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Design of tasks for the understanding of the concept of Linear Span

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Heraklion, November 2017

Στη μαμά μου, Λιάνα,

που με έκανε να αγαπήσω τα Μαθηματικά.

“Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.”

B. Russell

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Design of tasks for the understanding of the concept of Linear Span

Master's Thesis

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Ευχαριστίες

Θα ήθελα να ευχαριστήσω τον επιβλέποντα καθηγητή μου, κ. Χρήστο Κουρουνιώτη, για την βοήθεια του αλλά κυρίως που πίστεψε στα όνειρα μου. Θα ήθελα να ευχαριστήσω ακόμα την κ. Μαρία Λουκάκη που με βοήθησε να καταλάβω γιατί μου αρέσει τόσο πολύ η Άλγεβρα και την κ. Γιαννα Μαμωνα – Downs για την πολύτιμη βοήθεια στην εκπόνηση της εργασίας μου.

Ευχαριστώ μέσα από την καρδιά μου τους γονείς και την αδερφή μου που με στήριξαν και με στηρίζουν όλα αυτά τα χρόνια. Τους φίλους μου που άκουγαν και συζητούσαν τους προβληματισμούς μου ακόμα και αν η εμπειρία τους με τα μαθηματικά είχε τελειώσει στο σχολείο.

Ακόμα, χρωστάω ένα μεγάλο ευχαριστώ σε όλους του φοιτητές που μου έδιναν έμπνευση τα τελευταία δύο χρόνια. Τους φοιτητές που δέχτηκαν να συμμετέχουν στην έρευνα που έγινε στα πλαίσια της εργασίας και μου αφιέρωσαν πολύτιμο χρόνο. Τέλος, ιδιαίτερα ευχαριστώ τους φοιτητές δεθήκαμε λίγο περισσότερο, γίναμε φίλοι, με στήριξαν και μου έδωσαν υπέροχες ιδέες.

Περίληψη

Η έννοια της γραμμικής θήκης είναι μια από τις πρώτες αφηρημένες έννοιες που συναντούν οι φοιτητές σε ένα μάθημα Γραμμικής Άλγεβρας στο Πανεπιστήμιο. Στην παρούσα εργασία μελετώνται αρχικά οι δυσκολίες και οι παρανοήσεις που πιθανών να έχουν οι φοιτητές του τμήματος Μαθηματικών και Εφαρμοσμένων Μαθηματικών του Πανεπιστημίου Κρήτης, σχετικά με την έννοια της γραμμικής θήκης μέσα από τις απαντήσεις τους στην γραπτή εξέταση του μαθήματος "Γεωμετρία και Γραμμική Άλγεβρα". Στη συνέχεια χρησιμοποιώντας το ερευνητικό εργαλείο εννοιακή εικόνα και εννοιακός ορισμός (Tall & Vinner, 1981) σε συνδυασμό με παρατηρήσεις σχετικά με τη μάθηση και την διδασκαλία της Γραμμικής Άλγεβρας, παρουσιάζονται τρεις δραστηριότητες που έχουν σχεδιαστή για να εμπλουτίσουν τις εννοιακές εικόνες των φοιτητών σχετικά με την γραμμική θήκη. Οι δραστηριότητες αυτές θα μπορούσαν να συμπεριληφθούν σε ένα εργαστήριο προβλημάτων ενός εισαγωγικού πανεπιστημιακού μαθήματος στην Γραμμική Άλγεβρα. Κάθε μια από τις δραστηριότητες έχει σχεδιαστεί και/ή επιλεγεί έτσι ώστε να προσφέρει την ευκαιρία σε πιθανούς παράγοντες σύγκρουσης να έρθουν στην επιφάνεια και να αντιμετωπιστούν. Μια πρώτη αξιολόγηση των δραστηριοτήτων δείχνει ότι οι φοιτητές φαίνεται να τις δέχονται θετικά και ότι οι δραστηριότητες επιτυγχάνουν την αντιμετώπιση συγκεκριμένων παραγόντων σύγκρουσης.

Λέξεις κλειδιά: *Διδασκαλία και μάθηση Γραμμικής Άλγεβρας, Διδακτικές πρακτικές στο Πανεπιστήμιο, Γραμμική θήκη, Δραστηριότητες.*

Abstract

The concept of linear span is one of the first abstract notions that students encounter in a course on Linear Algebra at the University Level. This work focuses initially on the study of the difficulties and misconceptions which students at the Department of Mathematics and Applied Mathematics of the university of Crete might have regarding the notion of linear span through their answers in the written exams of the course "Geometry and Linear Algebra". Next, using the theoretical structure of concept image and concept definition (Tall & Vinner, 1981) along with observations about teaching and learning Linear Algebra, we present three tasks designed to enrich students' concept image regarding linear span. These tasks could be included in a problem workshop of an introductory university course on Linear Algebra. Each task is carefully created and/or selected so as to foster the ground for potential conflict factors to arise and be confronted. A preliminary evaluation shows that the tasks are well received by students and succeed in addressing certain conflicting factors.

Keywords: Teaching and learning of linear algebra; Teachers' and students' practices at university level; Linear span; Tasks.

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CHAPTER 1

Introduction

In recent years, the teaching of Linear Algebra at the undergraduate level attracts the interest of researchers in Mathematics Education. Linear Algebra is a subject with many applications in Mathematics and other sciences, but its teaching and learning proves to be demanding both for lecturers and students (Dorier et al., 2000). Research in teaching and learning Linear Algebra at the university level initially focused on identifying the reasons behind students' difficulties (see for example Dorier et al., 2000, Sierpinska, 2000) and key characteristics of the nature of Linear Algebra linked to its teaching (e.g. Hillel, 2000). Moreover, the role of Analytic Geometry on the teaching and learning of Linear Algebra was also an aspect that researcher debated on (Dorier et al, 1999; Harel, 2000; Gueudet-Chartier, 2004; Watson et al, 2003). More recent developments on the matter concern experimental studies and alternative approaches of teaching Linear Algebra (e.g. Stewart & Thomas, 2009; Warwo et al, 2013).

This work focuses on the teaching and learning of the notion of linear span and aims to contribute to the existing literature about teaching and learning Linear Algebra in two ways. Firstly, it identifies a misconception regarding the notions of linear combination and linear dependence in relation to the notion of linear span. Secondly, it aims to develop a set of tasks for an introductory course in Linear Algebra to enrich the understanding of the concept of linear span.

The choice of linear span as the focus of this study was not made at random. The concept of linear span is one of the fundamental notions in Linear Algebra, based solely on the definition of a vector space. In the literature, it has been mainly studied as part of larger inquiries regarding the most important concepts of Linear Algebra. We believe that there is a gap between studying the teaching and learning of the essential concepts of Linear Algebra and focusing on a particular concept as the center of attention. Moreover, other concepts such as linear combination, linear dependence and independence have been studied more extensively than linear span. These reasons support our decision to study how to improve the understanding of this particular notion.

This work addresses not only researchers in Mathematics Education but also university teachers of mathematics who are interested in creating tasks for students. We approach the subject from the perspective of a reflexive practitioner and our goal is to design tasks inspired by the needs of our students. We make observations about students' difficulties and misconceptions around the concept of linear span and the learning of Linear Algebra based on the way the subject is taught in the Department at the moment. Taking advantage of research in Mathematics Education is going to help in linking these observations to the state of the art. Finally, the theoretical constructs used in the study are presented and discussed in a way we believe educators can relate to, and perhaps find it inspiring for future use, adaptation or improvement.

In Chapter 2 we present the theoretical background of the study. We review the basic ideas of task design and present the main theoretical construct used in the study. We also review the literature related to the learning and teaching of Linear Algebra.

In Chapter 3 we present a preliminary study based on the written answers given by students of the "Geometry and Linear Algebra" course in response to a question in the final examination for the course. In the study we identify a common misconception of the students that will be used later, in the design of the tasks.

Chapters 4 and 5 are concerned with task design. Chapter 4 is dedicated to presenting the initial task design, including the framework, design principles and presentation of the tasks. Chapter 5 focuses on the initial evaluation of the tasks through interviews with seven students. In this chapter we analyze the students' reactions to each task and take note of some interesting findings.

CHAPTER 2

Theoretical Perspectives& Literature Review

This chapter is an overview of the theoretical background used in this work in order to develop the tasks. Firstly, we give some general information about task design. Secondly, we make a detailed report of the theoretical notions most used in this text. The theoretical notions that we use are believed to be easy to handle for Mathematicians that are not experienced in the research of Mathematics Education. The chapter ends with a literature review on the teaching and learning of Linear Algebra.

2.1 Task Design

Tasks play a very important role in teaching and learning mathematics. By its nature, mathematics includes the study of (unsolved) problems to produce new results by bringing together well established and new ideas. In simple terms, tasks can be viewed as the means by which students may explore the world of mathematics, engage into the activity of solving problems that are new to them, and acquire knowledge by making connections with prior knowledge. Through tasks students encounter concepts, ideas and/or common strategies and develop mathematical thinking and modes of inquiry (in Margolinas, 2013: 10).

Unlike what a mathematician might think, the words “task” and “activity” are often used to describe different things. To avoid confusing terminology, we use the term “activity”, in line with Christiansen & Walther (1986) and Mason & Johnston-Wilder (2004), as the interaction between the teacher, the student and the content. The term “task”, on the other hand, denotes the “devices for initiating activity” (Mason & Johnston-Wilder, 2004: 238). Therefore, task design can be seen as the process of developing tools to be used to generate activity. This process is usually prolonged and includes multiple cycles of design and evaluation.

At a first glance, each research informed task design is unique in many ways. For example, the researchers may choose or formulate different theories of teaching and learning to frame their work which will be reflected in the tasks. Moreover, the researchers may make different choices about the focus of the tasks, the time dedicated to the project or the use of tools. Aside from some differences, design typically shares basic characteristics among the different perspectives. Plomp (2009) identifies three phases in which designs agree, *preliminary research* (context analysis, literature review, development of a framework), *prototyping phase* (includes iterative stages of research and formative evaluation) and *assessment phase* (summative evaluation, recommendations for improvements). Nieeven et al (2006) included systematic reflection and documentation, as a last common stage.

One of the first concerns of the researchers during task design is to develop the framework and the principles. The term *theoretical framework* refers to the theoretical base of the task design, or a study in general. The theoretical framework consists of the notions and the theory that is used to support the task design; it presents the definitions of these notions, the reasoning and connection between them along with references to the relevant literature. In other words, it is the frame in which the design is developed. While the theoretical framework might illustrate the general ideas about the design, the *principles* are the “heuristic guidelines” (McKenney et al, 2006: 73) that include desired characteristics, based on the framework, and can be applied to the development of the tasks. They depict the most important characteristics of the tasks based on the theoretical framework or other important factors. The choice of framework and principles depends, among others, on institutional aspects, learning environment and researchers’ perspectives (Kieran et al, 2015). Moreover, task design is frequently part of a larger project (i.e. Curriculum development) in which case the theoretical framework and principles of the design may be part of a larger framework. Kieran et al (2015) also remark that, depending on the complexity of the factors involved, the design may involve a network or synthesis of theoretical frameworks and principles.

To determine the appropriate tasks one should take into account not only the theoretical framework and principles but also the priorities of the designers in respect to curricular aims or goals in mathematics (Kieran et al, 2015). Therefore, it is important that task design includes multiple cycles through which the developers choose, evaluate and improve their design. The type of tasks the designers choose depends heavily on the particular goals. For example, Fujii (2015) refers to four types of tasks typically used in Japanese Lesson Study (as identified by Doig et al (2011)). These four types are tasks that:

- Directly address a concept
- Develop mathematical processes
- Are chosen based on rigorous examination of scope and sequence
- Address a common misconception (Fujii, 2015: 279)

The above categorization of the tasks focuses on the main purpose of each task. The purpose of the task partially reveals the intentions of the designers and task users. Another characteristic of the tasks which should be taken into account is the amount of information or directions that are given to students through the tasks; this intention is usually described through the term *scaffolding*. In simple words, scaffolding is the process by which a tutor can support a student to solve a problem without rushing a correct answer. Wood, Bruner & Ross (1976) introduced the term in relation to the Vygotskian notion of the zone of proximal development. They identified six stages in the process of scaffolding: *recruitment*, *deduction in the degrees of freedom*, *direction maintenance*, *marking critical features*, *frustration control* and *demonstration*.

In relation to the role of scaffolding in the development of tasks, Burkhardt & Swan (2013) give a different classification of tasks based on types of performance:

> "Mathematical skills and practices can be taught and/or assessed partly in isolation, partly under scaffolded conditions, and partly when students face substantial problems without scaffolded support. We call tasks that assess these three different types of performance *novice, apprentice*, and *expert* tasks respectively" (p.434)

Both classifications are quite general and easy to understand; therefore we believe that they can be useful when making decisions upon the development of the tasks. In particular, when designing sequences of tasks, it is important for the resulting sequence to involve different types of tasks and maintain a balance across and within the tasks (Burkhardt & Swan, 2013; Ruthven, 2015).

Furthermore, we draw attention upon Burkhardt & Swan's (2013) observation that task difficulty is a parameter often ignored in task classification and design. Moreover, they identify *complexity, unfamiliarity, technical demand* and *student autonomy* as four notable factors that affect task difficulty. In our experience, these factors, very often, have an impact on the resulting activity.

In conclusion, task design is usually a complex procedure which includes the use of theoretical constructs and cycles of research to plan, develop and organize tasks. Regardless of the different theoretical perspectives the main purpose of task design is to aid teaching and learning mathematics.

2.2 Concept Image – Concept Definition

> *"…the axioms are no longer the "self-evident" truths of the Greeks, but concept definitions which are set-theoretically formulated abstractions."*
>
> *(Tall 1988b: 5)*

To illustrate the difference between the nature of mathematics and the nature of mathematical thinking, one can think of a mathematician presenting his/her findings. During the presentation, the truth of mathematics is displayed in a logical order based on abstract concepts of an axiomatic theory. On the other hand, the journey of the mathematician to discover this truth was far more complex. The mathematician worked months, or even years, moving back and forth, studying the roots of the problem, searching for links and gaps between his/her thoughts and testing hypotheses. In this process the mathematician was not always working in a logical order nor thinking about the mathematical entities as abstractions.

Tall & Vinner (1981) presented a cognitive model of mathematical thinking, highlighting the difference between a mathematical concept formally defined and the processes through which it is conceived, known as concept definition and concept image. Their theoretical construct became a paradigm for future cognitive research and grew to become one of the best known theories in mathematics education.

The notions of concept image and concept definition were introduced for the first time by Vinner & Herschowitz (1980) to analyze geometrical thinking of students. A year later, Tall & Vinner (1981) published a paper generalizing the theory beyond geometry and introduced the terminology as known today. We note that Tall and Vinner have different perceptions about the notions concept image and concept definition (Tall, 2003). In our work we are focusing more on the construct as it is presented in Tall &Vinner (1981) making comparisons where relevant.

According to Tall & Vinner (1981) *concept image* is “the total cognitive structure that is associated with the concept” (p. 152). For each individual a concept image includes all the mental pictures (graphs, symbols, formulas etc) generated about the concept, associated properties and processes. The concept image is unique for each student, “a result of his or her experience with examples and nonexamples of the concept” (Vinner and Dreyfus, 1989: 356) and is changing over time when the student meets new stimuli.

For example, in Linear Algebra students encounter the concept of matrix. At first, a matrix is typically introduced to the students as an algebraic object that is used to solve linear equations. At this point, a student’s concept image is likely to contain mental pictures of the form $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$(symbolic form) or the idea that the elements of the matrix represent the coefficients of a system of linear equations, properties of a matrix such as the operations of matrix addition, scalar and matrix multiplication and the process of Gaussian elimination. This imagery is not created spontaneously, it requires time and it is developed through different experiences and stimuli. Later, students encounter again matrices this time as elements of a vector space. This experience requires from the individual to enrich part of the concept image. For instance, up until that point a part of a student’s concept image is the ability to create linear combinations from the lines or the columns of a matrix. This image implies a particular connection between the concept of a matrix and the concept of linear combination. Encountering matrices as elements of a vector space, the student will come up against the idea of linear combinations between matrices as a whole. This requires a shift on this part of his/her concept image to fit the new information. Other parts of the concept image, such as Gaussian elimination, may remain undisturbed.

