University of Crete

Master thesis

# The singular limit of the Allen-Cahn equation

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*Dedicated to my beloved father Dimitris*

**Abstract**

We consider the AllenCahn equation with a bistable nonlinearity associated with a double-well potential with equal depths. The Allen-Cahn equation describes the process of phase separation in multi-component alloy systems in phase transitions. Given a general initial data $u_0$ (independent of $\varepsilon$), we study rigorously the generation and motion of the interface.

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# Contents

# 1 Introduction

This thesis is occupied with the singular limit of systems of parabolic non-linear partial differential equations, including a small parameter $\varepsilon$ with bistable non-linear reaction term and general initial data. These equations have arisen as math models in Biology and Materials Science for various phenomena.

We study the Allen-Cahn equation and a kind of reaction-diffusion system. Illustratively, we deal with systems and the corresponding interface limits, created in layered functions, in terms of space. We mainly study the motion of interfaces and how they interact.

Firstly, in section 2 we discuss the perturbed Allen-Cahn equation. It is standard that this equation has a unique smooth solution, which we denote by $u^\varepsilon$. As $\varepsilon \to 0$, a formal asymptotic analysis shows the following: in the very early stage, the diffusion term $\Delta u$ is negligible compared with the reaction term $\varepsilon^{-2}(f(u)-\varepsilon g^\varepsilon(x,t,u))$, and it follows that, in the rescaled time scale $\tau = t^2/\varepsilon$, the equation is well approximated by the ordinary differential equation $u_\tau = f(u) + O(\varepsilon)$. Hence $f$ has a bistable nonlinearity, the value of $u^\varepsilon$ becomes quickly close to either $\alpha_+$ or $\alpha_-$ in the most part of $\Omega$, creating a steep interface (transition layer) between the regions $\{u^\varepsilon \approx \alpha_-\}$ and $\{u^\varepsilon \approx \alpha_-\}$. Once such an interface develops, the diffusion term becomes large near the interface, and comes to balance with the reaction term.

In section 3, we present the derivation of such interface motion equation. Next in section 4, we assume the case of the single Allen-Cahn equation. We show that leaving from initial data, the solution becomes quickly close to a step function, apart from a small area of the initial interface, creating a steep transition layer (**generation of interface**). Moreover, we study the case where $g^\varepsilon \equiv 0$ and in section 4 the general case, where $g^\varepsilon \not\equiv 0$. In section 5, the interface starts to move and the solution remains close to the step function (**motion of interface**).

The equation in discussion has a "gradient structure". Also, the functional spaces are infinite dimensional, which provides important information for the evolution. Broadly, in order to find the generation and motion of the interface, the construction of sub- and super- solutions plays a very important role. More precisely, the sub- and super- solutions for the motion of interface are constructed by using the first two terms of the formal asymptotic expansion, while those for the generation of interface are constructed by modifying the solution of the equation in the absence of diffusion: $u_t = \varepsilon^{-2} f(u)$.

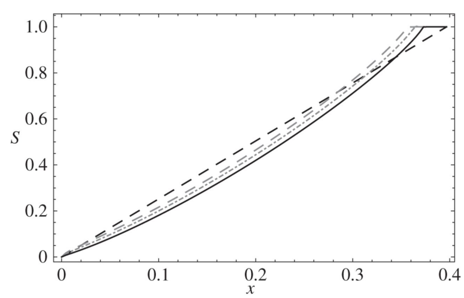
# 2 Preliminaries

We assume that there is a substance under certain conditions, in liquid and solid form. A model equation can be given by equation Allen-Cahn:

$$u_t = \Delta u + \varepsilon^{-2}(u - u^3),$$

where $\varepsilon$ is a small parameter.

Firstly, during the generation of interface, the diffusion term $\Delta u$ can be neglected before the reaction term $\varepsilon^{-2}(u - u^3)$. The solution $u_\varepsilon$ comprises as the solution of the ordinary differential equation $u_\tau = f(u)$, where $\tau = \frac{t}{\varepsilon^2}$, so the values of $u_\varepsilon$ quickly become close to one of the two stable equilibria 1 or -1 and a transition zone develops between two regions, $\{u_\varepsilon \approx 1\}$ and $\{u_\varepsilon \approx -1\}$. We consider that the area where $u_\varepsilon$ is about 1 outside of the solid form, the interval area is about -1 and the area where $u_\varepsilon$ is between 1 and -1 at the interface.




Now, the diffusion term can no longer be neglected and its combination with the reaction term induced, which provokes the motion of interface. Also, the thickness of the limits of the transition zone, creating interfaces, is related to the parameter $\varepsilon$. The equations that will be considered, they will be of this type. Broadly, it is an idea that the dynamics of the order parameter $u_\varepsilon$ develops in such a way that the value of the functional (free energy) to decline with the most effective way. To such an extent, the order parameter will be one of the points that energy takes the minimum value. Next we see the definition of bistability of $f$, where $f$ we say the term $(u - u^3)$.

### 2.0.1 Bistability

Generally, the Allen-Cahn equation :

$$u_t = \Delta u + \frac{1}{\varepsilon^2} f(u),$$

with the function f(u) of bistable type,with three zeros $\alpha_-, \alpha, \alpha_+$. Specifically, f(u)=0 only at $u = \alpha_-, \alpha, \alpha_+$,
$f'(u) > 0,\ \ u < \alpha_-^*$ or $u > \alpha_+^*$,
$f'(u) < 0, u \in (\alpha_-^*, \alpha_+^*)$,

where $\alpha_- < \alpha_-^* < \alpha < \alpha_+^* < \alpha_+$. We say that:
$u \in (\alpha_-, \alpha_-^*)$ is **metastable interval 1**;
$u \in (\alpha_-^*, \alpha_+^*)$ is **spinodal interval**;
$u \in (\alpha_+^*, \alpha_+)$ is **metastable interval 2**.

The parameter $\varepsilon$ is an interaction length, small compared to the characteristic dimensions. If the above equation is contained in the vessel $\Omega$, the equation should be supplemented with boundary conditions in the boundary $\partial\Omega$, such as:

$$\partial_v(u) = 0 \ \text{ on } \ \partial\Omega,$$

where $\partial_v$ denotes differentiation normal to $\partial\Omega$. The physical meaning of these two conditions is that none of the mixture can pass through the walls.

The first condition is the way to ensure that the total free energy of the mixture decreases in time (**requirement from thermodynamics**).

### 2.0.2 About Energy

As we mentioned, the order parameter is one of the points that the energy takes the minimum value. For this reason, we consider the properties of stability by using the functional Lyapunov from the theory of ordinary differential equations.

**Example**: An example of functional energy (Van der Waals type), let it be,

$$J_\varepsilon(u) = \int_\Omega (\frac{\varepsilon^2}{2} | \bigtriangledown u|^2 + W(u))dx,$$

where $W(u)$ is smooth and non-negative function, double well potential with two equal minima at $u = u_1$ and $u = u_2$, $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, and $\varepsilon$ a small, positive parameter.

The order parameter can be described by u. Physically, $\varepsilon$ gives the order of magnitude of the surface tendency and it is the main factor for the morphology of layers. We deal with bistable systems, the idea is that there is more than one desirable situation (order parameter). The gradient system that corresponds to $J_\varepsilon$ is

$$\frac{du}{dt} = -grad_{L^2} J_\varepsilon(u),$$

depending on the metric of $L_2$ manifold, where the gradient is calculated. This affects the direction that $J_\varepsilon$ decreasing. The most basic property of gradient systems is

$$\frac{d}{dt} J_\varepsilon(u) = - \|u_t\|^2_{L^2} .$$

The solutions of the energy are moving very close to +1 or -1 and we have the generation of layered profiles. There is a competition between W and gradient term in expression of $J_\varepsilon(u)$. The W term favors the separation and another the opposite. Thus, the generation of layers is the result of the conciliation of these two opposing situations. We set out the initial interface

$$\Gamma_\varepsilon(t) := \{ \, x \in \Omega, u_\varepsilon(x,t) = \alpha \, \} \, ,$$

and suppose is a $C^{3+\theta}$ hypersurface without boundary. Thus, we get the generation of one pattern in $\Omega$, which consists of large areas that are separated from thin zones. The solution in each area is almost -1 or +1 and we have sharp transition inside.
There are two models that describe the motion of interface: one that the interface does not have thickness sharp interface model and the other one that the interface is described as an inner layer with finite non-zero thickness. The Allen-Cahn equation is an example of the second model.

### 2.0.3 Phase change problems-thermostatics

The binary alloys are systems consisting of two types of elements A-B, in which the molecules of A and B create physical links leading to the creation/generation of a new mix. One such problem is Al-Si, Cu-Ni. Let m grams of binary alloy A-B which comprises of $m_A$ grams of kind A and $m_B$ grams of kind B, satisfying $m_A + m_B = m$. The fractions $c_A := \frac{m_A}{m}$, $c_B := \frac{m_B}{m}$ express the ratio of A and B in the corresponding total mass. Considering that $c_A + c_B = 1$, it is obvious that only one from these fractions is necessary to determine the concentration of the alloy. The phase changes are determined by the phase diagram, which describes the various phases that can coexist in thermodynamic equilibrium, the state of a thermodynamic system in which its properties do not change with time.

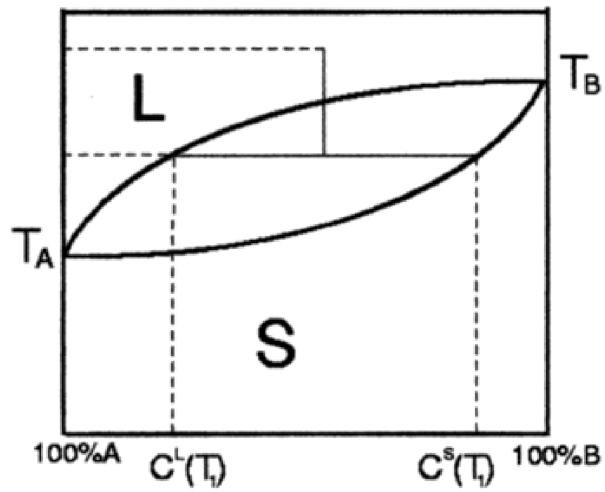



Figure 1: Phase diagram

Under constant pressure the thermodynamics state of the mixture is determined by two parameters: T= temperature, c= specific, $0 < c < 1$. $T_A$, $T_B$ are the melting temperatures of A and B. If a state $(c, T)$ is located between the liquidus and solidus lines, thus the alloy is contained by liquid with specific $c^L(T)$ and coexists with the solid with specific $c^S(T)$. So, liquid and solid coexist in thermodynamic equilibrium with different concentrations. If we let $\lambda$ the percentage of the liquid in liquid form, then from the law of conservation of mass

$$c = c^L(T_1) + (1 - \lambda)c^S(T_1).$$

As the temperature decreases in $(T_A, T_B)$, then we have redistribution of A and B. Complicated species of interfaces appear where the liquid and solid state coexist together. If the interface is smooth with respect to some macroscopic scale, then its motion is described by a system of differential equations.

The problems that we discuss are evolutionary equations. Some examples are the following:

1. **Allen-Cahn equation**:

$$\begin{cases} u_t = \varepsilon^2 \Delta u - W'(u) & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases},$$

2. **Modified Allen-Cahn equation**:

$$\begin{cases} u_t = \varepsilon^2 \Delta u - (W'(u) - \frac{1}{|\Omega|}\int_\Omega W'(u)dx) & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases},$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $N$= 1, 2, 3, $\frac{\partial}{\partial n}$ is the derivative over the unit normal vector exterior to $\partial\Omega$, $\varepsilon$ is related with the width of the interface and W(u) is the double well potential. The behavior changes according to the dimension $N$. Likewise, the equation should be provided with initial condition $u(x,0) = u_0(x)$, so we have to do with initial value problems.

**Proposition 1.** *$J_\varepsilon(u(\cdot,t))$ is a decreasing function with respect to t for the above Allen-Cahn equation.*

*Proof.* Let $J_\varepsilon(u(\cdot,t)) = \int_\Omega(\varepsilon^2|\nabla u| + W(u))dx$, we will prove $\frac{\partial}{\partial t}J_\varepsilon(u(\cdot,t)) \leq 0$. Let $u(\cdot,t)$ be a smooth function then,

$$\frac{\partial}{\partial t}J_\varepsilon(u(\cdot,t)) = \int_\Omega(\varepsilon^2\nabla u\nabla u_t| + W'(u)u_t)dx.$$

By Allen-Cahn equation and Green's Theorem we obtain that

$$-\int_\Omega(\varepsilon^2\Delta u - W'(u))u_t dx + \int_{\partial\Omega} u_t \cdot \frac{\partial u}{\partial n}dS,$$

since $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, we have

$$\int_\Omega(-\varepsilon^2\Delta u + W'(u))(\varepsilon^2\Delta u - W'(u))dx$$

$$= -\int_\Omega(\varepsilon^2\Delta u - W'(u))^2 dx$$

$$= -||u_t||^2_{L^2} \leq 0.$$

Therefore, for the case of Allen-Cahn equation we prove that $\frac{\partial}{\partial t}J_\varepsilon(u(\cdot,t)) \leq 0$ is valid. □

The Allen-Cahn equation is an evolutionary equation with dynamic compatible with the $\frac{\partial}{\partial t}J_\varepsilon(u(\cdot,t))$. Also, Allen-Cahn equation does not maintain the mass conservation.

## 2.1 Relating to the equation Allen-Cahn

We regard an Allen-Cahn type equation of the form

$$u_t = \Delta u + \varepsilon^{-2} f^{\varepsilon}(x,t,u), \tag{2.1}$$

where $\varepsilon > 0$ is a small parameter and $f^{\varepsilon}$ a bistable nonlinearity associated with a double-well potential, whose well-depths are unbalanced by order $\varepsilon$.
Given a general initial data $(u_0, v_0)$, we show that the component u develops a steep transition layer and that all the above-mentioned results remain true for the u-component of these systems.

## 2.2 Perturbed Allen-Cahn equation

In non-linear diffusion equations, solutions often develop sharp interval layers or interfaces that separate the spatial domain into different phase regions. Particularly, this happens when the diffusion coefficient is very small or the reaction term is very large. The motion of such interfaces is driven by their curvature.

A typical example is the Allen-Cahn equation:

$$u_t = \Delta u + \varepsilon^{-2} f(u), \tag{2.2}$$

where $\varepsilon > 0$ is a small parameter and $f(u)$ is a bistable non-linear function.

In this Chapter, we consider a perturbed Allen-Cahn equation of the form:

$$P^{\varepsilon}\begin{cases} u_t = \Delta u + \frac{1}{\varepsilon^2}(f(u) - \varepsilon g^{\varepsilon}(x,t,u)) & \text{in } \Omega \times (0,+\infty), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2.3}$$

we call $P^{\varepsilon}$ the above system and studying the behavior of layers near the sharp interface limit as $\varepsilon \to 0$.

- $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $(N \geq 2)$, and $\nu$ is the Euclidean unit normal vector exterior to $\partial\Omega$.
- The nonlinearity is given by $f(u) := -W'(u)$, where W(u) is a double-well potential with equal well-depths with global minimum value at $u = \alpha_-$ and $u = \alpha_+$. Also, we assume that f is of class $C^2$ on $\mathbb{R}$ and has exactly three zeros $\alpha_- < \alpha < \alpha_+$, such that
$f'(\alpha\pm) < 0$, $f'(\alpha) > 0$ (**bistable nonlinearity**)
and $\int_{-\alpha}^{+\alpha} f(u)du = 0$.

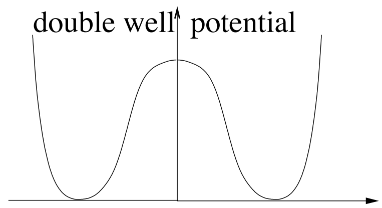



Figure 2: Double well potential, which is symmetry respect to u, $W(\alpha_-) = W(\alpha_+)$.

The conditions imply that the potential W(u) attains its local minima at $u = \alpha_-, \alpha_+$, and that $W(\alpha_-) = W(\alpha_+)$. As a result, $f$ has two stable zeros $\alpha_-, \alpha_+$ , having "balanced" stability.
A typical example is $f(u) = u(1 - u^2)$.

The term $g^\varepsilon$ represents a small perturbation, where $g^\varepsilon(x, t, u)$ is a function defined on $\overline{\Omega} \times [0, T] \times \mathbb{R}$. This breaks the balance of the two stable zeros sightly. In the special case where $g^\varepsilon \equiv 0$, Problem $(P^\varepsilon)$ reduces to the usual Allen-Cahn equation. We use this special case for our later results.

