Master's Diploma Thesis
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The Wasserstein metric and gradient flow theory in PDEs

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

In this work, we are interested in the Wasserstein gradient flow theory, which we investigate in a more applied framework, in Partial Differential Equations, Measure Theory and Probability. Our primary aim is to present the seminar parer by R. Jordan, D. Kinderlehrer and F. Otto [9], in which a discrete scheme in time is being used in order to regard the Fokker-Planck equation (2.1) as the gradient flow of a free energy functional. Then, we discuss another work by Otto [10], in which the theory of Wasserstein gradient flows is being applied in the porous medium equation (3.1). In order to present these works, we first introduce some elementary results in optimal transportation theory on probability measures and discuss a very important property of the optimal transport map, as it was first discovered by Y. Brenier [3].

1.2 Preliminaries

1.2.1 Transport of Measures

Let X_1, X_2 be separable metric spaces. Consider a probability measure on X_1, which we denote as $\mu \in \mathcal{P}(X_1)$ and $\mathbf{r} : X_1 \to X_2$ a Borel (or μ-measurable) map. We define the *image measure* or the *push-forward* measure of μ, which we write $\mathbf{r}\#\mu \in \mathcal{P}(X_2)$ as

$$\mathbf{r}\#\mu(A) := \mu(\mathbf{r}^{-1}(A)), \quad \text{for all Borel } A \subset X_2$$

and more generally, it holds

$$\int_{X_1} f(\mathbf{r}(x))\, d\mu(x) = \int_{X_2} f(y)\, d\mathbf{r}\#\mu(y), \quad \text{for all bounded Borel } f : X_2 \to \mathbb{R}.$$

For all probability measures $\mu, \nu \in \mathcal{P}(X_1)$, we observe that if μ is absolutely continuous with respect to ν, the push- forward of μ under $\mathbf{r}$ is again absolutely continuous with respect to the push-forward of ν under $\mathbf{r}$. One can also define the *composition* of the push-forward measures, using exactly the above formulation

$$(\mathbf{r} \circ \mathbf{s})\#\mu = \mathbf{r}\#(\mathbf{s}\#\mu),$$

where $\mathbf{s} : X_1 \to X_2$, $\mathbf{r} : X_2 \to X_3$ (here X_3 is again a separable metric space) and $\mu \in \mathcal{P}(X_1)$.

Let $\boldsymbol{\mu} \in \mathcal{P}(X)$, where X is the product of separable metric spaces $X := X_1 \times X_2 \times \cdots \times X_n$ for some $n \in \mathbb{N}$. We call *marginals* of $\boldsymbol{\mu}$ all the probability measures $\mu_i \in \mathcal{P}(X_i)$, for all $i \in \{1, 2, \ldots, n\}$ for which there holds

$$\int_X f(x_i)\, d\boldsymbol{\mu}(x_1, \ldots, x_n) = \int_{X_i} f(x_i)\, d\mu_i(x_i),$$

for all bounded Borel $f : X_i \to \mathbb{R}$. Formally, one can define the *projection operator* on X by

$$\pi^i : (x_1, \ldots, x_n) \mapsto x_i \in X_i, \quad \text{for all } i \in \{1, 2, \ldots, n\}$$

and then we have $\mu_i := \pi^i\#\boldsymbol{\mu}$. It is also convenient to define

$$\pi^{i,j} : (x_1, \ldots, x_n) \mapsto (x_i, x_j) \in (X_i \times X_j), \quad \text{for all } i, j \in \{1, 2, \ldots, n\}$$

and then we write $\boldsymbol{\mu}_{ij} := \pi^{i,j}\#\boldsymbol{\mu} \in \mathcal{P}(X_i \times X_j)$. If $\mu_i \in \mathcal{P}(X_i)$, we can define the class of *multiple plans* with marginal μ_i as

$$\Gamma(\mu_1, \mu_2, \ldots, \mu_n) := \{\boldsymbol{\mu} \in \mathcal{P}(X_1 \times X_2 \times \cdots \times X_n) : \pi^i\#\boldsymbol{\mu} = \mu_i, i = 1, 2, \ldots, n\}.$$

For $n = 2$, a measure $\boldsymbol{\mu} \in \Gamma(\mu_1, \mu_2)$ is called *transport plan* between μ_1 and μ_2. We can associate a transport plan to each couple of measures $\mu_1 \in \mathcal{P}(X_1)$ and $\mu_2 = \mathbf{r}\#\mu_1 \in \mathcal{P}(X_2)$ with $\mathbf{r} : X_1 \to X_2$ a Borel map

$$\boldsymbol{\mu} := (id. \times \mathbf{r})\#\mu_1,$$

where $id.$ is the identity map on X_1. We then say that this transport plan is *induced* by the transport map $\mathbf{r}$. In [1, §5.2] we find the following very useful Lemma:

Lemma 1 *Let* $(X_1, \mu_1), (X_2, \mu_2), (X_3, \mu_3)$ *be Radon separable metric spaces and* *consider the transport plans* $\boldsymbol{\gamma}_{12} \in \mathcal{P}(X_1 \times X_2)$ *and* $\boldsymbol{\gamma}_{13} \in \mathcal{P}(X_1 \times X_3)$ *such that*

$$\pi^1 \# \boldsymbol{\gamma}_{12} = \pi^1 \# \boldsymbol{\gamma}_{13} = \mu_1.$$

Then there exists

$$\boldsymbol{\gamma} \in \mathcal{P}(X_1 \times X_2 \times X_3) \quad \textit{such that} \quad \begin{cases} \pi^{1,2} \# \boldsymbol{\gamma} = \boldsymbol{\gamma}_{12} \\ \pi^{1,3} \# \boldsymbol{\gamma} = \boldsymbol{\gamma}_{13} \end{cases}$$

and we denote the set

$$\Gamma^1(\boldsymbol{\gamma}_{12}, \boldsymbol{\gamma}_{13}) := \{\boldsymbol{\gamma} \in \mathcal{P}(X_1 \times X_2 \times X_3) : \pi^{1,2} \# \boldsymbol{\gamma} = \boldsymbol{\gamma}_{12}, \pi^{1,3} \# \boldsymbol{\gamma} = \boldsymbol{\gamma}_{13}\}.$$

The above Lemma gives meaning to the following structure.

Remark 1 *(Composition of plans).* Consider the plans $\boldsymbol{\gamma}_{12} \in \mathcal{P}(X_1 \times X_2)$ and $\boldsymbol{\gamma}_{23} \in \mathcal{P}(X_2 \times X_3)$. We say that $\boldsymbol{\gamma} \in \Gamma^2(\boldsymbol{\gamma}_{12}, \boldsymbol{\gamma}_{23})$ when

$$\boldsymbol{\gamma} \in \mathcal{P}(X_1 \times X_2 \times X_3) \quad \text{and} \quad \begin{cases} \pi^{1,2} \# \boldsymbol{\gamma} = \boldsymbol{\gamma}_{12} \\ \pi^{2,3} \# \boldsymbol{\gamma} = \boldsymbol{\gamma}_{23} \end{cases}$$

while for $\Gamma^2(\boldsymbol{\gamma}_{12}, \boldsymbol{\gamma}_{23})$ to be nonempty, we require

$$\pi^1 \# \boldsymbol{\gamma}_{23} = \pi^2 \# \boldsymbol{\gamma}_{12} = \mu_2.$$

Observe that when $\boldsymbol{\gamma} \in \Gamma^2(\boldsymbol{\gamma}_{12}, \boldsymbol{\gamma}_{23})$, the push-forward $\pi^{1,3} \# \boldsymbol{\gamma} \in \Gamma(\mu_1, \mu_3)$ by construction. We call $\boldsymbol{\gamma}_{13}$ composition of $\boldsymbol{\gamma}_{12}$ and $\boldsymbol{\gamma}_{23}$ and denote it as $\boldsymbol{\gamma}_{12} \circ \boldsymbol{\gamma}_{23}$.

Remark 2 *(Radon spaces).* Given a Borel measure μ on X and a Borel subset $A \subset X$, we call the measure μ *outer regular* on A when

$$\mu(A) = \inf \big\{ \mu(O) :: A \subset O \text{ open} \big\}$$

and *inner regular* when

$$\mu(A) = \sup \big\{ \mu(K) :: A \supset K \text{ compact} \big\}.$$

We say that μ is a *Radon* measure on X if it is finite on all compact sets, outer regular on all Borel sets and inner regular on all open sets. When μ is finite, and inner regular then it is *tight* (see Propositions 7.5, 7.7 [6] and Theorem 5.1.3 [1]). Hence, tightness property is always guaranteed in a Radon space.

1.2.2 The Optimal Transportation Problem

The mass transference problem was introduced by Monge and then generalized by Kantrorovich. In its general form, given two Radon metric spaces X and Y, we consider the measures $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ and a cost function $c : X \times Y \to \mathbb{R}^+$. Then Monge's formulation reads as follows:

Find a transport map $\mathbf{s} : (X, \mu) \to (Y, \nu)$, such that $\mathbf{s}\#\mu = \nu$, and achieves the infimum of the transportation cost

$$\int_X c(x, \mathbf{s}(x))\, d\mu(x). \tag{1.1}$$

In Kantrorovich's formulation we seek a probability measure $\boldsymbol{\gamma} \in \Gamma(\mu, \nu)$ that minimizes the generalized transportation cost

$$\int_{X \times Y} c(x, y)\, d\boldsymbol{\gamma}(x, y). \tag{1.2}$$

Here $\boldsymbol{\gamma} \in \Gamma(\mu, \nu)$ means exactly that $\boldsymbol{\gamma} \in \mathcal{P}(X \times Y)$ with first marginal μ and second marginal ν *i.e.*

$$\int_X f(x)\, d\mu(x) = \int_{X \times Y} f(x)\, d\boldsymbol{\gamma}(x, y), \quad \text{for all bounded Borel } f : X \to \mathbb{R}$$
$$\int_Y g(y)\, d\nu(y) = \int_{X \times Y} g(y)\, d\boldsymbol{\gamma}(x, y), \quad \text{for all bounded Borel } g : Y \to \mathbb{R}.$$

This is the *Monge-Kantorovich problem* (MKP), which is a weak formulation of the original Monge problem. In that context, a minimizer $\boldsymbol{\gamma}^* \in \Gamma(\mu, \nu)$ is called an *optimal transport plan* between the probability measures μ and ν and the set of all optimal transport plans is denoted as $\Gamma_{opt}(\mu, \nu)$. Monge's formulation is well-posed only if there exists a transport map $\mathbf{s}$ such that $\nu = \mathbf{s}\#\mu$. For example, there is no such mapping when μ is a Dirac mass and ν is not. This problem is fixed by introducing the Kantrorovich's formulation, since $\Gamma(\mu, \nu)$ is always nonempty; it always contains the product measure defined as in (1.5). Nevertheless, Kantrorovich's formulation is, certainly, a relaxed version of the original problem in the sense that any admissible solution $\mathbf{s}$ to Monge's problem yields an admissible $\boldsymbol{\gamma} = (id. \times \mathbf{s})\#\mu$ for the MKP.

The existence of an optimal transport plan when c is lower semicontinuous is provided by the tightness of $\Gamma(\mu, \nu)$. Indeed, since X and Y are Radon spaces, the sequences $\{\mu_n\} \subset \mathcal{P}(X)$ and $\{\nu_n\} \subset \mathcal{P}(Y)$ are tight and thus (using *Prokhorov's* theorem - [1, §5.1]) narrowly convergent to some $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ respectively. This property is equivalent to the tightness of $\Gamma(\mu_n, \nu_n)$ and we deduce the existence of a subsequence, which we still label $\boldsymbol{\gamma}_n$, such that $\boldsymbol{\gamma}_n \rightharpoonup \boldsymbol{\gamma}^*$ narrowly, to some $\boldsymbol{\gamma}^* \in \Gamma(\mu, \nu)$. Now, the lower semicontinuity property reads

$$\liminf_{n\to\infty} \int_X g(x)\, d\mu_n(x) \geq \int_X g(x)\, d\mu(x), \tag{1.3}$$

for all $\{\mu_n\} \subset \mathcal{P}(X)$, $\mu_n \rightharpoonup \mu$ and for all lower semicontinuous functions $g : X \to (-\infty, +\infty]$ bounded from below. Briefly sketching the proof of (1.3), it is sufficient to take any subset $\mathcal{C}$ of bounded and continuous functions with dense span$\mathcal{C}$ in $C_b(X)$ and suppose that $\mathcal{C}_0 \subset \mathcal{C}$ satisfies the approximation property

$$\int_X f(x)\, d\mu(x) = \sup \left\{ \int_X h(x)\, d\mu(x) : h \in \mathcal{C}_0, h \leq f \right\},$$

for all $f \in \mathcal{C}$ (see [6],[1]). Then for all $f \in \mathcal{C}$ there holds

$$\begin{aligned} \liminf_{n\to\infty} \int_X f(x)\, d\mu_n(x) &\geq \sup_{\substack{h\in\mathcal{C}_0 \\ h\leq f}} \liminf_{n\to\infty} \int_X h(x)\, d\mu_n(x) \\ &= \sup_{\substack{h\in\mathcal{C}_0 \\ h\leq f}} \int_X h(x)\, d\mu(x) \\ &= \int_X f(x)\, d\mu(x), \end{aligned}$$

since $\mu_n \rightharpoonup \mu$ in X. Back to the problem of our interest, (1.3) gives

$$\liminf_{\substack{n\to\infty \\ \boldsymbol{\gamma}_n\in\Gamma(\mu_n,\nu_n)}} \int_{X\times Y} c(x,y)\, d\boldsymbol{\gamma}_n(x,y) \geq \int_{X\times Y} c(x,y)\, d\boldsymbol{\gamma}^*(x,y)$$

and provided that $\boldsymbol{\gamma}_n$ is a minimizing sequence *i.e.*

$$\lim_{n\to\infty} \int_{X\times Y} c(x,y)\, d\boldsymbol{\gamma}_n(x,y) = \inf_{\boldsymbol{\gamma}\in\Gamma(\mu,\nu)} \int_{X\times Y} c(x,y)\, dp(x,y),$$

we immediately have that

$$\int_{X\times Y} c(x,y)\, d\boldsymbol{\gamma}^*(x,y) \leq \inf_{\boldsymbol{\gamma}\in\Gamma(\mu,\nu)} \int_{X\times Y} c(x,y)\, d\boldsymbol{\gamma}(x,y).$$

Noticing that the reverse inequality holds trivially, we conclude. Note that there are some special cases where the existence of optimal maps is attained without any further assumptions on the cost function besides positivity and lower semicontinuity (see [1]).

Brenier in [3] introduced a projection problem associated to a particular MKP for the cost function $c = (1/2)|x-y|^2$, regarding absolutely continuous measures with respect to the Lebesgue measure. More precisely, he stated an MKP (along with its corresponding dual problem and the so-called *mixed* MKP) on $\mathbb{R}^d \times \bar{\Omega}$, where $\bar{\Omega} \subset \mathbb{R}^d$ is a bounded and normalized domain equipped with the

d-dimensional Lebesgue measure. Under the assumption that the probability measure $\boldsymbol{\gamma}$ has absolutely continuous marginals with respect to the Lebesgue measure, he stated and proved a polar factorization theorem for vector-valued functions. Later in this chapter, we will see in detail this formulation and give a proof for the polar factorization theorem by Brenier.

1.2.3 The Wasserstein Distance

The squared Wasserstein distance of order two, between two probability measures μ_1 and μ_2 on $\mathbb{R}^d$ with finite second moments is given by

$$W(\mu_1, \mu_2)^2 := \inf_{\boldsymbol{\gamma} \in \Gamma(\mu_1, \mu_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, \boldsymbol{\gamma}(dx, dy), \tag{1.4}$$

where $|\cdot|$ denotes the usual Euclidean norm on $\mathbb{R}^d$. We denote as $\mathcal{P}^2(\mathbb{R}^d)$ the set of all probability measures μ on $\mathbb{R}^d$ with finite second moments i.e.

$$\int_{\mathbb{R}^d} |x|^2 \, d\mu(x) < +\infty.$$

If μ_1 and $\mu_2 \in \mathcal{P}^2(\mathbb{R}^d)$, the set $\Gamma(\mu_1, \mu_2)$ contains all probability measures on $\mathcal{P}^2(\mathbb{R}^d \times \mathbb{R}^d)$ with first marginal μ_1 and second marginal μ_2. In other words, we require for all Borel $A \subset \mathbb{R}^d$ to hold

$$\boldsymbol{\gamma}(A \times \mathbb{R}^d) = \mu_1(A) \quad \text{and} \quad \boldsymbol{\gamma}(\mathbb{R}^d \times A) = \mu_2(A).$$

Now, $\Gamma(\mu_1, \mu_2)$ is nonempty, since it always contains the product measure $\mu_1 \times \mu_2$ which is

$$\boldsymbol{\gamma}(A \times B) = \mu_1(A)\mu_2(B), \quad \text{for all Borel } A, B \subset \mathbb{R}^d, \tag{1.5}$$

so $W(\mu_1, \mu_2) \in [0, +\infty]$ is well-defined. Also, the finite second moments requirement ensures that $W(\mu_1, \mu_2) \in [0, +\infty)$. Actually, the variational problem defined by the Wasserstein distance (1.4) is exactly a MKP, formulated for the particular cost function $c = |x - y|^2$. We can define in a similar way the Wasserstein distance of order p ($p \geq 1$).

It is easy to show that (1.4) defines a distance. Indeed, it is trivial to show that (1.4) is non-negative, symmetric in the arguments μ_1 and μ_2 and vanishes whenever $\mu_1 = \mu_2$. For the triangle inequality

$$W(\mu_1, \mu_3) \leq W(\mu_1, \mu_2) + W(\mu_2, \mu_3), \tag{1.6}$$

let $\boldsymbol{\gamma}_{12}$ and $\boldsymbol{\gamma}_{23}$ be optimal in the definition of the Wasserstein distance between

μ_1 and μ_2 and μ_2 and μ_3 respectively *i.e.*

$$W(\mu_1,\mu_2) = \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x_1 - x_2|^2 \,\boldsymbol{\gamma}_{12}(dx,dy)\right)^{1/2}$$
$$W(\mu_2,\mu_3) = \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x_2 - x_3|^2 \,\boldsymbol{\gamma}_{23}(dx,dy)\right)^{1/2}.$$

Since $\pi^1\#\boldsymbol{\gamma}_{23} = \pi^2\#\boldsymbol{\gamma}_{12} = \mu_2$, we use remark 1 to deduce the existence of a measure $\boldsymbol{\gamma} \in \mathcal{P}(\mathbb{R}^{3d})$ such that $\pi^{1,2}\#\boldsymbol{\gamma} = \boldsymbol{\gamma}_{12}$ and $\pi^{2,3}\#\boldsymbol{\gamma} = \boldsymbol{\gamma}_{23}$. Theefore

$$\begin{cases} W(\mu_1,\mu_2) &= \|x_1 - x_2\|_{L^2(\mathbb{R}^d,\boldsymbol{\gamma})} \\ W(\mu_2,\mu_3) &= \|x_2 - x_3\|_{L^2(\mathbb{R}^d,\boldsymbol{\gamma})} \end{cases}$$

and by construction $\boldsymbol{\gamma}_{12} \circ \boldsymbol{\gamma}_{23} = \boldsymbol{\gamma}_{13} = \pi^{1,3}\#\boldsymbol{\gamma} \in \Gamma(\mu_1,\mu_3)$. Since

$$\|x_1 - x_3\|_{L^2(\mathbb{R}^d,\boldsymbol{\gamma}_{13})} = \|x_1 - x_3\|_{L^2(\mathbb{R}^d,\boldsymbol{\gamma})},$$

we get (1.6) from the standard triangle inequality for the L^p distance.

