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HORAVA-LIFSHITZ GRAVITY AND MIRAGE COSMOLOGY

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Abstract

In the first part, we discuss the static and spherically symmetric solutions with zero shift vector, of the low energy Horava-Lifshitz gravity. We begin with a brief introduction to Horava-Lifshitz gravity, originally introduced in 2007 by Petr Horava. We continue with the construction of the most general power-counting renormalizable theory with anisotropic scaling. Then, we compute the large distance asymtpotics of the metric-fields, where for simplicity, we concentrate on the low energy effective action of the theory. Finally, we compare the results with those of General Relativity. In the second part, our three-dimensional space is treated as a hyper-surface in a ten-dimensional static and spherically symmetric background space. Due to its motion, we observe an effective cosmological evolution, that we compute. Using Freedman's equations, we define the effective matter density, that would produce the observed cosmological evolution in the context of General Relativity. Then, we apply the results in the case of the $AdS_5 \times S_5$ black hole background space. We calculate the asymptotics of the effective matter density and we show that for large distances, from the center of the AdS black hole, it takes a constant value. Therefore, we conclude that the motion of our three-dimensional space in the ten-dimensional background, could be the origin of dark energy. Lastly we generalize the results for a wide case of background metrics.

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1 PART I: Horava-Lifshitz Gravity

1.1 Introduction to Horava-Lifshitz gravity

In the last century there have been many attempts to quantize gravity. Nevertheless, it is still an open problem, due to difficulties in constructing a consistent renormalizable theory. Many proposed theories for a quantum theory of gravity have appeared, but none of these has been widely accepted by the scientific community. The most prominent approaches to the subject, are string theory [3], asymptotic safety [4], and loop quantum gravity [5].

In 2007, Petr Horava proposed a power counting renormalizable theory, as a candidate for a theory of quantum gravity [6], motivated by the improved scaling properties of anisotropic scaling theories. The main idea was to improve the UV behaviour of the theory, by adding higher derivative terms. The idea of adding such terms was not new. Since the late 70's many attempts have in this direction taken place [1, 2]. However, by just adding space and time derivatives, keeping general covariance, one can show that the theory contains ghosts, which implies that unitarity is violated. Thus, the idea was to add only higher space-derivatives, abandoning "full" general covariance. By "full", we mean that the theory is still invariant under a subgroup of general coordinate transformations, that is called diffeomorphism. Schematically, the graviton propagator that has four-momentum $k_\mu = (\omega, \vec{k})$ will be

$$\frac{1}{k^2}, \tag{1}$$

where $k = \sqrt{\omega^2 - \vec{k}^2}$. For example, by adding higher-derivative terms, we can achieve the improved UV behaviour

$$\frac{1}{k^2 - ak^4}, \tag{2}$$

where a is just a constant. Nevertheless, as we mentioned above such a theory contains ghosts. We can see this, by observing that

$$\frac{1}{k^2 - ak^4} = \frac{1}{k^2} - \frac{1}{k^2 - 1/a}. \tag{3}$$

We see that the propagator has two poles where the first is identified with a massless particle, and the second with a massive ghost. This means that the low energy effective Lagrangian will contain the following term

$$\mathcal{L} = \int d^4x \left(-(\partial c)^2 - \frac{1}{a} c^2 \right), \tag{4}$$

which describes a ghost field. Moreover, such a field with a minus sign in the kinetic term leads to negative norm states in the theory. This, implies the violation of unitarity.[1]Therefore, through adding just spatial derivatives we avoid this problem, but the price we have to pay is to abandon full general covariance.

In addition, the space and time coordinates must have different scaling dimensions, that is, we must have anisotropic scaling. The reason for this is that by sending the cut off to infinity, we want the higher spatial-derivative terms to be relevant. We know from Renormalization Group flow that when the cut-off is sent to infinity, the terms that survive are those whose dimensions are less or equal to the inverse dimensions of the volume element $-[dtd^3x]$. Therefore, in order for higher derivative terms to be relevant, we need to have anisotropic scaling. In this type of theories, the dimensions of the volume element will be increased, and we will be able to add higher derivative terms. Therefore, the theory will be constructed so as to be compatible with the anisotropic scaling

$$\vec{x} \to b\vec{x}, \quad t \to b^z t, \tag{5}$$

where z is the critical exponent that measures the degree of anisotropy. Observe that in such a theory the time-coordinate plays a special role.

Although a theory of this type will not have full general covariance, as mentioned above, we will restrict the theory to be invariant under the following sub-group of general coordinate transformations

$$\vec{x} \to \vec{x}'(\vec{x}, t), \quad t \to t'(t). \tag{6}$$

[1]In fact, although the appearance of ghosts implies that the theory will have negative norm states, it is not necessary that unitarity is violated. For instance, in Yang-Mills theories these are negative norm states but we can show that they cannot appear in "in" and "out" states, and consequently unitarity is preserved.

Now we are ready to construct the Lagrangian of the theory. We will use the language of the ADM formalism for this purpose, that is, we will parametrize the metric as

$$ds^2 = (-N^2 + N_i N^i)dt^2 + 2N_i dx^i dt + \gamma_{ij} dx^i dx^j, \tag{7}$$

where N and N_i are the lapse function and the shift vector respectively.

The kinetic term can be written covariantly in terms of the second fundamental form

$$K_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i). \tag{8}$$

Using the second fundamental form, we can construct two scalars, concluding that the kinetic term will be the following

$$S_K = \kappa \int dt d^3x \sqrt{\gamma} N (K_{ij} K^{ij} - \lambda K^2), \tag{9}$$

where λ and κ are just coupling constants. Note that $\lambda = 1$ corresponds to the case of General Relativity. Also, in the above equation we used the fact that $\sqrt{-g} = \sqrt{\gamma} N$. One can see this by writing the metric as

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_i N^i & N^i \\ N_i & \gamma_{ij} \end{pmatrix} = \begin{pmatrix} 1 & N^i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -N^2 & 0 \\ N_i & \gamma_{ij} \end{pmatrix}. \tag{10}$$

From (9) we see that the dimension of κ is

$$[\kappa] = \frac{z-3}{2}. \tag{11}$$

In order for the kinetic term to be relevant, z must be equal to 3. Therefore, we see that this requirement fixes the degree of anisotropy. This implies that the volume element has dimensions minus six, and according to the renormalization group flow, the potential term can contain all possible terms that have dimensions less than or equal to six. Having that in mind, we can write the most general power counting renormalizable action that is invariant under (6).

$$S = S_K + S_2 + S_3 + S_{rel} + S_3 + S_\alpha, \tag{12}$$

$$S_K = \kappa \int dt d^3x \sqrt{\gamma} N (K_{ij} K^{ij} - \lambda K^2), \tag{13}$$

$$S_{rel} = \int dt d^3x \sqrt{\gamma} N \left(\zeta_1 \varepsilon^{ijk} R_{il} \nabla_j R^l_k + \zeta_2 R_{ij} R^{ij} + \zeta_3 R^2 + \xi R + \sigma \right), \tag{14}$$

$$S_3 = \int dt d^3x \sqrt{\gamma} N \left(\beta_1 C_{ij} C^{ij} + \beta_2 R \nabla^i \nabla_j R + \zeta_3 R^2 + \beta_4 R^3 + \beta_4 R R_{ij} R^{ij} + \beta_5 R_{ij} R^{ik} R^j_k \right), \tag{15}$$

$$S_\alpha = \int dt d^3x \sqrt{\gamma} N \left(\alpha_1 (a_i a^i) + \alpha_2 (a_i a^i)^2 + \alpha_3 R^{ij} a_i a_j + \alpha_4 R \nabla_i a^i + \alpha_5 \nabla_i a_j \nabla^i a^j + \alpha_6 \nabla^i a_i (a^j a_j) + \ldots \right), \tag{16}$$

where C^{ij} is the Cotton tensor defined as

$$C^{ij} = \varepsilon^{ijk} \nabla_k \left(R^j{}_l - \frac{1}{4} R \delta^j{}_l \right), \tag{17}$$

a_i defined by

$$a_i \equiv \frac{\partial_i N}{N}. \tag{18}$$

and $\varepsilon^{ijk} = \frac{\epsilon^{ijk}}{\sqrt{\gamma}}$ is the standard covariant antisymmetric tensor. In the above expression, the ellipsis refers to dimension six terms involving a_i as well as curvatures. Note that although we can add terms containing a_i, we cannot add terms that are express as powers of the lapse function. For instance, we cannot add a term proportional to N^2 because it is not covariant under the transformation (6). N, N_i and γij transform under an infinitesimal diffeomorphism transformation $t' = t + k(t)$, $\vec{x}'^i = x^i + \epsilon^i(t, \vec{x})$ as

$$\delta\gamma_{ij} = \partial_i \epsilon^k \gamma_{jk} + \partial_j \epsilon^k \gamma_{ik} + \epsilon^k \partial_k \gamma_{ij} + k \dot{\gamma}_{ij} \tag{19}$$

$$\delta N_i = \partial_i \epsilon^j N_j + \partial_j \epsilon^j N_i + \dot{\epsilon}^j \gamma_{ij} + \dot{k} N_i + k \dot{N}_i, \quad \delta N = \epsilon^j \partial_j N + \dot{k} N + k \dot{N}. \tag{20}$$

1.2 The low-energy action

In this section, we will study a class of solutions of Horava-Lifshitz gravity, following the analysis in [8]. Unfortunately, even if we restrict on static and spherically symmetric solutions in the vacuum, the problem will be extremely difficult for us to solve. For this reason, we will concentrate on the low-energy effective theory. The terms that dominate at low energies are

$$S = \int dt d^3x \sqrt{\gamma} N \left[\alpha \left(K_{ij} K^{ij} - \lambda K^2 \right) + \xi R + \alpha_1 (a^i a_i) + \sigma \right]. \tag{21}$$

Setting $\lambda = 1$ in this effective action, we see that it reduces to the Hilbert action of General Relativity in the presence of a cosmological constant plus the term containing a_i. We note that this action, apart from the cosmological constant term, has isotropic scaling symmetry $t' \to bt$, $\vec{x} \to b\vec{x}$. Furthermore, we will concentrate on solutions of the form

$$ds^2 = -N^2(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{22}$$

that is static and spherically symmetric solutions with zero shift vector. In this situation, the first two terms containing K_{ij} are equal to zero. Varying the action (21) with respect to the gravitational field, we will obtain the equations of motion. The equation obtained after having varied N is

$$-\alpha\left(K_{ij}K^{ij}-\lambda K^2\right)+\xi R-\alpha_1(a^i a_i)+\sigma-2\alpha_1\nabla_i a^i=0. \tag{23}$$

The equation after having varied N_i is

$$\nabla_j K^{ij}-\lambda\nabla^i K=0. \tag{24}$$

Finally, the equation of motion after having varied g_{ij} is

$$\begin{aligned}
&\xi\left(\gamma_{ij}\nabla^i\nabla_i N-\nabla_i\nabla_j N+NR_{ij}\right)-\frac{N}{2}(\xi R+\sigma)\gamma_{ij}+2\alpha N\left(K_{ik}K^k{}_j-\lambda KK_{ij}\right)\\
&-\frac{\alpha N}{2}\left(K_{kl}K^{kl}-\lambda K^2\right)\gamma_{ij}+\frac{\alpha}{\sqrt{\gamma}}\gamma_{ik}\gamma_{jl}\frac{\partial}{\partial t}[\sqrt{\gamma}\left(K^{kl}-\lambda K\gamma^{kl}\right)]+\alpha\nabla^k[\left(K_{ik}K-\lambda KK_{ik}\right)N_j]\\
&+\alpha\nabla^k[\left(K_{jk}K-\lambda KK_{jk}\right)N_i]-\alpha\nabla^k[\left(K_{ij}K-\lambda KK_{ij}\right)N_k]=0.
\end{aligned} \tag{25}$$