We assume that $g^\varepsilon$ is $C^2$ in $x$ and $C^1$ in $t$, $u$, and that, for any $T > 0$, there exist $\theta \in (0, 1)$ and $C > 0$ such that, for all $(x, t, u) \in \overline{\Omega} \times [0, T] \times \mathbb{R}$,

$$|\Delta_x g^\varepsilon(x, t, u)| \le C\varepsilon^{-1} \quad \text{and} \quad |g_t^\varepsilon(x, t, u)| \le C\varepsilon^{-1}, \tag{2.4}$$

$$|g_u^\varepsilon(x, t, u)| \le C, \tag{2.5}$$

$$\|g^\varepsilon(\cdot, \cdot, u)\|_{C^{1+\theta, \frac{1+\theta}{2}}(\overline{\Omega} \times [0,T])} \le C. \tag{2.6}$$

Moreover, we assume that there exists a function $g(x, t, u)$ and a constant, which we denote again by $C$, such that

$$|g^\varepsilon(x, t, u) - g(x, t, u)| \le C\varepsilon, \tag{2.7}$$

for all small $\varepsilon > 0$. Note that the estimate 2.6 and the pointwise convergence $g^\varepsilon \to g$ (as $\varepsilon \to 0$) imply that $g$ satisfies the same estimate as 2.6.
For technical reasons we also assume that

$$\frac{\partial g^\varepsilon}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times [0, T] \times \mathbb{R}, \tag{2.8}$$

which implies the same Neumann boundary condition for $g$.

**Remark.** *The equation* $P^\varepsilon$ *can be expressed in the form:*

$$u_t = \Delta u + \frac{1}{\varepsilon^2}(f^\varepsilon(x,t,u),$$

*where* $f^\varepsilon$ *is* $C^2$ *in* $x$, $\varepsilon$ *and* $C^1$ *in* $t$, $u$. *Otherwise, by setting*

$$g^\varepsilon(x,t,u) = -\frac{f^\varepsilon(x,t,u) - f(u)}{\varepsilon}, \tag{2.9}$$

$$g(x,t,u) = -\frac{\partial f^\varepsilon}{\partial \varepsilon}(x,t,u) \quad \Big|_{\varepsilon=0}, \tag{2.10}$$

*the above equation is reduced to that in* ($P^\varepsilon$). *The conditions 2.4 and 2.11 then follows automatically from the above regularity assumptions on* $f^\varepsilon$. *The condition 2.6 holds if we impose slightly stronger regularity on* $f^\varepsilon$.

As for the initial datum $u_0(x)$ (independent of $\varepsilon$), we assume $u_0(x) \in C^2(\overline{\Omega})$. The constant $C_0$ will stand for the following quantity :

$$C_0 := \left\|u_0\right\|_{C^0(\overline{\Omega})} + \left\|\nabla u_0\right\|_{C^0(\overline{\Omega})} + \left\|\Delta u_0\right\|_{C^0(\overline{\Omega})}. \tag{2.11}$$

Furthermore, we define the initial interface $\Gamma_0$ by

$$\Gamma_0 := \{\, x \in \Omega, u_0(x) = \alpha \,\},$$

and suppose that $\Gamma_0$ is a $C^{3+\theta}$ hypersurface without boundary such that, $\nu$ being the outward unit normal vector to $\Gamma_0$,

$$\Gamma_0 \subset\subset \Omega \ \text{ and } \ \nabla u_0(x)\cdot\nu(x) \neq 0 \qquad \text{if } \ x \in \Gamma_0, \tag{2.12}$$

$$\tag{2.13}$$

$$u_0 > a \ \text{ in } \ \Omega_0^+, u_0 < \alpha \qquad \text{in } \ \Omega_0^-, \tag{2.14}$$

where $\Omega_0^-$ denotes the region enclosed by the hypersurface $\Gamma_0$ and $\Omega_0^+$ the region enclosed between the boundary of the domain $\partial\Omega$ and the hypersurface $\Gamma_0$.

# 3 Formal derivation of the interface motion equation

We derive the equation of interface motion corresponding to Problem $(P^\varepsilon)$ by using a formal asymptotic expansion, the method of matched asymptotic expansions for our problem. The resulting interface equation can be regarded as the singular limit of $(P^\varepsilon)$ as $\varepsilon \to 0$. Particularly, the first two terms of the asymptotic expansion determine the interface equation.

Let $u_\varepsilon$ be the solution of Problem $(P^\varepsilon)$. We recall that $\Gamma_t^\varepsilon := \{ x \in \Omega, u^\varepsilon(x,t) = \alpha \}$ is the interface at time t and call $\Gamma^\varepsilon := \bigcup_{t\geq 0}(\Gamma_t^\varepsilon \times \{t\})$ the interface. Let $\Gamma := \bigcup_{t\geq 0}(\Gamma_t^\varepsilon \times \{t\})$ be the solution of the limit geometric motion problem and $\widetilde{d}$ be the signed distance function to $\Gamma$ defined by:

$$\widetilde{d}(x,t) = \begin{cases} dist(x,\Gamma_t) & \text{for } \ x \in \Omega_t^+ \\ -dist(x,\Gamma_t) & \text{for } \ x \in \Omega^- \end{cases}, \tag{3.1}$$

where $dist(x,\Gamma_t)$ is the distance from x to the hypersurface $\Gamma_t$ in $\Omega$. We remark that $\widetilde{d} = 0$ on $\Gamma$ and that $|\nabla \widetilde{d}| = 1$ in a neighborhood of $\Gamma$. Then, we define

$$Q_T^+ = \bigcup_{0\leq t\leq T} (\Omega_t^+ \times \{t\}),\, Q_T^- = \bigcup_{0\leq t\leq T} (\Omega_t^- \times \{t\}).$$

We also assume that the solution $u_\varepsilon$ has the expansions

$$u^\varepsilon(x,t) = \alpha_\pm + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \cdots \tag{3.2}$$

away from the interface $\Gamma$ (the outer expansion) and

$$u^\varepsilon(x,t) = U_0(x,t,\xi) + \varepsilon U_1(x,t,\xi) + \varepsilon^2 U_2(x,t,\xi) + \cdots \tag{3.3}$$

near $\Gamma$ (the inner expansion). The functions $U_k(x,t,z)$, k= 0, 1, 2, $\cdots$, are defined for $x \in \overline{\Omega}$, $t \geq 0$, $z \in \mathbb{R}$ and by definition $\xi := \frac{\widetilde{d}(x,t)}{\varepsilon}$. The stretched space variable $\xi$ gives exactly the right spatial scaling to describe the rapid transition between the regions $\{u^\varepsilon \approx \alpha_-\}$ and $\{u^\varepsilon \approx \alpha_+\}$. We normalize $U_k$ in such a way that

$$U_0(x,t,0) = \alpha,\;\; U_k(x,t,0) = 0,$$

for all $k \geq 1$ (normalization conditions). To make the inner and outer expansions consistent, we require that

$$U_0(x,t,+\infty) = \alpha_+,\;\; U_k(x,t,+\infty) = 0, \tag{3.4}$$

$$U_0(x,t,-\infty) = \alpha_-,\;\; U_k(x,t,-\infty) = 0, \tag{3.5}$$

for all $k \geq 1$ (matching conditions). Then, we will replace the inner expansion 3.3 into parabolic equation of Problem $(P^\varepsilon)$ 2.3 and collect the $\varepsilon^{-2}$ and $\varepsilon^{-1}$ terms. For this purpose, we compute the needed terms of $(P^\varepsilon)$ and get:

$$u_t^\varepsilon = U_{0t} + U_{0z}\frac{\widetilde{d}_t}{\varepsilon} + \varepsilon U_{1t} + U_{1z}\widetilde{d}_t + \cdots$$

$$\nabla u^\varepsilon = \nabla U_0 + U_{0z}\frac{\nabla\widetilde{d}}{\varepsilon} + \varepsilon\nabla U_1 + U_{1z}\nabla\widetilde{d} + \cdots$$

$$\begin{aligned}\Delta u^\varepsilon &= \Delta U_0 + 2\frac{\nabla\widetilde{d}}{\varepsilon}\cdot\nabla U_{0z} + U_{0z}\frac{\Delta\widetilde{d}}{\varepsilon} + U_{0zz}\frac{|\nabla\widetilde{d}|}{\varepsilon^2} \\ &+\varepsilon\Delta U_1 + 2\nabla\widetilde{d}\cdot\nabla U_{1z} + U_{1z}\Delta\widetilde{d} + U_{1zz}\frac{|\nabla\widetilde{d}|^2}{\varepsilon} + \cdots ,\end{aligned}$$

where the functions $U_i (i = 0, 1)$, as well their derivatives, are taken at point $(x, t, \frac{\widetilde{d}(x,t)}{\varepsilon})$. Here, $\nabla U_0$ denotes the derivative with respect to x whenever we regard $U_0(x, t, z)$ as a function of three variables x, t and z. Similarly, we define the $\Delta U_0$ that applies to $U_{0z}$ and $U_{1zz}$. We also use the expansions

$$f(u^\varepsilon) = f(U_0) + \varepsilon f'(U_0)U_1 + O(\varepsilon^2),$$

$$\begin{aligned}g^\varepsilon(x, t, u^\varepsilon) &= g(x, t, u^\varepsilon) + \mathrm{O}(\varepsilon) \\ &= g(x, t, U_0) + \mathrm{O}(\varepsilon).\end{aligned}$$

Next we substitute the expressions above in the partial differential equation in Problem $(P^\varepsilon)$ 2.3.

$$\begin{aligned}U_{0t} + U_{0z}\frac{\widetilde{d}_t}{\varepsilon} + \varepsilon U_{1t} + U_{1z}\widetilde{d}_t + \cdots &= \\ \Delta U_0 + 2\frac{\nabla\widetilde{d}}{\varepsilon}\cdot\nabla U_{0z} + U_{0z}\frac{\Delta\widetilde{d}}{\varepsilon} + U_{0zz}\frac{|\nabla\widetilde{d}|^2}{\varepsilon^2} \\ +\varepsilon\Delta U_1 + 2\nabla\widetilde{d}\cdot\nabla U_{1z} + U_{1z}\Delta\widetilde{d} + U_{1zz}\frac{|\nabla\widetilde{d}|^2}{\varepsilon} \\ +\frac{1}{\varepsilon^2}(f(U_0) + \varepsilon f'(U_0)U_1 + O(\varepsilon^2) - \varepsilon(g(x, t, U_0) + O(\varepsilon))) \\ = \Delta U_0 + 2\frac{\nabla\widetilde{d}}{\varepsilon}\cdot\nabla U_{0z} + U_{0z}\frac{\Delta\widetilde{d}}{\varepsilon} + U_{ozz}\frac{|\nabla\widetilde{d}|^2}{\varepsilon^2} \\ 2\nabla\widetilde{d}\cdot\nabla U_{1z} + U_{1z}\Delta\widetilde{d} + U_{1zz}\frac{|\nabla\widetilde{d}|^2}{\varepsilon} + \frac{f(U_0)}{\varepsilon^2} \\ +\frac{f'(U_0)U_1}{\varepsilon} - \frac{g(x, t, U_0)}{\varepsilon} + \cdots ,\end{aligned}$$

and we observe that $U_0$ depends only on the variable z, we have $\nabla U_{0z} = 0$, which with the fact that $|\nabla\widetilde{d}| = 1$ near $\Gamma_t$. From the above we obtain that

$$\begin{aligned} U_{0t} + U_{0z}\frac{\widetilde{d}_t}{\varepsilon} + \varepsilon U_{1t} + U_{1z}\widetilde{d}_t + \cdots \\ = \Delta U_0 + U_{0z}\frac{\Delta\widetilde{d}}{\varepsilon} + U_{ozz}\frac{|\nabla\widetilde{d}|^2}{\varepsilon^2} + 2\nabla\widetilde{d}\cdot\nabla U_{1z} \\ +U_{1z}\Delta\widetilde{d} + U_{1zz}\frac{1}{\varepsilon} + \frac{f(U_0)}{\varepsilon^2} + \frac{f'(U_0)U_1}{\varepsilon} - \frac{g(x,t,U_0)}{\varepsilon} + \cdots \end{aligned}$$

Now we collect the $\varepsilon^{-2}$ terms and yield

$$U_{0zz} + f(U_0) = 0.$$

Taking into consideration the normalization and matching conditions, we now assert that $U_0(x,t,z) = U_0$,where $U_0(z)$ is the unique solution of the stationary problem

$$\begin{cases} U_0'' + f(U_0) = 0, \\ U_0(-\infty) = \alpha_-,\ U_0(0) = \alpha,\ U_0(+\infty) = \alpha_+. \end{cases} \tag{3.6}$$

This solution represents the first approximation of the profile of a transition layer around the interface observed in the stretched coordinates.

**Lemma 3.1.** *There exist positive constants $C$ and $\lambda$ such that the following estimates hold.*

$$\begin{aligned} 0 < \alpha_+ - U_0(z) \le Ce^{-\lambda|z|} \ \textit{for}\ \ z \ge 0, \\ 0 < U_0(z) - \alpha_- \le Ce^{-\lambda|z|} \ \textit{for}\ \ z \le 0. \end{aligned}$$

*In addition, $U_0$ is a strictly increasing function and,for j= 1, 2,*

$$|D^j U_0(z)| \le Ce^{-\lambda|z|} \ \textit{for}\ \ z \in \mathbb{R}. \tag{3.7}$$

Next we collect the $\varepsilon^{-1}$ terms and obtain

$$U_{1zz} + f'(U_0)U_1 = U_0'(\widetilde{d}_t - \Delta\widetilde{d}) + g(x,t,U_0), \tag{3.8}$$

where $U_0' = U_{0z}$. This problem can be seen as a linearized problem for 3.6 with an inhomogeneous term. The following lemma plays the key role in determining the equation of interface motion.

**Lemma 3.2** (Solvability condition)**.** *Let A(z) be a bounded function on $-\infty < z < \infty$. Then the problem*

$$\begin{cases} \psi_{zz} + f(U_0(z))\psi = A(z) \qquad z \in \mathbb{R}, \\ \psi(0) = 0, \qquad\qquad\qquad \psi \in L^\infty(\mathbb{R}), \end{cases} \tag{3.9}$$

*has a solution if and only if*

$$\int_{\mathbb{R}} A(z)U_0'(z)dz = 0. \tag{3.10}$$

*Moreover the solution, if it exists, is unique and satisfies for some constant $C > 0$,*

$$|\psi(z)| \leq C||A||_{L^\infty}, \tag{3.11}$$

*for all $z \in \mathbb{R}$.*

*Proof.* ($\Rightarrow$) First, multiplying the equation by $U_0'$ and integrating it by parts

$$U_0'\psi_{zz} + U_0'f'(U_0(z))\psi = U_0'A(z)$$

$$\int_{\mathbb{R}}(U_0'\psi_{zz} + U_0'f'(U_0(z))\psi)dz = \int_{\mathbb{R}} U_0'A(z)dz, \tag{3.12}$$

$$\int_{\mathbb{R}} U_0'\psi_{zz}dz = -\int_{\mathbb{R}} U_0''\psi_z dz + [U_0'(z)]_{-\infty}^{+\infty} = -\int_{\mathbb{R}} U_0''\psi_z dz,$$

$$\int_{\mathbb{R}} U_0'f'(U_0(z))\psi dz = \int_{\mathbb{R}} (f(U_0(z)))'\psi dz = -\int_{\mathbb{R}} f(U_0(z))\psi_z$$
$$+[f(U_0(z))\psi]_{-\infty}^{+\infty} = -\int_{\mathbb{R}} f(U_0(z))\psi_z + (f(\alpha_+)\psi - f(\alpha_-)\psi),$$

taking into consideration 3.4, 3.5 and that f is bistable type with three zeros at $\alpha_-$, $\alpha$, $\alpha_+$. Finally 3.12 is

$$\int_{\mathbb{R}}(-U_0''\psi_z - f(U_0(z))\psi_z)dz = \int_{\mathbb{R}} U_0'A(z)dz,$$
$$\int_{\mathbb{R}}(-U_0'' - f(U_0(z)))\psi_z dz = \int_{\mathbb{R}} U_0'A(z)dz,$$

by 3.6 we get $\int_{\mathbb{R}}(-U_0'' - f(U_0(z)))\psi_z dz = 0$. Therefore, the $\int_{\mathbb{R}} U_0'(z)A(z)dz = 0$ condition is necessary.
($\Leftarrow$) Conversely, suppose that this condition is satisfied. Then, since $U_0'$ is bounded positive solution to the homogeneous equation $\psi_{zz} + f(U_0(z))\psi = 0$, we can use the method of variation of constants to find the above solution $\psi$. More precisely, set $\varphi := U_0'$,

$$\psi(z) = \varphi(z)\int_0^z (\varphi^{-2}(\zeta)\int_{-\infty}^{\zeta} A(\xi)\varphi(\xi)d\xi)d\zeta \tag{3.13}$$

$$= -\varphi(z)\int_0^z (\varphi^{-2}(\zeta)\int_{\zeta}^{\infty} A(\xi)\varphi(\xi)d\xi)d\zeta. \tag{3.14}$$

The estimate 3.11 follows from the above expression and previous lemma. □

From the above lemma and multiplying with $U_0'$ the solvability condition for 3.8 is given by

$$\int_{\mathbb{R}} [U_0'^2(z)(\widetilde{d}_t - \Delta \widetilde{d})(x,t) + g(x,t,U_0(z))U_0'(z)]dz = 0 \text{ for all } (x,t) \in Q_T. \quad (3.15)$$

Then we get

$$\widetilde{d}_t - \Delta \widetilde{d} = -\frac{\int_{\mathbb{R}} g(x,t,U_0(z))U_0'(z)dz}{\int_{\mathbb{R}} U_0'^2(z)dz},$$

which gives

$$\widetilde{d}_t = \Delta \widetilde{d} - \frac{\int_{\alpha_-}^{\alpha_+} g(x,t,r)dr}{\int_{\mathbb{R}} U_0'^2(z)dz}. \quad (3.16)$$