As a result, different incentives may cause different parts of the concept image to be recalled at a specific time; Tall & Vinner (1981) named the activated parts of the concept image *the evoked concept image*. Moreover, the different parts of the concept image are not always coherent. As mentioned above, part of the concept image may be developed whereas others remain the same for the time being. Therefore, it is possible for different parts of the concept image to contain conflicting aspects. For instance, in Matrix Calculus students encounter for the first time a groundbreaking idea of Algebra, the idea of non-commutative operations. Until that point, students' experience in Algebra and real numbers may suggest that commutativity is inseparable from the concept of multiplication. Contrary to their prior knowledge and experience matrix multiplication is not commutative. Students are introduced to this seemingly strange fact through several examples of matrix multiplication and are expected to enrich their images of multiplication. Although in conflict with their prior knowledge, the new information does not cancel the previous one about commutativity, instead both ideas are "stored" in different parts of one's concept image which may be triggered by different stimuli. These conflicting aspects are called *potential conflict factors* (Tall &Vinner, 1981) and they are not evident to the individual until a stimulus causes the conflicting images to be evoked simultaneously and create confusion, in which case they are referred to as *conflict factors*.

The term *concept definition* is referring to "the form of words used to specify that concept" (Tall & Vinner, 1981: 152). The concept definition might be a reflection of an evoked concept image associated with the definition or a rote memorization of a given definition with little or no meaning to the student. Tall & Vinner (1981) stress the difference between the *personal concept definition* and the *formal definition* of a concept. The latter is the definition accepted by the mathematical community as a whole, whereas the personal concept definition is constructed by the individual. The personal concept definition might contain aspects not included in the formal definition and/or ignore others. Finally, the (personal) concept definition creates its own concept image, which is part of the concept image as a whole, called *concept definition image*.

It is worth mentioning that whereas Tall & Vinner (1981) and Tall (1988a) directly mention the difference between personal and formal concept definition, in Vinner & Hershkowitz (1980) and Vinner (1983) the distinction is not always clear. In addition, Tall (2003) states:

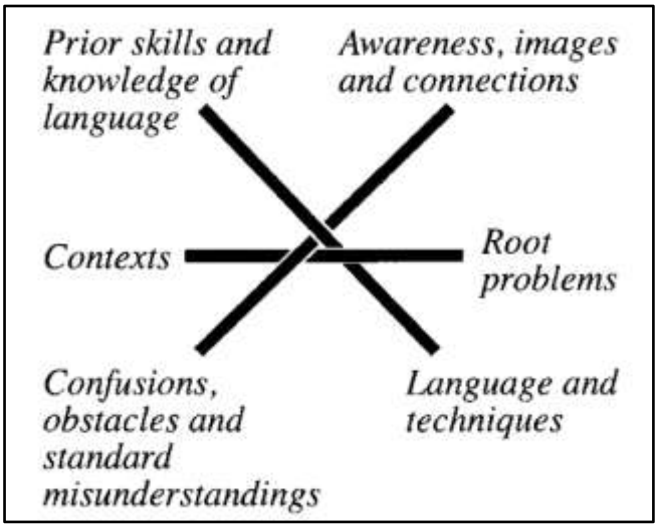
> *"Shlomo has always written about 'concept image' and 'concept definition' as being 'two distinct cells' which enables him to make subtle analyses of different ways of employing the two distinct ideas. As the concept definition is a form of words that can be written or spoken, I regard this as part and parcel of the total concept image in the mind/brain"*

This observation might explain the need for Tall to distinguish the formal definition and personal concept definition. In this case, only the personal concept definition can be linked with the concept image in the individual's mind. In this thesis, we make use of the distinction to help us design tasks that offer the opportunity to create coherent concept images including links between the formal definition of the concept of linear span. Furthermore, the distinction may allow more detailed analysis between students' personal definitions and the formal concept definition.

Upon studying Linear Algebra for the first time, students encounter many new formally defined concepts, i.e. subspace, linear combination, linear dependence, linear span etc. All the basic concepts in Linear Algebra are directly interlinked and create a net of ideas crucial for understanding both the need to study Linear Algebra and the structure of the theory. Tall & Vinner (1981) argue that potential conflict factors can be an obstacle in understanding the formal theory, especially the ones that are in contrast with the formal concept definition.

After its creation in the 1980s, the construct of concept image – concept definition became one of the most popular cognitive theories in Mathematics Education. The theory is primarily linked with cognitive studies in higher levels of education. Tall & Vinner (1981) focus on limits and continuity, since then there is a plethora of articles that made use of this construct to analyze notions in Calculus (e.g. Vinner & Dreyfus, 1989; Artigue, 1992; Biza & Zachariades, 2010). In Linear Algebra, Warwo et al (2011) investigated students' concept images of subspace and the links students create with the formal definition of a linear subspace. Even after more than 30 years, the construct is not only respected among scholars but continues to grow and adapt to contemporary trends.

Until 2000, concept image – concept definition was considered a purely cognitive framework. In other words, the structure of concept image – concept definition was interpreted as a tool to examine how notions are developed in an individuals' mind based solely on prior knowledge and stimuli. From then on, other aspects of the structure were taken into account too. Mason (2002) stressed the psychological aspects of the formation of the concept image. He breaks the concept image into "three interwoven dimensions corresponding to aspects of the psyche" (p.191).



2.1 The three dimensions of concept image. Source: Mason (2002)

According to Mason (2002) the first dimension is the "awareness, images and connections" together with "confusions, obstacles and standard misunderstandings". This dimension corresponds to cognition. The second dimension corresponds to behaviour and is all the prior and new skills and language connected with the concept. Finally, the third dimension represents emotions and contains the root problems and the range contexts in which the concept appears that will serve as motivation to the students.

In addition, Mason (2002) encourages tutors to use this interpretation of the concept image as a framework in preparing a tutoring session. In particular, he invites tutors to reflect upon his/her own mental images, known students' misconceptions and struggles and what may be the cause of them, determine the prior knowledge and new skills linked with the concept in question and consider a range of contexts and problem types. The language accompanying the topic is also important. One should take into account the prior knowledge of language together with the new terminology which students might not be familiar with.Related to that approach Watson & Thompson (2015) remark:

> “...mathematics cannot be presented as a linear accumulation of ideas with assumptions about prior learning, but instead task design needs to develop concept images and dispositions that will be sustainable across a range of mathematical activity and enable learning at several levels. That is, tasks need to be designed so there are multiple entry points, with options for extensions and adaptations.”

In conclusion, this framework can be viewed as a map that tutors may use to teach any mathematical topic. In this thesis we may use Mason’s (2002) framework of concept image to help us create a set of tasks that would correspond to the different aspects of one’s concept image.

Bingolbali & Monaghan (2008) demonstrated how the structure of concept image – concept definition can be used in socio-cultural research. They argued that although concept image is unique to the individual there are aspects that are shared among students. They link these aspects to teaching and shared experiences in the department they are studying. To be more precise, Bingolbali & Monaghan’s (2008) inquiry of first year Mathematics and Mechanical Engineering students show significant differences in the concept images of the derivative between prospective mathematicians and engineers which can be attributed to teaching and departmental orientations.

In this thesis we adopt the original concept image – concept definition framework (Tall & Vinner, 1981) along with its more recent developments (Mason, 2002 and Bingolbali & Monaghan, 2008) to create tasks that can enrich the understanding of the concept of linear span of undergraduate Mathematics students. One of our goals is to design tasks that could be integrated in the broader context of teaching Linear Algebra in a Mathematics Department. In addition, the tasks are meant to be used in situations which encourage interaction among students and tutors. Therefore we do not wish to ignore neither the psychological nor the socio-cultural aspects of the notions.

Finally, we believe that this framework can be easily understood and adopted by mathematicians. Since the tasks designed are of no use if they are not implemented, it is important that the lecturers and tutors, who might not have any particular knowledge of theories in mathematics education, will be able to relate to them. Nardi (2006) presents evidence from discussions with mathematicians which support the idea that the concept image – concept definition framework can be common ground among Mathematicians and Researchers in Mathematics Education. The article provides several examples where mathematicians use the construct explicitly or implicitly to describe students learning. Nardi (2006) also notes:

> "… it seems to be that the acceptance the CI/CD construct enjoys has emerged from its capacity to tell a part […] of the 'story' of learning mathematics."

In our experience, this(cognitive) "part of the story" is what makes mathematicians relate to this construct more than to others. It uses simple terminology which reflects valuable aspects of their personal journey of learning mathematics.

2.3 Examples

Selecting appropriate examples to illustrate a concept plays a very important role in the development of the concept image. The concept image is formed through the experience students have with the concept, examples, problems and prototypes (Vinner, 1992) both inside and outside of the classroom. Although it is impossible for a teacher to control the entire process of concept formation he or she can have some power over content presented in the classroom. Vinner (1983) argues:

> "… one has to provide the students with examples that form the desired concept image not only in the beginning of a chapter but throughout the whole period of learning"

However, the examples do not always depict the full picture; they are bound to particular circumstances and also contain irrelevant information which may become for students "key elements" of the concept (EMS, 2014). For instance, the vectors v= (1,2,2) , u = (1,1,1) and z= (0,1,1) are linearly dependent (v – u – z= 0). In this particular example, every vector is a linear combination of the remaining two. This aspect does not hold for every collection of linearly dependent vectors, but it may become part of the concept image of students if they encounter several examples containing the same information. In fact, this is a problem we discuss in more detail in chapter 3.

What is an example and when can an example be considered exemplary? One may imagine examples as the illustrations of a concept or a process which typically follow a definition or a theorem in a textbook or a lecture. Although these examples have a big part in teaching and learning, students also draw information about a concept or a theorem through tasks they may view as paradigms, models which justify a certain way of thinking. Mason & Watson (2008) use the term *example* to describe "anything from which a learner might generalize" (p.3) including illustrations of concepts and principles, placeholders used instead of general definitions and theorems, worked examples (questions worked out by a textbook or by the teacher for demonstration), exercises, representative of classes which are used to illustrate patterns and specific contextual situations. To us, the most important result of this categorization is that it illustrates the different ways in which an example can be integrated into the teaching of a concept. However, examples are not always exemplary. The term *exemplary* refers to specific situations which "represent a general class to which learners attention is to be drawn" (Liz, 2006: 127). In other words, examples can be exemplary only if they offer the opportunity to students to draw information that can be used to "appreciate a technical term, theorem proof, or structure" (Mason &Watson, 2008: 4).

For the purpose of this thesis, we are particularly interested in two types of examples: *exercises* and *learner generated examples*. To avoid confusion, we make explicit the distinction between these two terms with exercises referring to the questions that focus on the use and fluency with techniques (Mason & Watson, 2008) whereas learner generated examples are questions which prompt a learner to construct a specific example. Although exercises which focus on fluency might fail to trigger generalisations (Mason & Watson, 2008), they are important for the development of the concept image. Exercises provide an individual the opportunity to practice on a technique and create links between this procedure and the concept. For instance, exercises which focus on Gaussian elimination help the students to connect this process with different notions such as linear combination, column space etc.

Contributing to existing research about mathematical knowledge, Hiebert & Lefevre (1986) identified two distinct but closely related kinds of knowledge, conceptual and procedural knowledge. Conceptual knowledge is the kind of knowledge which is characterized by rich relations between "individual fact and propositions so that all pieces of information are linked to some network" (Hiebert & Lefevre, 1986: 4). On the other hand, procedural knowledge is the knowledge of language, symbols, rules and algorithms for completing mathematical tasks (Hiebert & Lefevre, 1986). Related to that, Skemp, 1976 talked about to different types of understanding, relational and instrumental understanding. The later type of understanding is the ability to apply a set of rules to produce a correct answer without understanding how and why these rules work. Relational understanding on the contrary is "knowing both what to do and why" (p. 20). In the case of learner generated examples the focus is moved away from techniques and students become more involved in the process of creating the material to support their learning. Research has shown that learner generated examples can help in conceptual understanding (Housman, 1997, as cited in Liz et al, 2006).

A closely related notion, which can complement the notion of concept image (Liz et al, 2006) is that of *example space*. Mason & Watson (2008) describe the example space metaphorically as a "larder" where the individual can search for an item needed for some purpose. When students are searching for an example based on some given principles, they do so by accessing a class of examples they think appropriate for the construction. The word "space" indicates that the set of examples the learner attends to is not just a list of available items; it is structured and it is "related to knowledge, experience and predisposition" (Mason & Watson, 2008: 57). Mason & Watson (2008) identify four principles concerning the construction of an example:

- Exemplification is individual and situational
- Perceptions of generality are individual
- Examples can be perceived or experienced as members of structured spaces
- Example spaces can be explored and extended by the learner, with or without external prompts. (p. 57)

In this study, we attempt to create a task that triggers potentially conflicting aspects of the notion of span by asking students to create their own examples with the hope of creating a more coherent concept image. We make use of these principles as a reminder of how students may work on and perceive a generated example.

Concluding, Liz et al (2006) remark that during task design one has to take into consideration the characteristics of each example, such as different representations and whether they trigger certain types of reasoning or cognitive conflict. For the purpose of this study, the characteristics of the examples have a crucial role. Since we wish to design tasks which may be used to enrich the concept images of students about the notion of linear span, the choice of examples is going to affect the learning outcome in certain ways, such as the choice of an algebraic or a more geometric approach and the connection of related concepts, processes and ideas.

2.4 Teaching and learning Linear Algebra: A literature review

A recent trend in research in mathematics education is teaching and learning at University level. For more than three decades, researchers have focused on students' difficulties, behaviours and teaching strategies in what is commonly called Advanced Mathematical Thinking (Tall, 1988b). Advanced Mathematical Thinking is typically used to refer to research concerning higher levels of education such as high school or University. Tall (1988b) described Advanced Mathematical Thinking as:

> "…any part of the complete process of mathematical problem-solving, from the creative processes involving deductive and associative resonances between previously unrelated, or even undefined, concepts, through to the final "precising" process of mathematical proof"

Through this line of inquiry, a growing interest in mathematics education at University Level emerged. Although, early research focused more on teaching and learning Calculus (e.g. Tall & Vinner, 1981; Artigue, 1992), the teaching of Linear Algebra attracted the attention of researchers and was recognized as a difficult subject to be taught. Even though to mathematicians the concepts of Linear Algebra might seem elementary, students face significant difficulties understanding "what the fuss is all about". For instance, Chandler

&Taylor (2008) mentioned that it felt like their students were viewing the topics disjointly and were trying to solve problems by manipulating symbols and following steps rather than trying to understand the concepts. Situations like this, makes Linear Algebra uncomfortable both to students and lecturers (Robert & Robinet, 1989 as cited in Dorier et al 2000).

The early literature emphasized the identification of students' difficulties in understanding Linear Algebra. One of the most recognized causes of difficulty is the abstract nature of the subject. Dorier et al (2000) called this "the obstacle of formalism", referring to the difficulties university students had with the use of formalism in the teaching of Linear Algebra in France at the time. The obstacle of formalism refers to difficulties that occur by the use of formal language in the theory of vector spaces in relation with the more intuitive context of geometry and systems of linear equations (Dorier et al, 2000). Their studies confirm that this obstacle is recognized, not only by students, but also by the teachers, as one of the most common causes of difficulties along with difficulties that are caused by students' inexperience with proofs, logic and set theory. Artigue (2001) notes that difficulties related to the obstacle of formalism are partially removed in countries where the first course of Linear Algebra is limited to the study of $\mathbb{R}^n$ with emphasis on matrix calculus and applications. On the other hand, she also noted other difficulties related to this approach supported by prior research.

There is no doubt that truly understanding Linear Algebra and appreciating its highly theoretical nature depends heavily on understanding the central ideas of subspace, span, linear combination and linear dependence/independence. Carlson (1993) points out some of the reasons why students struggle to understand these concepts. Firstly, Carlson (1993) believes that first year students are less sophisticated and therefore it is difficult for them to understand such concepts. He also notes that it is difficult to teach a concept to first year students, whose experience is almost entirely computational at this point. Moreover, working with these ideas requires different processes in different settings, for example finding a basis in R^n and finding a basis of a subspace in the vector space of functions. Finally, Carlson (1993) points out the lack of a variety of examples and applications of these concepts and the absence of substantial connection with students' prior knowledge.

Related to Carlson's remarks is, perhaps, the fact that understanding Linear Algebra demands a proper level of "cognitive flexibility" (Artigue et al, 2000; Dorier & Sierpinska, 2001) in manipulating the various languages, representations and settings of the concepts. Cognitive flexibility refers to the ability of students to distinguish these different representations, translate them from one type to another without confusion. Much like how Linear Algebra was born, from geometrical vectors to n-tuples, polynomials and functions, understanding its concepts requires a shift of perspective. Dorier & Sierpinska (2001) distinguish two stages in the construction of a concept:

- recognition of similarities between objects, tools and methods brings the unifying and generalizing concept into being;
- making the unifying and generalizing concept explicit as an object induces a reorganization of old competencies and elements of knowledge. (p. 257)

These stages are not in line with the prior experience of first year students who encounter a course of Linear Algebra in parallel to or immediately after a Calculus course which is typically computational, with little focus on formal proofs and abstract structures.