If μ_1 is absolutely continuous with respect to the Lebesgue measure, by *Radon-Nikodym* theorem, there exists a Lebesgue measurable function ρ_1 such that $d\mu_1(x) = \rho_1(x)dx$; ρ_1 is a probability density function. Then the set of all optimal transport plans between μ_1 and μ_2, $\Gamma_{opt}(\mu_1,\mu_2)$ is reduced to a single measure $\boldsymbol{\gamma}$, which is induced by a Borel mapping $\mathbf{r} : \mathbb{R}^d \to \mathbb{R}^d$ as

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,y)\,\boldsymbol{\gamma}(dx,dy) = \int_{\mathbb{R}^d} f(x,\mathbf{r}(x))\rho_1(x)\,dx,$$

for all $f \in C_b(\mathbb{R}^d)$. We call $\mathbf{r}$ the *optimal transport map* pushing μ_1 forward to μ_2. If μ_2 is also absolutely continuous with respect to the Lebesgue measure with density ρ_2, the optimal transport map $\mathbf{r}$ is $\mu_1 - a.e.$ essentially injective and its inverse $\mathbf{r}^{-1}$ is the optimal transport map pushing μ_2 forward to μ_1. Indeed, if $\mathbf{s}$ is the optimal transport map between μ_1 and μ_2 the uniqueness of optimal transport maps gives

$$\int_{\mathbb{R}^d} f(x,\mathbf{r}(x))\rho_1(x)\,dx = \int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,y)\,\boldsymbol{\gamma}(dx,dy) = \int_{\mathbb{R}^d} f(\mathbf{s}(y),y)\rho_2(y)\,dy,$$

for all $f \in C_b(\mathbb{R}^d)$. It follows that $\mathbf{s} \circ \mathbf{r} = id.$. Of course, when μ_1 and μ_2 are absolutely continuous with respect to the Lebesgue measure, and can be written in terms of densities, we denote the corresponding Wasserstein distance as $W(\rho_1,\rho_2)$ over the set of $\Gamma(\rho_1,\rho_2)$.

Next, we will see that given a probability measure with density $\rho \in \mathcal{P}_{reg}(\mathbb{R}^d)$, for all $\mathbf{r} \in L^p(\mathbb{R}^d,\rho)$ such that the push forward $\mathbf{r}\#\rho\ dx$ is again absolutely continuous with respect to the Lebesgue measure, we can write the transport map $\mathbf{r}$ as the gradient of a convex, lower semicontinuous function of a measure preserving map, performing *Brenier's polar factorization.*

1.2.4 Polar Factorization for Vector-Valued Functions

Consider a probability space (X, μ) and a bounded normalized domain $\Omega \subset \mathbb{R}^d$, equipped with the d-dimensional Lebesgue measure $|\cdot|$. Under some necessary assumptions on X and Ω, we have that for all non-singular $\mathbf{r} \in L^p(X, \mu; \mathbb{R}^d)$ (in the sense that $\mu(\mathbf{r}^{-1}(A)) = 0$, whenever $\mathcal{L}^d(A) = 0$) there exists a unique *polar factorization* of $\mathbf{r}$, *i.e.* $\mathbf{r} = \nabla\psi \circ s$, where $\psi \in \Omega$ is a convex function and $\mathbf{r} : (X, \mu) \to (\Omega, |\cdot|)$ is a measure preserving map.

Given a measure α on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} (1 + \|y\|)\, d\alpha(y) < +\infty, \tag{1.7}$$

we introduce a different aspect of the problems (1.1)-(1.2) (see [3], [12], [5]).

The primal MKP. We seek a function $\varphi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, \alpha)$ that minimizes

$$\int_{\mathbb{R}^d} \varphi\, d\alpha$$

and a function $\psi \in C(\Omega) \cap L^1(\Omega, \beta)$ that satisfies

$$\int_{\Omega} \psi\, d\beta = 0, \tag{1.8}$$

such that

$$\varphi(y) + \psi(z) \geq y \cdot z, \quad \text{for all } (y, z) \in \mathbb{R}^d \times \Omega. \tag{1.9}$$

The dual MKP. We seek a probability measure $\mathbf{p} \in \mathcal{P}(\mathbb{R}^d \times \bar{\Omega})$ that maximizes

$$\int_{\mathbb{R}^d \times \Omega} y \cdot z\, d\mathbf{p}(y, z),$$

under the conditions

$$\int_{\mathbb{R}^d \times \Omega} \|y\|\, d\mathbf{p}(y, z) < +\infty \tag{1.10}$$

and α and β are the marginals of $\mathbf{p}$ on $\mathbb{R}^d \times \bar{\Omega}$ respectively *i.e.*

$$\int_{\mathbb{R}^d\times\Omega} f(y)\, d\mathbf{p}(y,z) = \int_{\mathbb{R}^d} f(y)\, d\alpha(y), \quad \text{for all } f \in C(\mathbb{R}^d) : |f(y)| \leq c(1+\|y\|) \tag{1.11}$$

$$\int_{\mathbb{R}^d\times\Omega} g(z)\, d\mathbf{p}(y,z) = \int_{\Omega} g(z)\beta(z)\, dz, \quad \text{for all } g \in C(\bar{\Omega}), \tag{1.12}$$

for some constant c. Here β is a probability density (which we identify with the probability measure for simplicity: $d\beta(z) = \beta(z)\, dz$), so the probability measure β is absolutely continuous with respect to the Lebesgue measure.

The mixed MKP. Here, we find functions $\varphi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, \alpha)$, $\psi \in C(\Omega) \cap L^1(\Omega, \beta)$ and a probability measure $\mathbf{p} \in \mathcal{P}(\mathbb{R}^d \times \bar{\Omega})$ such that (1.7)-(1.12) hold and moreover

$$\int_{\mathbb{R}^d} \varphi\, d\alpha \leq \int_{\mathbb{R}^d\times\bar{\Omega}} y \cdot z\, d\mathbf{p}(y,z). \tag{1.13}$$

Note that for the dual and the mixed MKP we need $\mathbf{p}$ to be a tight probability measure on $\mathbb{R}^d \times \bar{\Omega}$ and $\mathbf{p}(\mathbb{R}^d \times \partial\Omega) = \beta(\partial\Omega) = 0$, for (1.10)-(1.12) to hold.

We can now introduce the set

$$K_0 = \Big\{ \psi \in W^{1,1}(\Omega,\beta) \cap C(\Omega) : \int_\Omega \psi\, d\beta = 0 \quad \text{and} \quad \exists \quad \tilde{\psi} : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$$
$$\text{convex and lower semicontinuous with} \quad \psi|_\Omega = \tilde{\psi} \Big\}$$

and for the weighted Sobolev space

$$W^{1,p}(\Omega,\beta) = \left\{ f \in L^p(\Omega,\beta) : \nabla f \in L^p(\Omega,\beta;\mathbb{R}^d) \right\},$$

the set

$$K = \{\nabla\psi : \psi \in W^{1,p}(\Omega,\beta) \quad \text{and } \psi \text{ is convex}\}.$$

Theorem 1 *(Polar Factorization). Let N be the set of all $\mathbf{r} \in L^p(X,\mu;\mathbb{R}^d)$ for which the following non-degeneracy condition fails*

$$\mu(\mathbf{r}^{-1}(A)) = 0, \quad \textit{for all } A \subset \mathbb{R}^d \textit{ Lebesgue null set.} \tag{1.14}$$

Then for all $\mathbf{r} \in L^p(X,\mu;\mathbb{R}^d)\backslash N$, there exists a unique pair of functions $(\nabla\psi, \mathbf{s})$ such that $\nabla\psi \in K$, $s : (X,\mu) \to (\bar{\Omega},\beta)$ is a measure preserving map and $\mathbf{r} = \nabla\psi \circ \mathbf{s}$.

Proof. We present this proof in four steps. The first two steps include the proving procedure of some very useful ‘a-priori’ results, given a solution for the mixed MKP. A less careful reader could take under consideration formulas (1.15)-(1.18) and (1.27)-(1.30) and proceed to the main body of the proof regarding the existence and uniqueness of the polar factorization, which follows in steps *3* and *4*.

Step 1. First we will show that if $(\psi, \varphi, \mathbf{p})$ is any solution of the mixed MKP, then

$$\psi \in K_0 \quad \text{and} \quad \|\nabla\psi\|_{L^1(\Omega,\beta)} = \int_{\mathbb{R}^d} \|y\| \, d\alpha(y), \tag{1.15}$$

$$\varphi = \psi^* = \sup_{z\in\Omega}\{y \cdot z - \psi(z)\} \quad \text{for all } y \in \mathbb{R}^d, \tag{1.16}$$

$$d\mathbf{p}(y, z) = \delta(y - \nabla\psi(z))\beta(z) \, dz, \tag{1.17}$$

$$\int_{\mathbb{R}^d} \varphi \, d\alpha = \int_{\mathbb{R}^d\times\bar{\Omega}} y \cdot z \, d\mathbf{p}(y, z) \quad \text{and} \quad \int_{\Omega} \psi \, d\beta = 0. \tag{1.18}$$

Define

$$\tilde{\varphi}(y) := \sup_{z\in\Omega}\{y \cdot z - \psi(z)\} \quad \text{for all } y \in \mathbb{R}^d, \tag{1.19}$$

and since $(\psi, \varphi, \mathbf{p})$ is a solution of the mixed MKP, (1.9) holds. Thus

$$\tilde{\varphi}(y) \le \varphi(y), \quad \text{for all } y \in \mathbb{R}^d. \tag{1.20}$$

Fix a $z_0 \in \Omega$, then (1.9) and (1.20) give

$$\tilde{\varphi}(y) \ge y \cdot z_0 - \psi(z_0), \quad \text{for all } y \in \mathbb{R}^d,$$

so $\tilde{\varphi}$ is finite everywhere. Also, notice that since Ω is bounded, $\bar{\Omega} \subset B(0, \ell)$, for some $\ell > 0$, then from (1.19) we have that $\tilde{\varphi}$ is convex and Lipschitz continuous with $Lip(\tilde{\varphi}) \le \ell$. Now define

$$\tilde{\psi}(z) := \sup_{y\in\mathbb{R}^d}\{y \cdot z - \tilde{\varphi}(y)\} \quad \text{for all } z \in \mathbb{R}^d. \tag{1.21}$$

Here $\tilde{\psi} : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a well-defined convex and lower semicontinuous function and $\tilde{\varphi}, \tilde{\psi}$ are convex conjugates *i.e.* $\tilde{\varphi} = \tilde{\psi}^*$ and $\tilde{\psi} = \tilde{\varphi}^*$. Now, (1.19) and (1.21) give

$$\tilde{\psi}(z) \le \psi(z) \quad \text{and} \quad \tilde{\psi}(z) \ge -\tilde{\varphi}(0) > -\infty. \quad \text{for all } z \in \Omega. \tag{1.22}$$

From (1.8) we have

$$\int_{\Omega} |1 + \psi(z)|\beta(z) \, dz < +\infty,$$

which combined with (1.7), the definition of mixed MKP and (1.22), give that $\tilde{\psi} \in L^1(\Omega, \beta)$, $\tilde{\varphi} \in L^1(\mathbb{R}^d, \alpha)$. In addition, we observe that (1.20), (1.22) and (1.21) give

$$\int_{\mathbb{R}^d} \tilde{\varphi}\, d\alpha \leq \int_{\mathbb{R}^d} \varphi\, d\alpha,$$
$$\int_{\Omega} \tilde{\psi}\, d\beta \leq \int_{\Omega} \psi\, d\beta,$$
$$\tilde{\varphi}(y) + \tilde{\psi}(z) \geq y \cdot z,$$

for all $y, z \in \mathbb{R}^d$ and since α and β are the marginals of $\mathbf{p}$, it follows

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^d \times \bar{\Omega}} (\tilde{\varphi}(y) + \tilde{\psi}(z) - y \cdot z)\, d\mathbf{p}(y, z) \\ &= \int_{\mathbb{R}^d} \tilde{\varphi}(y)\, d\alpha + \int_{\Omega} \tilde{\psi}(z)\, d\beta - \int_{\mathbb{R}^d \times \bar{\Omega}} y \cdot z\, d\mathbf{p}(y, z) \\ &\leq \int_{\mathbb{R}^d} \varphi(y)\, d\alpha + \int_{\Omega} \psi(z)\, d\beta - \int_{\mathbb{R}^d \times \bar{\Omega}} y \cdot z\, d\mathbf{p}(y, z). \end{aligned}$$

But

$$\int_{\mathbb{R}^d} \varphi(y)\, d\alpha + \int_{\Omega} \psi(z)\, d\beta - \int_{\mathbb{R}^d \times \bar{\Omega}} y \cdot z\, d\mathbf{p}(y, z) \leq 0,$$

because $(\psi, \varphi, \mathbf{p})$ is a solution to the mixed MKP and therefore (1.8) and (1.13) hold. So we have exactly that $\tilde{\varphi}(y) + \tilde{\psi}(z) = y \cdot z$, $\mathbf{p}$-*a.e.* on $\mathbb{R}^d \times \bar{\Omega}$ and moreover because of (1.20) and (1.22), we get

$$\varphi = \tilde{\varphi} = \sup_{z \in \Omega} \{y \cdot z - \psi(z)\} \quad \alpha - a.e$$

$$\psi = \tilde{\psi} = \sup_{y \in \mathbb{R}^d} \{y \cdot z - \varphi(y)\} \quad \beta - a.e$$

$$\int_{\mathbb{R}^d} \tilde{\varphi}\, d\alpha = \int_{\mathbb{R}^d} \varphi\, d\alpha = \int_{\mathbb{R}^d \times \Omega} y \cdot z\, d\mathbf{p}(y, z)$$

$$\int_{\Omega} \tilde{\psi}\, d\beta = \int_{\Omega} \psi\, d\beta = 0$$

Observe that by definition, $\psi \in C(\Omega)$ and since the convex function $\tilde{\psi} = \psi$ β-*a.e.*, $\tilde{\psi}$ is locally Lipschitz continuous on Ω. Therefore

$$\tilde{\psi} = \psi \quad \text{everywhere on } \Omega. \tag{1.23}$$

Moreover, $\nabla\psi : \Omega \to \mathbb{R}^d$ is well-defined up to a Lebesgue-null set as a Borel map and $\partial\psi(z) = \{\nabla\psi(z)\}$ *a.e.* on Ω. It follows that

$$z \in \partial\tilde{\varphi}(y), \quad y \in \partial\psi(z) \quad \mathbf{p}\text{-}a.e.\text{on } \mathbb{R}^d \times \bar{\Omega} \tag{1.24}$$

and since α and β are the marginals of $\mathbf{p}$ and β is absolutely continuous with respect to the Lebesgue measure

$$\mathbf{p}\left(\left\{(y,z) \in \mathbb{R}^d \times \bar{\Omega}; \partial\psi(z) \neq \{\nabla\psi(z)\}\right\}\right) = 0$$

because $\mathbf{p}(\mathbb{R}^d \times \partial\Omega) = \beta(\partial\Omega) = 0$ by definition. So we have

$$y = \nabla\psi(z), \quad \mathbf{p}\text{-}a.e.\text{on } \mathbb{R}^d \times \bar{\Omega}. \tag{1.25}$$

If we choose $f \in C(\mathbb{R}^d \times \bar{\Omega})$ such that $|f(y)| \leq c(1 + \|y\|)$, for some constant c for all $(y,z) \in \mathbb{R}^d \times \bar{\Omega}$, (1.10) gives that

$$\int_{\mathbb{R}^d \times \bar{\Omega}} (1 + \|y\|)\, d\mathbf{p}(y,z) < +\infty$$

so f is $\mathbf{p}$-integrable. The (1.25) gives

$$\int_{\mathbb{R}^d \times \Omega} f(y,z)\, d\mathbf{p}(y,z) = \int_{\mathbb{R}^d \times \Omega} f(\nabla\psi(z), z)\, d\mathbf{p}(y,z) = \int_{\Omega} f(\nabla\psi(z), z)\beta(z)\, dz, \tag{1.26}$$

so $d(y,z) = \delta(y - \nabla\psi(z))\beta(z)\, dz$. Now choose $f = \|y\|$; in such case, we get from (1.26)

$$\int_{\Omega} \|\nabla\psi(z)\|\beta(z)\, dz = \int_{\mathbb{R}^d} \|y\|\, d\alpha(y) < +\infty,$$

so $\psi \in W^{1,1}(\Omega, \beta)$ and from (1.23) we conclude that (1.15) indeed holds.

To conclude step *1*, consider the additional case where α is also absolutely continuous with respect to the Lebesgue measure. We will prove that

$$z = \nabla\psi^*(y) \quad \textit{and} \quad y = \nabla\psi(z), \quad \mathbf{p}\text{-}a.e. \text{ on } \mathbb{R}^d \times \bar{\Omega}, \tag{1.27}$$

$$z = \nabla\psi^*(\nabla\psi(z)), \quad \beta\text{-}a.e. \text{ on } \bar{\Omega}, \tag{1.28}$$

$$y = \nabla\psi(\nabla\psi^*(y)), \quad \alpha\text{-}a.e. \text{ on } \mathbb{R}^d, \tag{1.29}$$

$$d\mathbf{p}(y,z) = \delta(y - \nabla\psi^*(y))\, d\alpha(y). \tag{1.30}$$

Now, ψ^* is Lipschitz continuous on $\mathbb{R}^d$ and therefore $\nabla\psi^* : \mathbb{R}^d \to \mathbb{R}^d$ is well-defined *a.e.* as a Borel map. Also since the measure α is absolutely continuous with respect to the Lebesgue measure, the following set is $\alpha-$null

$$\left\{y \in \mathbb{R}^d : \partial\psi^*(y) \neq \{\nabla\psi^*(y)\}\right\}.$$

Therefore, $\partial\psi^*(y) = \{\nabla\psi^*(y)\}$ for $\mathbf{p} - a.e.(x, y) \in \mathbb{R}^d \times \bar{\Omega}$. This way we obtain (1.28), (1.29) and (1.30).