Moreover, multiplying equation 3.6 by $U_0'$ and integrating it from $-\infty$ to z, we obtain

$$0 = \int_{-\infty}^{z} (U_0''U_0' + f(U_0)U_0')(s)ds. \quad (3.17)$$

The first term from right side gives:

$$\int_{-\infty}^{z} U_0''U_0'(s)ds,$$

let $U_0 = U_0'(s)$ and $dU_0 = U_0''(s)ds$,
hence

$$\int_{-\infty}^{z} U_0''U_0'(s)ds = \int_{-\infty}^{z} U_0 dU_0 = [\frac{1}{2}U_0^2(s)]_{-\infty}^{z}$$

$$= \frac{1}{2}U_0'^2(z) - \frac{1}{2}U_0'^2(-\infty)$$

$$= \frac{1}{2}U_0'^2(z).$$

Next we set for the second term from right side, $U_0 = u$, $du = U_0'ds$, and we get

$$\int_{-\infty}^{z} f(U_0)U_0'(s)ds = \int_{-\infty}^{z} f(u)du$$

$$= -W(U_0(z)) + W(\alpha_-),$$

since we have defined $f(u) := -W'(u)$. Then 3.17 is equal with

$$\frac{1}{2}U_0'^2(z) - W(U_0(z)) + W(\alpha_-), \tag{3.18}$$

where we have also used the matching conditions 3.5. This implies that

$$U_0'(z) = \sqrt{2}(W(U_0(z)) - W(\alpha_-))^{1/2},$$

and therefore

$$\int_{\mathbb{R}} U_0'^2(z)dz = \int_{\mathbb{R}} U_0'(z)\sqrt{2}(W(U_0(z)) - W(\alpha_-))^{1/2}dz \tag{3.19}$$

$$= \sqrt{2}\int_{\alpha_-}^{\alpha+}(W(s) - W(\alpha_-))^{1/2}ds. \tag{3.20}$$

It then follows, that equation 3.16 becomes

$$\widetilde{d}_t = \Delta\widetilde{d} - \frac{\int_{\alpha_-}^{\alpha_+} g(x,t,r)dr}{\sqrt{2}\int_{\alpha_-}^{\alpha_+}(W(s) - W(\alpha_-))^{1/2}ds}. \tag{3.21}$$

We define the constant $c_0$ by

$$c_0 = [\sqrt{2}\int_{\alpha_-}^{\alpha_+}(W(s) - W(\alpha_-))^{1/2}ds]^{-1}, \tag{3.22}$$

with W the double- well potential associated with f :

$$W(s) = -\int_{\alpha}^{s} f(r)dr. \tag{3.23}$$

Consequently,

$$\widetilde{d}_t = \Delta\widetilde{d} - c_0\int_{\alpha_-}^{\alpha_+} g(x,t,r)dr. \tag{3.24}$$

Now, we derive the equation of interface motion. Since, $\nabla\widetilde{d}(= \nabla_x\widetilde{d}(x,t)$ coincides with the outward normal unit vector to the hypersurface $\Gamma_t$, we have $\widetilde{d}_t(x,t) = -V_n$, where $V_n$ is the normal velocity of the interface $\Gamma_t$. Also, we define that the mean of curvature $\kappa$ at each point of $\Gamma_t$, of the interface is equal to $\Delta\widetilde{d}/(N-1)$,

$$\kappa = \frac{\widetilde{d}}{(N-1)} \Rightarrow \widetilde{d} = (N-1)\kappa.$$

Thus, the equation of interface motion is given by:

$$V_n = -(N-1)\kappa + c_0\int_{\alpha_-}^{\alpha_+} g(x,t,r)dr \text{ on } \Gamma_t. \tag{3.25}$$

We define a step function $\widetilde{u}(x,t)$ by

$$\widetilde{u}(x,t) = \begin{cases} \alpha_+ & \text{in } \Omega_t^+ \\ \alpha_- & \text{in } \Omega_t^- \end{cases} \text{ for } \ t \in [0,T],$$

which represents the formal asymptotic limit of $u^\varepsilon$ (or the sharp interface limit) as $\varepsilon \to 0$. In our occasion, the solution $u^\varepsilon$ of Problem $(P^\varepsilon)$ satisfies

$$u^\varepsilon = \begin{cases} \alpha_+ & \text{in } Q_t^+ \\ \alpha_- & \text{in } Q_t^- \end{cases} \text{ as } \ \varepsilon \to 0,$$

we have formally proved that the boundary $\Gamma_t$ between $\Omega_t^-$ and $\Omega_t^+$ moves according to the law 3.25.
We give some basic estimates for $U_1(x,t,z)$, which we will need in later section to study the motion of interface. Substituting 3.16 into 3.8 gives

$$\begin{cases} U_{1zz} + f'(U_0(z))U_1 = g(x,t,U_0(z)) - \gamma(x,t)U_0'(z), \\ U_1(x,t,0) = 0, \qquad U_1(x,t,\cdot) \ \in \ L^\infty(\mathbb{R}), \end{cases} \tag{3.26}$$

where $\gamma := c_0 \int_{\alpha_-}^{\alpha_+} g(x,t,r)dr$. Thus $U_1()x,t,z)$ is a solution of 3.9 with

$$A = A_0(x,t,z) := g(x,t,U_0(z)) - \gamma(x,t)U_0'(z), \tag{3.27}$$

where the variables x, t are considered parameters. The problem 3.26 has a unique solution by virtue of Lemma 3.2. Next, since $A_0(x,t,z)$ remains bounded as (x, t, z) varies in $\overline{\Omega} \times [0,T] \times \mathbb{R}$, the estimate 3.11 implies that

$$|U_1(x,t,z)| \le M \text{ for } \ x \in \overline{\Omega}, \ t \in [0,T], \ z \in \mathbb{R}, \tag{3.28}$$

for some $M > 0$. Similarly, since $\nabla U_1$ is a solution of 3.9 with

$$A = \nabla_x A_0(x,t,z) := \nabla_x (g(x,t,U_0(z)) - \gamma(x,t)U_0'(z)),$$

and since g is assumed to be $C^1$ in x, we obtain

$$|\nabla_x U_1(x,t,z)| \le M \text{ for } \ x \in \overline{\Omega}, \ t \in [0,T], \ z \in \mathbb{R}, \tag{3.29}$$

for some $M > 0$. To obtain estimates $z \to \pm\infty$, we observe that Lemma 3.1 implies

$$A_0(x,t,z) - g(x,t,\alpha_\pm) = O(e^{-\lambda|z|}) \text{ as } \ z \to \pm\infty, \tag{3.30}$$

uniformly in $x \in \overline{\Omega}, t \in [0,T]$.

The following Lemma is the result from the two previous Lemmas.

**Lemma 3.3.** *Let the assumptions of Lemma 3.2 hold, and assume further that* $A(z) - A^{\pm} = O(e^{-\delta|z|})$ *as* $z \to \pm\infty$ *for some constants* $A^+$, $A^-$ *and* $\delta > 0$. *Then there exists a constant* $\lambda > 0$ *such that*

$$\psi(z) - \frac{A^{\pm}}{f'(\alpha_{\pm})} = O(e^{-\lambda|z|}), \quad |\psi'(z)| + |\psi''(z)| = O(e^{-\lambda|z|}), \tag{3.31}$$

*as* $z \to \pm\infty$.

From the above lemma and 3.30 we obtain the estimate

$$|U_{1z}(x,t,z)| + |U_{1zz}(x,t,z)| \leq Ce^{-\lambda|z|}, \tag{3.32}$$

for $x \in \overline{\Omega}$, $t \in [0,t]$, $z \in \mathbb{R}$. Similarly, since the definition of $A_0$ 3.27 and estimate 3.7 imply

$$(\nabla_x A_0)(x,t,z) - (\nabla_x g)(x,t,\alpha_{\pm}) = O(e^{-\lambda|z|}) \text{ as } \ z \to \pm\infty,$$

we can apply Lemma 3.3 to $\psi = \nabla_x U_1$, to obtain

$$|\nabla_x U_{1z}(x,t,z)| + |\nabla_x U_{1zz}(x,t,z)| \leq Ce^{-\lambda|z|},$$

for $x \in \overline{\Omega}$, $t \in [0,t]$, $z \in \mathbb{R}$. As a consequence, there is a constant, which we denote again by M, such that

$$|\nabla_x U_{1z}(x,t,z)| \leq M. \tag{3.33}$$

Eventually, we consider that 2.8 implies

$$\frac{\partial}{\partial \nu} A_0 = \frac{\partial}{\partial \nu}[g(x,t,U_0(z)) - \gamma(x,t)U_0'(z)] = 0 \text{ on } \ \partial\Omega \times [0,T] \times \mathbb{R}. \tag{3.34}$$

Consequently, from the expression 3.14 we get for the solution $U_1$ the corresponding expression

$$U_1(x,t,z) = U_0'(z) \int_0^z ((U_0'(\zeta))^{-2} \int_{-\infty}^{\zeta} A_0(x,t,\xi)U_0'(\xi)d\xi)d\zeta,$$

and we observe that

$$\frac{\partial U_1}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times [0,T] \times \mathbb{R}. \tag{3.35}$$

# 4 Generation of interface: special case

The aim of the present section is to study the limited behavior of the solution $u^\varepsilon$ of Problem $(P^\varepsilon)$ as $\varepsilon \to 0$. Our first main result describes the profile of the solution after a short initial period. It asserts that: given an arbitrary initial datum $u_0$, the solution $u^\varepsilon$ quickly becomes close to $\alpha_\pm$, except in a small neighborhood of the initial interface $\Gamma_0$, creating a steep transition layer around $\Gamma_0$ (**generation of interface**). The time needed to develop such a transition layer, which we will denote by $t^\varepsilon$, is of order $\varepsilon^2|ln\varepsilon|$. The theorem then states that the solution $u^\varepsilon$ remains close to the step function $\widetilde{u}$ on the time interval $[t^\varepsilon$, T] (**motion of interface**), in other words, the motion of transition layer is well approximated by the limit of interface equation and $\Gamma_t|_{t=0} = \Gamma_0$. For the time being, we focus on the special case where $g^\varepsilon = 0$. Furthermore, $\eta_0$ will stand for the following quantity:

$$\eta_0 := \frac{1}{2}\min(\alpha - \alpha_-, \alpha_+ - \alpha).$$

Our main result is the following.

**Theorem 4.1.** *Let $\eta \in (0, \eta_0)$ be arbitrary constant and define $\mu$ as the derivative of f(u) at the unstable equilibrium $u = \alpha$, that is*

$$\mu = f'(\alpha). \tag{4.1}$$

*Then there exist positive constants $\varepsilon_0$ and $M_0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $t^\varepsilon \le t \le T$, where $t^\varepsilon := \mu^{-1}\varepsilon^2|\ln\varepsilon|$,*

- *for all $x \in \Omega$,*

$$\alpha_- - \eta \le u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \le \alpha_+ + \eta, \tag{4.2}$$

- *for all $x \in \Omega$ such that $|u_0(x) - \alpha| \ge M_0\varepsilon$, we have that*

$$\text{if } u_0(x) \ge \alpha + M_0\varepsilon \quad \text{then } u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \ge \alpha_+ - \eta, \tag{4.3}$$

$$\text{if } u_0(x) \le \alpha - M_0\varepsilon \quad \text{then } u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \le \alpha_- + \eta. \tag{4.4}$$

## 4.1 The bistable ordinary differential equation

To prove these results for the generation of interface and the motion of interface that we will see in section 5, we construct two completely different pairs of sub- and super- solutions: one for the generation of interface and the other for the motion of interface. Fitting these pairs of sub- and super- solutions into each other, we estimate, in an optimal way, the thickness of transition layer, and its location. More precisely, the above sub- and super- solutions are constructed for the generation of interface by modifying the solution of the equation in the absence of diffusion term: $u_t = \varepsilon^{-2} f(u)$. We have the corresponding problem:

$$\overline{u}_t = \frac{1}{\varepsilon^2} f(\overline{u}), \;\; \overline{u}(x,0) = u_0(x). \tag{4.5}$$

This solution is written in the form

$$\overline{u}(x,t) = Y\left(\frac{t}{\varepsilon^2}, u_0(x)\right), \tag{4.6}$$

where $\mathrm{Y}(\tau, \xi)$ denotes the solution of the ordinary differential equation

$$\begin{cases} Y_\tau(\tau,\xi) = f(Y(\tau,\xi)) & \text{for} \;\; \tau > 0, \\ Y(0,\xi) = \xi. \end{cases} \tag{4.7}$$

Here $\xi$ ranges over the interval $(-2C_0, 2C_0)$, with $C_0$ being the constant defined in 2.11. We study the basic properties of Y to prove the previous Theorem.

**Lemma 4.2.** *We have $Y_\xi(\tau,\xi) > 0$, for all $\;\xi \in (-2C_0, 2C_0) \setminus \{\alpha_-, \alpha, \alpha_+\}$ and all $\tau > 0$. Furthermore,*

$$Y_\xi(\tau,\xi) = \frac{f(Y(\tau,\xi))}{f(\xi)}. \tag{4.8}$$

*Proof.* First, differentiating equation 4.7 with respect to $\xi$, we obtain

$$\begin{cases} Y_{\xi\tau} = Y_\xi f'(Y), \\ Y_\xi(0,\xi) = 1, \end{cases}$$

which can be integrated:

$$Y_\xi(\tau,\xi) = \exp\left[\int_0^\tau f'(Y(s,\xi))ds\right] > 0, \tag{4.9}$$

since $\tau > 0$. Then we differentiate equation 4.7 with respect to $\tau$ and we obtain

$$\begin{cases} Y_{\tau\tau} = Y_\tau f'(Y), \\ Y_\tau(0,\xi) = f(\xi), \end{cases}$$

which can be integrated:

$$\begin{aligned} Y_\tau(\tau,\xi) = f(\xi) \exp\left[\int_0^\tau f'(Y(s,\xi))ds\right] \\ = f(\xi) Y_\xi(\tau,\xi) \end{aligned}$$

and by 4.7, we take

$$Y_\xi(\tau,\xi) = \frac{f(Y(\tau,\xi))}{f(\xi)}.$$

□

We define a function $A(\tau,\xi)$ by

$$A(\tau,\xi) = \frac{f'(Y(\tau,\xi) - f'(\xi)}{f(\xi)}. \tag{4.10}$$

**Lemma 4.3.** *We have, for all* $\xi \in (-2C_0, 2C_0) \setminus \{\alpha_-, \alpha, \alpha_+\}$ *and all* $\tau > 0$,

$$A(\tau,\xi) = \int_0^\tau f''(Y(s,\xi))Y_\xi(s,\xi)ds. \tag{4.11}$$

*Proof.* Differentiating the equality of Lemma 4.2 4.8 with respect to $\xi$ leads to

$$Y_{\xi\xi} = \frac{f'(Y(\tau,\xi))Y_\xi f(\xi) - f(Y(\tau,\xi))f'(\xi)}{f^2(\xi)}$$

$$= \frac{f'(Y(\tau,\xi))Y_\xi f(\xi) - Y_\xi f(\xi) f'(\xi)}{f^2(\xi)}$$

$$= \frac{f'(Y(\tau,\xi)) - f'(\xi)}{f(\xi)} Y_\xi = A(\tau,\xi)Y_\xi,$$

whereas differentiating 4.9 with respect to $\xi$ yields

$$Y_{\xi\xi} = Y_\xi \int_0^\tau f''(Y(s,\xi))Y_\xi(s,\xi)ds.$$

Then,

$$A(\tau,\xi)Y_\xi = Y_\xi \int_0^\tau f''(Y(s,\xi))Y_\xi(s,\xi)ds,$$

and finally

$$A(\tau,\xi) = \int_0^\tau f''(Y(s,\xi))Y_\xi(s,\xi)ds.$$

□

Now we see some estimates on the growth of Y, A and their derivatives. We first consider the case where the initial value $\xi$ is far from the stable equilibria, more precisely when it lies between $\alpha_- + \eta$ and $\alpha_+ - \eta$.

