frac{y_i}{n}$
- SXX stands for the sum of squares for the x's, derived from $\sum(x_i - \bar{x})^2 = \sum(x_i - \bar{x})x_i$
- SYY stands for the sum of squares for the y's, derived from $\sum(y_i - \bar{y})^2 = \sum(y_i - \bar{y})y_i$
- SXY stands for the sum of cross –products, derived from $\sum(x_i - \bar{x})(y_i - \bar{y}) = \sum(x_i - \bar{x})y_i$

- SD_x stands for the sample standard deviation of the x's, derived from $\sqrt{\frac{SXX}{(n-1)}}$
- SD_y stands for the sample standard deviation of the y's, derived from $\sqrt{\frac{SYY}{(n-1)}}$
- r_{xy} stands for the sample correlation, derived from $\frac{s_{xy}}{(SD_x SD_y)}$, where s_{xy} is the sample covariance, derived from $\frac{SXY}{(n-1)}$

Then, if we want to obtain the estimator $\hat{\sigma}^2$, we must average the squared residuals, under the assumption of uncorrelated errors with zero means and a common variance of σ^2 as follows:

$$\hat{\sigma}^2 = \frac{RSS}{n-2} \tag{1.1.7}$$

where $RSS = \sum \hat{\sigma}_i^2$ and $n-2$ are the degrees of freedom (df), where residual df =number of cases minus the number of parameters in the mean function, so for the simple regression we will have residual $df = n-2$

The variance of the estimators is:

$$Var(\hat{\beta}_1) = \sigma^2 \frac{1}{SXX},$$
$$Var(\hat{\beta}_0) = \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{SXX}\right) \tag{1.1.8}$$

Where SXX is the sum of squares for the x: $\sum (x_i - \bar{x})^2 = \sum (x_i - \bar{x}) x_i$

The covariance of the two estimates is also given by:

$$Cov(\hat{\beta}_0, \hat{\beta}_1) = -\sigma^2 \frac{\bar{x}}{SXX} \tag{1.1.9}$$

And the correlation between the estimates is also given by:

$$\rho(\hat{\beta}_0, \hat{\beta}_1) = \frac{-\bar{x}}{\sqrt{\frac{SXX}{n} + \bar{x}^2}} = \frac{-\bar{x}}{\sqrt{\frac{(n-1)SD_x^2}{n} + \bar{x}^2}} \tag{1.1.10}$$

where SD_x^2 is the sample variance of x's, derived from $\frac{SXX}{n-1}$

The estimates for $Var(\hat{\beta}_0)$ and $Var(\hat{\beta}_1)$ are acquired by:

$$\widehat{Var}(\hat{\beta}_1) = \hat{\sigma}^2 \frac{1}{SXX},$$
$$\widehat{Var}(\hat{\beta}_0) = \hat{\sigma}^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{SXX}\right) \tag{1.1.11}$$

And the square root of an estimated variance is given by:

$$se(\hat{\beta}_1) = \sqrt{\widehat{Var}(\hat{\beta}_1)} \tag{1.1.12}$$

Where se is the standard error.

All the above functions are required for the analysis of the variance in which we can compare the fit of two or more mean functions for the same set of data. If we have a fitting the mean function:

$$E(Y|X = x) = \beta_0 \qquad (1.1.13)$$

We can say that this function is the same for all the values of X. If we can fit with this mean function we can find the best parallel to the horizontal axis of x. With the help of the OLS estimate we can have:

$$\widehat{E(Y|X)} = \hat{\beta}_0 \qquad (1.1.14)$$

Where $\hat{\beta}_0$is the value of β_0 that minimizes the $\sum(y_i - \beta_0)^2$ so we have:

$$\hat{\beta}_0 = \bar{y} \qquad (1.1.15)$$

And the residual sum of squares is:

$$\sum(y_i - \hat{\beta}_0)^2 = \sum(y_i - \bar{y})^2 = SYY \qquad (1.1.16)$$

With $n - 1$ df(degrees of freedom)

There are many categories of mean functions that can solve different problems. If we proceed with the multiple linear regression model we will have:

$$Y_i = E(Y|\boldsymbol{X} = x_i) = \beta_0 + \beta_1 x_1 + \ldots + \beta_m x_m \quad (1.1.17)$$

There is value in the study of the least squares criterion that is obtained based on the residuals. The residuals reflect the inherent asymmetry in the roles of the response and the predictor in regression problems. By applying the OLS method we can find the estimators and the values that minimize our function. The estimator for the variance is obtained by averaging the squared residuals. Here, an assumption that the errors are uncorrelated random variables with zero means and common variance of σ^2, is required. We can use the estimated mean function in order to obtain the values of the response for given values of the predictor. We must revise the residuals in order to verify if there is any failure of assumptions.

We can have polynomials problems with curved mean functions that can sometimes be included in a multiple linear regression model by adding polynomial terms in the predictor variables:

$$E(Y|\mathbf{X} = x_i) = \beta_0 + \beta_1 x + \beta_2 x^2 \qquad (1.1.18)$$

Here X is smooth but not straight. This quadratic mean function can be used then the mean is expected to have a minimum of a maximum in the range of the predictor, or when the mean function is curve but does not have neither a minimum nor a maximum within the range of the predictor. A general form of a polynomial function is:

$$E(Y|\boldsymbol{X} = x_i\,) \;=\; \beta_0 + \beta_1 x \;+\; \beta_2 x^2 \;+\; \dots \;+\; \beta_m x^m \qquad (1.1.19)$$

There can be interactions and other combinations of predictors, like the existence of a mass index that has many variables. There we can find joint effects of two or more variables. Furthermore, we have dummy variables and factors; a categorical predictor with more than one level. These factors accompany the linear regression model with the form of a dummy variable. A predictor like this can require several dummy variables. We must state here that a regression with m predictors could combine to give fewer than m terms or may require more than m terms. The interpretation of a value of a parameter that is estimated can depend on other terms in the mean function and can also change if these terms are replaced by a linear combination of theirs. So it is not always a good thing to have too many terms in a regression model that are not statistically important. The square of the correlation in a summary graph can be interpreted by R^2, in a multiple linear regression. But for a non-linear regression we must define the square of the correlation between the response and the fitted values to be R^2 and then use a summary graph in order to figure out if R^2 is useful or not. A non linear regression model can be described as below:

$$E(Y|\boldsymbol{X} \;= x_i) \;=\; \alpha_1 \;+\; \alpha_2\,(1 - exp\,(-\,\alpha_3 x_i)) \qquad (1.1.20)$$

Where $\alpha_1, \alpha_2, \alpha_3$ are the parameters we want to estimate and x_i is the predictor. The mean function is a non linear combination of parameters that makes eq.1.1.6 a non linear mean function.

If we use collinear predictors we can be leaded to variable estimated coefficients that are unacceptable when we compare them to problems with no co linearity. (Shewhart & Wilks, 2005) If we have $p > 2$ then the variance of the j-th coefficient is:

$$\mathrm{Var}(\widehat{\beta_j}) = \frac{\sigma_2}{1-R_j^2} - \frac{1}{SX_jX_j} \qquad (1.1.21)$$

Where $\frac{1}{1-R_j^2}$ is the $j-th$ variance inflation factor. If we keep SX_jX_j constant and take the X_j that make the $R_j^2 = 0$, an increase in variance caused by the correlation between the predictors and co linearity is represented by the variance inflation factor. Linear regression with variable selection is not the only approach to the problem of modeling a response as a function of a very large number of terms or predictors. (Shewhart & Wilks, 2005).

In general all generalized linear regression models have a distribution of a response Y and a given set of terms X that is distributed according to a form of exponential distribution. Moreover, a linear combination of $\acute{\beta}X$ is used to represent that the response Y is depending on a term X though this combination. The mean E(Y|**X** = x_i) = $\mu(\acute{\beta}X)$ for some kernel mean function is μ. For the multiple linear regression model μ is the identity function, and for a logistic regression model μ is the logistic function. (Shewhart & Wilks, 2005).

1.2. Quantile Regression

Quantile regression is a tool that can provide us with the power to analyze the whole conditional distribution of a response variable in terms of a set of explanatory variables, rather than confining us to the analysis of only the average, as is done in classical mean regression. Quantile regression has a unique feature that is able to deal with a variety of distributions and is able to eliminate dependence upon the normality assumptions, as well, setting all the problems that mean regression has to a more realistic and up to date framework. It takes the mean regression approach and extends it by lifting up the confinements of the specific location of the Y conditional distribution. With the quantile regression we can study the conditional distribution of Y on **X** anywhere, as far as the location of the distribution is concerned, having that way a global view of the interrelations between Y and **X**. Quantile regression was introduced by Koenker and Basset in 1978 as an extension of classical least squares estimation of conditional mean models to conditional quantile functions (Davino, Furno, & Vistocco, 2014). Quantile regression generalizes univariate quantiles for conditional distribution. If we want to define a quantile function we must first explain the relation between the mean and the median of a distribution. The asymmetry is defined by the comparison between mean and median as centers of a random variable distribution. If we accept that Y is a generic random variable with its mean to be the centre c of the distribution which minimizes the squared sum of deviations, we have the solution of a minimization problem as follows:

$$\mu = argmin_c \; E(Y-c)^2. \qquad (1.2.1)$$

The median minimizes the absolute sum of deviations, so in our minimization problem we have:

$$M_e = argmin_c \; E|Y-c|. \qquad (1.2.2)$$

If then we use a sample observations we can obtain that samples estimators $\hat{\mu}$ and $\widehat{M}_e$ for such centers.

The univariate quantiles are defined as specific locations of the distribution. The θ-th quantile is the value y that $P(Y \leq y) = \theta$. If we take the cumulative distribution function:

$$F_Y(y) = F(y) = P(Y \leq y) \qquad (1.2.3)$$

we can define the quantile function as its reverse:

$$Q_Y(\theta) = Q(\theta) = F_Y^{-1}(\theta) = inf\,\{y: F(y) > \theta\} \quad , for\ \theta \in [0,1] \qquad (1.2.4)$$

If F (.) is strictly increasing and continuous, then F^{-1} (θ) is the unique real number y such as that $F(y) = \theta$ (Gilchrist, 2000).

There are three distinct quantiles that are being used the most of the times in a quantile regression analysis; $\theta = \{0.25, 0.5, 0.75\}$. These quantiles can be the centre of a distribution, minimizing the weighted absolute sum of deviations (Davino, Furno, & Vistocco, 2014). If we want to define the θ-th quantile we can proceed as follow:

$$q_\theta = argmin_c \mathrm{E}\,[\varrho_\theta\,(Y-c)] \tag{1.2.5}$$

Where ϱ_θ (.) is the loss function that follows:

$$\varrho_\theta\,(y) = [\theta - I(y < 0)]y$$
$$= [(1-\theta)I(y \le c) + \theta I(y > c)]|y|$$

A loss function like this is an asymmetric absolute loss function, in other worlds it is a weighted sum of absolute deviations, where a (1 -θ) weight is assigned to the negative deviation and a θ weight is assigned for the positive deviation.

If now we have a discrete variable Y with a probability distribution like:
$f(y) = P(Y = y)$, then the minimization problem on our hands becomes:

$$q_\theta = argmin_c\, E\,[q_\theta(Y-c)]$$
$$= argmin_c\{(1-\theta)\,\textstyle\sum_{y\le c}|y-c|f(y) + \theta\sum_{y>c}|y-c|f(y)\}.$$

We can use the same criterion in the case of a continuous random variable substituting summation with integrals:

$$q_\theta = argminc\; E\,[q_\theta(Y-c)]$$
$$= argmin_c\,\{(1-\theta)\int_{-\infty}^{c}|y-c|f(y)d(y) + \theta\int_{c}^{+\infty}|y-c|f(y)d(y)\}$$

Here $f(y)$ is the probability density function of Y. We can obtain the sample estimator $\hat{q}_\theta$ for $\theta \in [0, 1]$ using the sample information from the above formula. Moreover, we can obtain the median solution defined in eq.1.2.2 for θ = 0,5. (Davino, Furno, & Vistocco, 2014).

The formulation of univariate quantiles can be used as the solutions of the minimization problem according to the Koenker. (Koenker & Xiao, Quantile autoregression, 2006). The main assumption is that Y is a continuous random variable, thus the expected value of the absolute sum of deviations from a given center c can be split into two terms, as follows:

$$E|Y-c| = \int_{y\in\mathbb{R}}|y-c|f(y)dx$$
$$= \int_{y<c}|y-c|f(y)dy + \int_{y>c}|y-c|f(y)dy$$

$$=\int_{y<c}|c-y|f(y)dy+\int_{y>c}|y-c|f(y)dy$$

Since the absolute value is a convex function, if we take the differential E|Y – c| with respect to c and set the partial derivatives to zero, we will have the solution for the minimum:

$$\frac{\partial}{\partial c}E|Y-c| = 0$$

This solution can be obtained by applying the derivatives and integrating per as part:

$$\{(c-y)\,f(y)\,|_{-\infty}^{c}+\int_{y<c}\frac{\partial}{\partial c}|c-y|f(y)dy\}+$$

$$\{(y-c)\,f(y)\,|_{c}^{+\infty}+\int_{y>c}\frac{\partial}{\partial c}|y-c|f(y)dy\}=0$$

With the follow restrain in mind that: $\lim_{x\to-\infty}f(x)=\lim_{x\to+\infty}f(x)=0$

If we want to well define the probability density function we will have to use the following integrand restricts in y=c; where E|Y-c| is minimized.

$$\{\underbrace{(c-y)f(y)}_{=0\ when\ y=0}|_{y=c}+\int_{y<c}f(y)dy\}+\{\underbrace{(y-c)f(y)}_{=0\ when\ y=0}|_{y=c}-\int_{y>c}f(y)dy\}$$

If we then use the cumulative distribution function from eq.1.2.3 we can have a reduced form of the above function as:

$$F(c)-[1-F(c)] = 0$$

Leading to:

$$2F(c)-1 = 0\Rightarrow F(c) = \frac{1}{2}\Rightarrow c = Me$$

So, the solution of the minimization problem from eq.1.2.2 is the median. The solution above does not change by multiplying the two components of $E|Y-c|$ by a constant θ and $(1-\theta)$ respectively. Allowing us to formulate the same problem for the generic θ quantile. If we use the same strategy for eq.1.2.5 we have:

$$\frac{\partial}{\partial c}E[\rho\theta(Y-c)]=\frac{\partial}{\partial c}\{(1-\theta)\int_{-\infty}^{c}|y-c|f(y)d(y)\ +\theta\int_{c}^{+\infty}|y-c|f(y)d(y)\}$$

If we repeat the argument above, we will have:

$$\frac{\partial}{\partial c}E[\rho\theta(Y-c)] = (1-\theta)F(c)-\theta(1-F(c)) = 0$$

And the q_θ will be the solution of the minimization problem as:

$F(c)-\theta F(c)-\theta+\theta F(c) = 0\Rightarrow F(c) = \theta\Rightarrow c = q_\theta$. (Davino, Furno, & Vistocco, 2014)

We now can use the quantile regressions for the solutions of the minimization problems as we have denoted the Y as a response variable and **X** as a set of predictor variables. This idea of the unconditional mean, as the minimizer of eq.1.2.1 can be extended to the estimation of the conditional mean function. Before we proceed with the multivariable quantile impulse response function, an introduction of the simplest model is mandatory. That will be a model with a quantitative response variable and a dummy predictor variable. This way we can see the differences in the response variable between two groups as determined by a dichotomous predictor variable. Take for example the price of a certain good in a country and compare the distribution of this good's price between the two groups of domestic and foreign company manufacturers for this specific good. The dummy variable will be the origin, which will take a unit value for foreign goods and zero value for domestic goods, and will be used as regressor. If we compare the dot plots of the two groups and emphasize in the differences that may have not in their means, which will be similar, but the tails of the distribution of the extreme quantiles that reside there, we will then have to compare the two datasets in the Q-Q plot, which will allow as to represent the quantiles of the first dataset an a x-axis versus the second datasets quantiles on the y-axis, along with a 45 degree reference line. If the two datasets share a common distribution, the points should fall along the reference line. If the points fall under the line, the corresponding set shows a shorter tail in that part of the distribution. Then exact opposite happens when the points fall above the reference line; a longer tail is shown in that part of the distribution. In a Q-Q plot the presentation offers the same information as a density plot, thus allowing us to deduct information that has to do with the shape, the shifts in location and in the scale, the present of outliers and differences in tail behavior. The Q-Q plot doesn't require any tuning, like a kernel width which is required in a density plot. As far as the model is concerned, we have the classical least square regression:

$$\widehat{Price} = \hat{\beta}_0 + \hat{\beta}_1 \text{ Origin}$$

That is like the mean comparison between the two groups of domestic and foreign manufactured products, which can provide us with the same results as the classical two sample t- test in case of interference. The estimation foe the quantile regression model is:

$$\widehat{Price}_\theta = \hat{\beta}_0(\theta) + \hat{\beta}_1(\theta)\text{Origin} \tag{1.2.6}$$

This model permits us to obtain an estimation of the Price quantities for the two groups of goods, for different values of $\theta \in [0,1]$. If we use the hypothesis that domestic = 0 and foreign = 1 for the Origin indicator variable in eq.1.2.6 then we have for the domestic:

$$\widehat{Price}_{\theta} = \hat{\beta}_0 (\theta) + \hat{\beta}_1 (\theta) \text{ x } 0 = \hat{\beta}_0 (\theta)$$

And for the foreign:

$$\widehat{Price}_{\theta} = \hat{\beta}_0 (\theta) + \hat{\beta}_1 (\theta) \text{ x } 1 = \hat{\beta}_0 (\theta) + \hat{\beta}_1 (\theta)$$

The $\hat{\beta}_0 (\theta)$ provides us the estimation of the conditional θ quantile of the price for the domestic goods and the $\hat{\beta}_0 (\theta) + \hat{\beta}_1 (\theta)$ provides us with the estimation of the conditional θ quantile of the price of the foreign goods. By varying the θ in the $[0,1]$ interval, the set of estimated intercepts offers an estimate of the price distribution for the domestic goods. As far as the foreign goods are concerned, the price is obtained though the sum of the sets of intercepts and slopes for the different θ. After that we have to construct the density and the $Q - Q\ plot$ perceptively in order to observe the coefficients correspondence to the differential of values of θ in order to describe the price distribution conditional on the two levels of the Origin variable. In this setting, such conditional distributions represent the estimation of the price distribution for the two different groups of goods computed using the predictor indicator variable.
The quantile regression is an extension of the classical estimation of conditional mean models to conditional quantile functions that provides us with the power to estimate the conditional quantiles of the distribution of a response variable Y in function of a set $\boldsymbol{X}$ of predictor variables. In a case of a generalized linear regression the quantile regression model for a given conditional quantile θ is:

$$Q_{\theta} (Y|\boldsymbol{X}) = \boldsymbol{X}\beta(\theta)$$

Where $0 < \theta < 1$ and $Q_{\theta} (.|.)$ is the conditional quantile function for the $\theta - th$ quantile. We can interpret the parameter estimates in quantile regression model the same way we interpret any other linear models, as rates of change. Thus, the $\beta_i(\theta)$ coefficient of a quantile regression model can be interpreted as the rate of change of the $\theta - th$ quantile of the dependent variable distribution per unit change in value of the $i - th$ regressor:

$$\beta_i(\theta) = \frac{\partial Q\theta(Y|\boldsymbol{X})}{\partial Xi}$$

The results can be graphical represented and inspected. A typical graphical representation of quantile regression coefficients can permit us to observe the different behaviors of the coefficients with respect to the different quantiles. (Koenker & Xiao, Quantile autoregression, 2006). Using the same rules of OLS regression, a categorical explanatory variable can also be included in the model by setting them as dummy variables except for one that is excluded and operates as the reference category in the interpretation of the results. (Scott, 1997)

As stated above, we can extend our view point on the whole conditional distribution of the response function by utilizing the quantile regression model. The idea of that the mean and the quantiles are specific centers of a distribution that can minimize a squared sum of deviations and a weighted absolute sum of deviations respectively, can be generalized to the regression in order to estimate conditional mean and conditional quantiles. Two groups determined by a dummy regressor can be compared by a simple linear regression model with a quantitative response variable and a dummy regressor that allows us to compare the mean and the quantiles between them. The parameter estimates in linear models are interpretable as rates of changes, both in classical regression and quantile regression, in the same way. The location, scale, and shape shift information on the conditional distribution of a response variable can by provided by the quantile regression. The quantile regression, allows us to approximate the whole distribution of a response variable conditional, on the values of a set of regressor. (Davino, Furno, & Vistocco, 2014).