Hillel (2000) studies the sources of conceptual difficulties in learning Linear Algebra concerning the existence of several languages or modes of descriptions, the problem of representation and the applicability of the general theory. These sources are interconnected and linked with the lack of cognitive flexibility. He distinguished three co-existing "modes of description" of the basic objects and operations in linear algebra:

1. The abstract mode - using the language and concepts of the general formalized theory, including: vector spaces, subspaces, linear span, dimension, operators, kernels.
2. The algebraic mode - using the language and concepts of the more specific theory of R^n, including: n-tuples, matrices, rank, solutions of systems of equations, row space.
3. The geometric mode - using the language and concepts of 2- and 3-space, including: directed line segments, points, lines, planes, geometric transformations (p. 192)

According to Hillel (2000) these modes are interchangeable but not equivalent. For example, to determine if a geometrical vector (geometric mode) is in a given plane, one can examine if it is possible to write it as a linear combination of two vectors spanning the plane (algebraic mode). At this point, the mode of description usually changes to coordinates, and the focus is placed on solving a system of linear equations. Although the procedure is algebraic the final result should be recognized as a geometric vector belonging (or not) to the given plane. Similarly, one is allowed to change the mode of description depending on the setting and procedures, but the different modes have always a different role in the process. Moreover it is possible to use different representations of vectors in the same mode, for instance geometric vectors may be described as directed line segments but it is also useful to think of a collection of vectors as points on a line, a plane or space.

This categorization makes explicit that cognitive flexibility is essential to cope with Linear Algebra. Hillel (2000) also attributes students' difficulties to understand the multiplicity of these representations to teachers and textbooks that constantly shift the mode of description without alerting the students or giving them time to think and discuss what remains invariant. Contributing to this indication, Stewart & Thomas (2010) used the terms of APOS theory and Tall's three worlds of mathematics (embodied, symbolic and formal) which is comparable to Hillel's modes of description. They observed that the students participating in their research often do not have the time and the opportunity to develop links between the three worlds.

In parallel to Hillel's work, Sierpinska distinguishes two different ways of thinking and identifies three modes of thinking that are essential to understanding Linear Algebra. In her work, Sierpinska (2000) describes two ways of thinking, inspired by a Vygotskian interpretation of scientific concepts, *Theoretical Thinking* and *Practical Thinking*. According to Sierpinska (2000) Theoretical Thinking is characterized by conscious reflection and the ability to express the reasoning behind an action. In Theoretical Thinking reasoning is based on systems of concepts by making logical and semiotic connections between the concepts of a system. Furthermore, a theoretical way of thinking means that the individual attends to the language and pays attention to contradictory thoughts. On the other hand Practical Thinking is “an auxiliary activity which accompanies and guides other activities” (Sierpinska, 2000: 212) rather than a specialized mental activity. It is expressed through “goal-oriented actions” (Sierpinska, 2000: 212) implementing reasoning on the basis of a concept's most typical examples. Sierpinska (2000) attributes students' difficulties in Linear Algebra to their practical way of thinking.

Aiming to identify the characteristics of students' ways of thinking in Linear Algebra, Sierpinska (2000) also identified three modes of thinking that coexist in the context of Linear Algebra: *synthetic-geometric*, *analytic-arithmetic* and *analytic-structural*. The difference between synthetic and analytic mode is that they give a different meaning to the notion in question. Synthetic mode describes ways of practical, geometric thinking. For example, determining if two vectors are linearly independent or dependent in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be achieved without the use or understanding of the algebraic definition of linear independence. On the contrary, in analytic thinking the student uses numbers or algebraic representations and tries to understand the objects through their definitions and properties (Sierpinska, 2000). There are two distinct modes of analytic thinking, arithmetic and structural. Sierpinska (2000) explains that analytic-arithmetic thinking aims at simplifying calculations whereas analytic-structural thinking focuses on extending the knowledge about the concept or concepts in question.

In line with Hillel's modes of descriptions, Sierpinska's modes of thinking co-exist and are interchangeable. It is important to note that none of the previously mentioned modes of description and thinking is the cause of failure in Linear Algebra. All modes are equally important and necessary. In fact, Bagley & Rabin (2016), following a similar framework, called the different types of thinking computational, abstract and geometric. In their study they found that despite its pitfalls computational thinking has also some very interesting affordances, including providing a general orientation to an unfamiliar problem and evaluating the applicability of known algorithms. Overall, developing these modes of thinking and taking advantage of the different modes of description is essential for students to face their failing understanding and overcome their difficulties.

Apart from classifying students' difficulties researchers of Mathematics Education have focused on the teaching of Linear Algebra. One of the first, and maybe the most studied and controversial teaching perspective is the use of geometry as an introduction to Linear Algebra. In Dorier et al (1999) the use of analytical geometry was proposed as a means to avoid formalism. Inspired by an epistemological analysis of Linear Algebra they suggested experimentation with "linear situations" such as geometry, linear systems and magic squares. The use of analytical geometry can potentially provide mental images for the basic concept of Linear Algebra (Dorier et al, 2000) and assist students to accept formalism as the final step toward the abstract theory (Dorier et al, 1999). Although the contribution of geometry in the development of the theory of vector spaces is indisputable, some researchers' findings suggest this approach has some downfalls or limitations (e.g. Harel, 2000; Gueudet-Chartier,2004). Despite the criticism, contemporary design research in Linear Algebra proposes interesting, meaningful and alternative ways to involve geometry in the introduction of Linear Algebra (Watson et al, 2003; Stewart & Thomas, 2009; Warwo et al, 2013).

Developing a curriculum in Linear Algebra at University level that would correspond to the pedagogical needs of students was a major concern early on. Harel (2000) pinpoints the fact that (American) high school education is not "geared towards the need of linear algebra" (p. 179). Although topics such as linear systems of equations, analytical geometry and Euclidian space are taught, they are treated in a superficial way that does not reflect the basic ideas of Linear Algebra (Harel, 2000). On the other hand, Harel (2000) observed that the authors of textbooks on elementary Linear Algebra assume that beginners are comfortable with abstract structures, basic ideas and ways of thinking unique to Linear Algebra. Driven by this observation and inspired by Piaget's philosophical theory of concept development, Harel (2000) formulated the three principles of teaching and learning Linear Algebra: *Concreteness*, *Necessity* and *Generalisability*. The Concreteness Principle states that in order for students to abstract mathematical structure from a particular model, such as geometrical vectors or $\mathbb{R}^n$- spaces, the context must be concrete to them. In other words, the abstract ideas of Linear Algebra should be built on a context familiar to the students which will allow them to make connections, develop a coherent concept image and lead to further abstractions (Harel, 2000).

The second principle, the Necessity Principle, as Harel (2000) calls it, states that in order to learn students have to be able to see an intellectual need for what they are taught. He also remarks:

> "the idea behind this principle is that instructional environments must include appropriate constraints by which students can reflectively abstract mathematical conceptions and, at the same time, keep the situation at hand realistic. The

> instructional activities must offer problematic situations that are realistic to and appreciated by the students. Through their activities, students must feel that what they do results in a solution of a problem (their own problem!) or in a resolution of a conflict (their own conflict!), and, if an idea (e.g., a definition of an operation, or a symbolization form of a concept) is initiated by their teacher, they must not feel that it was evoked arbitrarily." (p. 186)

Related to this principle, Warwo et al (2013) constructed a sequence of tasks using realistic situations to help students visualize basic concepts of Linear Algebra. Although to a mathematician a realistic situation might occur through established mathematical problems, these situations are rarely applicable to teaching first year students. Dorier et al (1999) denoted the difficulty of motivating students because the use of the theory will not be explicit until they are able to apply it to a wide range of situations. Moreover, Dorier &Sierpinska (2001) reported that the axiomatic approach in teaching linear algebra often seems unjustifiable to students because "all the linear problems that are within reach of first year students can be solved without using the theory of vectors spaces". Finally, the Generalisability Principle is addressed to the activities that students engage with by working on a 'concrete' model. These activities should enable students to "abstract concepts they learn in a specific model" (p. 187). These three principles describe the basic characteristics any educated and organized attempt to teaching Linear Algebra should have to fill the gap between high school and university mathematics.

Besides the well-documented difficulties in the basic concepts of Linear Algebra in general, contemporary research has revealed some intriguing findings about the teaching and learning of the concept of linear span and the ideas that are linked to it. In her thesis, Stewart (2008), developed a research-based framework, combining APOS theory and Tall's three worlds of mathematics which can be used both as an analytical tool, to interpret students understanding in Linear Algebra, and as a teaching approach. As part of her study, Stewart compared two groups of students; the first group was introduced to Linear Algebra through a course with emphasis on geometry, embodiment and linking of concepts whereas the second group attended a course with emphasis on symbolic algebra, matrices and concept definitions (Stewart and Thomas, 2010). Her analysis showed that students in the second group faced significant difficulties in understanding the concept of span compared to the group that was taught with focus on embodiment. Stewart and Thomas (2010) also observed that in the second group:

> "a small minority managed partially to connect their thoughts to the ideas of subspace and basis, no one mentioned that span is the set of all linear combinations of vectors (a process view), or referred in their descriptions to the strong embodied aspects of span, for example as a plane or other subspace in $\mathbb{R}^3$ (an object view)" (p. 182)

Similar results were observed concerning the concepts of basis and linear independence which strongly suggest that an embodied world approach contributes to the understanding of the concepts that are the essence of Linear Algebra by helping students making connections between different representations and related concepts.

A drastically different approach, but with equally positive results, is that of Warwo et al (2012). Based on the framework of Realistic Mathematics Education, Warwo et al (2012) created a sequence of tasks with algebraic and geometric approach on a realistic setting. They argue that introducing Linear Algebra through vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, making connections between their geometric and algebraic representations and studying their properties provided students "with rich geometric and algebraic imagery for linear independence/dependence, imagery that is strongly connected to the formal definitions" (p. 589). Concerning the notion of span, Warwo et al (2012) found that it was non-trivial for students to explore and develop a concept image of linear span as the set of all possible linear combinations. However, this approach showed positive results in motivating and developing the notions of span and linear independence and dependence as opposed to an introduction with systems of linear equations (Warwo et al, 2012; Warwo et al, 2013).

Finally, we would like to mention an important observation of Warwo et al (2011) related to this study. Using Tall and Vinner's structure of concept image and concept definition Warwo et al (2011) analyzed the concept images and the importance of concept definition of subspace. Among their findings was the observation of the recurring cognitive conflict they referred to as "nested subspaces". Students in their study expressed a conception of the vector space $\mathbb{R}^k$ as being a subspace of $\mathbb{R}^n$, where $k<n$. Warwo et al (2011) hypothesized based on their evidence that this confusion has roots in students thinking "some subspaces as "the same"" (p. 15). This "sameness" is essentially based on the concept of isomorphism,

with which students are unfamiliar at the early stages of learning Linear Algebra. Warwo et al (2011) remark that lecturers often make shifts between k-dimensional subspaces of $\mathbb{R}^n$ and $\mathbb{R}^k$ without substantial explanations, linking this observation to Hillel's modes of description. This can be confusing to students and perhaps falsely attribute this "sameness" to the geometric properties of $\mathbb{R}^2$ or $\mathbb{R}^3$ spaces due to inexperience with different modes of descriptions. However, Warwo et al (2011) proposed that these situations might offer the ground to motivate isomorphism as a tool and help better students understand this "sameness".

In conclusion, research on the teaching and learning Linear Algebra so far has brought to light some insights about the fundamental difficulties students face with the subject. The nature of Linear Algebra requires the development of different ways of thinking and of a reflexive attitude that often delays the understanding of essential concepts such as subspace, span, linear independence or basis. However, this situation is not out of hand. Experimental teaching and attention to students' needs has offered positive results so far. Although mathematics education cannot give a universal solution to the problem, continuing research contributes in making the teaching of this abstract subject richer and more appealing to students.

CHAPTER 3

Preliminary Study

This chapter is dedicated to the preliminary study conducted in order to support our task design. Firstly, we present some details about the mandatory courses in Linear Algebra offered in the Department, focusing on the teaching of linear span. In the second section, we present the purpose and the research question of the study. Next, we discuss the main features of the method used. To analyze and present our data we employed the grounded theory approach. This method is chosen in order to achieve a better understanding of students' needs regarding the teaching of linear span. Finally, we present the analysis of the data and formulate a hypothesis that will help us design the tasks.

3.1 The setting

In the context of this study, we analyze and present our findings regarding students' difficulties and/or misconceptions about the notion of span. These findings are linked to what students are expected to understand by the end of an introductory course in Linear Algebra and perhaps to the way the course is being taught. To give a full picture of what we are looking for in this study, we begin with a short description of the mandatory courses in Linear Algebra offered in the Department. This description is based on the curriculum of the Department and the lecture notes (Kourouniotis, 2014; Kourouniotis, 2016) provided to students each year.

The Department of Mathematics and Applied Mathematics of the University of Crete offers two mandatory courses in Linear Algebra. "Geometry and Linear Algebra" and "Linear Algebra I"[1]are designed to be taught[2] to students in the 1st and 2nd semester of their studies respectively. Both courses are typically taught through 4 hours of lectures and a two-hour problem workshop per week.

[1]The semester in question, the researcher was part of the group of tutors guiding the students through the
[2]Although designed for first year students, a number of people may be older students retaking the course.

The introductory course “Geometry and Linear Algebra” aims to acquaint students with different forms of mathematical representation of geometrical objects (vectors, lines, planes etc.) through problem solving, the study of $\mathbb{R}^n$, the use of matrices and the use of Gaussian elimination to solve systems of linear equations or study subspaces in $\mathbb{R}^n$. This approach is imposed in part by the fact that secondary education in Greece includes the teaching of notions such as vector on a plane and procedures like solving a simple system of linear equations but without highlighting key ideas of Linear Algebra. As a result, the goals of this introductory course are divided between introducing students to the basic concepts of Linear Algebra through the “model” spaces $\mathbb{R}^n$ and familiarization to problem solving methods using matrices.

Focusing on linear span, the notion is connected to those of subspace, linear combination and basis. It is one of the most central concepts in Linear Algebra. The concept is introduced in the course “Geometry and Linear Algebra”. As an intuitive introduction to the concept, students experiment with the idea in the Euclidian plane and 3-space. The notion is formally defined in the middle of the semester right after the definition of linear combination. We note that the notion of span in the context of this course is mostly referred to descriptively as “the subspace generated[3] by v_1, v_2, …,v_k”. In relation to the general goals of this course, students are expected to familiarize with the concept of linear span in subspaces of $\mathbb{R}^n$, be able to identify its geometrical representation in the case of $\mathbb{R}^2$ and $\mathbb{R}^3$ and be able to determine if a vector is in the span of a fixed set of vectors.

During the course “Linear Algebra I” students revisit concepts of Linear Algebra this time in an axiomatic manner. The course includes an introduction to the axiomatic study of vector spaces and the presentation of examples of more general vector spaces such as polynomial and function spaces. The study of more complicated problems is connected to the computational processes developed in the introductory course. In addition, it seeks to highlight the usefulness of the theoretical approach. Finally, in “Linear Algebra I” the concept of span is defined again in a similar manner for arbitrary vector spaces and more

[3]There is a difference between the translation of span as a verb and as a noun. In Greek the verb span is synonymous to the verb generate (in general) whereas the meaning of linear span is restricted in the context of mathematics.

examples are given to the students. In this course the students are expected to have some prior knowledge about the notion in order to explore some of its properties and use it properly in problem solving.

3.2 Methodology

3.2.1 The development of the research question

In addition to what is proposed in the literature, we attempted to identify specific difficulties and/or misconceptions regarding the notion of linear span that students in our Department struggle with. Bingolbali and Monaghan (2008) made a connection between an individual's concept image and teaching practices within different university departments. On this basis, we decided to investigate how students in the Department of Mathematics and Applied Mathematics of the University of Crete make use of the concept of linear span.

The main purpose of this preliminary study is to acknowledge key aspects of the notion that students may not fully comprehend. To this purpose, we have formulated the general research question:

- What key aspects of linear span students might overlook while solving a problem?

3.2.2 The method

To accomplish our purpose, it was decided to use a grounded theory approach to analyze the data we collected from the final exam of the course "Geometry and Linear Algebra". Grounded Theory is an experimental method where the researcher uses induction to generate a theory based on qualitative data. In contrast with other approaches, in grounded theory there is not a pre-existing hypothesis to put to test. Therefore, in grounded theory data are not "forced" to fit within a predetermined theory (Glaser and Strauss, 1967, as cited in Cohen et al, 2011). In this study, the approach was used to help us generate a hypothesis about students' difficulties and/or misconceptions about the concept of span. We later use this hypothesis to create tasks that could be used in the context of an introductory course in Linear Algebra. Since we had limited amount of time and data at the time, we proceed with an abbreviated version of grounded theory (Willig, 2001, p. 73).