Inserting the ansatz (22), equation(23) becomes

$$4\xi(f-1)-2\sigma r^2 f+8\xi f\frac{rN'}{N}+2(w+2\xi)f\frac{r^2N'^2}{N^2}=0, \tag{26}$$

and then, combining the equations obtained by substituting the ansatz (22) in equation(25), we have

$$4w^3\xi^2f^2-4w^2\xi f[2\xi(w+\xi)+(2w+3\xi)\sigma r^2-wr\xi f']+(w+2\xi)(2w\xi+2(w+\xi)\sigma r^2-wr\xi f')^2=0, \tag{27}$$

where $w=a_1-2\xi$ [2]. Even in this case, we cannot find exact solutions. However, we will study the asymptotic behaviour of the solution. In addition, we can simplify these two equations by defining the following function

$$f(r)=1+\frac{b}{a}+\frac{c}{ab}r^2+g(r), \tag{28}$$

where we also define

$$a=\frac{2w}{w+2\xi},\ b=\frac{4\xi}{w+2\xi},\ c=\frac{4\sigma}{w+2\xi},\ a+b=2. \tag{29}$$

Under the above substitution, equation (27) takes the form

$$r^2g'^2+2a(rg'+g)g+(2b+cr^2)g=0. \tag{30}$$

[2]The equations of motion in the presence of matter are extracted in [9]

Furthermore, having solved these equations we can compute the lapse function by writing equation (26) as

$$\frac{N'}{N} = -\frac{b}{2r}\left[1-\sqrt{1-\frac{2}{b}\left(1-\frac{1}{f}-\frac{c}{2b}r^2\right)}\right], \tag{31}$$

and then integrate it to obtain

$$\begin{aligned} N^2 =& C\exp\{-\int dr\frac{b}{r}\left[1-\sqrt{1-\frac{2}{b}\left(1-\frac{1}{f}-\frac{c}{2b}r^2\right)}\right]\} \\ =& Cr^{-b}\exp\{\int dr\frac{b}{r}\sqrt{1-\frac{2}{b}\left(1-\frac{1}{f}-\frac{c}{2b}r^2\right)}\}. \end{aligned} \tag{32}$$

Since we cannot solve equation (30) analytically, we will use perturbation theory to find the large distance asymptotics of $g(r)$. Firstly we will redefine the variables in (30) as

$$r = \sqrt{\frac{2b}{c}}\rho, \quad g = 2bh. \tag{33}$$

Then, we arrive at the following equation

$$\rho^2 h'^2 + 2a(\rho h' + h)h + (1+\rho^2)h = 0, \tag{34}$$

where prime now stands for a derivative with respect to ρ. In order to find the large distance asymptotics, we will expand $h(\rho)$ in powers of $\rho \gg 1$ as $h(\rho) = h_0(\rho) + h_1(\rho) + \ldots$ where $h_0(\rho) \gg h_1(\rho)$. According to our method, we will first obtain the differential equations for $h_0(\rho)$ and $h_1(\rho)$, and then find their asymptotics. Finally, we will use them to calculate the first two terms of the expansion of $g(r)$ and after $f(r)$ and $N^2(r)$.

Large distance expansion

For large distances, $r >> \sqrt{\frac{2b}{c}}$ or equivalent $\rho \gg 1$, the leading terms of (34) are

$$\rho^2 h_0'^2 + 2a(rh_0' + h_0)h_0 + \rho^2 h_0 = 0, \tag{35}$$

and the subleading terms

$$(\rho h_0' + ah_0)\rho h_1' + a(\rho h_0' + 2h_0 + \frac{1}{2a}\rho^2)h_1 + \frac{1}{2}h_0 = 0. \tag{36}$$

Equation (35) can be solved exactly for $\rho(h_0)$ and then we proceed as follows. First, we perform the following transformation

$$h_0 = -\frac{\rho^2 Y^2}{1 + abY^2}, \tag{37}$$

and (35) becomes

$$2\rho Y' + (1 + abY^2)(1 - (a+2)Y) = 0. \tag{38}$$

Integrating this equation we get

$$\frac{\rho}{\rho_0} = \frac{|1 + \sqrt{|\alpha|b}Y|^{\alpha_+}}{|1 - \sqrt{|\alpha|b}Y|^{\alpha_-}} |(|a| - 2)Y + 1|^{\frac{|a|-2}{2-3|\alpha|}}, \tag{39}$$

where

$$a_\pm = \frac{\sqrt{|a|b} \pm (|a| - 2)}{6|a| - 4}. \tag{40}$$

We see that when $\rho \to \infty$ the right-hand side of (39) must also go to infinity. As was shown in [7], linear stable perturbations correspond to $a < 0$. In that paper, the authors considered perturbations around the flat metric and showed that are stable only in this case. Consequently, we conclude that always $a_+ > 0$. Therefore, we expect two types of behaviour.

1. The denominator in (39) will go to zero for large ρ. This occurs for $|a| > \frac{2}{3}$ ($a_- > 0$) and
$$Y \to \frac{1}{\sqrt{|a|b}} \tag{41}$$
2. and the factor raised to the $\frac{|a|-2}{2-3|\alpha|}$ power will go to infinity for large ρ. This occurs for $|a| > 2$ or $|a| < \frac{2}{3}$ and
$$Y \to \frac{1}{2 - |a|} \tag{42}$$

In both cases, for large ρ ,$Y(\rho)$ goes to a constant value. Thus, the subleading terms will be of the form $A\rho^{-\beta}$ where A is a function of a and $\beta > 0$. Submitting this in (39), we find that

1. in first the case where $|a| > \frac{2}{3}$
$$\beta = \frac{1}{a_-}, \quad A = \frac{1}{\sqrt{|a|b}} \left(2^{a_+} |1 - \frac{a+2}{\sqrt{|a|b}}|^{-\frac{a+2}{3a+2}} \rho_0 \right)^{\frac{1}{a_-}}, \tag{43}$$

   and therefore
$$Y(\rho) = \frac{1}{\sqrt{|a|b}} + A\rho^{-\frac{1}{a_-}}, \tag{44}$$

2. and in the other case where $|a|>2$ or $|a|<\frac{2}{3}$

$$\beta=\frac{3|a|-2}{|a|-2},\quad B=\pm\frac{\rho_0^{\frac{3|a|-2}{|a|-2}}}{|2-|a||^2}\frac{(1+|a|)^{a'_-}}{(3+|a|)^{a'_+}},\quad a'_\pm=\frac{\sqrt{|a|b}\pm(|a|-2)}{|a|-2},\tag{45}$$

and therefore

$$Y(\rho)\approx\frac{1}{2-|a|}+A\rho^{-\frac{3|a|-2}{|a|-2}}.\tag{46}$$

Using these results we can find the leading terms for h_0.

1. When $|a|>\frac{2}{3}$, using (37) we find that the expansion of h_0, for large ρ is

$$h_0(\rho)\approx\frac{1}{2\sqrt{(|a|b)^{3/2}}A}\rho^{2+\frac{1}{a_-}}-\rho^2.\tag{47}$$

Note that $2+\frac{1}{a_-}$ is always bigger that 2, since $|a|>\frac{2}{3}$ implies $a_->0$.

2. and in the case $|a|>2$ or $|a|<\frac{2}{3}$ the first two terms of g_0 are

$$h_0(\rho)\approx-\frac{1}{6a+4}\rho^2-\frac{2(a+2)}{6a+4}B\rho^{\frac{2-a}{2+a}}\tag{48}$$

where B is given by (45). Note that $\frac{2-a}{2+a}=2-\frac{2-3|a|}{2-|a|}<2$, and therefore the second term is always smaller than the first one.

In the first case where $|a|>\frac{2}{3}$, $h(\rho)$ goes like $\rho^{2+\frac{1}{a_-}}$. Thus, both $g(r)$ and $f(r)$ will behave as $r^{2+\frac{1}{a_-}}$. These situations cannot give the same asymptotic expansion as in GR, where $f(r)\sim r^2$. On the other hand, in the second case where $|a|>2$ or $|a|<\frac{2}{3}$, $f(r)$ is going to have the same asymptotic behaviour as in GR. So, for this case, having already found the leading term of h_0, the next task is to find the first order correction of $h(\rho)$, namely the function $h_1(\rho)$. Substituting (48) in (36) we obtain

$$-\frac{2+a}{6a+4}\rho h_1'+\frac{2-a}{6a+4}h_1=\frac{1}{2}\frac{1}{6a+4.}\tag{49}$$

The solution of this equation is

$$h_1(\rho)\approx\frac{1}{4-2a}+C\rho^{\frac{2-a}{2+a}}.\tag{50}$$

In this regime, h_1 grows slower than h_0, so perturbation theory works fine.

We can also check the validity of perturbation theory by comparing the quadratic terms in h_1 that appear in (30), and ensure that they will grow slower than the linear terms. These terms are

$$r^2 h_1'^2 + 2a(rh_1' + h_1)h_1 + h_1 \tag{51}$$

and indeed are negligible compared with those in (49). Therefore, the expansion of $h(\rho)$ is

$$h(\rho) \approx -\frac{1}{6a+4}\rho^2 + \frac{1}{4-2a} + \left(C - \frac{2(a+2)}{6a+4}B\right)\rho^{\frac{2-a}{2+a}}. \tag{52}$$

and using (33), the expansion for $g(r)$ is

$$g(r) \approx 2bh\left(\sqrt{\frac{c}{2b}}r\right) = -\frac{1}{6a+4}cr^2 + 1 - \frac{2(a+2)}{6a+4}Bcr^{\frac{2-a}{2+a}}. \tag{53}$$

Lastly, the solution to the blackness function $f(r)$ is

$$f(r) = 2 - \frac{2\xi}{2\xi - a_1} + \frac{(\xi - a_1)^2}{\xi(3\xi - 2a_1)(2\xi - a_1)}\sigma r^2 + \dots \tag{54}$$

in terms of the original couplings. Therefore, in the regime where $|a| > 2$ or $|a| < \frac{2}{3}$, taking the limit $a_1 = 0$ ($a \to -\infty$) that corresponds to General Relativity, we get the usual behaviour

$$f(r) = 1 + \frac{\sigma}{6\xi}r^2. \tag{55}$$

On the other hand, from (32) the leading term in the expansion of the lapse function is

$$N(r) = Cr^{-b}e^{\sqrt{c}r+\dots}. \tag{56}$$

Consequently, the lapse function does not have the same asymptotics as in GR $N^2 \sim f(r)$, in the limit $a_1 \to 0$, since both c and b go to infinity. Therefore, following the above simple analysis, we conclude that the theory does not lead to solutions with the same asymptotic behaviour as in GR with a cosmological constant. The expansion of the function $f(r)$ differs from the GR result in the subleading terms, and the power depends on the parameter a. On the other hand, the leading term in the expansion for N has an exponential behaviour and therefore, it exposes completely different behaviour from General Relativity.