Step 2. Here, consider a $\psi \in K_0$ such that

$$\int_{\Omega} f(\nabla\psi(z))\beta(z)\, dz = \int_{\mathbb{R}^d} f(y)\, d\alpha(y), \tag{1.31}$$

for all $f \in C(\mathbb{R}^d)$ such that $|f(y)| \leq c(1 + \|y\|)$, for some constant c. Then the MKP has a unique solution $(\phi, \psi, \mathbf{p})$ for which (1.16) and (1.17) hold. In order to prove that claim, remember that since $\psi \in K_0$ it follows that $\nabla\psi \in L^1(\Omega, \beta; \mathbb{R}^d)$, hence $\mathbf{p}$ is well defined by (1.17), while (1.26) holds for all $f \in C(\mathbb{R}^d \times \bar{\Omega})$ such that $|f(x, y)| \leq c(1 + \|y\|)$, for some constant c. Using the latter, (1.15) gives

$$\int_{\mathbb{R}^d \times \bar{\Omega}} (1 + \|y\|)\, d\mathbf{p}(y, z) = \int_{\bar{\Omega}} (1 + \|\nabla\psi(z)\|)\beta(z)\, dz < +\infty.$$

But (1.31) and (1.17) indicate that the marginal of $\mathbf{p}$ on $\bar{\Omega}$ is β and on $\mathbb{R}^d$ is α, while because $\psi \in K$ the convexity property holds

$$\psi(z) + \nabla\psi(z) \cdot (\tilde{z} - z) \leq \psi(\tilde{z}), \quad \text{for all } \tilde{z} \in \Omega, \text{ and all } z \in \Omega \setminus E, \tag{1.32}$$

where E is a Lebesgue-null subset of Ω. Also, there exists a $\mathbf{p}-$null subset of $\mathbb{R}^d \times \bar{\Omega}$, we call F, such that $y = \nabla\psi(z)$, for all $(y, z) \in \mathbb{R}^d \times \bar{\Omega} \setminus F$. So (1.32) gives

$$\psi(z) + y(\tilde{z} - z) \leq \psi(\tilde{z}), \quad \text{for all } \tilde{z} \in \Omega, \text{ and all } (y, z) \in (\mathbb{R}^d \times (\Omega \setminus E)) \setminus F.$$

But observe that the set $A := (\mathbb{R}^d \times (\Omega \setminus E)) \setminus F$ has $\mathbf{p}-$measure 1, since

$$1 - \mathbf{p}(A) \leq \mathbf{p}\left(\mathbb{R}^d \times (\partial\Omega \cup E)\right) \mathbf{p}(F) = \beta(\partial\Omega \cup E) = 0,$$

so

$$\phi(y) = \psi^*(y) = \sup_{\tilde{z} \in \Omega}\{y \cdot \tilde{z} - \psi(\tilde{z})\}, \quad \text{for all } y \in \mathbb{R}^d$$

satisfies

$$\phi(y) + \psi(z) \leq y \cdot z, \quad \mathbf{p} - a.e. \text{ on } \mathbb{R}^d \times \bar{\Omega}.$$

It follows that

$$\int_{\mathbb{R}^d} \phi \, d\alpha + \int_\Omega \psi \, d\beta \leq \int_{\mathbb{R}^d \times \bar{\Omega}} y \cdot z \, d\mathbf{p}(y, z)$$

and because of (1.9), we have exactly that

$$\phi(y) + \psi(z) = y \cdot z, \quad \mathbf{p} - a.e. \text{ on } \mathbb{R}^d \times \bar{\Omega}.$$

So $(\phi, \psi, \mathbf{p})$ for $\phi = \psi^*$ is indeed a solution to the mixed MKP.

For the uniqueness part, consider a solution $(\hat{\phi}, \hat{\psi})$ to the primal MKP and a solution $\hat{\mathbf{p}}$ to the dual MKP. Then $(\phi, \psi, \hat{\mathbf{p}})$ and $(\hat{\phi}, \hat{\psi}, \mathbf{p})$ solve the mixed MKP. Observe that since $\hat{\mathbf{p}}$ solves the dual MKP, it maximizes $\int_{\mathbb{R}^d \times \bar{\Omega}} y \cdot z \, d\mathbf{p}(y, z)$, while (1.9) holds for the solution $(\hat{\phi}, \hat{\psi})$, $\hat{\mathbf{p}}(\mathbb{R}^d \times \partial\Omega) = \beta(\partial\Omega) = 0$ and by definition $\hat{\phi}$ minimizes $\int_{\mathbb{R}^d} \phi \, d\alpha$. We then obtain

$$\begin{aligned}
\int_{\mathbb{R}^d \times \bar{\Omega}} y \cdot z \, d\mathbf{p}(y, z) &\leq \int_{\mathbb{R}^d \times \bar{\Omega}} y \cdot z \, d\hat{\mathbf{p}}(y, z) \\
&\leq \int_{\mathbb{R}^d \times \Omega} \hat{\phi}(y) + \hat{\psi}(z) \, d\hat{\mathbf{p}}(y, z) \\
&= \int_{\mathbb{R}^d} \hat{\phi} \, d\alpha + \int_\Omega \hat{\psi} \, d\beta \\
&= \int_{\mathbb{R}^d} \hat{\phi} \, d\alpha \\
&\leq \int_{\mathbb{R}^d} \phi \, d\alpha = \int_{\mathbb{R}^d \times \bar{\Omega}} y \cdot z \, d\mathbf{p}(y, z), \quad \text{by (1.18).}
\end{aligned}$$

So, we actually have equalities, and thus $(\phi, \psi, \hat{\mathbf{p}})$ and $(\hat{\phi}, \hat{\psi}, \mathbf{p})$ solve the mixed MKP. It follows that (1.15), (1.16) and (1.27) hold and therefore $\hat{\mathbf{p}} = \mathbf{p}$, $\nabla\hat{\psi} = \nabla\psi$ *a.e.* on $\bar{\Omega}$. Moreover, because $\hat{\psi}, \psi \in K$ and Ω supposed to be connected, eventually $\hat{\psi} = \psi$ and $\hat{\phi} = \hat{\psi}^* = \psi^* = \phi$, $\alpha - a.e.$.

We can now proceed to the main body of the proof for the polar factorization theorem.

Step 3. (Existence of a Polar Factorization). Let $\mathbf{u} \in L^1(X, \mu; \mathbb{R}^d)$ such that (1.14) is satisfied. This means exactly that the probability measure α defined as

$$\int_{\mathbb{R}^d} f(y) \, d\alpha(y) = \int_X f(\mathbf{u}(y)) \, d\mu(x), \quad \text{for all } f \in C_c(\mathbb{R}^d) \tag{1.33}$$

is absolutely continuous with respect to the Lebesgue measure and $d\alpha(y) = \alpha(y) \, dy$, where α is a non-negative Lebesgue integrable function on $\mathbb{R}^d$. Combining (1.16), (1.17) and (1.30) we get

$$d\mathbf{p}(y, z) = \delta\left(y - \nabla\psi(z)\right) \beta(z) \, dz = \delta\left(z - \nabla\phi(y)\right) \alpha(y) \, dy, \tag{1.34}$$

and since $\mathbf{u} : (X, \mu) \to \mathbb{R}^d$ and $\nabla\phi : \mathbb{R}^d \to \mathbb{R}^d$ are Borel maps $\mathbf{s}(x) := \nabla\phi(\mathbf{u}(x))$ defines a Borel map. Now, $\mathbf{s}$ is measure preserving since using (1.33) and (1.34) we have

$$\begin{aligned}\int_X f(\mathbf{s}(x))\, d\mu(x) &= \int_X f(\nabla\phi(\mathbf{u}(x)))\, d\mu(x) \\ &= \int_{\mathbb{R}^d} f(\nabla\phi(y))\alpha(y)\, dy \\ &= \int_{\mathbb{R}^d\times\bar{\Omega}} f(z)\, d\mathbf{p}(y,z) \\ &= \int_{\bar{\Omega}} f(z)\beta(z)\, dz, \quad \text{for all } f \in C_c(\mathbb{R}^d) \end{aligned} \tag{1.35}$$

and this can extended to any $f \in L^1(\Omega, \beta)$.

To prove existence, it is enough to show that $\mathbf{u}(x) = \nabla\psi(\mathbf{s}(x))$, for $\mu - a.e.$ $x \in X$. Now, we define the set

$$M := \{x \in X : \mathbf{u}(x) \neq \nabla\psi(\mathbf{s}(x))\}$$

and prove that is $\mu-$negligible. Indeed,

$$\begin{aligned} M &= \{x \in X : \mathbf{u}(x) \neq \nabla\psi(\mathbf{s}(x))\} \\ &= \{x \in X : \mathbf{u}(x) \neq \nabla\psi(\nabla\phi((x)))\} \\ &= \mathbf{u}^{-1}\left(\left\{y \in \mathbb{R}^d : \nabla\psi(\nabla\phi(y))\right\}\right). \end{aligned}$$

So, by (1.33), (1.29) and (1.17)

$$\begin{aligned} \mu(M) &= \alpha\left(\left\{y \in \mathbb{R}^d : \nabla\psi(\nabla\phi(y))\right\}\right) \\ &= \mathbf{p}\left(\left\{(y,z) \in \mathbb{R}^d \times \bar{\Omega} : \nabla\psi(\nabla\phi(y))\right\}\right) = 0. \end{aligned}$$

Step 4. (Uniqueness of the Polar Factorization). Assuming that u can be written as $\nabla\psi' \circ \mathbf{s}'$, where $\mathbf{s}'$ is an element of the convex hull of K and $\psi \in K$, we will show that $\psi' = \psi$ and $\mathbf{s}' = \mathbf{s} = \nabla\phi \circ \mathbf{u}$, $\mu - a.e.$ on X. For all $f \in C(\mathbb{R}^d)$ with $|f(y)| \leq c(1 + \|y\|)$, for some constant c, by (1.33) and assuming that $\mathbf{s}'$ is measure preserving it follows

$$\begin{aligned} \int_{\mathbb{R}^d} f(y)\alpha(y)\, dy &= \int_X f(\mathbf{u}(x))\, d\mu(x) \\ &= \int_X f(\nabla\psi'(\mathbf{s}'(x)))\, d\mu(x) \\ &= \int_{\bar{\Omega}} f(\nabla\psi'(z))\beta(z)\, dz. \end{aligned}$$

Then, from step *2.* we deduce that

$$\psi = \psi'. \tag{1.36}$$

Finally, notice that $\mathbf{s}' = \nabla\phi \circ \mathbf{u} = \mathbf{s}$, $\mu - a.e$ is equivalent to $\mathbf{s}' = \nabla\phi \circ \nabla\psi' \circ \mathbf{s}'$. Indeed, (1.35) and (1.36) give

$$\begin{aligned}\mu\left(\{x \in X : \mathbf{s}'(x) \neq \nabla\phi(\nabla\psi'(\mathbf{s}'(x)))\}\right) &= \beta\left(\left\{z \in \bar{\Omega} : z \neq \nabla\phi(\nabla\psi(z))\right\}\right) \\ &= \mathbf{p}\left(\left\{(y,z) \in \mathbb{R}^d \times \bar{\Omega} : z \neq \nabla\phi(\nabla\psi(z))\right\}\right) \\ &= 0,\end{aligned}$$

which concludes the proof. ■

Remark 3 A significantly simplified and more straightforward proof was given by W. Gangbo in [7]. He based his analysis in examining the dual problem of (1.2) (namely (1.37)-(1.38)), using elements of convex analysis.

Remark 4 *(The duality formula).* The dual problem of (1.2) is to find the

$$\sup\left\{\int_X \phi(x)\, d\mu(x) + \int_Y \psi(y)\, d\nu(y)\right\} \tag{1.37}$$

over all continuous and bounded functions such that $\phi(x) + \psi(y) \leq c(x,y)$, which in the case of our interest becomes

$$\phi(x) + \psi(y) \leq \frac{1}{2}|x-y|^2. \tag{1.38}$$

After an appropriate change of variables setting

$$\begin{aligned}\check{\phi}(x) &= \frac{1}{2}|x|^2 - \phi(x) \\ \check{\psi}(x) &= \frac{1}{2}|x|^2 - \psi(y),\end{aligned}$$

the problem (1.37)-(1.38) is transformed into minimizing

$$\int_X \check{\phi}(x)\, d\mu(x) + \int_Y \check{\psi}(y)\, d\nu(y),$$

subject to $\check{\phi}(x) + \check{\psi}(y) \geq x \cdot y$ which, according to Brenier, is the primal MKP.

Chapter 2

A Variational Formulation for the Fokker-Planck Equation

In this chapter we discuss the pioneering work by F. Otto, R. Jordan and D. Kinderlehrer [9], where the solution of the Fokker-Planck equation

$$\frac{\partial}{\partial t}\rho = \nabla \cdot (\rho \nabla \Psi) + \beta^{-1} \Delta \rho \tag{2.1}$$

is considered as the limit of a sequence of solutions for a discrete, time-dependent variational scheme. A normalized solution for a given Fokker-Planck equation is a probability density $\rho(t, x)$ defined for almost every fixed time t, that describes the position or the velocity of a moving particle according to the associated Ito stochastic differential equation namely

$$\begin{cases} dX(t) = -\nabla \Psi(X(t))\, dt + \sqrt{2\beta^{-1}}\, dB(t) \\ X(0) = X_0, \end{cases}$$

where $B(t)$ denotes the standard n-dimensional Brownian motion, β is the inverse temperature, while X_0 is an n-dimensional random variable with probability density ρ_0. In that work, the drift coefficient in the Fokker-Planck equation is considered as the gradient of a smooth, non-negative potential $\Psi(x) : \mathbb{R}^d \to [0, +\infty)$, namely $\Psi \in C_c^\infty(\mathbb{R}^d)$, $\Psi(x) \geq 0$, for all $x \in \mathbb{R}^d$ and furthermore

$$|\nabla \Psi(x)| \leq c(\Psi(x) + 1), \quad \text{for all } x \in \mathbb{R}^d \tag{2.2}$$

for some finite constant c.

Under that assumptions, the introduced discrete scheme is constructed based on the minimization of the functional

$$\mathfrak{R}(\rho) := \frac{1}{2} W(\rho^{(k-1)}, \rho)^2 + hF(\rho) \tag{2.3}$$

whose minimizers $\rho^{(k)}$, via an appropriate interpolation and some additional assumptions we shall mention later, converge to the solution of the Fokker-Planck equation. Here $W(\rho^{(k-1)}, \rho)^2$ is the squared second-order Wasserstein distance (1.4) between the probability densities $\rho^{(k-1)}$ and ρ, h denotes the time step size, while $F(\rho)$ is the *free energy* functional

$$F(\rho) = E(\rho) + \beta^{-1} S(\rho), \tag{2.4}$$

where $E(\rho)$ is the *energy* functional defined as

$$E(\rho) = \int_{\mathbb{R}^d} \Psi \rho \, dx \tag{2.5}$$

and $S(\rho)$ denotes the negative of *Gibbs-Boltzmann entropy* functional

$$S(\rho) = \int_{\mathbb{R}^d} \rho \log \rho \, dx. \tag{2.6}$$

In that point, let us emphasize that we seek for minimizers for (2.3), with finite second moments under the assumption that $\rho^0 = \rho^{(0)}$, which is a given probability density on $\mathbb{R}^d$. More precisely, ρ^0 belongs to the set of admissible densities

$$\mathcal{K} := \left\{ \rho : \mathbb{R}^d \to [0, \infty) \quad \text{measurable} \,\middle|\, \int_{\mathbb{R}^d} \rho(x) dx = 1, M(\rho) < \infty \right\}, \tag{2.7}$$

where

$$M(\rho) = \int_{\mathbb{R}^d} |x|^2 \rho(x) \, dx,$$

while we minimize over all probability densities $\rho \in \mathcal{K}$.

As the authors explain in [9], the choice of (2.3) comes in a natural way, in order to be able to consider the Fokker-Planck equation as a gradient flow of the free energy functional with respect to the Wasserstein distance. To be more specific, according to [13], there exists a unique stationary solution for the Fokker-Planck equation

$$\rho_s(x) = Z^{-1} \exp\{\beta \Psi(x)\},$$

where

$$Z = \int_{\mathbb{R}^d} \exp\{\beta \Psi(x)\} \, dx$$

is the partition function and in order $Z < +\infty$, we require Ψ to grow rapidly enough. The Gibbs distribution ρ_s minimizes over all probability densities on $\mathbb{R}^d$ the free energy functional $F(\rho)$ and even when ρ_s cannot be defined, we are still able to define $F(\rho)$, for a $\rho(t,x)$ which satisfies the Fokker-Planck equation, assuming that $F(\rho^0) < +\infty$ (this is also a requirement in theorem 2). It is also known that the free energy functional decreases in time for any solution for the Fokker-Planck equation [13]. In that context, Otto *et al.* show that at each time, the solution $\rho(t,\cdot)$ of the Fokker-Planck equation follows the direction of the gradient flow of the free energy functional, with respect to the Wasserstein distance (1.4), on the set of probability densities in $\mathcal{K}$ and in order to do so, they use a discrete formulation in time.

We present here those results, starting with the proof regarding the existence and uniqueness of the minimizer $\rho^{(k)}$ for (2.3). Then, we will discuss the proof where the sequence of minimizers $\{\rho^{(k)}\}_{k\in\mathbb{N}}$ converges to the unique solution of (2.1), $\rho(t,x)$ in $L^1((0,T),\mathbb{R}^d)$, for all finite time intervals ($T < +\infty$), as we let the time step size $h \to 0$.

2.1 Minimizer for the Discrete Scheme

In this section, we examine existence and uniqueness of the solution for the minimizing scheme associated with (2.3).

Proposition 1 *Given $\rho^0 \in \mathcal{K}$, there exists a unique $\rho^{(k)}$ that minimizes (2.3), over all $\rho \in \mathcal{K}$.*

Proof. We divide the proof into three steps. In the first step, we argue that the functional (2.6) is well-defined on the space of admissible densities $\mathcal{K}$; then having established that our problem is well-posed, we proceed in proving existence and uniqueness for the minimizer, in steps *2.* and *3.* respectively.

Step 1. Now, we show that S is well-defined on $\mathcal{K}$: for all $\alpha \in (\frac{d}{d+2}, 1)$ there exist a constant $C = C(d) < +\infty$ such that

$$S(\rho) \geq -C(M(\rho)+1)^{\alpha}, \quad \text{for all } \rho \in \mathcal{K}. \tag{2.8}$$

Actually, a finer estimate can been proven: there exists a constant $C = C(\alpha, d) < +\infty$ such that for all $R > 0$ and $\rho \in \mathcal{K}$ we have

$$\int_{\mathbb{R}^d - B_R} |\min\{\rho\log\rho, 0\}| dx \leq C\left(\frac{1}{R^2+1}\right)^{\frac{1}{2}(\alpha(2+d)-d)} (M(\rho)+1)^{\alpha}, \tag{2.9}$$

where B_R denotes the ball of radius R centered at zero. Let us begin with observing that when fixing an $\alpha < 1$, there holds

$$|\min\{z\log z, 0\}| \leq Cz^{\alpha} \quad \text{for all } z \geq 0.$$

Using *Hölder's* inequality we have the estimate

$$\begin{aligned}\int_{\mathbb{R}^d - B_R} |\min\{\rho\log\rho, 0\}| dx &\le C\int_{\mathbb{R}^d - B_R} \rho^\alpha \, dx \\ &= C\int_{\mathbb{R}^d - B_R} \rho^\alpha \left(\frac{|x|^2+1}{|x|^2+1}\right)^\alpha dx \\ &\le C\left(\int_{\mathbb{R}^d - B_R} \left(\frac{1}{|x|^2+1}\right)^{\frac{\alpha}{1-\alpha}} dx\right)^{1-a} \left(\int_{\mathbb{R}^d - B_R} \rho(|x|^2+1)\, dx\right)^\alpha \\ &\le C\left(\int_{\mathbb{R}^d - B_R} \left(\frac{1}{|x|^2+1}\right)^{\frac{\alpha}{1-\alpha}} dx\right)^{1-\alpha} (M(\rho)+1)^\alpha.,\end{aligned}$$

But for $\frac{\alpha}{1-\alpha} > \frac{d}{2}$ there holds

$$\int_{\mathbb{R}^d - B_R} \left(\frac{1}{|x|^2+1}\right)^{\frac{\alpha}{1-\alpha}} dx \le C\left(\frac{1}{R^2+1}\right)^{\frac{\alpha}{1-\alpha}-\frac{d}{2}}$$

and this way we obtain (2.9).