We gathered written answers on a task given to students in the final exam of the course "Geometry and Linear Algebra" during the winter semester of 2016-17. The task given to the students was the following:

> *"Εξετάστε αν το διάνυσμα z=(1, –1, 1) ανήκει στο χώρο V που παράγεται από τα διανύσματα u=(2, 1, –1) και w=(3, 2, –2)."*
>
> ***Translation:*** *"Determine if the vector z=(1, –1, 1) belongs to the space V spanned by the vectors u=(2, 1, –1) and w=(3, 2, –2)."*

The task was chosen by one of the lecturers and was given only to his students (about half of the students that took the exam). It was part of a two-question task and served as a scaffold to answer the second one. This question examines the ability of students to decide whether a vector lies in the span of some other given vectors. It also checks the ability of students to follow a standardized procedure. There is more than one correct approach that one could use to answer this question. The goal is for one to conclude that z can be written as a linear combination of u and w. We note that the vectors u and v are linearly independent so they form a basis for V. This fact was unintentional but eventually it was proven very fruitful in our study.

The sample consisted of the 129 students who attempted the task. These students were in various years of their studies in the Department. To be more precise, the following table lists the number of students that answered the question by year and stream of study.

	1st year	2nd year	3rd year	4th year	> 4th year
Mathematics	21	17	9	4	11

Applied Mathematics	21	16	14	14	2

The decision to collect data from an official exam was made mainly to ensure that the answers represent more accurately the students' thoughts about the notion. Even though, the responses were not anonymous we tried to warrant confidentiality in the context of the Code of Ethics of the University of Crete (2002).

To analyze the data we used coding and constant comparison. Coding processes in general require moving back and forth in the data so that the researcher can identify similarities and differences between categories. Constant comparative analysis "ensures that the coding process maintains its momentum" (Willig, 2001, p. 71) throughout the coding process. By the end of this process a core category was identified in order to answer our research question.

3.3 Analysis of the data

This section is concerned with the analysis and the interpretation of the answers provided by 129 students on the question discussed earlier. At first, we present the success rates as based on the lecturer's grades. Then, we give a detailed analysis of our findings in regard of the difficulties and misconceptions that students might have about the notion of span. It is worth noticing that our focus was on students' reasoning throughout the process. Therefore we did not take into account arithmetic mistakes as long as the reasoning was not shifted afterwards.

The following figure shows the success rates based on the lecturer's grades on the question. The data are divided into four categories, namely A, B, C and D. Each of the categories represents a class of grades from 0 to 8, where 8 was the maximum grade for this question. The category "A" stands for grades 8 and 7, "B" for grades 6 and 5, "C" for 4 and 3, finally "D" stands for grades less than 3.

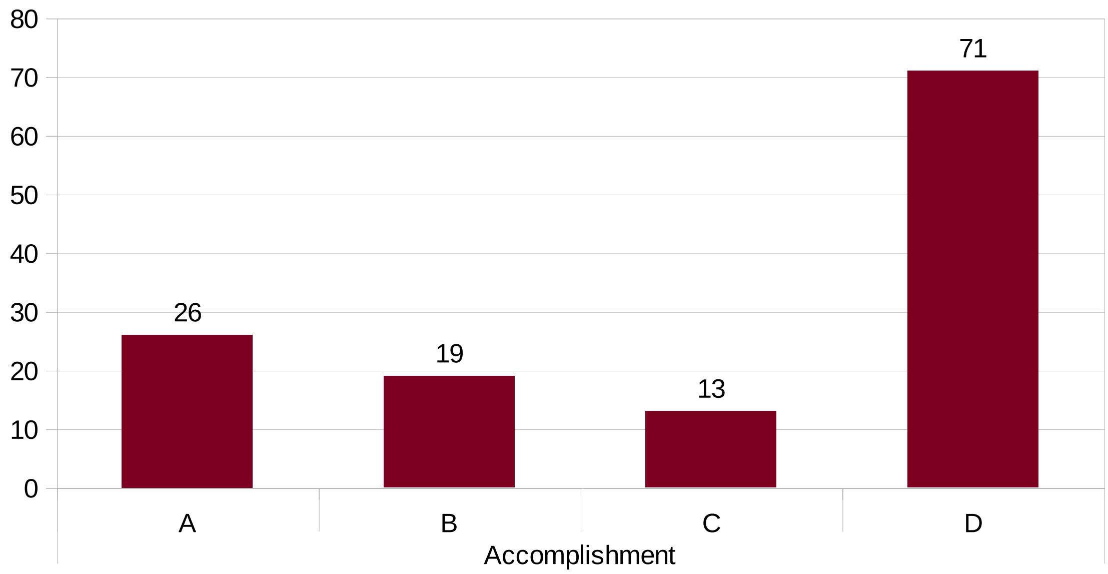


Figure 3.1

Focusing on the category “D”, 71 of the students failed to fulfill the requirements of the task which was essentially to determine if z can be written as a linear combination of u and w. This category includes answers that had incorrect reasoning paired with insufficient procedures and answers that were either completely unrelated to the question or just yes or no statements without proper justification. On the other hand only 26 out of 129 students managed to give a satisfactory answer. The rest of the students, 19 in the category B and 13 in the category C, were found to have conceptual or computational difficulties. Over all it is reasonable to assume, at this point, that most of the students eitherstruggle with some aspects of the notion of span or have difficulties in the procedure. To investigate further this trend, we tried to analyze as best as possible the answers by looking for gaps in the line of reasoning and searching for connections between students’ attempts.

After careful coding and constant comparison we determined that students use different notions and/or procedures to achieve their goal. The figure below depicts our results regarding the use of the different concepts. The code that represents each of them is an abbreviation of the main goal. The code “LinC”, standing for linear combination, was used in the cases where students either stated that in order for z to be in the span of u and w, z must be written as a linear combination of u and w, or whenever the results were consistent with this approach. Similarly, the codes “LinD” and “LinIn” were chosen for cases where the dominant idea was that z lies in the span of u and w if these three vectors are linearly dependent or linearly independent respectively. Finally, there were 6 cases where the goal was vague denoted as “notC” (not clear) and 26 where students used unrelated procedures or just gave a yes or no answer, referred to as “Other”.

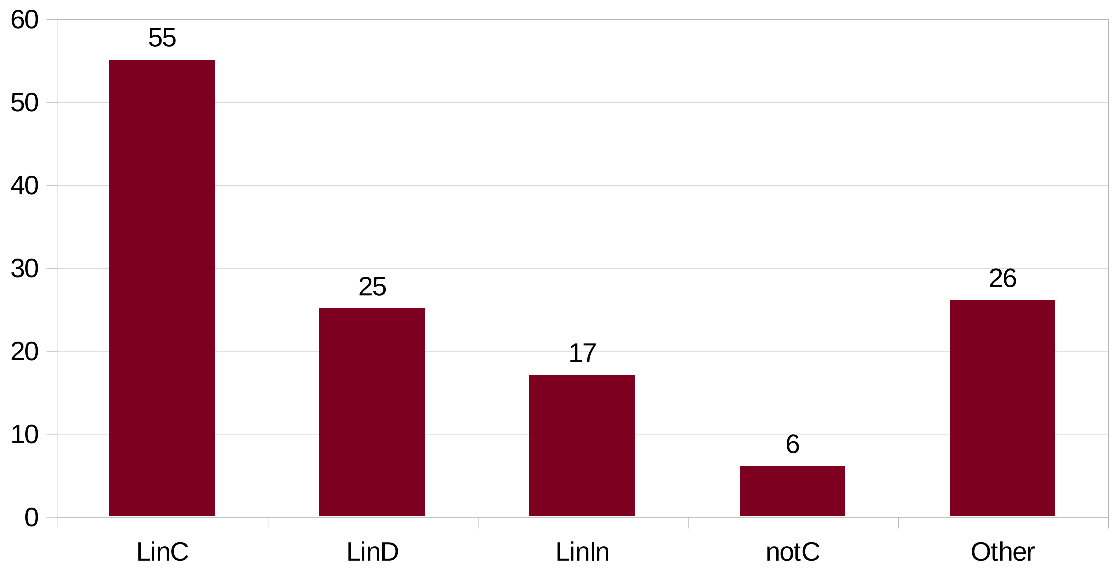


Figure 3.2

Only one of the concepts mentioned above is appropriate for solving the problem. Linear span of a set of vectors is defined to be the vector space of all the linear combinations produced by the set. On this basis, z belongs to the span of u and w if and only if z can be expressed as a linear combination of u and w. In other words if and only if there are real numbers a, b such as $z=au+bw$. Since z actually belongs to the span of the given vectors, the right way for someone to answer this question is to reach to the conclusion that z can be written as a linear combination of those two. Therefore, 55 of the students (LinC) seem to have set the right goal to solve the problem.

On the contrary, the goal LinD is not enough to answer the question fully. If a vector z is in the span of some other vectors $v_1, v_2, \ldots, v_k$ then the set $\{z, v_1, v_2, \ldots, v_k\}$ is linearly dependent. However, the opposite is not always true. For example the set {(1,0), (0,1), (0,2)} is linearly dependent but (1,0) is not in $\langle(0,1), (0,2)\rangle$. In our case, u and w are linearly independent therefore the fact that the set {z, u, w} is linearly dependent implies that z can be written as a linear combination of u and w. Either way, the fact that z, u and w are linearly dependent is not enough. There must be a statement about the status of u and w. In our case, none of the 25 students who used the concept of linear dependency made such a statement, therefore their answers cannot be considered sufficient. There were, also, 17 answers based on the belief that “z is in the span of u and w if z, u and w are linearly independent”. This could partially be interpreted as confusion between the concept of span and the basis.

In order to have an overall view for each of the main categories LinC, LinD and LinIn, we further defined three subcategories based on how clearly was the goal stated each time. To do so, we used three new codes. “Clear” which applies to the cases where the goal was explicitly stated or supported by the conclusion. “Hazy” refers to answers where the explanation given shares aspects with more than one concepts. For example an answer is coded as Hazy LinC if the student refers to the concept of linear combination but later uses a procedure that is typically used in determining linear dependence (e.g. solving Ax=0). Finally the subcategory “Instrumental” refers to the answers where the student follows a known procedure without reasoning for his/her actions. We note that, although this is essentially a procedural task, the code “Instrumental” refers only to the way some students seem to follow that procedure without questioning their practices.

Out of the 55 answers making a reference to the notion of linear combination, 14 are classified as “Clear”, 21 as “Hazy” and 20 as “Instrumental”. In contrast with the other two main categories, most of the answers coded as LinC are either purely procedural or show signs of confusion. Also interesting is the fact that in category LinD 13 out of 25 students clearly stated that the vectors have to be linearly dependent. This supports the interpretation that the concept of linear combination in relation to that of linear dependence is confusing to many of our students.

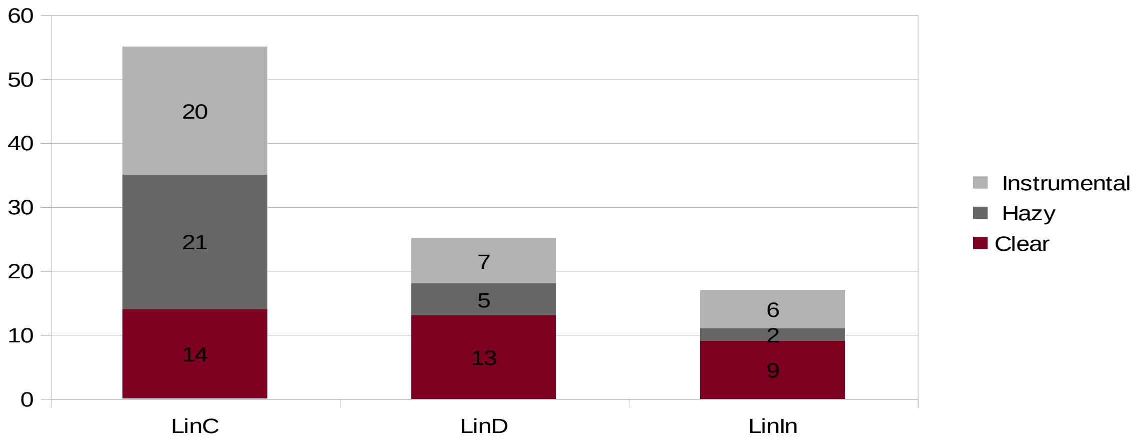


Figure 3.3

3.4 Conclusion & the hypothesis

By thorough examination of the available data, we observed that most of the students have difficulties determining when a vector lies in the span of some set. Those difficulties vary from total misuse of the concept of span to misconceptions on some aspects of it. In either case, a need for more attention on the teaching of the concept is apparent.

Among the goals of the course “Geometry and Linear Algebra” regarding the concept of span is familiarizing with the notion and being able to examine if a vector belongs to a linear span. Our analysis shows many instances where these goals are not achieved. Less than half of the students used the right approach in trying to answer this question, but most of the time they also showed signs of confusion or instrumental understanding.

Perhaps the most interesting finding is that there are some students who attempted to give an answer by examining if the three vectors are linearly dependent. As we discussed earlier, this approach is not the most appropriate to solve this problem. Although in this particular case being linearly dependent implies that z is in the span of v and w, a full answer requires the students to notice that the set of generators is linearly independent. In our data we observed that the students did not make an explicit or an implicit reference to this fact. Therefore we are led to believe that some students may ignore or have difficulties in understanding the difference between linear dependence and the existence of a particular linear combination. The notions of linear combination and linear dependence may be quite alike to the eyes of a student. Apart from the similarities in the algebraic representations of the two concepts, a deeper root for this problem may be the false assumption that the two notions are equivalent. Typically, in “Geometry and Linear Algebra” students are introduced to the concept of linear dependence through the existence of a linear combination between some of the vectors in a given set. At this stage, because of their inexperience with logic or lack of evidence and limited example space, it is possible that some students may assume that every vector in a linearly dependent collection can be expressed as a linear combination of the others. In the context of linear span the notion of linear combination is a very important aspect and serves a very different purpose than the notion of linear dependence. Even though this difficulty is not explicitly related to the notion of span it is important to confront it. Examining the notion of linear span may offer the opportunity to confront such difficulties in a meaningful way. Therefore we believe that it is important to take this observation into account when designing and evaluating the tasks.

Summarizing, in this mini study we focused on difficulties and/or misconceptions about the notion of span that students of the Department might have. The greater purpose of this inquiry is to create a set of tasks in the context of an introductory course in Linear Algebra that could help future students in better understanding the notion.

CHAPTER 4

Task Design

In this chapter we present in detail a sequence of text-based tasks designed to enrich students' understanding of the concept of linear span. The tasks are designed to project key ideas about the concept of linear span. First we provide some information about the purpose of the design. Then we make some brief notes about the most important aspects of the concept of linear span. Finally, we discuss in detail the theoretical framework, design principles and the development of the tasks.

4.1.1 The purpose

Our aim is to create a sequence of tasks that could be used in the context of an introductory course in Linear Algebra to assist students create coherent images about the concept of linear span. The need for these tasks arose from the participation in the problem workshops of the course where we witnessed firsthand the benefits of this process and also the efforts and the struggles of students to comprehend aspects of Linear Algebra.

Typically the majority of the tasks in the problem workshop of the course "Geometry and Linear Algebra" is computational and aims to develop mathematical procedures. We would like to create tasks that can be integrated in a problem workshop about linear span, complementing these standard computational exercises in a way that would promote conceptual understanding and theoretical thinking. This intention corresponds to the way problem workshops operate in the Department. Nevertheless, the set of tasks is designed so that it could easily fit in similar circumstances with little or no alterations.

4.1.2 The concept of linear span

A subspace can be described either as the set of solutions of a homogenous linear equation (equivalently, as the kernel of a linear transformation) or as the linear span of a set called the set of generators. The second approach is, ultimately, more fundamental since the set of solutions can be expressed as the linear span of a set of special solutions.