Finally, it is important to examine the case without a cosmological constant. Setting $c = 0$, equations (30) and (32) become

$$r^2 g'^2 + 2a(rg' + g)g + 2bg = 0, \tag{57}$$

and

$$N^2 = Cr^{-b}\exp\{\int dr \frac{b}{r}\sqrt{1 - \frac{2}{b}\left(1 - \frac{1}{f}\right)}\}. \tag{58}$$

Using the same method as before, we find the following large distance expansion for $g(r)$

$$g(r) = \frac{b}{|a|} - \frac{2b}{|a| + \sqrt{|a|b}} \left(\frac{\sqrt{b} - \sqrt{|a|}}{\sqrt{b} + \sqrt{|a|}} \right)^{\frac{1}{|a|+\sqrt{|a|b}}} \frac{r_0}{r} + \mathcal{O}(r^{-2}). \tag{59}$$

Using this result we can find the asymptotic behaviour of the metric components. Using equation (28) with $c = 0$, we obtain

$$f(r) = 1 - \frac{2b}{|a| + \sqrt{|a|b}} \left(\frac{\sqrt{b} - \sqrt{|a|}}{\sqrt{b} + \sqrt{|a|}} \right)^{\frac{1}{|a|+\sqrt{|a|b}}} \frac{r_0}{r} + \mathcal{O}(r^{-2}). \tag{60}$$

and using (58) we obtain

$$N^2(r) = 1 - \frac{2b}{|a| + \sqrt{|a|b}} \left(\frac{\sqrt{b} - \sqrt{|a|}}{\sqrt{b} + \sqrt{|a|}} \right)^{\frac{1}{|a|+\sqrt{|a|b}}} \frac{r_0}{r} + \mathcal{O}(r^{-2}). \tag{61}$$

Therefore, we see that in the absence of a cosmological constant, the metric fields have the same behaviour as in General Relativity plus corrections of lower order.

2 PART II: Mirage Cosmology

2.1 Brane Geodesics

In this section, we will consider the motion of a D3-brane in a higher-dimensional curved space. Observers restricted on the D-brane see that the geometry of the brane changes as the brane moves along geodesics. This change is due to the motion of the brane, in contrast to the cosmological evolution predicted by General Relativity, that is due to the energy density within the brane. Thus, observers see an effective cosmological evolution in their universe. It is obvious that such an effective cosmological evolution can take place only in a curved background space, otherwise the brane moves from area to area with the same geometry, so they cannot feel any geometrical change. In the general case, the higher-dimensional geometry may be an arbitrary curved space and also depend on the motion of the D-brane itself. However, for the sake of simplicity, we will study the motion of a D3-brane in a 10-dimensional spherically symmetric space, neglecting the back-reaction of the D-brane. Therefore, the problem is reduced to finding the geodesics equation for a D3-brane and then examine the cosmological evolution that the observers see.

The 10-dimensional brane may be parametrized as

$$ds_{10} = g_{00}(r)dt^2 + g(r)(d\vec{x})^2 + g_{rr}(r)dr^2 + g_s d\Omega_5, \tag{62}$$

where $g_{00}(r) < 0$ and $d\Omega_5 = h_{ij}\varphi^j\varphi^j$ is the part of the 5-sphere. To leading order, the motion of the brane is governed by the Nambu-Goto action[10].

$$S = T_3 \int d^4\xi \sqrt{-det(G_{\alpha\beta})}\epsilon^{-\phi} - T_3 \int d^4\xi C(r), \tag{63}$$

where $G_{\alpha\beta}$ is the induced metric defined as,

$$G_{\alpha\beta} = G_{\mu\nu}\frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial x^\nu}{\partial \xi^\beta}, \tag{64}$$

$\varphi(r)$ is the dilaton field, $C(r)$ is the RR field and $G_{\mu\nu}$ is the above ten-dimensional metric. As we know, this action is invariant under reparametrization. In order to fix it, we choose the static gauge $X^\alpha = \xi^\alpha$ for $\alpha = 0, 1, 2, 3$. Moreover, we will concentrate on motion only along the time direction, that is every point on the brane will have the same r coordinate at each moment. In other words, the brane does not wimple. In this regime, the induced metric is written as

$$G_{\alpha\beta} = \begin{pmatrix} g_{00} + g_{rr}\dot{r}^2 + g_s h_{ij}\dot{\varphi}^j\dot{\varphi}^i & 0 \\ 0 & g\mathbb{1}_{3x3} \end{pmatrix} \tag{65}$$

In the static gauge, the action (63) in term of the metric components is written as

$$\begin{aligned} S &= T_3 V_3 \int d\xi^0 \left(e^{-\phi}\sqrt{-g^3 g_{00} - g^3 g_{rr}\dot{r}^2 - g^3 g_s h_{ij}\dot{\varphi}^j\dot{\varphi}^i} - C(r) \right) \\ &= T_3 V_3 \int d\xi^0 \left(\sqrt{A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i} - C(r) \right), \end{aligned} \tag{66}$$

where since the integrand has only ξ^0-dependence, we performed the integration over the remaining coordinates, that gives the volume of the world sheet V_3. Also we defined

$$A = g^3|g_{00}|e^{-2\phi}, \quad B = g^3 g_{rr} e^{-2\phi}, \quad D = g^3 g_s e^{-2\phi}. \tag{67}$$

In this regime, the problem is effectively one-dimensional with the following Lagrangian

$$L(r, \dot{r}, \varphi^i, \dot{\varphi}^j) = \sqrt{A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i} - C(r). \tag{68}$$

Rather than finding the equation of motion using the Lagrange equations, one can more easily calculate the integrals of motion, and then find the trajectory of the brane. Since the Lagrangian has no explicit dependence on the time coordinate, the Hamiltonian will be conserved. Also, as it is shown in the Appendix A the total angular momentum is conserved. Therefore, having two constant quantities we are able to determine the motion of the brane. Firstly, the momenta are given by

$$p_r = -\frac{B\dot{r}}{\sqrt{A - B\dot{r}^2 - Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i}} \tag{69}$$

$$p_i = -\frac{Dh_{ij}\dot{\varphi}^j}{\sqrt{A - B\dot{r}^2 - Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i}}. \tag{70}$$

Proceeding now in computing the Hamiltonian we find

$$\mathcal{H} = C - \frac{A}{\sqrt{A - B\dot{r}^2 - Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i}}. \tag{71}$$

and for total angular momentum

$$\ell^2 = h^{ij}p_ip_j = \frac{D^2h_{ij}\dot{\varphi}^j\dot{\varphi}^i}{A - B\dot{r}^2 - Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i}. \tag{72}$$

In addition, solving the last equation for $h_{ij}\dot{\varphi}^j\dot{\varphi}^i$, we have that

$$h_{ij}\dot{\varphi}^j\dot{\varphi}^i = \frac{\ell^2\,(A - B\dot{r}^2)}{D(D+\ell^2)}. \tag{73}$$

Combining equations (71) and (73), we arrive at the following equations

$$\dot{r}^2 = \frac{A}{B}\left(1 - \frac{A}{(\mathcal{H}+C)^2}\frac{D+\ell^2}{D}\right), \quad h_{ij}\dot{\varphi}^j\dot{\varphi}^i = \frac{\ell^2A^2)}{D^2(C+\mathcal{H})^2}. \tag{74}$$

In a generic space of the form (62), one can solve (74) and determine the motion of the brane in the r-direction. In order to compute the effective cosmological evolution we proceed as follows. The induced four-dimensional metric on the D3-brane universe is

$$ds^2 = \left(g_{00} + g_{rr}\dot{r}^2 + g_sh_{ij}\dot{\varphi}^j\dot{\varphi}^i\right)dt^2 + g(r)(\vec{dx}^2). \tag{75}$$

Furthermore, using the second equation in set (74) and the definitions (67), it becomes

$$ds^2 = -\frac{g_{00}^2g^3e^{-2\phi}}{(C+\mathcal{H})^2}dt^2 + g(r)(\vec{dx}^2). \tag{76}$$

We can define the cosmic time as

$$d\eta = \sqrt{b}dt, \tag{77}$$

and (75) takes a FLRW form

$$ds^2 = -d\eta^2 + a^2(\eta)(\vec{dx}^2) \quad , \quad a^2(\eta) = g(r(\eta)) \tag{78}$$

The cosmic time is the same as the proper time in the brane universe. Equation (78) is the standard form of a flat expanding universe, where its evolution is governed by the usual Friedman equations. The first of them (in flat space) is written as

$$\left(\frac{\partial_\eta a}{a}\right)^2 = \frac{8\pi}{3}\rho_{eff}. \tag{79}$$

By analogy, we can define the effective matter density from the brane's point of view as

$$\frac{8\pi}{3}\rho_{eff} = \left(\frac{\partial_\eta a}{a}\right)^2 = H^2, \tag{80}$$

where H is the usual Hubble function. This effective matter density would be responsible for the observed cosmological evolution, in the context of General Relativity. In terms of the metric components, the effective matter density is

$$\frac{8\pi}{3}\rho_{eff} = \frac{(C+\mathcal{H})^2 g_S e^{2\phi} - |g_{00}|(g_S g^3 + \ell^2 e^{2\phi})}{4|g_{00} g_{rr} g_S g^3}\left(\frac{g'}{g}\right)^2. \tag{81}$$

2.2 The Addition of the Ricci scalar

The modification here is the addition of the four-dimensional Ricci scalar of the D3-brane. Then, the motion of the brane will be governed by the action

$$S = T_3 \int d^4\xi \sqrt{-det(G_{\alpha\beta})}\left(e^{-\phi} - \frac{2}{3}\Lambda\hat{R}\right) - T_3 \int d^4\xi C(r), \tag{82}$$

where $\hat{R}$ is the Ricci scalar of the four-dimensional world sheet and Λ is just a coupling constant. The 2/3 factor is introduced just for future convenience. From now on, we will refer to the part in the action without the term $-\frac{2}{3}\Lambda\hat{R}$ as the "Naumbu-Goto part" while the rest will be mentioned the "Einstein part".

In order to fix the reparametrization invariance we choose again the static gauge $X^\alpha = \xi^\alpha$ for $\alpha = 0, 1, 2, 3$. Once more, we will concentrate on motion without wimpling.

