University of Crete
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An Information – theoretic view of Game theory

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Περίληψη

Η Θεωρία Παιγνίων είναι ο τομέας των εφαρμοσμένων μαθηματικών ο οποίος ασχολείται με την στρατηγική σκέψη σε καταστάσεις συγκρούσεων και συνεργασίας. Μέχρι τώρα η Θεωρία Παιγνίων είχε να κάνει με θεωρίες ισορροπιών, όπως την ισορροπία του Nash και διάφορες παραλλαγές τους. Όμως, αυτές οι ισορροπίες, γνωστές σε όλους τους παίκτες, είναι σαν μια υπόδειξη της Θεωρίας Παιγνίων στους παίκτες για το τι θα πρέπει να παίξουν σε ένα παίγνιο. Στην πραγματικότητα όμως όλες αυτές οι θεωρίες δεν μπορούν να πουν στους παίκτες τι θα πρέπει να περιμένουν σε ένα παίγνιο σε πραγματικό περιβάλλον. Έτσι, το 2008 ο Robert Aumann και ο Jacques Dreze εισήγαγαν την έννοια των Rational Expectations στα παίγνια κατά την οποία οι παίκτες παίζουν το παίγνιο κάτω από μία κατάσταση παιγνίου (πραγματικό περιβάλλον) – πεποιθήσεων για τις πράξεις των άλλων (διατυπωμένων με πιθανοθεωρητικούς όρους). Κάθε παίκτης έχει την δική του πληροφορία (εκτίμηση πιθανοτήτων) και προσπαθεί να βελτιστοποιήσει το αναμενόμενο κέρδος του. Αυτό μας οδήγησε να δούμε αυτές τις καταστάσεις παιγνίων από την σκοπιά της Θεωρίας Πληροφοριών και να προσπαθήσουμε να βρούμε τι μπορούν να περιμένουν οι παίκτες σε διαφορετικά περιβάλλοντα πληροφορίας.

Abstract

Game theory is the branch of applied mathematics which deals with the strategic thinking in conflict and cooperation situations. So far, Game theory deals with equilibrium theories such as Nash Equilibrium and different refinements of them. However, these equilibriums, which are known to every player, are like a recommendation of the Game theory to the players what they should play in a game. Actually, none of these theories can tell to the players what they can expect in a game with a real life context. So, in 2008 Robert Aumann and Jacques Dreze introduced the concept of Rational Expectations in Games in which the players play a game under a game situation (the real life context) – beliefs about the actions of the others. Every player has his information and tries to optimize his expectation. This led us to see these game situations from an Information – theoretic point of view and try to find what the players will expect in different Information environments.

Ευχαριστίες

Πρώτα από όλα θα ήθελα να ευχαριστήσω τον καθηγητή μου Απόστολο Τραγανίτη για την υποστήριξη και την πίστη του σε μένα στα χρόνια των σπουδών μου, αισθάνομαι τυχερός που γνώρισα αυτόν τον άνθρωπο και επιστήμονα.

Στην συνέχεια θα ήθελα να ευχαριστήσω τους γονείς μου Γιάννη και Ευαγγελία και την αδερφή μου Στέλλα για την στήριξή τους σε όλα τα βήματα της ζωής μου, χωρίς αυτούς δε θα βρισκόμουν στο σημείο να γράφω αυτό το κείμενο.

Τέλος θα ήθελα να δώσω ένα μεγάλο ευχαριστώ στην Λίνα για τα χρόνια που είναι συνεχώς δίπλα μου.

Στους γονείς μου

Ευαγγελία, Γιάννη

Contents

Introduction

The Game theory and the Shannon Information theory are two important scientific disciplines with a deep impact in many scientific, financial and social activities of mankind in the last 60 years.

The scope of the Game theory is to analyse situations of conflict and cooperation in which many players with different preferences try to optimize their payoffs. Thus, the contributions of game theory are in many scientific fields such as in economics, in psychology, in philosophy, in engineering, in computer science, in political science, in management, in biology etc. On the other hand, the main contributions of Information theory are in communication networks, in signal processing, in cryptography in applications such as coding, data compression etc., giving us for instance the maximum capacity of communication or the best rate of compression.

During the years, there were many contributors to the modern Game theory development. The first was John von Neumann who published the minimax theorem in [Vo28] in his attempt to analyse the real life bluffs. According to this theorem, in every two – person zero – sum game (see text for definition) there is a mixed strategy for every player which can guarantee a specific minimum payoff.

In [Na51], John F. Nash gave the equilibrium concept in every finite game, a stable point from which none has any incentive to deviate, known as the Nash Equilibrium. For this work, Nash was awarded with the Nobel Prize in economics in 1994.

In [Au74], Robert Aumann introduced another solution concept of the games, the Correlated Equilibrium, which is a generalization of the Nash Equilibrium and describes an equilibrium over a probability distribution.

All these theories deal with equilibriums, known to every player, resembling recommendations of the Game theory about what they should play in a game. However, none of these theories can tell to the players what they can expect in a game in a real life situation. So, in [Au08], Robert Aumann and Jacques Dreze presented the rational expectations approach of the games in which the players have beliefs about the behavior of the other players and try to optimize their choices based on these beliefs (estimation of the probabilities of the actions of the other players). These beliefs and the game constitute a game situation in which every player has his information (expressed by these probabilities) and tries to optimize his expectation. However, the players must learn their beliefs (estimate their probabilities), so they must read some form of information about them. It seems that the game situation inserts information on the games, so this led us to search the role of the Information Theory in the rational expectations of the players, that is to look the Game Theory from an Information Theoretic point of view.