**Lemma 4.4.** *Let $\eta \in (0, \eta_0)$ be arbitrary. Then there exist positive constants $\widetilde{C}_1 = \widetilde{C}_1(\eta)$, $\widetilde{C}_2 = \widetilde{C}_2(\eta)$ and $C_3 = C_3(\eta)$ such that, for all $\tau > 0$,*

- *if $\xi \in (\alpha, \alpha_+ - \eta)$ then, for every $\tau > 0$ such that $Y(\tau, \xi)$ remains in the interval $(\alpha, \alpha_+ - \eta)$, we have*
$$\widetilde{C}_1 e^{\mu\tau} \leq Y_\xi(\tau, \xi) \leq \widetilde{C}_2 e^{\mu\tau}, \tag{4.12}$$
*and*
$$\mid A(\tau, \xi) \mid \leq C_3(e^{\mu\tau} - 1), \tag{4.13}$$
- *if $\xi \in (\alpha_- + \eta, \alpha)$ then, for every $\tau > 0$ such that $Y(\tau, \xi)$ remains in the interval $(\alpha_- + \eta, \alpha)$ and the above 4.12, 4.13 hold as well, where $\mu$ is the constant that defined as $\mu = f'(\alpha)$.*

*Proof.* We take $\xi \in (\alpha, \alpha_+ - \eta)$ and suppose that for $s \in (0, \tau)$, $Y(s, \xi)$ remains in the interval $(\alpha, \alpha_+ - \eta)$. Integrating the equality 4.7

$$Y_\tau(s, \xi) = f(Y(s, \xi)) \Rightarrow \frac{Y_\tau(s, \xi)}{f(Y(s, \xi))} = 1$$

from 0 to $\tau$ yields

$$\int_0^\tau \frac{Y_\tau(s, \xi)}{f(Y(s, \xi))} ds = \tau.$$

By the change of variable $q = Y(s, \xi)(dq = Y(s, \xi)ds, s \in (0, \tau) \Rightarrow$ the integral now ranges from $\xi$ to $Y(\tau, \xi))$, and we get

$$\int_\xi^{Y(\tau,\xi)} \frac{dq}{f(q)} = \tau. \tag{4.14}$$

Moreover, the equality of Lemma 4.2, 4.8 leads to

$$\ln Y_\xi(\tau, \xi) = \int_\xi^{Y(\tau,\xi)} \frac{f'(q)}{f(q)} dq$$

$$= \int_\xi^{Y(\tau,\xi)} \frac{f'(\alpha) + f'(q) - f'(\alpha)}{f(q)} dq$$

$$= \int_\xi^{Y(\tau,\xi)} \left[ \frac{f'(\alpha)}{f(q)} + \frac{f'(q) - f'(\alpha)}{f(q)} \right]$$

$$= \mu\tau + \int_\xi^{Y(\tau,\xi)} h(q) dq,$$

since from 4.14 and where $h(q) = \frac{f'(q)-\mu}{f(q)}$.

And from means value theorem $h(q) \to \frac{f''(\alpha)}{f'(\alpha)}$ as $q \to \alpha$, h is continuous on $[\alpha, \alpha_+ - \eta]$. Hence we can define

$$H = H(\eta) :=\| h \|_{L^\infty(\alpha,\alpha_+-\eta)} .$$

Since $| Y(\tau,\xi) - \xi |$ takes its values in the interval $[0, \alpha + -\alpha - \eta] \subset [0, \alpha_+ - \alpha]$. It follows from the above integrals that

$$\mu\tau - H(\alpha_+ - \alpha) \leq \ln Y_\xi(\tau,\xi) \leq \mu\tau + H(\alpha_+ - \alpha),$$

gives

$$\widetilde{C}_1 e^{\mu\tau} \leq Y_\xi(\tau,\xi) \leq \widetilde{C}_2 e^{\mu\tau}.$$

Next by Lemma 4.3 and the above estimate yield

$$| A(\tau,\xi) |=| \int_0^\tau f''(Y(s,\xi))Y_\xi(s,\xi)ds |$$

$$\leq \sup_{z\in[\alpha_-,\alpha_+]} | f''(z) | \int_0^\tau \widetilde{C}_2 e^{\mu s} ds$$

$$\leq C^3(e^{\mu\tau} - e^0) \leq C^3(e^{\mu\tau} - 1),$$

which completes the proof. The case where $\xi$ and $Y(\tau,\xi)$ are in $(\alpha_- + \eta, )\alpha$ is similar. □

**Corollary 4.4.1.** *Let $\eta \in (0, \eta_0)$ be arbitrary. Then there exist positive constants $C_1 = C_1(\eta)$ and $C_2 = C_2(\eta)$ such that, for all $\tau$,*

- *if $\xi \in (\alpha, \alpha_+ - \eta)$ then, for every $\tau$ such that $Y(\tau,\xi)$ remains in the interval $(\alpha, \alpha_+ - \eta)$, we have*

$$C_1 e^{\mu\tau}(\xi - \alpha) \leq Y(\tau,\xi) - \alpha \leq C_2 e^{\mu\tau}(\xi - \alpha), \tag{4.15}$$

- *if $\xi \in (\alpha_- + \eta, \alpha)$ then, for every $\tau > 0$ such that $Y(\tau,\xi)$ remains in the interval $(\alpha_- + \eta, \alpha)$, we have*

$$C_2 e^{\mu\tau}(\xi - \alpha) \leq Y(\tau,\xi) - \alpha \leq C_1 e^{\mu\tau}(\xi - \alpha). \tag{4.16}$$

*Proof.* Since

$$\frac{f(q) - f(\alpha)}{(q-\alpha)} = \frac{f(q)}{(q-\alpha)} \to f'(\alpha) = \mu \text{ as } \ \mu \to \alpha,$$

by means value theorem and $f(u) = 0$ at $u = \alpha_-, \alpha, \alpha_+$. It is possible to find constants $B_1 = B_1(\eta) > 0$ and $B_2 = B_2(\eta) > 0$ such that, for all $q \in (\alpha, \alpha_+ - \eta)$,

$$B_1(q - \alpha) \leq f(q) \leq B_2(q - \alpha). \tag{4.17}$$

We write this inequality for $\alpha \leq Y(\tau, \xi) \leq \alpha_+ - \eta$ to obtain

$$B_1(Y(\tau, \xi) - \alpha) \leq f(Y(\tau, \xi)) \leq B_2(Y(\tau, \xi) - \alpha).$$

We also write this inequality for $\alpha \leq \xi \leq \alpha_+ - \eta$ to obtain

$$B_1(\xi - \alpha) \leq f(\xi) \leq B_2(\xi - \alpha).$$

Next we use the equality of Lemma 4.2 to conclude that

$$\frac{B_1}{B_2}(Y(\tau, \xi) - \alpha) \leq (\xi - \alpha) Y_\xi(\tau, \xi) \leq \frac{B_2}{B_1}(Y(\tau, \xi) - \alpha),$$

which, in view of 4.12, implies that

$$\frac{B_1}{B_2}\widetilde{C}_1 e^{\mu\tau}(\xi - \alpha) \leq Y(\tau, \xi) - \alpha \leq \frac{B_2}{B_1}\widetilde{C}_2 e^{\mu\tau}(\xi - \alpha),$$

since $\int_\alpha^\xi Y_s(\tau, s) ds = Y(\tau, \xi) - Y(\tau, \alpha) = Y(\tau, \xi) - \alpha$. This proves the first estimate, the second is similar. □

We now present estimates in the case where the initial value $\xi$ is smaller than $\alpha_- + \eta$ or larger than $\alpha_+ - \eta$.

**Lemma 4.5.** *Let $\eta \in (0, \eta_0)$ and $M > 0$ be arbitrary. Then there exists a positive constant $C_4 = C_4(\eta, M)$ such that*

- *if $\xi \in [\alpha_+ - \eta, \alpha_+ + M]$, then, for all $\tau > 0$, $Y(\tau, \xi)$ remains in the interval $[\alpha_+ - \eta, \alpha_+ + M]$ and*

$$| A(\tau, \xi) | \leq C_4 \tau \quad \text{for} \quad \tau > 0, \tag{4.18}$$

- *if $\xi \in [\alpha_- - M, \alpha_- + \eta]$, then, for all $\tau > 0$, $Y(\tau, \xi)$ remains in the interval $\xi \in [\alpha_- - M, \alpha_- + \eta]$ and the above estimate hold as well.*

Now we choose the constant M in the above lemma sufficiently large so that $[-2C_0, 2C_0] \subset [\alpha_- - M, \alpha_+ + M]$, and fix M hereafter. Then $C_4$ only depends on $\eta$. Using the fact that $\tau = O(e^{\mu\tau} - 1)$ for $\tau > 0$, one can easily deduce from 4.13 and 4.18 the following general estimate.

**Lemma 4.6.** *Let $\eta \in (0, \eta_0)$ be arbitrary and let $C_0$ be the constant defined in 2.11. Then there exists positive constant $C_5 = C_5(\eta)$ such that, for all $\xi \in (-2C_0, 2C_0)$ and all $\tau > 0$,*

$$| A(\tau, \xi) | \leq C_5(e^{\mu\tau} - 1) \tag{4.19}$$

## 4.2 Construction of sub- and super- solutions

Now we construct the sub- and super- solutions for the study of generation of interface. For simplicity, we first consider the case where

$$\frac{\partial u_0}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega. \tag{4.20}$$

In this case, our sub- and super- solutions are given by

$$w_\varepsilon^\pm(x,t) = Y(\frac{t}{\varepsilon^2}, u_0(x) \pm \varepsilon^2 C_6(e^{\mu t/\varepsilon^2} - 1)). \tag{4.21}$$

In the general case where 4.20 does not necessarily hold, we have to slightly modify $w_\varepsilon^\pm(x,t)$ near the boundary $\partial\Omega$, which we see later.

**Lemma 4.7.** *Assume 4.20. Then there exist positive constants $\varepsilon_0$ and $C_6$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, $(w_\varepsilon^-, w_\varepsilon^+)$, is a pair of sub- and super- solutions for Problem ($P^\varepsilon$), in the domain*

$$\{ \ (x,t) \in Q_T, \ x \in \Omega, \ 0 \le t \le \mu^{-1}\varepsilon^2 \mid \ln \varepsilon \mid \ \},$$

*satisfying $w_\varepsilon^-(x,0) = w_\varepsilon^+(x,0) = u_0(x)$. Consequently*

$$w_\varepsilon^-(x,t) \le u^\varepsilon(x,t) \le w_\varepsilon^+(x,t) \quad \textit{for} \ \ x \in \overline{\Omega}, \ 0 \le t \le \mu^{-1}\varepsilon^2 \mid \ln \varepsilon \mid \tag{4.22}$$

*Proof.* The assumption 4.20 implies

$$\frac{\partial w_\varepsilon^\pm}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega \times (0, +\infty).$$

Now we define the operator $L_0$ by

$$L_0 u := u_t - \Delta u - \frac{1}{\varepsilon^2} f(u),$$

since $g^\varepsilon \equiv 0$ and we prove that $L_0 w_\varepsilon^+ \ge 0$. Straightforward computations yield

$$L_0 w_\varepsilon^+ = \frac{1}{\varepsilon^2} Y_\tau + C_6 \mu e^{\mu t/\varepsilon^2} Y_\xi - \Delta u_0 Y_\xi - \mid \nabla u_0 \mid^2 Y_{\xi\xi} - \frac{1}{\varepsilon^2} f(Y),$$

since from

$$(w_\varepsilon^+)_t = \frac{1}{\varepsilon^2} Y_\tau + \varepsilon^2 C_6 \left(\frac{\mu\tau}{\varepsilon^2}\right)' e^{\mu\tau/\varepsilon^2},$$

$$\Delta w_\varepsilon^+ = \Delta u_0 Y_\xi + \mid \nabla u_0 \mid^2 Y_{\xi\xi},$$

$$\frac{1}{\varepsilon^2} f(w_\varepsilon^+) = \frac{1}{\varepsilon^2} f(Y),$$

where $w_\varepsilon^+(x,t) = Y(\frac{t}{\varepsilon^2}, u_0(x) + \varepsilon^2 C_6(e^{\mu\tau/\varepsilon^2} - 1))$, for $\tau = \frac{t}{\varepsilon^2}$ and $\xi = u_0(x) + \varepsilon^2 C_6(e^{\mu t/\varepsilon^2} - 1)$. Therefore, in view of 4.7 we have that $Y_\tau(\tau,\xi) = f(Y(\tau,\xi))$ and we obtain

$$L_0 w_\varepsilon^+ = \left[ C_6 \mu e^{\mu t/\varepsilon^2} - \Delta u_0 - \mid \nabla u_0 \mid^2 \frac{Y_{\xi\xi}}{Y_\xi} \right] Y_\xi$$

We note that, the range $0 \leq t \leq \mu^{-1}\varepsilon^2 \mid \ln \varepsilon \mid$, we have, for $\varepsilon_0$ sufficiently small,

$$0 \leq \varepsilon^2 C_6(e^{\mu t/\varepsilon^2} - 1) \leq \varepsilon^2 C_6(\varepsilon^{-1} - 1) \leq C_0,$$

where $C_0$ is the constant that defined in 2.11. Hence,

$$\xi := u_0(x) \pm \varepsilon^2 C_6(e^{\mu t/\varepsilon^2} - 1) \ \in \ (-2C_0, 2C_0),$$

so that we can use the estimate of Lemma 4.6 and $A = \frac{Y_{\xi\xi}}{Y_\xi}$, then we obtain

$$L_0 w_\varepsilon^+ \geq \left[ C_6 \mu e^{\mu t/\varepsilon^2} - \mid \Delta u_0 \mid - C_5(e^{\mu t/\varepsilon^2} - 1) \mid \nabla u_0 \mid^2 \right] Y_\xi$$

$$\geq \left[ (C_6\mu - C_5 \mid \nabla u_0 \mid^2) e^{\mu t/\varepsilon^2} - \mid \Delta u_0 \mid - C_5 \mid \nabla u_0 \mid^2 \right] Y_\xi.$$

Since $Y_\xi > 0$, this inequality implies that, for $C_6$ large enough,

$$L_0 w_\varepsilon^+ \geq \left[ C_6\mu - C_5 C_0^2 - C_0 \right] Y_\xi \geq 0.$$

Hence $w_\varepsilon^+$ is a super solution for the problem $(P^\varepsilon)$. Similarly $w_\varepsilon^-$ is a sub- solution. Obviously,

$$w_\varepsilon^-(x,0) = w_\varepsilon^+(x,0) = Y(0, u_0(x)) = u_0(x).$$

□

In the more general case where 4.20 is not necessarily valid, one can proceed as follows, we have defined "the initial interface" $\Gamma_0$ by

$$\Gamma_0 := \{ \, x \in \Omega, u_0(x) = \alpha \, \},$$

and having supposed that $\Gamma_0$ is a $C^{3+\theta}$ hypersurface without boundary such that, n being the outward unit normal vector to $\Gamma_0$,

$$\begin{array}{ll} \Gamma_0 \subset\subset \Omega \ \text{ and } \ \nabla u_0(x) \cdot n(x) \neq 0 & \text{if } \ x \in \Gamma_0, \\ u_0 > \alpha \text{ in } \ \Omega_0^+, \ u_0 < \alpha & \text{in } \ \Omega_0^-, \end{array}$$

where $\Omega_0^-$ denotes the region enclosed by the hypersurface $\Gamma_0$ and $\Omega_0^+$ the region enclosed between the boundary of the domain $\partial\Omega$ and the hypersurface $\Gamma_0$. Then there exist positive constants $d_1, \rho$ such that $u_0(x) \geq \alpha + \rho$ if $d(x, \partial\Omega) \leq d_1$. Let $\chi$ be a smooth cut- off function defined on $[0, +\infty)$ such that $0 \leq \chi \leq 1$, $\chi(0) = \chi'(0) = 0$ and $\chi(z) = 1$ for $z \geq d_1$. Then we define

$$u_0^+(x) = \chi(d(x, \partial\Omega))u_0(x) + [1 - \chi(d(x, \partial\Omega))] \max_{x \in \overline{\Omega}} u_0(x), \tag{4.23}$$

$$u_0^-(x) = \chi(d(x, \partial\Omega))u_0(x) + [1 - \chi(d(x, \partial\Omega))](\alpha + \rho) \tag{4.24}$$

Clearly, $u_0^- \leq u_0 \leq u_0^+$, and both $u_0^-$ and $u_0^+$ satisfy 4.20. Now we set

$$w_\varepsilon^\pm(x,t) = Y\left(\frac{t}{\varepsilon^2}, u_0^\pm(x) \pm \varepsilon^2 C_6(e^{\mu t/\varepsilon^2} - 1)\right). \tag{4.25}$$

Then the same argument as in Lemma 4.7 shows that $(\widetilde{w}_\varepsilon^-, \widetilde{w}_\varepsilon^+)$ is a pair of sub- and super- solutions for the Problem $(P^\varepsilon)$. Furthermore, since $\widetilde{w}_\varepsilon^-(x,0) = u_0^-(x) \leq u_0(x) \leq u_0^+(x) = \widetilde{w}_\varepsilon^+(x,0)$, the comparison principle asserts that

$$\widetilde{w}_\varepsilon^-(x,t) \leq u^\varepsilon(x,t) \leq \widetilde{w}_\varepsilon^+(,t) \text{ for } x \in \overline{\Omega}, 0 \leq t \leq \mu^{-1}\varepsilon^2 \mid \ln \varepsilon|. \tag{4.26}$$

## 4.3 Proof of Theorem 4.1

To prove Theorem 4.1 we first present a key estimate on the function $Y(\tau, \xi)$ after a time interval of order $\tau \sim\mid \ln \varepsilon \mid$.