For simplicity, let us write the induced metric (65) as

$$G_{\alpha\beta} = \begin{pmatrix} -b & 0 \\ 0 & a\mathbb{1}_{3x3} \end{pmatrix} \tag{83}$$

Then, the Ricci scalar is

$$\hat{R} = \frac{6\ddot{a}b - 3\dot{a}\dot{b}}{2b^2 a} = \frac{1}{\sqrt{b}a^{\frac{3}{2}}}\left(3\frac{d}{dt}(\sqrt{\frac{a}{b}}\dot{a}) - \frac{3}{2}\frac{\dot{a}^2}{\sqrt{ab}}\right), \tag{84}$$

and the action takes the form

$$S = T_3 \int d^4\xi \left(e^{-\phi}\sqrt{b}a^{\frac{3}{2}} - 2\Lambda\frac{d}{dt}(\sqrt{\frac{a}{b}}\dot{a}) + \Lambda\frac{\dot{a}^2}{\sqrt{ab}}\right) - T_3 \int d^4\xi C. \tag{85}$$

We can neglect the second term because it is a total derivative, and the action becomes

$$S = T_3 V_3 \int d\xi^0 \left(e^{-\phi}\sqrt{b}a^{\frac{3}{2}} + \Lambda \frac{\dot{a}^2}{\sqrt{ab}} \right) = T_3 V_3 \int d\xi^0 \left(e^{-\phi}\sqrt{ba^3} + \Lambda \frac{a\dot{a}^2}{\sqrt{a^3 b}} - C \right), \tag{86}$$

Again we can perform the integration over the ξ^i coordinates, that gives the volume of the world sheet V_3. Then, using the definitions for a and b, in terms of the ten-dimensional metric, the action is written as

$$\begin{aligned} S &= T_3 V_3 \int d\xi^0 \left(e^{-\phi}\sqrt{-g^3 g_{00} - g^3 g_{rr}\dot{r}^2 - g^3 g_s h_{ij}\dot{\varphi}^j\dot{\varphi}^i} + \Lambda \frac{g g'^2 \dot{r}^2}{\sqrt{-g^3 g_{00} - g^3 g_{rr}\dot{r}^2 - g^3 g_s h_{ij}\dot{\varphi}^j\dot{\varphi}^i}} - C(r) \right) \\ &= T_3 V_3 \int d\xi^0 \left(\sqrt{A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i} + \frac{E\dot{r}^2}{\sqrt{A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i}} - C(r) \right) \end{aligned} \tag{87}$$

where

$$A = g^3 |g_{00}| e^{-2\phi}, \quad B = g^3 g_{rr} e^{-2\phi}, \quad D = g^3 g_s e^{-2\phi}, \quad E = \Lambda g'^2 g e^{-\phi}. \tag{88}$$

The problem effectively is also one-dimensional with the following Lagrangian

$$L(r, \dot{r}, \varphi^i, \dot{\varphi}^j) = \sqrt{A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i} + \frac{E\dot{r}^2}{\sqrt{A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i}} - C(r). \tag{89}$$

Proceeding further, the momenta are

$$\begin{aligned} p_r &= -\frac{B\dot{r}}{\sqrt{A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i}} + \frac{2E\dot{r}}{\sqrt{A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i}} + \frac{BE\dot{r}^3}{(A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i)^{\frac{3}{2}}} \\ &= \frac{1}{\sqrt{A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i}} \left(2E\dot{r} - B\dot{r} + \frac{BE\dot{r}^3}{A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i} \right), \end{aligned} \tag{90}$$

$$\begin{aligned} p_i &= -\frac{D h_{ij}\dot{\varphi}^j}{\sqrt{A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i}} + \frac{E\dot{r}^2 D h_{ij}\dot{\varphi}^j}{(A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i)^{\frac{3}{2}}} \\ &= \frac{D h_{ij}\dot{\varphi}^j}{\sqrt{A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i}} \left(\frac{E\dot{r}^2}{A - B\dot{r}^2 - D h_{ij}\dot{\varphi}^j\dot{\varphi}^i} - 1 \right). \end{aligned} \tag{91}$$

We can now compute the Hamiltonian as usual

$$
\begin{aligned}
\mathcal{H} &= p_r\dot{r} + p_i\dot{\varphi}^i - L = \\
&= \frac{1}{\sqrt{A - B\dot{r}^2 - Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i}}\Bigg(2E\dot{r}^2 - B\dot{r}^2 + \frac{BE\dot{r}^4}{A - B\dot{r}^2 - Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i} + \frac{E\dot{r}^2 Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i}{A - B\dot{r}^2 - Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i} \\
&\quad - Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i - (A - B\dot{r}^2 - Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i) - E\dot{r}^2\Bigg) + C(r) \\
&= \frac{A}{\sqrt{A - B\dot{r}^2 - Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i}}\left(\frac{E\dot{r}^2}{A - B\dot{r}^2 - Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i} - 1\right) + C(r),
\end{aligned}
\tag{92}
$$

and the total angular momentum as

$$
\ell^2 = h^{ij}p_ip_j = \frac{D^2h_{ij}\dot{\varphi}^j\dot{\varphi}^i}{A - B\dot{r}^2 - Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i}\left(\frac{E\dot{r}^2}{A - B\dot{r}^2 - Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i} - 1\right)^2. \tag{93}
$$

Additionally, we can simplify the problem by defining the following parameters

$$
x = \frac{Dh_{ij}\dot{\varphi}^j\dot{\varphi}^i}{A - B\dot{r}^2}, \quad \epsilon = \frac{\ell^2}{D}, \quad \zeta = -\frac{E\dot{r}^2}{A - B\dot{r}^2} = -\Lambda\frac{g'^2ge^{-\phi}\dot{r}^2}{A - B\dot{r}^2}. \tag{94}
$$

Then, the equations for the Hamiltonian and the total angular momentum take the following form

$$
\mathcal{H} = C(r) - \frac{A}{\sqrt{A - B\dot{r}^2}}\frac{1}{\sqrt{1-x}}\left(\frac{\zeta}{1-x} + 1\right), \tag{95}
$$

$$
\epsilon = \frac{x}{1-x}\left(1 + \frac{\zeta}{1-x}\right)^2, \tag{96}
$$

respectively. While $y = 1 - x$ satisfies

$$
y^3(\mathcal{H} + C)^2 = \frac{A^2}{A - B\dot{r}^2}(\zeta + y)^2, \tag{97}
$$

$$
y^3\epsilon = (1 - y)(\zeta + y)^2. \tag{98}
$$

These equations can be solved for y. Dividing the second with the first, what we obtain is

$$
y = 1 - \frac{\epsilon A^2}{(\mathcal{H} + C)^2(A - B\dot{r}^2)}. \tag{99}
$$

After that, inserting this expression in (98), we reach to a differential equation for $\dot{r}$

$$
\left(A - \frac{\epsilon A^2}{(\mathcal{H} + C)^2} - B\dot{r}^2\right)^3 = \frac{A^2}{(\mathcal{H} + C)^2}\left(A - \frac{\epsilon A^2}{(\mathcal{H} + C)^2} - (B + E)\dot{r}^2\right)^2. \tag{100}
$$

Also, in order to bring this equation in a more concrete form, we define the following quantities

$$z(r) = \frac{A}{B} - \frac{\epsilon A^2}{B(\mathcal{H}-C)^2} = \frac{|g_{00}|}{g_{rr}} - \frac{\ell^2 g_{00}^2}{g_s g_{rr}(\mathcal{H}+C)^2}, \tag{101}$$

$$h(r) = \frac{A^2}{B(\mathcal{H}+C)^2} = \frac{g^3 g_{00}^2 e^{-2\phi}}{g_{rr}(\mathcal{H}+C)^2}, \tag{102}$$

$$e(r) = 1 + \frac{E}{B} = 1 + \Lambda \frac{g'^2 e^{\phi}}{g^2 g_{rr}}, \tag{103}$$

while equation (100) takes the form

$$\left(z(r) - \dot{r}^2\right)^3 = h(r)\left(z(r) - e(r)\dot{r}^2\right)^2. \tag{104}$$

This equation can have up to three possible solutions for $\dot{r}^2$, as a function of the metric components, depending on the sign of its discriminant. The general solution is discussed in the Appendix B. In a generic space of the form (62), one can solve (104) and determine the motion of the brane in the r-direction. We could have also obtained the other equations that specify the motion in the other directions, but it is not necessary since our aim is to find the effective cosmological evolution observed by the "habitants" of the brane.

In terms of the 10-dimensional metric the Humble function is

$$H^2 = \left(\frac{\partial_\eta a}{a}\right)^2 = \frac{1}{4b}\frac{\dot{r}^2 g'^2}{g^2}. \tag{105}$$

In addition, we can express b as a function of r and $\dot{r}$ only, as

$$\begin{aligned}
b &= |g_{00}| - g_{rr}\dot{r}^2 - g_s h_{ij}\varphi^i\varphi^j \\
&= \frac{e^{2\phi}}{g^3}\left(A - B\dot{r}^2 - D h_{ij}\varphi^i\varphi^j\right) \\
&= \frac{e^{2\phi}}{g^3}(A - B\dot{r}^2)(1-x) \\
&= \frac{e^{2\phi}}{g^3}(A - B\dot{r}^2)y \\
&= \frac{e^{2\phi}}{g^3}(A - B\dot{r}^2)\left(1 - \frac{\epsilon A^2}{(\mathcal{H}+C)^2(A - B\dot{r}^2)}\right) \\
&= \frac{e^{2\phi}}{g^3}\left(A - B\dot{r}^2 - \frac{\epsilon A^2}{(\mathcal{H}+C)^2}\right) \\
&= |g_{00}| - g_{rr}\dot{r}^2 - \frac{\ell^2 |g_{00}|^2}{g_s(\mathcal{H}+C)^2},
\end{aligned} \tag{106}$$

and inserting it in the previous expression for H we find

$$H^2 = \frac{1}{4}\frac{g_s(\mathcal{H}+C)^2\dot{r}^2}{g_s(\mathcal{H}+C)^2(|g_{00}|-g_{rr}\dot{r}^2)-l^2g_{00}^2}\left(\frac{g'}{g}\right)^2, \tag{107}$$

where the dot stands for the derivative with respect to the cosmic time, while the prime stands for the derivative with respect to r. In addition, we can write the Hubble function in terms of the auxiliary functions $z(r), h(r), e(r)$

$$H^2 = \frac{e^{-\phi}}{4\Lambda}\frac{\dot{r}^2}{z(r)-\dot{r}^2}(e(r)-1). \tag{108}$$

We see that, once we have obtained the function $\dot{r}$ from the solutions of (104), we can use it in (107) and compute the Hubble function. Finally, we can obtain the effective cosmological matter density using (80). In addition, we can find the scale factor, as a function of cosmic time, by solving the next differential equation

$$\left(\frac{\partial_\eta a}{a}\right)^2 = H^2(a). \tag{109}$$

Now, having built our machinery for determining the effective density, first by solving (104) and then using (107), we are going to examine the case of $\mathrm{AdS}_5 \times S_5$ black hole background.

2.3 AdS black hole

In this section we will apply the above procedure in the case of the AdS_5 black hole background metric, i.e

$$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + (\vec{dx})^2\right) + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2, \tag{110}$$

where the RR field is $C = \frac{r^4}{L^4}$. In the first case (without the Einstein term), we can use (81) and calculate easily the effective matter density

$$\frac{8\pi}{3}\rho_{eff} = \frac{1}{L^2}\left(\frac{2\mathcal{H}}{a^4} - \frac{\ell^2}{L^2}\frac{1}{a^6} + \frac{\mathcal{H}^2}{a^8}\right). \tag{111}$$

Far away from the black hole $\rho_{eff} \sim a^{-4}$. In this regime, the motion of the brane produces an effective matter density like the one due to the radiation on the brane. On the other hand, at sort distances from the black hole, the term $\sim a^{-8}$ dominates corresponding to dilute matter with $p = w\rho$ and $w = 5/3$.

In the second case, where the Einstein term is present, more work is needed. An analytic solution to the problem still exists, however it is not very elegant and gives no physical intuition about the implications of the model. Firstly, we will show that equation (104) has positive solutions for $\dot{r}^2$ for each value of $\gamma = \frac{4\Lambda}{L^2}$. Then, ensuring that a solution that covers the whole space exists, i.e. without any discontinuities, we will study the asymptotic behaviour of the effective density. Afterwards, we will compare them with those of the model in the presence of either only the Einstein term or just the Naumbu-Goto term.