Outlining the thesis, in the second chapter we give an introduction of the Game theory giving the basic topics of this theory. In the third chapter, we describe the Rational Expectations approach of Robert Aumann and Jacques Dreze. In the fourth chapter we give some elementary topics in Shannon Information theory and in the fifth chapter give an Information theoretic aspect of Game theory, giving our basic results four theorems in which we determine the rational expectations of the players in different Information environments. Finally, we give some conclusions for our thesis.

The Mathematical Theory of Games

The Game theory is the mathematical approach to describe and analyse a conflict or a cooperation situation – a game. In this way, we can give answers about the best choice in situations in which rational players choose their strategies to optimize their preferences taking into account the strategies of the other players. For example, questions like "What is the best strategy in a poker game?", "How can one optimally bid in an auction?", "How must one drive on a crossroad?", "How can one allocate the bandwidth fairly?" are just a small sample of the questions which the Game theory can answer.

Strategic form of a game

The strategic form of a game is the formal description of a static game in which the players act simultaneously without any knowledge of the strategies that are being chosen by the other players (one – shot game).

The elements which describe a strategic game are the players, the strategies and the utilities. Formally, we define a strategic game as the triplet $\langle N, S_i, u_i \rangle$, where

I. *N is the set of the players* $N = \{1,..,n\}$

II. S_i *is the set of player* i *strategies*

III. $u_i : S \to R$ *is the utility function of player* i, *where* $S = \prod_{i=1}^{N} S_i$ *is the total strategy profile*

Also, the total strategy profile of all players other than player i is $S_{-i} = \prod_{\substack{j=1 \\ j \neq i}}^{N} S_j$.

Every rational player tries to optimize his utility function, so we say that a player prefers the strategy s_i more than the strategy $s_i^{'}$ when $u_i\left(s_i, s_{-i}\right) \geq u_i\left(s_i^{'}, s_{-i}\right)$ for $\forall s_{-i} \in S_{-i}$.

Example

The Chicken Game

In a cross – road there are two drivers $N = \{1, 2\}$, every driver i has two strategies $S_i = \{Stop, Go\}$ and the utilities of the driver i for every strategy profile are $u_i\left(Stop, Stop\right) = 6$, $u_i\left(Stop, Go\right) = 2$, $u_i\left(Go, Stop\right) = 7$ and $u_i\left(Go, Go\right) = 0$. Every driver wants to cross the road but he must consider what the other driver is going to do. On the one hand, if the other player will stop, it is better for the driver to go and on the other hand if the other player will go it is better for the driver to stop. All these decisions are made simultaneously. The matrix below describes this static game and we can see that in every strategy profile the payoffs of the two players are depicted, for example in the $\left(Stop, Go\right)$ profile the 2 is the payoff of the row player and the 7 the payoff of the column player. This game in bibliography is called as “the Chicken Game”.

	Stop	Go
Stop	6,6	2,7
Go	7,2	0,0

The Chicken Game

Zero – sum games

There is a specific class of 2 – player games which are named zero – sum games or constant – sum games. The characteristic of these games is that the sum of the payoffs of the players is zero or constant in every strategy profile, so the profit of the one player is the loss for the other.

Example

The Penalty Game

A classical zero – sum game is the penalty game in which there are two players the penalty kicker and the goal keeper $N = \{1, 2\}$. The penalty kicker is the row player and the goal keeper is the column player. The players play the game simultaneously, the penalty kicker tries to score a goal and the goal keeper tries to save the goal. Every player has two choices – strategies $S_i = \{Left, Right\}$, the kicker can shoot left or right and the goal keeper can move left or right. If the kicker shoots in the same direction with the keeper then the kicker loses 1 point and the keeper wins one point, else the kicker wins 1 and the keeper loses 1.

	Left	Right
Left	-1,1	1,-1
Right	1,-1	-1,1

The Penalty Game

Dominant strategy

A strategy k of a player i is dominant to another strategy k' if $u_i\left(k, s_{-i}\right) \geq u_i\left(k', s_{-i}\right)$ for $\forall s_{-i} \in S_{-i}$. In other words, the strategy k is a better choice than the strategy k' in any case. Furthermore, a strategy k is strictly dominant to another strategy k' if $u_i\left(k, s_{-i}\right) \succ u_i\left(k', s_{-i}\right)$ for $\forall s_{-i} \in S_{-i}$.

Example

If we change the payoffs of the row player in the Chicken game we can easily see that the *Stop* strategy is a dominant strategy for this player.

	Stop	Go
Stop	8,6	2,7
Go	7,2	0,0

An alternative Chicken Game

Strictly dominated strategy

A strategy k of a player i is strictly dominated by another strategy k' if $u_i\left(k, s_{-i}\right) \prec u_i\left(k', s_{-i}\right)$. In other words, the strategy k is worse choice than the strategy k' in any case.

Example

In the previous example, we can easily see that the *Go* strategy is strictly dominated by the *Stop* strategy.

	Stop	Go
Stop	8,6	2,7
Go	7,2	0,0

An alternative Chicken Game

Weakly dominated strategy

A strategy k of a player i is weakly dominated by another strategy k' if $u_i\left(k, s_{-i}\right) \leq u_i\left(k', s_{-i}\right)$ for all s_{-i} and $u_i\left(k, s_{-i}\right) \prec u_i\left(k', s_{-i}\right)$ for some s_{-i}.

Example

If we change the payoffs of the row player in the Chicken game we can easily see that the *Go* strategy is weakly dominated strategy by the *Stop* strategy for this player.

	Stop	Go
Stop	7,6	2,7
Go	7,2	0,0

An alternative Chicken Game

Solution concepts

The Game theory attempts to predict the result of the game when the players are rational and they have the total knowledge of the game. Some solution concepts are the pure **N**ash **E**quilibriums, the mixed **N**ash **E**quilibriums and the **C**orrelated **E**quilibriums.