**Lemma 4.8.** *Let $\eta \in (0, \eta_0)$ be arbitrary, there exist positive constants $\varepsilon_0$ and $C_7$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

- *for all $\xi \in (-2C_0, 2C_0)$ ,*

$$\alpha_- - \eta \leq Y(\mu^{-1} \mid \ln \varepsilon \mid, \xi) \leq \alpha_+ + \eta, \tag{4.27}$$

- *for all $\xi \in (-2C_0, 2C_0)$ such that $\mid \xi - \alpha \mid \geq C_7\varepsilon$, we have that*

$$\textit{if } \ \xi \geq \alpha + C_7\varepsilon \qquad \textit{then } \ Y(\mu^{-1} \mid \ln \varepsilon \mid, \xi) \geq \alpha_+ - \eta \tag{4.28}$$

$$\textit{if } \ \xi \leq \alpha - C_7\varepsilon \qquad \textit{then } \ Y(\mu^{-1} \mid \ln \varepsilon \mid, \xi) \leq \alpha_- + \eta. \tag{4.29}$$

*Proof.* We first prove 4.28. For $\xi \geq \alpha + C_7\varepsilon$, as long as $Y(\tau, \xi)$ has not reached $\alpha_+ - \eta$, we can use 4.15 to deduce that

$$\begin{aligned} Y(\tau, \xi) &\geq \alpha + C_1 e^{\mu\tau}(\xi - \alpha) \\ &\geq \alpha + C_1 C_7 e^{\mu\tau}\varepsilon \\ &\geq \alpha_+ - \eta, \end{aligned}$$

since $(\xi - \alpha) \geq C_7\varepsilon$, solving with respect to $\tau$ in previous estimate and we have

$$\alpha + C_1 C_7 e^{\mu\tau}\varepsilon \geq \alpha_+ - \eta, \tag{4.30}$$

$$e^{\mu\tau} \geq \frac{\alpha_+ - \alpha - \eta}{C_1 C_7 \varepsilon}, \tag{4.31}$$

$$\tau \geq \mu^{-1} \ln \frac{\alpha_+ - \alpha - \eta}{C_1 C_7 \varepsilon}, \tag{4.32}$$

then we define $\tau^\varepsilon$ and provided that $\tau$ satisfies

$$\tau \geq \tau^\varepsilon =: \mu^{-1} \ln \frac{\alpha_+ - \alpha - \eta}{C_1 C_7 \varepsilon}.$$

Choosing

$$C_7 = \frac{\max(\alpha - \alpha_-, \alpha_+ - \alpha) - \eta}{C_1},$$

we see that $\mu^{-1} \mid \ln\varepsilon \mid \geq \tau^\varepsilon$, which completes the proof of 4.28.
Now for 4.29 we use the second estimate of Corollary, the 4.16, and get

$$\begin{aligned} Y(\tau,\xi) &\leq \alpha + C_1 e^{\mu\tau}(\xi - \alpha) \\ &\leq \alpha + C_1 C_7 e^{\mu\tau}\varepsilon \\ &\leq \alpha_- + \eta, \end{aligned}$$

and similarly

$$\alpha + C_1 C_7 e^{\mu\tau}\varepsilon \leq \alpha_- + \eta,$$

$$e^{\mu\tau} \leq \frac{\alpha_- - \alpha + \eta}{C_1 C_7 \varepsilon},$$

$$\tau \leq \mu^{-1} \ln \frac{\alpha_- - \alpha + \eta}{C_1 C_7 \varepsilon},$$

provided that $\tau$ satisfies

$$\tau \leq \tau^\varepsilon =: \mu^{-1} \ln \frac{\alpha_- - \alpha + \eta}{C_1 C_7 \varepsilon}.$$

Now choosing

$$C_7 = \frac{\min(\alpha - \alpha_-, \alpha_+ - \alpha) + \eta}{C_1},$$

and from $\mu^{-1} \mid \ln\varepsilon \mid \leq \tau^\varepsilon$, which completes the proof of 4.29.
Next we prove the 4.27. First by the bistable assumptions of f, if we leave from a initial value $\xi \in [\alpha_- - \eta, \alpha_+ + \eta]$ then $Y(\tau, \xi)$ will remain in $[\alpha_- - \eta, \alpha_+ + \eta]$.

Now suppose that $\alpha_+ + \eta \leq \xi \leq 2C_0$. We check bellow that $Y(\mu^{-1} \mid \ln\varepsilon \mid, \xi) \leq \alpha_+ + \eta$. First we recall that valid the following

$$f'(\alpha_\pm) < 0,\ f'(\alpha) > 0,\ \int_{\alpha_-}^{\alpha_+} f(u)du = 0$$

and we can find $p > 0$ such that

$$\text{if}\ \ \alpha_+ \leq u \leq 2C_0 \qquad \text{then}\ \ f(u) \leq p(\alpha_+ - u) \tag{4.33}$$

$$\text{if}\ \ 2C_0 \leq u \leq \alpha_- \qquad \text{then}\ \ f(u) \geq -p(u - \alpha_-). \tag{4.34}$$

We then use the 4.7 to obtain, as long as $\alpha_+ + \eta \leq Y \leq 2C_0$, the inequality

$$Y_\tau \leq f(Y) \leq p(\alpha_+) - Y.$$

It follows that

$$\frac{Y_\tau}{Y - \alpha_+} \leq -p.$$

Integrating this inequality from 0 to $\tau$ leads to

$$\int_0^\tau \frac{Y_\tau}{Y - \alpha_+} ds \leq -\int_0^\tau p ds,$$

$$[\ln(Y(s,\xi)) - \alpha_+]_0^\tau \leq -p\tau,$$

$$\ln\left(\frac{Y(\tau,\xi) - \alpha_+}{Y(0,\xi) - \alpha_+}\right) \leq -p\tau,$$

$Y(0,\xi) = \xi$ from the initial condition of 4.7 and we get

$$\ln\left(\frac{Y(\tau,\xi) - \alpha_+}{\xi - \alpha_+}\right) \leq -p\tau,$$

finally

$$Y(\tau,\xi) \leq \alpha_+ + (\xi - \alpha_+)e^{-p\tau}$$

$$\leq \alpha_+ + (2C_0 - \alpha_+)e^{-p\tau}.$$

Since $(2C_0 - \alpha_+)e^{-p\mu^{-1}|\ln\varepsilon|} = (2C_0 - \alpha_+)e^{-\infty} \to 0$ as $\varepsilon \to 0$, where $\lim_{x\to 0^+} x = -\infty$, the above inequality proves that, for $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 = \varepsilon_0(\eta)$ sufficiently small and we obtain

$$Y(\mu^{-1} \mid \ln\varepsilon \mid, \xi) \leq \alpha_+ + \eta,$$

which completes the proof of 4.27. □

Now we are ready to prove Theorem 4.1. By setting $\tau = \mu^{-1}\varepsilon^2 \mid \ln \varepsilon \mid$ in 4.26, we obtain

$$Y(\mu^{-1} \mid \ln \varepsilon \mid, u_0^-(x) - (C_6\varepsilon - C_6\varepsilon^2)) \tag{4.35}$$
$$\leq u^\varepsilon(x, \mu^{-1}\varepsilon^2 \mid \ln \varepsilon \mid) \leq Y(\mu^{-1} \mid \ln \varepsilon \mid, u_0^+(x) + C_6\varepsilon - C_6\varepsilon^2). \tag{4.36}$$

Furthermore, by the definition of $C_0$, we have, for $\varepsilon_0$ small enough,

$$-2C_0 \leq u_0^{\pm} \pm (C_6\varepsilon - C_6\varepsilon^2) \leq 2C_0, \text{ for } x \in \Omega. \tag{4.37}$$

Thus the assertion 4.2 of Theorem 4.1 is direct consequence of 4.27 and 4.36. Next we prove the second assertion of the Theorem. We choose $M_0$ large enough so that $M_0\varepsilon - C_6\varepsilon + C_6\varepsilon^2 \leq C_7\varepsilon$. Then for any $x \in \Omega$ such that $u_0^-(x) \geq \alpha + M_0\varepsilon$, we have

$$u_0^-(x) - (C_6\varepsilon - C_6\varepsilon^2) \geq \alpha + M_0\varepsilon - C_6\varepsilon + C_6\varepsilon^2 \geq \alpha_+ + C_7.$$

Combining this, 4.36 and 4.28, we see that

$$u^\varepsilon(x, \mu^{-1}\varepsilon^2 \mid \ln \varepsilon \mid) \geq \alpha_+ - \eta,$$

for any x satisfies $u_0^-(x) \geq \alpha + M_0\varepsilon$ . From the definition of $u_0^-$, it is clear that

$$u_0^-(x) \geq \alpha + M_0\varepsilon \text{ if and only if } u_0(x) \geq \alpha + M_0\varepsilon,$$

provided that $\varepsilon$ is small enough. This proves 4.3. The inequality 4.4 can be shown the same way. This completes the proof of Theorem 4.1.

## 4.4 Optimality of the generation time

To conclude this section we show that the generation time $\tau := \mu^{-1}\varepsilon^2 \mid \ln \varepsilon \mid$ that appears in Theorem 4.1 is optimal. In other words, the interface will not be fully developed much before $t^\varepsilon$.

**Proposition 2.** *Denote by $t^\varepsilon_{\min}$ the smallest time such that for all $\varepsilon \in (0, \varepsilon_0)$ and for constant $C$*

$$u^\varepsilon(x,t) \in \begin{cases} [\alpha_- - \eta, \alpha_+ + \eta] & \text{if } x \in N_{C\varepsilon}(\Gamma_t) \\ [\alpha_- - \eta, \alpha_- + \eta] & \text{if } x \in \Omega_t^- \setminus N_{C\varepsilon}(\Gamma_t) \\ [\alpha_+ - \eta, \alpha_+ + \eta] & \text{if } x \in \Omega_t^+ \setminus N_{C\varepsilon}(\Gamma_t), \end{cases}$$

*where $N_r(\Gamma_t) := \{ x \in \Omega, dist(x, \Gamma_t) < r \}$ denotes the $r$- neighborhood of $\Gamma_t$, holds for all $t \in [t^\varepsilon_{\min}, T]$. Then there exists a constant $b = b(C)$ such that*

$$t^\varepsilon_{\min} \geq \mu^{-1}\varepsilon^2(\mid \ln \varepsilon \mid - b) \text{ for all } \varepsilon \in (0, \varepsilon_0).$$

*Proof.* For simplicity, we deal with the case where

$$\frac{\partial u_0}{\partial \nu} = 0 \text{ on } \partial\Omega$$

is valid. In that case, the 4.22 hold for all small $\varepsilon_0 > 0$. For each $b > 0$, we set

$$t^\varepsilon(b) := \mu^{-1}\varepsilon^2(|\ln\varepsilon| - b),$$

and evaluate $u^\varepsilon(x, t^\varepsilon(b))$ at point $x \in \Omega_0^+$ where $dist(x, \Gamma_0) = C\varepsilon$. Since $u_0 = a$ on $\Gamma_t$ and since $|\nabla u_0| \leq C_0$ by definition of $C_0$, we have

$$u_0(x) \leq \alpha + C_0 C\varepsilon. \tag{4.38}$$

It follows from this and from 4.15 that

$$\begin{aligned} w_\varepsilon^+(x, t^\varepsilon(b)) &= Y(\mu^{-1}(|\ln\varepsilon| - b), u_0(x) + \varepsilon C_6 e^{-b} - \varepsilon^2 C_6) \\ &\leq \alpha + C_2 e^{|\ln\varepsilon| - b}(u_0(x) + \varepsilon C_6 e^{-b} - \varepsilon^2 C_6 - \alpha) \\ &\leq \alpha + C_2\varepsilon^{-1}e^{-b}(C_0 C\varepsilon + \varepsilon C_6 e^{-b}) \\ &= \alpha + C_2 e^{-b}(C_0 C\varepsilon + C_6 e^{-b}). \end{aligned}$$

Now choose b to be sufficiently large, so

$$\alpha + C_2 e^{-b}(C_0 C\varepsilon + C_6 e^{-b}) < \alpha_+ - \eta.$$

Then the above estimate and 4.22 yield

$$u^\varepsilon(x, t^\varepsilon(b)) \leq w_\varepsilon^+(x, t^\varepsilon(b)) < \alpha_+ - \eta.$$

This implies that the $u^\varepsilon(x, t)$ from the assumption of proposition does not hold at $t = t^\varepsilon(b)$, hence $t^\varepsilon(b) = t^\varepsilon_{\min}$. The proposition is proved. □

# 5 Generation of interface in the general case

Our goal in this section is to extend Theorem 4.1 to the case where $g^\varepsilon \neq 0$. The underlying ideas are the same, but we point out the main differences.

## 5.1 The perturbed bistable ordinary differential equation

We first assume a slightly perturbed nonlinearity:

$$f_\delta(u) = f(u) + \delta,$$

where $\delta$ is any constant. For $\mid \delta \mid$ small enough, this function is still bistable. We claim that $f_\delta$ has the following properties.

**Lemma 5.1.** *Let $\delta_0$ be small enough. Then for all $\delta \in (-\delta_0, \delta_0)$,*

- *$f_\delta$ has exactly three zeros, namely $\alpha_-(\delta) < \alpha(\delta) < \alpha_+(\delta)$ and there exists a positive constant $C$ such that*

$$\mid \alpha_-(\delta) - \alpha_- \mid + \mid \alpha(\delta) - \alpha \mid + \mid \alpha_+(\delta) - \alpha_+ \mid \leq \; C \mid \delta \mid . \tag{5.1}$$

- *We have that*

$$f_\delta \;\text{ is strictly positive in }\; (-\infty, \alpha_-(\delta)) \cup (\alpha(\delta), \alpha_+(\delta)), \tag{5.2}$$
$$f_\delta \;\text{ is strictly negative in }\; (\alpha_-(\delta), \alpha(\delta)) \cup (\alpha_+(\delta), +\infty). \tag{5.3}$$

- *Set*

$$\mu(\delta) := f'_\delta(\alpha(\delta)) = f'(\alpha(\delta)),$$

  *then there exists a positive constant, which denote again by $C$, such that*

$$\mid \mu(\delta) - \mu \mid \leq C \mid \delta \mid \tag{5.4}$$

  *or $\mid f'(\alpha(\delta)) - \mu \mid \leq C \mid \delta \mid$.*

We define $Y(\tau, \xi; \delta)$ for each $\delta \in (-\delta_0, \delta_0)$ as the solution of the following ordinary differential equation :

$$\begin{cases} Y_\tau(\tau, \xi; \delta) = f_\delta(Y(\tau, \xi; \delta)) & \text{for} \;\; \tau > 0, \\ Y(0, \xi; \delta) = \xi, \end{cases} \tag{5.5}$$

where $\xi$ varies in $(-2C_0, 2C_0)$ with $C_0$ being the constant that we have defined. To prove Theorem 4.1, we will construct a pair of sub- and super- solutions for $(P^\varepsilon)$ by simply replacing the function $Y(\tau, \xi)$ in 4.21 by $Y(\tau, \xi; \delta)$, with an appropriate choice of $\delta$. We have to check that the basic properties of $Y(\tau, \xi)$

in subsection 4.1 carry over to $Y(\tau,\xi;\delta)$. It is clear that all the differential and integral identities in subsection 4.1 are still valid for 5.5. Particularly, Lemmas 4.2 and 4.3 remain to hold if we replace $Y(\tau,\xi)$ by $Y(\tau,\xi;\delta)$, $f$ by $f_\delta$ and $A(\tau,\xi)$ by $A(\tau,\xi;\delta)$, where

$$A(\tau,\xi;\delta) = \frac{f'_\delta(Y(\tau,\xi;\delta)) - f'_\delta(\xi)}{f_\delta(\xi)}. \tag{5.6}$$

Next we show the basic estimates from section 4 that are also valid for the function $Y(\tau,\xi;\delta)$. The following lemma is fundamental(analogue of 4.4)

**Lemma 5.2.** *Let $\eta \in (0,\eta_0)$ be arbitrary. Then there exist positive constants $\delta_0 = \delta_0(\eta)$, $\widetilde{C}_1 = \widetilde{C}_1(\eta)$, $\widetilde{C}_2 = \widetilde{C}_2(\eta)$ and $C_3 = C_3(\eta)$ such that, for all $\delta \in (-\delta_0,\delta_0)$, for all $\tau > 0$,*

- *if $\xi \in (\alpha(\delta), \alpha_+ - \eta)$ then, for every $\tau > 0$ such that $Y(\tau,\xi;\delta)$ remains in the interval $(\alpha(\delta), \alpha_+ - \eta)$, we have*

$$\widetilde{C}_1 e^{\mu(\delta)\tau} \leq Y_\xi(\tau,\xi;\delta) \leq \widetilde{C}_2 e^{\mu(\delta)\tau}, \tag{5.7}$$

*and*

$$| A(\tau,\xi;\delta) | \leq C_3(e^{\mu(\delta)\tau} - 1), \tag{5.8}$$

- *if $\xi \in (\alpha_- + \eta, \alpha(\delta))$ then for every $\tau > 0$ such that $Y(\tau,\xi;\delta)$ remains in the interval $(\alpha_- + \eta, \alpha(\delta))$, and the above hold as well.*

*Proof.* In view of 5.1, we can choose a small constant $\delta_0 = \delta_0(\eta)$ such that $(\alpha(\delta), \alpha_+ - \eta) \subset (\alpha(\delta), \alpha_+(\delta))$ for every $\delta \in [-\delta_0, \delta_0]$. Therefore $f_\delta(q)$ does not change sign in the interval $(\alpha(\delta), \alpha_+ - \eta)$. We just have to write again the proof of Lemma 4.4, simply replacing $Y(\tau,\xi)$ by $Y(\tau,\xi;\delta)$. Instead, we explain why $\widetilde{C}_1$, $\widetilde{C}_2$ and $C_3$ are independent of $\delta$, in view of the proof of Lemma 4.1, it is sufficient to estimate, for $q \in [\alpha(\delta), \alpha_+ - \eta]$, the modulus of the quantity

$$h_\delta(q) = \frac{f'_\delta(q) - f'_\delta(\alpha(\delta))}{f_\delta(q)}$$

by a constant depending on $\eta$ but not on $\delta \in [-\delta_0, \delta_0]$. Since

$$h_\delta(q) \to \frac{f''_\delta(\alpha(\delta))}{f'_\delta(\alpha(\delta))} = \frac{f''(\alpha(\delta))}{f'(\alpha(\delta))} \quad \text{as} \quad q \to \alpha(\delta),$$

since having used the means value theorem, we also see that the function $(q,\delta) \mapsto h_\delta(q)$ is continuous in the compact region $\{ \mid \delta \mid \leq \delta_0, \ \alpha(\delta) \leq q \leq \alpha_+ - \eta \}$. Then, it follows that $| h_\delta(q) |$ is bounded as $(q,\delta)$ varies in this region. □