In a given vector space U over a field K the linear span of a set of vectors S is defined[4] to be the subspace of all the linear combinations of elements of S. In the course "Geometry and Linear Algebra" students are given a slightly modified version of this definition limited to the spaces R^n. In Kourouniotis (2014) the definition is given as such:

> *"Θεωρούμε τον γραμμικό υπόχωρο V του R^n. Τα διανύσματα $w_1, w_2, \ldots, w_k$ του R^n παράγουν τον υπόχωρο $V \subseteq R^n$ εάν:*
>
> *a) $w_j \in R^n$ για κάθε $j=1,2,\ldots,k$ και*
> *b) κάθε διάνυσμα του V εκφράζεται σαν γραμμικός συνδυασμός των $w_1, w_2, \ldots, w_k$, δηλαδή για κάθε $v \in V$ υπάρχουν αριθμοί $c_1, c_2, \ldots, c_k$ τέτοιοι ώστε $v=c_1w_1+c_2w_2+\ldots+c_kw_k$."*
>
> **Translation:** *Let V be a subspace of R^n. The vectors $w_1, w_2, \ldots, w_k$ of R^n span the sunspace $V \subseteq R^n$ if:*
>
> *a) $w_j \in R^n$ for every $j=1,2,\ldots,k$ and*
> *b) every vector in V can be expressed as a linear combination of $w_1, w_2, \ldots, w_k$, that is for every $v \in V$ there are real numbers $c_1, c_2, \ldots, c_k$ such that $v=c_1w_1+c_2w_2+\ldots+c_kw_k$.*

In both cases, namely the general and the limited definition, the most important aspects of the concept are comprised. The first is the aspect of *closure* under the operations of a vector space. The linear span of a set S is a *subspace* of U (or R^n). Every element of S and every linear combination of them is an element in the subspace spanned by S. The second important aspect is that *every* element in this subspace is a linear combination of some vectors in S. This is stated very clearly in the definition. The final aspect is also very important and typically overlooked. There is *no limitation* in a choice of the set of generators S. In contrast to the concept of basis where the vectors have to be linearly independent, the set S may contain linearly dependent vectors. The findings in our

[4]There is an equivalent definition of linear span as the intersection of all the subspaces in U containing S.

preliminary research support the claim that this aspect is confusing for some students perhaps due to misleading examples or lack of attention.

These three aspects make the concept of linear span remarkable both for mathematical and didactical reasons. For mathematicians abstracting and generalizing ideas is the key to develop new theory. The notion of linear span can be used in the context of Linear Algebra in abstract cases where it is difficult to determine a basis, for instance in the vector space of real functions. The essence of these three aspects is that they depict the notion of span based only on the axioms of a vector space as an algebraic structure in contrast with the notion of the basis. As a result, the concept of the set of generators, which is closely linked to that of linear span, can be generalized to other algebraic structures. In contrast with the concept of span, the concept of basis can be generalized only in special cases (e.g. modules over rings). From a didactical standpoint, we argue that if someone's image of linear span includes these aspects they would set good foundations for understanding key ideas in Algebra generally.

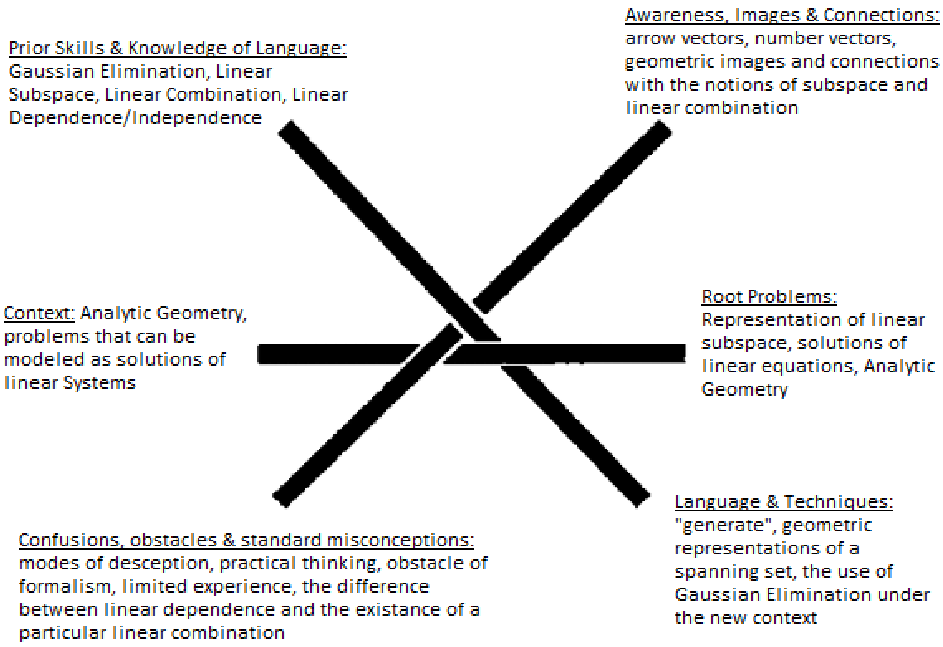
4.1.3 The setting

Problem workshops are an important part in the teaching of the mandatory courses in the Department. Apart from lectures, students are encouraged to attend weekly problem workshops in each mandatory course. Typically, students are expected to work in groups on selected problems of the subject taught that week with the guidance of the lecturer and a number of postgraduate or senior undergraduate students. Problem workshops are designed to assist students in their learning process, promote discussion and create a learning culture.

4.1.4 The framework

The design is based on the theoretical construct of concept image and concept definition (Tall & Vinner, 1981). We adopt a more recent approach to this theory proposed by Bingolbali & Monaghan (2008). This approach takes into account the relations between the development of one's concept image, the teaching and the departmental affiliations. By the use of this theoretical structure we attempt to design tasks that could help students to develop concept images about linear span consistent with the key aspects of the concept (as presented in the previous section). These aspects are related to the goals of the introductory course in Linear Algebra as taught in the Department of Mathematics and Applied Mathematics of the University of Crete.

As a starting point for the design, we use of Mason's (2002) concept image framework. During this process we took the time to reflect upon different aspects of the concept of linear span that are of importance. The following diagram shows our interpretation of the three dimensions of the concept image corresponding to aspects of the psyche (Mason, 2002) in regard to the notion of span.



This diagram depicts briefly all the points we need to take into account while designing the tasks. The first dimension, which corresponds to cognition, contains the desired images that students should acquire for the notion of span along with known potential conflict factors. In correlation with the goals of the course these images should include connections between algebraic and geometric representation of the notion of span and the related notions of vectors, subspace and linear combination. Moreover, we have to take into account the difficulties proposed by the literature as well as the misconception we identified in the preliminary study regarding the notions of linear combination and linear dependence/independence.

The second dimension corresponds to the skills and language that students should use and develop in the frame of linear span. Since our focus is more on conceptual understanding and the creation of a coherent concept image about the notion of span we are interested in skills that require a level of cognitive flexibility and not just computational skills. Therefore, the students should be able to "read through" the results of the process of the Gaussian Elimination, be aware of the different contexts in which they can use this process and make connections between algebraic and geometric representations of vectors. We are aware that these are skills that typically students have difficulties with, thus we have to take into account potential conflict factors that might emerge.

Finally, the third dimension depicts the different contexts in which the notion appears and can act as motivation to the students. The notion of linear span arose from the need to solve linear problems. It gives a way to describe the solutions of a system of linear equations and ultimately represent a linear subspace. In this frame, the notion of linear span is essential to the theory of Linear Algebra. In "Geometry and Linear Algebra" the students encounter problems that can be modeled by $\mathbb{R}^n$ therefore the most appropriate context might be Analytic Geometry.

The choice of this framework was made because of our belief that it is simple to understand and make good use of it. Since this work aims to be used by mathematicians teaching at the university we do not want to use a more complicated construct which might be dismissed by some. Our choice is supported by Nardi (2006) where she gives several examples of mathematicians' perspectives about the matter. Moreover, the theory of concept image and concept definition is considered a "solid" tool in mathematics education (EMS, 2014).

In addition, we take into account Sierpinska's (2000) remarks about theoretical thinking and Harel's (2000) principles of teaching and learning Linear Algebra. To be more specific, the task should have characteristics that correspond to theoretical thinking, such as opportunities for conscious reflection, connections between related concepts or different representations and attention to contradictory thoughts. Moreover, the tasks should include familiar concepts (concreteness principle), justifying the need of linear span (necessity principle) and allow generalization of the key ideas (generalizability principle).

4.1.5 The design principles

We identify the following principles based on the theoretical framework, the concept of span as thought in the course "Geometry and Linear Algebra" as well as the needs of our students.

Include key aspects of linear span	• Closure under the operations of a vector space • Every vector is a linear combination of the set of generators • No limitation in the choice of the set of generators
Tackle potential conflict factors	• Modes of representation (Hillel, 2000) • The difference between linear combination and linear dependence
Promote theoretical thinking (Sierpinska, 2000)	• Reflection • Connections between different representations • Attention to contradictory thoughts
Concreteness principle (Harel, 2000)	• Include familiar concepts • Connection with prior knowledge and language
Necessity principle (Harel, 2000)	• Justifying the need of linear span • Root problems
Generalizability principle (Harel, 2000)	
Promote discussion	

4.2 The tasks

This work proposes a sequence of tasks that could be included in the problem workshops of an introductory university course on Linear Algebra, such as the course "Geometry and Linear Algebra". The sequence is inquiry-based and attempts to enrich the concept image of students regarding the concept of span. In this section, we present the tasks giving details about the structure and their purpose. Each task is carefully created and/or selected so as to foster the ground for potential conflict factors to arise and be confronted.

4.2.1 Task 1

The first task is based on an exercise from the book "Linear Algebra: Concepts and Methods" by Antony and Harvey (2012). The syntax and notation was slightly altered to fit that of the course notes (Kourouniotis, 2014).We also included some extra details for guidance or reflection. This task can be classified as an apprentice task (Burkhardt & Swan, 2013), it aims to provide connections with prior knowledge, known processes and language under the new context and introduce to students basic ideas linked with the concept. The context of this task is considered appropriate for students to make connections between algebraic and geometric representations of the notion of span. At the same time, it contains prompting for reflection that is expected to promote discussion. Through this task, students can examine the effect that different choices of vectors have on the outcome. The task is presented below together with more details about each sub-question.

Δίνονται τα διανύσματα:

$$v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, w_1 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, w_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

i. Δείξτε ότι το w_1μπορεί να εκφραστεί σαν γραμμικός συνδυασμός των v_1και v_2, αλλά τοw_2δεν μπορεί να εκφραστεί σαν γραμμικός συνδυασμός των v_1και v_2.
ii. Εξηγήστε ποιος είναι ο υπόχωρος του $\mathbb{R}^3$που παράγεται από τα v_1, v_2 και w_1.
Εξηγήστε ποιος είναι ο υπόχωρος του $\mathbb{R}^3$που παράγεται από τα v_1, v_2 και w_2.
Τι παρατηρείτε;

iii. Δείξτε ότι τα διανύσματα v_1, v_2, w_1 και w_2 παράγουν τον $\mathbb{R}^3$, δηλαδή ότι για κάθε u=(x,y,z) υπάρχουν a, b, c, d τέτοια ώστε:

$$u = av_1 + bv_2 + cw_1 + dw_2$$

Ακόμα δείξτε ότι κάθε διάνυσμα $u \in \mathbb{R}^3$ μπορεί να εκφραστεί ως γραμμικός συνδυασμός των v_1, v_2, w_1 και w_2 με άπειρους τρόπους.

Translation: *Consider the vectors:*

$$v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, w_1 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, w_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

i. *Show that w_1 can be expressed as a linear combination of v_1 and v_2, but w_2 cannot be expressed as a linear combination of v_1 and v_2.*

ii. *Explain what subspace of $\mathbb{R}^3$ is spanned by v_1, v_2 and w_1.*
Explain what subspace of $\mathbb{R}^3$ is spanned by v_1, v_2 and w_2.
What do you observe?

iii. *Show that the vectors v_1, v_2, w_1 and w_2 span $\mathbb{R}^3$, that is for every u=(x, y, z) there are a, b, c, d such that:*

$$u = av_1 + bv_2 + cw_1 + dw_2$$

Show also that every vector $u \in \mathbb{R}^3$ can be expressed as a linear combination of v_1, v_2, w_1 and w_2 in infinitely many ways.

The task is divided into three interconnected subtasks as a scaffolding strategy to support students. The first subtask is referring to the notion of linear combination. At this point, students are expected to have some knowledge on how to determine if a vector is a linear combination of some other given vectors. It is an introductory task aiming to guide students to the right direction in subtask (ii). This subdivision may also help to concentrate on theoretical thinking in the following subtask by limiting its focus on calculations.

Subtask (ii) is chosen in an attempt to enrich students' image of linear span and make connections between the algebraic and geometrical representations of the concept in

$\mathbb{R}^3$.The task is designed to promote the "no limitation" aspect in the notion of span by paying attention to contradictory thoughts. Here students are anticipated to discuss the effects of linear dependence and independence of the set of generators on the notion of span. By the way the course is structured students have already seen at that point that the subspaces of $\mathbb{R}^3$ are the entire space, planes and lines that pass through the origin, and the zero subspace. Therefore, students are expected to be led to assume that the linear span of the given vector is one of those subspaces (in the first case a plane and in the second the whole space). The proposition "what do you observe?" is added as an encouragement for reflection and generalization of the idea for arbitrary vectors in $\mathbb{R}^3$. Possible outcomes of that are to make connections between linear span and linear dependence and to enrich the images about different representations of geometrical objects. This subtask could also help students create links between linear span and the concept of dimension that follows. Moreover, this subtask may motivate students to seek a deeper connection between Analytic Geometry and Linear Algebra.

The third subtask is the most instructional among them because it gives specific outlines for the appropriate procedure. Its goal is to give an example on how to determine if a set of vectors spans a given vector space using Gaussian elimination. Moreover, it aims to create a link between the relation of the given vectors and the number of ways arbitrary vectors can be expressed as linear combination of the elements in the set. Originally, it was not intended to include the part "that is for every u=(x, y, z) there are a, b, c, d such that: $\mathrm{u} = \mathrm{av}_1 + \mathrm{bv}_2 + \mathrm{cw}_1 + \mathrm{dw}_2$" in the text. We decided to add that partin an attempt to help students pick a suitable approach for two main reasons. The first reason is that it is possible for someone to answer the first part of subtask (iii) by stating that since v_1, v_2 and w_1 span $\mathbb{R}^3$ and $w_2 \in \mathbb{R}^3$ all four of them also span $\mathbb{R}^3$. This approach will not allow students to give an explanation of why any vector in $\mathbb{R}^3$ can be expressed as a linear combination of those four vectors in infinitely many ways. Even if the students determine that the vectors are linearly dependent through Gaussian Elimination, they are not very keen on "reading through" the process, which may be an obstacle. For a trained eye the process of Gaussian Elimination may reveal much more information about the set of vectors, such as the existence of particular linear combinations. At this stage students are not aware of this possibility unless the matrix is arranged in a certain way associated with finding a linear combination. The second reason is that we had previously witnessed students struggle with similar problems that require the use of arbitrary vectors.

4.2.2 Task 2

In the previous chapter we concluded that potential conflict factors in the image of linear span may be the relationship between linear combination and linear dependence in the context of linear span. The second task was created as an attempt to urge these potential conflict factors to stand out through the generation of an example. The idea for this task was based on our goal to promote theoretical thinking and discussion.

Before deciding on this task, we considered adding a procedural task that would include examples where the idea of linear dependence is not sufficient to give a successful answer, but for that to be achieved effectively it would be necessary to use examples not only in $\mathbb{R}^3$but also in higher dimensions. We believe that this idea might seem obvious in $\mathbb{R}^3$ and therefore it might be ignored by the students. In addition, by working in higher dimensions students do not have the opportunity to use different representations. Therefore, this approach was abandoned because we thought it wouldn't meet the principles of our design. We acknowledge that such tasks are also important in the process of learning and we encourage anyone who would like to use this sequence in his/her course to include an exercise that deals with this idea in a rote way as a ground for the task we are about to present.

After consideration, we finally created the following task:

Έστω v_1, v_2 και w γραμμικά εξαρτημένα διανύσματα στον $\mathbb{R}^3$. Το w είναι δυνατόν να μην ανήκει στον χώρο που παράγεται από τα v_1και v_2 παρόλο που τα v_1, v_2 και w είναι γραμμικά εξαρτημένα. Δώστε ένα παράδειγμα. Γιατί πιστεύετε ότι συμβαίνει αυτό;

Translation: *Let v_1, v_2 and w be linearly dependent vectors in $\mathbb{R}^3$. It is possible for w not to be in the space spanned by v_1 and v_2 although v_1, v_2 and w are linearly dependent. Give an example. Why do you think this can happen?*

The task is presented as a challenge. The conflict is given to the student as a statement and the goal is to find an example to support the given proposition. It is expected that students will first use a trial and error approach by reaching for appropriate vectors in their example space (Mason & Watson, 2008). This approach will probably fail if students are not able to identify what are the key relations between v_1, v_2 and w in the proposition. Moreover this task promotes all three key aspects of the concept. Closure is reflected in the fact that the linear span of v_1 and v_2 must be one of the known subspaces of$\mathbb{R}^3$. Because w is not in the span of v_1 and v_2, it cannot be expressed as a linear combination of them, finally the "no limitation" aspect is the most important. If one's concept image includes conflicting ideas about the status of vectors in a set of generators, it might be difficult to find anexample without careful prompting and discussion.