First of all, equation (104) in this background (setting for simplicity $\ell = 0$) becomes

$$\left(\left(\frac{r}{L}\right)^4 - \dot{r}^2\right)^3 = \left(\frac{r}{L}\right)^{12} \frac{1}{\left(\mathcal{H} + \left(\frac{r}{L}\right)^4\right)^2} \left(\left(\frac{r}{L}\right)^4 - (1 + \frac{4\Lambda}{L^2})\dot{r}^2\right)^3 . \tag{112}$$

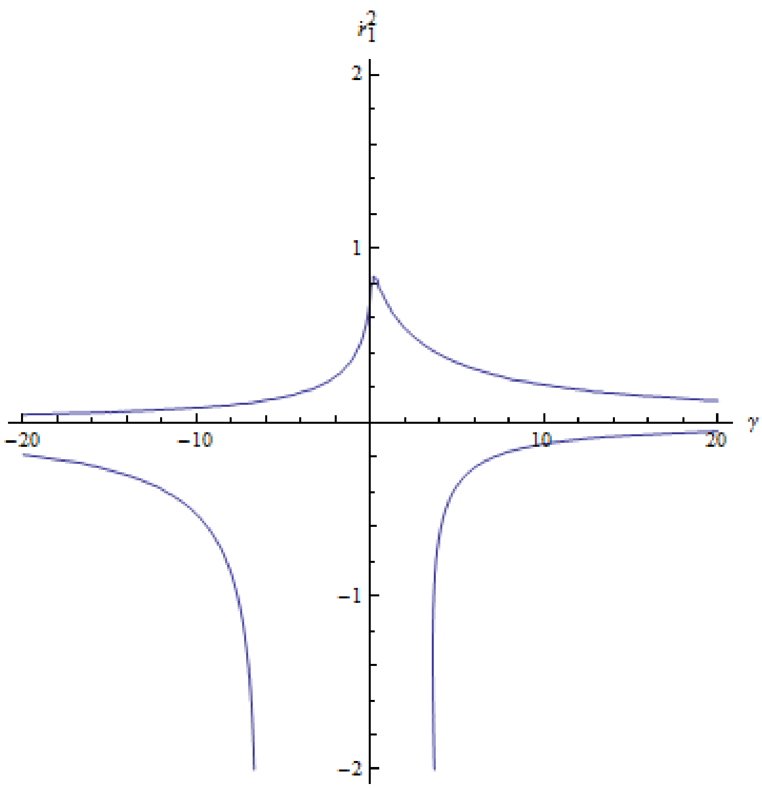
Then, rescaling r ans t as $r \to (\mathcal{H}^{1/4}L)r$ and $t \to (\mathcal{H}^{-1/4}L)t$ we get

$$\left(r^4 - \dot{r}^2\right)^3 = \frac{r^{12}}{(1+r^4)^2} \left(r^4 - (1+\gamma)\dot{r}^2\right)^2 , \tag{113}$$

where $\gamma = \frac{4\Lambda}{L^2}$. Setting $\dot{r}^2 = 0$ in this equation, we see that $\dot{r}^2$ vanishes only at $r = 0$. Thus, for every r the solutions have the same sign. Then, by examining its sign for a specific value of r we can fix its sign everywhere. The simplest choice is $r = 1$. In this case, the above equation becomes

$$\left(1 - \dot{r}_1^2\right)^3 = \frac{1}{4} \left(1 - (1+\gamma)\dot{r}_1^2\right)^2 , \tag{114}$$

where $\dot{r}_1$ is the solution of $\dot{r}$ for this value of r. The solution is plotted in the following graph



From this plot we see that for each value of γ, $\dot{r}^2$ has a positive solution at $r = 1$. Since equation (113) is a third order polynomial, there is always a real solution. Therefore, there is always a positive solution for each r and γ. Next, we can easily find the asymptotics of (113), observing that the right hand side of the equation is always positive. For $r \to 0$, $\dot{r}^2$ has to start as r^n, where $n \geq 4$. Substituting the following expansion

$$\dot{r}^2 = Ar^4 + \sum_{i=1} a_i r^{n_i}, \tag{115}$$

with $n_{i+1} > n_i > 4$ into (113), for $\gamma \neq 0$ we obtain

$$\dot{r}^2 = r^4 - \gamma^{\frac{2}{3}} r^{\frac{20}{3}} + \frac{2}{3}\gamma^{\frac{1}{3}}(1+\gamma) r^{\frac{28}{3}} + \frac{2}{3}\gamma^{\frac{2}{3}} r^{\frac{32}{3}} - \frac{1}{3}(1+\gamma)^2 r^{12} + \mathcal{O}(r^{\frac{40}{3}}), \quad r \to 0. \tag{116}$$

While for $\gamma = 0$ there is an exact solution of (113)

$$\dot{r}^2 = r^4 - \frac{r^{12}}{(1+r^4)^2} = r^4 - r^{12} + 2r^{16} + \mathcal{O}(r^{20}), \quad r \to 0. \tag{117}$$

The solution with γ equal to zero corresponds to the case with only the Namdu-Goto term. Following the same procedure, in the other asymptotic region, i.e. for $r \to \infty$, we find three possible cases (where now we demand $n_i < 4$)

For $\gamma \leq 0$

$$\dot{r}^2 \simeq \frac{2}{1-2\gamma} - \left[\frac{4(4+\gamma^2)}{(1-2\gamma)^3} + \frac{8(1+\gamma)}{(1-2\gamma)^2} + \frac{3}{(1-2\gamma)}\right]\frac{1}{r^4}, \quad r \to \infty, \tag{118}$$

for $0 < \gamma < \frac{1}{2}$

$$\dot{r}^2 \simeq \begin{cases} \frac{2}{1-2\gamma} - \left[\frac{4(4+\gamma^2)}{(1-2\gamma)^3} + \frac{8(1+\gamma)}{(1-2\gamma)^2} + \frac{3}{(1-2\gamma)}\right]\frac{1}{r^4}, & r \to \infty \\ \text{or} \\ Ar^4 + \frac{2(1-A)(1-A(1+\gamma))}{1-2\gamma-A(1+\gamma)}\frac{1}{r^4}, & r \to \infty \end{cases} \tag{119}$$

and for $\gamma \geq \frac{1}{2}$

$$\dot{r}^2 \simeq Ar^4 + \frac{2(1-A)\left(1-A(1+\gamma)\right)}{1-2\gamma-A(1+\gamma)}\frac{1}{r^4}, \quad r \to \infty, \tag{120}$$

where A is the solution of

$$(1-A)^3 = (1-A(1+\gamma))^2. \tag{121}$$

This equation imposes some restrictions on A. First of all, A must be positive in order for $\dot{r}^2$ to be positive too. Moreover, writing this equation as a quadratic polynomial for γ and requiring the discriminant to be positive, we get that A is less than one. Consequently, we have that

$$0 < A \leq 1. \tag{122}$$

Finally, the non-zero solutions for A are

$$A = \frac{1}{2}\left(2 - 2\gamma - \gamma^2 - \gamma^{\frac{3}{2}}\sqrt{4+\gamma}\right),\ A = \frac{1}{2}\left(2 - 2\gamma - \gamma^2 + \gamma^{\frac{3}{2}}\sqrt{4+\gamma}\right). \tag{123}$$

In addition, for $0 < A \leq 1$ we conclude that $\gamma \geq 0$. Note that $\gamma = 0$ corresponds to $A = 1$.

Knowing the possible behaviours of $\dot{r}^2$ in the asymptotic regions, we can compare them with those as if we had only one of the two terms in the action. We already know the asymptotics when we have only the Nambu-Goto term, by setting $\gamma = 0$ in the above expressions. That is

$$\dot{r}^2 \simeq \begin{cases} r^4 - r^{12} + 2r^{16}, & r \to 0 \\ 2 - \frac{3}{r^4}, & r \to \infty \end{cases} \tag{124}$$

In order to find $\dot{r}^2$ when only the Einstein term is present, we write equation (100) in this background

$$\begin{aligned} \left(\frac{|g_{00}|}{g_{rr}} - \dot{r}^2\right)^3 &= \frac{g^3 g_{00}^2 e^{-2\phi}}{g_{rr}(\mathcal{H}+C)^2}\left(\frac{|g_{00}|}{g_{rr}} - \left(1 + \Lambda\frac{g'^2 e^{\phi}}{g^2 g_{rr}}\right)\dot{r}^2\right)^2 \\ &= \frac{g^3 g_{00}^2}{g_{rr}(\mathcal{H}+C)^2}\left(e^{-\varphi}\left(\frac{|g_{00}|}{g_{rr}} - \dot{r}^2\right) - \Lambda\frac{g'^2}{g^2 g_{rr}}\dot{r}^2\right)^2, \end{aligned} \tag{125}$$

(126)

and then taking the limit $e^{-\varphi} \to 0$, $\frac{C}{H} \to 0$. Then, rescaling r and t to $r \to (\mathcal{H}^{1/4}L)r$ and $t \to (\mathcal{H}^{-1/4}L)t$ we arrive to the desired equation

$$\left(r^4 - \dot{r}^2\right)^3 = \gamma^2 r^{12}\dot{r}^4. \tag{127}$$

Following again by the same method, we find that

$$\dot{r}^2 \simeq \begin{cases} r^4 - \gamma^{\frac{2}{3}} r^{\frac{20}{3}} + \frac{2}{3}\gamma^{\frac{4}{3}} r^{\frac{28}{3}} & , \quad r \to 0 \\ \frac{1}{|\gamma|} - \frac{3}{2\gamma^2}\frac{1}{r^4} & , \quad r \to \infty. \end{cases} \tag{128}$$

Note that this equation matches with (118) for $\gamma \to -\infty$. Using these results, we are able to find the effective density in each case for large and small r. The first step is to write the Hubble function in this background. From equations (107) we have that

$$H^2 = \frac{1}{L^2}\frac{\dot{r}^2}{r^4 - \dot{r}^2}. \tag{129}$$

We can now present the asymtpotics of the effective density as a function of the scale factor. Firstly, we will rescale back the r coordinate ($r \to \mathcal{H}^{-\frac{1}{4}}L^{-1}r$) and then we will use the relation between the scale factor and r ($a^2 = g(r) = \left(\frac{r}{L}\right)^2$). In the following table we present the results for the three cases of γ mentioned above.

where

$$B(\gamma) = \frac{2}{1-\gamma} - \left[\frac{4(4+\gamma^2)}{(1-2\gamma)^3} + \frac{8(1+\gamma)}{(1-2\gamma)^2} + \frac{3}{(1-2\gamma)}\right] \tag{130}$$

$$B'(\gamma) = \frac{2\left(1 - A(\gamma)(1+\gamma)\right)\left(2 + A(\gamma)\right)}{\left(1 + A(\gamma)\right)\left(1 - 2\gamma - A(\gamma)(1+\gamma)\right)} \tag{131}$$

$$A(\gamma) = \frac{1}{2}\left(2 - 2\gamma - \gamma^2 \pm \gamma^{\frac{3}{2}}\sqrt{4+\gamma}\right) \tag{132}$$