**Corollary 5.2.1.** *Let $\eta \in (0, \eta_0)$ be arbitrary. Then there exist positive constants $\delta_0 = \delta_0(\eta)$, $C_1 = C_1(\eta)$ and $C_2 = C_2(\eta)$ such that, for all $\delta \in (-$ $delta_0, \delta_0)$, for all $\tau > 0$,*

- *if $\xi \in (\alpha(\delta), \alpha_+ - \eta)$ then, for every $\tau > 0$ such that $Y(\tau, \xi; \delta)$ remains in the interval $(\alpha(\delta), \alpha_+ - \eta)$, we have*

$$C_1 e^{\mu(\delta)\tau}(\xi - \alpha(\delta)) \leq Y(\tau, \xi; \delta) - \alpha(\delta) \leq C_2 e^{\mu(\delta)\tau}(\xi - \alpha(\delta)), \tag{5.9}$$

- *if $\xi \in (\alpha_- + \eta, \alpha(\delta))$ then, for every $\tau > 0$ such that $Y(\tau, \xi; \delta)$ remains in the interval $(\alpha_- + \eta, \alpha(\delta))$, we have*

$$C_2 e^{\mu(\delta)\tau}(\xi - \alpha(\delta)) \leq Y(\tau, \xi; \delta) - \alpha(\delta) \leq C_1 e^{\mu(\delta)\tau}(\xi - \alpha(\delta)). \tag{5.10}$$

*Proof.* In order to prove that $C_1$ and $C_2$ are independent of $\delta$, all we have to do is to find constants $B_1 = B_1(\eta) > 0$ and $B_2 = B_2(\eta) > 0$ such that, for all $q \in (\alpha(\delta), \alpha_+ - \eta)$,

$$B_1(q - \alpha) \leq f(q) \leq B_2(q - \alpha). \tag{5.11}$$

In view of 5.4, we can choose $\delta_0 > 0$ small enough so that, for all $\delta \in [-\delta_0, \delta_0]$, we have $\mu(\delta) \geq \mu/2 > 0$. Since

$$\frac{f_\delta(q) - f(\alpha(\delta))}{q - \alpha(\delta)} = \frac{f_\delta(q)}{q - \alpha(\delta)} \to f'(\alpha(\delta)) = \mu(\delta) \text{ as } \; q \to \alpha(\delta),$$

by means value theorem and bistability of $f_\delta$. It follows that $(q, \delta) \mapsto f_\delta(q)/(q - \alpha(\delta))$ is a strictly positive and continuous function on the compact region $\{ \mid \delta \mid \leq \delta_0, \; \alpha(\delta) \leq q \leq \alpha_+ - \eta \}$, which insures the existence of the constants $B_1$ and $B_2$ We write this inequality for $\alpha(\delta) \leq Y(\tau, \xi) \leq \alpha_+ - \eta$ to obtain

$$B_1(Y(\tau, \xi) - \alpha(\delta)) \leq f(Y(\tau, \xi)) \leq B_2(Y(\tau, \xi) - \alpha(\delta)).$$

We also write this inequality for $\alpha(\delta) \leq \xi \leq \alpha_+ - \eta$ to obtain

$$B_1(\xi - \alpha(\delta)) \leq f(\xi) \leq B_2(\xi - \alpha(\delta)).$$

Next we use the equality $Y_\xi = f_\delta(Y)/f_\delta(\xi)$ to conclude that

$$\frac{B_1}{B_2}(Y(\tau, \xi) - \alpha(\delta)) \leq (\xi - \alpha(\delta))Y_\xi(\tau, \xi) \leq \frac{B_2}{B_1}(Y(\tau, \xi) - \alpha(\delta)),$$

which, in view of 5.7, implies that

$$\frac{B_1}{B_2}\widetilde{C}_1 e^{\mu\tau}(\xi - \alpha(\delta)) \leq Y(\tau, \xi) - \alpha(\delta) \leq \frac{B_2}{B_1}\widetilde{C}_2 e^{\mu\tau}(\xi - \alpha(\delta)),$$

since $\int_{\alpha(\delta)}^{\xi} Y_s(\tau, s)ds = Y(\tau, \xi) - Y(\tau, \alpha(\delta)) = Y(\tau, \xi) - \alpha(\delta)$. This proves the first estimate, the second is similar. □

Now we establish an analogue of Lemmas 4.5 and 4.6 with constants independent of $\delta$. We claim that:

**Lemma 5.3.** *Let $\eta \in (0, \eta_0)$ and $M > 0$ be arbitrary. Then there exist positive constants $\delta_0 = \delta_0(\eta, M)$ and $C_4 = C_4(\eta, M)$ such that, for all $\delta \in (-\delta_0, \delta_0)$,*

- *if $\xi \in [\alpha_+ - \eta, \alpha_+ + M]$, then, for all $\tau > 0$, $Y(\tau, \xi; \delta)$ remains in the interval $[\alpha_+ - \eta, \alpha_+ + M]$ and*

$$| A(\tau, \xi; \delta) | \leq C_4 \tau \quad \textit{for} \quad \tau > 0, \tag{5.12}$$

- *if $\xi \in [\alpha_- - M, \alpha_- + \eta]$, then, for all $\tau > 0$, $Y(\tau, \xi; \delta)$ remains in the interval $\xi \in [\alpha_- - M, \alpha_- + \eta]$ and the above estimate hold as well.*

Now we choose the constant M in the above lemma sufficiently large so that $[-2C_0, 2C_0] \subset [\alpha_- - M, \alpha_+ + M]$, and fix M hereafter. Then $C_4$ only depends on $\eta$. Using the fact that $\tau = O(e^{\mu\tau} - 1)$ for $\tau > 0$, one can easily deduce from 5.12 and 5.8 the following general estimate.

**Lemma 5.4.** *Let $\eta \in (0, \eta_0)$ be arbitrary and let $C_0$ be the constant defined in 2.11. Then there exist positive constants $\delta_0 = \delta_0(\eta)$ $C_5 = C_5(\eta)$ such that, for all $\delta \in (-\delta_0, \delta_0)$, for all $\xi \in (-2C_0, 2C_0)$ and all $\tau > 0$,*

$$| A(\tau, \xi; \delta) | \leq C_5 (e^{\mu(\delta)\tau} - 1). \tag{5.13}$$

## 5.2 Construction of sub- and super- solutions

We now use $Y(\tau, \xi; \delta)$, the solution of the ordinary differential equation 5.5, to construct the sub- and super- solutions for the study of generation of interface. The same cut-off argument as in subsection 4.2 enable us to assume

$$\frac{\partial u_0}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega.$$

for simplicity. We set our sub- and super- solutions are given by

$$w_\varepsilon^{\pm}(x, t) = Y\left(\frac{t}{\varepsilon^2}, u_0(x) \pm \varepsilon^2 r\left(\pm\varepsilon G, \frac{t}{\varepsilon^2}\right); \pm\varepsilon G\right), \tag{5.14}$$

where the function $r(\delta, \tau)$ is given by

$$r(\delta, \tau) = C_6 (e^{\mu(\delta)\tau} - 1),$$

and the constant G is chosen such that, for all small $\varepsilon > 0$,

$$| g^\varepsilon(x, t, u) | \leq G \text{ for all } \ (x, t, u) \in \overline{\Omega} \times [o, T] \times \mathbb{R},$$

which in view of 2.6, is clearly possible.

**Lemma 5.5.** *Assume 4.20. Then there exist positive constants* $\varepsilon_0$ *and* $C_6$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$, $(w_\varepsilon^-, w_\varepsilon^+)$, *is a pair of sub- and super- solutions for Problem* $(P^\varepsilon)$*, in the domain*

$$\{ \, (x,t) \in Q_T, \; x \in \Omega, \; 0 \le t \le \mu^{-1}\varepsilon^2 \mid \ln \varepsilon \mid \, \},$$

*satisfying* $w_\varepsilon^-(x,0) = w_\varepsilon^+(x,0) = u_0(x)$. *Consequently*

$$w_\varepsilon^-(x,t) \;\le\; u^\varepsilon(x,t) \;\le\; w_\varepsilon^+(x,t) \quad \textit{for} \;\; x \in \overline{\Omega}, \; 0 \le t \le \mu^{-1}\varepsilon^2 \mid \ln \varepsilon \mid \qquad (5.15)$$

*Proof.* The assumption 4.20 implies that $w_\varepsilon^\pm$ satisfy the homogeneous Neumann boundary condition

$$\frac{\partial w_\varepsilon^\pm}{\partial \nu} = 0 \text{ space on } \;\partial\Omega \times (0, +\infty).$$

Now we define the operator $L$ by

$$Lu := u_t - \Delta u - \frac{1}{\varepsilon^2}(f(u) - \varepsilon g^\varepsilon(x,t,u)),$$

and we prove that $Lw_\varepsilon^+ \ge 0$. Straightforward computations yield

$$Lw_\varepsilon^+ = \frac{1}{\varepsilon^2} Y_\tau + Y_\xi \left[ C_6 \mu e^{\mu t/\varepsilon^2} - \Delta u_0 - \mid \nabla u_0 \mid^2 \frac{Y_{\xi\xi}}{Y_\xi} \right] - \frac{1}{\varepsilon^2} f(Y) + \frac{1}{\varepsilon} g^\varepsilon(x,t,Y),$$

since

$$(w_\varepsilon^+)_t = \frac{1}{\varepsilon^2} Y_\tau + \varepsilon^2 C_6 \left( \frac{\mu(\varepsilon G)\tau}{\varepsilon^2} \right)' e^{\mu(\varepsilon G)\tau/\varepsilon^2},$$

$$\Delta w_\varepsilon^+ = \Delta u_0 Y_\xi + \mid \nabla u_0 \mid^2 Y_{\xi\xi},$$

$$\frac{1}{\varepsilon^2} f(w_\varepsilon^+) = \frac{1}{\varepsilon^2} f(Y),$$

$$\frac{1}{\varepsilon} g^\varepsilon(x,t,w_\varepsilon^+) = \frac{1}{g^\varepsilon}(x,t,Y),$$

where $w_\varepsilon^+(x,t) = Y(\frac{t}{\varepsilon^2}, u_0(x) + \varepsilon^2 r(\varepsilon G, \frac{t}{\varepsilon^2}))$, for $\tau = \frac{t}{\varepsilon^2}$ and $\xi = u_0(x) + \varepsilon^2 r(\varepsilon G, \frac{t}{\varepsilon^2})$. Therefore, in view of 5.5 we have that $Y_\tau(\tau,\xi) = f(Y(\tau,\xi)) + \varepsilon G$ and we obtain

$$Lw_\varepsilon^+ = \left[ C_6 \mu(\varepsilon G) e^{\mu(\varepsilon G)t/\varepsilon^2} - \Delta u_0 - \mid \nabla u_0 \mid^2 \frac{Y_{\xi\xi}}{Y_\xi} \right] Y_\xi + \frac{1}{\varepsilon} \left[ g^\varepsilon(x,t,Y) + G \right].$$

We note that $\pm\varepsilon G \in (-\delta_0, \delta_0)$ and that, in the range $0 \leq t \leq \mu^{-1}\varepsilon^2 \mid \ln\varepsilon \mid$, we have, for $\varepsilon_0$ sufficiently small,

$$0 \ \leq \mid \varepsilon^2 C_6(e^{\mu(\pm\varepsilon G)t/\varepsilon^2} - 1) \mid \leq \mid \varepsilon^2 C_6(\varepsilon^{-\mu(\pm\varepsilon G)/\mu} - 1) \ \leq \ C_0,$$

which implies that

$$u_0(x) \pm \varepsilon^2 r(\pm\varepsilon G, \frac{t}{\varepsilon^2})$$

where $C_0$ is the constant that defined in 2.11. Hence,

$$\xi := u_0(x) \pm \varepsilon^2 r(\pm\varepsilon G, \frac{t}{\varepsilon^2}),$$

with $\tau = \frac{t}{\varepsilon}$ and $\delta = \varepsilon G$. By the choice of G the second term is positive. Using the estimate of $A = \frac{Y_{\xi\xi}}{Y_\xi}$ in Lemma 5.4, then we obtain, for a constant $C_5$ that is independent of $\varepsilon$,

$$Lw_\varepsilon^+ \geq \left[C_6\mu(\varepsilon G)e^{\mu(\varepsilon G)t/\varepsilon^2} - \mid \Delta u_0 \mid - C_5(e^{\mu(\varepsilon G)t/\varepsilon^2} - 1) \mid \nabla u_0 \mid^2\right] Y_\xi$$

$$\geq \left[(C_6\mu(\varepsilon G) - C_5 \mid \nabla u_0 \mid^2)e^{\mu(\varepsilon G)t/\varepsilon^2} - \mid \Delta u_0 \mid + C_5 \mid \nabla u_0 \mid^2\right] Y_\xi.$$

In view of 5.4, this inequality implies that, for $\varepsilon \in (0, \varepsilon_0)$ for $C_6$ large enough,

$$Lw_\varepsilon^+ \geq \left[C_6\frac{1}{2}\mu - C_5C_0^2 - C_0\right] Y_\xi \geq 0.$$

Hence $w_\varepsilon^+$ is a super solution for the problem $(P^\varepsilon)$. Similarly $w_\varepsilon^-$ is a sub- solution. Obviously,

$$w_\varepsilon^-(x,0) = w_\varepsilon^+(x,0) = Y(0, u_0(x); \pm\varepsilon G) = u_0(x).$$

□

## 5.3 Proof of Theorem 4.1 for the general case

As in subsection 4.3, we first present a key estimate on the function $Y(\tau, \xi; \delta)$ after a time interval of order $\tau \sim\mid \ln\varepsilon \mid$. A perturbation $\delta$ of order $\varepsilon$ does not affect the result of Lemma 4.7.

**Lemma 5.6.** *Let $\eta \in (0, \eta_0)$ be arbitrary, there exist positive constants $\varepsilon_0$ and $C_7$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

- *for all $\xi \in (-2C_0, 2C_0)$ ,*

$$\alpha_- - \eta \leq Y(\mu^{-1} \mid \ln\varepsilon \mid, \xi; \pm\varepsilon G) \leq \alpha_+ + \eta, \tag{5.16}$$

- *for all* $\xi \in (-2C_0, 2C_0)$ *such that* $\mid \xi - \alpha \mid \geq C_7\varepsilon$*, we have that*

$$\text{if } \xi \geq \alpha + C_7\varepsilon \quad \text{then } Y(\mu^{-1} \mid \ln \varepsilon \mid, \xi; \pm\varepsilon G) \geq \alpha_+ - \eta \tag{5.17}$$
$$\text{if } \xi \leq \alpha - C_7\varepsilon \quad \text{then } Y(\mu^{-1} \mid \ln \varepsilon \mid, \xi; \pm\varepsilon G) \leq \alpha_- + \eta. \tag{5.18}$$

*Proof.* In view of 5.1, we have, for $C_7$ large enough,

$$\alpha + C_7\varepsilon \geq \alpha(\pm\varepsilon G) + \frac{1}{2}C_7\varepsilon,$$

for all $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0$ sufficiently small. Hence for $\xi \geq \alpha + C_7\varepsilon$, as long as $Y(\tau, \xi; \pm\varepsilon G)$ has not reached $\alpha_+ - \eta$, we can use 5.9 to deduce that

$$\begin{aligned} Y(\tau, \xi; \pm\varepsilon G) &\geq \alpha(; \pm\varepsilon G) + C_1 e^{\mu(;\pm\varepsilon G)\tau}(\xi - \alpha(; \pm\varepsilon G)) \\ &\geq \alpha(; \pm\varepsilon G) - \varepsilon CG + \frac{1}{2}C_1 C_7 e^{\mu(;\pm\varepsilon G)\tau}\varepsilon \\ &\geq \alpha_+ - \eta, \end{aligned}$$

since $(\xi - \alpha(; \pm\varepsilon G)) \geq C_7\varepsilon$, solving with respect to $\tau$ in previous estimate and we have

$$\tau \geq \frac{1}{\mu(; \pm\varepsilon G)} \ln \frac{m_0 - \eta + CG\varepsilon}{\frac{1}{2}C_1 C_7 \varepsilon} =: \mu^{-1}(\varepsilon) \mid \ln \varepsilon \mid \tag{5.19}$$

A simple computation shows that

$$\mu^{-1} \mid \ln \varepsilon \mid - \mu^{-1}(\varepsilon) \mid \ln \varepsilon \mid = \frac{\mu(\pm\varepsilon G) - \mu}{\mu(\pm\varepsilon G)} \mid \ln \varepsilon \mid - \frac{1}{\mu(\pm\varepsilon G)} \ln \frac{m_0 - \eta + CG\varepsilon}{\frac{1}{2}C_1} + \frac{1}{\mu(\pm\varepsilon G)} \ln C_7.$$