Step 2. The functional (2.3) is bounded below, since given the optimal $\boldsymbol{\gamma} \in \Gamma(\rho_0, \rho_1)$, the inequality $|x_1|^2 \le 2|x_0|^2 + 2|x_0 - x_1|^2$ implies

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} |x_1|^2 \, \boldsymbol{\gamma}(dx_0, dx_1) &\le 2\int_{\mathbb{R}^d\times\mathbb{R}^d} |x_0|^2 + |x_0 - x_1|^2 \, \boldsymbol{\gamma}(dx_0, dx_1) \\ \int_{\mathbb{R}^d} |x_1|^2\rho_1 \, dx_1 &\le 2\int_{\mathbb{R}^d} |x_0|^2\rho_0 \, dx_0 + 2\int_{\mathbb{R}^d\times\mathbb{R}^d} |x_0 - x_1|^2 \, \boldsymbol{\gamma}(dx_0, dx_1) \\ \therefore M(\rho_1) &\le 2M(\rho_0) + 2W(\rho_0, \rho_1)^2, \end{aligned} \tag{2.10}$$

for all $\rho_0, \rho_1 \in \mathcal{K}$. Under the assumption that $\Psi(x) \ge 0$ for all $x \in \mathbb{R}^d$, it follows that, using (2.10) and (2.8), one may obtain the bound

$$\begin{aligned}\frac{1}{2}W(\rho^{(k-1)}, \rho)^2 + hF(\rho) &\ge \frac{1}{4}M(\rho) - \frac{1}{2}M(\rho^{(k-1)}) + hS(\rho) \\ &\ge \frac{1}{4}M(\rho) - \frac{1}{2}M(\rho^{(k-1)}) - Ch(M(\rho)+1)^\alpha. \end{aligned} \tag{2.11}$$

Consider a minimizing sequence $\{\rho_n\}$ for $\mathfrak{R}(\rho)$. Then

$$\{S(\rho_n)\} \quad \text{is bounded above,} \tag{2.12}$$

while, because of (2.11), the sequence

$$\{M(\rho_n)\} \quad \text{is bounded,} \tag{2.13}$$

which combined with (2.9), gives that

$$\left\{ \int_{\mathbb{R}^d - B_R} |\min\{\rho_n \log \rho_n, 0\}|\ dx \right\}$$

is bounded. The latter together with (2.12) imply that

$$\int_{\mathbb{R}^d - B_R} \max\{\rho_n \log \rho_n, 0\}\ dx \quad \text{is bounded} \tag{2.14}$$

too. Let us now consider the mapping $z \mapsto g(z) := \max\{z \log z, 0\}$, for $z \in [0, \infty)$ which is convex, has super-linear growth and moreover

$$\lim_{z\to\infty} \frac{g(z)}{z} = +\infty.$$

This elementary observation combined with (2.14), allows us to use the *de la Vallé-Poussin* theorem to confirm uniform integrability for $\{\rho_n\}$. Then because of (2.13) it is not difficult to verify that $\{\rho_n\}$ is uniformly bounded too. Consequentially, the *Dunford-Pettis* theorem (see [4]) implies the existence of a density $\rho^{(k)} \in \mathcal{K}$ such that extracting a subsequence if necessary (which we will still label $\{\rho_n\}$ for simplicity)

$$\rho_n \rightharpoonup \rho^{(k)} \quad \text{weakly in } L^1(\mathbb{R}^d). \tag{2.15}$$

It remains to show the lower semicontinuity conditions

$$S(\rho^{(k)}) \leq \liminf_{n\to\infty} S(\rho_n), \tag{2.16}$$

$$E(\rho^{(k)}) \leq \liminf_{n\to\infty} E(\rho_n), \tag{2.17}$$

$$W(\rho^{(k-1)}, \rho^{(k)})^2 \leq \liminf_{n\to\infty} W(\rho^{(k-1)}, \rho_n)^2. \tag{2.18}$$

For (2.16), owning (2.15), we use the convexity of the maps $z \mapsto z \log z$ and $z \mapsto g(z) \geq 0$, to derive

$$\int_{B_R} \rho^{(k)} \log \rho^{(k)}\ dx \leq \liminf_{n\to\infty} \int_{B_R} \rho_n \log \rho_n\ dx, \tag{2.19}$$

$$\int_{\mathbb{R}^d - B_R} \max\{\rho^{(k)} \log \rho^{(k)}, 0\}\ dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^d - B_R} \max\{\rho_n \log \rho_n, 0\}\ dx, \tag{2.20}$$

for any $R < \infty$. From (2.9) and (2.13), sending $R \to \infty$ we obtain

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d - B_R} |\min\{\rho_n \log \rho_n, 0\}|\ dx = 0. \tag{2.21}$$

Finally, combining

$$S(\rho^{(k)}) \leq \int_{B_R} \rho^{(k)} \log \rho^{(k)} \, dx + \int_{\mathbb{R}^d - B_R} \max\{\rho^{(k)} \log \rho^{(k)}, 0\} \, dx$$

with (2.19), (2.20) and (2.21) we obtain (2.16). For (2.17), we use (2.15) and *Fatou's* lemma. Now, moving to (2.18), first choose an admissible transport plan $\boldsymbol{\gamma}_n \in \Gamma(\rho^{(k-1)}, \rho_n)$, such that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \boldsymbol{\gamma}_n(dx, dy) \leq W(\rho^{(k-1)}, \rho_n)^2 + \frac{1}{n}.$$

Then, (2.13) implies that the sequence of probability measures $\{\rho_n \mathcal{L}^d\}$ is tight or precompact with respect to the narrow convergence in the space of probability measures on $\mathbb{R}^d$ (see [2, chapter 5]). Likewise, since $M(\rho^{(k-1)}) < \infty$, we deduce that $\{\rho^{(k-1)} \mathcal{L}^d\}$ is tight too and Lemma 5.2.2 [1] allows us to pass the tightness property on $\{\boldsymbol{\gamma}_n\} \in \Gamma(\rho^{(k-1)}, \rho_n)$. Thus, invoking *Prokhorov's* theorem, there exists a subsequence of $\{\boldsymbol{\gamma}_n\}$, (which we will still label $\{\boldsymbol{\gamma}_n\}$ for simplicity) such that

$$\boldsymbol{\gamma}_n \rightharpoonup \boldsymbol{\gamma} \qquad \text{narrowly to some } \boldsymbol{\gamma} \in \Gamma(\rho^{(k-1)}, \rho^{(k)}).$$

Finally, consider a continuous function $\varphi : \mathbb{R}^d \to [0, 1]$ such that

$$\varphi_R(x) = \begin{cases} 1 & \text{when inside of } B_R \\ 0 & \text{outside of } B_{2R}. \end{cases}$$

Then a straightforward analysis reveals that for all fixed $R < \infty$

$$\begin{aligned} &\int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_R(x) \varphi_R(y) |x-y|^2 \boldsymbol{\gamma}(dx, dy) \\ &\qquad = \lim_{n \to \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_R(x) \varphi_R(y) |x-y|^2 \boldsymbol{\gamma}_n(dx, dy) \\ &\qquad \leq \liminf_{n \to \infty} W(\rho^{(k-1)}, \rho_n)^2, \end{aligned}$$

while using the *Monotone Convergence* theorem

$$W(\rho^{(k-1)}, \rho^{(k)})^2 \leq \lim_{R \to \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_R(x) \varphi_R(y) |x-y|^2 \boldsymbol{\gamma}(dx, dy).$$

Step 3. For the uniqueness of the minimizer, we argue that $\mathcal{K}$ is convex, while (2.3) is strictly convex: S is strictly convex, $\rho \mapsto W(\rho^{(k-1)}, \rho)$ is convex over $\mathcal{K}$ and E is linear. ■

2.2 Convergence to the solution of the Fokker-Planck equation

We are now ready to discuss upon convergence to the solution of (2.1) introducing the associated variational scheme: Given a finite time interval $(0,T)$, we discretize it uniformly and at each time step we consider the unique minimizer of (2.3). After an appropriate interpolation in time, this sequence of minimizers converge narrowly in $L^1(\mathbb{R}^d)$ and eventually for all times $t \in (0,+\infty)$, to the unique solution $\rho(x,t)$ of (2.1); moreover, ρ satisfies the initial condition of the discrete scheme and for any finite time horizon T, $M(\rho)$ and $E(\rho) \in L^\infty((0,T))$.

Theorem 2 *Given $\rho^0 \in \mathcal{K}$ with $F(\rho^0) < \infty$ and $h > 0$, consider the sequence of minimizers $\{\rho_h^{(k)}\}$ of (2.3), over all $\rho \in \mathcal{K}$ and define $\rho_h : (0,+\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ as $\rho_h(t) := \rho_h^{(k)}$ for all $t \in [kh,(k+1)h)$, $k \in \mathbb{N}_0$. Then*

$$\rho_h(t) \xrightarrow[h\to 0^+]{} \rho(t) \quad \textit{weakly in } L^1(\mathbb{R}^d), \textit{ for all } t \in (0,+\infty), \tag{2.22}$$

where $\rho \in C^\infty((0,+\infty)\times\mathbb{R}^d)$ is the unique solution of the Fokker-Planck equation (2.1) and satisfies

$$\rho(t) \xrightarrow[t\to 0^+]{} \rho^0 \quad \textit{in } L^1(\mathbb{R}^d), \tag{2.23}$$

$$M(\rho), E(\rho) \in L^\infty((0,T)), \quad \textit{for all } T < +\infty. \tag{2.24}$$

Proof. We organize this proof into four steps. The first part is about calculating the discrete derivative of $\mathfrak{R}(\rho)$ with respect to time, in order to write a formula in relation to its first variation and then suitably estimate the associated gradient flow of the free energy functional. In the second part of the proof, after constructing some useful bounds, we deduce a weaker formulation of (2.22), for a finite time horizon. As for the third part, we work to derive (2.22) which allow us to take the limit as $h \to 0^+$ in the gradient flow inequality, we constructed in the first step. Finally, we show that each solution for this variational problem is smooth, satisfies the Fokker-Planck equation and the initial condition and it is unique.

To simplify the notation, let $\beta \equiv 1$.

Step 1. Consider a smooth vector field with bounded support $\boldsymbol{\xi} \in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$ and for $\tau \in \mathbb{R}$ define the *flow* $\{\Phi_\tau\}$ by

$$\partial_\tau \Phi_\tau = \boldsymbol{\xi}(\Phi_\tau), \quad \text{for all } \tau \in \mathbb{R},\ \Phi_0 = id. \tag{2.25}$$

Then for any $\tau \in \mathbb{R}$, let $(\rho_\tau \mathcal{L}^d) := \Phi_\tau \#(\rho^{(k)} \mathcal{L}^d)$ *i.e.*

$$\int_{\mathbb{R}^d} \varphi(y)\rho_\tau(y)\, dy = \int_{\mathbb{R}^d} \varphi(\Phi_\tau(y))\rho^{(k)}(y)\, dy, \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^d), \tag{2.26}$$

where $\mathcal{L}^d$ denotes the $d-$dimensional Lebesgue measure. Assuming that $\det \nabla\Phi_\tau \neq 0$, we use the change of variables formula

$$\int_{\mathbb{R}^d} \varphi(y)\rho_\tau(y)\, dy = \int_{\mathbb{R}^d} \varphi(\Phi_\tau(x))\rho_\tau(\Phi_\tau(x)) \det \nabla\Phi_\tau(x)\, dx$$

so the right hand side equals the right hand side of (2.26). Then

$$\int_{\mathbb{R}^d} \varphi(\Phi_\tau(x))(\rho^{(k)}(x) - \rho_\tau(\Phi_\tau(x)) \det \nabla\Phi_\tau(x))\, dx = 0$$
$$\therefore \det \nabla\Phi_\tau \rho_\tau \circ \Phi_\tau(x) = \rho^{(k)}(x). \tag{2.27}$$

If the measure $\rho^{(k)}$ is optimal for the functional (2.3), given admissible $\rho^{(k-1)}, \rho_\tau$ then $\mathfrak{R}(\rho^{(k)}) \leq \mathfrak{R}(\rho_\tau)$, for all $\tau > 0$. Consequentially

$$\frac{1}{\tau}\left(\left(\frac{1}{2}W(\rho^{(k-1)}, \rho_\tau)^2 + hF(\rho_\tau)\right) - \left(\frac{1}{2}W(\rho^{(k-1)}, \rho^{(k)})^2 + hF(\rho^{(k)})\right)\right) \geq 0, \tag{2.28}$$

for all $\tau > 0$. Now, we want to investigate the limit as $\tau \to 0$. In order to do so, first we calculate respectively the discrete derivatives for $E(\rho_\tau)|_{\tau=0}$, $S(\rho_\tau)|_{\tau=0}$ and then we estimate the limit as $\tau \to 0$, of the difference

$$\frac{1}{\tau}\left(\frac{1}{2}W(\rho^{(k-1)}, \rho_\tau)^2 - \frac{1}{2}W(\rho^{(k-1)}, \rho^{(k)})^2\right). \tag{2.29}$$

Using (2.26) for $\varphi = \Psi$ and (2.5) we have

$$E(\rho_\tau) - E(\rho^{(k)}) = \int_{\mathbb{R}^d} \rho^{(k)}(y)(\Psi(\Phi_\tau(y)) - \Psi(y))\, dy$$

and if we multiply both sides with $1/\tau$ then

$$\frac{d}{d\tau}E(\rho_\tau)|_{\tau=0} = \int_{\mathbb{R}^d} \nabla\Psi(y) \cdot \boldsymbol{\xi}(y)\rho^{(k)}(y)\, dy. \tag{2.30}$$

Observe that from (2.26) and (2.27) we have

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_\tau(y)\log(\rho_\tau(y))\, dy &= \int_{\mathbb{R}^d} \rho^{(k)}(y)\log(\rho_\tau(\Phi_\tau(y)))\, dy \\ &= \int_{\mathbb{R}^d} \rho^{(k)}(y)\log\left(\frac{\rho^{(k)}(y)}{\det \nabla\Phi_\tau(y)}\right)\, dy\end{aligned}$$

then
$$\frac{1}{\tau}(S(\rho_\tau) - S(\rho^{(k)})) = -\frac{1}{\tau}\int_{\mathbb{R}^d} \rho^{(k)}(y)\log(\det \nabla\Phi_\tau(y))\, dy$$

which together with

$$\frac{d}{d\tau}\det \nabla\Phi_\tau(y) = \nabla\Phi_\tau(y)\nabla\cdot\boldsymbol{\xi}$$

gives that
$$\frac{d}{d\tau}S(\rho_\tau)|_{\tau=0} = -\int_{\mathbb{R}^d}\rho^{(k)}\nabla\cdot\boldsymbol{\xi}\, dy. \tag{2.31}$$

recalling that $\Phi_0 = id.$. In order to estimate (2.29) consider the optimal plan $\boldsymbol{\gamma}\in\Gamma_{opt}(\rho^{(k-1)},\rho^{(k)})$ and define a plan $\boldsymbol{\gamma}_\tau\in\Gamma(\rho^{(k-1)},\rho_\tau)$ such that $(\rho_\tau\mathcal{L}^d) = \Phi_\tau\#(\rho^{(k)}\mathcal{L}^d)$ *i.e.*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(x,y)\,\boldsymbol{\gamma}_\tau(dx,dy) = \int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(x,\Phi_\tau(y))\,\boldsymbol{\gamma}(dx,dy),$$

for all $\varphi\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$. Therefore

$$\frac{1}{\tau}\left(\frac{1}{2}W(\rho^{(k-1)},\rho_\tau)^2 - \frac{1}{2}W(\rho^{(k-1)},\rho^{(k)})^2\right) \le \int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{\tau}\left(\frac{1}{2}|\Phi_\tau(y)-x|^2 - \frac{1}{2}|y-x|^2\right)\,\boldsymbol{\gamma}(dx,dy)$$

and by the definition of because Φ_τ it follows

$$\limsup_{\tau\to 0}\frac{1}{\tau}\left(\frac{1}{2}W(\rho^{(k-1)},\rho_\tau)^2 - \frac{1}{2}W(\rho^{(k-1)},\rho^{(k)})^2\right) \le \int_{\mathbb{R}^d\times\mathbb{R}^d}(y-x)\cdot\boldsymbol{\xi}(y)\,\boldsymbol{\gamma}(dx,dy).$$

Combining now the above results (2.30), (2.31) with the latter and because of (2.28) and the symmetry $\boldsymbol{\xi} = -\boldsymbol{\xi}$, we obtain

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}(y-x)\cdot\boldsymbol{\xi}\,\boldsymbol{\gamma}(dx,dy) + h\int_{\mathbb{R}^d}(\nabla\Psi\cdot\boldsymbol{\xi} - \nabla\cdot\boldsymbol{\xi})\rho^{(k)}\, dy = 0,$$

so for $\boldsymbol{\xi} = \nabla\varphi$

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}(y-x)\cdot\nabla\varphi\,\boldsymbol{\gamma}(dx,dy) + h\int_{\mathbb{R}^d}(\nabla\Psi\cdot\nabla\varphi - \Delta\varphi)\rho^{(k)}\, dy = 0. \tag{2.32}$$

We use $\varphi(y) - \varphi(x) \leq \nabla_y \varphi(x-y)$ to estimate

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} (\rho^{(k)} - \rho^{(k-1)}) \varphi(y)\, dy - \int_{\mathbb{R}^d \times \mathbb{R}^d} (y-x) \cdot \nabla \varphi(y)\, \boldsymbol{\gamma}(dx, dy) \right| \\
&\quad = \left| \int_{\mathbb{R}^d} \varphi(y)\, \boldsymbol{\gamma}(dx, dy) - \int_{\mathbb{R}^d} \varphi(x)\, \boldsymbol{\gamma}(dx, dy) - \int_{\mathbb{R}^d \times \mathbb{R}^d} (y-x) \cdot \nabla \varphi(y)\, \boldsymbol{\gamma}(dx, dy) \right| \\
&\quad \leq \frac{1}{2} \sup_{\mathbb{R}^d} |\nabla^2 \varphi| \int_{\mathbb{R}^d \times \mathbb{R}^d} |y-x|^2 \cdot \nabla \varphi(y)\, \boldsymbol{\gamma}(dx, dy) \\
&\quad = \frac{1}{2} \sup_{\mathbb{R}^d} |\nabla^2 \varphi|\, W(\rho^{(k-1)}, \rho^{(k)})^2
\end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. Consequently, (2.32) gives

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \frac{1}{h} (\rho^{(k)} - \rho^{(k-1)}) \varphi + (\nabla \Psi \cdot \nabla \varphi - \Delta \varphi) \rho^{(k)}\, dy \right| \\
&\qquad\qquad \leq \frac{1}{2} \sup_{\mathbb{R}^d} |\nabla^2 \varphi| \frac{1}{h}\, W(\rho^{(k-1)}, \rho^{(k)})^2, \qquad (2.33)
\end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$.

