Heraklion 2012

Master Thesis

Non-Relativistic Holography

---

by

Anna-Maria Taki

Supervisor: Elias Kiritsis

Abstract

Our work is concentrated on the study of Non-Relativistic Conformal symmetries and geometries for scale-invariant theories. Attempting to find a bulk dual of the Non-Relativistic Conformal theories in order to realize the Schrödinger symmetry geometrically, in the context of the Anti-de Sitter space/Conformal field theory correspondence, we examine whether the isometries of the given metrics, comprise the $(d+1)$- dimensional Schrödinger group. After having tested the invariance of the AdS-Schwartzschild metric in light-cone coordinates we continue working with this background in order to compute holographically a DC conductivity related to transport of baryon number charge. The whole process relies on the work done by E. Kiritsis and B.S. Kim. The method we follow in order to compute the conductivity is the Karch-O'Bannon probe brane method presented in detail in [19], [20].

Contents

1 Basic Arguments on AdS/CFT

The considered relation between string theory and confining gauge theories has provided many lines of research over the years, extending the conceptual background of theoretical physics. At first sound, it seems prodigious to correlate a theory with gravity to a theory without gravity. String theory provides a consistent framework to compute finite quantum corrections to classical general relativity, but the full non-perturbative structure of the theory is not officially formulated since the theory is strongly coupled at high energies. On the other hand, the perturbative gauge theory has a well defined description in the high-energy regime, while obstacles emerge at low energies where the coupling constant of our theory gets a larger value.

The expectation that it could be possible to study a gauge theory in the context of a string theory was motivated at different levels, giving rise to the development of relevant dualities, according to which a single theory has (at least) two different descriptions, such that when one description is weakly coupled the other is strongly coupled and vice versa. One essential aspect of this reformulation, is the anti-de Sitter/conformal field theory correspondence (AdS/CFT), which constitutes one of the most significant achievements of string theory and was originally proposed by Juan Maldacena [17]. It expresses the conjectured equivalence between a conformal field theory in flat space and a string theory in a higher-dimensional space.

The most well-understood example illustrating this idea is the duality between type IIB string theory compactified on $AdS_5 \times S^5$, and four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The abbreviations AdS_5 , S^5 stand for an anti-de Sitter space in five dimensions and a five-dimensional sphere, respectively. Anti-de Sitter spaces are maximally symmetric solutions of the Einstein equations with a negative curvature in Minkowskian signature.Apart from these backgrounds, the AdS/CFT correspondence has been generalized to other dualities where neither the string theory lives on an anti-de Sitter space, nor the dual field theory is conformal.

The AdS/CFT correspondence relies on the following basic considerations. The first of these is the large-N expansion, suggested by Gerhard t'Hooft [2], which defines a particular perturbative analysis of quantum field theories with an internal symmetry group. In particular, it states that the gauge theory in the limit where the number of colours, N, is large is equivalent to a string theory, and as a result the simplified theory enjoys a systematic expansion in terms of the parameter 1/N. The latter argument is related to the holographic principle, originating in the study of the thermodynamics of black holes. Jacob Bekenstein [3] showed that the maximal entropy contained in a closed region with surface area A is given by $S_{max} = A/4G_N$, where G_N stands for the Newton's constant. In its simplest form, the principle asserts that the degrees of freedom of a quantum system with gravity can be represented by a theory on the boundary. The AdS/CFT correspondence is holographic because it declares that quantum gravity in five dimensions is equivalent to a local field theory in four dimensions.

To develop the correspondence, one considers a configuration of N parallel D3 branes in type IIB string theory, sufficiently close to each other. The D3 branes are extended along a (3+1)-dimensional plane in (9+1)-dimensional Minkowski spacetime. String theory on this background contains two kinds of perturbative excitations, closed strings and open strings. The closed strings are the excitations of empty space and the open strings end on the D-branes and describe their excitations. To derive the correspondence, one considers a suitable low energy limit of the system, which includes taking $l_s \to 0$, where l_s denotes the string scale. The closed string description reduces to string theory on $AdS_5 \times S^5$, whereas the open string description reduces to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We notice that both from the point of view of a field theory residing on the brane, and the supergravity's perspective, we obtain two decoupled theories in the low-energy regime. Therefore, the AdS/CFT duality arises as a consequence of the duality between closed and open strings.