Pure Nash Equilibriums

In the total strategy profile, there may be some interesting points, the pure **N**ash **E**quilibriums, which can be characterized as a prediction of the game. A pure **N**ash **E**quilibrium is a strategic profile in which no one has any incentive to deviate from this. In other words, this point is preferable from everyone and everyone is happy in this point. So, a strategy profile $\left(s_1^*, s_2^*, ..., s_n^*\right)$ is a pure **NE** if and only if for every player i

$$u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i^{'}, s_{-i}^*\right)$$

for $\forall s_{i^{'}} \in S_i$.

Example of pure NE

Prisoners' dilemma

The story is that two prisoners are arrested by the police for a crime. The police have insufficient evidence for a conviction and separate both prisoners into different cells to make them a deal. The deal is that if someone confesses and the other does not, then the first will be free and the other will receive a full 20 – year sentence. If both confess, then they will receive a 10 – year sentence. Finally, if no one confesses, then they will be sentenced to only one year in jail for a minor charge. Every choice is secret before the end of the investigation. How should the prisoners act?

	No Confess	Confess
No Confess	1,1	20,0
Confess	0,20	10,10

The Prisoners' dilemma

Every prisoner selfishly tries to minimize the years in the jail. We can see that the Confess strategy is the best correspondence for every choice of the other prisoner. This strategy is a dominant strategy. So, the (Confess, Confess) point is a pure **NE**.

	No Confess	Confess
No Confess	1,1	20,0
Confess	0,20	**10,10**

Pure **NE** point of the Prisoners' dilemma

A very interesting point in the prisoners' dilemma game is that the best strategy profile for both players is the (No Confess, No Confess) point but every player does not know the choice of the other player so they selfishly try to minimize their payoff and they go to the (Confess, Confess) point.

Example of pure NE

Pure **NE**s of the Chicken Game

We can see that if the one player plays *Stop* then the other player prefers to *Go* and if the one player plays *Go* then the other player prefers to *Stop*. So, the Chicken Game has two pure **NE**s the (Stop, Go) and (Go, Stop) strategy profiles.

	Stop	Go
Stop	6,6	**2,7**
Go	**7,2**	0,0

Pure **NE**s of the Chicken Game

Mixed Nash Equilibriums

Sometimes the players independently draw a dice and take decisions about their strategies. This is called mixed strategy because they mix their choices according to a dice. If we symbolize as $\sum_i$ all distributions of player i over S_i, then a mixed strategy is symbolized as $\sigma_i \in \sum_i$, this is the probability mass of the pure strategies. We can see that the mixed strategies are a generalization of the pure strategies. In this case, every rational player i tries to optimize the utility

$$u_i(\sigma) = \sum_{s \in S} \left(\prod_{j=1} \sigma_j(s_j) \right) u_i(s)$$

, where j is the other players.

Thus, the profile of the mixed strategies $(\sigma_1^*, \sigma_2^*, ..., \sigma_n^*)$ is a mixed **NE** if and only if for every player i

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i^{'}, \sigma_{-i}^*)$$

for $\forall \sigma_{i^{'}} \in \Sigma_i$.

The main property of a mixed strategy is that it creates indifference between the pure strategies which belong to the support of this distribution. A pure strategy belongs to the support of a mixed strategy if the probability of this pure strategy is non – zero, $\sigma_i(s_i) > 0$.

Example of mixed NE

We try to find the mixed **NE** of the chicken game.

	Stop	Go
Stop	6,6	2,7
Go	7,2	0,0

The Chicken Game

Assume that p_1 is the probability of the row player to play the *Stop* strategy and the probability of the column player is p_2. If the column player plays the mixed **NE**, then the row player is indifferent between his strategies. If p_2^* is the mixed **NE** strategy of column player then his payoffs are

$$Stop: 6p_2^* + 2(1-p_2^*) = 4p_2^* + 2$$

$$Go: 7p_2^*$$

The two payoffs are equal if and only if $p_2^* = 2/3$. So, the $(2/3, 2/3)$ point is a mixed **NE** because of the game symmetry.

Example of mixed NE in zero – sum games

The Jamming Game

In a wireless network, a transmitter wants to send a packet to a receiver. There are two channels and the transmitter must choose only one of them to send the data.

However, a malicious jammer wants to disrupt the transmission. If we consider that this is a zero – sum game, what is the best strategy for every node?

	Ch1	Ch2
Ch1	-1,1	1,-1
Ch2	1,-1	-1,1

The Jamming Game

We can see that if one of the nodes chooses only one channel, then the other will have a deterministic choice, so the first player will lose the game. Therefore, we conclude that the players must randomize their strategies between the channels. However, which is the best frequency of randomization?

The mutual best frequencies of randomization are the frequencies of the mixed **NE** strategies, playing these strategies both players are happy. We will use the indifference property of the mixed **NE** to find them. The probability of player 1 to choose the channel 1 is p_1 and for player 2 is p_2. If player 2 plays the mixed **NE**, then player 1 is indifferent between the channels. If p_2^* is the mixed **NE** strategy of player 2 then the payoffs of player 1 are

$$Ch_1 : 1-2p_2^*$$

$$Ch_2 : 2p_2^*-1$$

The two payoffs are equal if and only if $p_2^* = 1/2$. So, the $(1/2, 1/2)$ point is a mixed **NE** because of the game symmetry.

Mixed NE of the zero – sum games as a maxminimizer linear program

An interesting point here is that the mixed **NE** of the zero – sum games is the solution of a maxminimizer linear program. We can see again the Jamming Game.