One can give an answer using either a geometric or an algebraic approach to linear dependence and linear combination in the context. We presume that this task is going to give opportunities for theoretical thinking and discussion. Because of the nature of the problem, students would want to cross-examine their findings or get some guidance. In the end, students are again asked to reflect upon their findings and make generalizations. This question was added to ensure that students will give more attention to the purpose of the task.

4.2.3 Task 3

The final task is a set of true or false questions. In this task students are expected to reflect on what they have learnt about the concept through the previous tasks. It is designed to promote theoretical thinking and reflection upon the notion of linear span and especially on the aspects included in task 1 and task 2. The propositions are chosen in such a way as to draw the attention to important information about the concept included in the previous tasks. Namely, each proposition examines the outcome of different relations between the set of generators, these relations are implied in the previous tasks. This task could also give space for more potential conflict factors to arise and be discussed. The task is formed as follows:

Χαρακτηρίστε τις παρακάτω προτάσεις με Σωστό ή Λάθος:

i. Εάν τα v_1, v_2, v_3, v_4 παράγουν τον $\mathbb{R}^3$ τότε και τα v_1, v_2, v_3 παράγουν τον $\mathbb{R}^3$.
ii. Εάν τα v και w είναι γραμμικά ανεξάρτητα και το σύνολο {v, w, z} είναι γραμμικά εξαρτημένο, τότε το z ανήκει στον χώρο που παράγεται από τα v και w.
iii. Εάν το σύνολο των διανυσμάτων $S = \{v_1, v_2, \ldots, v_k\}$ παράγει τον V, και $w \in V$, το σύνολο $T = \{v_1, v_2, \ldots, v_k, w\}$ επίσης παράγει τον V.

Translation: *Describe the following propositions as True or False:*

i. If v_1, v_2, v_3, v_4 span $\mathbb{R}^3$ then v_1, v_2, v_3 span $\mathbb{R}^3$.
ii. If v and w are linearly independent and the set {v, w, z} is linearly dependent, z is in the space spanned by v and w.
iii. If the set of vectors $S = \{v_1, v_2, \ldots, v_k\}$ spans V, and $w \in V$, the set $T = \{v_1, v_2, \ldots, v_k, w\}$ also spans V.

The first proposition questions when three vectors span $\mathbb{R}^3$. Students have already seen that any vector in $\mathbb{R}^3$ can be expressed as a linear combination of any three linearly independent vectors. The most important aspect one can see here is that there is no limitation in the choice of vectors. Students are expected to make a connection between these facts and identify the difference. Namely that there is no information about the linear dependence of the vectors v_1, v_2 and v_3 and thus the statement is false. One good example of that can be found in task 1. Supposing that the four initial vectors are the ones given in task 1, students will have already seen the different outcomes.

The second proposition is closely connected to task 2. Here students are presented with a similar idea as that of task 2 but with a very different outcome. This is expected to provide a second chance for discussion and further clarification. Finally, the third proposition examines the understanding of all three key aspects of the definition. The basic idea that this proposition implies is that it is possible for a linear subspace to be spanned by different sets of vectors under the conditions. This idea links the notion of linear span with the concept of basis. Moreover, it will be useful especially in the course “Linear Algebra I” where they will encounter that idea in more detail. The students may link this proposition with task 1 where they engaged with two different sets of vectors that span $\mathbb{R}^3$. Also, this observation may lead to further reflection upon the relation and the results of the different set.

4.3 Conclusion

In this chapter we presented three tasks designed to help in the development of students' concept images about the notion of linear span. The tasks are designed so that they can give space for potential conflict factors to be confronted. In addition, the tasks are not meant as homework, we believe that these tasks work best in circumstances where students can discuss their progress and their difficulties with each other. Finally, we hope that these tasks will provide inspiration for lecturers and tutors to create their own tasks in a similar fashion for more concepts in Linear Algebra.

CHAPTER 5

Task Evaluation

In this chapter we present the results of a preliminary evaluation of the tasks in a group of 7 students. The students were asked to review the tasks and give us their opinion on them. The first part includes information about the method of the evaluation, the participants and other important information about the process. The second part of this chapter is the analysis of the data collected in order to assess and improve the tasks.

5.1 Methodology

5.1.1 Purpose and Questions

> *"Expecting specific tasks to have given learning outcomes is highly unrealistic, even if you include in the task specific stimuli for reflection…"*
>
> *(Mason, 2002: 130)*

Following the creation of the tasks, our greater concern was to receive feedback. The first phase of the design (creating the sequence) was completed half way through the spring semester and due to time constraints it was impossible to wait six months to put the tasks to test. Therefore, we decided to carry out a preliminary evaluation by giving the tasks to students for review. This way we could get an insight on how students might perceive them in a problem workshop and what can be improved about the design. Also, this could give opportunities to try different prompts that could be used or avoided in tutoring.

Our general research questions are:

- What students can tell us regarding the tasks?
- Does studying the concept of linear span through this sequence of tasks improve students' conceptual understanding?

More specifically:

- What is the students' reaction to the tasks?
- Do students find the tasks interesting and/or useful?
- Do the tasks promote discussion?
- Is the language used in the tasks clear?
- Are there potential difficulties with the tasks?
- Do students find the reflection useful?
- In what extent could the tasks help with conceptual understanding?

We were particularly interested in students' reactions at the time since the tasks are meant to be used in situations where they are encouraged to work and discuss their findings not only with tutors but mostly with each other. Therefore, it was important for the tasks to be appealing to students.

5.1.2 The method

To meet our purpose, interviews were carried out with students who had attended the course "Geometry and Linear Algebra" the previous semester. This choice was made in the hope that the students would reflect on their prior experience, feelings and behaviours during the course. Moreover, at the time of the interviews the majority of the selected students were attending the course "Linear Algebra I"; this fact might have helped students to make links between the tasks and aspects of the concept of linear span encountered later.

For the selection of the participants we contacted students who had answered the test question analyzed in the preliminary research in Chapter 4. This approach was chosen in order to have some control over the selection by putting two constraints. The first one was to select at least one participant from each of the categories LinD, LinC, LinIn and Other, as indicated in the preliminary research. The reasoning behind this constraint was to ensure that our sample wasn't biased. Apart from that, we don't wish to make any cross comparisons to their test responses and the interviews. The second constraint was that the participants should represent both degree orientations, namely Mathematics and Applied Mathematics, to ensure diversity among students' opinions. The initial number of students that agreed to participate was eight. However, one of the students could not follow through with the interview for personal reasons. As it was not possible to replace the 8th student, due to time constraints, only seven interviews were carried out.

The participants were first year or older students following the degree orientation in Mathematics or Applied Mathematics. Five of the participants are studying for the degree in Mathematics and two for the degree in Applied Mathematics. To be more precise, there were five female participants (three of them were first year students) and two male students (one in the first and the other in the second year of studies). The following table summarizes the information about the seven participants.

	Mathematics			**Applied Mathematics**	
	1st Year	*2nd Year*	*3rd Year*	*1st Year*	*2nd Year*
Male	0	1	0	1	0
Female	3	0	1	0	1

All interviews were videotaped with the signed agreement of the participants. To ensure confidentiality each student was assigned and referred to with an alias (Ariadne, Minos, Artemis, Helen, Pasiphae, Hector and Andromeda) during data analysis and presentation of the results. The processes of interviewing and analysis of the data follow the restrictions of the Code of Ethics of the University of Crete (2002).

The interviews were semi structured. This type of interview is based on a predetermined general plan that allows comparison between the data and at the same time leaves space for the interviewer to investigate any particular point that may come up during the process (Bogdan & Biklen 2006; Cohen et al, 2011). The plan typically includes two or three ice-breaking questions, up to eight key questions paired with transitive questions and finally an overview of the most important parts of the interview (Johnson & Rowland, 2012 as cited in

Mann 2016). This type of interview was selected so as to help us in the coding process and the quantification of the results. Prior to the interviews, each participant was given a folder[5]. The folder included the tasks, information about the study and instructions about the process. Also, each folder contained an excerpt of the course notes (Kourouniotis, 2014). The students were given one week to attempt and review the tasks before the interviews.

The students were informed that if they had any problem with solving the tasks they would be able to discuss it with the interviewer, similarly to what they would do in the context of a problem workshop. We note that in problem workshops students are encouraged to work in groups. In contrast, the students were interviewed individually. Although the process might differ from what one encounters in a problem workshop, our focus at this time was more towards students' personal opinion about the tasks rather than observing how the tasks can work in peer groups.

The data collected from the interviews (videos and students' notes) were analyzed using the qualitative data analysis software ATLAS.ti. We tried to apply consistent codes to all seven interviews in order to compare and quantify the results where that was possible. Also we identified some unexpected situations that occurred during some of the interviews. Those situations will be discussed separately at the end of the following section.

5.2 Data Analysis

In this section we present our interpretation of the data as it emerged from the coding process. First we give an overview of students' reaction to the tasks and then we discuss each task in detail. Finally, we attempt to analyze some extra information we obtained from some interviews. The results are quantified and presented in tables accompanied with quotations from the interviews.

[5]The folder as given to the students (in Greek) can be found in the Appendix.

5.2.1 Overview

To begin with first impressions, students seemed content with the tasks for a variety of reasons. Namely students referred to coherence between the tasks, inclusion of the basic aspects and a more theoretical approach. We briefly quote fragments from participants to indicate different reasons why they found them useful.

Ariadne: […] τα συνδέεις όλα στο μυαλό σου, αυτό… που σου έλεγα είναι… υπάρχει μία αλληλουχία.

Minos: […] απ' όλα είχε, τα βασικά νομίζω… και παραπάνω από τα βασικά.

Artemis: […] Μου άρεσαν, να σου πω γιατί. Ήταν κάπως πιο θεωρητικές […] βοηθάει πιστεύω γιατί όταν είναι πιο θεωρητική άσκηση ο άλλος σκέφτεται πιο πολλά […]

Translation:

Ariadne: […] they link everything in your mind, that… I told you… there is ατα συνδέεις όλα στο μυαλό σου, αυτό… που σου έλεγα είναι… υπάρχει μία coherence.

Minos: […] They include a bit of everything, the basics I think… and more than the basics.

Artemis: […] I enjoyed them, let me tell you why. They were a little bit more theoretical […] this is helpful, I think, because in theoretical assessments one thinks more.

All the students seemed satisfied with the tasks for one or more of these reasons. Most of them indicated that the tasks depict all the basic aspects of the notion. The following table quantifies this observation.

Coherence	Inclusion	Approach
1	5	2

This interpretation, although pleasing, does not give particular information about the tasks. Next, we discuss in more detail results from each task separately.

5.2.2 Task 1

The table below shows the number of students who completed each subtask of Task 1 without additional help from the interviewer. Subtask (i) was completed without difficulty or errors from all the students participating in the inquiry. Five of the seven students completed subtask (ii) and only two of them subtask (iii).

Task 1i	Task 1ii	Task 1iii
7	5	2

The students who did not complete (ii) appeared to have trouble with methodology. In both cases the problem was resolved through discussion in reasonable amount of time and the two students managed to complete the subtask without further problems. Apart from that, six out of the seven students found the question *"what do you observe?"* useful. In particular, three of them indicated that they might not have given a second thought to their result if it wasn't for this question. On the other hand one of the students, Pasiphae, found the question stressful. She had successfully answered the question but she seemed genuinely worried because she thought that her observation was not the right one. She said:

Pasiphae: Εμένα σαν ερώτηση με αγχώνει… προσωπικά. Νομίζω ότι θέλει πάντα απάντηση και… αυτό, δεν το ΄χουν όλοι με το να αναλύ[σουν]… να γράψουν αναλυτικά. Ας πούμε, εγώ αυτό που σκέφτηκα δεν ξέρω καν κατά πόσο βασίζεται σε… επίπεδο απόδειξης, δηλαδή… το είχα δει σε ασκήσεις παλαιότερα, αλλά δεν ξέρω κατά πόσο είναι απλά μια σκέψη ή μπορεί όντως να αποδειχθεί[…]

<u>Translation:</u>

Pasiphae: As a question it makes me feel anxious ... personally. I think it always need an answer and ... this, it is not easy for everyone to analy[ze] ... write in detail. Let's say, I didn't even know if what I thought was based on ... a level of proof, that is ... I had seen it in exercises in the past, but I do not know whether it is just a thought or it can actually be proved [...]

Although none of the other students felt the same, the thoughts of this student are significant to us. The student seems confused about the role of reflection. Perhaps she is used to more instrumental tasks from her prior experience. We have unofficially observed related thoughts from students in problem workshops in the past. Math anxiety is a factor that we need to take into account. Clute (1984) made the observation that college students' achievements are related to mathematical anxiety. Moreover, her research showed that students with higher anxiety levels can benefit more from instrumental approaches whereas students with low anxiety from relational approaches. Lazarus (1974) believed that the source of mathematics anxiety is in secondary school education, where the problem is less obvious because many students are able to memorize formulas and rules for short periods of time, until they are tested on them. Relational questions are not frequent in secondary school education therefore it is reasonable to assume that some students would have difficulty (and in some cases anxiety) answering such questions.

Further observation of subtask (ii) revealed that the fact that the vectors v_1 and v_2 are linearly independent was overlooked during the interviews and possibly in the process of the design. This fact might seem obvious to a trained eye; students should not be expected to make this observation by themselves. During the interviews, there was no significant discussion about that, in some cases there was just a brief reference to the fact but without further discussion we cannot be sure that the assumption that the vectors v_1 and v_2 are linearly independent was because students observed this from the data or they assume it is true because w_1 is a linear combination of these two vectors. The latter is directly related to the conflict identified in Chapter 4 and if overlooked it might create further confusion to the students in a problem workshop environment.

The final subtask of Task 1 was the one completed by the least number of students. They mostly struggled with the second part of the question. The first part of the question can be answered following the same reasoning used in subtask (ii), but this will not help answering the second part which requires from students to solve a system of linear equations. Four of the five students that didn't complete (iii), tried to solve it using the same approach as in (ii). In each case the task was completed with the help of the interviewer but we find that subtask (iii) required more instruction from the part of the interviewer compared to subtask (ii). The fifth student managed to solve the required linear system but she could not make a connection between the infinite number of solutions and the fact that the four vectors are more than enough to describe any vector in $\mathbb{R}^3$. Instead she used a known proposition which justifies the statement that the four vectors span $\mathbb{R}^3$ but does not explain why any vector in $\mathbb{R}^3$ can be written as a linear combination of v_1, v_2, w_1 and w_2 in infinitely many ways. Her approach may imply that her understanding of the situation is for the most part procedural.

Before the interviews we made some speculations why subtask (iii) might be difficult for students. We identified two possible reasons, the first one being the complex structure of the question and the second might be the difficulty of some students to identify the random vector u = (x, y, z) as a parameter of the problem and not as a variable. Three of the five students faced a difficulty making use of the proposition *"for every u=(x, y, z) there are a, b, c, d such that* $u = av_1 + bv_2 + cw_1 + dw_2$*"*. In an attempt to resolve this problem, the interviewer suggested to one of these students to try and solve the problem with any particular vector *u* she chose. This approach helped the student, as indicated by her to answer the original question.

Helen: Ναι νομίζω είναι πιο κατανοητό […] όταν θα το έχεις τελειώσει θα έχεις καταλάβει κάτι παραπάνω.

<u>Translation:</u>

Helen: Yes I think is more conceivable […] when it's all over one will have understood something more.

Another student who solved the system of linear equation by using an abstract vector u=(x,y,z) but had trouble connecting her solution to the required answer also indicated that a similar approach could help students who have difficulties:

Interviewer: Εε, πιστεύεις ότι αν είχε ένα ερώτημα το οποίο να έλεγε ότι ένα συγκεκριμένο διάνυσμα u γράφεται σαν γραμμικός συνδυασμός των υπολοίπων…

Pasiphae: Ναι… Ναι θα ήταν πιο εύκολο.

Interviewer: Και μετά ζητούσε να κάνεις αυτό (δείχνει το ερώτημα (iii)) που ουσιαστικά είναι μια γενίκευση… πιστεύεις ότι θα βοηθούσε ή θα το έκανε πιο κουραστικό;

Pasiphae: Όχι πιστεύω θα βοηθούσε, ακόμα και εγώ που θεωρώ τον εαυτό μου εξοικειωμένο με τα… x, y, z… ότι αν πρώτα είναι να αποδείξω κάτι συγκεκριμένο και μετά κάτι πιο γενικό συνήθως αυτό με βοηθάει στις ασκήσεις γενικότερα.

Translation:

Interviewer: Em, do you think that if there was a question saying that a particular vector u is written as a linear combination of the others...

Pasiphae: Yes ... It would have been easier.