2 Motivation and Applications

Physics falls within the purview of the natural sciences and is qualified as an academic discipline in virtue of its attempt to explain natural events in such a way as to make useful predictions about them. In the spirit of this perspective, one can not help but wonder why it is important to extend the holographic approach to non-relativistic conformal field theories and whether this description could be able to offer insight into some of the physical processes that have been a major focus of theoretical and experimental work. Various types of answers come to mind.

It is widely acceptable that the Anti-de Sitter/conformal field theory correspondence is an outstanding approach to strongly coupled field theories, since it renders the latter amenable to a certain computational treatment that leads to their conceptual clarification. One basic aspect of this physical motivation appears in condensed matter systems that can be engineered and studied in table-top experiments. In fact, observations in materials including strongly correlated electrons are of conspicuous technological interest, while conducing to the apprehension of significant concepts of high energy theory, as in the case of magnetic materials that constitutes one of the most conceivable examples of quantum phase transitions. For further details we refer the reader to the excellent account in [15]. We are mainly interested in quantum critical theories that arise at continuous phase transitions at zero temperature. [1] Quantum critical points are characterized by a spacetime scale invariance that provides a strong kinematic connection to the simplest versions of the AdS/CFT correspondence. A particularly interesting example of subjection to this scaling invariance, is the case of fermionic atoms at unitarity, which are conjectured to experimentally realize the Schrödinger symmetry, using trapped cold atoms at the Feshbach resonance. [2] These are scale-invariant in the

[1] A zero temperature phase transition is a non-analyticity in the ground state of an infinite system as a function of some parameter, such as the pressure of a magnetic field. [7]

[2] A Feshbach resonance is a resonance of a many-body system in which a bound state is achieved if the coupling(s) between an internal degree(s) of freedom and the reaction coordinates which lead to dissociation vanish.

following sense. Near a Feshbach resonance we can, theoretically, tune the scattering length [3] to any desirable value, and thus we can gain a great amount of control over atomic interactions. The material is dilute enough to render the effective range of the potential negligible. Despite the fact that the mass of the atoms is a dimensionful parameter, the dependence of the relevant physical observables on the mass is determined by the symmetry algebra. Also, these cold fermionic atoms provide another naturally realized example of a system with low viscosity similar to the Quark Gluon Plasma.

3 Non-Relativistic Conformal Symmetries

Let us present here the basic properties of the non-relativistic conformal groups, focusing on the structure of the Schrödinger algebra.

In the non-relativistic case, instead of the more familiar Lorentz scale invariance which arises in the conformal group, an anisotropic space and time scaling is being employed, which introduces the dynamical critical exponent z, as indicated in the following formula:

$$t \to \lambda^z t, \quad \mathbf{x} \to \lambda \mathbf{x} \tag{1}$$

This scaling symmetry furnishes part of the so-called Lifshitz algebra which contains in addition to the dilatations D, rotations M_{ij}, spatial translations P_i and time translations H. In terms of the generators for these symmetries, applied in the case of a dynamical system in d-spatial dimensions [4], we obtain the following commutation relations [8]:

$$\begin{gathered}
[M_{ij}, M_{gh}] = i(\delta_{ig} M_{jh} - \delta_{jg} M_{ih} + \delta_{ih} M_{gj} - \delta_{jh} M_{gi}), \\
[M_{ij}, P_g] = i(\delta_{ig} P_j - \delta_{jg} P_i), \quad [M_{ij}, H] = 0, \\
[H, P_i] = [P_i, P_j] = 0, \\
[D, H] = izH, \quad [D, P_i] = iP_i, \quad [D, M_{ij}] = 0
\end{gathered} \tag{2}$$

The Lifshitz algebras (for each value of z) may be extended to the Galilean Scaling Algebra by comprising the Galilean boosts K_i, and the scalar mass operator M. Apart from the commutators given in (2) we have to fill in the details for the generators K_i,M :

$$[M_{ij}, K_h] = i(\delta_{ih} K_j - \delta_{jh} K_i), \quad [M_{ij}, M] = 0,$$

$$[P_i, K_j] = -iM, \quad [H, K_i] = -iP_i, \quad [D, K_i] = i(1-z)K_i,$$

[3]The scattering of two particles with sort-range interactions at sufficiently low energy is determined by their S-wave scattering length [4]. By low energy, we mean energy close to the threshold for the two particles. The energy is sufficiently low if the de Broglie wavelengths of the particles are large compared to the range of the interaction. The scattering length is important not only for the two-body systems, but also for few-body and many-body systems.