1.3. Quantiles in Time Series

As noted above, with quantile regression we can estimate models of conditional quantile functions that can give us various systematic strategies for the examination of the influence that the covariates have upon the location, shape and scale of a response distribution as a whole, by studying the various heterogeneity that can been found in response dynamics. (Galvao, Montes-Rojas, & Park, 2013). A quantile autoregressive distributed lag model QADL can describe the asymmetric dynamics in time series by exposing the importance of the heterogeneity in lagged regressors and exogenous covariates. A QADL model has a stationary process because of the use of quantile regression in standard linear time series context, that model the conditional quantile function as linear and to be depended on past values of the dependable variable, rather than modeling themselves as an autoregressive process. The work with time series can provide us with estimations of the conditional quantile functions of a particular variable along time, such as GDP, consumption, index numbers, output gap, inflation and Fed Funds rates. We can define the different phases of a business cycle as the conditional quantiles at a given time. There, the definition will depend on the height of the quantiles of the conditional distribution. For high quantiles of the conditional distribution the price returns will correspond to increasing prices and for low quantiles of the conditional distribution the price returns will correspond to periods of decreasing price. The same interpretation can also be used for value at risk, consumption growth, output gap, inflation and Fed Funds rates applications. We can define an autoregressive distributed lag model as:

$$Y_t = \mu + \sum_{j=1}^{p} \alpha_j y_{t-j} + \sum_{l=0}^{q} \acute{x}_{t-l} \vartheta_l + \varepsilon_t \qquad (1.3.1)$$

Where $t = (1, \ldots, n)$, y_t is the response variable, y_{t-j} is the lag of the response variable, x_t is a dim(x)-dimensional vector of covariates, ε_t is the innovation. While α and ϑ are some unknown functions $[0,1] \rightarrow \mathbb{R}$ that we want to estimate. The main focus of this model is the short run dynamic structure. (Galvao, Montes-Rojas, & Park, 2013). The heterogeneity that can be found in the impact of the shocks in a given time series cannot be described with an efficient way by the least squares models, but a QADL model can describe them in a better way. The θ th conditional quantile function of y_t can be written as:

$$Qy_t(\theta|\Im_t) = \mu(\theta) + \sum_{j=1}^{p} \alpha_j(\theta) y_{t-j} + \sum_{l=0}^{q} x_{t-q}^{\prime} \vartheta_l(\theta) \qquad (1.3.2)$$

Where $\Im_t$ is the σ-field generated by {y_s, x_s, s$\leq t$}. This is a quantile autoregressive distributed lag model of order p and q (QADL (p, q)). A requirement for this model is that $Qy_t(\theta|\Im_t)$ to be a monotone increasing in θ for all $\Im_t$, so we can write it as:

$$Qy_t(\theta|\Im_t) = z_t^{'}\beta(\theta) \qquad (1.3.3)$$

Where $z_t=(1, y_{t-1}, \dots , y_{t-p}x_t, \dots , x_{t-q})'$

and $\beta(\theta) = (\mu(\theta), \alpha_1(\theta), \dots, \alpha_p(\theta), \vartheta_0^{'}(\theta), \dots, \vartheta_q^{'}(\theta))'$

It is required that $Qy_t(\theta|\Im_t)$ to be a monotonic function in θ in a specific region of the $\Im_t$ space. The estimated conditional quantile function $\hat{Q}y_t(\theta|\Im_t) = z_t^{'}\hat{\beta}(\theta)$ is ensured to be monotone in aθ at $z_t = \bar{z}$. (Koenker & Xiao, Quantile autoregression, 2006). But, the monotonicity in θ is not a given for other values of z. Moreover, because of the usage of a linear model there will be a crossing far away from$\bar{z}$, but it is not something unusual as one can find a linear reparametrization of the model that does exhibit co-monotonicity over some specific space as Koenker and Xiao (2006) mention. The estimation procedure is based on a standard linear quantile regression. The selection of an appropriate model is done with the help of the BIC criterion; due to the great importance of the choice of the parameters p and q hold. This criterion is based on the Asymmetric Laplace Distribution. As the median the BIC criterion is used as described below:

$$\text{BIC} = \text{n}\log\hat{\sigma} + \frac{1+p+(1+q)\times\dim(x)}{2}\log n \qquad (1.3.4)$$

Where $\hat{\sigma} = n^{-1}\sum|y_t - z_t^{'}\hat{\beta}(1/2)|$ (Galvao, Montes-Rojas, & Park, 2013)

General hypotheses tests on the vector $\beta(\theta)$ can be done by Wald type tests as (Galvao, Montes-Rojas, & Park, 2009) describe. That way we can test for equality of many slope coefficients across several quantiles.

1.4. Multivariate Quantiles in Time Series (VAR)

In contrast with the case in which we have only one variable in the quantile function and a specific model, in the multivariable case there is no unique definition about the multivariable quantile function. Borrowing the Monte-Rojas (2007) analysis, we can consider an m-dimensional process $\mathbf{Y}_\theta = (Y_{1\theta}, \ldots, Y_{m\theta})^T$ and that for all $\theta \in \{0,1,\ldots\}$ and $\mathbf{Y}_\theta \in \mathcal{Y} \subseteq \mathbb{R}^m$. Then, we can assume that we have $k \times 1$vector of covariates $\mathbf{X}_\theta \in \mathbb{R}^{k.}$. If we exploit the covariates generated by the σ-field given by $\{ Y_s : s < \theta \}$and all the information available at time t, and take all of them into account, we find ourselves with a VARQ quantile model. For an autoregressive model of order p, $X_{\theta-1} = (Y_{\theta-1}^T, \ldots, Y_{\theta-p}^T)^T$ and $k = mp$. We can index VARQ models according to the lag order, $VARQ(p)$.

We set $\theta = (\theta_1, \ldots, \theta_m)$ as an index of the $\mathbb{R}^m$ space, having that be an element of the open unit ball in $\mathbb{R}^m$, deprived of the origin $T^m = \{z \in \mathbb{R}^m : 0 < \|z\| < 1\}$. We let $\|.\|$ be the Euclidean norm. A reduced form vector directional quantile (VDQ) model is:

$$Qy_t\ (\theta|X_{t-1} = x_{t-1}) = B(\theta)x_{t-1} + A(\theta) \qquad (1.4.1)$$

Here Q is an $m \times 1$ vector which corresponds to the multivariate quantiles of the m random variables. $B(\theta) = (B_1(\theta), \ldots, B_m(\theta_t))$ is an $m \times k$ matrix of coefficients with $B_j(\theta)$ for each $j \in \{1, \ldots, m\}, k \times 1$ vector coefficients of the j-th element in $\mathbf{Y}$. $A(\theta)$ is an $m \times 1$ vector coefficients. If we set $B_{.h}(\theta) = B_{1h}(\theta), \ldots, B_{mh}(\theta))$ as the h-lag coefficients for all the endogenous variables models, for $h = 1, \ldots, p$, we will have Q as a map $X \times T^m \longrightarrow \mathcal{Y}$. The VDQ applied to an autoregressive model is then the VARQ model that Monte-Rojas proposes. (Montes-Rojas, 2019)

In order to operate in a time series he defines the lag polynomials $B(\theta, L)$. L is the lag operator leading to:

$$B(\theta)X_{t-1} = B(\theta, L)Y_t = \sum_{k=1}^{p} B_{.k}(\theta)L^k Y_t$$

and

$$Qy_t\ (\theta|x_{t-1}) = B(\theta, L)y_t + A(\theta) \qquad (1.4.2)$$

here y_t denotes the value of Y_t to be used in the equation. In order to construct the VARQ model he defines $Qy_t\ (\theta|x_{t-1}) := \{q_1(\theta|x_{t-1}), \ldots q_m(\theta|x_{t-1})\}^T$ from the system of equations below:

[Πληκτρολογήστε κείμενο]

$$\left.\begin{array}{l} q_1(\theta|x_{t-1}) := c_1(\theta_1)^T q_{-1}(\theta|x_{t-1}) + b_1(\theta_1)^T x_{t-1} + \alpha_1(\theta_1) \\ \vdots \quad := \quad \vdots \quad + \quad \vdots \quad + \quad \vdots \\ q_m(\theta|x_{t-1}) := c_m(\theta_m)^T q_{-m}(\theta|x_{t-1}) + b_m(\theta_m)^T x_{t-1} + \alpha_m(\theta_m) \end{array}\right\} \quad (1.4.3)$$

$\{c_j(\theta_j)\}_{j=1}^{m}$ is the vector of dimension $(m-1)\times 1$, $\{b_j(\theta_j)\}_{j=1}^{m}$ is the vector of dimension $k \times 1$, $\{\alpha_j(\theta_j)\}_{j=1}^{m}$ are scalars, $\{q_j(\theta|x_{t-1})\}_{j=1}^{m}$is the individual time series quantile regression model of each Y_{jt} ; j component, on all the others Y_{-jt} ; - j components. X_t is the lags where all the components are simultaneously evaluated at $Q(\theta|x_{t-1})$. These directional quantiles are used for a fix orthonormal basis and the $VARQ$ estimator is a fixed point solution to a system of equations. (Hallin, Paindaveine, & Šiman, 2010).

We can define a $VARQ$ model as:

$$Qy_t\ (\theta|x_{t-1}) = \{I_m - C(\theta)\}^{-1}\ \{b(\theta)x_{t-1} + \alpha(\theta)\} := B(\theta)x_{t-1} + A(\theta) \qquad (1.4.4)$$

I_m is the m-dimensional identity matrix, $B(\theta) := \{I_m - C(\theta)\}^{-1}\, b(\theta)$, $A(\theta) := \{I_m - C(\theta)\}^{-1}\, \alpha(\theta)$. For eq1.4.4 to be constructed we must consider that $C(\theta) := \{C_1(\theta_1), \dots, C_m(\theta_m)\}^T$ to be a matrix based on eq1.4.3 of $m \times m$ dimensions. And that in that matrix the $\{C_j(\theta_j)\}_{j=1}^{m} m \times 1$-dimensional vectors contain all the elements of the $m-1$ vector coefficients $\{c_j(\theta_j)\}_{j=1}^{m}$ that have a 0 in the corresponding j-th component. And finally that, $b(\theta) = \{b_1(\theta_1), \dots, b_m(\theta_m)\}^T$ to be a $m \times k$ matrix and $\alpha(\theta) = \{\alpha_1(\theta_1), \dots, \alpha_m(\theta_m)\}^T$ to be a $m \times 1$ vector. For a fixed θ, the number of parameters to be estimated is that of a structural mean based VAR model. (Montes-Rojas, 2019).

The multivariate random variables conditional on the past can be described by a $VARQ$ model by way of modeling the simultaneous responses. The conditional performance of the $j-th$ endogenous variable conditional on the values of the others and the available past information can be described by θ_j quantile model, for each of the j equations. The individual contribution of every endogenous variable in the system after considering the effect of all the others can be represented by the θ. A quantile autoregressive distributed lag model can be corresponded by an individual equation. (Galvao, Montes-Rojas, & Park, 2013). A simultaneous solution of all equations for a fixed collection of individual univariate quantile indexes by θ can be a $VARQ$ model, which corresponds to a reduced form of a VAR model that is a functional in θ. The $VARQ$ model is constructed for stationary processes, but the unit root process can be detected if we look at their dynamic behavior. (Koenker & Xiao, Quantile autoregression, 2006)

We saw how Monte-Rojas defines a $VARQ$ model. But a $VARQ$ model defines one period ahead forecasting for the entire distribution of Y_{t+1}, if we have all the information available to us in time t, as shown below:

$$Qy_{t+1}(\theta|x_t) = QY_{t+1}(\theta|\{y_t, y_{t-1}, \dots, y_{t-p}\}) := B(\theta,L)y_{t+1} + A(\theta) \quad (1.4.5)$$

If we take all the information available at time t we can define the one period ahead forecast as $Q_1(\theta|x_t) = Qy_{t+1}(\theta|x_t)$.

For two periods ahead forecast the indexes becomes $t+2$ at quantiles θ_2. That will depend on the response from the past which is at $t+1$ and the implicit quantile θ_1. In turn this will depend on both quantiles (θ_2, θ_1). That way we have given the definition of a two period quantile path, where a potential path of the system of endogenous variables is corresponding with the collection of indexes:

$$Q_2\{(\theta_1, \theta_2)|x_t\} := Q_2[\theta_2|\{Q_1(\theta_1|x_t). y_t, \dots, y_{t-p+1}\}] \quad (1.4.6)$$

For a $VARQ(1)$ model this will be:

$$Q_2\{(\theta_1, \theta_2)|x_t\} := B(\theta_2)B(\theta_1)x_t + B(\theta_2)A(\theta_1) + A(\theta_2) \quad (1.4.7)$$

In general the h-periods ahead forecast can be written as a function of the forecast of the previous quantiles:

$$Q_h\{(\theta_h, \dots, \theta_1)|x_t\} := B(\theta_h,L)Q_k\{(\theta_h, \dots, \theta_1)|x_t\} + A(\theta_h) \quad (1.4.8)$$

Where $Q_h(.|.) = y_{t-k}$ if $L^k(t+h) \leq t$ and $(\theta_h, \dots, \theta_1)$, $k = 1, \dots, h-1$ is the k-periods quantile path. So, in a more general form we have the framework for forecasting different quantile paths:

$$Q_h\{(\theta_h, \dots, \theta_1)|x_t\} = \{\prod_{k=1}^{h} B(\theta_k)\}x_t + \sum_{k=1}^{h-1} \{\prod_{j=k+1}^{h} B(\theta_j)\}A(\theta_k) + A(\theta_h). (1.4.9)$$

If we want to evaluate the future values on the conditional median values of the endogenous variables we can proceed with a canonical case of this forecast that fixes $\theta_i = (0.5, \dots, 0.5)$ for all $i = 1, \dots, h$. the estimates that derive from this procedure are similar with the estimates that derive from a mean based VAR forecast in general. In this case each realization is evaluated at the conditional median and the $h-$periods ahead forecast is also constructed by using $h-1, \dots, 1$ values at the median. If this procedure generalized for any $\theta_i = (\theta, \dots, \theta)$ for all $i = 1, \dots, h$, a case with high values of θ will correspond to the persistent occurrence of the θ conditional quantile in all endogenous variables. The same θ quantile is no necessarily needed for all the endogenous variables equations. (Galvao, Montes-Rojas, & Park, 2013).

2.1. The $VARQ$ model and $QIRF$ analysis

One of the main reasons for the study of VAR models are the effects on the endogenous variables of a system that occur from a sudden distortion upon the system. We study the impulse responses. These exogenous shocks must be uncorrelated with each other and they must have economic meaning. Another characteristic of these shocks is that they should be exogenous to the other current and lagged endogenous variables of a model. Moreover, they should represent either unanticipated movements in exogenous variables or clues about future movements in exogenous variables. (Ramey, 2016). The measuring shocks on time series models is being done by VAR models, where a shock refers to a change in the residual of a conditional model and identifying exogenous changes in a structural model. But, the multivariable quantile does not have a structural model or a residual system in a reduced form as an additive model. We have though, a replica of the simultaneous movements in the endogenous variables that we notate as θ. So, a $VARQ$ model is a reduced form model that is eligible for forecasting and we can do impulse response analysis, after the shocks that have been constructed based on a mean based structural VAR model. If we then compute a counterfactual change $\delta \in \mathcal{Y} \subseteq \mathbb{R}^m$ in y_t, we can evaluate the transmission of those shocks in the multivariable distribution of the m-variate process. We can define the IRF from the comparison of the multivariate quantiles at $x_t^{\delta}: = (y_t + \delta, y_{t-1}, \ldots , y_{t-p})$ with the quantiles at $x_t = (y_t + y_{t-1}, \ldots , y_{t-p})$. If we have a shock at time t, $\delta \in \mathcal{Y} \subseteq \mathbb{R}^m$ we can define the θ-quantile IRF $(QIRF)$ at $t+1$ as described below:

$$\mathrm{Qirf}_1(\theta,\delta|x_t) = Q_1(\theta\, |x_t^{\delta}\,) - Q_1(\theta\, |x_t\,) = B_{.1}(\theta)\delta$$

Where Q_1 is the one period ahead forecast.

If we have a two periods ahead IRF with $t+2$ at θ_2 quantiles, the quantile path will be:

$$\mathrm{Qirf}_{2(1)}\{(\theta_2, \theta_1)\delta|x_t\} = Q_2(\theta_2, \theta_1|x^{\delta}_t\,) - Q_2(\theta_2, \theta_1|x_t\,)$$
$$= \begin{cases} (B_{.2}(\theta_2) + B_{.1}(\theta_2)\, B_{.1}(\theta_1))\delta & p > 1 \\ \\ (B_{.1}(\theta_2)\, B_{.1}(\theta_1))\, \delta & p = 1 \end{cases} \quad (1.2.1)$$

This $QIRF$ is constructed for different quantile paths, in which each forecast is evaluated at a given multivariate quantile index and for a fixed quantile index used for the previous endogenous variables forecasts. (Montes-Rojas, 2019). If we integrate out θ_1 by using $\theta_1 \sim U(0,1)^m$ we could have a two period ahead forecast that may not depend on the implicit quantile that is used for the one step forecast:

$$\mathrm{Qirf}_2(\theta,\delta|x_t) = Q_2(\theta|x_t^{\delta}) - Q_2(\theta|x_t)$$

$$= \begin{cases} (B_{.2}(\theta) + B_{.1}(\theta)\,\bar{B}_{.1})\delta \; p > 1 \\ \\ (B_{.1}(\theta)\,\bar{B}_{.1}\,\delta \; p = 1 \end{cases} \quad (1.2.2)$$

The difference between eq1.2.1 and eq1.2.2 is that eq1.2.1 corresponds to a particular path of assumed realizations of the multivariate process and eq1.2.2 focuses on the two period ahead distribution for a forecast value of one period ahead. If we generalize for h-periods ahead IRFs we will have:

$$Qirf_{h(h-1,\dots,1)}\{(\theta_h, \theta_{h-1}, \dots, \theta_1), \delta|_t^x\} =$$

$$Q_h\{(\theta_h, \theta_{h-1}, \dots, \theta_1), |x_t^{\delta}\} - Q_h\{(\theta_h, \theta_{h-1}, \dots, \theta_1), |_t^x\} \quad (1.2.3)$$

For shock δ at time t and for a given path of multivariate quantiles $(\theta_h, \theta_{h-1}, \dots, \theta_1)$ we will also have:

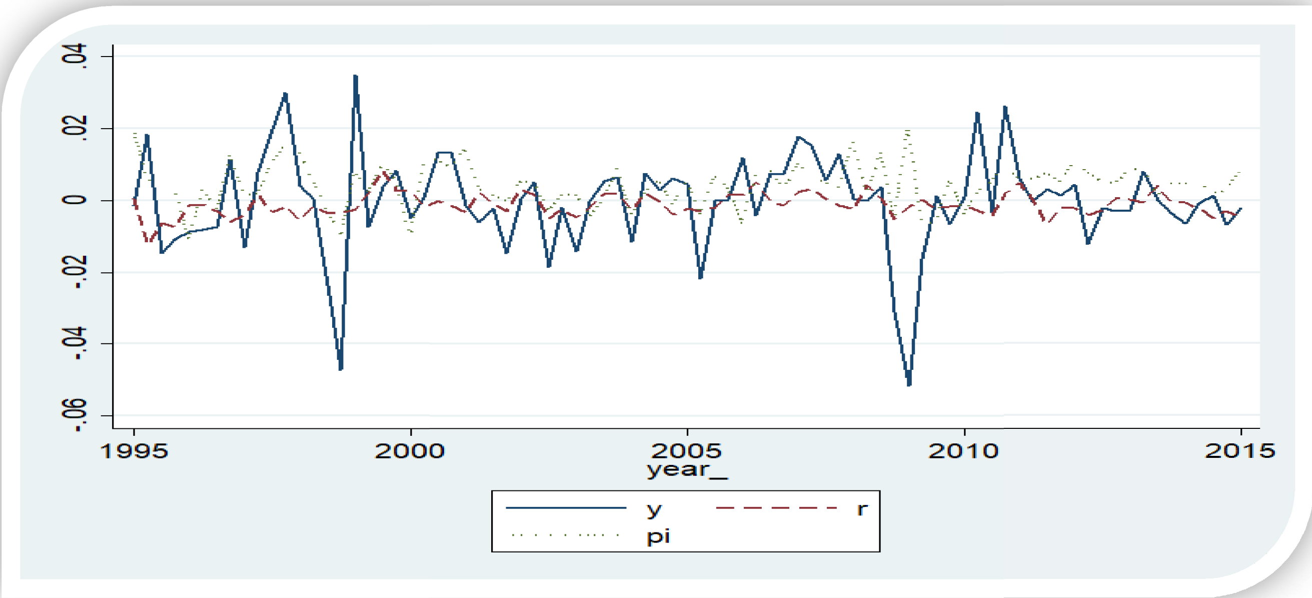
$$Qirf_h(\theta, \delta|x_t) = Q_h(\theta|x_t^{\delta}) - Q_h(\theta|x_t) \quad (1.2.4)$$

The mean based VAR analysis differs from this analysis. Here, the effect on h periods ahead is the result of the conditional expectations in the previous periods, by using the iterated expectations property.