	Ch1	Ch2
Ch1	-1,1	1,-1
Ch2	1,-1	-1,1

The Jamming Game

If the probability of the row player to choose the Ch_1 is p then his expected payoff when the other player chooses Ch_1 is

$$Exp(p, ch_1) = -p + 1 - p = -2p + 1$$

and when the other player chooses Ch_2

$$Exp(p, ch_2) = p - 1 + p = 2p - 1$$

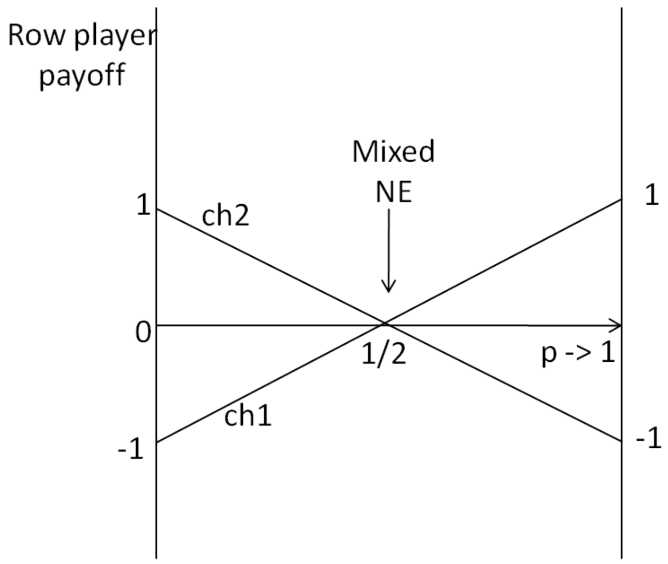

fig. 1. Maxminimizer

We can see that, if the column player randomizes his strategy between the two channels the expected payoff of the row player is in the interval $[-1,1]$. However, there is a strategy $p = 1/2$ that can guarantee a minimum payoff and in this strategy he is indifferent about the strategy of the other player. This strategy is called minimax strategy and if both players play this strategy, this is a mixed **NE**. The payoff is called the value of the zero – sum or constant sum game, in this example this value is zero.

Correlated Equilibriums

In [Au74], Robert Aumann introduced the **C**orrelated **E**quilibrium concept. This solution concept is a generalization of the **NE**, so every **NE** is a **CE**. The difference is that a **NE** is only produced by a product distribution. So, a **CE** which is based on a product distribution is a **NE.** The idea is that the players decide about their actions according to a trusty signal. This information source chooses a strategy profile from a probability distribution and secretly informs every player his strategy from which no one has any incentive to deviate. For example, in a cross – road a traffic light recommends every driver what to do according to a public distribution.

	Stop	Go
Stop	6,6	2,7
Go	7,2	0,0

The Chicken Game

	Stop	Go
Stop	1/3	1/3
Go	1/3	0

The traffic light distribution

If one player will be informed to stop, then he knows that the other player will uniformly play one of the strategies and his expected payoff is $\frac{1}{2}6+\frac{1}{2}2=4$. If he deviates, his expected payoff is $3,5$. So, he has no incentive to deviate. On the other hand, if the player will be informed to go, he knows that the other player will surely stop and the best strategy for him is to go. This probability distribution is called **CE**. Every **CE** can be characterized as a specific number of linear inequalities depending on the number of the strategies. These linear inequalities create the polytope of the **CE**.

Formal description of a Correlated Equilibrium

A **CE** is a probability distribution $\{p_s\}$ over the strategy profiles of a game. For every strategy k of a player i the expected payoff playing this strategy is no smaller than every other strategy k'.

$$\sum_{s\in S_{-i}}\left(u_i\left(k,s\right)-u_i\left(k',s\right)\right)p_{ks}\geq 0$$

So, every player has no incentive to deviate from every recommendation which is based on a **CE**.

Rational Expectations in Games

In [Au08], Robert Aumann and Jacques Dreze introduced the **R**ational **E**xpectations approach in games. In this approach, every player has beliefs about the other player and chooses a strategy in order to maximize his expectation based on these beliefs and on the abstract formulation of the game (the strategic form of the game).

The main reason for following this approach is that if one knows only the abstract formulation of a game, it is not sure that the players will play a **NE**. So, one cannot reasonably expect a **NE** payoff, even in the case in which there is only one **NE**. This is because in real life there are many parameters which can affect the result of the game besides the abstract formulation of the game (the players, the strategies and the payoffs). So, the real life context creates different situations in the same game. For example, we can imagine a road with two lanes and two players with two different directions. The same game will be played differently in Greece and in UK because the traffic regulations about the directions in the lanes are different in these two countries. So, the real life context creates two different situations in the same game.

We can understand that the abstract formulation is poor to lead the players into a reasonable expectation. Someone must also take into account the “game situation” (the real life context) in which the game will be played.

Every game situation is the beliefs of the players, the belief hierarchy of each player, what a player believes about the actions of the other player, about what the other player believes that he will play, about what the other player believes about that and so on. This belief hierarchy creates a tree of beliefs see fig. 2.

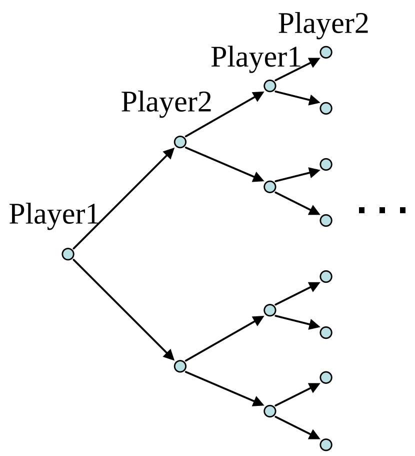

fig. 2. An example of belief hierarchy

Belief hierarchy and belief system

We define formally the belief hierarchy of the game $\langle N, S_i, u_i \rangle$. A belief hierarchy is described by a belief system B which consists of

I. *A set* $T_i = \{t_1, .., t_k\}$ *which is the set of player* i *'s types*

II. *For every type* t_i *of player* i *there is*

1. *A pure strategy* $s_i(t_i)$ *in the game which maximizes his expected payoff*

2. *A probability distribution on the types of other players which is called the theory of this type*

These belief hierarchies are symbolized as a **C**ommon **P**rior, a distribution over the players' types $T_1 \times ... \times T_n$ (known to everyone) and the theory of every type t_i is the conditional probability of this distribution for this type. So, according to this approach every player secretly knows his type (his information) and chooses a strategy to maximize his expected payoff based on this theory.