Step 2. In this step, we consider the sequence of minimizers $\{\rho_h^{(k)}\}$ of (2.3) over $\mathcal{K}$. In order to pass to the limit $h \to 0^+$ for (2.33),we need, first, to construct some helpful estimates. Working with a finite time horizon $T < +\infty$, we will prove that there exist a constant $C < +\infty$ for which for all $N \in \mathbb{N}, h \in [0,1]$ such that $Nh \leq T$, there holds

$$M(\rho_h^{(N)}) \leq C \qquad (2.34)$$

$$\int_{\mathbb{R}^d} \max\{\rho_h^{(N)} \log \rho_h^{(N)}, 0\} dx \leq C \qquad (2.35)$$

$$E(\rho_h^{(N)}) \leq C \qquad (2.36)$$

$$\sum_{k=1}^{N} W(\rho_h^{(k-1)}, \rho_h^{(k)})^2 \leq Ch. \qquad (2.37)$$

For (2.34), assume that $\rho_h^{(k-1)}$ satisfies the variational principle for determining the minimizer $\rho_h^{(k)}$ of (2.3); then

$$\frac{1}{2} W(\rho_h^{(k-1)}, \rho_h^{(k)})^2 + hF(\rho_h^{(k)}) \leq hF(\rho_h^{(k-1)})$$

and if we sum over k, we get

$$\sum_{k=1}^{N} \frac{1}{2h} W(\rho_h^{(k-1)}, \rho_h^{(k)})^2 \leq F(\rho^0) - F(\rho_h^{(N)}). \qquad (2.38)$$

Then using (2.10), *Cauchy-Schwarz* inequality, (2.38) and (2.8) we have that

$$
\begin{aligned}
M(\rho_h^{(N)}) &\leq 2N \sum_{k=1}^{N} W(\rho_h^{(k-1)}, \rho_h^{(k)})^2 + 2M(\rho^0) \\
&\leq 4hN(F(\rho^0) - F(\rho_h^{(N)})) + 2M(\rho^0) \\
&\leq 4T \left(F(\rho^0) + C(M(\rho_h^{(N)}) + 1)^\alpha \right) + 2M(\rho^0).
\end{aligned}
$$

For (2.35), we use (2.9) for $R = 0$ and (2.38):

$$
\begin{aligned}
\int_{\mathbb{R}^d} \max\{\rho_h^{(N)} \log \rho_h^{(N)}, 0\} \, dx &\leq S(\rho_h^{(N)}) + C(M(\rho_h^{(N)}) + 1)^\alpha \\
&\leq F(\rho_h^{(N)}) + C(M(\rho_h^{(N)}) + 1)^\alpha \\
&\leq F(\rho^0) + C(M(\rho_h^{(N)}) + 1)^\alpha,
\end{aligned}
$$

while, in order to obtain (2.36), we use (2.5) and (2.38):

$$
\begin{aligned}
E(\rho_h^{(N)}) &= F(\rho_h^{(N)}) - S(\rho_h^{(N)}) \\
&\leq F(\rho_h^{(N)}) + C(M(\rho_h^{(N)}) + 1)^\alpha \\
&\leq F(\rho^0) + C(M(\rho_h^{(N)}) + 1)^\alpha.
\end{aligned}
$$

Finally for (2.37), inequalities (2.38) and (2.8) give

$$
\begin{aligned}
\sum_{k=1}^{N} W(\rho_h^{(k-1)}, \rho_h^{(k)})^2 &\leq 2h(F(\rho^0) - F(\rho_h^{(N)})) \\
&\leq 2hF(\rho^0) + 2hC(M(\rho_h^{(N)}) + 1)^\alpha.
\end{aligned}
$$

Arguing as in the second step of the previous proof (there we used (2.14) and (2.13)), we find out that the estimates (2.34) and (2.35) imply the existence of a measurable $\rho \in L^1((0,T), \mathbb{R}^d)$ and a subsequence of $\{\rho_h\}$, (which we will not relabel) such that

$$
\rho_h \rightharpoonup \rho \quad \text{weakly in } L^1((0,T) \times \mathbb{R}^d) \text{ for all } T < +\infty. \tag{2.39}
$$

Observe, as well, that the previous estimations also imply that $\rho(t) \in \mathcal{K}$, $M(\rho)$ and $E(\rho) \in L^\infty((0,T))$, for all $T < +\infty$.

Step 3. We will now work with (2.39) to derive (2.22). Using estimate (2.37) and the *Cauchy-Schwarz* inequality we have that for all $T < +\infty$ there exist a constant $C < +\infty$ such that for all $N, \tilde{N} \in \mathbb{N}$ and all $h \in [0,1]$ with $Nh, \tilde{N}h \leq T$, there holds

$$
\begin{aligned}
W(\rho_h^{(\tilde N)}, \rho_h^{(N)})^2 &\le \left(\sum_{k=\min\{\tilde N, N\}}^{\max\{\tilde N, N\}} W(\rho_h^{(k-1)}, \rho_h^{(k)}) \right)^2 \\
&\le |\tilde N - N| \sum_{k=\min\{\tilde N, N\}}^{\max\{\tilde N, N\}} W(\rho_h^{(k-1)}, \rho_h^{(k)})^2 \\
&\le |\tilde N - N| \sum_{k=1}^{\max\{\tilde N, N\}} W(\rho_h^{(k-1)}, \rho_h^{(k)})^2 \\
&\le C|\tilde N h - N h|. \qquad (2.40)
\end{aligned}
$$

For all $\rho, \tilde\rho \in \mathcal{K}$ such that $\boldsymbol{\gamma} \in \Gamma(\rho, \tilde\rho)$ and for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ we estimate the difference

$$
\begin{aligned}
\left| \int_{\mathbb{R}^d} \varphi \tilde\rho \, dx - \int_{\mathbb{R}^d} \varphi \rho \, dx \right| &= \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} (\varphi(x) - \varphi(y)) \, \boldsymbol{\gamma}(dx, dy) \right| \\
&\le \sup_{\mathbb{R}^d} |\nabla \varphi| \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| \, \boldsymbol{\gamma}(dx, dy) \\
&\le \sup_{\mathbb{R}^d} |\nabla \varphi| \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \, \boldsymbol{\gamma}(dx, dy) \right)^{1/2} \\
&\le \sup_{\mathbb{R}^d} |\nabla \varphi| \, W(\rho, \tilde\rho)
\end{aligned}
$$

thus via (2.40) and the definition of ρ_h the latter becomes

$$
\left| \int_{\mathbb{R}^d} \varphi \rho_h(\tilde t) \, dx - \int_{\mathbb{R}^d} \varphi \rho_h(t) \, dx \right| \le C \sup_{\mathbb{R}^d} |\nabla \varphi| (|\tilde t - t| + h)^{1/2}, \qquad (2.41)
$$

for all $\tilde t, t \in (0,T)$ and all $\varphi \in C_c^\infty(\mathbb{R}^d)$.

Let $t \in (0,T)$ and fix a $\varphi \in C_c^\infty(\mathbb{R}^d)$. Then for all $\varepsilon > 0$ we write

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^d} \varphi \rho_h(t) \, dx - \int_{\mathbb{R}^d} \varphi \rho(t) \, dx \right| \\
&\quad \le \left| \int_{\mathbb{R}^d} \varphi \rho_h(t) \, dx - \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \int_{\mathbb{R}^d} \varphi \rho_h(s) \, dx ds \right| \\
&\quad + \left| \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \int_{\mathbb{R}^d} \varphi \rho_h(s) \, dx ds - \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \int_{\mathbb{R}^d} \varphi \rho(s) \, dx ds \right| \qquad (2.42) \\
&\quad + \left| \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \int_{\mathbb{R}^d} \varphi \rho(s) \, dx ds - \int_{\mathbb{R}^d} \varphi \rho(t) \, dx \right|.
\end{aligned}
$$

We treat each term in right hand side of (2.42) separately. The first term according to (2.41) is bounded by $C \sup_{\mathbb{R}^d} |\nabla \varphi| (\varepsilon + h)^{1/2}$. The second term because of (2.39), converges to 0 as $h \to 0$, for all $\varepsilon > 0$ fixed. As for the third term, we will demonstrate later that ρ and all the derivatives of ρ belong to $L^p_{loc}((0,\infty)\times\mathbb{R}^d)$; thus using the *Lebesgue Convergence* theorem we conclude it converges to 0, as $\varepsilon \to 0$. Consequentially

$$\int_{\mathbb{R}^d} \varphi\rho_h(t)\,dx \to \int_{\mathbb{R}^d} \varphi\rho(t)\,dx \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^d)$$

and since (2.34) and (2.35) hold, the above is equivalent to

$$\rho_h \rightharpoonup \rho \quad \text{weakly in } L^1(\mathbb{R}^d) \text{ or}$$
$$\int_{\mathbb{R}^d} \varphi\rho_h(t)\,dx \to \int_{\mathbb{R}^d} \varphi\rho(t)\,dx, \quad \text{for all } \varphi \in L^\infty(\mathbb{R}^d).$$

We can now return to (2.33) and refer (2.37) and (2.22) to obtain

$$-\int_{(0,\infty)\times\mathbb{R}^d} \rho(\partial_t\varphi - \nabla\Psi\cdot\nabla\varphi + \Delta\varphi)\,dtdx = \int_{\mathbb{R}^d} \rho^0\varphi(0)\,dx, \tag{2.43}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$, with $M(\rho), E(\rho) \in L^\infty((0,T))$, for all $T<\infty$.

Step 4. To conclude the proof it remains to show

1. Each solution of (2.43) is smooth and satisfies the Fokker-Planck equation (2.1).
2. Each solution of (2.43) with $M(\rho), E(\rho) \in L^\infty((0,T))$, $T<\infty$ satisfies the initial condition (2.23).
3. There exists at most one such solution.

1. Observe that (2.43) for $\varphi = \varphi(x, t+t_1)$ and $\varphi = \varphi(x, t+t_0)$ gives

$$\int_{\mathbb{R}^d} \rho(t_1)\varphi(t_1)\,dx - \int_{t_0}^{t_1}\int_{\mathbb{R}^d} \rho(\partial_t\varphi - \nabla\Psi\cdot\nabla\varphi + \Delta\varphi)\,dxdt = \int_{\mathbb{R}^d} \rho(t_0)\varphi(t_0)\,dx,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ and *a.e.* $0<t<1$. Now, let $\eta \in C_c^\infty(\mathbb{R}^d)$ be a fixed (cut-off) function, then the above implies

$$\begin{aligned}\int_{\mathbb{R}^d} \eta\rho(t_1)\varphi(t_1)\,dx &- \int_{t_0}^{t_1}\int_{\mathbb{R}^d} \eta\rho(\partial_t\varphi + \Delta\varphi)\,dxdt \\ &= \int_{t_0}^{t_1}\int_{\mathbb{R}^d} \eta\rho(\Delta\eta - \nabla\Psi\cdot\nabla\eta)\varphi\,dxdt \\ &+ \int_{t_0}^{t_1}\int_{\mathbb{R}^d} \rho(2\nabla\eta - \eta\nabla\Psi)\cdot\nabla\varphi\,dxdt \\ &+ \int_{\mathbb{R}^d} \eta\rho(t_0)\varphi(t_0)\,dx,\end{aligned} \tag{2.44}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and *a.e.* $0 \le t_0 < t_1$. Fix also a point $(t_1, x_1) \in (0,\infty) \times \mathbb{R}^d$ and for all $\varepsilon > 0$, define

$$\varphi_\varepsilon(t,x) := G(t_1 + \varepsilon - t, x - x_1), \qquad \text{where}$$
$$G(t,x) = t^{-d/2} g(t^{-1/2}x), \quad \text{and} \quad g(x) = (2\pi)^{d/2} \exp^{-1/2|x|^2}.$$

Then if we replace φ_ε in (2.44) and take the limit $\varepsilon \to 0$ we have for *a.e.* $0 \le t_0 < t_1$

$$\begin{aligned}(\rho\eta)(t_1) = &\int_{t_0}^{t_1} (\rho(t)(\Delta\eta - \nabla\Psi \cdot \nabla\eta)) * G(t_1 - t)\, dt \\ &+ \int_{t_0}^{t_1} (\rho(t)(2\nabla\eta - \eta\nabla\Psi)) * \nabla G(t_1 - t))\, dt \\ &+ (\rho\eta)(t_0) * (t_1 - t_0),\end{aligned} \tag{2.45}$$

where " $*$ " denotes the convolution with respect to x. Taking the L^p norm we have for *a.e.* $0 \le t_0 < t_1$

$$\begin{aligned}\|(\rho\eta)(t_1)\|_{L^p} = &\int_{t_0}^{t_1} \|\rho(t)(\Delta\eta - \nabla\Psi \cdot \nabla\eta)\|_{L^1} \|G(t_1 - t)\|_{L^p}\, dt \\ &+ \int_{t_0}^{t_1} \|\rho(t)(2\nabla\eta - \eta\nabla\Psi))\|_{L^1} \|\nabla G(t_1 - t)\|_{L^p}\, dt \\ &+ \|(\rho\eta)(t_0)\|_{L^1} \|(t_1 - t_0)\|_{L^p}.\end{aligned}$$

Since

$$\|G(t)\|_{L^p} = t^{(\frac{1}{p}-1)\frac{d}{2}} \|g\|_{L^p} \quad \text{and} \quad \|\nabla G(t)\|_{L^p} = t^{\frac{1}{p}\frac{d}{2} - \frac{d+1}{2}} \|\nabla g\|_{L^p}$$

the latter becomes

$$\begin{aligned}\|(\rho\eta)(t_1)\|_{L^p} = &\operatorname*{ess\,sup}_{(t_0,t_1)} \|\rho(t)(\Delta\eta - \nabla\Psi \cdot \nabla\eta)\|_{L^1} \int_{t_0}^{t_1} t^{(\frac{1}{p}-1)\frac{d}{2}} \|g\|_{L^p}\, dt \\ &+ \operatorname*{ess\,sup}_{(t_0,t_1)} \|\rho(t)(2\nabla\eta - \eta\nabla\Psi))\|_{L^1} \int_{t_0}^{t_1} t^{\frac{1}{p}\frac{d}{2} - \frac{d+1}{2}} \|\nabla g\|_{L^p}\, dt \\ &+ \|(\rho\eta)(t_0)\|_{L^1} \|(t_1 - t_0)\|_{L^p},\end{aligned}$$

for *a.e.* $0 \le t_0 < t_1$. Note that for $p < \frac{d}{d-1}$ the integrals with respect to t are finite. We conclude using the usual bootstrap arguments that since $\rho \in L^p_{loc}((0,\infty) \times \mathbb{R}^d)$ then all its derivatives will be in $L^p_{loc}((0,\infty) \times \mathbb{R}^d)$.

2. Notice that for all $t > 0$, using (2.45), we can also express the difference

$$\|(\rho\eta)(t_1) - (\rho^0\eta) * G(t_1)\|_{L^1} = \operatorname*{ess\,sup}_{(0,t_1)} \|\rho(t)(\Delta\eta - \nabla\Psi\cdot\nabla\eta)\|_{L^1} \int_0^{t_1} \|g\|_{L^1}\, dt$$
$$+ \operatorname*{ess\,sup}_{(0,t_1)} \|\rho(t)(2\nabla\eta - \eta\nabla\Psi))\|_{L^1} \int_0^{t_1} t^{-1/2}\|\nabla g\|_{L^1}\, dt,$$

for all $t_1 > 0$. Consequentially

$$(\rho\eta)(t) - (\rho^0\eta) * G(t) \xrightarrow[t\to 0]{} 0 \quad \text{in } L^1(\mathbb{R}^d),$$
$$(\rho^0\eta) * G(t) \xrightarrow[t\to 0]{} \rho^0\eta \quad \text{in } L^1(\mathbb{R}^d),$$
$$\therefore (\rho\eta)(t) \xrightarrow[t\to 0]{} \rho^0\eta \quad \text{in } L^1(\mathbb{R}^d)$$

and since $M(\rho(t)) < \infty$ we have the convergence in $L^1(\mathbb{R}^d)$.

3. Consider ρ_1 and ρ_2 two solutions of (2.1), that additionally satisfy respectively

$$\rho_1 \xrightarrow[L^1(\mathbb{R}^d)]{t\to 0^+} \rho^0 \quad \text{and } E(\rho_1), M(\rho_1) \in L^\infty(\mathbb{R}^d), \text{ for all } T < \infty$$
$$\rho_2 \xrightarrow[L^1(\mathbb{R}^d)]{t\to 0^+} \rho^0 \quad \text{and } E(\rho_2), M(\rho_2) \in L^\infty(\mathbb{R}^d), \text{ for all } T < \infty.$$

Then, it is easy to see that the difference $\rho := \rho_1 - \rho_2$ also satisfies (2.1). Here, we make of use a convex and smooth enough approximation of the $|\cdot|$ function, lets say $\phi_\varepsilon(x) := (x^2 + \varepsilon^2)^{1/2}$, where, of course, we are interested of the limit $\{\phi_\varepsilon\}_{\varepsilon\to 0^+}$. Multiplying (2.1) for $\rho := \rho_1 - \rho_2$ (assuming always that $\beta \equiv 1$) with $\phi'_\varepsilon(\rho)$ we have

$$\frac{\partial}{\partial t}(\phi_\varepsilon(\rho)) - \nabla\cdot\{\phi_\varepsilon(\rho)\nabla\Psi + \nabla(\phi_\varepsilon(\rho))\} = -\phi''_\varepsilon(\rho)|\nabla\rho|^2 + (\phi'_\varepsilon(\rho)\rho - \phi_\varepsilon(\rho))\,\Delta\Psi$$
$$\leq (\phi'_\varepsilon(\rho)\rho - \phi_\varepsilon(\rho))\,\Delta\Psi,$$

using the convexity property of ϕ_ε. Multiplying with a test function $\eta \in C_c^\infty(\mathbb{R}^d)$ and then integrating on $\mathbb{R}^d$, the latter gives

$$\frac{d}{dt}\int_{\mathbb{R}^d} \eta\phi_\varepsilon(\rho)\, dx + \int_{\mathbb{R}^d} \nabla\eta\,(\phi_\varepsilon(\rho)\nabla\Psi + \nabla(\phi_\varepsilon(\rho)))\, dx$$
$$\leq \int_{\mathbb{R}^d} \eta\,(\phi'_\varepsilon(\rho)\rho - \phi_\varepsilon(\rho))\,\Delta\Psi\, dx$$
$$\therefore \quad \frac{d}{dt}\int_{\mathbb{R}^d} \eta\phi_\varepsilon(\rho)\, dx + \int_{\mathbb{R}^d} \phi_\varepsilon(\rho)(\nabla\eta\nabla\Psi - \Delta\eta)\, dx$$
$$\leq \int_{\mathbb{R}^d} \eta\,(\phi'_\varepsilon(\rho)\rho - \phi_\varepsilon(\rho))\,\Delta\Psi\, dx.$$

Choosing $t \in (0, +\infty)$ and integrating over $(0, t)$

$$\int_{\mathbb{R}^d} \eta \phi_\varepsilon(\rho)\, dx + \int_0^t \int_{\mathbb{R}^d} \phi_\varepsilon(\rho)(\nabla\eta\nabla\Psi - \Delta\eta)\, dx\, dt$$
$$\leq \int_0^t \int_{\mathbb{R}^d} \eta \left(\phi_\varepsilon'(\rho)\rho - \phi_\varepsilon(\rho)\right) \Delta\Psi\, dx\, dt$$

and passing to the limit as $\varepsilon \to 0$

$$\int_{\mathbb{R}^d} \eta|\rho(t)|\, dx + \int_0^t \int_{\mathbb{R}^d} |\rho(t)|(\nabla\eta\nabla\Psi - \Delta\eta)\, dx\, dt \leq 0$$

and since (2.2) holds for all $x \in \mathbb{R}^d$, and $E(\rho), M(\rho) \in L^\infty(\mathbb{R}^d)$, for all $T < \infty$, we conclude that ρ and $\rho\nabla\Psi$ are integrable on $\mathbb{R}^d$. So, replacing $\eta \leftarrow \eta_R(x/R)$, where

$$\eta_R(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| < 2 \end{cases}$$

and letting $R \to \infty$, we get that

$$\int_{\mathbb{R}^d} |\rho(t)|\, dx = 0.$$

■

Remark 5 In the following chapter we will discuss the case of the porous medium equation. It is interesting to compare the gradient flow formulation for the Fokker-Planck equation versus the one induced by the underlying Riemannian structure in the porous medium case. It shall be obvious that using Riemannian calculus can be stiff, but still, we will not need come up with a discrete scheme in order to give meaning to the gradient flow for the associated energy functional. The crucial observation that allows us to use Riemannian geometry, is that the Wasserstein distance can be regarded as the induced distance for a metric tensor.