Therefore, we conclude that the Einstein term dominates for small r. In this regime, $\rho_{eff} \sim a^{-8/3}$ that corresponds to matter with $w = 1/9$. On the other hand, for large r, we have two possible asymptotic behaviours. The first one

$$\frac{8\pi L^2}{3}\rho_{eff} = \frac{2}{1-2\gamma}\frac{\mathcal{H}}{a^4} + B(\gamma)\frac{\mathcal{H}^2}{a^8}, \tag{133}$$

for $\gamma \to \infty$ gives the Einstein contributions, whereas for small γ we get the Nambu-Goto contribution. The other possible asymptotic behaviour

$$\frac{8\pi L^2}{3}\rho_{eff} = \frac{A(\gamma)}{1-A(\gamma)} + B'(\gamma)\frac{\mathcal{H}^2}{a^8}, \tag{134}$$

takes a constant value for large r. At this point a crucial observation must be made. We see that by adding the Einstein term to the model studied in [10], the effective density takes

		Nambu-Goto	Einstein	Both terms
$r \to 0$	$0 < \gamma < \frac{1}{2}$	$\frac{\mathcal{H}^2}{a^8} + \frac{2\mathcal{H}}{a^4}$	$\frac{\mathcal{H}^{\frac{2}{3}}}{\gamma^{2/3}a^{8/3}} - \frac{1}{3}$	$\frac{\mathcal{H}^{\frac{2}{3}}}{\gamma^{2/3}a^{8/3}} - \frac{1}{3}$
$r \to \infty$	$\gamma \leq 0$	$\frac{2\mathcal{H}}{a^4} - \frac{\mathcal{H}^2}{a^8}$	$\frac{1}{\lvert\gamma\rvert}\frac{\mathcal{H}}{a^4} - \frac{2\mathcal{H}^2}{3\gamma^2 a^8}$	$\frac{2}{1-2\gamma}\frac{\mathcal{H}}{a^4} + B(\gamma)\frac{\mathcal{H}^2}{a^8}$
	$0 < \gamma < \frac{1}{2}$	$\frac{2\mathcal{H}}{a^4} - \frac{\mathcal{H}^2}{a^8}$	$\frac{1}{\lvert\gamma\rvert}\frac{\mathcal{H}}{a^4} - \frac{2\mathcal{H}^2}{3\gamma^2 a^8}$	$\frac{2}{1-2\gamma}\frac{\mathcal{H}}{a^4} + B(\gamma)\frac{\mathcal{H}^2}{a^8}$ or $\frac{A(\gamma)}{1-A(\gamma)} + B'(\gamma)\frac{\mathcal{H}^2}{a^8}$
	$\gamma \geq \frac{1}{2}$	$\frac{2\mathcal{H}}{a^4} - \frac{\mathcal{H}^2}{a^8}$	$\frac{1}{\lvert\gamma\rvert}\frac{\mathcal{H}}{a^4} - \frac{2\mathcal{H}^2}{3\gamma^2 a^8}$	$\frac{A(\gamma)}{1-A(\gamma)} + B'(\gamma)\frac{\mathcal{H}^2}{a^8}$

Table 1: The asyptotics of the effective density $\frac{8\pi L^2}{3}\rho_{eff}$, in the asymptotic regions

a constant positive value for large r and $\gamma > 0$. This behaviour fits the solution predicted by General Relativity in the regime of vacuum dominance ($w = -1$). Also note, that a similar solution is not present even in the situation with only the Einstein term. In this sense, the Nambu-Goto term can be interpreted as the origin of dark energy. Next, we plot three examples for the three domains of γ.

Observe that in the first diagram ($\gamma = \frac{1}{3}$) there are three possible asymptotic behaviours. The two of them come from the solution (133) that goes to zero, and the other two from (134), one for every $A_\pm(\gamma)$.

In the above analysis we studied the motion of a D3-brane along geodesics that initially have zero angular momentum and non zero energy. Also, we can study the case with vanishing initial energy or with non-zero angular momentum. Writing equation (104) in the AdS_5 background, without setting $\ell = 0$ we get

$$\left(\left(\frac{r}{L}\right)^4-\frac{\ell^2}{L^2}\left(\frac{r}{L}\right)^6\frac{1}{\left(\mathcal{H}-\left(\frac{r}{L}\right)^4\right)^2}-\dot{r}^2\right)^3=\left(\frac{r}{L}\right)^{12}\frac{1}{\left(\mathcal{H}+\left(\frac{r}{L}\right)^4\right)^2}\left(\left(\frac{r}{L}\right)^4-\frac{\ell^2}{L^2}\left(\frac{r}{L}\right)^6\frac{1}{\left(\mathcal{H}-\left(\frac{r}{L}\right)^4\right)^2}-(1+\gamma)\dot{r}^2\right)^3. \tag{135}$$

We see that in both in the two asymptotic regions the new term disappears. Consequently, we conclude that the presence of non-zero angular momentum does not affect the asymptotic behaviour of the solution. Finally, in the following discussion we present the cases with zero energy.

• In the case with $\ell = 0 = \mathcal{H}$, equation (113) becomes

$$\left(r^4 - \dot{r}^2\right)^3 = r^4 \left(r^4 - (1+\gamma)\dot{r}^2\right)^2, \tag{136}$$

having the following exact solution

$$\dot{r} = Ar^4, \tag{137}$$

with $A = 0, \frac{1}{2}\left(2 - 2\gamma - \gamma^2 \pm \gamma^{\frac{3}{2}}\sqrt{4+\gamma}\right)$. As before, this behaviour gives the following result for the effective density

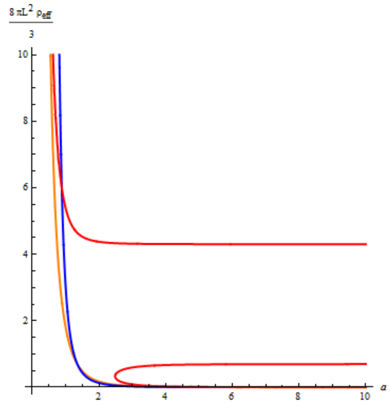
$$\frac{8\pi}{3}\rho_{eff} = \frac{A}{1-A}. \tag{138}$$

This result is not interesting, since it gives constant effective density everywhere.

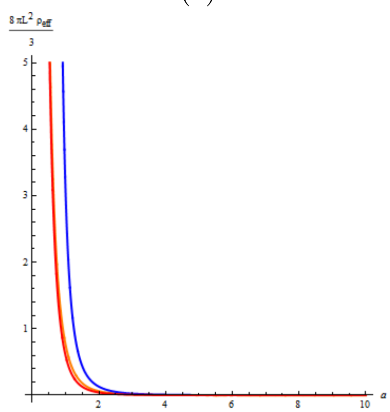
• In the case with $\ell \neq 0, \mathcal{H} = 0$, equation (135) becomes

$$\left(\left(\frac{r}{L}\right)^4 - \frac{\ell^2}{L^2}\left(\frac{L}{r}\right)^2 - \dot{r}^2\right)^3 = \left(\frac{r}{L}\right)^4\left(\left(\frac{r}{L}\right)^4 - \frac{\ell^2}{L^2}\left(\frac{L}{r}\right)^2 - (1+\gamma)\dot{r}^2\right)^3. \tag{139}$$

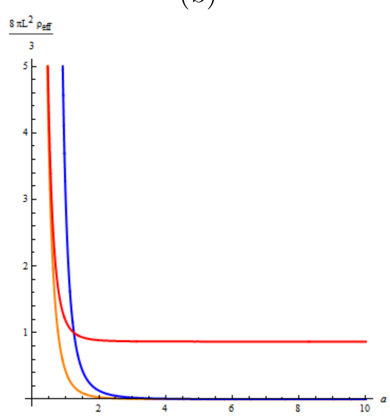
This equation does not have a positive asymptotic at $r \to 0$. Therefore, it does not give a solution that is valid for every r. So this case is of no in particular importance. To summarize,



(a)



(b)



(c)

Figure 1: Effective density as a function of the scale factor for (a) $\gamma = \frac{1}{3}$, (b) $\gamma = -1$ and (c) $\gamma = 2$. The blue line in the solution with only the Einstein term, the orange one is with only the Nambu-Goto term, and the red one with both terms.

we computed the asymptotic behaviour of the effective matter density in the case with $\ell = 0$ and $H \neq 0$. The results are exhibited in Table (1). We saw that the Hamiltonian affects only the scale of r, and that the behaviour of ρ_{eff} is controlled just by γ. Moreover, we concluded that the presence of a non-zero angular momentum would not affect the asymptotics of ρ_{eff}. Finally, in the case with zero Hamiltonian we proved that for $r \to 0$ we cannot obtain a positive solution for $\dot{r}^2$ and therefore we do not have a solution for every r.

2.4 Generalization to a class of background metrics

In this section, we will generalize the results for the effective density for a wide class of background metrics. More specifically, we will study the case where the ten-dimensional metric is such, that in the asymptotic regions the auxiliary functions (101,102,103) have a power law behaviour and are positive. In addition, we will concentrate on motion with zero angular momentum. With these restrictions, we can find the asymtpotics for $\dot{r}$ and consequently the asymtpotics of ρ_{eff}. In other words, we will compute the effective density in the case where $r \to 0$

$$\begin{aligned} z(r) &= A_z r^{n_z} + \ldots,\ A_z > 0 \\ h(r) &= A_h r^{n_h} + A'_h r^{n'_h} + \ldots,\ A_h > 0,\ n'_h > n_h \\ e(r) &= 1 + A_e r^{n_e} + \ldots \end{aligned} \tag{140}$$

and the brane is moving with zero angular momentum. Respectively, for $r \to \infty$ we will consider that

$$\begin{aligned} z(r) &= D_z r^{k_z} + \ldots,\ D'_z > 0 \\ h(r) &= D_h r^{k_h} + D'_h r^{k'_h} \ldots,\ D_h > 0,\ k'_h < k_h \\ e(r) &= 1 + D_e r^{k_e} + \ldots \end{aligned} \tag{141}$$

In the first asymptotic region ($r \to 0$) the equation (100) becomes

$$\left(A_z r^{n_z} - \dot{r}^2 + \ldots\right)^3 = \left(A_h r^{n_h} + A'_h r^{n'_h} + \ldots\right)\left(A_z r^{n_z} - (1 + A_e r^{n_e})\dot{r}^2 + \ldots\right)^2 \tag{142}$$

As in the case of the AdS space, $\dot{r}^2$ necessarily must have an expansion of the following form (since the right hand side of the equation is always positive)

$$\dot{r}^2 = B r^{n_z} + \sum_i B_i r^{m_i},\ m_i > n_z,\ B \leq A_z. \tag{143}$$

In the following analysis, we will compute only the leading term of the effective density. The computation of the subleading terms is straightforward. Inserting this expansion in (142) we get

$$\{(A_z-B)r^{n_z}-B_1r^{m_1}+\ldots\}^3=(A_hr^{n_h}+A'_hr^{n'_h}+\ldots)\{(A_z-B)r^{n_z}-B_1r^{m_1}-A_eBr^{n_z+n_e}+\ldots\}^2 \tag{144}$$

We distinguish the following situations

- $B\neq A_z$, $B\neq 0$:
  - $n_e>0$: In this situation, equating the leading terms of the two sides in (144) gives

    $$(A_z-B)^3r^{3n_z}=A_hr^{n_h}(A_z-B)^2r^{2n_z}$$

    that can be satisfied iff $n_z=n_h$ and $B=A_z-A_h$.
  - $n_e=0$: In this case we get

    $$(A_z-B)^3r^{3n_z}=A_hr^{n_h+2n_z}\{A_z-(1+A_e)B\}^2$$

    that can be satisfied iff $n_z=n_h$ and B given by $(A_z-B)^3=A_h\{A_z-(1+A_e)B\}^2$.
  - $n_e<0$: And in this case

    $$(A_z-B)^3r^{3n_z}=A_hr^{n_h}(A_er^{n_e}Br^{n_z})^2$$

    that can be satisfied iff $n_z=n_h+2n_e$ and B given by $(A_z-B)^3=A_hA_e^2B^2$.