2.2. Data Analysis

In our analysis we estimate a model with three variables the Output gap, which is generated by the first difference of the Hodrick-Prescott[1] linear filter with linear trend, using the Nominal Gross domestic product, seasonally adjusted[2], denoted y_t, the inflation rate, which is the log difference of the GDP deflator, seasonally adjusted[3], denoted π_t, the Fed Funds rate as the fiscal policy instrument corresponds to the first difference of the quarterly Government Bonds[4], denoted r_t. So we have $Y_t = (y_t, \pi_t, r_t)$. The plots that are described here are shown on Graphs 1.1 to 1.4 and their summary statistics report is on Tables 1.1.1 to 1.4.3 for each country.

[1] Hodrick-Prescott filter has a forward forecasting ability thus it may perceive some economic activities as simple trends and not as significant changes, such as the recession of 2008 in Greece.
[2] in Domestic currency, source International Financial Statistics, Metadata by Country, Gross Domestic Product and Components selected indicators, IFS
[3] source: International Financial Statistics, Metadata by Country, Gross Domestic Product and Components selected indicators, IFS
[4] source: International Financial Statistics, Metadata by Country, Interest Rates selected Indicators, IFS



Graph1.1: Series 1995q1-2015q1. Notes: Output gap, inflation rate and interest rate series for Finland

Table 1.1.1 Summary statistics for the series 1995q1-2015q1 for Finland

Variable	Obs	Mean	Std. Dev	Min	Max
y	81	−0,0003178	0,0138842	−0,515775	0,348388
π	81	0,004436	0,0063623	−0,0111005	0,0206033
r	81	−0,001195	0,0032122	−0,0122667	0,0083333

Table 1.1.2 Correlations (y_t, π_t, r_t).

	y	π	r
y	1,0000		
π	0,3302	1,0000	
r	0,1930	0,0466	1,0000

Table 1.1.3 Correlations (y_t, π_t, r_t) mean based VAR residuals

	y	π	r
y	1,0000		
π	0,3208	1,0000	
r	0,1316	−0,0021	1,0000

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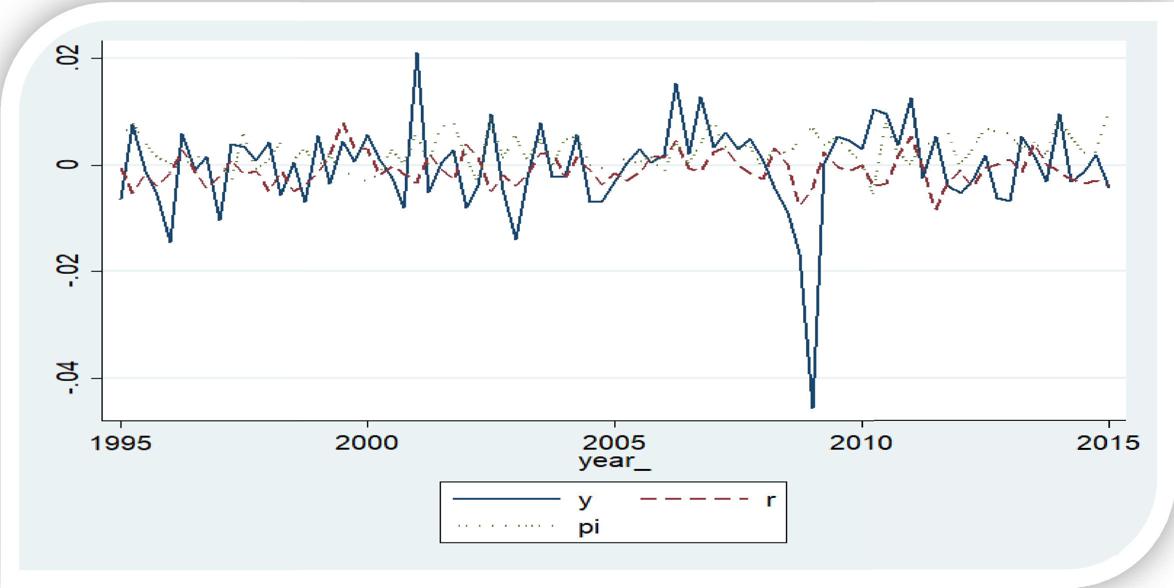


Graph1.2: Series 1995q1-2015q1. Notes: Output gap, inflation rate and interest rate series for Germany

Table 1.2.1 Summary statistics for the series 1995q1-2015q1 for Germany

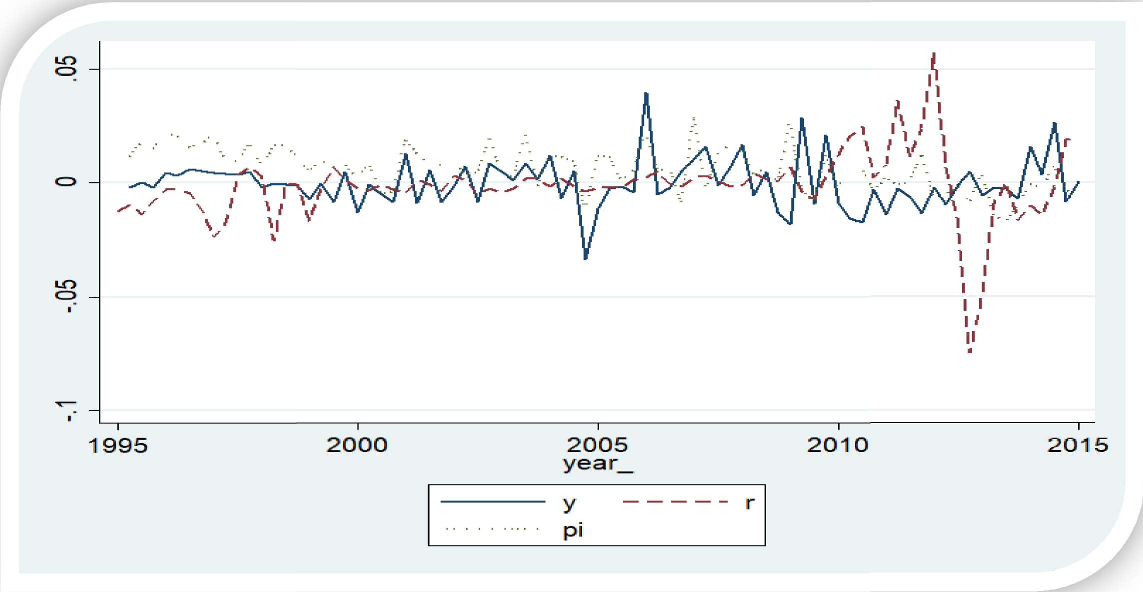
Variable	Obs	Mean	Std. Dev	Min	Max
y	81	$-0{,}002216$	0,0083667	$-0{,}0456853$	0,0210145
π	81	0,0026471	0,0031855	$-0{,}0055431$	0,0094304
r	81	$-0{,}0008877$	0,002927	$-0{,}0083667$	0,0079333

Table 1.2.2 Correlations (y_t, π_t, r_t).

	y	π	r
y	1,0000		
π	$-0{,}0207$	1,0000	
r	0,2030	$-0{,}2385$	1,0000

Table 1.2.3 Correlations (y_t, π_t, r_t) mean based VAR residuals

	y	π	r
y	1,0000		
π	0,0589	1,0000	
r	0,1523	$-0{,}1960$	1,0000



Graph 1.3: Series 1995q1-2015q1. Notes: Output gap, inflation rate and interest rate series for Greece

Table 1.3.1. Summary statistics for the series 1995q1-2015q1 for Greece

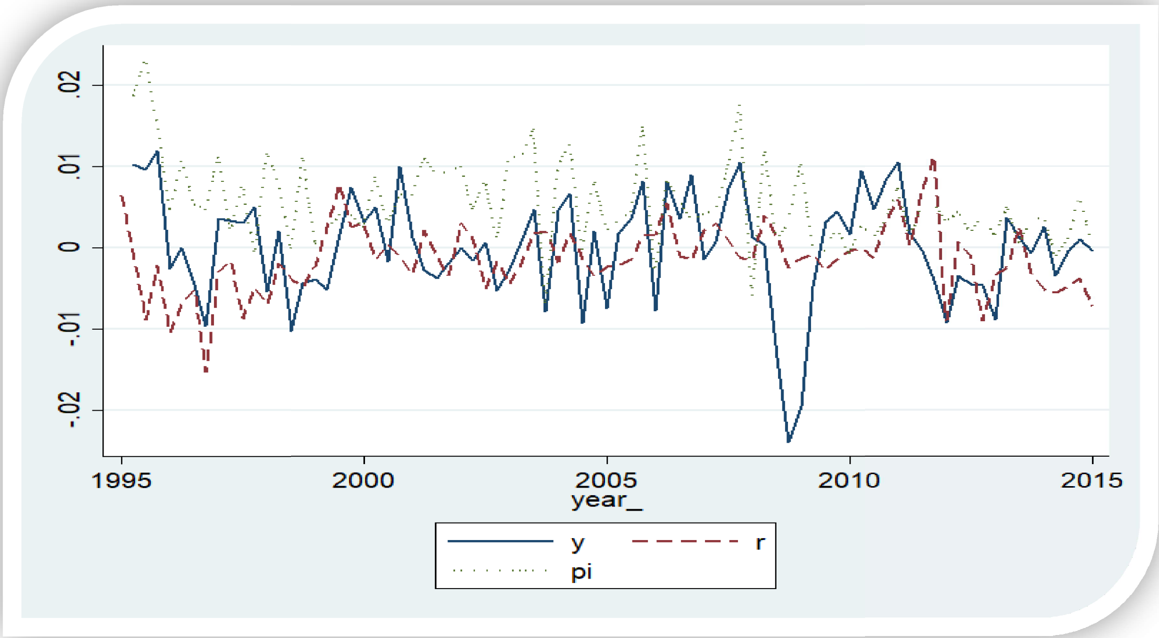
Variable	Obs	Mean	Std. Dev	Min	Max
y	80	0,0000412	0,0108035	−0,0332788	0,0396173
π	80	0,006371	0,0098346	−0,0171075	0,0289579
r	81	−0,0012255	0,0155478	−0,0752667	0,0570667

Table 1.3.2 Correlations (y_t, π_t, r_t).

	y	π	r
y	1,0000		
π	0,2835	1,0000	
r	−0,1522	−0,0257	1,0000

Table 1.3.3 Correlations (y_t, π_t, r_t) mean based VAR residuals

	y	π	r
y	1,0000		
π	0,2353	1,0000	
r	−0,1205	0,0068	1,0000



Graph 1.4: Series 1995q1-2015q1. Notes: Output gap, inflation rate and interest rate series for Italy

Table 1.4.1. Summary statistics for the series 1995q1-2015q1 for Italy

Variable	Obs	Mean	Std. Dev	Min	Max
y	80	0,0000747	0,0066389	−0,0239279	0,0118604
π	80	0,0055292	0,0054386	−0,0071405	0,023252
r	81	−0,0013062	0,0042909	−0,0153333	0,0112

Table 1.4.2 Correlations (y_t, π_t, r_t).

	y	π	r
y	1,0000		
π	0,3808	1,0000	
r	0,1905	−0,0453	1,0000

Table 1.4.3 Correlations (y_t, π_t, r_t) mean based VAR residuals

	y	π	r
y	1,0000		
π	0,4227	1,0000	
r	0,2087	−0,0174	1,0000

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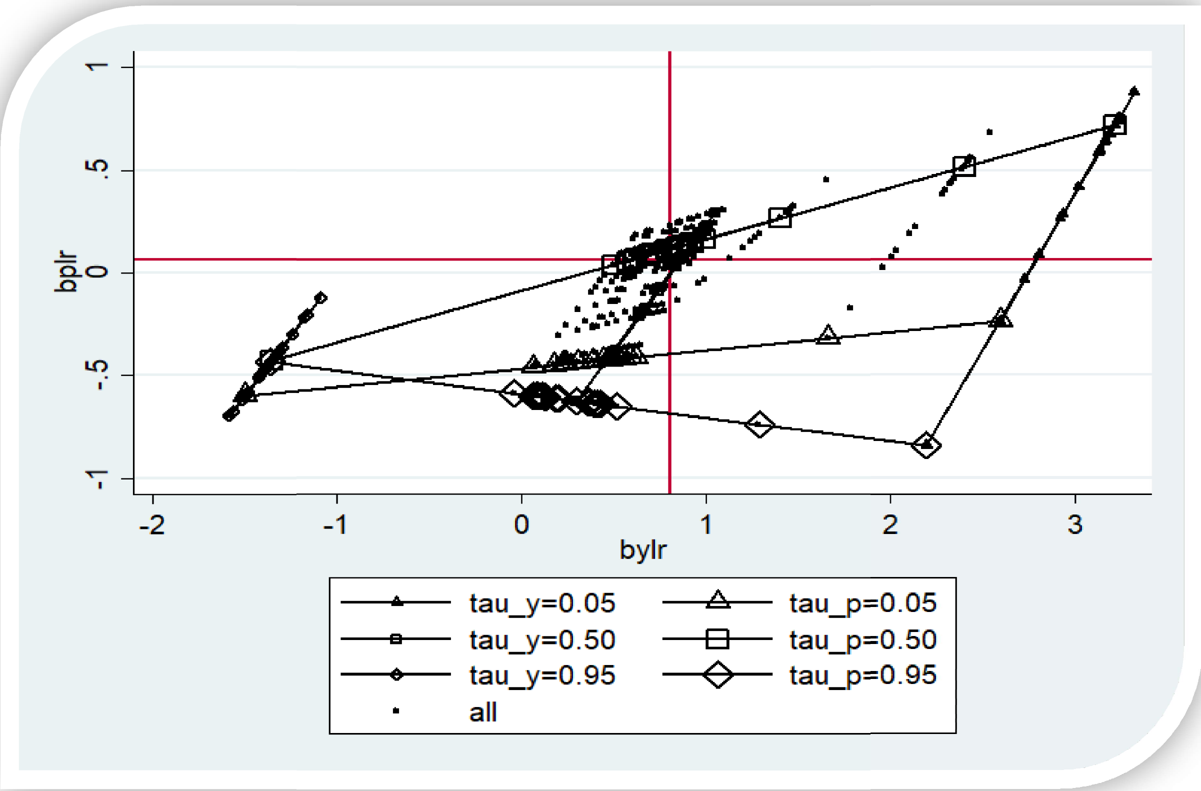
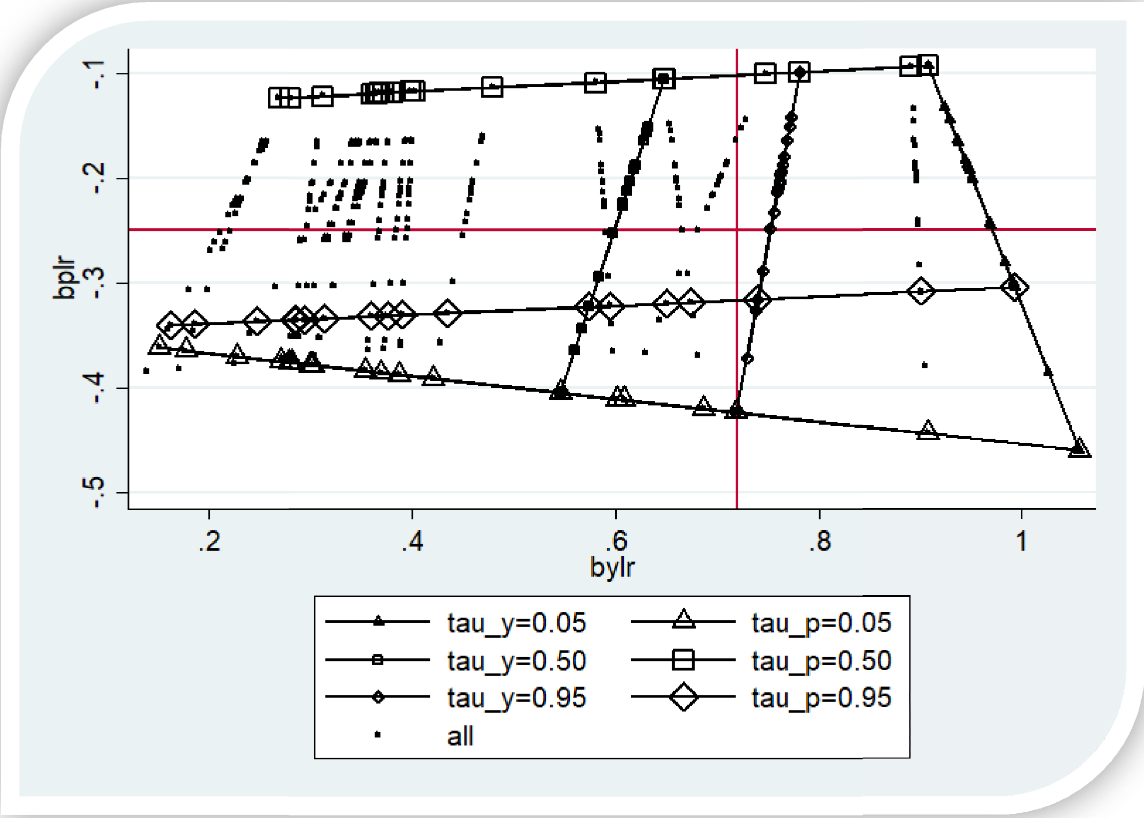
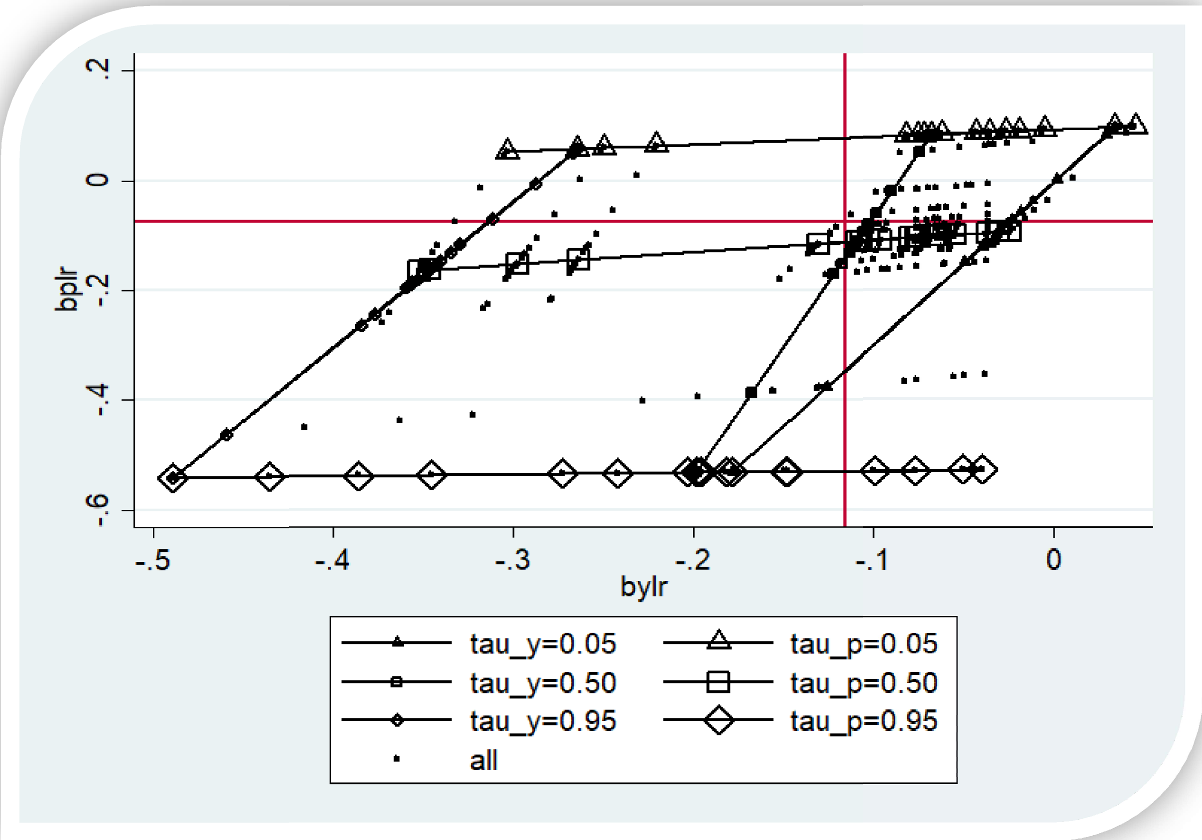