Chapter 3

A gradient flow formulation for the porous medium equation

The present chapter follows along the lines of [10], where the author introduces a new approach of regarding the porous medium equation

$$\frac{\partial \rho}{\partial t} - \Delta \rho^m = 0, \quad x \in \mathbb{R}^d,\ t \in [0, +\infty),\ \rho(x,t) \geq 0 \tag{3.1}$$

as a gradient flow for a given energy functional defined on a Riemannian manifold. The author uses elements of Riemannian calculus to justify the choice of the corresponding gradient flow representation, while the main aim of this work is the examination of the asymptotic behavior of an arbitrary solution for (3.1), which converges to a so-called Barenblatt solution. Then, this result is extended to convergence of the weak solution of (3.1). On the other hand, we are more interested in confirming that the underlying Riemannian framework offers, indeed, a great variety of tools, altering the way we handle the gradient flow representation (especially in comparison with the proving procedure we followed in the previous chapter). Another key ingredient that makes this approach quite appealing, is that we identify the mathematical object called induced distance with the Wasserstein distance, as we will see in detail.

We organize this chapter as follows: In section 3.1 after introducing the framework we work with, we derive the porous medium equation. In section 3.2 we discuss about the convergence of an arbitrary and a weak solution to the Barenblatt solution, while in section 3.3 we prove that the induced metric is the Wasserstein. Finally, the last section 3.4 is about a time discrete variational scheme which characterizes the gradient flow dynamical system.

3.1 Gradient flow representation of the porous medium equation

A Riemannian manifold $(\mathcal{M}, g)$ consists of a C^∞ manifold $\mathcal{M}$, which we equip with a metric tensor g_ρ on each of the tangent spaces $T_\rho\mathcal{M}$ of $\mathcal{M}$. The Riemannian metric g is a symmetric, nondegenerate, positive definite, smooth, covariant 2-tensor and it is an element of $T_p^*\mathcal{M}$, that is, the dual space of the tangent space $T_\rho\mathcal{M}$, for all $\rho \in \mathcal{M}$. We use the notation $g(v, w)$ dropping the subscript ρ with the understanding that we evaluate g at each point $v, w \in T_\rho\mathcal{M}$ are defined. When we define a functional $E = E(\rho)$ on $\mathcal{M}$ we may write the evolution equation

$$\frac{\partial \rho}{\partial t} = -\text{grad}E_\rho \tag{3.2}$$

which is the gradient flow of E -evaluated at $\rho \in \mathcal{M}$- on $(\mathcal{M}, g)$. We work with the manifold

$$\mathcal{M} = \left\{ \rho : \mathbb{R}^d \to [0, \infty) \middle| \int_{\mathbb{R}^d} \rho(y) dy = 1 \right\},$$

whose tangent space is defined as

$$T_\rho\mathcal{M} = \left\{ s : \mathbb{R}^d \to \mathbb{R}^d \middle| \int_{\mathbb{R}^d} s(y) dy = 0 \right\}$$

while the elliptic equation

$$-\nabla \cdot (\rho \nabla p) = s \tag{3.3}$$

defines the identification

$$T_\rho\mathcal{M} \cong \left\{ \text{all functions } p \text{ on } \mathbb{R}^d \right\} \tag{3.4}$$

up to an additive constant.

Remark 6 *(Properties of the metric tensor g)* [8, §2.2, §3.1]. Consider a Riemannian manifold $(\mathcal{M}, g)$ and a smooth function $f : \mathcal{M} \to \mathbb{R}$. The *differential* of f diff$f \in T_p^*\mathcal{M}$, is a (1-0) tensor field which at each point $p \in \mathcal{M}$ assigns the tensor $(df)_p : T_p\mathcal{M} \to \mathbb{R}$. The real number $(df)_p.v$ is the directional derivative of f at p along the vector v, for all $v \in T_p\mathcal{M}$.

A Riemannian metric $g(\cdot, \cdot)$ on $\mathcal{M}$ determines a linear isomorphism Φ_g : $T_p\mathcal{M} \to T_p^*\mathcal{M}$, for all $p \in \mathcal{M}$, that is, $\Phi(v)(w) = g(v, w)$ for all $v, w \in T_\rho\mathcal{M}$. We then define the *gradient* grad$f$ of f to be the vector field associated to the differential diff$f$ through the isomorphism determined by g, namely

$$g(v, \text{grad}f) = \text{diff}f.v, \quad \text{for all } v \text{ in the tangent bundle } T\mathcal{M}.$$

So, indeed, the metric tensor g connects the differential $\text{diff} f \in T^*\mathcal{M}$, an element of the cotangent space with the gradient $\text{grad} f \in T\mathcal{M}$, which is a tangent vector field. The gradient clearly depends on the metric g and it is not possible to define an invariant notion of the gradient without g (see [11, §1.1.1]).

In our case, we have

$$g(\text{grad}E, s) = \text{diff}E_\rho.s, \quad \text{for all vector fields } s \in T\mathcal{M}, \tag{3.5}$$

which combined with (3.2) gives

$$g_\rho\left(\frac{d\rho}{dt}, s\right) + \text{diff}E_\rho.s = 0 \quad \text{for all vector fields } s \text{ along } \rho. \tag{3.6}$$

Also observe that (3.6) implies

$$\frac{d}{dt}E(\rho) = \text{diff}E_\rho.\left(\frac{d\rho}{dt}\right) = -g_\rho\left(\frac{d\rho}{dt}, \frac{d\rho}{dt}\right). \tag{3.7}$$

Here, the metric tensor is given by

$$g_\rho(s_1, s_2) = \int \rho\, \nabla p_1 \cdot \nabla p_2, \quad \text{for all } s_1 \text{ and } s_2 \text{ tangent vectors at } \rho, \tag{3.8}$$

where according to (3.3), $-\nabla \cdot (\rho \nabla p_1) = s_1$ and $-\nabla \cdot (\rho \nabla p_2) = s_2$. Then we may rewrite (3.8) as

$$g_\rho(s_1, s_2) = \int s_1\, p_2. \tag{3.9}$$

Having all that in mind, we will show that the gradient flow for the functional

$$E(\rho) = \begin{cases} \dfrac{1}{m-1}\displaystyle\int \rho^m, & m \neq 1 \\ \displaystyle\int \rho \log \rho, & m = 1 \end{cases} \tag{3.10}$$

indeed coincides with the porous medium equation. To do that, first, we calculate the differential of (3.10) with respect to s

$$\text{diff}E_\rho.s = \frac{d}{d\varepsilon}E(\rho + \varepsilon s)|_{\varepsilon=0} = \begin{cases} \dfrac{m}{m-1}\displaystyle\int (\rho + \varepsilon s)^{m-1} s\Big|_{\varepsilon=0}, & m \neq 1 \\ \displaystyle\int s \log(\rho + \varepsilon s) + s\Big|_{\varepsilon=0}, & m = 1 \end{cases}$$

$$\therefore \text{diff}E_\rho.s = \begin{cases} \dfrac{m}{m-1}\displaystyle\int \rho^{m-1} s, & m \neq 1 \\ \displaystyle\int (\log \rho + 1) s, & m = 1. \end{cases}$$

Then, using that according to (3.9)

$$g_\rho\left(\frac{d\rho}{dt}, s\right) = \int \frac{\partial \rho}{\partial t}\, p, \tag{3.11}$$

equation (3.6) becomes

$$\begin{cases} \displaystyle\int \frac{\partial \rho}{\partial t}\, p + \frac{m}{m-1}\int \rho^{m-1} s = 0, & m \neq 1 \\ \displaystyle\int \frac{\partial \rho}{\partial t}\, p + \int (\log \rho + 1) s = 0, & m = 1 \end{cases}$$

and because of (3.3)

$$\begin{cases} \displaystyle\int \frac{\partial \rho}{\partial t}\, p - \frac{m}{m-1}\int \rho^{m-1} \nabla\cdot(\rho\nabla p) = 0, & m \neq 1 \\ \displaystyle\int \frac{\partial \rho}{\partial t}\, p - \int (\log \rho + 1)\nabla\cdot(\rho\nabla p) = 0, & m = 1. \end{cases}$$

After an integration by parts we finally obtain the porous medium equation in the form

$$\int \left(\frac{\partial \rho}{\partial t} - \Delta \rho^m\right) p = 0.$$

3.2 The Barenblatt solution and the asymptotic results

We call a Barenblatt solution a self-similar solution for (3.1) of the form

$$\varrho(t,x) = \frac{1}{t^{d\alpha}}\tilde{\varrho}\left(\frac{x}{t^\alpha}\right),$$

where $\alpha = (d(m-1)+2)^{-1}$ and $\tilde{\varrho}$ is given implicitly by

$$\mathsf{e}'(\tilde{\varrho}(y)) = \begin{cases} \displaystyle\frac{m}{m-1}\tilde{\varrho}(y)^{m-1} = \max\left\{\lambda - \frac{\alpha}{2}|y|^2, 0\right\}, & m > 1 \\ \log\tilde{\varrho}(y) + 1 = \lambda - \displaystyle\frac{\alpha}{2}|y|^2, & m = 1 \\ \displaystyle\frac{m}{m-1}\tilde{\varrho}(y)^{m-1} = \lambda - \frac{\alpha}{2}|y|^2, & m < 1 \end{cases} \tag{3.12}$$

where

$$\mathsf{e}(z) = \begin{cases} \displaystyle\frac{m}{m-1} z^m, & m \neq 1 \\ z\log z, & m = 1 \end{cases} \quad \text{and } \lambda \text{ such that } \quad \int \tilde{\varrho} = 1.$$

The Barenblatt solution describes the asymptotic behavior of an arbitrary solution ρ, in the sense that, when setting

$$\begin{cases} x = t^\alpha y \\ t = e^\tau \end{cases} \quad \text{so} \quad \rho(x,t) = \frac{1}{t^{d\alpha}}\tilde{\rho}\left(\log t, \frac{x}{t^\alpha}\right), \tag{3.13}$$

then $\tilde{\rho}$ approaches $\tilde{\varrho}$ for large times.

3.2.1 Convergence of an arbitrary solution to the Barenblatt solution

Now, the first asymptotic result is that $\tilde{\rho}$ converges to $\tilde{\varrho}$ with rate $\alpha-$ exponential with respect to t, or polynomial with respect to τ. This is realized according to the following set of inequalities

$$\frac{d}{d\tau}\left(e^{2\alpha\tau}|\mathrm{grad}F_{\tilde{\rho}}|^2\right) \leq 0 \tag{3.14}$$

$$\frac{d}{d\tau}\left(e^{2\alpha\tau}(F(\tilde{\rho}) - F(\tilde{\varrho}))\right) \leq 0 \tag{3.15}$$

$$\frac{d}{d\tau}\left(e^{2\alpha\tau}d(\tilde{\rho},\tilde{\varrho})^2\right) \leq 0 \tag{3.16}$$

where the augmented functional F is

$$F(\tilde{\rho}) = E(\tilde{\rho}) + \alpha M(\tilde{\rho}),$$

with

$$M(\tilde{\rho}) = \frac{1}{2}\int |y|^2\tilde{\rho}(y)\ dy.$$

Remark 7 *(Induced distance).* In (3.16), d denotes the *induced distance* on $(\mathcal{M}, g)$. Generally speaking, the notion of the induced distance is tightly related to what we call geodesics, that is constant speed curves - for a formal definition see Remark 10. To be more precise, take a C^∞ curve $\gamma : I \to \mathcal{M}$, where $I \subset \mathbb{R}$. If γ has zero acceleration $\ddot{\gamma} = 0$, or equivalently constant speed $|\dot{\gamma}| = \sqrt{g(\dot{\gamma},\dot{\gamma})} =$ *const.*, then we call γ a geodesic and it holds

$$\frac{d}{dt}g(\dot{\gamma},\dot{\gamma}) = 2g(\ddot{\gamma},\dot{\gamma}) = 0. \tag{3.17}$$

The notion of distance between two points p and q on a Riemannian manifold $(\mathcal{M}, g)$ makes sense in conjunction with measuring the length of a curve that connects them. In the definition for the length functional, it is sufficient to use C^∞ piecewise curves *i.e.* $\gamma : [0,1] \to \mathcal{M}$, continuous and such that $\gamma|_{[a_i,a_{i+1}]} \in C^\infty$ for any partition $0 = a_1 < a_2 < \cdots < a_k = 1$ of $[0,1]$, and any $i =$

$1, 2, \ldots, k-1$. Then the *length functional* $\ell(\gamma)$ is given by

$$\ell(\gamma) = \int_0^1 |\dot{\gamma}| \, dt = \int_0^1 \sqrt{g(\dot{\gamma}, \dot{\gamma})} \, dt \tag{3.18}$$

and the induced distance by

$$d(p,q)^2 = \inf \left\{ \ell(\sigma) :: [0,1] \ni \sigma \mapsto \mathcal{M}, \ \sigma \text{ is piecewise } C^\infty \text{ and } \begin{Bmatrix} \sigma(0) = p \\ \sigma(1) = q \end{Bmatrix} \right.$$

which is indeed a metric. The energy functional

$$E(\gamma) = \frac{1}{2} \int_0^1 |\dot{\gamma}|^2 \, dt \tag{3.19}$$

measures the total kinetic energy of a particle moving along γ and has the same minima with $\ell(\gamma)$ for constant speed curves (see chapter 5 in [11]). So indeed, we can redefine the induced distance as

$$d(p,q)^2 = \inf \left\{ E(\sigma) :: [0,1] \ni \sigma \mapsto \mathcal{M}, \ \sigma \text{ is piecewise } C^\infty \text{ and } \begin{Bmatrix} \sigma(0) = p \\ \sigma(1) = q \end{Bmatrix} \right. .$$

Back to the problem of our interest, the asymptotic result formulated in (3.14)-(3.16) is a consequence of the following properties

$$\tilde{\rho} \quad \text{satisfies} \quad \frac{d\tilde{\rho}}{d\tau} = -\text{grad} F_{\tilde{\rho}} \tag{3.20}$$

$$\tilde{\varrho} \quad \text{satisfies} \quad F(\tilde{\rho}) - F(\tilde{\varrho}) \geq 0, \quad \text{for all } \tilde{\rho} \in \mathcal{M} \tag{3.21}$$

$$F \quad \text{satisfies} \quad g(s, \text{Hess} F(\tilde{\rho})) \geq \alpha |s|^2, \quad \text{for all } s \in T_{\tilde{\rho}}\mathcal{M} \text{ and } \tilde{\rho} \in \mathcal{M}. \tag{3.22}$$

First we should verify that (3.20), (3.21) and (3.22) indeed hold. For (3.20), we calculate the differential of $F(\tilde{\rho})$ with respect to s

$$\text{diff} F_{\tilde{\rho}}.s = \begin{cases} \int \left(\frac{m}{m-1} \tilde{\rho}^{m-1} + \frac{\alpha}{2} |y|^2 \right) s, & m \neq 1 \\ \int \left(\log \tilde{\rho} + 1 + \frac{\alpha}{2} |y|^2 \right) s, & m = 1, \end{cases}$$

then because of (3.11), equation (3.6) becomes

$$\begin{cases} \int \frac{\partial \tilde{\rho}}{\partial \tau} + \int \left(\frac{m}{m-1} \tilde{\rho}^{m-1} + \frac{\alpha}{2} |y|^2 \right) s = 0, & m \neq 1 \\ \int \frac{\partial \tilde{\rho}}{\partial \tau} + \int \left(\log \tilde{\rho} + 1 + \frac{\alpha}{2} |y|^2 \right) s = 0, & m = 1 \end{cases}$$

so when substituting (3.3) it further becomes

$$\begin{cases} \int \frac{\partial \tilde{\rho}}{\partial \tau} - \int \left(\frac{m}{m-1} \tilde{\rho}^{m-1} + \frac{\alpha}{2}|y|^2 \right) \nabla \cdot (\tilde{\rho} \nabla p) = 0, & m \neq 1 \\ \int \frac{\partial \tilde{\rho}}{\partial \tau} - \int \left(\log \tilde{\rho} + 1 + \frac{\alpha}{2}|y|^2 \right) \nabla \cdot (\tilde{\rho} \nabla p) = 0, & m = 1 \end{cases}$$

$$\begin{cases} \int \frac{\partial \tilde{\rho}}{\partial \tau} - \nabla \cdot \left(\tilde{\rho}\, \nabla \left(\frac{m}{m-1} \tilde{\rho}^{m-1} + \frac{\alpha}{2}|y|^2 \right) \right) p = 0, & m \neq 1 \\ \int \frac{\partial \tilde{\rho}}{\partial \tau} - \nabla \cdot \left(\tilde{\rho}\, \nabla \left(\log \tilde{\rho} + 1 + \frac{\alpha}{2}|y|^2 \right) \right) p = 0, & m = 1 \end{cases}$$

$$\begin{cases} \frac{\partial \tilde{\rho}}{\partial \tau} - \nabla \cdot \left(\tilde{\rho}\, \nabla \left(\frac{m}{m-1} \tilde{\rho}^{m-1} + \frac{\alpha}{2}|y|^2 \right) \right) = 0, & m \neq 1 \\ \frac{\partial \tilde{\rho}}{\partial \tau} - \nabla \cdot \left(\tilde{\rho}\, \nabla \left(\log \tilde{\rho} + 1 + \frac{\alpha}{2}|y|^2 \right) \right) = 0, & m = 1. \end{cases}$$

$$\therefore\ \frac{\partial \tilde{\rho}}{\partial \tau} + \Delta \tilde{\rho}^m - \alpha \nabla \cdot (\tilde{\rho}\, y) = 0.$$

Observing that when $\tilde{\rho}$ satisfies the latter, we have equivalently that ρ satisfies the porous medium equation, we conclude.