- $B=A_z$: In this case equation (142) takes the following form

  $$(-B_1r^{m_1}+\ldots)^3=(A_hr^{n_h}+\ldots)\left\{B_1r^{m_1}+A_eA_zr^{n_e+n_z}+\ldots\right\}^2 \tag{145}$$

  - $n_e<m_1-n_z$ (>0): In this case the second tern in the curly brackets in the right-hand side dominates and gives $B_1=-\sqrt[3]{A_hA_e^2A_z^2}$ and $m_1=\frac{n_h+2n_e+2n_z}{3}$ that implies the condition $n_z<n_h-n_e$. This result is consistent with the restriction $m_1>n_z$ only when $n_z<n_h+2n_e$. Note that the last inequality is automatically satisfied if $n_e>0$ and $n_z<n_h-n_e$ hold.
  - $n_e=m_1-n_z$: In this case B_1 is given by $-B_1^3=A_h(B_1+A_eA_z)^2$ and $m_1=n_h$. This implies the condition $n_z=n_h-n_e$. Also, consistency with the restriction $m_1>n_z$ gives $n_h>n_z$.
  - $n_e>m_1-n_z$: Finally in this case, we get $B_1=-A_h$ and $m_1=n_h$ that implies $n_z>n_h-n_e$. Again, consistency with the restriction $m_1>n_z$ gives $n_h>n_z$.

- $B=0$: In this case (142) becomes

  $$(A_zr^{n_z}-B_1r^{m_1}+\ldots)^3=(A_hr^{n_h}+A'_hr^{n'_h}+\ldots)\{A_zr^{n_z}-(1+A_er^{n_e})B_1r^{m_1}+\ldots\}^2 \tag{146}$$

- $n_e > 0$: The leading terms of both sides give $n_z = n_h$ and $A_z = A_h$, while the first subleading term gives $m_1 = n'_h$ and $B_1 = -\frac{A'_h A_z}{2A_h - 3A_z} = A'_h$.
- $n_e = 0$: The leading terms of both sides give $n_z = n_h$ and $A_z = A_h$, while the subleading terms give $m_1 = n'_h$ and $B_1 = -\frac{A'_h A_z}{2A_h(1+A_e)-3A_z} = \frac{A'_h}{1-2A_e}$.
- $n_e < 0$: The leading terms of both terms give $n_z = n_h$ and $A_z = A_h$, while the subleading terms give $m_1 = n'_h$ and $B_1 = -\frac{A'_h}{3}$. The restriction $m_1 > n_z$ is automatically satisfied here.

In the following tables, we summarize the asympotics of $\dot{r}^2$. Note that in the case where $n_e = 0$, for $n_z = n_h$, it seems that we have four solutions. Nevertheless, the second solution is valid only when $A_z = A_h$. In this case the equation

$$(A_z - B)^3 = A_h\{A_z - (1 + A_e)B\}^2 \tag{147}$$

has a zero solution. Thus, we still have three asymptotics for $\dot{r}^2$, as it should be, since (142) is a cubic equation. Moreover, we can use these results to compute the asymptotics of the effective density. We will compute it for $\phi = 0$. In this case, from (108), the effective density for $r \to 0$ is written as

$$\frac{8\pi}{3}\rho_{eff} = \frac{1}{4\Lambda}\frac{\dot{r}^2}{A_z r^{n_z} - \dot{r}^2} A_e r^{n_e}. \tag{148}$$

and its asympotics are summarized in Tables (4) and (5).

Finally we can write the effective density as a function of the scale factor, inverting the relation $g(r) = a^2$. For the sake of simplicity, we do not present the results. Given a background metric, one can find the expansion of the auxiliary functions in the asymptotic regions, then check to which situation this metric fits. Finally, inverting $g(r) = a^2$ one can express the effective density in terms of a. From this analysis, we conclude that we can have all kinds of asymptotic behaviour for ρ_{eff}, depending on the details of the background metric.

	$n_z > n_h$	-
	$n_z = n_h$	$\dot{r}^2 \simeq (A_z - A_h) r^{n_z}$ **or** $\dot{r}^2 \simeq A'_h r^{n'_h}$, $A_z = A_h$
$n_e > 0$	$n_h > n_z > n_h - n_e$	$\dot{r}^2 \simeq A_z r^{n_z} - A_h r^{n_h}$
	$n_z = n_h - n_e$	$\dot{r}^2 \simeq A_z r^{n_z} + B_1 r^{n_h}$, $-B_1^3 = A_h (B_1 + A_e A_z)^2$
	$n_z < n_h - n_e$	$\dot{r}^2 \simeq A_z r^{n_z} - \sqrt[3]{A_h A_e^2 A_z^2}\; r^{\frac{n_h + 2n_e + 2n_z}{3}}$
	$n_z > n_h$	-
$n_e = 0$	$n_z = n_h$	$\dot{r}^2 \simeq B r^{n_z}$, $(A_z - B)^3 = A_h \{A_z - (1 + A_e) B\}^2$ **or** $\dot{r}^2 \simeq \frac{A'_h}{1 - 2A_e} r^{n'_h}$, $A_z = A_h$
	$n_z < n_h$	$\dot{r}^2 \simeq A_z r^{n_z} - \sqrt[3]{A_h A_e^2 A_z^2}\; r^{\frac{n_h + 2n_z}{3}}$
	$n_z > n_h$	-
	$n_z = n_h$	$\dot{r}^2 \simeq A_z r^{n_z} - \frac{A'_h}{3} r^{n_h}$, $A_z = A_h$
$n_e < 0$	$n_h > n_z > n_h + 2n_e$	-
	$n_z = n_h + 2n_e$	$\dot{r}^2 \simeq B r^{n_z}$, $(A_z - B)^3 = A_h (A_e B)^2$
	$n_z < n_h + 2n_e$	$\dot{r}^2 \simeq A_z r^{n_z} - \sqrt[3]{A_h A_e^2 A_z^2}\; r^{\frac{n_h + 2n_e + 2n_z}{3}}$

Table 2: The asymptotics of $\dot{r}^2$ for $r \to 0$

	$k_z < k_h$	-
	$k_z = k_h$	$\dot{r}^2 \simeq (D_z - D_h) r^{k_z}$ **or** $\dot{r}^2 \simeq D'_h r^{k'_h}$, $D_z = D_h$
$k_e < 0$	$k_h < k_z < k_h - k_e$	$\dot{r}^2 \simeq D_z r^{k_z} - D_h r^{k_h}$
	$k_z = k_h - k_e$	$\dot{r}^2 \simeq D_z r^{k_z} + B_1 r^{k_h}$, $-B_1^3 = D_h (B_1 + D_e D_z)^2$
	$k_z > k_h - k_e$	$\dot{r}^2 \simeq D_z r^{k_z} - \sqrt[3]{D_h D_e^2 D_z^2}\; r^{\frac{k_h + 2k_e + 2k_z}{3}}$
	$k_z < k_h$	-
$k_e = 0$	$k_z = k_h$	$\dot{r}^2 \simeq B r^{k_z}$, $(D_z - B)^3 = D_h \{D_z - (1 + D_e) B\}^2$ **or** $\dot{r}^2 \simeq \frac{D'_h}{1 - 2D_e} r^{k'_h}$, $D_z = D_h$
	$k_z > k_h$	$\dot{r}^2 \simeq D_z r^{k_z} - \sqrt[3]{D_h D_e^2 D_z^2}\; r^{\frac{k_h + 2k_z}{3}}$
	$k_z < k_h$	-
	$k_z = k_h$	$\dot{r}^2 \simeq D_z r_{k_z} - \frac{D'_h}{3} r^{k_h}$, $D_z = D_h$
$k_e > 0$	$k_h < k_z < k_h + 2k_e$	-
	$k_z = k_h + 2k_e$	$\dot{r}^2 \simeq B r^{k_z}$, $(D_z - B)^3 = D_h (D_e B)^2$
	$k_z > k_h + 2k_e$	$\dot{r}^2 \simeq D_z r^{k_z} - \sqrt[3]{D_h D_e^2 D_z^2}\; r^{\frac{k_h + 2k_e + 2k_z}{3}}$

Table 3: The asymptotics of $\dot{r}^2$ for $r \to \infty$

$n_e > 0$	$n_z > n_h$	-
	$n_z = n_h$	$\frac{1}{4\Lambda}\frac{(A_z-A_h)A_e}{A_h}r^{n_e}$ **or** $\frac{1}{4\Lambda}\frac{A'_h A_e}{A_z}r^{n'_h+n_e-n_h}$, $A_z{=}A_h$
	$n_h > n_z > n_h - n_e$	$\frac{1}{4\Lambda}\frac{A_z A_e}{A_h}r^{n_z+n_e-n_h}$
	$n_z = n_h - n_e$	$-\frac{1}{4\Lambda}\frac{A_z A_e}{B_1}r^{n_z+n_e-n_h}$, $-B_1^3{=}A_h(B_1{+}A_e A_z)^2$
	$n_z < n_h - n_e$	$\frac{1}{4\Lambda}\frac{A_z A_e}{\sqrt[3]{A_h A_e^2 A_z^2}}r^{\frac{n_z+n_e-n_h}{3}}$
$n_e = 0$	$n_z > n_h$	-
	$n_z = n_h$	$\frac{1}{4\Lambda}\frac{BA_e}{A_z-B}$, $(A_z{-}B)^3{=}A_h\{A_z{-}(1{+}A_e)B\}^2$ **or** $\frac{1}{4\Lambda}\frac{A'_h}{A_z(1-2A_e)}r^{n_h-n'_h}$, $A_z{=}A_h$
	$n_z < n_h$	$\frac{1}{4\Lambda}\frac{A_z A_e}{\sqrt[3]{A_h A_e^2 A_z^2}}r^{\frac{n_z-n_h}{3}}$
$n_e < 0$	$n_z > n_h$	-
	$n_z = n_h$	$\frac{1}{4\Lambda}\frac{3A_z A_e}{A'_h}r^{n_z+n_e-n_h}$
	$n_h > n_z > n_h + 2n_e$	-
	$n_z = n_h + 2n_e$	$\frac{1}{4\Lambda}\frac{BA_e}{A_z-B}r^{n_e}$, $(A_z{-}B)^3{=}A_h(A_e B)^2$
	$n_z < n_h + 2n_e$	$\frac{1}{4\Lambda}\frac{A_z A_e}{\sqrt[3]{A_h A_e^2 A_z^2}}r^{\frac{n_z+n_e-n_h}{3}}$