An expectation is rational, if it is based on a **C**ommon **P**rior (**CP**) and if there is **C**ommon **K**nowledge of **R**ationality (**CKR**), everyone is rational, everyone knows it and everyone knows that and so on.

This expectation is expressed as a conditional **CE** payoff of a game close to the initial game. In particular, in order to find the **RE**s based on a **CP** we use a **CE** distribution of a game $G^{'}$ very close to the initial game G. The only difference in both games is that there are the same strategies but the game $G^{'}$ has the same number of the copies of the strategies as the number of the player's types in the **CP** which correspond to a strategy. We change the game because we want a **CE** distribution which corresponds to the **CP** types and the **CE** payoffs of the games are different in the two cases. In the **RE** approach we consider only pure strategies and we look for the payoff which a player can expect when he chooses a pure strategy.

Example

We consider a situation of the Chicken Game and we try to discover the **RE**s in this situation. We know that according to the abstract formulation of the game this game has two pure **NE**s (Stop, Go), (Go, Stop) strategic profiles and one mixed **NE** (2/3 Stop, 2/3 Stop). However, if there is a **CP** distribution such as in our example below it is not sure that the players will play a pure **NE**. The players have their private type – information and try to optimize their choice according to this information. We can see that the **CP** distribution is a **CE** over the game (every strategy corresponds to only one type).

	Stop	Go
Stop	6, 6	2, 7
Go	7, 2	0, 0

The Chicken Game

	Stop	Go
Stop	1/3	1/3
Go	1/3	0

A Common Prior

	Stop	Go
Stop	1/2	1/2
Go	1	0

Row Player beliefs

	Stop	Go
Stop	1/2	1
Go	1/2	0

Column Player beliefs

If the row player's type is *Stop* then he believes that the column player is $\frac{1}{2}$ Stop and $\frac{1}{2}$ Go and he believes that if the column player is *Stop* then column player believes that the row player is $\frac{1}{2}$ Stop and $\frac{1}{2}$ Go and so on. This is the belief hierarchy which is described by the **CP**.

If the row player's type is *Stop* then the strategy which maximizes his payoff is to stop and his expectation is $\frac{1}{2}6+\frac{1}{2}2=4$. If the column player is of type *Stop* then the mutual **RE**s are $(4,4)$. However, if the column player 2 is of type *Go* then we see that the mutual **RE**s are $(4,7)$, this payoff profile is out of the feasible payoff area. So, we realise that the **RE**s may be mutually inconsistent because of the different information.

Example

We consider a particular situation of the Chicken Game again and we try to discover the **RE**s in this situation. In this case, we assume that there are more types *Go* in the **CP** which correspond to the same strategy *Go.*

	Stop	Go
Stop	6, 6	2, 7
Go	7, 2	0, 0
Go	7,2	0,0

The Chicken Game $G^{'}$

	Stop	Go
Stop	1/3	1/3
Go1	1/6	0
Go2	1/8	1/24

A Common Prior

	Stop	Go
Stop	1/2	1/2
Go1	1	0
Go2	3/4	1/4

Row Player beliefs

	Stop	Go
Stop	8/15	8/9
Go1	4/15	0
Go2	3/15	1/9

Column Player beliefs

In order to find the **RE**s we just duplicate the *Go* strategy and take the game $G^{'}$ because we can see that in **CP** we have two types *Go1* and *Go2* which correspond to the same strategy.

Example of zero – sum game

We consider a situation of the Jamming Game and we try to discover the **RE**s in this situation. However, the Jamming Game is a zero – sum game and the only feasible **CE – CP** is the product distribution of a Mixed **NE**.

	Ch1	Ch2
Ch1	-1,1	1,-1
Ch2	1,-1	-1,1

The Jamming Game

	Ch1	Ch2
Ch1	1/4	1/4
Ch2	1/4	1/4

A Common Prior

	Ch1	Ch2
Ch1	1/2	1/2
Ch2	1/2	1/2

Row Player beliefs

	Ch1	Ch2
Ch1	1/2	1/2
Ch2	1/2	1/2

Column Player beliefs

We can see that the only **RE** in this example is the zero which is the value of this zero – sum game.

The main theorems of the **RE** approach are

I. *The* ***RE*** *of any two – person zero – sum game situation with* ***CKR*** *and* ***CP*** *is the value of the game.*

II. *The* ***RE****s in a game G are the conditional* ***CE*** *payoffs in the game 2G.*

The first theorem tells us that the only reasonable expectation in a zero – sum game is the value of the game because the only feasible game situation **CE** is the product distribution of the mixed **NE**. It is a very reasonable result because we can think that in every zero – sum game the profit of one player is the loss for the other and every player has a guarantee minimax strategy which is the mixed **NE**. So, it is only rational that the players will expect the value of the game.

In the **RE**s approach, the players don't have a guaranteed payoff, they just expect something rational, the **RE** payoff. The difference between this approach and equilibrium theories is that the equilibriums are like a recommendation of the Game theory which is known to everyone. However, in the **RE**s approach every player has private information and tries to play optimally according to this information. However, how can a player learn his type? How can a player learn what he believes? It seems that the game situation inserts Information on the game which every player must read in order to learn his type. We try to discover the role of this Information in these game theoretic situations from an Information theoretic angle.

Information Theory

In [Sh48], Claude Shannon gave answers about the maximum capacity of a communication channel and the optimum data compression. Claude Shannon defined the quantity of the Information which is contained in a random event as the average uncertainty of this event. For instance, if an unexpected event happens, it will give us more Information than an expected event because we did not expect it!