[4]In this paper d always expresses the number of spatial dimensions , so d=3 corresponds to a real physical system.

$$[D, M] = i(z-2)M, \quad [H, M] = [P_i, M] = [K_i, M] = 0. \tag{3}$$

A priori z can take any positive value. In the special case, in which z=2, the group is further extended as to incorporate a special conformal transformation with generator C. The additional commutation relations are then given by:

$$[C, P_i] = iK_i, \quad [C, K_i] = 0,$$
$$[M_{ij}, C] = 0, \quad [D, C] = -2iC, \quad [H, C] = -iD \tag{4}$$

and the resulting group is called the Schrödinger group. For the previously chosen value of z, the generator M plays the role of a central term; it only appears in the commutator $[P_i, K_j]$.

The Schrödinger symmetry group in (d+1)-dimensional spacetime is identified as a non-relativistic counterpart of the conformal symmetry group and it stands for the group of symmetries of the free Schrödinger wave operator in (d+1)-dimensions. In particular, it is generated by those transformations that commute with the operator $S = i\partial_t + \frac{1}{2M}\partial_i^2$.

The Schrödinger group can be represented by the following set of transformations [8]:

$$\mathbf{x} \to \mathbf{x}' = \frac{\boldsymbol{\Re x} + \boldsymbol{v}t + \boldsymbol{\alpha}}{\psi t + \omega}, \quad t \to t' = \frac{\varphi t + \varrho}{\psi t + \omega} \tag{5}$$

with $\varphi\omega - \varrho\psi = 1$. The symmetry group contains the Galilean group with its central extension which is interpreted as the non-relativistic mass and corresponds to the symmetry of the Schrödinger equation under a phase transformation. Along with the spatial translations which are intimated by $\boldsymbol{\alpha}$, the rotations labelled by $\boldsymbol{\Re}$ and the Galilean boosts with velocity $\boldsymbol{v}$, there are two more generators D,C which correspond to a dilatation operator and the temporal component of the special conformal transformations, respectively.

Let us now re-deduce the Schrödinger algebra for $z = 2$ by following a different approach. We are familiar with the fact that the Poincaré algebra with (d+1) spatial dimensions and one timelike dimension is reduced to the Galilean algebra in d spatial dimensions on a light-cone

$$x^{\pm} = x^{\circ} \pm x^{d+1}, \tag{6}$$

where x° denotes the temporal direction in the relativistic theory. Likewise, the Schrödinger algebra is obtained by introducing the light-cone coordinates and maintaining the subalgebra commuting with the momentum in the light-cone direction P_-, which is interpreted as the mass operator M. The identification between the generators is displayed on the following matrix:

Generator	*Galilean*	*Conformal*
Mass Operator	M	P_-
Hamiltonian	H	P_+
Momenta	P_i	P_i
Angular Momenta	M_{ij}	M_{ij}
Galilean Boosts	K_i	M_{i-}
Dilatation	D	$D + M_{+-}$
Special Conformal	C	K_-

The above is a well-known example of embedding in the context of discrete light-cone quantization (DLCQ). [5]

Apparently, by retaining only the generators that commute with the mass operator P_-, we exclude some of the relativistic conformal generators. Specifically, there is no generator in the Schrödinger algebra that corresponds to the spatial components of the special conformal transformations. As a result, we are dealing with a smaller group compared to the relativistic conformal group. [6] The full algebra is then given by as

$$[M_{ij}, M_{gh}] = i(\delta_{ig}M_{jh} - \delta_{jg}M_{ih} + \delta_{ih}M_{gj} - \delta_{jh}M_{gi}),$$

$$\begin{aligned} [M_{ij}, P_g] &= i(\delta_{ig}P_j - \delta_{jg}P_i), \\ [M_{ij}, K_h] &= i(\delta_{ih}K_j - \delta_{jh}K_i), \\ [M_{ij}, H] = [M_{ij}, D] &= [M_{ij}, C] = 0, \\ [P_i, P_j] &= [K_i, K_j] = 0, \\ [P_i, K_j] &= -iM, \\ [H, P_i] = 0, &\quad [H, K_i] = -iP_i, \\ [D, P_i] = iP_i, &\quad [D, K_i] = -iK_i, \\ [C, P_i] = iK_i, &\quad [C, K_i] = 0, \\ [D, H] = 2iH, &\quad [D, C] = -2iC, \\ [H, C] &= -iD. \end{aligned} \tag{7}$$