For (3.21), consider the functional

$$H(\tilde{\rho}_1, \tilde{\rho}_0) := \int \mathsf{e}(\tilde{\rho}_1) - \mathsf{e}(\tilde{\rho}_0) - \mathsf{e}'(\tilde{\rho}_0)(\tilde{\rho}_1 - \tilde{\rho}_0) \geq 0 \tag{3.23}$$

then it is sufficient to show that

$$F(\tilde{\rho}) - F(\tilde{\varrho}) \begin{cases} \geq H(\tilde{\rho}, \tilde{\varrho}), & m > 1 \\ = H(\tilde{\rho}, \tilde{\varrho}), & m \leq 1. \end{cases}$$

First consider the case $m \neq 1$. Then

$$\begin{aligned} H(\tilde{\rho}, \tilde{\varrho}) &= \int \frac{1}{m-1}(\tilde{\rho}^m - \tilde{\varrho}^m) - \frac{m}{m-1} \tilde{\varrho}^{m-1}(\tilde{\rho} - \tilde{\varrho}) \\ &= E(\tilde{\rho}) - E(\tilde{\varrho}) - \int \frac{m}{m-1} \tilde{\varrho}^{m-1}(\tilde{\rho} - \tilde{\varrho}) \end{aligned}$$

so rearranging the terms and adding $M(\tilde{\rho})$ and $M(\tilde{\varrho})$ to both sides of the equation we have, indeed, that

$$F(\tilde{\rho}) = F(\tilde{\varrho}) + H(\tilde{\rho}, \tilde{\varrho}) + \int \left(\frac{m}{m-1} \tilde{\varrho}^{m-1} + \frac{\alpha}{2}|y|^2 \right) (\tilde{\rho} - \tilde{\varrho}). \tag{3.24}$$

Now, when $m < 1$, we read from (3.12) that

$$\lambda = \frac{m}{m-1} \tilde{\varrho}^{m-1} + \frac{\alpha}{2}|y|^2$$

so that

$$F(\tilde{\rho}) = F(\tilde{\varrho}) + H(\tilde{\rho}, \tilde{\varrho}) + \lambda \int (\tilde{\rho} - \tilde{\varrho})$$

but since

$$\lambda \int (\tilde{\rho} - \tilde{\varrho}) = \int \left(\frac{m}{m-1} \tilde{\varrho}^{m-1} + \frac{\alpha}{2}|y|^2 \right) (\tilde{\rho} - \tilde{\varrho})$$

the above in comparison with (3.24) gives that

$$F(\tilde{\rho}) = F(\tilde{\varrho}) + H(\tilde{\rho}, \tilde{\varrho}).$$

For $m > 1$ according to (3.12)

$$\frac{m}{m-1} \tilde{\varrho}^{m-1} \geq \lambda - \frac{\alpha}{2}|y|^2$$

$$\therefore \left(\frac{m}{m-1} \tilde{\varrho}^{m-1} + \frac{\alpha}{2}|y|^2 \right) (\tilde{\rho} - \tilde{\varrho}) \geq \lambda(\tilde{\rho} - \tilde{\varrho}), \quad \text{for all } y \in \mathbb{R}^d$$

so that

$$F(\tilde{\rho}) \geq F(\tilde{\varrho}) + H(\tilde{\rho}, \tilde{\varrho}) + \lambda \int (\tilde{\rho} - \tilde{\varrho}) = F(\tilde{\varrho}) + H(\tilde{\rho}, \tilde{\varrho}).$$

Finally, when $m = 1$, from the definition of $F(\tilde{\rho})$ and (3.10) we get

$$F(\tilde{\rho}) = \int \tilde{\rho} \log \tilde{\rho} + \int \frac{\alpha}{2}|y|^2 \tilde{\rho} \tag{3.25}$$

but in that case, from (3.12) we deduce that

$$\frac{\alpha}{2}|y|^2 = \lambda - \log \tilde{\varrho} + 1$$

and then

$$F(\tilde{\rho}) = \int (\log \tilde{\rho} + \lambda - \log \tilde{\varrho})\, \tilde{\rho}. \tag{3.26}$$

Now, observe that the definition of $\mathtt{e}(\cdot)$ and (3.23) imply that

$$\begin{aligned} H(\tilde{\rho}, \tilde{\varrho}) &= \int \tilde{\rho} \log \tilde{\rho} - \tilde{\varrho} \log \tilde{\varrho} - (\log \tilde{\varrho} + 1)(\tilde{\rho} - \tilde{\varrho}) \\ &= \int \tilde{\rho}(\log \tilde{\rho} - \log \tilde{\varrho}) - (\tilde{\rho} - \tilde{\varrho}) \\ &= \int (\log \tilde{\rho} - \log \tilde{\varrho})\, \tilde{\rho}, \end{aligned}$$

since both $\tilde{\rho}$ and $\tilde{\varrho}$ are probability densities. Then (3.26) becomes

$$F(\tilde{\rho}) = H(\tilde{\rho}, \tilde{\varrho}) + \lambda,$$

so that combining equation (3.25) with (3.26) formulated for $\tilde{\varrho}$, indeed get

$$F(\tilde{\rho}) - F(\tilde{\varrho}) = H(\tilde{\rho}, \tilde{\varrho}).$$

Property (3.22) which we may rewrite as

$$\text{Hess}F(\tilde{\rho}) \geq \alpha\, id. \quad \text{for all } \tilde{\rho} \in \mathcal{M},$$

follows from the properties

$$\text{Hess}E(\tilde{\rho}) \geq 0 \quad \text{and} \quad \text{Hess}M(\tilde{\rho}) = id. \quad \text{for all } \tilde{\rho} \in \mathcal{M} \tag{3.27}$$

which are established in [10, §4.4] via the computation of the corresponding Hessian matrices.

Remark 8 For $E(\tilde{\varrho})$ to be well-defined and convex and $M(\tilde{\varrho})$ to be well-defined on $(\mathcal{M}, g)$ we require $m \geq 1 - (1/d)$ and $m \geq d/(d+2)$.

Remark 9 Observe that (3.21) implies that $\tilde{\varrho}$ minimizes F on $(\mathcal{M}, g)$ and as a minimizer, it is of course a stationary point too

$$-\text{grad}F(\tilde{\varrho}) = 0. \tag{3.28}$$

Remark 10 *(Covariant differentiation)* [8, §3.1, §3.2]. Let us now explain how to take the derivative of one vector field in the direction of another vector field. This is the notion of covariant derivative. Let $\mathcal{M}$ be a differentiable manifold and consider the set $\mathfrak{X}(\mathcal{M})$ of all vector fields on $\mathcal{M}$. An *affine connection* on $\mathcal{M}$ is a map $\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ such that

1. $\nabla_{fX+gY} Z = f\nabla_Y Z + g\nabla_Y Z$,
2. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,
3. $\nabla_X(fY) = (X \cdot f)Y + f\nabla_X Y$,

for all vector fields $X, Y, Z \in \mathfrak{X}(\mathcal{M})$ and $f, g \in C^\infty(\mathcal{M})$ while $\nabla_X(Y)$ stands for the covariant derivative of Y along the vector field X.

In a similar way, we can take the covariant derivative of a vector field V along a differentiable curve $\gamma : I \to \mathcal{M}$, $I \subset \mathbb{R}$, with $\dot{\gamma} \neq 0$; that is, a vector field defined along γ, given by

$$\frac{DV}{dt}(t) := \nabla_{\dot{\gamma}(t)} V = (\nabla_X Y)_{\gamma(t)},$$

for any vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$ such that $X_{\gamma(t)} = \dot{\gamma}(t)$ and $Y_{\gamma(s)} = V(s)$, with $s \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. We say that V is *parallel* along γ if

$$\frac{DV}{dt}(t) = 0, \quad \text{for all } t \in I$$

and the curve γ is called *geodesic* of the connection ∇, if $\dot{\gamma}$ is parallel along γ, *i.e.*

$$\frac{D\dot{\gamma}}{dt}(t) = 0, \quad \text{for all } t \in I.$$

From now on, we can adopt the framework of Remark 7 and yet use the covariant derivatives with the understanding that $\ddot{\gamma} = \nabla_{\dot{\gamma}}\dot{\gamma}$.

Now we are ready to attack (3.14) and (3.16)-the equivalent result (3.15) is established in [10, §3.5]. For (3.14), according to Remarks 7 and 10, we calculate

$$\begin{aligned}
|\mathrm{grad}F(\tilde{\rho})|^2 &= g(\mathrm{grad}F_{\tilde{\rho}}, \mathrm{grad}F_{\tilde{\rho}}),\\
\therefore \frac{d}{d\tau}|\mathrm{grad}F_{\tilde{\rho}}|^2 &= 2g\left(\mathrm{grad}F_{\tilde{\rho}}, \frac{D}{D\tau}\mathrm{grad}F_{\tilde{\rho}}\right)\\
&= 2g\left(\mathrm{grad}F_{\tilde{\rho}}, \mathrm{Hess}F_{\tilde{\rho}}\frac{d\tilde{\rho}}{d\tau}\right)\\
&= -2g\left(\mathrm{grad}F_{\tilde{\rho}}, \mathrm{Hess}F_{\tilde{\rho}}\, \mathrm{grad}F_{\tilde{\rho}}\right)\\
&\geq -2\alpha|\mathrm{grad}F_{\tilde{\rho}}|^2.
\end{aligned}$$

Here, $\dfrac{D}{D\tau}$ is the covariant derivative along $\tilde{\rho}$, while we also used the definitive property of the Hessian matrix (see [11, §2.1.3]) and properties (3.20),(3.22). For (3.16), first notice that according to Remark 7 the induced distance is given by

$$d(\tilde{\rho}_0, \tilde{\rho}_1)^2 := \inf\left\{\int_0^1 \left|\frac{d\check{\tilde{\rho}}}{d\sigma}\right|^2 d\sigma :: [0,1] \ni \sigma \mapsto \check{\tilde{\rho}}(\sigma) \in \mathcal{M} \text{ with } \begin{Bmatrix} \check{\tilde{\rho}}(0) = \tilde{\rho}_0 \\ \check{\tilde{\rho}}(1) = \tilde{\rho}_1 \end{Bmatrix}\right\}. \tag{3.29}$$

Hence, if we take the curve of *least energy* between $\tilde{\rho}_0$ and $\tilde{\rho}_1$ that is $[0,1] \ni \sigma \mapsto \check{\tilde{\rho}}(\sigma) \in \mathcal{M}$ such that

$$d(\tilde{\rho}_0, \tilde{\rho}_1)^2 = \int_0^1 \left|\frac{d\check{\tilde{\rho}}}{d\sigma}\right|^2 d\sigma \tag{3.30}$$

and $[0,1] \ni \sigma \mapsto \check{\tilde{\rho}}(\sigma)$ is a geodesic *i.e.*

$$\frac{D}{d\sigma}\frac{d\check{\tilde{\rho}}}{d\sigma} = 0 \tag{3.31}$$

then there holds

$$\frac{d}{d\sigma}\left|\frac{d\check{\tilde{\rho}}}{d\sigma}\right|^2 = 2\,g\left(\frac{d\check{\tilde{\rho}}}{d\sigma}, \frac{D}{\partial\sigma}\frac{d\check{\tilde{\rho}}}{d\sigma}\right) = 0. \tag{3.32}$$

Then also notice that since (3.20) and (3.28) hold, when we define $\tilde{\rho}_0 := \tilde{\varrho}$, $\tilde{\rho}_0$ solves the equation

$$\frac{d\tilde{\rho}_0}{d\tau} = -\mathrm{grad}F_{\tilde{\rho}_0}$$

and if $\tilde{\rho}_1$ denotes a second solution, it is enough to check that

$$\frac{d^+}{d\tau} d(\tilde{\rho}_1, \tilde{\rho}_0)^2 := \limsup_{\tau \to \tau_0} \frac{d(\tilde{\rho}_1, \tilde{\rho}_0)^2|_\tau - d(\tilde{\rho}_1, \tilde{\rho}_0)^2|_{\tau_0}}{\tau - \tau_0} \leq 2\,\alpha\, d(\tilde{\rho}_1, \tilde{\rho}_0)^2$$

where the dependence on τ and respectively τ_0 is considered as follows: Given a fixed τ_0 and for all τ, take a curve $[0,1] \ni \sigma \mapsto \check{\tilde{\rho}}(\sigma, \tau) \in \mathcal{M}$ such that $\check{\tilde{\rho}}(0,\tau) =: \tilde{\rho}_0(\tau)$, $\check{\tilde{\rho}}(1,\tau) =: \tilde{\rho}_1(\tau)$ and such that to be the curve of least energy when $\tau = \tau_0$. In that case, by definition we have that

$$d(\tilde{\rho}_0, \tilde{\rho}_1)^2 \begin{cases} = \displaystyle\int_0^1 \left| \frac{d\check{\tilde{\rho}}}{d\sigma} \right|^2 d\sigma, & \text{when } \tau = \tau_0 \\ \leq \displaystyle\int_0^1 \left| \frac{d\check{\tilde{\rho}}}{d\sigma} \right|^2 d\sigma, & \text{for all other } \tau \end{cases} \tag{3.33}$$

so that, if $[0,1] \ni \sigma \mapsto \check{\tilde{\rho}}(\sigma, \tau)$ is a smooth function of τ, we may calculate

$$\begin{aligned}
\frac{d^+}{d\tau} d(\tilde{\rho}_1, \tilde{\rho}_0)^2 \Big|_{\tau_0} &\leq \frac{d}{d\tau} \left\{ \int_0^1 \left| \frac{d\check{\tilde{\rho}}}{d\sigma} \right|^2 d\sigma \right\} \Big|_{\tau_0} \\
&= 2 \int_0^1 g\left(\frac{\partial \check{\tilde{\rho}}}{\partial \sigma}, \frac{D}{\partial \tau}\Big|_{\tau_0} \frac{\partial \check{\tilde{\rho}}}{\partial \sigma} \right) d\sigma \\
&= 2 \int_0^1 g\left(\frac{\partial \check{\tilde{\rho}}}{\partial \sigma}, \frac{D}{\partial \sigma} \frac{\partial \check{\tilde{\rho}}}{\partial \tau}\Big|_{\tau_0} \right) d\sigma \\
&= 2 \int_0^1 \frac{d}{d\sigma} \left\{ g\left(\frac{\partial \check{\tilde{\rho}}}{\partial \sigma}, \frac{\partial \check{\tilde{\rho}}}{\partial \tau}\Big|_{\tau_0} \right) - g\left(\frac{D}{\partial \sigma} \frac{\partial \check{\tilde{\rho}}}{\partial \sigma}, \frac{\partial \check{\tilde{\rho}}}{\partial \tau}\Big|_{\tau_0} \right) \right\} d\sigma \\
&= 2 \int_0^1 \frac{d}{d\sigma}\, g\left(\frac{\partial \check{\tilde{\rho}}}{\partial \sigma}, \frac{\partial \check{\tilde{\rho}}}{\partial \tau}\Big|_{\tau_0} \right) d\sigma \\
&= 2 \left\{ g\left(\frac{\partial \check{\tilde{\rho}}}{\partial \sigma}\Big|_{\sigma=1}, \frac{\partial \tilde{\rho}_1}{\partial \tau} \right) - g\left(\frac{\partial \check{\tilde{\rho}}}{\partial \sigma}\Big|_{\sigma=0}, \frac{\partial \tilde{\rho}_0}{\partial \tau} \right) \right\} \\
&= -2 \left\{ g\left(\frac{\partial \check{\tilde{\rho}}}{\partial \sigma}\Big|_{\sigma=1}, \mathrm{grad} F_{\tilde{\rho}_1} \right) - g\left(\frac{\partial \check{\tilde{\rho}}}{\partial \sigma}\Big|_{\sigma=0}, \mathrm{grad} F_{\tilde{\rho}_0} \right) \right\} \\
&\leq -2\,\alpha\, d(\tilde{\rho}_0, \tilde{\rho}_1)^2.
\end{aligned}$$

Here, we first used (3.33)and (3.17)in combination with Remark 10, then a *Koszul type* formula for mixed partial derivatives [11, §5.1] and finally property(3.31), followed by (3.20) and (3.22).

3.2.2 Computation of grad$F$

In this paragraph we prove a variation of (3.36) namely

$$|\text{grad}F(\tilde{\rho})|^2 = \int \tilde{\rho}|\nabla p|^2 \;\text{ where }\; p(y) = \mathsf{e}'(\tilde{\rho}) + \frac{\alpha}{2}|y|^2. \tag{3.34}$$

Since the metric tensor g is positive definite we have that $g(v-s, v-s) \geq 0$, for all $v, s \in T_{\tilde{\rho}}\mathcal{M}$, so there holds $\frac{1}{2}g(v,v) \geq g(v,s) - \frac{1}{2}g(s,s)$, for all $v, s \in T_{\tilde{\rho}}\mathcal{M}$. Therefore, fixing $v = \text{grad}F$ and using the equivalent of (3.5) for F, we have

$$\frac{1}{2}g_{\tilde{\rho}}(\text{grad}F, \text{grad}F) = \sup_{s \in T_{\tilde{\rho}}\mathcal{M}} \left\{ \text{diff}F_{\tilde{\rho}}.s - \frac{1}{2}g(s,s) \right\}. \tag{3.35}$$

We already have calculated the differential of F, that is

$$\text{diff}F_{\tilde{\rho}}.s = \int ps,$$

when substituting p as in (3.34), while, according to (3.8) and (3.9)

$$\int ps - \frac{1}{2}g(s,s) = \int \tilde{\rho}\, \nabla p \cdot \nabla q - \frac{1}{2}\int |\nabla q|^2 \tilde{\rho}, \quad \text{for } s = -\nabla \cdot (\tilde{\rho}\nabla q).$$

Substituting in (3.35) we get

$$\begin{aligned} \frac{1}{2}g_{\tilde{\rho}}(\text{grad}F, \text{grad}F) &\overset{(3.4)}{=} \sup_{p} \left\{ \int \tilde{\rho}\, \nabla p \cdot \nabla q - \frac{1}{2}\int |\nabla q|^2 \tilde{\rho} \right\} \\ &= \frac{1}{2}\int |\nabla p|^2 \tilde{\rho} \end{aligned}$$

(since the $\max_x\{ax - \frac{1}{2}x^2\}$ is achieved for $x = a$).

3.2.3 Convergence of a weak solution to the Barenblatt solution

Convergence for a weak solution of (3.1) is established in [10, §5] and it is formulated in correspondence with (3.14)-(3.16). The theorem reads as follows.

Theorem 3 *If m satisfies Remark 8 and ρ is a **weak** solution of (3.1) with initial datum ρ_0 such that*

$$\rho_0 : \mathbb{R}^d \to [0, +\infty), \;\textit{measurable with}\; \int \rho_0 = 1, \;\left(\frac{1}{m-1}\int \rho_0^{m+1} \right), M(\rho_0) < \infty$$

define the function $\tilde{\rho}$ on $(-\infty, +\infty) \times \mathbb{R}^d$ via (3.13). Then (3.14),(3.15) and

(3.16) hold for $\tilde{\rho}$ in the distributional sense. Here we consider

$$|gradF_{\tilde{\rho}}|^2 = \int \frac{1}{\tilde{\rho}} |\nabla \pi(\tilde{\rho}) + \alpha\, \tilde{\rho}\, y|^2 \, dy, \tag{3.36}$$

$$F(\tilde{\rho}) - F(\tilde{\varrho}) = \begin{cases} \left(\int \mathsf{e}(\tilde{\rho}) + \alpha M(\tilde{\rho})\right) - \left(\int \mathsf{e}(\tilde{\varrho}) + \alpha M(\tilde{\varrho})\right), & m > 1 \\ H(\tilde{\rho}, \tilde{\varrho}), & m \leq 1 \end{cases}$$

where $\pi(z) := z\mathsf{e}'(z) - \mathsf{e}(z)$ and $H(\tilde{\rho}, \tilde{\varrho})$ is given by (3.23).

3.3 The induced distance on $(\mathcal{M}, g)$

The porous medium equation describes the diffusion of gas particles through a porous medium and there are two alternative ways to consider this system; via Eulerian or Lagrangian coordinates. Here, the Eulerian description is realized using the manifold $(\mathcal{M}, g)$ and we understand the evolution of the system as the evolution of the associated particle densities, while, the Lagrangian description corresponds to a flat Riemannian manifold $(\mathcal{M}^*, g^*)$ and each state of the system is understood using particle coordinates, or a flow map Φ, which we define as in (2.25) from the previous chapter.