Interviewer: And then asking you to do this (she points to subtask (iii)) which is essentially a generalization ... do you think it would help or make it more tiring?

Pasiphae: No I believe it would help, even for me that I consider myself familiar with ... x, y, z ... that if it is first to prove something specific and then something more general this usually helps me in the exercises in general.

Her response may also imply that Pasiphae would have managed to give a full answer if she had encountered the same idea in a specific example prior to this question. Even though these opinions cannot be generalized they might indicate some improvements for the task and a potentially good approach for tutoring this task in the future. Moreover, three of the students when asked if they had any problem with u being a random vector said that they had none because they imagine it as a particular vector in the process. This is an approach that is expected to be acquired through the course “Geometry and Linear Algebra”.

5.2.3 Task 2

Task 2 is the central task of the sequence, designed as an attempt to provide an opportunity to create a more coherent concept image of the notion of span and resolve the cognitive conflict identified in the preliminary research. Only one student had found an example of three vectors fulfilling the requirements of the task before the interview. It was expected that some students might experience some difficulties with the task due to the nature of the task and the possibility their concept images were in conflict with what the task is proposing. Three of the students said that they felt a little nervous when they couldn’t find an example but later concluded that if they had seen such a task in problem workshops where they would have had help from others they would not have had a problem.

Perhaps the most important observation made by analyzing the discussions about task 2 was that in four of the seven cases conflicting images emerged. This reinforces our preliminary hypothesis that students struggle with identifying the difference between the notions of linear combination and linear dependence. Furthermore, it might be an indication that Task 2 can help potential conflict factors to emerge and be resolved in a controlled environment. The following quotations capture these observations.

Minos: Λοιπόν αυτό που σκέφτηκα είναι ότι μπορώ να έχω δύο διανύσματα… τα οποία θα είναι γραμμικά ανεξάρτητα που θα μου παράγουν ένα επίπεδο στον $\mathbb{R}^3$. Μπορώ βέβαια… είμαι σίγουρος ότι μπορώ να βρω άλλο ένα τρίτο διάνυσμα που δεν θα ανήκει σε αυτό το επίπεδο αλλά να ισχύει η σχέση…. Να είναι γραμμικά εξαρτημένα αυτά τα τρία διανύσματα.

Translation:

Minos: So, what I thought was that I can have two vectors ... which will be linearly independent that will span a plane in $\mathbb{R}^3$. I can of course ... I am sure that I can find another third vector that will not belong in the plane but the relationship to be true .... these three vectors to be linearly dependent.

Minos preferred to imagine the vector geometrically and was thinking of the subspace spanned from v_1 and v_2 as a plane, which requires the vectors to be linearly independent. Moreover, he appears certain that the vectors should be linearly independent. This might be an indication of an evoked concept image of a subspace spanned by two vectors as a plane.

Interviewer: Εεμ.. όταν λέω ότι κάποια διανύσματα παράγουν ένα χώρο σημαίνει ότι τα διανύσματα που ανήκουν σε αυτό το χώρο γράφονται σαν γραμμικός συνδυασμός τους [των πρώτων].

Pasiphae: Ναι … ναι…

Interviewer: Ωραία οπότε για να μην ανήκει το w στον χώρο που παράγουν τα άλλα δύο δεν θα πρέπει να γράφεται σαν γραμμικός συνδυασμός τους…

Pasiphae: Ναι… ναι… ωραία… Μα όμως πως θα είναι γραμμικά εξαρτημένα; Είναι όλα μεταξύ τους γραμμικά εξαρτημένα…

Translation:

Interviewer: Em.. When I say that some vectors span a space it means that the vectors belonging to that space can be expressed as a linear combination of those vectors [spanning the space].

Pasiphae: Yes … Yes…

Interviewer: Well, so for w not to belong in the span of the two other vectors it could not be written as a linear combination of them…

Pasiphae: Yes… yes… well… But then how can they be linearly dependent? They are all together linearly dependent…

In this fraction, the interviewer and Pasiphae are discussing about the basic ideas of Task 2. Pasiphae seems to struggle with the idea of three vectors being linearly dependent and at the same time one of them cannot be expressed as a linear combination of the others. She was probably thinking that the notions of linear combination and linear dependence are equivalent or that a set of generators is a linearly independent set.

Interviewer: Το ότι το w δεν ανήκει στο χώρο που παράγουν τα v_1 και v_2 τι σημαίνει;

Andromeda: Ότι θα είναι γραμμικά ανεξάρτητα. [σκέφτεται] Όχι….

Translation:

Interviewer: The fact that w does not belong to the vector space spanned by v_1 and v_2, what this means?

Andromeda: That they are linearly independent. [she thinks] No…

Andromeda's first thought of a vector not being in the span of the others was for them to be linearly independent. She then thinks about it and understands that there is something wrong with this assumption but she is not immediately aware of the conflict. We note that if three, or more, vectors are linearly independent then any one of them is not in the linear span of the others. The converse is not true.

Ariadne: Καταρχάς εμένα μου φάνηκε παράλογο την πρώτη φορά… γιατί τι μου λέει; Μου λέει ότι είναι γραμμικά εξαρτημένα οπότε αν λύσω ως προς το w θα βρω έναν γραμμικό συνδυασμό οπότε με βάση τη θεωρία ανήκει στον υπόχωρο που παράγουν το v_1 και το v_2.

<u>Translation:</u>

Ariadne: To begin with, to me it seemed absurd at first ... because… what does it tell me? It tells me that they are linearly dependent, so if I solve for w, I will find a linear combination, so based on the theory it belongs to the subspace spanned by v_1 and v_2.

Ariadne presents in detail her reasoning. Therefore, it is easy to observe the logical mistake. Later she also says that she thinks the notion of linear dependence and linear combination to be the same even though she successfully refers to the (personal) definitions for both concepts.

Particularly in this task, all six students reported that the discussion was very useful and task 2 is important for understanding the concept. Three of them said that this was the task that made them the biggest impression out of the three. Lastly four of the students, including the one who found the example without discussion suggested that it would be better if this task was presented to them in a problem workshop after a sequence of related more instrumental tasks.

This final suggestion can be related to the attitude of some students toward this kind of tasks. To be more precise, three of the students indicated that they are not very keen on tasks which require them to find examples. One of them gave a very detailed account about the reason why she does not feel convinced about examples.

Ariadne: Κοίτα σίγουρα είναι πιο εύκολο να βρεις παράδειγμα, αλλά θα σου πει ο άλλος… μα είναι ένα παράδειγμα… Οπότε, καλά άποψή μου βέβαια αυτό… δηλαδή εγώ όταν *βρίσκω μια ειδική περίπτωση… εε δεν αρκούμαι, λέω και αν για άλλα νούμερα ας πούμε βρω κάτι άλλο; Αυτό. Οπότε… καλά αυτό μου το έχουν κολλήσει από το λύκειο να σου πω την αλήθεια… οπότε θα το χω καλύτερα να το πάω με γενικό τρόπο.*

Translation:

Ariadne: Look, it's easier to find an example, but someone would tell you ... that it's only an example ... So, well that's my opinion of course, when I find a special case ... I am not convinced, I think can I find something else? This. So ... well, I've got stuck with this kind of thinking in high school to tell you the truth ... so I'd better see it in a general way.

The student is not necessarily aware that an example cannot always be exemplary. Her attitude is mostly connected to her experience with high school mathematics where most of the time students encounter exercises that require proofs and the examples are mostly given by the teacher or a textbook.

We are closing this section with some observations about the different approaches used in the interviews to discuss Task 2. These are based on the line of thinking of the students, but they are also influenced by the interviewer. Therefore, it is not certain that students would use the same approach in different circumstances. The following table shows the different approaches used in the six interviews where students discussed the task during the interview. In one case, task 2 was discussed and solved both algebraically and geometrically.

Algebraic approach	Geometric approach	Trial and error
4	2	1

In each interview the final example was found by the students by trying different numbers. The categorization above is made by the main approach of the notions during the discussion.

5.2.4 Task 3

The main purpose of Task 3 was to make students reflect upon what they had encountered about the concept of linear span in the previous tasks. The analysis of the interviews showed that some students made connections between Task 1, Task 2 and Task 3 in answering the true – false questions.

Task 3i	Task 3ii	Task 3iii
7	5	6

The table above illustrates the number of students that answered correctly to the true or false questions beforehand. Proposition (i) was successfully answered by everyone and only one student mentions having some difficulty with this proposition. The second proposition, related to Task 2, was the one answered successfully by the least number of students. The two students who gave the wrong answer easily corrected their answers by reflecting upon Task 2. Moreover, four of the seven students saw the relation between proposition (ii) and Task 2. Finally, only one student gave a wrong answer in proposition (iii) but he quickly corrected himself by making a connection with Task 1 after the indication of the interviewer that the initial answer was wrong.

Because of the nature of this task it did not give much information about students thinking. Ideally, these true of false questions are supposed to work as a mechanism to check if the students are able to reflect upon what they have learned about the notion of span and make generalizations. In this case the students' concept images were already rich with examples and experience from previous and current courses in Linear Algebra. Therefore it was not possible to determine if the propositions can produce reflection upon the specific tasks unless the students have stated otherwise.

5.2.5 Additional findings

Nested subspaces

During Task1 ii an unexpected discussion emerged with two of the students. These students interestingly answered that the span of the vectors v_1, v_2 and $w_1$is the vector space $\mathbb{R}^2$. The correct answer to this question is that the span is a plane in $\mathbb{R}^3$. The vector space $\mathbb{R}^2$ might be conceptualized as a 2-dimentional subspace of $\mathbb{R}^3$. This conflict factor is called by Wawro et al (2011: p. 13) as "nested subspaces". Warwo et al (2011) hypothesized based on their evidence that this confusion has roots in students thinking "some subspaces as "the same"" (p. 15) and suggested that lecturers must be aware of this as a potential conflict factor. Their hypothesis is confirmed in our case. We further hypothesize that this view of "sameness" might be linked with a mental image of 3-dimentional real space as three intersecting perpendicular planes. Ariadne tries to explain this image:

Ariadne: Αν το σκεφτείς ως εικόνα και ξεχάσεις λίγο τις πράξεις, έχεις στην ουσία… σαν να παίρνεις ότι ο ένας τοίχος είναι ένα επίπεδο, ο άλλος τοίχος είναι το άλλο επίπεδο και αυτός εδώ (δείχνει το πάτωμα) είναι το άλλο επίπεδο. Οπότε σκέφτεσαι ότι ο χώρος αποτελείται από τρία επίπεδα εντός εισαγωγικών. Οπότε είναι λογικό ένα επίπεδο να είναι υπόχωρος του χώρου. […]

Translation:

Ariadne: If you think of it as an image and forget about calculations for a moment, you really have ... as if you take this wall is a plane, the other wall is the other plane and here (showing the floor) is the other plane. So you think that the space consists of three planes in a manner of speaking. So it is reasonable for a plane to be submerged in space. [...]

She imagines 3-dimentional space as a "semi-open" room with the (x,y), (x,z), (y,z)- planes being the walls and the floor. This imagery is commonly used in high school to give a first more intuitive image of space but it possibly creates a conflicting image of space being the surface of these three planes as indicated by the student.

The interviewer prompted the students to think about their conflicting images by adopting an algebraic approach of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. We cannot claim that the conflicts where resolve by the discussion but the tactic of making comparisons between geometrical and algebraic vectors was successful in making this potential conflict factor arise. This observation is important because it gives information about potential conflict factors which we did not take into account initially but which might surface while solving these tasks.

Concept Image – Concept Definition

At the end of the interview each of the participants was asked a question regarding the use of definition and the use of examples in completing these Tasks. Five out of the seven students said that having access to the formal definition of linear span is useful in solving the task. Four of them reported looking at the formal definition while answering Task 3. The fifth student indicated that although she used what she already knew in solving the tasks she thinks that having the formal definition alongside the tasks would be useful in a problem workshop. The other two indicated that they prefer to have seen the formal definition and some examples beforehand and then work on problems. The following quotes depict their thoughts about the matter:

Ariadne: […] αν έχεις τον ορισμό μπροστά σου, βλέπεις τον ορισμό την λύνεις. Αυτό δεν σημαίνει όμως ότι τα χεις καταλάβει. Αυτό.

Translation:

Ariadne: […] if you have the definition in front of you, you look at it and solve the problem. That does not mean that you have understood it. That.

Ariadne feels that by accessing the definition one can solve an exercise without noticing particular elements of the exercise. Moreover when she was asked if she uses definitions in solving the tasks she immediately referred to her personal definition of the notion of linear dependence as an example, which reinforces the assumption that she prefers to work from her concept image.

Minos: […] Δεν χρειάζεται ντε και καλά να κοιτάς τον ορισμό. Αν τον έχεις στο μυαλό σου [και] έχεις δει και δύο ασκήσεις… οκ!

Translation:

Minos: [...] There is no need to look at the definition. If you have it in your mind [and] you have seen two exercises ... ok!

In Minos case it is apparent that he does not see a necessity of using the formal definition. He also indicates that he cannot remember a formal definition therefore he only tries to remember the basics. In this case, Minos consciously distinguishes the formal definition from his personal definition in contrast to Ariadne.

Also there were some students who talked about the importance of examples and tasks in understanding a concept and therefore the formation of the concept image. Four of the students reported that they understand a notion better through examples and tasks than just by studying the definition. The way that students' concept image is formed through model examples and experience, is of course well known. What is important here is the fact that the students are aware of this happening. Although they refer to it in their own words, based on their experience of studying, and not as a researcher of mathematics education might do, using some theoretical framework, it is clear that students are aware that they understand a notion from examples and exercises. In university, students are faced with an extreme number of new ideas in a short time compared to secondary school; this can force some of them to pay more attention to their learning processes and possibly to notice the ways in which they learn more effectively.

This last observation is an indication why it is important to pay attention to the examples and tasks used in any course. There are students who are consciously depending on them and expect to understand the "mysterious" concepts that the lecturer is talking about through them. Therefore, the lecturer has the obligation to choose carefully the examples and the tasks, if he/she wants to help students form a coherent concept image.

5.3 Results and Discussion

The analysis of the interviews gave us very important information about how tasks can be improved and used in a problem workshop for the course "Geometry and Linear Algebra". Although all students indicated that they find the tasks useful they gave us opportunities to reflect upon our design and experiment with different tactics which can be used by tutors in an attempt to make the most out of these tasks.

Beginning with the first task, students appeared to have particular difficulty in subtask (iii). One reason might be that (iii) requires a shift in thinking and cannot be fully answered by using the same approach as in subtask (ii). In an attempt to resolve this issue we are proposing a slightly different version of this part of the task that forces students to begin with the shifted approach as follows:

Δείξτε ότι για κάθε u=(x,y,z) υπάρχουν a, b, c, d τέτοια ώστε:

$$u = av_1 + bv_2 + cw_1 + dw_2$$

Ακόμα δείξτε ότι κάθε διάνυσμα $u \in \mathbb{R}^3$ μπορεί να εκφραστεί ως γραμμικός συνδυασμός των v_1, v_2, w_1 και w_2 με άπειρους τρόπους. Συμπεράνετε ότι τα v_1, v_2, w_1 και w_2 παράγουν τον $\mathbb{R}^3$.

<u>Translation:</u>

Show that for every u = (x, y, z) there exist a, b, c, d such that:

u = av1 + bv2 + cw1 + dw2

Conclude that v_1, v_2, w_1 and w_2 span $\mathbb{R}^3$. Moreover, show that every vector $u \in \mathbb{R}^3$ can be expressed as a linear combination of v_1, v_2, w_1 and w_2 in infinitely many ways.

It was also indicated by students that it might be easier to answer (iii) if they had first seen a similar task with a known vector u. We decided not to include an extra step in this task because we believe that it could make it appear difficult or boring. Also this task is designed to be included in a longer sequence of tasks that would include similar numerical problems. We encourage tutors to try prompting the students solving this task using the same tactic as in the interviews.

Tactic: Encourage students who struggle with subtask (iii) to choose a specific vector to prove the result first. This could help making apparent that the same is true for any vector they choose.

One more observation we made by discussing task 1 with students was that of "nested subspaces". This is another conflict factor we didn't take into account at first and realized it only when interviewing the students. Our observation is in line with the hypothesis of Warwo et al (2011). Moreover, we propose that this misconception might be linked with a mental image of $\mathbb{R}^3$ as three intersecting perpendicular planes. This potential conflict factor should be taken into account by the tutors.

Task 2 was fruitful both in terms of meaningful discussion and reflection. Students found Task 2 important for understanding the concept of span and successfully reflected upon it when thinking about proposition (ii) in Task 3. We also observed manifestations of cognitive conflict which indicates that the task can be used as a means to resolve potential conflict factors. Different approaches can be used to discuss these conflicts with students (algebraically, geometrically or by trial and error). A useful tactic might be to discuss the conflicting factors using more than one representation of vectors with the same group of students.