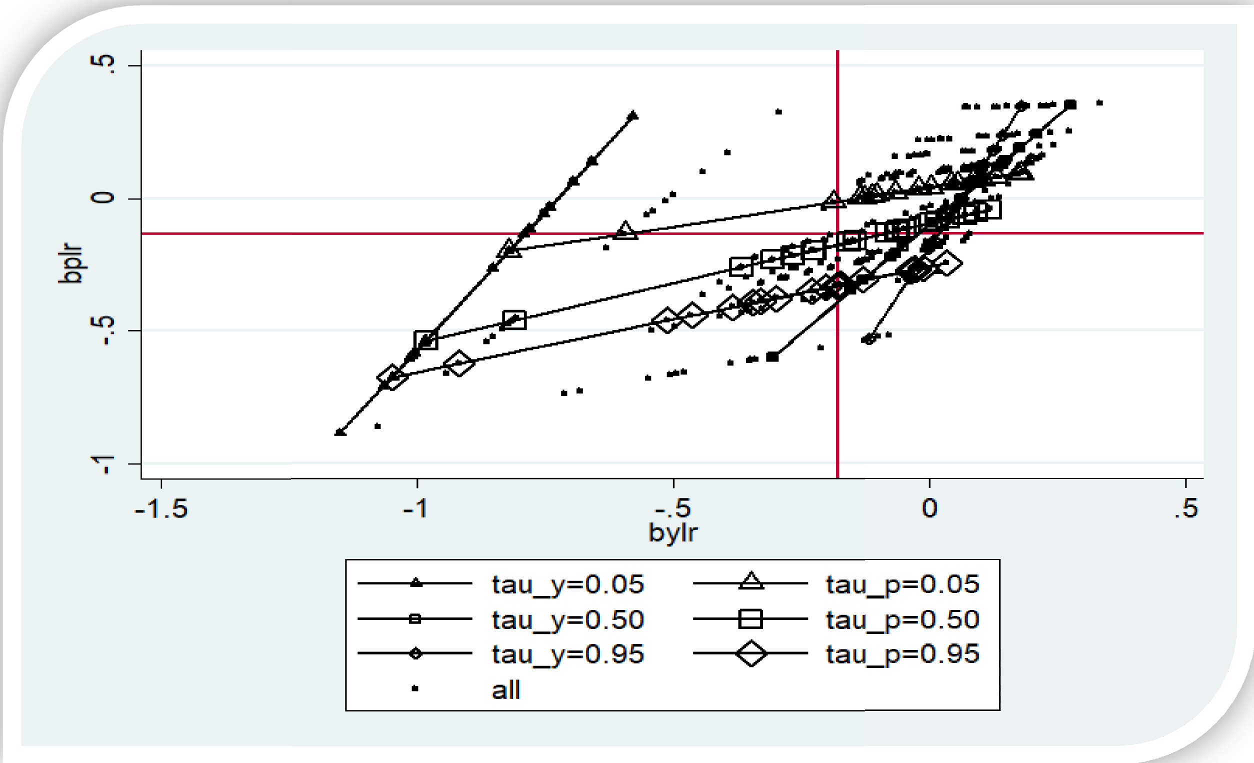
Graph 2.1 for Finland: The VARQ coefficients are for $\tau_y \in \{0.05,0.10,\dots,0.95\}$, $\tau_\pi \in \{0.05,0.10,\dots,0.95\}$, $\tau_\pi \in \{0.05,0.10,\dots,0.95\}$ *and* $\tau_r = 0.50$. Notes: In this Graph we can see the heterogeneity in the effect of the QR coefficients of a lagged change in the interest rate on output, which is the *bylr* in the horizontal axis and the inflation which is the *bplr* in the vertical axis. The vertical and the horizontal lines show the mean based VAR effects. The lines with the small triangles show the VARQ coefficients with$\tau_y = 0.05$,$\tau_\pi \in \{0.05,0.10,\dots,0.95\}$,$\tau_r = 0.50$. The lines with the small squares show the VARQ coefficients with$\tau_y = 0.50$,$\tau_\pi \in \{0.05,0.10,\dots,0.95\}$,$\tau_r = 0.50$. The lines with the small diamonds show the VARQ coefficients with$\tau_y = 0.95$,$\tau_\pi \in \{0.05,0.10,\dots,0.95\}$,$\tau_r = 0.50$. The lines with the large triangles show the VARQ coefficients with $\tau_y \in \{0.05,0.10,\dots,0.95\}$, $\tau_\pi = 0.05$, $\tau_r = 0.50$. The lines with the large squares show the VARQ coefficients with $\tau_y \in \{0.05,0.10,\dots,0.95\}$, $\tau_\pi = 0.50$, $\tau_r = 0.50$ and the lines with the large diamonds show the VARQ coefficients with $\tau_y \in \{0.05,0.10,\dots,0.95\}$, $\tau_\pi = 0.95$, $\tau_r = 0.50$



Graph 2.2 for Germany: The VARQ coefficients are for $\tau_y \in \{0.05, 0.10, \ldots, 0.95\}$, $\tau_\pi \in \{0.05, 0.10, \ldots, 0.95\}$, $\tau_\pi \in \{0.05, 0.10, \ldots, 0.95\}$ and $\tau_r = 0.50$. Notes: In this Graph we can see the heterogeneity in the effect of the QR coefficients of a lagged change in the interest rate on output, which is the $bylr$ in the horizontal axis and the inflation which is the $bplr$ in the vertical axis. The vertical and the horizontal lines show the mean based VAR effects. The lines with the small triangles show the VARQ coefficients with$\tau_y = 0.05$,$\tau_\pi \in \{0.05, 0.10, \ldots, 0.95\}$,$\tau_r = 0.50$. The lines with the small squares show the VARQ coefficients with$\tau_y = 0.50$,$\tau_\pi \in \{0.05, 0.10, \ldots, 0.95\}$,$\tau_r = 0.50$. The lines with the small diamonds show the VARQ coefficients with$\tau_y = 0.95$,$\tau_\pi \in \{0.05, 0.10, \ldots, 0.95\}$,$\tau_r = 0.50$. The lines with the large triangles show the VARQ coefficients with $\tau_y \in \{0.05, 0.10, \ldots, 0.95\}$, $\tau_\pi = 0.05$, $\tau_r = 0.50$. The lines with the large squares show the VARQ coefficients with $\tau_y \in \{0.05, 0.10, \ldots, 0.95\}$, $\tau_\pi = 0.50$, $\tau_r = 0.50$ and the lines with the large diamonds show the VARQ coefficients with $\tau_y \in \{0.05, 0.10, \ldots, 0.95\}$, $\tau_\pi = 0.95$, $\tau_r = 0.50$



Graph 2.3 for Greece: The VARQ coefficients are for $\tau_y \in \{0.05, 0.10, \ldots, 0.95\}$, $\tau_\pi \in \{0.05, 0.10, \ldots, 0.95\}$, $\tau_\pi \in \{0.05, 0.10, \ldots, 0.95\}$ *and* $\tau_r = 0.50$. Notes: In this Graph we can see the heterogeneity in the effect of the QR coefficients of a lagged change in the interest rate on output, which is the *bylr* in the horizontal axis and the inflation which is the *bplr* in the vertical axis. The vertical and the horizontal lines show the mean based VAR effects. The lines with the small triangles show the VARQ coefficients with$\tau_y = 0.05$,$\tau_\pi \in \{0.05, 0.10, \ldots, 0.95\}$,$\tau_r = 0.50$. The lines with the small squares show the VARQ coefficients with$\tau_y = 0.50$,$\tau_\pi \in \{0.05, 0.10, \ldots, 0.95\}$,$\tau_r = 0.50$. The lines with the small diamonds show the VARQ coefficients with$\tau_y = 0.95$,$\tau_\pi \in \{0.05, 0.10, \ldots, 0.95\}$,$\tau_r = 0.50$. The lines with the large triangles show the VARQ coefficients with $\tau_y \in \{0.05, 0.10, \ldots, 0.95\}$, $\tau_\pi = 0.05$, $\tau_r = 0.50$. The lines with the large squares show the VARQ coefficients with $\tau_y \in \{0.05, 0.10, \ldots, 0.95\}$, $\tau_\pi = 0.50$, $\tau_r = 0.50$ and the lines with the large diamonds show the VARQ coefficients with $\tau_y \in \{0.05, 0.10, \ldots, 0.95\}$, $\tau_\pi = 0.95$, $\tau_r = 0.50$



Graph 2.4 for Italy: The VARQ coefficients are for $\tau_y \in \{0.05,0.10,\dots,0.95\}$, $\tau_\pi \in \{0.05,0.10,\dots,0.95\}$, $\tau_\pi \in \{0.05,0.10,\dots,0.95\}$ *and* $\tau_r = 0.50$. Notes: In this Graph we can see the heterogeneity in the effect of the QR coefficients of a lagged change in the interest rate on output, which is the *bylr* in the horizontal axis and the inflation which is the *bplr* in the vertical axis. The vertical and the horizontal lines show the mean based VAR effects. The lines with the small triangles show the VARQ coefficients with$\tau_y = 0.05$,$\tau_\pi \in \{0.05,0.10,\dots,0.95\}$,$\tau_r = 0.50$. The lines with the small squares show the VARQ coefficients with$\tau_y = 0.50$,$\tau_\pi \in \{0.05,0.10,\dots,0.95\}$,$\tau_r = 0.50$. The lines with the small diamonds show the VARQ coefficients with$\tau_y = 0.95$,$\tau_\pi \in \{0.05,0.10,\dots,0.95\}$,$\tau_r = 0.50$. The lines with the large triangles show the VARQ coefficients with $\tau_y \in \{0.05,0.10,\dots,0.95\}$, $\tau_\pi = 0.05$, $\tau_r = 0.50$. The lines with the large squares show the VARQ coefficients with $\tau_y \in \{0.05,0.10,\dots,0.95\}$, $\tau_\pi = 0.50$, $\tau_r = 0.50$ and the lines with the large diamonds show the VARQ coefficients with $\tau_y \in \{0.05,0.10,\dots,0.95\}$, $\tau_\pi = 0.95$, $\tau_r = 0.50$

Graphs 2.1 to 2.4 show the effect of a unit change in r keeping (y, π) unchanged, on the coefficients$B_{(\pi r)1}(\tau_y, \tau_\pi, \tau_r)$, denoted as $bplr$ in the graph and $B_{(yr)1}(\tau_y, \tau_\pi, \tau_r)$ denoted as $bylr$ in the graph. We include least squares (OLS) estimate given by a regression model of y_t and π_t on $(y_{t-1}, \pi_{t-1}, r_{t-1})$. We hypothesize different scenarios with$\tau_y \in \{0.05, 0.10, \dots, 0.95\}$, $\tau_\pi \in \{0.05, 0.10, \dots, 0.95\}$ and $\tau_r = 0.50$ for which we make evaluations of the effect of a unit change in r. Also, Graphs 2.1 to 2.4 show in general heterogeneity as far as output and inflation is concerned and their reactions to a change in the interest rate. As a general conclusion from these Graphs, the OLS and the median effects are small while the highest effects are derived from low τ_y and τ_π quantiles.

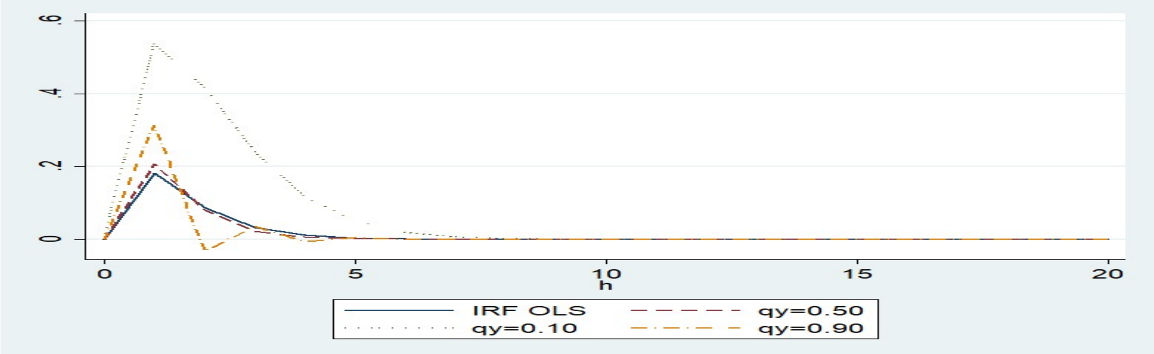
But in more details, we can see in Finland that follows this rule having the highest effects to correspond to low τ_y and τ_π quantiles, almost identical with Greece's and Italy's behavior, with the later to have an upward tendency for all the indicators as well as the highest OLS and median effects of the four countries. In Germany the highest effects correspond to medium τ_y and τ_π quantiles while the lowest effects correspond to low τ_y and τ_π quantiles.

Moving forward, we make IRFs following the Cholesky identification procedure (Eichenbaum, Christiano, & Evans, 1996) and having the same assumptions as (Montes-Rojas, 2019); we use the residuals from the VAR model assuming that r has no simultaneous effect on y and π. That π has an effect on r but no effect on y. And that y affects both π and r. As an economic interpretation that means that shocks to the Fed Funds rate probably has no simultaneous effect on the other economic variables. Furthermore, we make an evaluation upon the effect of the shock in r, calculated as the standard deviation of this structural shock, on output gap and inflation, which are being standardized by the standard deviation of their corresponding structural shocks. (Montes-Rojas, 2019).

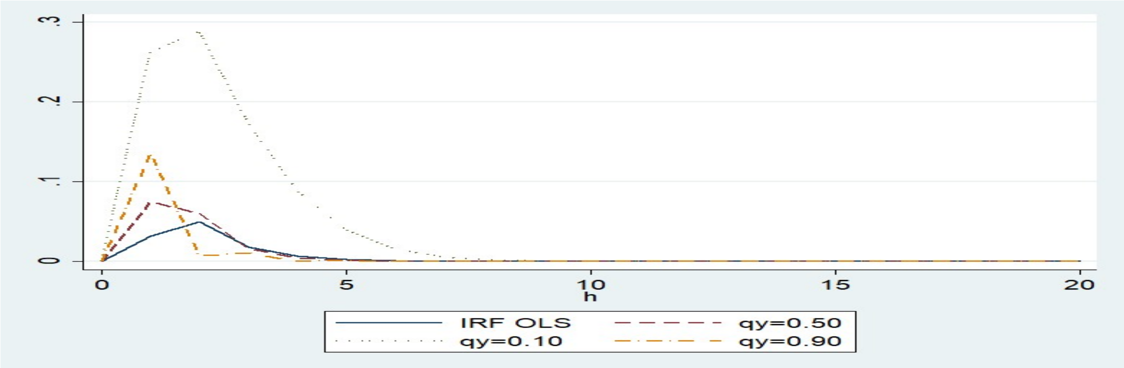
In general we can conclude that the mean effects are more powerful than the median ones (see also the appendix tables 1.1 to 1.4).