The Schrödinger algebra is not the only potential candidate for describing non-relativistic systems in a proper way. As illustrated in [9], another valid option arises in the case of the Galilean Conformal Algebra (GCA) in d spatial dimensions. This algebra can be obtained by a parametric contraction of the relativistic conformal group in (d+1)-dimensions. The key differences between the Galilean Conformal Algebra and the Schrödinger algebra are reflected in the facts that the former has the same number of generators as the relativistic symmetry group, and it does not allow for the central extension, depicting in some sense the symmetry of a "massless" non-relativistic system. To conclude, the two algebras share a common Galilean subgroup and are different in all other respects.

We will now return to the Schrödinger algebra which will be our primary focus for the rest of our analysis.

[5]DLCQ refers to compactification on a light-like circle,

$$(x^+, x^-, x^i) \cong (x^+, x^- + 2\pi R, x^i)$$

with fixed nonzero momentum $P_- = N/R$, where R,N denote the radius of the circle and the number of quanta of the light-cone momentum, respectively.

[6]For instance, in the special case of (3+1)-dimensions the Schrödinger algebra contains 12 generators and the additional central term, whereas the relativistic conformal group includes 15 generators.

4 Geometries for scale-invariant theories

4.0.1 Constructing a general metric

At this point we attempt to find a bulk dual of non-relativistic conformal field theories in order to realize the Schrödinger symmetry geometrically. To do so, in the context of the Anti-de Sitter space/conformal field theory correspondence, we focus on examining spacetime metrics in -at least- one higher dimension than the corresponding field theory. Keeping that in mind, we initiate the following set of metrics based on considerations regarding the algebra of the generators which appears in (7),

$$ds^2 = L^2(-\frac{dt^2}{r^{2z}} + \frac{dx^i dx^i}{r^2} + \frac{dr^2}{r^2}). \tag{8}$$

Setting $z = 1$ means that we are dealing with the familiar case of the Anti-de Sitter space, which is invariant under the whole conformal group. The AdS metric in Poincaré coordinates takes the form [5]:

$$ds^2 = L^2(-\frac{dt^2}{r^2} + \frac{dx^i dx^i}{r^2} + \frac{dr^2}{r^2}). \tag{9}$$

The generators of the conformal group correspond to the following infinitesimal coordinate transformations that keep the metric unchanged:

$$P^\mu : x^\mu \to x^\mu + \alpha^\mu,$$

$$\begin{aligned} D : x^\mu \to (1-\alpha)x^\mu, \quad r \to (1-\alpha)r, \\ K^\mu : x^\mu \to x^\mu + \alpha^\mu(r^2 + x \cdot x) - 2x^\mu(\alpha \cdot x) \end{aligned} \tag{10}$$

For $z > 1$ these spaces are candidate duals to non-relativistic field theories. The impossibility of achieving an applied correspondence between the whole algebra of a d-dimensional Schrödinger invariant field theory and the isometries of a (d+1)-dimensional spacetime, as denoted in [7], motivated D.T. Son[5] and K. Balasubramanian and J. Mc Greevy[6] to slightly weaken the rules of AdS/CFT and deform the metric in order to preserve its invariance under the underlying symmetry group. More specifically, the Lifshitz metric in (8), is modified to the one that is displayed below,

$$ds^2 = L^2(-\frac{(dx^{+^2})}{r^{2z}} + \frac{-2dx^+ dx^- + dx^i dx^i + dr^2}{r^2}) \tag{11}$$

Especially, for $z = 2$, we derive the so-called Schrödinger metric, while the symmetry of the spacetime, and hence of the dual scale-invariant theory is substantially enhanced. As it was particularly stated in section 3, besides rotations, spacetime translations and dilatations, the theory enjoys special conformal symmetry and Galilean boost symmetries. Accordingly, the following metric satisfies this condition and we intend to verify that it exhibits a full Schrödinger symmetry.

$$ds^2 = L^2(-\frac{(dx^{+^2})}{r^4} + \frac{-2dx^+dx^- + dx^idx^i + dr^2}{r^2}) \tag{12}$$

where in this construction, as it was prementioned in section 3, x^- is a compact null circle, with the mass of the non-relativistic conformal field theory identified with momentum along this direction under a DLCQ .