These two interpretations are connected through a differentiable isometric mapping $\boldsymbol{\pi} : \mathcal{M}^* \to \mathcal{M}$ which is a *submersion, i.e.* its differential at each $\rho \in \mathcal{M}$ is surjective. We define $(\mathcal{M}^*, g^*)$ as follows:

$$\mathcal{M}^* = \left\{ \text{all diffeomorphisms } \Phi \text{ on } \mathbb{R}^d \right\},$$

namely all differentiable maps on $\mathbb{R}^d$, which are bijective with differentiable inverse. Then given a fixed reference density $\rho^0 \in \mathcal{M}$, we write any $\rho := \boldsymbol{\pi}(\Phi) \in \mathcal{M}$ as the push-forward $\rho = \Phi \# \rho^0$. The tangent space of $\mathcal{M}^*$ is

$$T_\Phi \mathcal{M}^* = \left\{ \text{all vector fields } v \text{ on } \mathbb{R}^d \right\},$$

and the corresponding Riemannian metric is

$$g^*_\Phi(v_1, v_2) = \int v_1 \cdot v_2 \, \rho^0, \quad \text{for all } v_1, v_2 \in T_\Phi \mathcal{M}^*$$

which is the ρ^0 weighted L^2-inner product.The fact that $\boldsymbol{\pi} : \mathcal{M}^* \to \mathcal{M}$ is an isometric submersion is interpreted in terms of the map

$$T_\Phi \boldsymbol{\pi} : T_\Phi \mathcal{M}^* \to T_\rho \mathcal{M}, \quad \text{for all } \Phi \in \mathcal{M}^*,$$

as

$$g_\rho(s,s) = \inf_{T_\Phi \boldsymbol{\pi}(v)=s} g_\Phi(v,v), \quad \text{for all } s \in T_\rho\mathcal{M},\ \rho = \boldsymbol{\pi}(\Phi). \tag{3.37}$$

The induced distance on $(\mathcal{M}^*, g^*)$ is the ρ^0 weighted L^2-norm namely

$$d^*(\Phi_0, \Phi_1) = \int |\Phi_0 - \Phi_1|^2 \rho^0.$$

Now, the key ingredient is to characterize the induced distance on $(\mathcal{M}, g)$ in terms of the induced distance on $(\mathcal{M}^*, g^*)$. We will actually see that

$$d(\rho^0, \rho)^2 = \inf_{\boldsymbol{\pi}(\Phi)=\rho} d^*(id., \Phi)^2, \tag{3.38}$$

which is exactly

$$\begin{aligned} d(\rho^0, \rho)^2 &= \inf_{\rho = \Phi\#\rho^0} \int |id. - \Phi|^2 \rho^0 \\ &= W(\rho^0, \rho)^2, \quad \text{where } \ \rho := \Phi\#\rho^0. \end{aligned}$$

For (3.38), first we prove there holds

$$d(\rho^0, \rho)^2 \le d^*(id., \Phi)^2, \quad \text{for all } \Phi \in \mathcal{M}^* \text{ such that } \boldsymbol{\pi}(\Phi) = \rho. \tag{3.39}$$

Let $[0,1] \ni \sigma \mapsto \check{\Phi}(\sigma) \in \mathcal{M}^*$, be any curve connecting $id. =: \check{\Phi}(0)$ and $\Phi =: \check{\Phi}(1)$ and consider its image under $\boldsymbol{\pi}$, that is $[0,1] \ni \sigma \mapsto \check{\rho}(\sigma) \in \mathcal{M}$, where by definition $\check{\rho}(0) = \rho^0$ and by assumption $\boldsymbol{\pi}(\Phi) = \rho$. Then

$$\begin{aligned} d(\rho^0, \rho)^2 &= \inf \int_0^1 \left| \frac{d\check{\rho}}{d\sigma} \right|^2 \, d\sigma \\ &= \inf \int_0^1 g_{\check{\rho}} \left(\frac{d\check{\rho}}{d\sigma}, \frac{d\check{\rho}}{d\sigma} \right) \, d\sigma \\ &\le \int_0^1 g_{\check{\rho}} \left(\frac{d\check{\rho}}{d\sigma}, \frac{d\check{\rho}}{d\sigma} \right) \, d\sigma \\ &\le \int_0^1 g^*_{\check{\Phi}} \left(\frac{d\check{\Phi}}{d\sigma}, \frac{d\check{\Phi}}{d\sigma} \right) \, d\sigma \\ &\le d^*(id., \Phi)^2. \end{aligned}$$

Where, the inequality

$$\int_0^1 g_{\check{\rho}} \left(\frac{d\check{\rho}}{d\sigma}, \frac{d\check{\rho}}{d\sigma} \right) \, d\sigma \le \int_0^1 g^*_{\check{\Phi}} \left(\frac{d\check{\Phi}}{d\sigma}, \frac{d\check{\Phi}}{d\sigma} \right) \, d\sigma$$

is a consequence of the property (3.37), in the following sense: let $[0,1] \ni \sigma \mapsto$

$\Phi(\sigma) \in \mathcal{M}^*$ be any curve on $\mathcal{M}^*$ and take its image under $\boldsymbol{\pi}$, that is a curve $[0,1] \ni \sigma \mapsto \rho(\sigma) \in \mathcal{M}$, such that $\rho(\sigma) = \boldsymbol{\pi}(\Phi(\sigma))$. Then

$$\frac{d\rho}{d\sigma} \in T_\rho\mathcal{M} \;\text{ and }\; \frac{d\Phi}{d\sigma} \in T_\Phi\mathcal{M}^*,$$

in particular

$$\frac{d\rho}{d\sigma} = T_\Phi\boldsymbol{\pi}\left(\frac{d\Phi}{d\sigma}\right).$$

So indeed, (3.37) holds with equality if and only if

$$\frac{d\Phi}{d\sigma} \in \big(\ker(T_\Phi\boldsymbol{\pi})\big)^\perp, \tag{3.40}$$

where

$$\ker(T_\Phi\boldsymbol{\pi}) = \big\{v \in T_\Phi\mathcal{M}^* \;\text{ such that }\; T_\rho\mathcal{M} \ni T_\Phi\boldsymbol{\pi}(v) = 0\big\}.$$

Now, it is sufficient to show that

$$\text{there exists a } \Phi \text{ with } \boldsymbol{\pi}(\Phi) = \rho, \text{ such that } \; d(\rho^0,\rho)^2 \geq d^*(id., \Phi)^2. \tag{3.41}$$

For, consider the curve of least energy connecting $\rho^0 = \rho(0)$ and ρ that is $[0,1] \ni \sigma \mapsto \check{\rho}(\sigma) \in \mathcal{M}$, which is a geodesic. Then there exists a geodesic $[0,1] \ni \sigma \mapsto \check{\Phi}(\sigma) \in \mathcal{M}^*$, with $\check{\Phi}(0) = id.$, $\boldsymbol{\pi}(\check{\Phi}(\sigma)) = \check{\rho}(\sigma)$, for which (3.40) holds and moreover $\sigma \mapsto \check{\rho}(\sigma)$ is its image under $\boldsymbol{\pi}$ (§4.2 [10]). Observe that if $\Phi := \check{\Phi}(1)$ so that $\boldsymbol{\pi}\Phi = \boldsymbol{\pi}(\check{\Phi}(1)) = \check{\rho}(1) =: \rho$, we have

$$\begin{aligned} d(\rho^0,\rho)^2 &= \int_0^1 g_{\check{\rho}}\left(\frac{d\check{\rho}}{d\sigma}, \frac{d\check{\rho}}{d\sigma}\right)\, d\sigma \\ &= \int_0^1 g^*_{\check{\Phi}}\left(\frac{d\check{\Phi}}{d\sigma}, \frac{d\check{\Phi}}{d\sigma}\right)\, d\sigma \\ &\geq d^*(id., \Phi)^2. \end{aligned}$$

Hence, combining (3.39) and (3.41), we get (3.38).

3.4 A time discretization for the gradient flow

In this final section we will show that the gradient flow representation (3.2) is compatible with a time discrete variational scheme which relies on the minimization of (2.3), reformulated as

$$\mathfrak{R}(\rho) := \frac{1}{2h} d(\rho^{(k-1)}, \rho)^2 + E(\rho). \tag{3.42}$$

First of all, we calculate the first variation of (3.42) and since $d(\rho^{(k-1)}, \rho)^2$ is given by (3.29), we will make of use the following Remark, where we calculate the first variation of (3.19).

Remark 11 *(First variation formula for the energy functional).* Consider a curve $\gamma : I \to \mathcal{M}$, $I \subset \mathbb{R}$. A *variation* of γ is a family of curves $t \mapsto \bar{\gamma}(s,t) : (-\varepsilon, \varepsilon) \times [a,b] \to \mathcal{M}$, for all $\varepsilon > 0$, such that $\bar{\gamma}(0,t) = \bar{\gamma}(t)$, for all $t \in [a,b]$. It is sufficient to consider piecewise smooth variations, such that the curves $t \mapsto \bar{\gamma}(s,t)$ are piecewise smooth and the curves $s \mapsto \bar{\gamma}(s,t)$ are smooth. Here, we prove the formula for smooth variations which is

$$\frac{dE(\bar{\gamma})}{ds} = g\left(\frac{\partial\bar{\gamma}}{\partial s}, \frac{\partial\bar{\gamma}}{\partial t}\right)\Big|_{(s,b)} - g\left(\frac{\partial\bar{\gamma}}{\partial s}, \frac{\partial\bar{\gamma}}{\partial t}\right)\Big|_{(s,a)} - \int_a^b g\left(\frac{\partial\bar{\gamma}}{\partial s}, \frac{\partial^2\bar{\gamma}}{\partial t^2}\right)\, dt.$$

Indeed we have

$$\begin{aligned}
\frac{dE(\bar{\gamma})}{ds} &= \frac{d}{ds}\frac{1}{2}\int_a^b g\left(\frac{\partial\bar{\gamma}}{\partial t}, \frac{\partial\bar{\gamma}}{\partial t}\right)\, dt \\
&= \frac{1}{2}\int_a^b \frac{\partial}{\partial s} g\left(\frac{\partial\bar{\gamma}}{\partial t}, \frac{\partial\bar{\gamma}}{\partial t}\right)\, dt \\
&= \int_a^b g\left(\frac{\partial^2\bar{\gamma}}{\partial s\partial t}, \frac{\partial\bar{\gamma}}{\partial t}\right)\, dt \\
&= \int_a^b g\left(\frac{\partial^2\bar{\gamma}}{\partial t\partial s}, \frac{\partial\bar{\gamma}}{\partial t}\right)\, dt \\
&= \int_a^b \frac{\partial}{\partial t} g\left(\frac{\partial\bar{\gamma}}{\partial s}, \frac{\partial\bar{\gamma}}{\partial t}\right)\, dt - \int_a^b \frac{\partial}{\partial t} g\left(\frac{\partial\bar{\gamma}}{\partial s}, \frac{\partial^2\bar{\gamma}}{\partial t^2}\right)\, dt \\
&= g\left(\frac{\partial\bar{\gamma}}{\partial s}, \frac{\partial\bar{\gamma}}{\partial t}\right)\Big|_a^b - \int_a^b \frac{\partial}{\partial t} g\left(\frac{\partial\bar{\gamma}}{\partial s}, \frac{\partial^2\bar{\gamma}}{\partial t^2}\right)\, dt.
\end{aligned}$$

The first equality comes from the observation that when substituting (3.18) into (3.19), we indeed have

$$E(\bar{\gamma}) = \frac{1}{2}\int_a^b g\left(\frac{\partial\bar{\gamma}}{\partial t}, \frac{\partial\bar{\gamma}}{\partial t}\right)\, dt,$$

while the third from the Koszul type formula for mixed partials (§5.1 [11]).

Now, given $\rho^{(k)}$ that minimizes $\mathfrak{R}(\rho)$ over all $\rho \in \mathcal{M}$, first consider the curve of least energy connecting $\rho^{(k-1)}$ and $\rho^{(k)}$, that is $[0,1] \ni \sigma \mapsto \check{\rho}^{(k)}(\sigma) \in \mathcal{M}$. Then according to (3.30), (3.31) and (3.32) there holds

$$d(\rho^{(k-1)}, \rho^{(k)})^2 = \int_0^1 \left|\frac{d\check{\rho}^{(k)}}{d\sigma}\right|^2\, d\sigma,$$

$$\frac{D}{d\sigma}\frac{d\check{\rho}^{(k)}}{d\sigma} = 0,$$
$$\frac{d}{d\sigma}\left|\frac{d\check{\rho}^{(k)}}{d\sigma}\right|^2 = 2\,g\left(\frac{d\check{\rho}^{(k)}}{d\sigma}, \frac{D}{\partial\sigma}\frac{d\check{\rho}^{(k)}}{d\sigma}\right) = 0.$$

Then, we consider a variation $\rho_\varepsilon^{(k)}$ of $\rho^{(k)}$, that is $\varepsilon \mapsto \rho_\varepsilon^{(k)} \in \mathcal{M}$, a curve that passes through $\rho^{(k)}$ for $\varepsilon = 0$ and we define the curve connecting $\rho^{(k-1)}$ and $\rho_\varepsilon^{(k)}$, that is $[0,1] \ni \sigma \mapsto \check{\rho}_\varepsilon^{(k)}(\sigma) \in \mathcal{M}$, which coinsides with $[0,1] \ni \sigma \mapsto \check{\rho}^{(k)}(\sigma) \in \mathcal{M}$ for $\varepsilon = 0$. So, when we rewrite (3.42) for $\rho = \rho^{(k)}$ we have

$$\begin{aligned}\mathfrak{R}(\rho^{(k)}) &= \frac{1}{2h}\int_0^1 \left|\frac{d\check{\rho}^{(k)}}{d\sigma}\right|^2 d\sigma + E(\rho^{(k)}) \\ &\leq \frac{1}{2h}d(\rho^{(k-1)}, \rho_\varepsilon^{(k)})^2 + E(\rho_\varepsilon^{(k)}) \\ &\leq \frac{1}{2h}\int_0^1 \left|\frac{d\check{\rho}_\varepsilon^{(k)}}{d\sigma}\right|^2 d\sigma + E(\rho_\varepsilon^{(k)}).\end{aligned}$$

We can now calculate

$$\begin{aligned}0 = &\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\left(\frac{1}{2h}\int_0^1 \left|\frac{d\check{\rho}_\varepsilon^{(k)}}{d\sigma}\right|^2 d\sigma + E_{\rho_\varepsilon^{(k)}}\right) \\ &= \frac{1}{h}\int_0^1 g\left(\frac{d\check{\rho}^{(k)}}{d\sigma}, \left.\frac{D}{d\varepsilon}\right|_{\varepsilon=0}\frac{d\check{\rho}_\varepsilon^{(k)}}{d\sigma}\right) d\sigma + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{\rho_\varepsilon^{(k)}}\ \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\rho_\varepsilon^{(k)} \\ &= \frac{1}{h}\int_0^1 g\left(\frac{d\check{\rho}^{(k)}}{d\sigma}, \frac{D}{d\sigma}\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\check{\rho}_\varepsilon^{(k)}\right) d\sigma + g\left(\mathrm{grad}E_{\rho^{(k)}}, \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\rho_\varepsilon^{(k)}\right) \\ &= \frac{1}{h}\int_0^1 \left\{\frac{d}{d\sigma}g\left(\frac{d\check{\rho}^{(k)}}{d\sigma}, \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\check{\rho}_\varepsilon^{(k)}\right) - g\left(\frac{D}{d\sigma}\frac{d\check{\rho}^{(k)}}{d\sigma}, \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\check{\rho}_\varepsilon^{(k)}\right)\right\} d\sigma \\ &\quad + g\left(\mathrm{grad}E_{\rho^{(k)}}, \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\rho_\varepsilon^{(k)}\right) \\ &= \frac{1}{h}g\left(\left.\frac{d}{d\sigma}\right|_{\sigma=1}\check{\rho}^{(k)}, \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\check{\rho}_\varepsilon^{(k)}\right) + g\left(\mathrm{grad}E_{\rho^{(k)}}, \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\rho_\varepsilon^{(k)}\right).\end{aligned} \tag{3.43}$$

For the second equality we used that $[0,1] \ni \sigma \mapsto \check{\rho}_\varepsilon^{(k)}$ coincides with $[0,1] \ni \sigma \mapsto \check{\rho}^{(k)}$ for $\varepsilon = 0$ and the definition of the induced distance; for the third equality, we used (3.5), while the fourth equality is a result of the metric property of the covariant differentiation (by the Fundamental Theorem of Riemannian Geometry [11, §2.1.2]) namely

$$\begin{aligned}\frac{d}{d\sigma}g\left(\frac{d\check{\rho}^{(k)}}{d\sigma}, \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\check{\rho}_\varepsilon^{(k)}\right) =& g\left(\frac{d\check{\rho}^{(k)}}{d\sigma}, \frac{D}{d\sigma}\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\check{\rho}_\varepsilon^{(k)}\right) \\ &+ g\left(\frac{D}{d\sigma}\frac{d\check{\rho}^{(k)}}{d\sigma}, \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\check{\rho}_\varepsilon^{(k)}\right).\end{aligned}$$

Finally, for the fifth equality we used that $[0,1] \ni \sigma \mapsto \check{\rho}^{(k)}$ is a geodesic.

Hence, since (3.43) holds for all $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \rho_\varepsilon^{(k)} \in T_\rho\mathcal{M}$, the first variation of $\mathfrak{R}(\cdot)$ is

$$\frac{1}{h} \left.\frac{d}{d\sigma}\right|_{\sigma=1} \check{\rho}^{(k)} + \mathrm{grad} E_{\rho^{(k)}} = 0.$$

Now, remember that in the previous chapter we interpolated in time defining $\rho_h(t) := \rho^{(k-1)}$ for all $t \in [(k-1)h, kh)$, $k \in \mathbb{N}_0$, $h \in [0,1]$, so $\rho_h(t,\cdot)$ was a piecewise constant function with respect to time. Here, we have constructed in advance the curve of least energy $[0,1] \ni \sigma \mapsto \check{\rho}^{(k)} \in \mathcal{M}$ between $\rho^{(k-1)}$ and $\rho^{(k)}$, where, of course, $\check{\rho}^{(k)}(\sigma = 1) = \rho^{(k-1)}$ and $\check{\rho}^{(k)}(\sigma = 0) = \rho^{(k)}$, thus we instead define $\rho_h(t) := \check{\rho}^{(k)}\left(\frac{t}{h} - (k-1)\right)$, for all $t \in [(k-1)h, kh]$. Then the curve $[0,+\infty) \ni t \mapsto \rho_h(t)$ is continuous and piecewise differentiable with $\rho_h((k-1)h) = \rho^{(k-1)}$, $\rho_h(kh) = \rho^{(k)}$ and

$$\left.\frac{d\rho_h}{dt}\right|_{t=kh} = \frac{1}{h} \left.\frac{d\check{\rho}^{(k)}}{dt}\right|_{t=kh} = \frac{1}{h} \left.\frac{d\check{\rho}^{(k)}}{d\sigma}\right|_{\sigma=1}.$$

Therefore

$$\left.\frac{d\rho_h}{dt}\right|_{t=kh} = -\mathrm{grad} E_{\rho_h}\Big|_{t=kh}.$$

Bibliography

[1] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.

[2] Patrick Billingsley. *Probability and measure*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, third edition, 1995. A Wiley-Interscience Publication.

[3] Yann Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Communications on Pure and Applied Mathematics*, 44(4):375–417, 1991.

[4] Lawrence C. Evans. *Weak convergence methods for nonlinear partial differential equations*, volume 74 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1990.

[5] Lawrence C. Evans. Partial differential equations and Monge-Kantorovich mass transfer. In *Current developments in mathematics, 1997 (Cambridge, MA)*, pages 65–126. Int. Press, Boston, MA, 1999.

[6] Gerald B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[7] Wilfrid Gangbo. An elementary proof of the polar factorization of vector-valued functions. *Arch. Rational Mech. Anal.*, 128(4):381–399, 1994.

[8] Leonor Godinho and José Natário. *An introduction to Riemannian geometry*. Universitext. Springer, Cham, 2014. With applications to mechanics and relativity.

[9] Richard Jordan, David Kinderlehrer, and Felix Otto. The variational formulation of the Fokker-Planck equation. *SIAM J. Math. Anal.*, 29(1):1–17, 1998.

[10] Felix Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.

[11] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[12] S. T. Rachev. The Monge-Kantorovich problem on mass transfer and its applications in stochastics. *Teor. Veroyatnost. i Primenen.*, 29(4):625–653, 1984.

[13] H. Risken. *The Fokker-Planck equation*, volume 18 of *Springer Series in Synergetics*. Springer-Verlag, Berlin, second edition, 1989. Methods of solution and applications.