QIRF (for Finland)

Output gap (Graph 3.1.1)

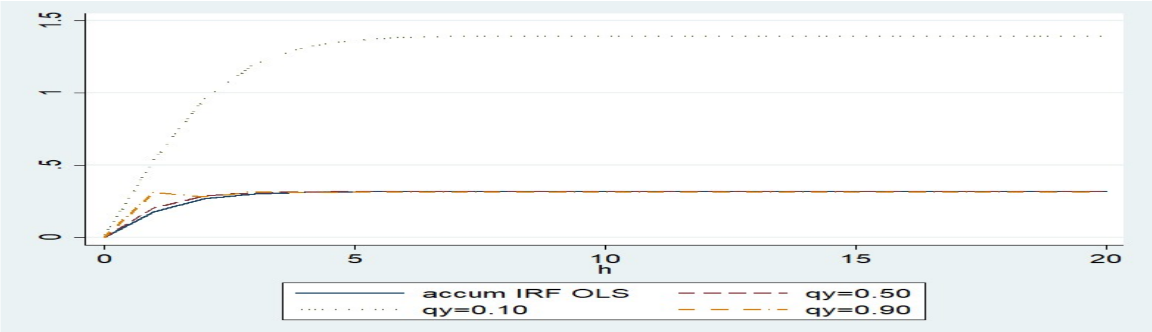


Inflation (Graph 3.1.2)

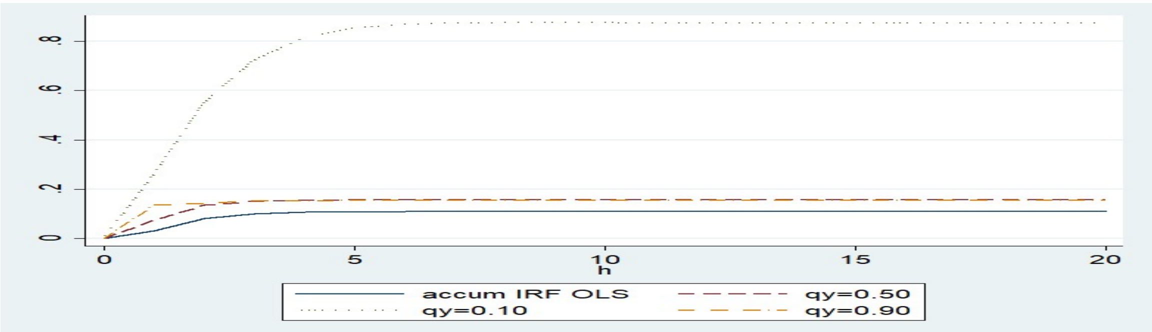


Accumulated QIRF (for Finland)

Output gap (Graph 3.1.3)



Inflation (Graph 3.1.4)



Graphs 3.1.1 to 3.1.4 for Finland: QIRF for different τ_y. Notes: QIRF on output gap and inflation of a standard deviation shock in r_t for $\tau_y \in \{0.10, 0.50, 0.90\}, \tau_\pi = 0.50$ *and* $\tau_r = 0.50$

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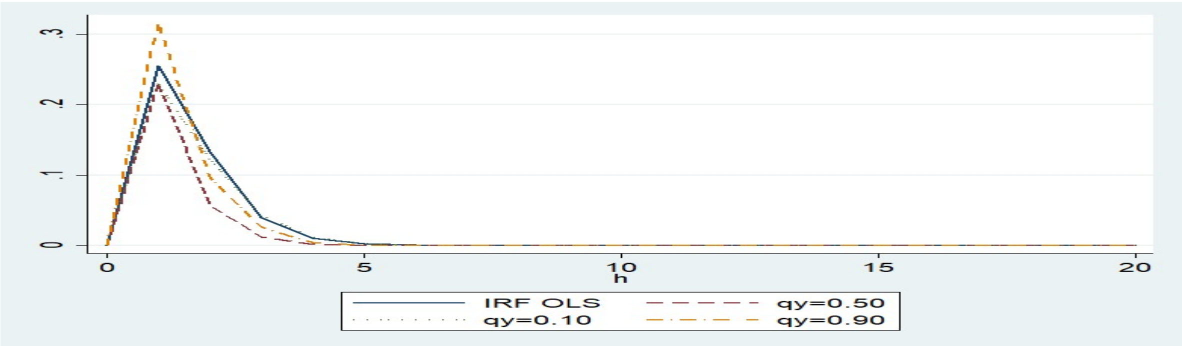
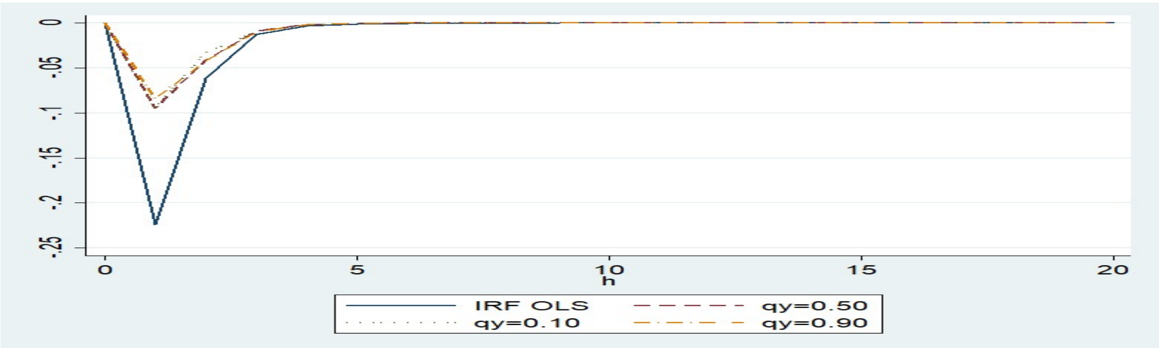
QIRF (for Germany)

Output gap (Graph 3.2.1) &3.2.2

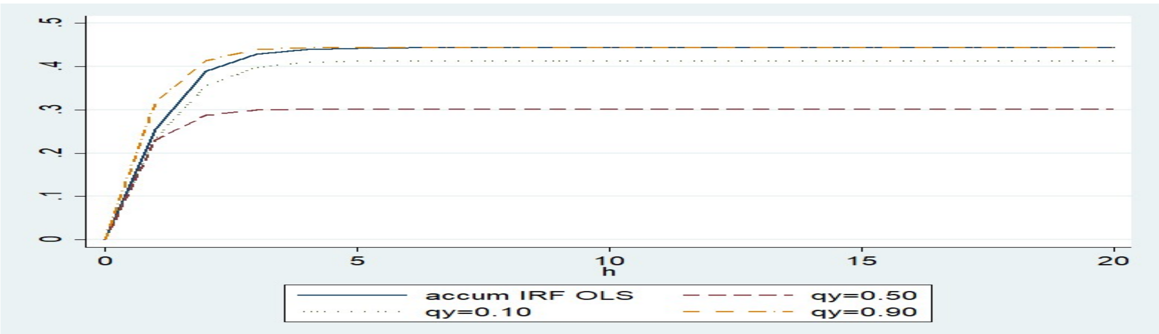


Inflation (Graph 3.2.2)

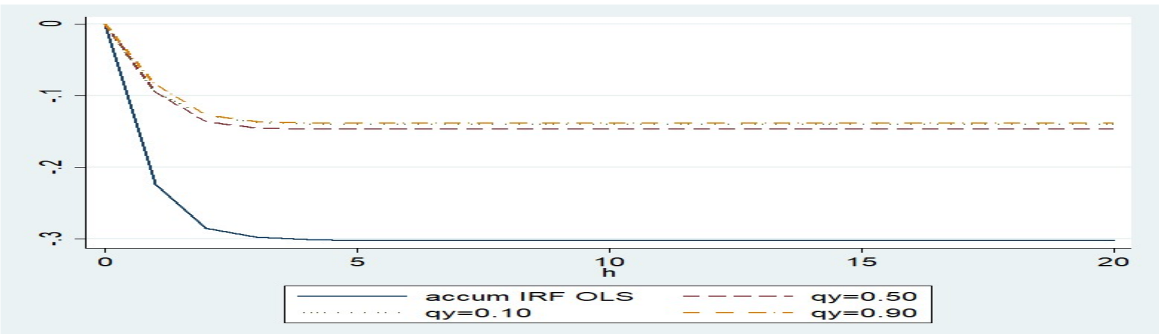


Accumulated QIRF (for Germany)

Output gap (Graph 3.2.3)



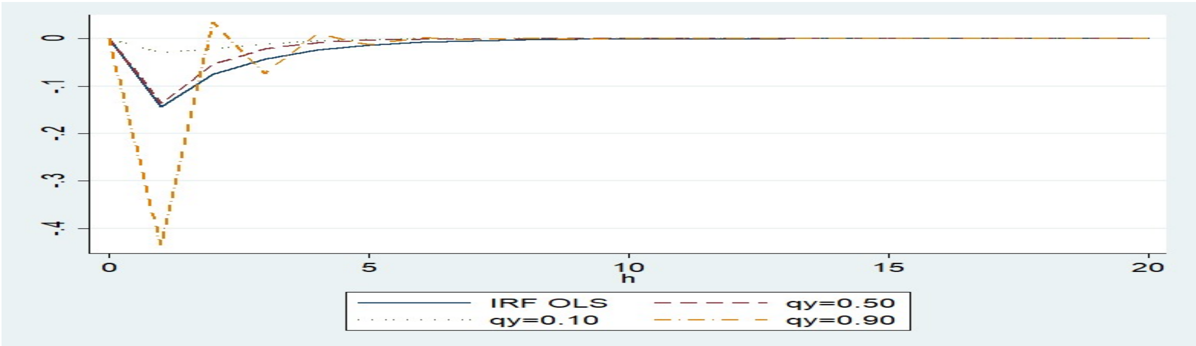
Inflation (Graph 3.2.4)



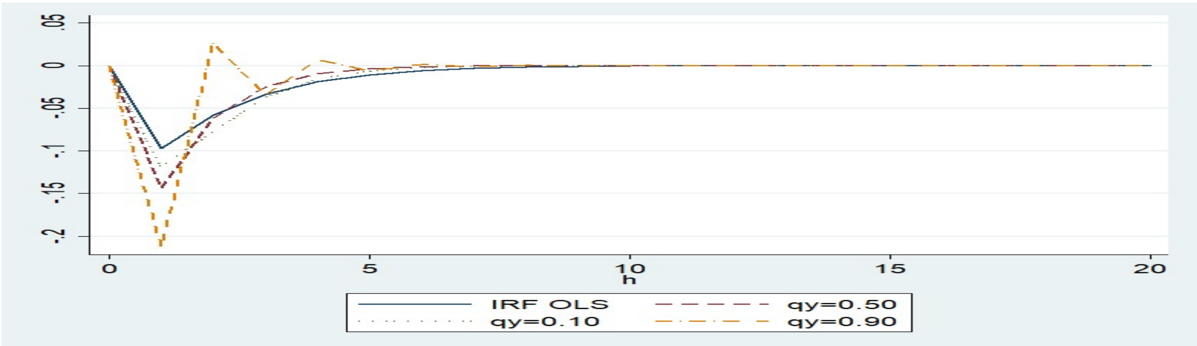
Graphs 3.2.1 to 3.2.4 for Germany: QIRF for different τ_y. Notes: QIRF on output gap and inflation of a standard deviation shock in r_t for $\tau_y \in \{0.10, 0.50, 0.90\}, \tau_\pi = 0.50\ and\ \tau_r = 0.50$

QIRF (for Greece)

Output gap (Graph 3.3.1)

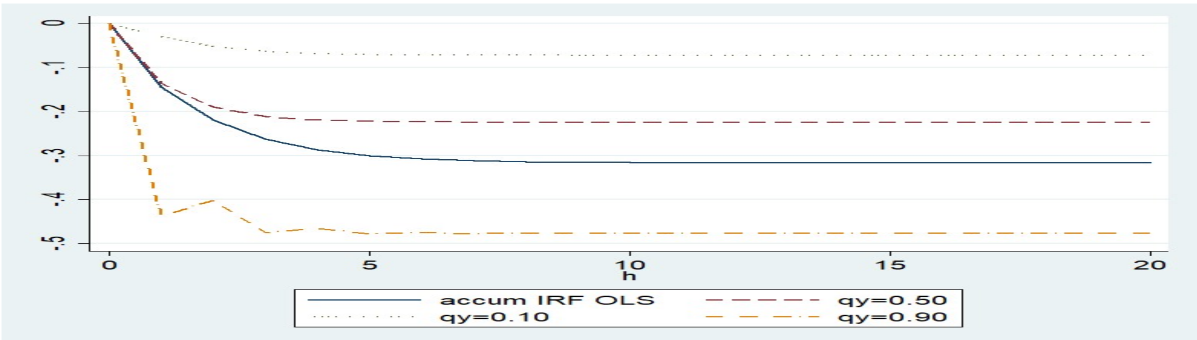


Inflation (Graph 3.3.2)

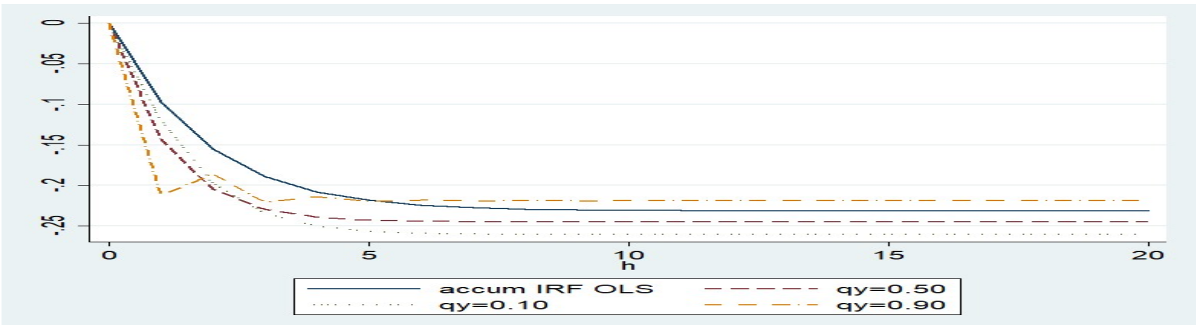


Accumulated QIRF (for Greece)

Output gap (Graph 3.3.3)



Inflation (Graph 3.3.4)



Graphs 3.3.1 to 3.3.4 for Greece: QIRF for different τ_y. Notes: QIRF on output gap and inflation of a standard deviation shock in r_t for $\tau_y \in \{0.10, 0.50, 0.90\}, \tau_\pi = 0.50$ *and* $\tau_r = 0.50$

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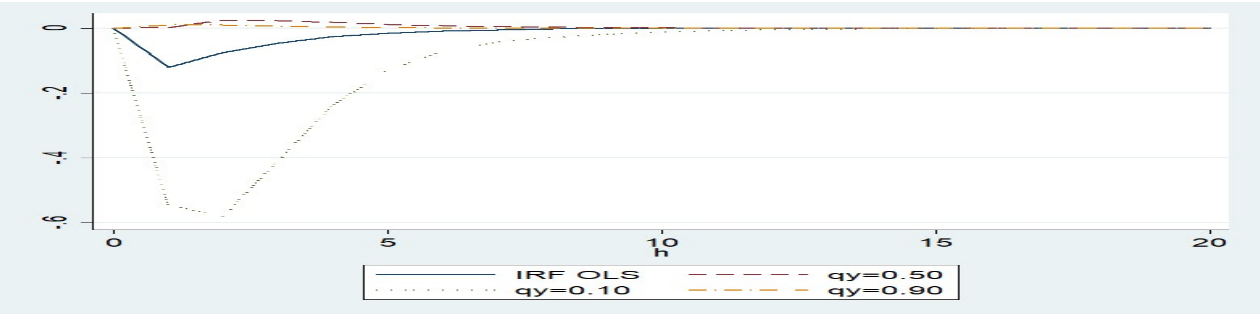
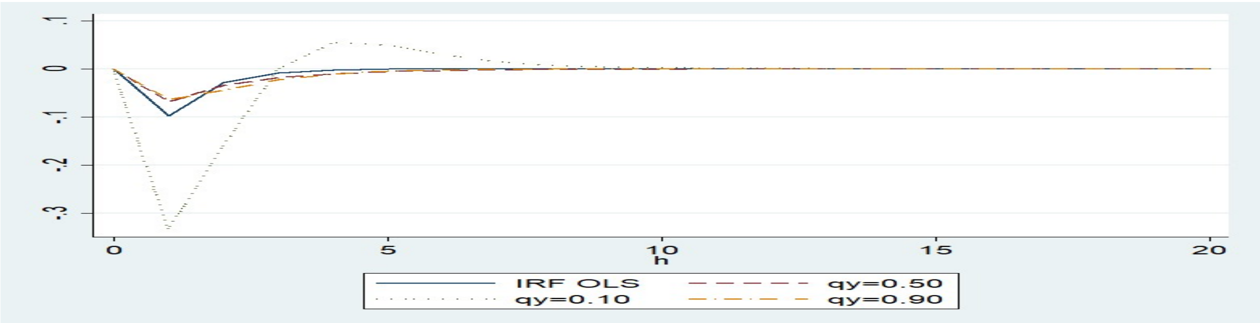
QIRF (for Italy)

Output gap (Graph 3.4.1)

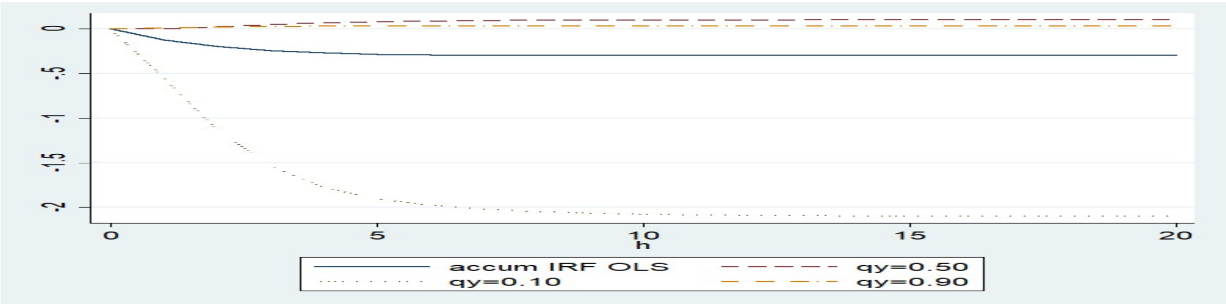


Inflation (Graph 3.4.2)

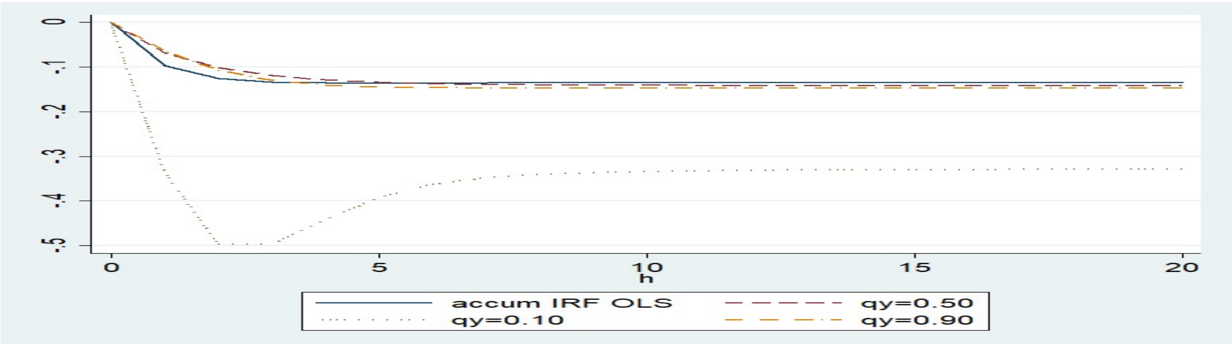


Accumulated QIRF (for Italy)

Output gap (Graph 3.4.3)



Inflation (Graph 3.4.4)



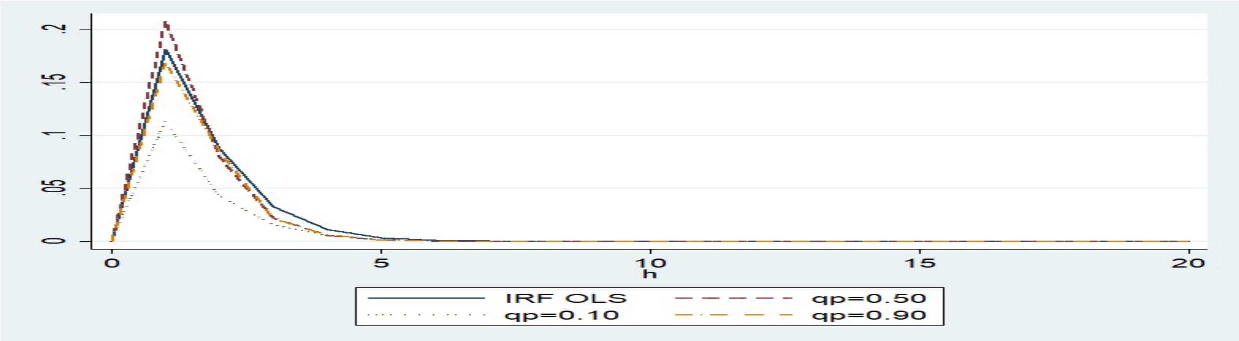
Graphs 3.4.1 to 3.4.4 for Italy: QIRF for different τ_y. Notes: QIRF on output gap and inflation of a standard deviation shock in r_t for $\tau_y \in \{0.10, 0.50, 0.90\}, \tau_\pi = 0.50$ *and* $\tau_r = 0.50$