In particular, we aim to show that the following isometries of the above metric comprise the (d+1)-dimensional non-relativistic conformal group with algebra (7). The corresponding isometries are :

$$P^i : x^i \to x^i + \alpha^i \tag{13}$$

$$H : x^+ \to x^+ + \alpha \tag{14}$$

$$M : x^- \to x^- + \alpha \tag{15}$$

$$K^i : x^i \to x^i - \alpha^i x^+,\ x^- \to x^- - \alpha^i x^i \tag{16}$$

$$D : x^i \to (1-\alpha)x^i,\ r \to (1-\alpha)r,\ x^+ \to (1-\alpha)^2 x^+,\ x^- \to x^- \tag{17}$$

$$C : r \to (1-\alpha x^+)r,\ x^i \to (1-\alpha x^+)x^i,\ x^+ \to (1-\alpha x^+)x^+,\ x^- \to x^- - \frac{\alpha}{2}(x^ix^i + r^2). \tag{18}$$

Let us now perform each of the above transformations to the metric (12) taken under consideration.

- For the spatial translations in (13) we have:

$$\begin{aligned} d\tilde{s}^2 &= L^2[-\frac{(dx^{+^2})}{r^4} + \frac{-2dx^+dx^- + d\tilde{x^i}d\tilde{x^i} + dr^2}{r^2}] \\ &= ds^2 \end{aligned} \tag{19}$$

  since α^i is a constant vector.

- Under the action of the Hamilton operator in (14) and the Mass operator in (15), the metric remains as it is, since α is a constant parameter.

- As far as the galilean boosts ,K^i, in (16), are concerned, we have:

$$d\tilde{s}^2 = L^2[-\frac{(dx^{+^2})}{r^4} + \frac{-2dx^+d\tilde{x^-} + d\tilde{x^i}d\tilde{x^i} + dr^2}{r^2}] \tag{20}$$

$$= L^2[-\frac{(dx^{+^2})}{r^4} + \frac{dr^2}{r^2} + \frac{-2dx^+(dx^-\alpha^idx^i) + (dx^i - \alpha^idx^+)(dx^i - \alpha^idx^+)}{r^2}] \tag{21}$$

$$= L^2[-\frac{(dx^{+^2})}{r^4} + \frac{-2dx^+dx^- + dx^idx^i + dr^2}{r^2} + \frac{2\alpha^idx^idx^+ - 2\alpha^idx^idx^+}{r^2} + \frac{(\alpha^idx^+)^2}{r^2}] \tag{22}$$

$$= ds^2 + \mathcal{O}(\alpha^2) \tag{23}$$

- The metric under the dilatation operator in (17) takes the form:

$$
\begin{aligned}
d\tilde{s}^2 &= L^2[-\frac{(d\tilde{x}^{+^2})}{\tilde{r}^4} + \frac{-2d\tilde{x}^+ d\tilde{x^-} + d\tilde{x^i} d\tilde{x^i} + d\tilde{r}^2}{\tilde{r}^2}] && (24)\\
&= L^2[-\frac{(1-\alpha)^4 dx^{+^2}}{(1-\alpha)^4 r^4} + \frac{-2(1-\alpha)^2 dx^+ dx^- + (1-\alpha)^2 dx^i dx^i + (1-\alpha)^2 dr^2}{(1-\alpha)^2 r^2})] && (25)\\
&= L^2[-\frac{(dx^{+^2})}{r^4} + (1+2\alpha)(1-\alpha)^2 \frac{-2dx^+ dx^- + dx^i dx^i + dr^2}{r^2} + && (26)\\
&= L^2(-\frac{(dx^{+^2})}{r^4} + (1+2\alpha)(1-2\alpha)\frac{-2dx^+ dx^- + dx^i dx^i + dr^2}{r^2}] + \mathcal{O}(\alpha^2)) && (27)\\
&= ds^2 + \mathcal{O}(\alpha^2) && (28)
\end{aligned}
$$