The Shannon Information theory has many applications in almost all scientific fields. For example, in communications, in statistics, in portfolio theory, in mathematics of gambling, in computer science, in mechanics etc. In this chapter, we give the basic elements of the Shannon Information theory.

Shannon Entropy

The main concept of the Shannon Information theory is the entropy. Entropy is the average uncertainty of a random variable. Entropy tells us how many units of Information are carried by a random variable. If we consider a discrete random variable with an alphabet X and a probability mass function $p(x) = \Pr[X = x], x \in X$, then the entropy is

$$H(X) = -\sum_{x \in X} p(x) \log p(x)$$

If the $\log$ is to the base 2, then the unit of entropy is the bit (binary unit).

Example

In a coin flip with a fair coin, the entropy of the result X $H(X) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1$ bit (binary unit) we need 1 bit (binary digit) to describe the result. If the probability mass function is $p(head) = \frac{3}{4}$, then $H(X) = -\frac{3}{4}\log_2\frac{3}{4} - \frac{1}{4}\log_2\frac{1}{4} = 0.8113$ bits (binary units).

Joint Entropy

If we have a pair of random variables (X,Y) with a joint distribution $p(x,y)$, then the entropy of this distribution is called joint entropy. This entropy is defined as

$$H(X,Y) = -\sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2 p(x,y)$$

Example

We consider the joint distribution $p(x,y)$ and we try to find the joint entropy.

	y_1	y_2
x_1	0	$1/2$
x_2	$1/2$	0

Joint distribution $p(x,y)$

$$H(X,Y) = \frac{1}{2}\log_2(2) + \frac{1}{2}\log_2(2) = 1\,\text{bit}$$

We need only 1 bit of Information in order to describe the result of the experiment.

Conditional Entropy

The condition entropy $H(Y/X)$ is described as

$$H(Y/X) = -\sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2 p(y/x)$$

Example

We consider the joint distribution $p(x,y)$ and we try to find the conditional entropy $H(Y/X)$.

	y_1	y_2
x_1	$1/4$	$1/4$
x_2	$1/4$	$1/4$

Joint distribution $p(x,y)$

	y_1	y_2
x_1	$1/2$	$1/2$
x_2	$1/2$	$1/2$

Conditional distribution $p(y/x)$

$$H(Y/X) = \frac{1}{2} H(Y/X = x_1) + \frac{1}{2} H(Y/X = x_2) = \frac{1}{2} + \frac{1}{2} = 1 \text{ bit}$$

This means that if we know the X we want on average 1 bit more in order to find the Y.

Mutual Information

When we have two random variables, we can measure the amount of Information which one variable gives about other. This can be calculated by the mutual Information $I(X;Y)$. Intuitively, the mutual Information is the distance (Kullback distance) between the joint distribution and the product distribution.

$$I(X;Y) = \sum_{x \in X} \sum_{y \in Y} p(x,y) \log \frac{p(x,y)}{p(x)\,p(y)}$$

Example

We consider the joint distribution $p(x,y)$ and we try to find the mutual Information.

	y_1	y_2
x_1	0	$1/2$
x_2	$1/2$	0

Joint distribution $p(x,y)$

$$I(X;Y)=\sum_{x\in X}\sum_{y\in Y} p(x,y)\log\frac{p(x,y)}{p(x)\,p(y)}=1 \text{ bit}$$

This means that when we know something about the value of X distribution then we know the value of Y. In this particular case, we can tell that both distributions are fully correlated and we just want only one bit to describe the total result because with only one bit we can describe one of the distributions.

The relationship between the entropy, the joint entropy, the conditional entropy and the mutual Information is depicted on the figure 3.

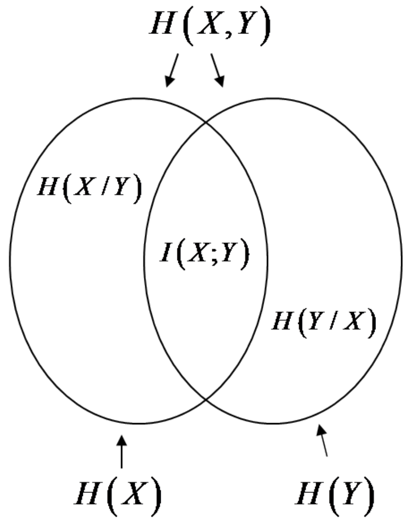

fig. 3. The decomposition of the total average Information

An Information – theoretic view of Games

How can a player learn his type in a game situation? How can a player learn what he believes? It seems that the game situation inserts information into the game (in the form of probability distributions) which every player must read in order to learn his type.

Thus, a game situation is characterized by an amount of information on the game. In the following we will look at this from an Information theoretic point of view and will try to determine the role of the Shannon Information Theory on the **R**ational **E**xpectations of the games.

The total average Information of a game situation equals with the minimum number of bits of Information which are needed to describe the result of the game. When we try to find the **RE**s, we use a **CE** of the types of players. This **CE** is a joint probability distribution of the types of both players. So, the total Information of the game situation is the joint entropy of the joint distribution of the types of the players, the entropy of **CE**.

If X is the random variable of player 1's types and Y is the random variable of player 2's types then the total Information of the game situation is the entropy of the **CE**

$$H(X,Y) = -\sum_{x \in X} \sum_{y \in Y} p(x,y) \log_2 p(x,y)$$

The minimum number of bits which player 1 needs to learn his type is

$$H(X) = H(X/Y) + I(X;Y)$$

, where $H(X/Y)$ is the private Information, the average extra bits which player 1 knows when player 2 is of a specific type and $I(X;Y)$ is the public information.

For player 2

$$H(Y) = H(Y/X) + I(X;Y)$$

, where $H(Y/X)$ is the private and $I(X;Y)$ is the public information.