[Πληκτρολογήστε κείμενο]

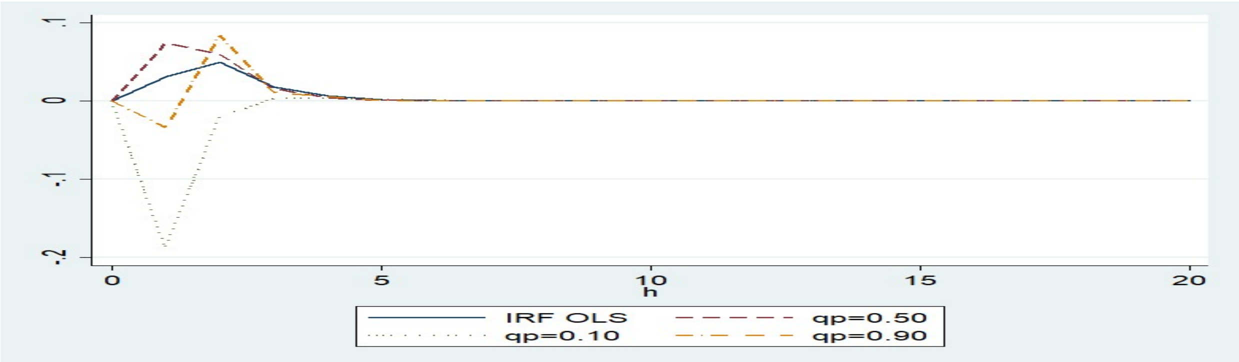
Graphs 3.1.1 to 3.4.4 and 4.1.1 to 4.4.4 plots the QIRF of the r shock on output gap and inflation dynamics for the VAR-OLS model and for indexes $\tau = (\tau_y, \tau_\pi = 0.5, \tau_r = 0.5)$ with $\tau_y = 0.10, 0.50, 0.90$ for the former and $\tau = (\tau_y = 0.5, \tau_\pi, \tau_r = 0.5)$ with $\tau_\pi = 0.10, 0.50, 0.90$. The potential response of y and π if the VARQ model is evaluated at fixed τ for all $h = 1,2, \dots ,20$ is represented by the quantile curves. For example in Graphs 3.1.1 to 3.4.4 there is a case that $\tau_y = 0.10$ corresponds to the simulation of what would be the response of output and inflation to a change only in the interest rate if output response were to remain at the bottom 10% conditional quantile. This could correspond to an extreme event like an unusual depression as it is known that persistent low quantiles could be related with such kind of events. Furthermore, the case with the $\tau_y = 0.90$ correspond to a case of output response always is in the upper 10% quantile, which is connected with an extraordinary growth compared to the estimation samples in each case in general. Moreover, in Graphs 4.1.1 to 4.4.4 we analyze the case of persistent conditional high ($\tau_\pi = 0.90$) or low($\tau_\pi = 0.10$) inflation for each country respectively.

QIRF (for Finland)

Output gap (Graph 4.1.1)

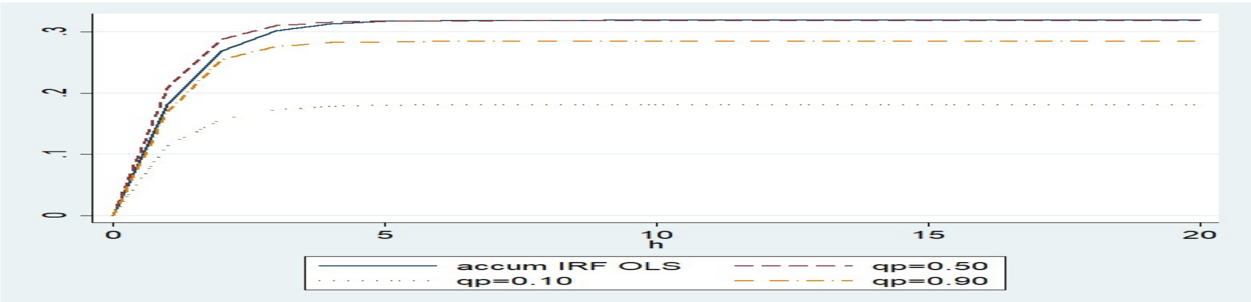


Inflation (Graph 4.1.2)

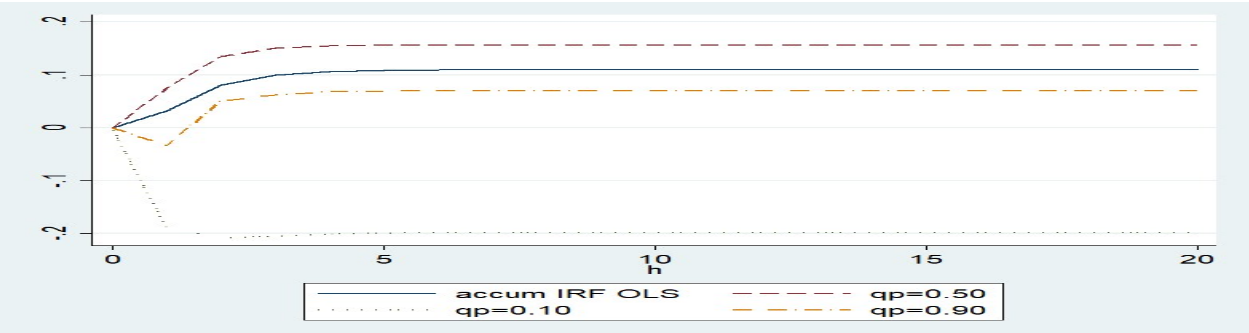


Accumulated QIRF (for Finland)

Output gap (Graph 4.1.3)



Inflation (Graph 4.1.4)



Graphs 4.1.1 to 4.1.4 for Finland: QIRF for different τ_p. Notes: QIRF on output gap and inflation of a standard deviation shock in r_t for $\tau_p \in \{0.10, 0.50, 0.90\}, \tau_y = 0.50$ and $\tau_r = 0.50$

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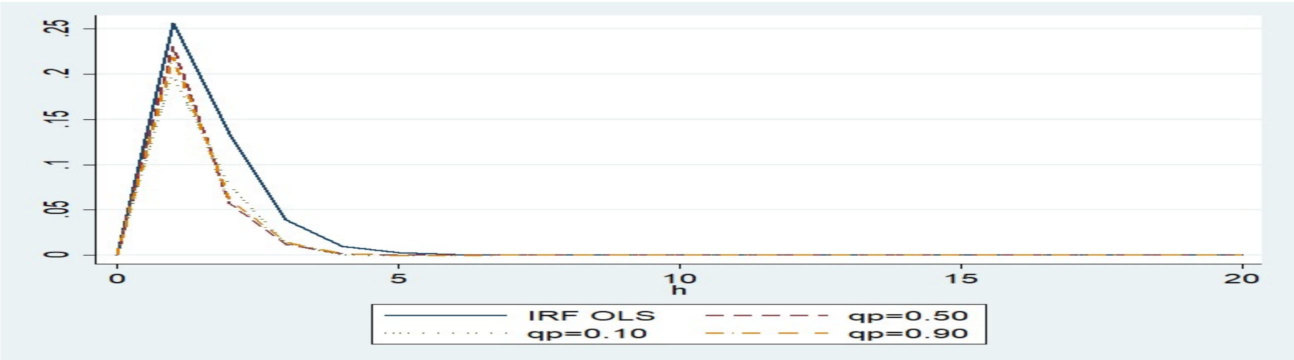
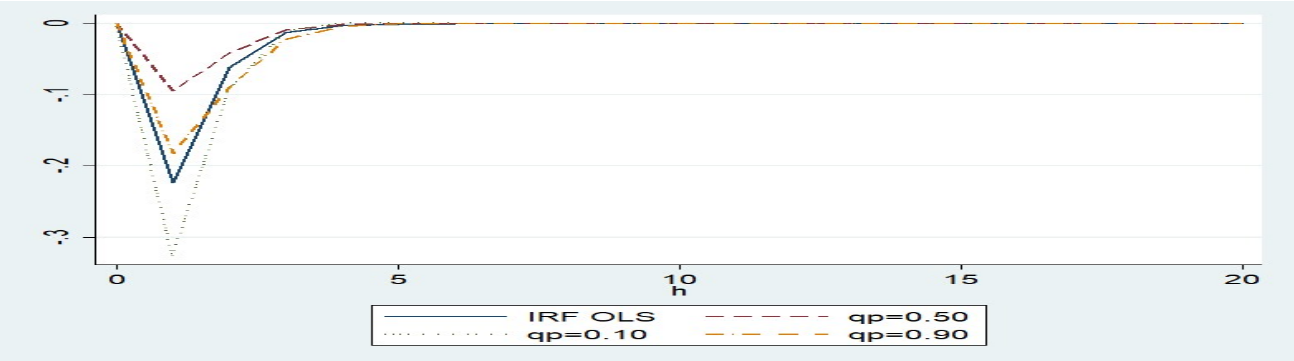
QIRF (for Germany)

Output gap (Graph 4.2.1)

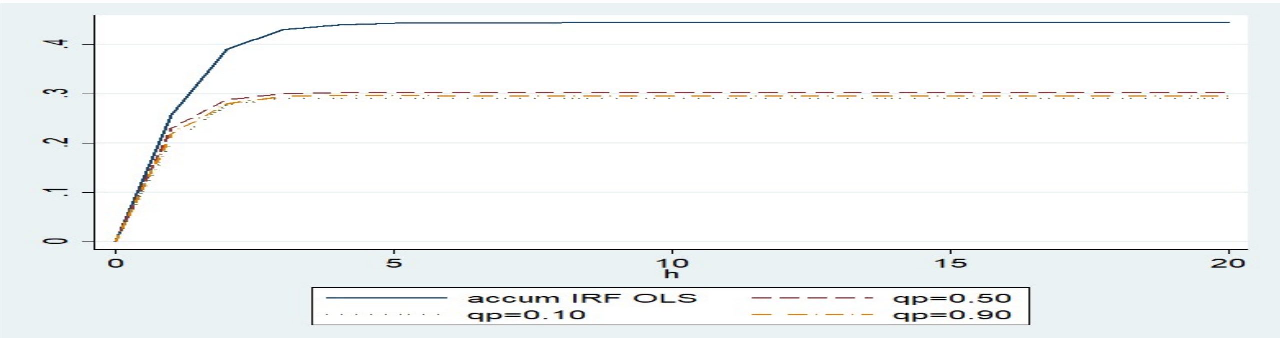


Inflation (Graph 4.2.2)

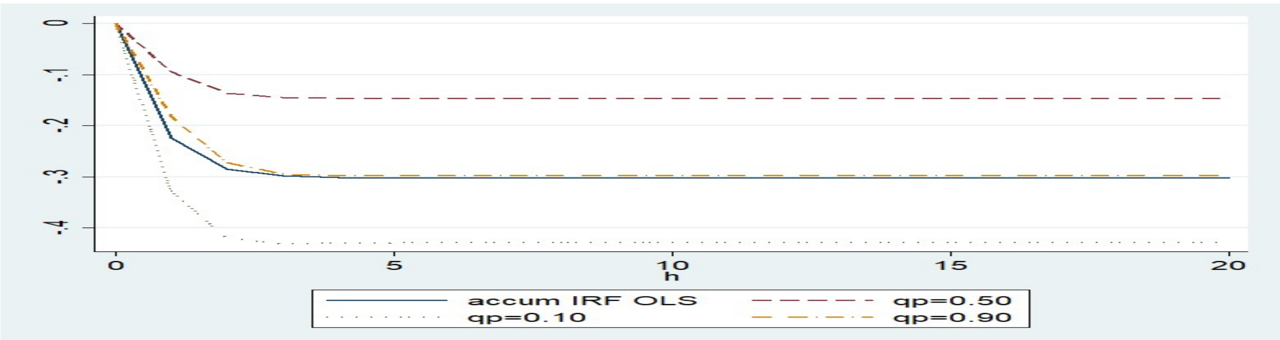


Accumulated QIRF (for Germany)

Output gap (Graph 4.2.3)



Inflation (Graph 4.2.4)

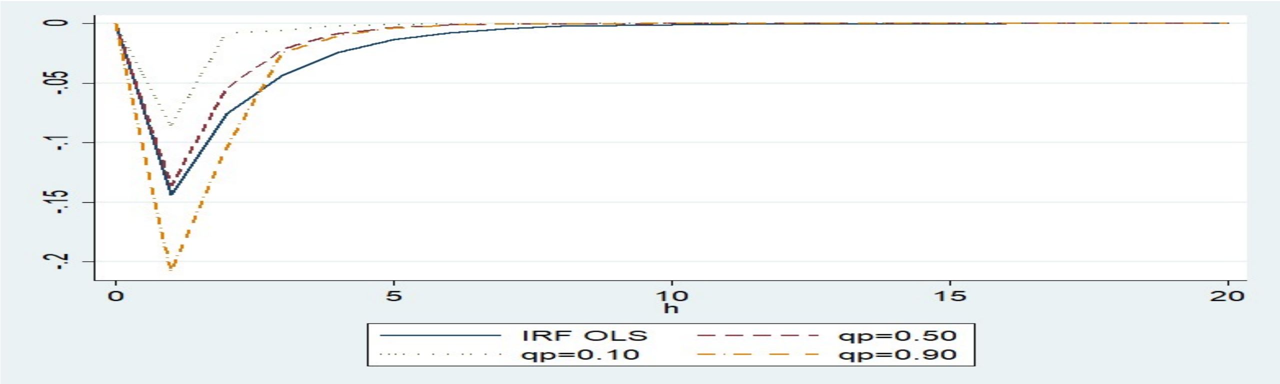


Graphs 4.2.1 to 4.2.4 for Germany: QIRF for different τ_p. Notes: QIRF on output gap and inflation of a standard deviation shock in r_t for $\tau_p \in \{0.10, 0.50, 0.90\}, \tau_y = 0.50\ and\ \tau_r = 0.50$

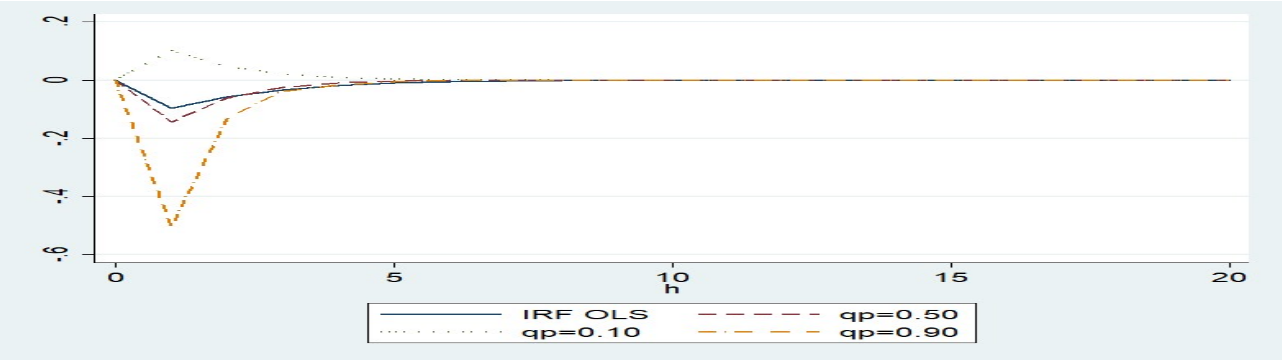
[Πληκτρολογήστε κείμενο]

QIRF (for Greece)

Output gap (Graph 4.3.1)

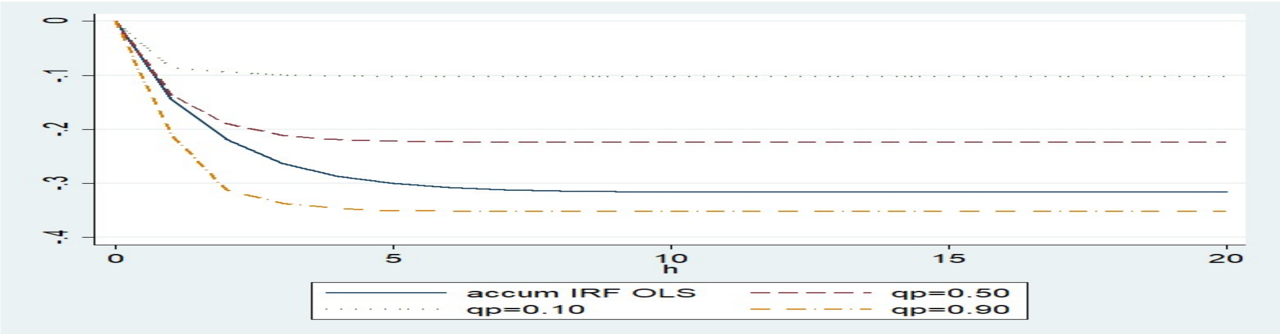


Inflation (Graph 4.3.2)

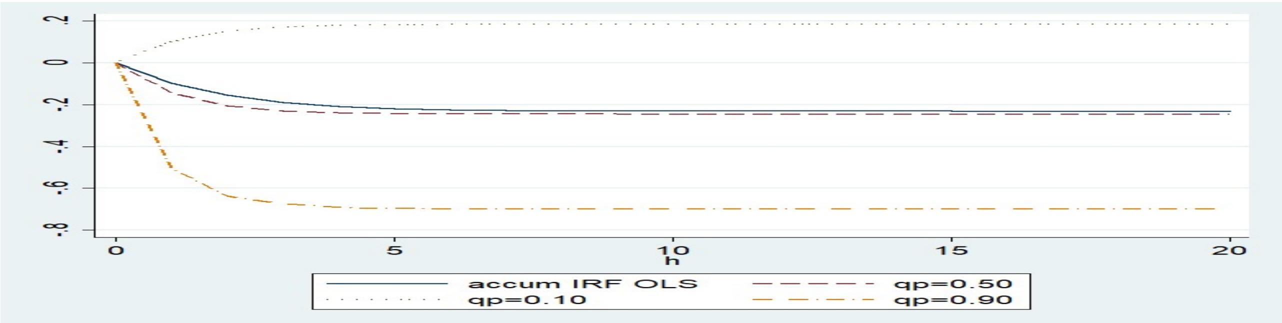


Accumulated QIRF (for Greece)

Output gap (Graph 4.3.3)



Inflation (Graph 4.3.4)