Tactic: Propose to students that there is more than one way to think about the problem and encourage them to discuss it from different standpoints.

The third task did not give as much information as the others, partially because the participants were more experienced than a student who encounters the concept for the first time. Nonetheless, there were some occasions where students reflected upon the previous tasks to support their answer. Either way, students indicated that they used either reflection upon prior knowledge or reviewing the definition to answer Task 3.

Finally, the indications about the need of examples and tasks in achieving conceptual understanding, was of great importance. This fact depicts the necessity of well thought examples and tasks in order to help students create a coherent concept image.

CHAPTER 6

Conclusion

This work focuses on enriching the understanding of the notion of linear span through a set of inquiry based tasks. The tasks are designed for first year Mathematics students. A starting point for the design was a study of the written answers given by students of the “Geometry and Linear Algebra” course, in response to a question in the final examination for the course. The question asked them to determine whether a vector belongs to the subspace spanned by two other vectors. In answering the question, students had to use their insight on how to apply the concept of linear span in problem solving. Analyzing their answers we studied differences and similarities in students evoked concept image triggered by this particular question.

The findings of this study led us to believe that some students may have the misconception that in a linearly dependent set of vectors, every vector can be expressed as a linear combination of the others. This misconception was found to affect students’ understanding of linear span and to be a potential conflict factor. Examining the notion of linear span may offer the opportunity to confront such difficulties in a meaningful way. A vector belongs to a given linear span if and only if the vector can be expressed as a linear combination of the vectors in the spanning set. On the other hand, linear dependence of the spanning set together with an extra vector is not sufficient to conclude that this vector is in the span. Therefore we value this observation and include it in the design and initial evaluation the tasks.

The designing process began by developing a design framework that would include chances to promote conceptual understanding and discussion. Concept image – concept definition (Tall & Vinner, 1981) is a well established (EMS, 2014) theoretical structure that was used to study students’ understanding of the concept and reflect upon different aspects of teaching the notion. The design principles adopted were an amalgam of the use of this theoretical structure and findings regarding teaching and learning Linear Algebra (Sierpinska, 2000; Harel, 2000). Based on them we developed three tasks that we believe will aid in the understanding of the concept of linear span.

The tasks were created and/or chosen to reflect the different aspects of linear span that are of value in an introductory course. The first task is structured in such way as to lead students to create links between the geometrical representation of the concept and a more algebraic approach. The second task was developed to confront the potential conflict factors that were identified in the preliminary study. Finally the third task was designed with the aim to aid students make generalizations based on the other two tasks.

A preliminary evaluation of the tasks was based on interviews with seven students. The analysis of the answers of the students to the tasks and the comments about their attempts during the interviews, gave the opportunity to identify details we overlooked in the initial task design. Moreover, it suggested ways to improve the tasks before introducing them to students in a problem workshop.

During this initial evaluation we made some further observations. Firstly, through the interviews we had the chance to analyze further the misconception identified in the preliminary study regarding the notions of linear combination and linear dependence. Students' reaction to Task 2 is an indication that this particular misconception can be confronted and possibly resolved through this task. Secondly, we observed the misconception of "nested subspaces", noted by Warwo et al. (2011). Finally, we propose a tactic that may be used in a problem workshop to assist students' learning without giving them an answer. These tactics were a product of reflection on the discussions with students. More general tactics can be found in Mason (2002).

Overall, the approach used for task design and evaluation of the task was partially based on and inspired by the writer's involvement in problem workshops in the Department. Therefore, it combines aspects of the research in Mathematics Education and observations through her experience in the Department. If educators wish to use the design or the tasks, they may need to adapt them depending on the different circumstances and curriculum of their Department.

In conclusion, a big limitation factor was time. It lasted only six months, therefore the time of development and the evaluation of the tasks were affected. In the future the tasks should be tested further. Receiving feedback from lecturers would be beneficial for further development of the tasks. In addition, we suggest conducting interviews with a focus group of 5 to 6 students to study the effects of the task in groups where students may discuss their difficulties and findings. This would allow an in depth analysis of the results of the task in situations similar to those of a problem workshop. Moreover, the tasks must be tested in a problem workshop and be compared to other contiguous tasks. The analysis of the results would provide us with information regarding the proper use and inclusion of the task in the workshops. Even further, it would allow a more accurate quantification of students' achievement in the tasks. Finally and most importantly, the effects of the tasks should be evaluated in the long run. Therefore, studying the understanding of students regarding the notion of linear span after the introduction of the tasks is crucial.

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Appendix A

Solutions to the Tasks

<u>*Task 1*</u>

Consider the vectors:

$$v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, w_1 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, w_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

i. *Show that w_1 can be expressed as a linear combination of v_1 and v_2, but w_2 cannot be expressed as a linear combination of v_1 and v_2.*

ii. *Explain what subspace of $\mathbb{R}^3$ is spanned by v_1, v_2 and w_1.*
Explain what subspace of $\mathbb{R}^3$ is spanned by v_1, v_2 and w_2.
What do you observe?

iii. *Show that the vectors v_1, v_2, w_1 and w_2 span $\mathbb{R}^3$, that is for every $u=(x, y, z)$ there are a, b, c, d such that:*

$$u = av_1 + bv_2 + cw_1 + dw_2$$

Show also that every vector $u \in \mathbb{R}^3$ can be expressed as a linear combination of v_1, v_2, w_1 and w_2 in infinitely many ways.

<u>*Solution:*</u>

i. *There should be a solution to the system:*

$$\begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \textit{(1)}$$

The solution is $a = 2$, $b = 1$. Therefore, $\boldsymbol{w_1 = 2v_1 + v_2}$.

There should <u>not</u> be a solution to the system:

$$\begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Or equivalently, the vectors v_1, v_2, w_2 should be linearly independent.

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} \mathbf{-1} & 1 & -1 \\ 0 & \mathbf{2} & 2 \\ 0 & 0 & \mathbf{2} \end{bmatrix}$$

The vectors v_1, v_2, w_2 are linearly independent. Therefore w_2 cannot be expressed as a linear combination of v_1 and v_2.

ii. $V_1 = \{u \in \mathbb{R}^3 | u = a_1v_1 + a_2v_2 + a_3w_1,\ a_1, a_2, a_3 \in \mathbb{R}^3\}$. *From (1) we know that v_1 and v_2 are linearly independent and* $w_1 = 2v_1 + v_2$. *Therefore,*

$$V_1 = \{u \in \mathbb{R}^3 | u = (a_1 - 2)v_1 + (a_2 - 1)v_2, a_1, a_2 \in \mathbb{R}^3\}$$
$$= \{u \in \mathbb{R}^3 | u = sv_1 + tv_2, s, t \in \mathbb{R}^3\}$$

The geometrical representation of V_1 is a plane in $\mathbb{R}^3$.

$V_2 = \{u \in \mathbb{R}^3 | u = a_1v_1 + a_2v_2 + a_3w_3\}$. *We know that the vectors v_1, v_2, w_2 are linearly independent. Therefore,* ***the geometrical representation of V_2 is the space $\mathbb{R}^3$***

We observe that a set of three vectors is possible to span vector spaces of different dimensions.

iii. Let $u = (x, y, z)$ *a vector in $\mathbb{R}^3$.*

$$u = av_1 + bv_2 + cw_1 + dw_2$$

Equivalently,

$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 2 & 2 \\ 1 & 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} -1 & 1 & -1 & 1 & x \\ 0 & 2 & 2 & 2 & y \\ 1 & 3 & 5 & 5 & z \end{array}\right] \sim \left[\begin{array}{cccc|c} -1 & 1 & -1 & 1 & x \\ 0 & 2 & 2 & 2 & y \\ 0 & 4 & 4 & 6 & z + x \end{array}\right] \sim \left[\begin{array}{cccc|c} \mathbf{-1} & 1 & -1 & 1 & x \\ 0 & \mathbf{2} & 2 & 2 & y \\ 0 & 0 & 0 & \mathbf{2} & z + x - 2y \end{array}\right]$$

There are infinitely many solutions to the system. ***There for every vector in $\mathbb{R}^3$ can be expressed as a linear combination of v_1, v_2, w_1 and w_2*** *in infinitely many ways.* ***The set $\{v_1, v_2, w_1, w_2\}$ spans $\mathbb{R}^3$.***

<u>Task 2</u>

Let v_1, v_2 and w be linearly dependent vectors in $\mathbb{R}^3$. It is possible for w not to be in the space spanned by v_1 and v_2 although v_1, v_2 and w are linearly dependent. Give an example. Why do you think this can happen?

<u>Solution:</u>

There should be vectors v_1, v_2, w such that v_1, v_2 and w are linearly dependent but w not to be expressed as a linear combination of v_1 and v_2.

In order this to be true. The vectors v_1 and v_2 should be linearly dependent (they span a line in $\mathbb{R}^3$. For example $\boldsymbol{v_1} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}$ and $\boldsymbol{v_2} = \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \\ \mathbf{0} \end{bmatrix}$.Then for any choice of w the set {v_1, v_2, w} is linearly dependent. For w not to be expressed as a linear combination of v_1 and v_2 one should choose a vector w that does not belong on the line spanned by v_1 and v_2 . For example $\boldsymbol{w} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix}$.

Task 3

Describe the following propositions as True or False:

i. If v_1, v_2, v_3, v_4 span $\mathbb{R}^3$ then v_1, v_2, v_3 span $\mathbb{R}^3$.

ii. If v and w are linearly independent and the set {v, w, z} is linearly dependent, z is in the space spanned by v and w.

iii. If the set of vectors S = {v_1, v_2, …,v_k} spans V, and w ∈ V, the set T = {v_1, v_2, …, v_k, w} also spans V.

Solution:

*i. **False** (For example in task 1 the set task 1 the set {v_1, v_2, w_1, w_2} spans $\mathbb{R}^3$ but the set {v_1, v_2, w_1} does not).*

*ii. **True** (The only way for v and w to be linearly independent is for w to be a linear combination of them).*

*iii. **True** ($w \in V \Leftrightarrow v = b_1v_1 + b_2v_2 + \cdots + ba_kv_k$*

$< T >= \{v \in \mathbb{R}^3 | v = a_1v_1 + a_2v_2 + \cdots + a_kv_k + dw\} =$
$\{v \in \mathbb{R}^3 | v = (a_1 + db_1)v_1 + (a_2 + db_2)v_2 + \cdots + (a_k + db_k)v_k\} =$
$\{v \in \mathbb{R}^3 | v = c_1v_1 + c_2v_2 + \cdots + c_kv_k\} =< S >$ *)*

Appendix B

The folder

ΟΔΗΓΙΕΣ

Στο επόμενο φύλλο θα βρείτε τρεις δραστηριότητες που έχουν στόχο να βοηθήσουν στην καλύτερη κατανόηση της έννοιας της παραγωγής υποχώρου. Οι δραστηριότητες δημιουργήθηκαν στα πλαίσια της μεταπτυχιακής μου εργασίας με τίτλο ”Δημιουργία δραστηριοτήτων για την κατανόηση της έννοιας της Γραμμικής Θήκης” (επιβλέπων Χρήστος Κουρουνιώτης) έτσι ώστε να μπορούν να αποτελέσουν μέρος των ασκήσεων του εργαστηρίου στα πλαίσια του μαθήματος ”Γεωμετρία και Γραμμική Άλγεβρα”. Για το λόγο αυτό θα ήθελα την βοήθειά σας. Είναι σημαντικό οι δραστηριότητες να ανταποκρίνονται στις ανάγκες σας και να μπορούν να βοηθήσουν ουσιαστικά.

Οι έννοιες αυτές περιλαμβάνονται στις σημειώσεις του κ. Κουρουνιώτη για το μάθημα ”Γεωμετρία και Γραμμική Άλγεβρα” στις σελίδες 127, 128. Για την διευκόλυνση σας οι σελίδες αυτές δίνονται μαζί με τις δραστηριότητες. Αφιερώστε χρόνο να τις λύσετε ώστε να μπορέσετε να βοηθήσετε στην αξιολόγηση τους. Για να μπορεί να γίνει καλύτερα η αξιολόγηση των δραστηριοτήτων σας δίνονται οι εξής οδηγίες:

- *Όλες οι ασκήσεις μπορούν να λυθούν χωρίς την αναφορά της έννοιας της βάσης.* Η έννοια της παραγωγής υποχώρου προηγείται της έννοιας της βάσης. Προσπαθήστε να μπείτε στην θέση κάποιου που βλέπει την έννοια για πρώτη φορά. Για περισσότερες πληροφορίες μπορείτε να ανατρέξετε στις σημειώσεις του μαθήματος, εκεί μπορείτε να δείτε ποιες έννοιες προηγούνται και ποιες ακολουθούν.

- *Μην αγχωθείτε αν κάπου κολλήσετε!* Στη συνέντευξη θα έχουμε χρόνο να συζητήσουμε ότι σας δυσκόλεψε και να το λύσουμε μαζί όπως θα γινόταν και στο εργαστήριο. Οι δραστηριότητες έχουν βοηθητικό και όχι εξεταστικό χαρακτήρα.

- Τέλος, την ημέρα που θα γίνει η συνέντευξη **θα πρέπει να κρατάτε μαζί σας τις απαντήσεις που δώσατε**. Αυτό θα βοηθήσει στην καλύτερη αξιολόγηση των δραστηριοτήτων. Με αυτόν τον τρόπο θα μπορείτε ευκολότερα να ανατρέξετε σε ότι σας δυσκόλεψε ή ότι σας άρεσε σχετικά με τις δραστηριότητες.

Εάν χρειάζεστε επιπλέον πληροφορίες μπορείτε να επικοινωνήσετε μαζί μου στο **papadaki.evie@gmail.com**

Ευχαριστώ πολύ για τον χρόνο και το ενδιαφέρον σας. Η βοήθειά σας είναι πολύ σημαντική στην προσπάθεια δημιουργίας δραστηριοτήτων που να ανταποκρίνονται στις ανάγκες των φοιτητών.

ΔΡΑΣΤΗΡΙΟΤΗΤΕΣ

1. Δίνονται τα διανύσματα:

$$v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \ w_1 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, \ w_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}.$$

   (α') Δείξτε ότι το w_1 μπορεί να εκφραστεί σαν γραμμικός συνδυασμός των v_1 και v_2, αλλά το w_2 δεν μπορεί να εκφραστεί σαν γραμμικός συνδυασμός των v_1 και v_2.

   (β') Εξηγήστε ποιός είναι ο υπόχωρος του $\mathbb{R}^3$ που παράγεται από τα $v_1,\ v_2$ και w_1.
   Εξηγήστε ποιός είναι ο υπόχωρος του $\mathbb{R}^3$ που παράγεται από τα $v_1,\ v_2$ και w_2.
   Τι παρατηρείτε;

   (γ') Δείξτε ότι τα διανύσματα $v_1,\ v_2,\ w_1$ και w_2 παράγουν τον $\mathbb{R}^3$, δηλαδή οτι για κάθε $u = (x,\ y,\ z)$ υπάρχουν $a,\ b,\ c,\ d$ τέτοια ώστε:

$$u = av_1 + bv_2 + cw_1 + dw_2\,.$$

   Ακόμα δείξτε ότι κάθε διάνυσμα $u \in \mathbb{R}^3$ μπορεί να εκφραστεί ως γραμμικός συνδυασμός των $v_1,\ v_2,\ w_1$ και w_2 με άπειρους τρόπους.

2. Έστω $v_1,\ v_2$ και w γραμμικά εξαρτημένα διανύσματα στον $\mathbb{R}^3$. Το w είναι δυνατό να μην ανήκει στον χώρο που παράγεται από τα v_1 και v_2 παρόλο που τα $v_1,\ v_2,\ w$ είναι γραμμικά εξαρτημένα. Δώστε ένα παράδειγμα.
   Γιατί πιστεύετε ότι συμβαίνει αυτό.

3. Χαρακτηρίστε τις παρακάτω προτάσεις με Σωστό ή Λάθος.

   (α') Εάν τα $v_1,\ v_2,\ v_3,\ v_4$ παράγουν τον $\mathbb{R}^3$ τότε και τα $v_1,\ v_2,\ v_3$ παράγουν τον $\mathbb{R}^3$.

   (β') Εάν v και w είναι γραμμικά ανεξάρτητα και το σύνολο $\{v,\ w,\ z\}$ είναι γραμμικά εξαρτημένο, τότε το z ανήκει στο χώρο που παράγεται από τα v και w.

   (γ') Εάν το σύνολο διανυσμάτων $S = \{v_1, v_2, ..., v_k\}$ παράγει τον V, και $w \in V$, το σύνολο $T = \{v_1, v_2, ..., v_k, w\}$ επίσης παράγει τον V.