- Finally,we perform the special conformal transformation ,C, in (18)

$$
\begin{aligned}
d\tilde{s}^2 &= L^2\{-\frac{(d\tilde{x}^{+^2})}{\tilde{r}^4} + \frac{-2d\tilde{x}^+ d\tilde{x^-} + d\tilde{x^i} d\tilde{x^i} + d\tilde{r}^2}{\tilde{r}^2}\} && (29)\\
&= L^2\{-\frac{(1-2\alpha x^+)^2 dx^{+^2}}{r^4(1-\alpha x^+)^4} + \frac{-2(1-2\alpha x^+)dx^+[dx^- - \alpha(x^i dx^i + r dr)]}{(1-\alpha x^+)^2 r^2}\\
&+ \frac{[dx^i(1-\alpha x^+) - \alpha x^i dx^+]^2}{(1-\alpha x^+)^2 r^2} + \frac{[dr(1-\alpha x^+) - \alpha r dx^+]^2}{(1-\alpha x^+)^2 r^2}\} && (30)\\
&= L^2\{-\frac{dx^{+^2}(1-2\alpha x^+)^2}{(1-2\alpha x^+)^2 r^4} + \frac{-2(1-2\alpha x^+)(1+2\alpha x^+)(dx^+ dx^- - \alpha x^i dx^i dx^+ - \alpha r dr dx^+)}{r^2}\\
&+ \frac{(1+2\alpha x^+)[dx^{i^2}(1-2\alpha x^+) - 2\alpha x^i dx^+(1-\alpha x^+)dx^i]}{r^2})\\
&+ \frac{(1+2\alpha x^+)[dr^2(1-2\alpha x^+) - 2\alpha r dx^+(1-\alpha x^+)dr]}{r^2}\} && (31)\\
&= L^2\{-\frac{(dx^{+^2})}{r^4} + \frac{-2(dx^+ dx^- - \alpha x^i dx^i dx^+ - \alpha r dr dx^+)}{r^2}\\
&+ \frac{[dx^{i^2} - 2\alpha x^i dx^i dx^+(1+\alpha x^+)]}{r^2} + \frac{[dr^2 - 2\alpha r dr dx^+(1+\alpha x^+)]}{r^2}\} + \mathcal{O}(\alpha^2) && (32)\\
&= ds^2 + L^2\{\frac{2\alpha(x^i dx^i + r dr)dx^+}{r^2} + \frac{-2\alpha(x^i dx^i + r dr)dx^+}{r^2}\} + \mathcal{O}(\alpha^2) && (33)\\
&= ds^2 + \mathcal{O}(\alpha^2) && (34)\\
& && (35)
\end{aligned}
$$

To recapitulate what we have explicitly shown above, we note that the parameter α is the one which changes the coordinates, while for $\alpha = 0$, the transformed coordinates coincide with the initial ones. Transformations of this kind correspond to symmetries of the metric,as long as the latter, expressed in terms of the transformed coordinates, doesn't change numerically up to first order with respect to the parameter α. In our case, the deformed metric in (12) conforms to the Schrödinger symmetry, as implied by the above transformation properties.

As far as the connection of the Schrödinger metric (12) to the real world physical systems is concerned, we note that this metric has been proposed [5, 6] as a potential gravity dual of the unitary Fermi gas. Of considerable interest in the active research field, is the dynamics of Fermi superfluids which can be made to undergo a cross-over point between a Bose condensation phase and a standard BCS superconductivity phase. This is widely known as the system of fermions at unitarity, since at this cross-over point the s-wave scattering cross section saturates the unitarity bound. This unitarity bound is commonly referred to as the limit of a diverging scattering length $\alpha \to \infty$, and an effective range of the interaction $r \to 0$. A Fermi gas at this limit attains universality: particles with short-range interactions and a large scattering length have universal low energy properties that depend neither on the details of their interactions, nor their interactions at short distances. By low energy, we mean energy close to the scattering threshold for the interacting particles.