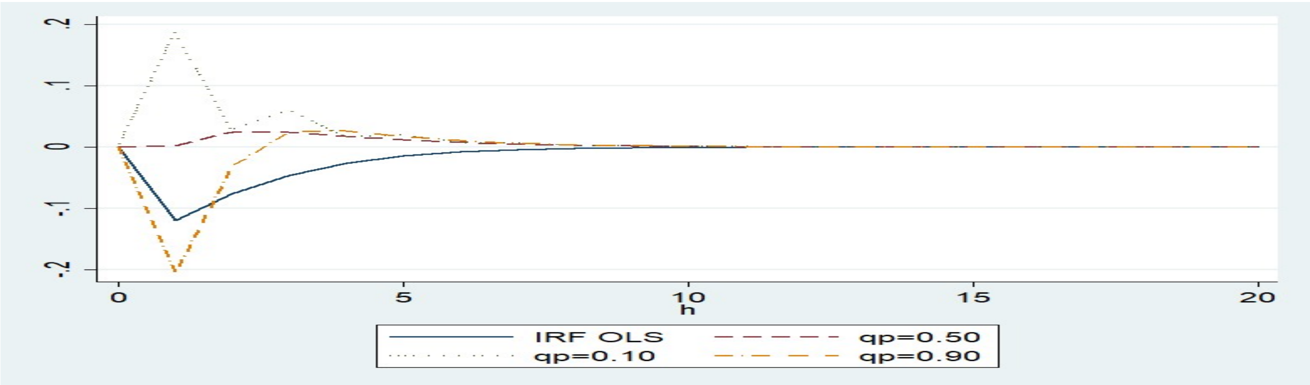


Graphs 4.3.1 to 4.3.4 for Greece: QIRF for different τ_p. Notes: QIRF on output gap and inflation of a standard deviation shock in r_t for $\tau_p \in \{0.10, 0.50, 0.90\}, \tau_y = 0.50$ *and* $\tau_r = 0.50$

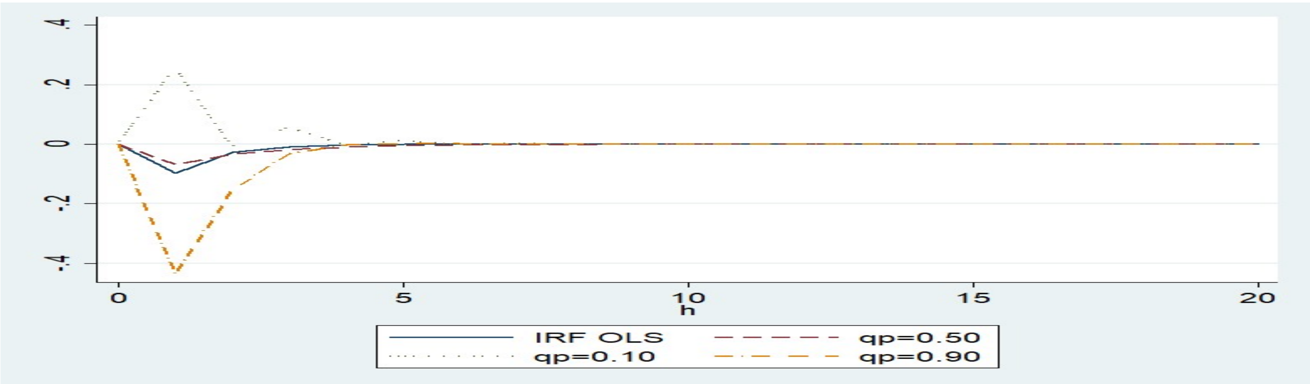
[Πληκτρολογήστε κείμενο]

QIRF (for Italy)

Output gap (Graph 4.4.1)

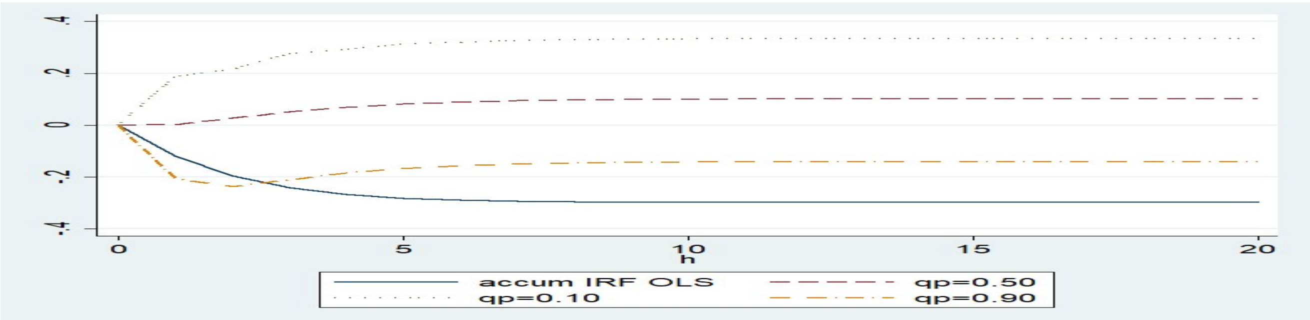


Inflation (Graph 4.4.2)

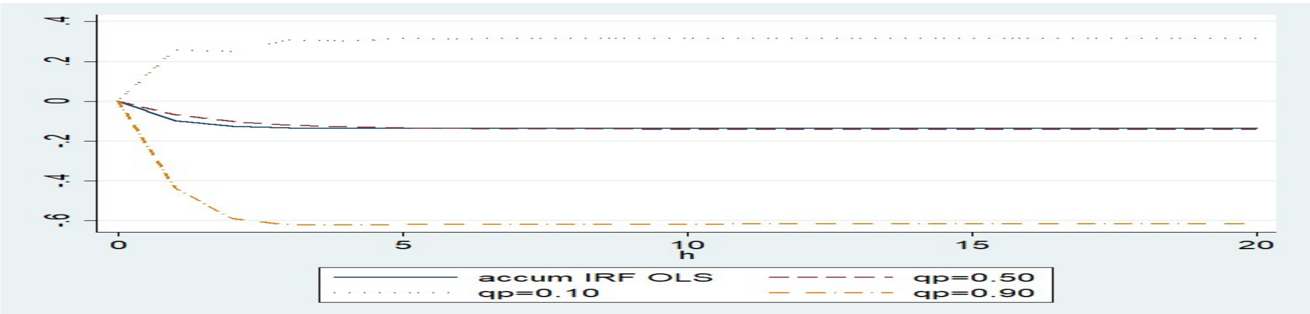


Accumulated QIRF (for Italy)

Output gap (Graph 4.4.3)



Inflation (Graph 4.4.4)



Graphs 4.4.1 to 4.4.4 for Italy: QIRF for different τ_p. Notes: QIRF on output gap and inflation of a standard deviation shock in r_t for $\tau_p \in \{0.10, 0.50, 0.90\}, \tau_y = 0.50$ *and* $\tau_r = 0.50$

[Πληκτρολογήστε κείμενο]

As far as Finland is concerned, for output gap both responses are very similar and positive, but we cannot say the same for inflation.

The $\tau_y = 0.90$ output quantile path effects for both output and inflation although they start having positive curse, after the 5th period they come close to zero after that, leaving an in general positive but low and stable accumulated result. This determines that a dynamic path of extraordinary growth would not be affected by changes in the interest rate except from the short run period of this growth.

The $\tau_y = 0.10$ output quantile path follows a positive and persistent effect on both output and on inflation, especially on output gap where the cumulative effect is by far the largest from all the countries in our analysis. So if Finland's economy were to stay in a state of permanent recession, as given by persistent realizations in the lower 10th conditional quantile in output gap, increasing the interest rate by 1 standard deviation would increase the output by about 1,4 standard deviation in the long run as compared to the value if the interest rate would not be changed.

When computing the same graphs for different fixed values of τ_p we observe considerable differences between the cases for high and low inflation quantile paths. Although that in the case for $\tau_p = 0.90$ we have the same behavior of stability after the 5th period, we can see that there is a negative starting effect followed by a positive effect and then we have the stability, as far as inflation is concerned with a positive accumulated resold. On the other hand, as far as output gap is concerned, we can see a big positive starting effect in the short run that stabilizes thereafter, with positive accumulated results. When we use the $\tau_p = 0.10$ we have a big negative starting spike on inflation that leads on a negative accumulated result of about -0,2.

Continuing with Germany, for output gap both responses are very similar and positive, and for inflation the responses are very similar but negative.

The $\tau_y = 0.90$ output quantile path effects are positive for output gap but negative for inflation. The $\tau_y = 0.10$ output quantile path follows a persistence effect which is positive for the output gap and negative for the inflation as shown in the accumulated results.

When computing the same graphs for different fixed values of τ_p we observe differences between the cases for output gap and inflation quantile paths, with the results of the low quantiles $\tau_p = 0.10$ to be bigger than the effects of the high quantiles $\tau_p = 0.90$, as far as inflation is concerned. In the case for $\tau_p = 0.90$ we have negative results leading to negative accumulated effects on inflation in contrast with the positive effects on output gap with the same quantile path. The same principal applies to the results in the case for$\tau_p = 0.10$; with a negative effect on inflation and positive ones on the output gap.

As far as Greece in concerned, we can see some interesting results. For output gap the responses have some fluctuation as well as in the inflation, leading to negative results. For $\tau_y = 0.90$ output quantile path effects are negative in general despite the fluctuation on a positive curse on both output gap and inflation. These rises are too small and are also depicted on the accumulated results. The $\tau_y = 0.10$ output quantile path although negative is smoother for the output gap and has no fluctuations for inflation.

When computing the same graphs for different fixed values of τ_p we observe considerable differences between the cases for high and low inflation quantile paths, for inflation while the effects on the output gap are negative. In the case for $\tau_p = 0.90$ we have negative results for the output gap a s well as inflation and the accumulated results are very negative. On the other hand, for $\tau_p = 0.10$ output gap we can see negative but smaller results compared to the $\tau_p = 0.90$ for output gap. But the results for inflation are positive, although they have only $\frac{1}{3}$ of the power of the negative effects on inflation compared to the case of $\tau_p = 0.90$.

Finally for Italy we have the most impressive results.

The $\tau_y = 0.90$ output quantile path effects are close to zero and for the output gap and slightly negative for the inflation resulting to overall slightly negative accumulated effects. But the $\tau_y = 0.10$ output quantile path follows a very negative and persistent path for output gap leading to big negative effects and for inflation despite some fluctuations the negative effects prevail that of the positive ones, as shown on the accumulated results.

In the case for $\tau_p = 0.90$ we can see overall negative effects on both inflation and output gap. On the other hand, in the case for $\tau_p = 0.10$, we can see positive effects on both output gap and inflation which result to positive cumulative effects.

The models thus conclude that there is potential asymmetry in the dynamic propagation of shocks. We can see that high and low quantiles associated with more persistence and larger in general effects of a given fiscal shock. These simple linear models that use different quantile paths can be used for the evaluation of extreme events despite the fact that we lack the observations to evaluate correctly a structural change. In general QR models are used to evaluate heterogeneous effects that have derived from unobserved factors. So, omitted variables can result in having different quantile paths. In a case of a country such as Italy, low quantiles combined with a positive shock on the interest rate can have as a result the attraction of foreign capital and can generate a positive impact on output with idle productive capacity. (Montes-Rojas, 2019)

Tables 2.1 to 2.4 show the dynamic stability of all different specifications for the respected countries.

Table 2.1: VAR system stability for Finland

Model	Eiegn1	Eiegn2	Eiegn3
VAR-OLS	0,316	0,173	0,126
VARQ(ty=0,5 , tp=0,1 , tr=0,5	0,013	0,300	0,300
VARQ(ty=0,1 , tp=0,5 , tr=0,5	0,141	0,429	0,429
VARQ(ty=0,5 , tp=0,5 , tr=0,5	0,110	0,237	0,126
VARQ(ty=0,5 , tp=0,9 , tr=0,5	0,220	0,234	0,140
VARQ(ty=0,9 , tp=0,5 , tr=0,5	0,282	0,365	0,156

Table 2.2: VAR system stability for Germany

Model	Eiegn1	Eiegn2	Eiegn3
VAR-OLS	0,294	0,149	0,149
VARQ(ty=0,5 , tp=0,1 , tr=0,5	0,038	0,213	0,213
VARQ(ty=0,1 , tp=0,5 , tr=0,5	0,026	0,267	0,267
VARQ(ty=0,5 , tp=0,5 , tr=0,5	0,191	0,109	0,191
VARQ(ty=0,5 , tp=0,9 , tr=0,5	0,245	0,128	0,245
VARQ(ty=0,9 , tp=0,5 , tr=0,5	0,215	0,227	0,227

Table 2.3: VAR system stability for Greece

Model	Eiegn1	Eiegn2	Eiegn3
VAR-OLS	0,086	0,181	0,553
VARQ(ty=0,5 , tp=0,1 , tr=0,5	0,151	0,107	0,405
VARQ(ty=0,1 , tp=0,5 , tr=0,5	0,366	0,168	0,222
VARQ(ty=0,5 , tp=0,5 , tr=0,5	0,374	0,020	0,189
VARQ(ty=0,5 , tp=0,9 , tr=0,5	0,356	0,186	0,186
VARQ(ty=0,9 , tp=0,5 , tr=0,5	0,440	0,186	0,379

Table 2.4: VAR system stability for Italy

Model	Eiegn1	Eiegn2	Eiegn3
VAR-OLS	0,048	0,524	0,370
VARQ(ty=0,5 , tp=0,1 , tr=0,5	0,483	0,597	0,462
VARQ(ty=0,1 , tp=0,5 , tr=0,5	0,452	0,634	0,452
VARQ(ty=0,5 , tp=0,5 , tr=0,5	0,404	0,015	0,597
VARQ(ty=0,5 , tp=0,9 , tr=0,5	0,227	0,598	0,227
VARQ(ty=0,9 , tp=0,5 , tr=0,5	0,393	0,393	0,282

All the Eigen-values modules are inside the unit circle for all countries. In none of the countries' cases the system is close to the unit root, as a result we are going to have stationary behavior in the QIRF for all these cases.
Starting our QIRF analysis we can see that mean based OLS and median responses show similar dynamic behavior in both inflation cases as well as output gap cases but with the mean effects to be larger than the median ones in general.

The following tables 3.1 and 3.2 correspond to the summary of the effects on the cases of persistent conditional high($\tau_y = 0.90$) or low($\tau_y = 0.10$) output gap and on the cases of persistent conditional high($\tau_\pi = 0.90$) or low($\tau_\pi = 0.10$) inflation for each country respectively

Table 3.1: summary effects

Response of y	**Finland**	**Germany**	**Greece**	**Italy**
Case of low quantiles $\tau_y = 0.10$				
On output gap	Positive	Positive	Negative-Close to 0	Negative
On inflation	Positive	Negative	Negative	Generally-Negative
Accumulated QIRF				
On output gap	Positive	Positive	Negative-Close to 0	Negative
On inflation	Positive	Negative	Negative	Negative with fluctuations
Case of high quantiles $\tau_y = 0.90$				
On output gap	Positive	Positive	Generally-Negative	Positive -Close to 0
On inflation	Positive	Negative	Generally-Negative	Negative-Close to 0
Accumulated QIRF				
On output gap	Positive	Positive	Negative with fluctuations	Close to 0
On inflation	Positive	Negative	Negative with fluctuations	Negative

Table 3.2: summary effects

Response of π	**Finland**	**Germany**	**Greece**	**Italy**
Case of low quantiles $\tau_{\pi} = 0.10$				
On output gap	Positive	Positive	Negative	Positive with fluctuations
On inflation	Negative	Negative	Positive	Positive with fluctuations
Accumulated QIRF				
On output gap	Positive	Positive	Negative	Positive with fluctuations
On inflation	Negative	Negative	Positive	Positive with fluctuations
Case of high quantiles $\tau_{\pi} = 0.90$				
On output gap	Positive	Positive	Negative	Negative
On inflation	Positive with fluctuations	Negative	Negative	Negative
Accumulated QIRF				
On output gap	Positive	Positive	Negative	Negative
On inflation	Positive with fluctuations-Close to 0	Negative	Negative	Negative

[Πληκτρολογήστε κείμενο]

Epilogue

Though this attempt of construction IRFs using multivariate semi-parametric directional quantiles we can explore various dynamic heterogeneities in the potential effects of a shock into the future performance of series.

As a result we can see that some countries have the same response in a given shock, for example Finland and Germany have the same response on output gap when a shock applied on both low and high quantiles. On the other hand, some countries like Greece and Italy don't follow the same pattern as others. Other counties can have the exact opposite behavior like Greece and Finland, or their own unique response to the shock like Italy which behavior doesn't much any of the previous countries. The reason why these differences in the responses of the different counties occur may lay in the unique characteristics of each county's economic structure.

Although we have used only linear QR models, we can have an evaluation a potential structural break in a country's economy due to an extreme or unusual effect or shock. This is a useful way to forecast future events for which we do not have enough observational data to analyze by drawing our conclusions and future predictions for these events based on extreme past events that correspond to different quantile paths of high or low conditional output and inflation.

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Appendix

Nomenclature

- $\boldsymbol{x}$ (lower bold letters): stands for the vectors.
- n Subscript: denotes the vector dimension, where the notation x_n is used
- $\boldsymbol{X}$ (Upper bold letters): stands for the matrices.
- $n \times p$ Subscript: denotes the matrix dimensions, where the notation $X_{n\times p}$ is used
- T: stands for the transpose operator (x^T)
- X: stands for random variables
- $F_y(y)$: stands for the cumulative distribution function
- Y Subscript: denotes the variables on which the function is computed
- $F(y)$: stands for the shortened notation of the above function and is used when there is no risk of ambiguity
- $Q_Y(\theta)$: stands for the quantile function
- Y Subscript: denotes the variables on which the quantile is computed
- $Q(\theta)$: stands for the shortened notation of the above function and is used when there is no risk of ambiguity
- x_i: stands for the i-th vector element
- $\boldsymbol{x_i}$: stands for the i-th matrix row
- 0: stands for the null vector
- 1: stands for the identity vector
- I: stands for the identity matrix
- n: stands for the sample size
- p: stands for the number of regressors
- θ: stands for the quantile
- k: stands for the number of estimated quantiles
- $\beta(\theta)$: stands for the quantile regression parameter
- $\hat{\beta}(\theta)$: stands for the quantile regression estimate
- $Q_\theta(y|x) = x\beta(\theta) + e$: stands for the simple quantile regression model
- $Q_\theta(y|X) = X\beta(\theta) + e$: stands for the multiple quantile regression model
- $\rho_\theta(y)$: stands for the loss or check function
- $y = \beta_0 + \beta_1 x + e$: stands for the simple regression model

Table 1.1: Details about Finland's coefficients.

y

	Percentiles	Smallest		
1%	-.0515775	-.0515775		
5%	-.0217667	-.0472362		
10%	-.0145565	-.0309543	Obs	81
25%	-.0064537	-.0237283	Sum of Wgt.	81
50%	.0002679		Mean	-.0003178
		Largest	Std. Dev.	.0138842
75%	.0062149	.0246939		
90%	.0135659	.0263092	Variance	.0001928
95%	.0198908	.030159	Skewness	-.8001586
99%	.0348388	.0348388	Kurtosis	5.948132

pi

	Percentiles	Smallest		
1%	-.0111005	-.0111005		
5%	-.0063355	-.0102191		
10%	-.004071	-.0098219	Obs	81
25%	.0015208	-.00637	Sum of Wgt.	81
50%	.0049428		Mean	.004436
		Largest	Std. Dev.	.0063623
75%	.0082467	.0157988		
90%	.0116707	.0165505	Variance	.0000405
95%	.0143533	.019914	Skewness	-.0930134
99%	.0206033	.0206033	Kurtosis	3.303639

r

	Percentiles	Smallest		
1%	-.0122667	-.0122667		
5%	-.0061	-.0073667		
10%	-.0047667	-.0065813	Obs	81
25%	-.0029667	-.0063333	Sum of Wgt.	81
50%	-.0015333		Mean	-.001195
		Largest	Std. Dev.	.0032122
75%	.000627	.0042489		
90%	.0025667	.0049522	Variance	.0000103
95%	.003928	.005	Skewness	-.0683973
99%	.0083333	.0083333	Kurtosis	4.191033

Table 2.1: Stability condition for Finland

```
. varstable,graph

   Eigenvalue stability condition
```

Eigenvalue	Modulus
.320778 + .2762494*i*	.423335
.320778 - .2762494*i*	.423335
-.30646 + .1878548*i*	.359454
-.30646 - .1878548*i*	.359454
.1692244 + .2081755*i*	.26828
.1692244 - .2081755*i*	.26828

```
   All the eigenvalues lie inside the unit circle.
   VAR satisfies stability condition.
```