We know that $H(X,Y) = H(X/Y) + I(X;Y) + H(Y/X)$, this means that if one wants to know the result of the game then he must know the private Information of each player and the public Information of the game.

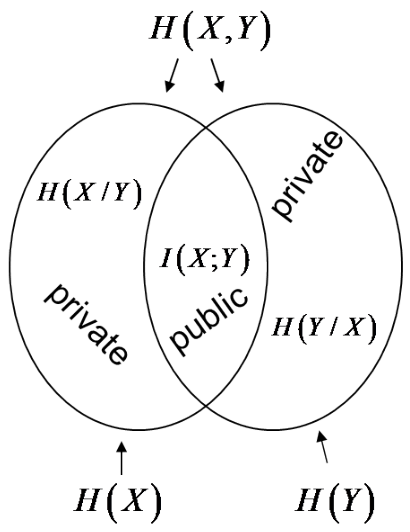

fig. 4. The total average Information of the game situation

Example

We can consider again the situation of the Chicken Game and we try to discover the total Information in this game. The **CE** which we use to find the **RE**s is the same as the **CP** because every strategy corresponds to only one type.

The Chicken Game

	Stop	Go
Stop	6,6	2,7
Go	7,2	0,0

A Common Prior

	Stop	Go
Stop	1/3	1/3
Go	1/3	0

Player 1 beliefs

	Stop	Go
Stop	1/2	1/2
Go	1	0

Player 2 beliefs

	Stop	Go
Stop	1/2	1
Go	1/2	0

The total average Information of this game is

$$H(X,Y) = \log(3) \text{ bits}$$

, this means that we need $\log(3)$ bits in order to describe the result of the game.

According to the **CP**, the distribution of player 1's types is $[Stop, Go] = [2/3, 1/3]$. So, player 1 needs $H(X)$ bits to learn his strategy.

$$H(X) = 0.9183 \text{ bits}$$

Similarly, the distribution of player 2's types is $[Stop, Go] = [2/3, 1/3]$. He needs on average $H(Y)$ bits to learn his strategy.

$$H(Y) = 0.9183 \text{ bits}$$

In our example, the private Information of each player is zero.

$$H(X/Y) = 2/3 \text{ bits}$$

$$H(Y/X) = 2/3 \text{ bits}$$

The public Information is

$$I(X;Y) = 0.253 \text{ bits}$$

Based on these observations we can formulate and prove our four Theorems.

THEOREM A:

In a game situation with **C**ommon **P**rior and **C**ommon **K**nowledge of **R**ationality and with no total Information the players play a pure **NE**.

Proof.

In this case, the players don't need Information to learn their types. Everyone knows everything and the players choose the strategies which maximize their payoffs. This is a pure **NE**. The result of the game is deterministic.

Example

In the Chicken Game we consider a situation in which it is sure that the players will play one only strategy profile.

	Stop	Go
Stop	6, 6	2, 7
Go	7, 2	0, 0

The Chicken Game

	Stop	Go
Stop	0	1
Go	0	0

A Common Prior

	Stop	Go
Stop	0	1
Go	0	0

Player 1 beliefs

	Stop	Go
Stop	0	1
Go	0	0

Player 2 beliefs

The total average Information of this game is

$$H\left(X,Y\right)=0 \text{ bits}$$

The result of the game is deterministic.

According to the **CP**, the distribution of player 1's types is $\left[Stop, Go\right] = \left[1, 0\right]$. So, player 1 needs $H\left(X\right)$ bits to learn his strategy.

$$H\left(X\right) = 0 \text{ bits}$$

Similarly, the distribution of player 2's types is $\left[Stop, Go\right] = \left[0,1\right]$. He needs on average $H\left(Y\right)$ bits to learn his strategy.

$$H\left(Y\right) = 0 \text{ bits}$$

In our example, the private Information of each player is zero.

$$H\left(X / Y\right) = 0 \text{ bits}$$

$$H\left(Y / X\right) = 0 \text{ bits}$$

The public Information is

$$I\left(X ; Y\right) = 0 \text{ bits}$$

We see that the players play a pure **NE**.

THEOREM B:

In a game situation with **CP** and **CKR** in which there is only public Information, the players play a pure **NE**.

Proof.

In this case, there is no private Information, everyone knows the same public Information. So, if a player learns his type then he learns the type of other player. Furthermore, we know that both players play strategies which maximize their expected payoff according to their types, so these strategies constitute a pure **NE**.

Example

In the Chicken Game we consider a situation in which there is only public Information.

	Stop	Go
Stop	6,6	2,7
Go	7,2	0,0

The Chicken Game

	Stop	Go
Stop	0	1/2
Go	1/2	0

A Common Prior

	Stop	Go
Stop	0	1
Go	1	0

Player 1 beliefs

	Stop	Go
Stop	0	1
Go	1	0

Player 2 beliefs

The total average Information of this game is

$$H(X,Y) = 1 \text{ bit}$$

According to the **CP**, the distribution of player 1's types is $[Stop, Go] = [1/2, 1/2]$.

So, player 1 needs $H(X)$ bits to learn his strategy.

$$H(X) = 1 \text{ bit}$$

Similarly, the distribution of player 2's types is $\left[Stop, Go\right] = \left[1/2, 1/2\right]$. He needs on average $H\left(Y\right)$ bits to learn his strategy.

$$H\left(Y\right) = 1 \text{ bit}$$

In our example, the private Information of each player is zero.

$$H\left(X/Y\right) = 0$$

$$H\left(Y/X\right) = 0$$

The public Information is

$$I\left(X;Y\right) = 1 \text{ bit}$$

We see that the players reach a pure **NE**.

THEOREM C:

In a game situation with **CP** and **CKR** and with only private Information the players expect a mixed **NE** payoff.

Proof.