Table 4: The asymptotics of ρ_{eff} for $r \to 0$

$k_e < 0$	$k_z < k_h$	-
	$k_z = k_h$	$\frac{1}{4\Lambda}\frac{(D_z-D_h)D_e}{D_h}r^{k_e}$ **or** $\frac{1}{4\Lambda}\frac{D'_h D_e}{D_z}r^{k'_h+k_e-k_h}$, $D_z{=}D_h$
	$k_h < k_z < k_h - k_e$	$\frac{1}{4\Lambda}\frac{D_z D_e}{D_h}r^{k_z+n_e-k_h}$
	$k_z = k_h - k_e$	$-\frac{1}{4\Lambda}\frac{D_z D_e}{B_1}r^{k_z+k_e-k_h}$, $-B_1^3{=}D_h(B_1{+}D_e D_z)^2$
	$k_z > k_h - k_e$	$\frac{1}{4\Lambda}\frac{D_z D_e}{\sqrt[3]{D_h D_e^2 D_z^2}}r^{\frac{k_z+k_e-k_h}{3}}$
$k_e = 0$	$k_z < k_h$	-
	$k_z = k_h$	$\frac{1}{4\Lambda}\frac{BD_e}{D_z-B}r^{k_e}$, $(D_z{-}B)^3{=}D_h\{D_z{-}(1{+}D_e)B\}^2$ **or** $\frac{1}{4\Lambda}\frac{D'_h}{D_z(1-2D_e)}r^{k'_h-k_h}$, $D_z{=}D_h$
	$k_z > k_h$	$\frac{1}{4\Lambda}\frac{D_z D_e}{\sqrt[3]{D_h D_e^2 D_z^2}}r^{\frac{k_z-k_h}{3}}$
$k_e > 0$	$k_z < k_h$	-
	$k_z = k_h$	$\frac{1}{4\Lambda}\frac{3D_z D_e}{D'_h}r^{k_z+k_e-k_h}$
	$k_h < k_z < k_h + 2k_e$	-
	$k_z = k_h + 2k_e$	$\frac{1}{4\Lambda}\frac{BD_e}{D_z-B}r^{k_e}$, $(D_z{-}B)^3{=}D_h(D_e B)^2$
	$k_z > k_h + 2k_e$	$\frac{1}{4\Lambda}\frac{D_z D_e}{\sqrt[3]{D_h D_e^2 D_z^2}}r^{\frac{k_z+k_e-k_h}{3}}$

Table 5: The asymptotics of ρ_{eff} for $r \to \infty$

3 Conclusions

In the first part of this paper, we studied the static and spherically symmetric solutions of the low-energy Horava-Lifshitz gravity. We shortly discussed the motivation of the theory, that is the construction of a power counting renormalizable theory of quantum gravity with anisotropic scaling. At high energies, the dispersion relation of gravitons is modified to $\omega \sim \vec{k}^6$. This modification improves the UV behaviour of the theory, rendering the theory to be power counting renormalizable. On the other hand, at low energies (tuning the parameters) the action is reduced to that of General Relativity with a cosmological constant plus a new term which contains derivatives of the lapse function.

Then, we focused on the low energy effective action, and we studied its static and spherically symmetric solutions with zero shift vector. We showed that in the absence of the cosmological constant term, the solution has the same asymptotic behaviour with the Schwarzschild solution at large distances plus corrections of lower order. However, in the presence of a cosmological constant term, the solutions do not have the same asymptotic behaviour like the de Sitter-Schwarzschild solution. Instead, the lapse function acquires an exponential behaviour.

In the second part, we studied a scenario of mirage cosmology. In this model, our three-dimensional universe is treated as a three-dimensional hypersurface (a brane), moving in a higher dimensional background space. We showed that the residents of the brane observe an effective cosmological evolution even when the background space is static. Initially, we studied this cosmological evolution while the motion of the brane was governed by the Nambu-Goto action, originally done in [10]. Applying the results in the case of the AdS_5 black hole background, we found that far away from the black hole $\rho_{eff} \sim a^{-4}$, and the effective cosmological expansion is indistinguishable from that one produced by radiation on the brane. On the other hand, near the center of the black hole $\rho_{eff} \sim a^{-8}$, and the effective matter density corresponds to dilute matter with $w = 5/3$.

Also, we studied the addition of the Ricci scalar of the brane in the action, and we recomputed the effective matter density. In the case of AdS_5 background, we found that $\rho_{eff} \sim a^{-8/3}$. This behaviour in the context of General Relativity, corresponds to exotic matter, with $w = 1/9$. Nevertheless, far away from the center of the black hole, the effective matter density reaches a constant value. This result is very interesting because $\rho_{eff} = const$ resembles the case of vacuum dominance in General Relativity. As we know, the experimental observations have proved that in this period our universe is in this phase. However, there is not known matter, having equation of state with $w = -1$, that is the notorious problem of dark energy. Therefore, we conclude that a possible explanation of this problem can be given by this model. In other words, we can explain that the matter density takes a constant value effectively, due to the motion of our three-dimensional universe in a higher AdS_5 black hole background.

Finally, in the last section we generalized the results for the effective matter density in the case of a general class of background metrics. We proved that ρ_{eff} can have all possible power law behaviours, and can also reach a constant value either for small or large r, depending on the details of the background metric.

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APPENDIX

A Conservation of $h^{ij}p_ip_j$

Consider a Lagrangian of the form

$$L = L(r, \dot{r}, x), \quad x = h_{ij}\dot{\varphi}^i\dot{\varphi}^j \tag{149}$$

Then, the quantity $h^{ij}p_ip_j$ is conserved, where p_i is the canonical momentum with respect to φ^i given by

$$p_i = \frac{\partial L}{\partial \dot{\varphi}^i} = \frac{\partial L}{\partial x} 2h^{ij}\dot{\varphi}^j. \tag{150}$$

Taking the time derivative of this quantity we have

$$\frac{d}{dt}(h^{ij}p_ip_j) = \frac{\partial h^{ij}}{D\varphi^k}\dot{\varphi}^k p_ip_j + 2h^{ij}p_i\dot{p}_j \tag{151}$$

$$= \frac{\partial h^{ij}}{\partial \varphi^k}\dot{\varphi}^k p_ip_j + 2h^{ij}p_i\frac{\partial L}{\partial \varphi^j}. \tag{152}$$

Additionally, the first term is equal to

$$\frac{\partial h^{ij}}{\partial\varphi^k}\dot\varphi^k p_i p_j = \frac{\partial h^{ij}}{\partial\varphi^k}\dot\varphi^k\left(2\frac{\partial L}{\partial x}\right)^2 h_{i\lambda}\dot\varphi^\lambda h_{j\alpha}\dot\varphi^\alpha \tag{153}$$

$$= -\left(2\frac{\partial L}{\partial x}\right)^2 \frac{\partial h_{mn}}{\partial\varphi^k} h^{mi}h^{nj}h_{i\lambda}h_{j\alpha}\dot\varphi^k\dot\varphi^\lambda\dot\varphi^\alpha \tag{154}$$

$$= -\left(2\frac{\partial L}{\partial x}\right)^2 \frac{\partial h_{mn}}{\partial\varphi^k}\delta^m_\lambda\delta^n_\alpha\dot\varphi^k\dot\varphi^\lambda\dot\varphi^\alpha \tag{155}$$

$$= -\left(2\frac{\partial L}{\partial x}\right)^2 \frac{\partial h_{mn}}{\partial\varphi^k}\dot\varphi^k\dot\varphi^m\dot\varphi^n, \tag{156}$$

and the second term

$$2h^{ij}p_i\frac{\partial L}{\partial\varphi^j} = 2h^{ij}p_i\frac{\partial L}{\partial x}\frac{\partial x}{\partial\varphi^j} \tag{157}$$

$$= 2h^{ij}2\frac{\partial L}{\partial x}h_{ik}\dot\varphi^k\frac{\partial L}{\partial x}\frac{\partial h_{mn}}{\partial\varphi^j}\dot\varphi^m\dot\varphi^n \tag{158}$$

$$= \left(2\frac{\partial L}{\partial x}\right)^2 h^{ij}h_{ik}\dot\varphi^k\dot\varphi^m\dot\varphi^n\frac{\partial h_{mn}}{\partial\varphi^j} \tag{159}$$

$$= \left(2\frac{\partial L}{\partial x}\right)^2 \dot\varphi^j\dot\varphi^m\dot\varphi^n\frac{\partial h_{mn}}{\partial\varphi^j}. \tag{160}$$

So the two terms cancel and the total angular momentum is conserved.

B The real solutions of the cubic equation

Consider that we have an equation of the form

$$\left(z(r)-\dot r^2\right)^3 = h(r)\left(z(r)-e(r)\dot r^2\right)^2. \tag{161}$$

Using Cardano's method for solving cubic equations, what we get is

$$\dot r^2 = \frac{1}{6}\left(-2e^2h+6z-2^{2/3}(u+v)\right), \tag{162}$$

where

$$u^3 = \xi_+, \tag{163}$$

$$v^3 = \xi_-, \tag{164}$$

satisfying the constraint $uv\in\mathbb{R}$. Also, $\xi_\pm$ is given by

$$\xi_\pm = h\left(2e^6h^2-18(e-1)e^3hz+27(e-1)^2z^2\right)\pm 3\sqrt{3}\sqrt{(1-e)^3h^2z^3\left(4e^3h-27(e-1)z\right)}. \tag{165}$$

We have to analyze two distinct situations. The first is when the discriminant is non-negative

$$\Delta = (1-e)^3 h^2 z^3 \left(4e^3 h - 27(e-1)z\right) \geq 0. \tag{166}$$

In this case, there is a single real solution (there are also two complex conjugates coming from the cubic roots in (163) and (164)) that are written as

$$\begin{aligned}
\dot{r}^2 &= \frac{1}{6}\left(-2e^2 h + 6z - 2^{2/3}(\xi_+^{1/3} + \xi_-^{1/3})\right) \\
&= \frac{1}{6}\left(-2e^2 h + 6z - 2^{2/3}\xi_+^{1/3} - 2^{2/3}\frac{(\xi_-\xi_+)^{1/3}}{\xi_+^{1/3}}\right) \\
&= \frac{1}{6}\left(-2e^2 h + 6z - 2^{2/3}\xi_+^{1/3} - \frac{2^{4/3} eh(e^3 h + 6z - 6ez)}{\xi_+^{1/3}}\right).
\end{aligned} \tag{167}$$

In the opposite case, i.e. when the discriminant is negative, the $\xi_\pm$ are complex numbers and according to De Moivre's theorem

$$u = |\xi_+|^{\frac{1}{3}} e^{i\frac{\theta + 2n\pi}{3}}, \tag{168}$$

$$v = |\xi_+|^{\frac{1}{3}} e^{i\frac{-\theta + 2m\pi}{3}}, \tag{169}$$

where $n, m = 0, 1, 2$. In this way, satisfying the constraint $uv \in \mathbb{R}$

$$u + v = 2|\xi_+|^{\frac{1}{3}} \cos\left(\frac{-\theta + 2m\pi}{3}\right), \tag{170}$$

and the solution for $\dot{r}^2$ is

$$\dot{r}^2 = \frac{1}{6}\left(-2e^2 h + 6z - 2^{2/3} 2|\xi_+|^{\frac{1}{3}} \cos\left(\frac{\theta + 2m\pi}{3}\right)\right), \tag{171}$$

where for each value of m we have one real solution. These are the general solutions of (104) in the two cases. When the discriminant (166) is non-negative the solution is (167), while when the discriminant is negative the solutions are given by (171).

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