Thanks to 5.4, as $\varepsilon \to 0$, the first above term is of order $\varepsilon \mid \ln \varepsilon \mid$ and the second one of order 1. For $C_7$ large enough, the upper quantity is positive for all $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0$ sufficiently small. Similarly proved the second one.
Now we prove the estimate. First, by taking $\varepsilon_0$ sufficiently small, we can assume that the stable equilibria of $f_{\pm\varepsilon G}$, namely $\alpha_-(\pm\varepsilon G)$ and $\alpha_-(\pm\varepsilon G)$, are in $[\alpha_- - \eta, \alpha_+ + \eta]$. Hence, $f_{\pm\varepsilon G}$ being a bistable function, if we leave from a $\xi \in [\alpha_- - \eta, \alpha_+ + \eta]$ then $Y(\tau, \xi; \pm\varepsilon G)$ will remain in the interval $[\alpha_- - \eta, \alpha_+ + \eta]$. Now suppose that $\alpha_+ + \eta \leq \xi \leq 2C_0$. We check bellow that $Y(\mu^{-1} \mid \ln \varepsilon \mid, \xi; \pm\varepsilon G) \leq \alpha_+ + \eta$. As done in the proof of Lemma 4.8, as long as $\alpha_+ + \eta \leq Y \leq 2C_0$, 4.33, 4.34 leads to the inequality $Y_\tau \leq p(\alpha_+ - Y) + \varepsilon G$. It follows that

$$\frac{Y_\tau}{Y - \alpha_+} \leq -p + \varepsilon \frac{G}{\eta},$$

which implies, by integration from 0 to $\tau$, that

$$Y(\tau, \xi; \pm\varepsilon G) \leq \alpha_+ + (2C_0 - \alpha_+)e^{(-p+\varepsilon\frac{G}{\eta})\tau}.$$

Since $(2C_0 - \alpha_+)e^{(-p+\varepsilon G\eta^{-1})\mu^{-1}|\ln\varepsilon|} \to 0$ as $\varepsilon \to 0$, the above inequality proves that, for $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 = \varepsilon_0(\eta)$ sufficiently small, $Y(\mu^{-1} \mid \ln \varepsilon \mid, \xi; \pm\varepsilon G) \leq \alpha_+ + \eta$, which completes the proof. □

We are now ready to prove Theorem 4.1 in the general case. By setting $t = \mu^{-1}\varepsilon^2 \mid \ln\varepsilon \mid$ in 5.15, we get

$$\begin{aligned} &Y(\mu^{-1} \mid \ln\varepsilon \mid, u_0(x) - \varepsilon^2 r(-\varepsilon G, \mu^{-1} \mid \ln\varepsilon \mid); -\varepsilon G) \\ \leq u^\varepsilon(x, \mu^{-1}\varepsilon^2 \mid \ln\varepsilon \mid) &\leq Y(\mu^{-1} \mid \ln\varepsilon \mid, u_0(x) + \varepsilon^2 r(\varepsilon G, \mu^{-1} \mid \ln\varepsilon \mid); +\varepsilon G). \end{aligned}$$

The point is that, in view of 5.4,

$$\lim_{\varepsilon\to 0} \frac{\mu - \mu(\pm\varepsilon G)}{\mu} \ln\varepsilon = 0.$$

Hence we have, for $\varepsilon_0$ small enough,

$$\varepsilon^2 r(\pm\varepsilon G, \mu^{-1} \mid \ln\varepsilon \mid) = C_6\varepsilon(\varepsilon^{(\mu-\mu(\pm\varepsilon G))/\mu} - \varepsilon) \in \left(\frac{1}{2}C_6\varepsilon, \frac{3}{2}C_6\varepsilon\right).$$

Hence, as in subsection 4.3, the result 4.2 of Theorem 4.1 is a direct consequence of 5.16 and the above estimate. Next we prove 4.3. We take $x \in \Omega$ such that $u_0(x) \geq \alpha + M_0\varepsilon$ so that

$$\begin{aligned} u_0(x) - \varepsilon^2 r(-\varepsilon G, \mu^{-1} \mid \ln\varepsilon \mid) &\geq \alpha + M_0\varepsilon - \frac{3}{2}C_6\varepsilon \\ &\geq \alpha + C_7\varepsilon, \end{aligned}$$

if we choose $M_0$ large enough. Using the above estimate and 5.17 we obtain 4.3.

# 6 Motion of interface

In the previous sections, we proved that the solution $u^\varepsilon$ develops a clear transition layer within a very short time. The aim of the present section is to show that, once such a clear transition layer is formed, it persists for the rest of time and that its law of motion is well approximated by the interface equation $(P^0)$.
We formulate the above assertion by taking the first two terms of the formal asymptotic expansion 3.3 we get a formal approximation of the solution $u^\varepsilon$ up to order $\varepsilon$:

$$u^\varepsilon(x,t) \approx \widetilde{u}^\varepsilon(x,t) := U_0\left(\frac{\widetilde{d}(x,t)}{\varepsilon}\right) + \varepsilon U_1\left(x,t,\frac{\widetilde{d}(x,t)}{\varepsilon}\right). \tag{6.1}$$

Here $U_0,\ U_1$ are defined in 3.6 and 3.26. The right side has a clear transition layer which lies exactly on $\Gamma_t$. Our goal is to show that this function is a good approximation of the real solution, more precisely:

If $u^\varepsilon$ becomes rather close to $\widetilde{u}^\varepsilon$ at some time moment $t = t_0$, then it stays close to $\widetilde{u}^\varepsilon$ for the rest of time. Consequently, $\Gamma_t^\varepsilon$ evolves roughly like $\Gamma_t$.

In order to prove such a result, we will construct a pair of sub- and super-solution $u_\varepsilon^-$ and $u_\varepsilon^+$ for Problem $(P^\varepsilon)$ by slightly modifying the above function $\widetilde{u}^\varepsilon$. It then follows that, if the solution $u^\varepsilon$ satisfies

$$u_\varepsilon^-(x,t_0) \leq u^\varepsilon(x,t_0) \leq u_\varepsilon^+(x,t_0),$$

for some $t_0 \geq 0$, then

$$u_\varepsilon^-(x,t) \leq u^\varepsilon \leq u_\varepsilon^+(x,t),$$

for $t_0 \leq t \leq T$. As a result, since both $u_\varepsilon^+, u_\varepsilon^-$ stay close to $\widetilde{u}^\varepsilon$, the solution $u^\varepsilon$ also stays close to $\widetilde{u}^\varepsilon$ for $t_0 \leq t \leq T$. That is why we should construct these sub- and super- solutions.

## 6.1 A modified signed distance function

We define a cut- off signed distance function d as follows. We recall that $\widetilde{d}$ is defined in 3.1 as the usual distance function. First, choose $d_0 > 0$ small enough so that $\widetilde{d}(\cdot,\cdot)$ is smooth in the tubular neighborhood of $\Gamma$

$$\{\ (x,t) \in \overline{Q_T}, \mid \widetilde{d}(x,t) \mid < 3d_0\ \}$$

and such that

$$dist(\Gamma_t, \partial\Omega) \geq 3d_0 \ \text{ for all } \ t \in [0, T]. \tag{6.2}$$

Next let $J(s)$ be a smooth increasing function on $\mathbb{R}$ such that

$$J(s) = \begin{cases} s & \text{if} \ \mid s \mid \leq \ d_0, \ (-d_0 \leq s \leq d_0) \\ -2d_0 & \text{if} \ s \ \leq \ -2d_0 \\ 2d_0 & \text{if} \ s \ \geq \ 2d_0. \end{cases}$$

We then define the cut- off signed distance function d by

$$d(x,t) = J\left(\widetilde{d}(x,t)\right).$$

Therefore,

$$d(x,t) = J\left(\widetilde{d}(x,t)\right) = \begin{cases} \widetilde{d}(x,t) & \text{if} \ \mid \widetilde{d}(x,t) \mid \leq d_0 \\ -2d_0 & \text{if} \ \widetilde{d}(x,t) \leq -2d_0 \\ 2d_0 & \text{if} \ \widetilde{d}(x,t) \geq 2d_0, \end{cases}$$

where

$$\widetilde{d}(x,t) = \begin{cases} dist(x, \Gamma_t) & \text{for} \ \ x \in \Omega_t^+ \\ -dist(x, \Gamma_t) & \text{for} \ \ x \in \Omega_t^-. \end{cases}$$

Note that $\mid \nabla d \mid = 1$ in the region $\{ \ (x,t) \in \overline{Q_T}, \mid \widetilde{d}(x,t) \mid < d_0 \ \}$ and that, in view of and the above definition, $\nabla d = 0$ in a neighborhood of $\partial\Omega$. Note also that the equation of motion $(P^0)$, which is equivalent to 3.24, is now written as

$$d_t = \Delta d - \gamma(x,t) \ \text{on} \ \ \Gamma_t, \tag{6.3}$$

where we recall that

$$\gamma(x,t) = c_0 \int_{\alpha_-}^{\alpha_+} g(x,t,r)dr. \tag{6.4}$$

## 6.2 Construction of sub- and super- solutions

The sub- and super- solutions for the motion of interface are constructed by using the first two terms of the formal asymptotic expansion 3.3. Now, we construct these solutions by modifying the function $\widetilde{u}_\varepsilon$ in 6.1. Concerning the second term $U_1$, which define in 3.26, the terms $\Delta U_1$ and $U_{1t}$ do not make sense as we only assume that $g(\cdot,\cdot,u) \in C^{1+\theta,\frac{1+\theta}{2}}$. In order to cope with this lack of smoothness, as $g^\varepsilon(\cdot,\cdot,u) \in C^{2,1}$ for $\theta = 1$, we replace $U_1$ by a more smooth function $U_1^\varepsilon$, which defined by

$$\begin{cases} U_{1zz}^{\varepsilon} + f'(U_0(z))U_1^{\varepsilon} = g^{\varepsilon}(x,t,U_0(z)) - \gamma^{\varepsilon}(x,t)U_0'(z), \\ U_1^{\varepsilon}(x,t,0), \qquad U_1^{\varepsilon}(x,t,\cdot) \in L^{\infty}(\mathbb{R}), \end{cases} \tag{6.5}$$

where

$$\gamma^{\varepsilon}(x,t) = c_0 \int_{\alpha_-}^{\alpha_+} g^{\varepsilon}(x,t,r)dr. \tag{6.6}$$

Thus $U_1^{\varepsilon}(x,t,z)$ is a solution of 3.9 with

$$A = \nabla_x A_0^{\varepsilon}(x,t,z) := g^{\varepsilon}(x,t,U_0(z)) - \gamma^{\varepsilon}(x,t)U_0'(z)), \tag{6.7}$$

where the variables $x, t, \varepsilon$ are considered parameters. Using 2.6 and the same arguments as in the end of the section 3, we obtain estimates analogous to 3.28 and 3.29, with a constant $M$ independent of $\varepsilon$:

$$\mid U_1^{\varepsilon}(x,t,z) \mid \leq M, \ \ \mid \nabla_x U_1^{\varepsilon}(x,t,z) \mid \leq M. \tag{6.8}$$

Moreover, $g^{\varepsilon}$ being $C^2$ in $x$ and $C^1$ in $t$, $\nabla_x U_1^{\varepsilon}$ and $U_{1t}^{\varepsilon}$ are solutions of 3.9 with $A = \nabla_x A_0^{\varepsilon}$ and $A = A_{0t}^{\varepsilon}$, respectively. Thus, in view of 2.4, we obtain

$$\mid \Delta_x U_1^{\varepsilon}(x,t,z) \mid \leq C/\varepsilon, \ \ \mid U_{1t}^{\varepsilon}(x,t,z) \mid \leq C/\varepsilon, \tag{6.9}$$

for a constant C independent of $\varepsilon$. Similarly, 2.5, 2.6 and Lemma 3.3 yield estimates analogous to 3.32 and 3.33 for $U_1^{\varepsilon}$, for constants C and M independent of $\varepsilon$:

$$\mid U_{1z}^{\varepsilon}(x,t,z) \mid + \mid U_{1zz}^{\varepsilon}(x,t,z) \mid \leq Ce^{-\lambda|z|}, \tag{6.10}$$

$$\mid \nabla_x U_{1z}(x,t,z) \mid \leq M. \tag{6.11}$$

In the rest of this section, $C$ and $M$ will stand for the constants that appear in inequalities 6.8- 6.11. Also, by the same arguments used to obtain 3.35, we see that 2.8 implies the homogeneous Neumann boundary condition for $U_1^{\varepsilon}$:

$$\frac{\partial U_1^{\varepsilon}}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega \times [0,T] \times \mathbb{R}. \tag{6.12}$$

We look for a pair of sub- and super- solutions $u_{\varepsilon}^{\pm}$ for $(P^{\varepsilon})$ of the form

$$u_{\varepsilon}^{\pm}(x,t) = U_0\left(\frac{\widetilde{d}(x,t) \pm \varepsilon p(t)}{\varepsilon}\right) + \varepsilon U_1^{\varepsilon}\left(x,t,\frac{\widetilde{d}(x,t) \pm \varepsilon p(t)}{\varepsilon}\right) \pm q(t), \tag{6.13}$$

where

$$p(t) = -e^{-\beta t/\varepsilon^2} + e^{Lt} + K, \tag{6.14}$$

$$q(t) = \sigma(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}). \tag{6.15}$$

Note that $q = \sigma\varepsilon^2 p_t$. It is clear from the definition of $u_\varepsilon^\pm$ that

$$\lim_{\varepsilon\to 0} u_\varepsilon^\pm(x,t) = \begin{cases} \alpha_+ & \text{for all } (x,t) \in Q_T^+ \\ \alpha_- & \text{for all } (x,t) \in Q_T^-. \end{cases} \tag{6.16}$$

The main result in this section is the following.

**Lemma 6.1.** *Choose $\beta$, $\sigma > 0$ appropriately. For any $K > 1$ we can find positive constants $\varepsilon_0$ and $L$ such that, for any $\varepsilon \in (0,\varepsilon_0)$, the functions $(u_\varepsilon^-, u_\varepsilon^+)$ are a pair of sub- and super- solutions for the Problem $(P^\varepsilon)$ in the range $x \in \overline{\Omega}$, $0 \le t \le T$. In other words, $u_\varepsilon^-$ and $u_\varepsilon^+$ satisfy the homogeneous Neumann boundary condition and*

$$Lu_\varepsilon^- \le 0 \le Lu_\varepsilon^+,$$

*in the range $x \in \overline{\Omega}$, $0 \le t \le T$, where we recall that the operator $L$ is defined by*

$$Lu := u_t - \Delta u - \varepsilon^{-2}(f(u) - \varepsilon g^\varepsilon(x,t,u)).$$

## 6.3 Proof of Lemma 6.1

By virtue of 6.12 and the fact that $\nabla d = 0$ near $\partial\Omega$, we have the homogeneous boundary condition

$$\frac{\partial u_\varepsilon^\pm}{\partial \nu} = 0 \text{ on } \partial\Omega \times [0,T].$$

In the following we prove inequality $Lu_\varepsilon^+ \ge 0$, the inequality $Lu_\varepsilon^- \le 0$ following the same argument.