In a game situation where the public Information is zero the mutual Information $I\left(X;Y\right)$ is zero. When the mutual Information is zero, the types' distributions are independent, but a **CE** which is a product distribution it is a mixed Nash Equilibrium. So, in these situations players expect a mixed **NE** payoff.

Example

In the Chicken Game we consider a situation in which there is no public Information.

	Stop	Go
Stop	6,6	2,7
Go	7,2	0,0

The Chicken Game

	Stop	Go
Stop	4/9	2/9
Go	2/9	1/9

A Common Prior

	Stop	Go
Stop	2/3	1/3
Go	2/3	1/3

Player 1 beliefs

	Stop	Go
Stop	2/3	2/3
Go	1/3	1/3

Player 2 beliefs

The total average Information of this game is

$$H(X,Y) = 1.8366 \text{ bits}$$

According to the **CP**, the distribution of player 1's types is $[Stop, Go] = [2/3, 1/3]$.

So, player 1 needs $H(X)$ bits to learn his strategy.

$$H(X) = 0.9183 \text{ bits}$$

Similarly, the distribution of player 2's types is $[Stop, Go] = [2/3, 1/3]$.

He also needs on average $H(Y)$ bits to learn his strategy.

$$H(Y) = 0.9183 \text{ bits}$$

In our example, the private Information of each player is

$$H(X/Y) = H(X)$$

$$H(Y/X) = H(Y)$$

The public Information is

$$I(X;Y) = 0 \text{ bits}$$

We see that the players expect a mixed **NE**.

THEOREM D:

In any two – player zero – sum game with **CP** and **CKR**, the only possible "game situation" has private Information.

Proof.

In any two – player zero – sum game the only possible game situation is a game situation with only private Information. The reason is that the only **CE** distribution over a two – player zero – sum game is the distribution of a mixed **NE**. Intuitively, in a two – player zero – sum game every player wins the loss of the other player, therefore it is reasonable in these games for a player to not know anything about the type of the other player. So, the only possible Information environment is a situation with only private Information. This theorem is almost the same as the theorem A in [Au08] because if there is only private Information then the only **RE** is the value of the game. But in our approach our interest lies with the Information environment of these games.

A simple 2 - player Poker Game

The poker game is a fundamental situation of conflict and we realise that the Information theory plays a crucial role in the game because in every decision making the players must take into account the bits of Information on the table. So, we present a simple poker situation from an Information theoretic point of view of game theory.

We consider a 2 – player poker Game, the two players read Information from the table (the public hands, the movements of the players) and they understand if they have the best hand or not. In every movement they must think if they have the best hand or not in order to play as they have it or not. So, we construct a very simple game in which every player chooses if he has the best hand or not. If both players simultaneously choose that they have the best hand or the worst then the payoff for everyone is 0 in these cases none has any advantage in the poker game. If someone chooses that he has the best hand and the other the worst then the first takes 2 points and the second 1 point because the player with the best hand has advantage in the game for example he can bluff easier than the other player.

	Best hand	Worst hand
Best hand	0,0	2,1
Worst hand	1,2	0,0

Best or worse hand Game

This game has two pure **NE** the (Worst, Best) and the (Best, Worst) and one mixed **NE** the (2/3 Best, 2/3 Best) in which they randomize their strategies with a probability 2/3 best hand.

Case 1

If we assume that the total Information of the game is zero, the result of the game is deterministic everyone knows that the column player has the best hand on the table.

	Best	Bad
Best	0, 0	2,1
Bad	1, 2	0, 0

The best – bad hand Game

	Best	Bad
Best	0	1
Bad	0	0

A Common Prior

	Best	Bad
Best	0	1
Bad	0	0

Player 1 beliefs

	Best	Bad
Best	0	1
Bad	0	0

Player 2 beliefs

Case 2

If there is only mutual Information then everyone knows the same Information so the result of the game will be one of the two pure **NE**.

	Best	Bad
Best	0, 0	2,1
Bad	1, 2	0, 0

The best – bad hand Game

	Best	Bad
Best	0	1/2
Bad	1/2	0

A Common Prior

	Best	Bad
Best	0	1
Bad	1	0

Player 1 beliefs

	Best	Bad
Best	0	1
Bad	1	0

Player 2 beliefs

Case 3

If there is only private Information then they must expect only a mixed **NE** payoff.

	Best	Bad
Best	0, 0	2,1
Bad	1, 2	0, 0

The best – bad hand Game

	Best	Bad
Best	4/9	2/9
Bad	2/9	1/9

A Common Prior

	Best	Bad
Best	2/3	1/3
Bad	2/3	1/3

Player 1 beliefs

	Best	Bad
Best	2/3	2/3
Bad	1/3	1/3

Player 2 beliefs

We can see that the information has a crucial role in decision making when someone knows only his private Information the only rational result is a mixed **NE**. On the other hand, if everything are known as the two first cases the players know everything in the table so they will reach a pure **NE**.

Conclusions

In our Theorems, we give four cases of feasible information environments which result in a **NE**.

The result of the first theorem is not surprising because if everyone knows the action of the other player the rational players will play a **NE**. This is the main point of the Nash equilibrium theory.

The result of the second theorem is like a generalization of the first theorem because if there is only public information in the game situation this means that in every case when a player reads information about his type then automatically knows what type the other player is. So, if everyone knows what the other play will play then the result will be a pure **NE**.

The result of the third theorem tells us that in every Information environment in which every player knows the private bits for his type the only rational expectations of the players is a mixed **NE** payoff. So, when a player does not know anything about the type of the other player then the players will expect a payoff of a mixed **NE**.

Finally, the last theorem is a specific case of the third in the meaning that in the zero – sum games the only Information environment is with only private information. Each player does not have any information about the type of the other player(s), so the only rational expectation in this case is the mixed NE payoff.

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