Graph:5.1: Unit circle for Finland

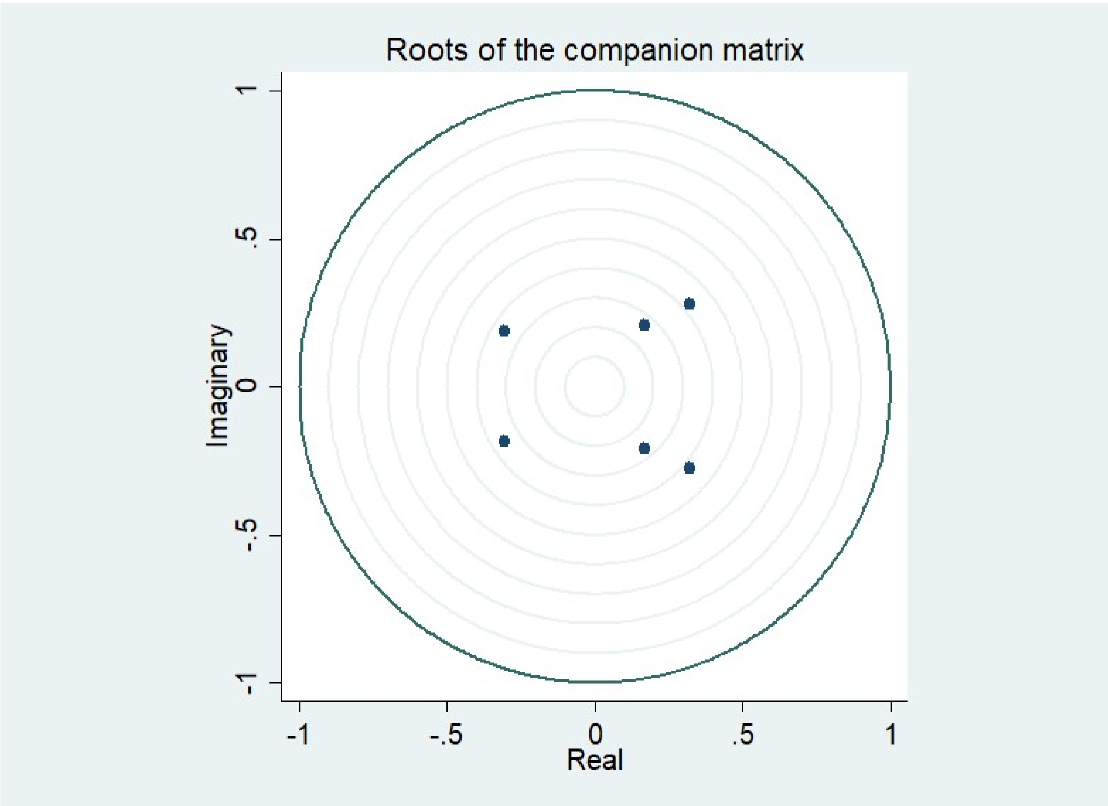


Table 1.2: Details about Germany's coefficients

```
. sum y pi r,detail

                              y
-------------------------------------------------------------
      Percentiles      Smallest
 1%    -.0456853      -.0456853
 5%    -.0103918      -.0166253
10%    -.0070307      -.0146042       Obs                  81
25%    -.0041895       -.013946       Sum of Wgt.          81

50%     .0005216                      Mean          -.0002216
                        Largest       Std. Dev.      .0083667
75%     .0044287       .0126083
90%     .0079487       .0127869       Variance         .00007
95%     .0104214       .0153034       Skewness      -1.807215
99%     .0210145       .0210145       Kurtosis       12.50463

                             pi
-------------------------------------------------------------
      Percentiles      Smallest
 1%    -.0055431      -.0055431
 5%    -.0018644      -.0035584
10%    -.0008996      -.0031703       Obs                  81
25%     .0003642      -.0028311       Sum of Wgt.          81

50%     .0024349                      Mean           .0026471
                        Largest       Std. Dev.      .0031855
75%     .0049182       .0084455
90%     .0069294       .0085118       Variance       .0000101
95%     .0084326       .0089311       Skewness       .0480101
99%     .0094304       .0094304       Kurtosis       2.603384

                              r
-------------------------------------------------------------
      Percentiles      Smallest
 1%    -.0083667      -.0083667
 5%        -.005      -.0076333
10%    -.0041333      -.0052333       Obs                  81
25%    -.0029333          -.005       Sum of Wgt.          81

50%    -.0012333                      Mean          -.0008877
                        Largest       Std. Dev.       .002927
75%     .0013333       .0038667
90%     .0029333          .0046       Variance       8.57e-06
95%     .0038333          .0054       Skewness       .2509386
99%     .0079333       .0079333       Kurtosis       3.289448
```

Table 2.2: Stability condition for Germany

```
.    varstable,graph

  Eigenvalue stability condition
```

Eigenvalue	Modulus
.5663072	.566307
.2027514 + .3859042i	.435925
.2027514 - .3859042i	.435925
-.1350349 + .2444756i	.27929
-.1350349 - .2444756i	.27929
-.2401043	.240104

```
  All the eigenvalues lie inside the unit circle.
  VAR satisfies stability condition.
```

Graph:5.2: Unit circle for Germany

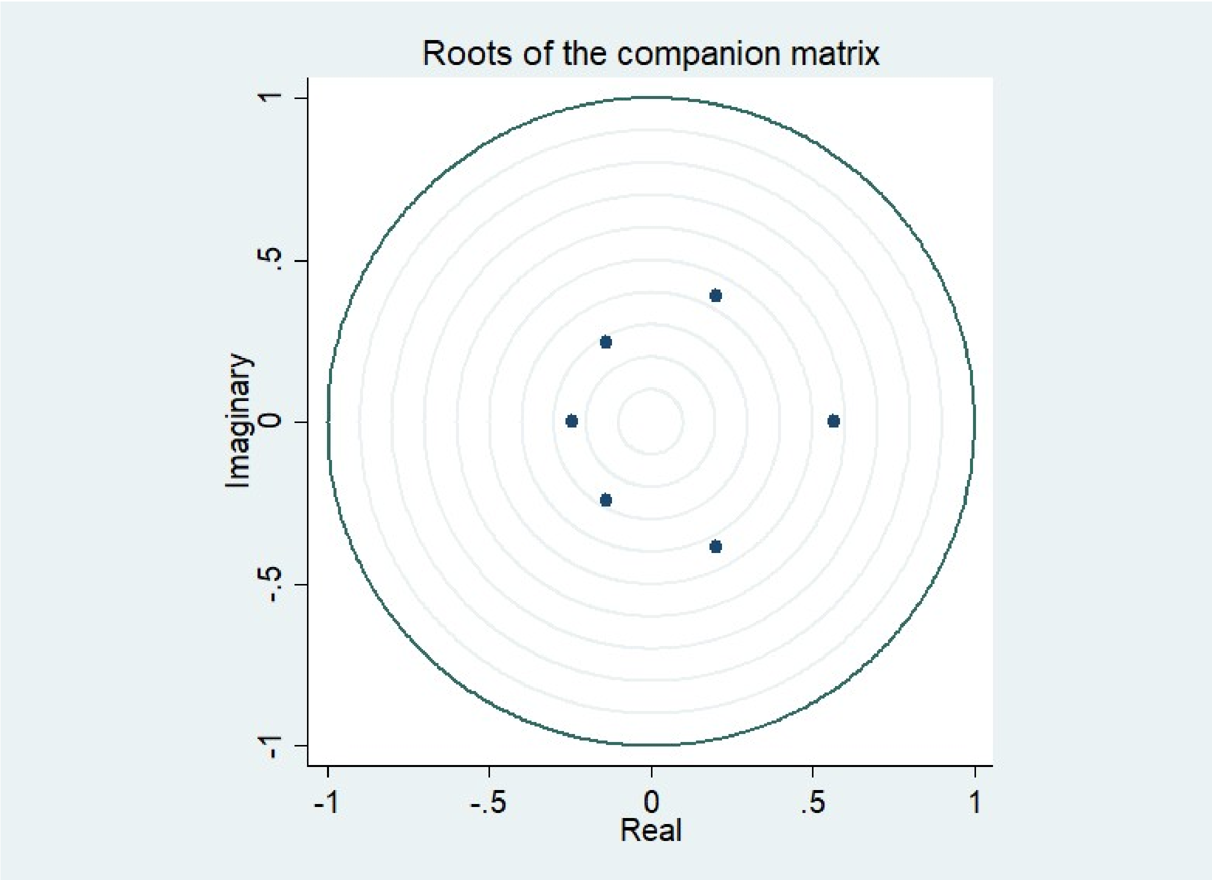


Table 1.3: Details about Greece’s coefficients

```
. sum y pi r,detail

                              y
-------------------------------------------------------------
      Percentiles      Smallest
 1%    -.0332788      -.0332788
 5%    -.0147444      -.0182195
10%    -.0119883      -.0172766       Obs                  80
25%    -.0067256      -.0155572       Sum of Wgt.          80

50%    -.0010648                      Mean           .0000412
                        Largest       Std. Dev.      .0108035
75%     .0049424         .02113
90%     .0125328       .0265368       Variance       .0001167
95%      .018914       .0287479       Skewness       .6478869
99%     .0396173       .0396173       Kurtosis       5.470269

                             pi
-------------------------------------------------------------
      Percentiles      Smallest
 1%    -.0171075      -.0171075
 5%    -.0095074      -.0141957
10%    -.0069241       -.012314       Obs                  80
25%    -.0011583       -.010473       Sum of Wgt.          80

50%     .0067233                      Mean            .006371
                        Largest       Std. Dev.      .0098346
75%     .0125854       .0218087
90%     .0197175       .0219433       Variance       .0000967
95%     .0214501       .0275764       Skewness      -.0686368
99%     .0289579       .0289579       Kurtosis       2.574734

                              r
-------------------------------------------------------------
      Percentiles      Smallest
 1%    -.0752667      -.0752667
 5%    -.0173667         -.0502
10%    -.0142333      -.0254667       Obs                  81
25%        -.004         -.0239       Sum of Wgt.          81

50%       -.0012                      Mean          -.0012255
                        Largest       Std. Dev.      .0155478
75%     .0026667       .0242333
90%     .0111333       .0249333       Variance       .0002417
95%        .0206          .0364       Skewness      -.8418862
99%     .0570667       .0570667       Kurtosis        11.2359
```

Table 2.3: Stability condition for Greece

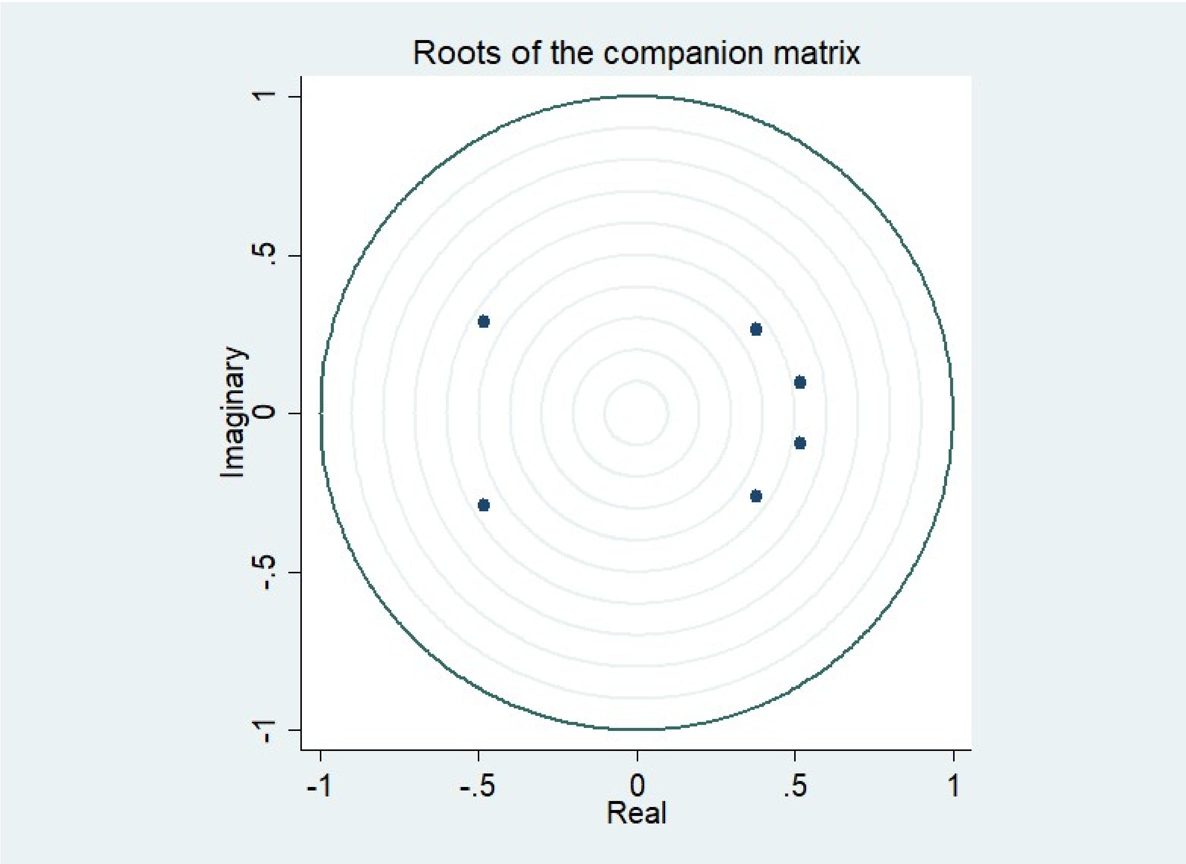
```
.    varstable,graph

  Eigenvalue stability condition
```

Eigenvalue	Modulus
-.4825746 + .2888633*i*	.562424
-.4825746 - .2888633*i*	.562424
.5195036 + .0965613*i*	.528401
.5195036 - .0965613*i*	.528401
.3811427 + .2622768*i*	.462665
.3811427 - .2622768*i*	.462665

```
  All the eigenvalues lie inside the unit circle.
  VAR satisfies stability condition.
```

Graph:5.3: Unit circle for Greece



[Πληκτρολογήστε κείμενο]

Table 1.4: Details about Italy's coefficients

```
. sum y pi r,detail

                              y
-------------------------------------------------------------
      Percentiles      Smallest
 1%    -.0239279      -.0239279
 5%    -.0099329       -.019395
10%    -.0082981      -.0134136       Obs                  80
25%    -.0038804      -.0102729       Sum of Wgt.          80

50%     .0007863                      Mean           .0000747
                        Largest       Std. Dev.      .0066389
75%       .00411       .0102661
90%     .0085958       .0104967       Variance       .0000441
95%     .0101435       .0105701       Skewness       -.808229
99%     .0118604       .0118604       Kurtosis       4.467926

                             pi
-------------------------------------------------------------
      Percentiles      Smallest
 1%    -.0071405      -.0071405
 5%    -.0011357      -.0059421
10%    -.0003599      -.0032746       Obs                  80
25%     .0018586      -.0011687       Sum of Wgt.          80

50%     .0044594                      Mean           .0055292
                        Largest       Std. Dev.      .0054386
75%     .0092206        .014884
90%     .0121586       .0176021       Variance       .0000296
95%     .0148514       .0187308       Skewness       .5913071
99%      .023252        .023252       Kurtosis       3.691316

                              r
-------------------------------------------------------------
      Percentiles      Smallest
 1%    -.0153333      -.0153333
 5%       -.0089         -.0104
10%    -.0067667         -.0091       Obs                  81
25%    -.0034333          -.009       Sum of Wgt.          81

50%       -.0014                      Mean          -.0013062
                        Largest       Std. Dev.      .0042909
75%     .0015667          .0064
90%        .0031       .0068667       Variance       .0000184
95%        .0059          .0078       Skewness      -.1476238
99%        .0112          .0112       Kurtosis       4.167732
```

Table 2.4: Stability condition for Italy

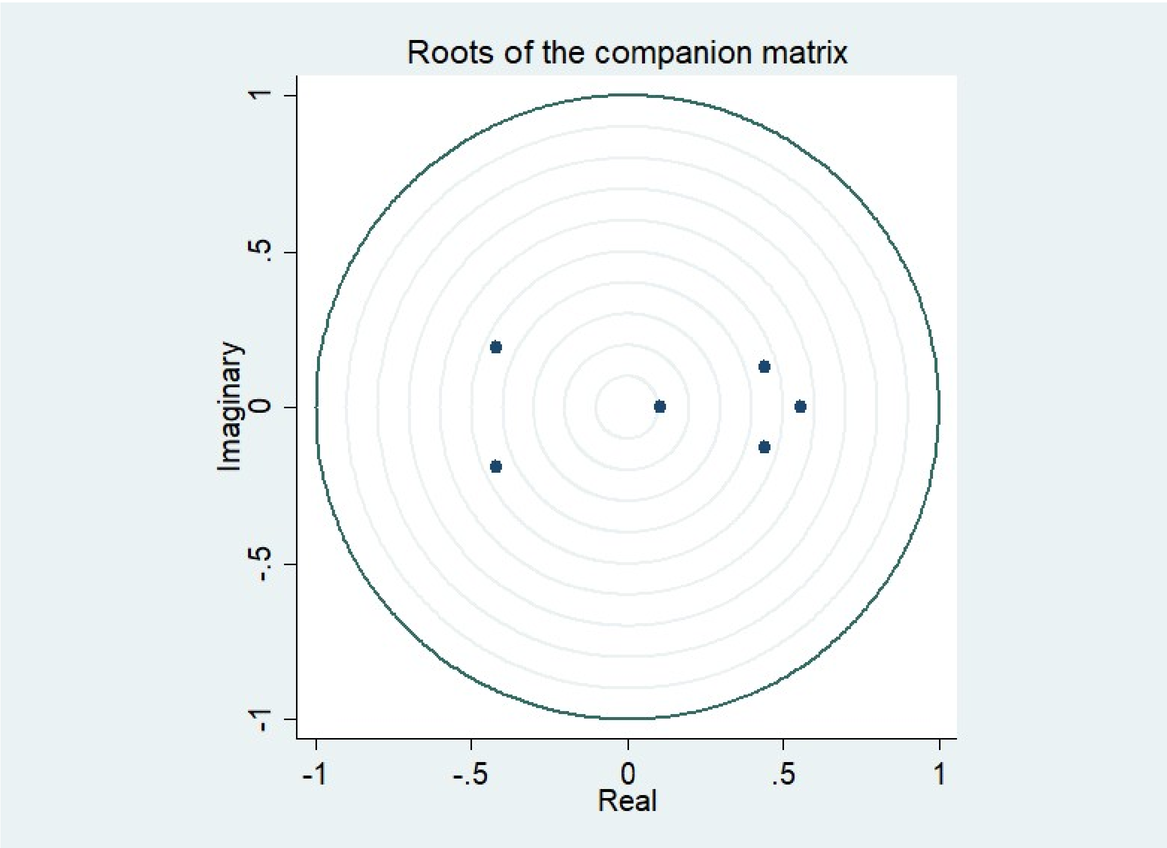
```
.    varstable,graph

  Eigenvalue stability condition
```

Eigenvalue	Modulus
.5577894	.557789
.440582 + .1289842*i*	.459075
.440582 - .1289842*i*	.459075
-.4169255 + .1915409*i*	.458819
-.4169255 - .1915409*i*	.458819
.106957	.106957

```
  All the eigenvalues lie inside the unit circle.
  VAR satisfies stability condition.
```

Graph:5.4: Unit circle for Italy



[Πληκτρολογήστε κείμενο]