**Computation of $Lu_\varepsilon^+$:**
Straightforward computation yields

$$\left(u_\varepsilon^+\right)_t = U_0'\left(\frac{d_t}{\varepsilon} + p_t\right) + \varepsilon U_{1t}^\varepsilon + U_{1z}^\varepsilon\,(d_t + \varepsilon p_t) + q_t,$$

$$\nabla u_\varepsilon^+ = U_0'\frac{\nabla d}{\varepsilon} + \varepsilon\nabla U_1^\varepsilon + \nabla U_{1z}^\varepsilon \nabla d,$$

$$\Delta u_\varepsilon^+ = U_0''\frac{|\nabla d|^2}{\varepsilon^2} + U_0'\frac{\Delta d}{\varepsilon} + \varepsilon\Delta U_1^\varepsilon + 2\nabla U_{1z}^\varepsilon \cdot \nabla d + U_{1zz}^\varepsilon\frac{|\nabla d|^2}{\varepsilon} + U_{1z}^\varepsilon \Delta d,$$

where the function $U_0$, as well its derivatives, is evaluated at $(x, t(d(x,t) + \varepsilon p(t))/\varepsilon)$. Here, $\nabla U_1^\varepsilon$ denotes the derivative with respect to $x$ whenever we regard $U_1^\varepsilon(x,t,z)$

as a function of three variables x, t and z. The symbol $\Delta U_1^\varepsilon$ is defined similarly. We also expand the reaction terms

$$f(u_\varepsilon^+) = f(U_0) + (\varepsilon U_1^\varepsilon + q)f'(U_0) + \frac{1}{2}(\varepsilon U_1^\varepsilon + q)^2 f''(\theta),$$

$$g(x,t,u_\varepsilon^+) = g(x,t,U_0) + (\varepsilon U_1^\varepsilon + q)g_u(x,t,\omega),$$

where $\theta(x,t)$ and $\omega(x,t)$ are some functions satisfying $U_0 < \theta < u_\varepsilon^-$, $U_0 < \omega < u_\varepsilon^+$. Writing $g^\varepsilon = g + g^\varepsilon - g$ and combining the above expressions with equations 6.5 and 3.6.
We recall that 3.6 is solution of the ordinary differential equation

$$\begin{cases} U_0'' + f(U_0) = 0, \\ U_0(-\infty) = \alpha_-,\ U_0(0) = \alpha,\ U_0(+\infty) = \alpha_+, \end{cases}$$

and 6.5 :

$$\begin{cases} U_{1zz}^\varepsilon + f'(U_0(z))U_1^\varepsilon = g^\varepsilon(x,t,U_0(z)) - \gamma^\varepsilon(x,t)U_0'(z), \\ U_1^\varepsilon(x,t,0), \qquad U_1^\varepsilon(x,t,\cdot) \in L^\infty(\mathbb{R}), \end{cases}$$

where

$$\gamma^\varepsilon(x,t) = c_0 \int_{\alpha_-}^{\alpha_+} g^\varepsilon(x,t,r)dr.$$

Finally, we have

$$Lu_\varepsilon^+ = U_0'\left(\frac{d_t}{\varepsilon} + p_t\right) + \varepsilon U_{1t}^\varepsilon + U_{1z}^\varepsilon\,(d_t + \varepsilon p_t) + q_t - U_0''\frac{|\ \nabla d\ |^2}{\varepsilon^2}$$

$$-U_0'\frac{\Delta d}{\varepsilon} - \varepsilon \Delta U_1^\varepsilon - 2\nabla U_{1z}^\varepsilon \cdot \nabla d - U_{1zz}^\varepsilon \frac{|\ \nabla d\ |^2}{\varepsilon} - U_{1z}^\varepsilon \Delta d$$

$$-\frac{1}{\varepsilon^2}f(U_0) - \frac{1}{\varepsilon^2}(\varepsilon U_1^\varepsilon + q)f'(U_0) - \frac{1}{\varepsilon^2}\frac{1}{2}(\varepsilon U_1^\varepsilon + q)^2 f''(\theta)$$

$$+\frac{1}{\varepsilon}g(x,t,U_0) + \frac{1}{\varepsilon}(\varepsilon U_1^\varepsilon + q)g_u(x,t,\omega) + \frac{1}{\varepsilon}g^\varepsilon(x,t,U_0)$$

$$+\frac{1}{\varepsilon}(\varepsilon U_1^\varepsilon + q)g_u^\varepsilon(x,t,\omega) - \frac{1}{\varepsilon}g(x,t,U_0) - \frac{1}{\varepsilon}(\varepsilon U_1 + q)g_u(x,t,\omega).$$

We obtain

$$Lu_\varepsilon^+ = E_1 + E_2 + \cdots + E_7,$$

where:

$$E_1 = -\frac{1}{\varepsilon^2} q \left( f'(U_0) + \frac{1}{2} q f''(\theta) \right) + U_0' p_t + q_t,$$

$$E_2 = \left( \frac{U_0''}{\varepsilon^2} + \frac{U_{1zz}^\varepsilon}{\varepsilon} \right) (1 - \mid \nabla d \mid^2),$$

$$E_3 = \left( \frac{U_0'}{\varepsilon} + U_{1z}^\varepsilon \right) (d_t - \Delta d + \gamma),$$

$$E_4 = \varepsilon U_{1z}^\varepsilon p_t + \frac{1}{\varepsilon} q (g_u(x,t,\omega) - U_1^\varepsilon f''(\theta)),$$

$$E_5 = -\gamma U_{1z}^\varepsilon - \frac{1}{2} (U_1^\varepsilon)^2 f''(\theta) + U_1^\varepsilon g_u(x,t,\omega) - 2\nabla U_{1z}^\varepsilon \cdot \nabla d,$$

$$E_6 = \varepsilon U_{1t}^\varepsilon - \varepsilon \Delta U_1^\varepsilon,$$

$$E_7 = \frac{1}{\varepsilon}(g^\varepsilon - g)(x,t,u_\varepsilon^+) - \frac{1}{\varepsilon}(g^\varepsilon - g)(x,t,U_0) + \frac{1}{\varepsilon}(\gamma^\varepsilon - \gamma)(x,t){U_0}'.$$

Since having substituted where, $U_0'' = -f(U_0)$ and $U_{1zz}^\varepsilon = -f(U_0(z))U_1^\varepsilon + g^\varepsilon - \gamma\varepsilon U_0'$. We estimate each of the above terms so as to estimate the $Lu_\varepsilon^+$. Before starting to estimate the terms, we see some useful inequalities. First, by bistability assumption $f'(\alpha_\pm) < 0,\ f'(\alpha) > 0$, there exist positive constants b, m such that

$$f'(U_0(z)) \leq -m \ \text{ if } \ U_0(z) \in [\alpha_-, \alpha_- + b] \cup [\alpha_+ - b, \alpha_+]. \tag{6.17}$$

On the other hand, since the region $\{\ z \in \mathbb{R} \mid U_0(z) \in [\alpha_- + b, \alpha_+ - b]\ \}$ is compact and since $U_0' > 0$ on $\mathbb{R}$, there exists a constant $a_1$ such that

$$U_0'(z) \geq a_1 \ \text{ if } \ U_0(z) \in [\alpha_- + b, \alpha_+ - b]\ \}. \tag{6.18}$$

We set

$$\beta = \frac{m}{4}, \tag{6.19}$$

and choose $\sigma$ that satisfies

$$0 < \sigma \leq \min(\sigma_0, \sigma_1, \sigma_2), \tag{6.20}$$

where

$$\sigma_0 := \frac{a_1}{m + F_1},\ \sigma_1 := \frac{1}{\beta + 1},\ \sigma_2 := \frac{4\beta}{F_2(\beta + 1)},$$

with the constant $F_1$ and $F_2$ defined by

$$F_1 := \max_{\alpha_- \leq u \leq \alpha_+} |\ f'(u)\ |,\ F_2 := \max_{\alpha_- - 2 \leq u \leq \alpha_+ + 2} |\ f''(u)\ |\ .$$

Combining 6.17 and 6.18, and considering that $\sigma \leq \sigma_0$, we obtain

$$\sigma \leq \sigma_0 \ \Rightarrow\ \sigma \leq \frac{a_1}{m + F_1}.$$

Then,

$$U_0'(z) - \sigma f'(U_0(z)) \geq \sigma m \quad \text{for} \quad -\infty < z < \infty. \tag{6.21}$$

Now let $K > 1$ arbitrary. In what follows we will show that $Lu_\varepsilon^+ \geq 0$ provided that the constants $\varepsilon_0$ and $L$ are appropriately chosen. We recall that $\alpha_- < U_0 < \alpha_+$. We continue under the following assumptions

$$\varepsilon_0 M \leq 1, \quad \varepsilon_0^2 L e^{LT} \leq 1. \tag{6.22}$$

It follows from 6.8 that, for all $\varepsilon \in (0, \varepsilon_0)$, we have $\varepsilon \mid U_1^\varepsilon(x, t, z) \mid \leq 1$. Moreover,

$$\sigma \leq \sigma_1 \ \Rightarrow\ \sigma \leq \frac{1}{\beta + 1} \ \Rightarrow\ \sigma \leq \frac{1}{\frac{m}{4} + 1}$$

implies that $0 \leq q(t) \leq 1$, so that, in view of 6.13,

$$\alpha_- - 2 \leq u_\varepsilon^\pm(x, t) \leq \alpha_+ + 2. \tag{6.23}$$

**The term $E_1$:**

$$E_1 = -\frac{1}{\varepsilon^2} q \left( f'(U_0) + \frac{1}{2} q f''(\theta) \right) + U_0' p_t + q_t.$$

By substituting 6.14, 6.15 in $E_1$, we obtain

$$E_1 = -\frac{1}{\varepsilon^2}\sigma \left(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}\right) \left( f'(U_0) + \frac{1}{2}\sigma(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}) f''(\theta) \right)$$
$$+ U_0' \left( \frac{\beta}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + L e^{Lt} \right) - \sigma \frac{\beta^2}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + \varepsilon^2 L^2 e^{Lt},$$

where

$$p_t = \frac{\beta}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + Le^{Lt},$$
$$q_t = \sigma \frac{\beta^2}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + \varepsilon^2 L^2 e^{Lt}.$$

We set

$$I = U_0' - \sigma f'(U_0) - \frac{\sigma^2}{2} f''(\theta)(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 Le^{Lt}).$$

And we end up in the following relation

$$E_1 = \frac{\beta}{\varepsilon^2} e^{-\beta t/\varepsilon^2}(I - \sigma\beta) + Le^{Lt}(I + \varepsilon^2 \sigma L).$$

In virtue of 6.21 and 6.23, we have

$$I \geq \sigma m - \frac{\sigma^2}{2} f''(\theta)(\beta + \varepsilon^2 Le^{LT}).$$

Combining this, 6.22 and the inequality

$$\sigma \leq \sigma_2 \;\Rightarrow\; \sigma \leq \frac{m}{F_2(\frac{m}{4} + 1)}.$$

We obtain

$$I \geq 2\sigma\beta.$$

Consequently, we have

$$E_1 \geq \frac{\sigma\beta^2}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + 2\sigma\beta Le^{LT}.$$

**The term $E_2$:**

$$E_2 = \left( \frac{U_0''}{\varepsilon^2} + \frac{U_{1zz}^{\varepsilon}}{\varepsilon} \right) (1 - \mid \nabla d \mid^2).$$

First, in the region where $\mid d \mid < d_0$, we have $\mid \nabla d \mid = 1$, hence $E_2 = 0$. Next we consider the region where $\mid d \mid \geq d_0$. We deduce from Lemma 3.1 and from 6.10 that:

$$|E_2| \leq C \left( \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \right) \left(1 + \|\nabla d\|_\infty^2\right) e^{-\lambda |d + \varepsilon p|/\varepsilon}$$

$$\leq \frac{2C}{\varepsilon^2} \left(1 + \|\nabla d\|_\infty^2\right) e^{-\lambda (d_0/\varepsilon - |p|)}.$$

By definition of $p$ in 6.14, we have that $0 < K - 1 \leq p \leq e^{LT} + K$. Consequently, if we assume

$$e^{LT} + K \leq \frac{d_0}{2\varepsilon_0}, \tag{6.24}$$

then $\frac{d_0}{\varepsilon} - |p| \geq \frac{d_0}{2\varepsilon}$, so that

$$|E_2| \leq \frac{2C}{\varepsilon^2}\left(1 + \|\nabla d\|_\infty^2\right) e^{-\lambda d_0/(2\varepsilon)}$$

$$\leq C_2 := \frac{32C}{(e\lambda_0)^2}\left((1 + \|\nabla d\|_\infty^2\right).$$

**The term** $E_3$:

$$E_3 = \left(\frac{U_0'}{\varepsilon} + U_{1z}^\varepsilon\right)(d_t - \Delta d + \gamma).$$

We recall that

$$(d_t - \Delta d + \gamma)(x,t) = 0 \quad \text{on} \quad \Gamma_t = \{\, x \in \Omega, d(x,t) = 0 \,\}.$$

By equality 6.4 and assumption 2.6, we see that $\gamma$ is in $C^{1+\theta,\frac{1+\theta}{2}}$ so that the interface $\Gamma$ is of class $C^{3+\theta,\frac{3+\theta}{2}}$. Therefore both $\Delta d$ and $d_t$ are Lipschitz continuous near $\Gamma_t$. It follows, from the mean value theorem applied separately on both sides on $\Gamma_t$ that there exists a constant $N_0 > 0$ such that:

$$|(d_t - \Delta d + \gamma)(x,t)| \leq N_0\,|d(x,t)| \quad \text{for all} \quad (x,t) \in Q_T.$$

Applying Lemma 3.1 and estimate 6.10 we deduce that

$$|E_3| \leq 2N_0C\frac{|\ d(x,t)\ |}{\varepsilon}e^{-\lambda|d(x,t)/\varepsilon + p(t)|}$$

$$\leq 2N_0C \max_{y\in\mathbb{R}} |y|\, e^{-\lambda|y+p(t)|}$$

$$\leq 2N_0C\max(|p(t)|\,, \frac{1}{\lambda})$$

$$\leq 2N_0C(|p(t)| + \frac{1}{\lambda}),$$

by setting $y = \frac{d(x,t)}{\varepsilon}$.
Thus recalling that

$$|p(t)| \leq e^{Lt} + K,$$

we obtain

$$|E_3| \leq C_3(e^{Lt} + K) + C_3{}',$$

where $C_3 := 2N_0C$ and $C_3{}' = 2N_0C/\lambda$.

**The term** $E_4$:

$$E_4 = \varepsilon U_{1z}^{\varepsilon} p_t + \frac{1}{\varepsilon} q(g_u(x,t,\omega) - U_1^{\varepsilon} f''(\theta)).$$

In view of 2.5 and 6.10, both $g_u$ and $|U_{1z}^{\varepsilon}|$ are bounded by some constant C. Hence, substituting the expression for $p_t$ and $q$, we obtain

$$|E_4| = \varepsilon U_{1z}^{\varepsilon}(\frac{\beta}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + Le^{Lt}) + \frac{1}{\varepsilon}\sigma(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 Le^{Lt})(g_u - U_1^{\varepsilon} f''(\theta)),$$

we observe that $\mid g_u \mid \leq C$. Then

$$|E_4| \leq (\frac{1}{\varepsilon}\beta e^{-\beta t/\varepsilon^2} + \varepsilon Le^{Lt}),$$

where $C_4 := C + \sigma(C + MF_2)$.

**The term** $E_5$:

$$E_5 = -\gamma U_{1z}^{\varepsilon} - \frac{1}{2}(U_1^{\varepsilon})^2 f''(\theta) + U_1^{\varepsilon} g_u(x,t,\omega) - 2\nabla U_{1z}^{\varepsilon} \cdot \nabla d.$$

In view of 6.4, the term $|\gamma|$ is bounded by $c_0(\alpha_+ - \alpha_-)C$ on $\overline{\Omega} \times [0,T]$. Using 6.8, 6.10, 6.11 and 2.5, we obtain

$$|E_5| \leq c_0(\alpha_+ - \alpha_-)CM + \frac{1}{2}M^2 F_2 + MC + 2M \left\|\nabla d\right\|_{\infty}^2 =: C_5.$$

**The term** $E_6$:

$$E_6 = \varepsilon U_{1t}^{\varepsilon} - \varepsilon \Delta U_1^{\varepsilon}.$$

We use 6.9 to deduce that

$$|E_6| \leq 2C =: C_6.$$

**The term** $E_7$:

$$E_7 = \frac{1}{\varepsilon}(g^\varepsilon - g)(x,t,u_\varepsilon^+) - \frac{1}{\varepsilon}(g^\varepsilon - g)(x,t,U_0) + \frac{1}{\varepsilon}(\gamma^\varepsilon - \gamma)(x,t){U_0}'.$$

We recall that $|g^\varepsilon - g| \leq C\varepsilon$ so that $|\gamma^\varepsilon| \leq c_0(\alpha_+ - \alpha_-)C\varepsilon$. We also observe that

$$\frac{1}{\varepsilon}(g^\varepsilon - g)(x,t,u_\varepsilon^+) - \frac{1}{\varepsilon}(g^\varepsilon - g)(x,t,U_0) \leq 2C.$$

It then follows, in view of Lemma 3.1, that

$$|E_7| \leq 2C + c_0(\alpha_+ - \alpha_-)C^2 =: C_7.$$

**Completion of the proof**: So collecting the above estimates of $E_1 - E_7$ terms gives

$$Lu_\varepsilon^+ \geq (\frac{\sigma\beta^2}{\varepsilon^2} - \frac{C_4\beta}{\varepsilon})e^{-\beta t/\varepsilon^2} + (2\sigma\beta L - C_3 - \varepsilon C_4 L)e^{Lt} - C_8, \tag{6.25}$$

where

$$C_8 := C_2 + KC_3 + {C_3}' + C_5 + C_6 + C_7.$$

Now we set

$$L := \frac{1}{T}\ln\frac{d_0}{4\varepsilon_0},$$

which, for $\varepsilon_0$ small enough, validates assumptions 6.22 and 6.24. If $\varepsilon_0$ is chosen sufficiently small, also L large enough, we have for all $0 < \varepsilon < \varepsilon_0$ that the first term of the right side of 6.25 is positive, and that

$$\begin{aligned} Lu_\varepsilon^+ &\geq [\sigma\beta L - C_3]\, e^{Lt} - C_8 \\ &\geq \frac{1}{2}\sigma\beta L - C_8 \\ &\geq 0. \end{aligned}$$

With the choice of the constants $\beta,\ \sigma$ as in 6.19 and 6.20 the proof of Lemma 6.1 is complete.

# References

[1] Matthieu Alfaro, Danielle Hilhorst, and Hiroshi Matano. The singular limit of the allencahn equation and the fitzhughnagumo system. *Journal of Differential Equations*, 245(2):505 – 565, 2008.

[2] Lawrence C Evans. Partial differential equations. 2010.

[3] Paul C Fife. Models for phase separation and their mathematics. *Electron. J. Differential Equations*, 48(2000):1–26, 2000.

[4] K-I Nakamura, H Matano, D Hilhorst, and R Schätzle. Singular limit of a reaction-diffusion equation with a spatially inhomogeneous reaction term. *Journal of statistical physics*, 95(5):1165–1185, 1999.

[5] Yasumasa Nishiura. *Far-from-equilibrium Dynamics*, volume 209. American Mathematical Soc., 2002.