4.0.2 Studying the Schwartzschild metric

As in the relativistic case, a strong-coupling expansion of a conformal field theory in the non-relativistic context, is expected to be expressed in terms of a dual geometry whose isometries reproduce the symmetries of the non-relativistic CFT. As illustrated in subsection 4.0.1, such geometries were introduced in [5, 6] as a new application of gauge/gravity duality to non-relativistic CFTs in which the usual AdS_{d+1} space is replaced by the Sch_{d+2} geometry,

$$ds^2 = L^2(-\frac{(dx^{+^2})}{r^4} + \frac{-2dx^+dx^- + dx^i dx^i + dr^2}{r^2}). \tag{36}$$

In [5, 6] this spacetime was shown to satisfy the equations of motion of Einstein gravity coupled to a gauge field of mass $m_A^2 = \frac{z(z+d)}{L^2}$ and a cosmological constant $\Lambda = \frac{(d+1)(d+2)}{L^2}$, and was argued to be dual to a non-relativistic CFT at zero temperature and zero density. In order to proceed to finite temperature generalizations we need to place a black hole inside this geometry. Pursuing this direction, the corresponding black hole solution with the asymptotics of (36) has been considered in [10, 13].

Our main interest now, is to examine the invariance of the so-called Schwartzschild metric in light-cone coordinates

$$\begin{aligned} ds^2 &= g_{++}dx^{+^2} + 2g_{+-}dx^+dx^- + g_{--}dx^{-^2} + g_{yy}dy^2 + g_{zz}dz^2 + g_{rr}dr^2 \qquad (37) \\ &= \left(\frac{r}{L}\right)^2 \{\frac{1-h}{4b^2}dx+^2 - (1+h)dx^+dx^- + (1-h)b^2dx-^2 + dy^2 + dz^2\} \\ &+ \left(\frac{L}{r}\right)^2 h^{-1}dr^2 \quad with \quad h := 1 - (\frac{r_H}{r})^4, \qquad (38) \end{aligned}$$

where $r = r_H$ is the location of the horizon, while the boundary is set at some large (with respect to r_H) fixed value of r. The light-cone coordinates are

$$x^+ = b(t+x), \quad x^- = \frac{1}{2b}(t-x). \tag{39}$$

We know from the former analysis in subsection 4.0.1, that the generators of the Schrödinger algebra correspond to the following isometries of the metric:

$$P^i : x^i \to x^i + \alpha^i \tag{40}$$

$$H : x^+ \to x^+ + \alpha \tag{41}$$

$$M : x^- \to x^- + \alpha \tag{42}$$

$$K^i : x^i \to x^i - \alpha^i x^+,\ x^- \to x^- - \alpha^i x^i \tag{43}$$

$$D : x^i \to (1-\alpha)x^i,\ r \to (1+\alpha)r,\ x^+ \to (1-\alpha)^2 x^+,\ x^- \to x^-,\ b \to (1-\alpha)b \tag{44}$$

$$C : r \to (1+\alpha x^+)r,\ x^i \to (1-\alpha x^+)x^i,\ x^+ \to (1-\alpha x^+)x^+,\ x^- \to x^- - \frac{\alpha}{2}(x^i x^i + \frac{1}{r^2}) \tag{45}$$

and we perform each of the above infinitesimal transformations to the expression given in (38).

- Once we apply the transformation that is implied by the momenta in (40) we obtain:

$$\begin{aligned} d\tilde{s}^2 &= \left(\frac{r}{L}\right)^2 \{\frac{1-h}{4b^2} dx^{+^2} - (1+h)dx^+ dx^- + (1-h)b^2 dx^{-^2} + dy^2 + dz^2\} + \left(\frac{L}{r}\right)^2 h^{-1} dr^2 \\ &= ds^2 \end{aligned} \tag{46}$$

  since α^i is a constant vector.

- Due to the constancy of the underlying parameter,α, the operation of the transformations which are generated by the Hamilton operator in (41) and the Mass operator in (42), yields the original metric.

- Next we consider the galilean boosts ,K^i, in (43) and the corresponding outcome is:

$$d\tilde{s}^2 = \left(\frac{r}{L}\right)^2 \{\frac{1-h}{4b^2}dx^{+^2} - (1+h)dx^+ d\tilde{x}^- + (1-h)b^2 d\tilde{x}^{-^2} + d\tilde{y}^2 + d\tilde{z}^2\} + \left(\frac{L}{r}\right)^2 h^{-1}dr^2 \tag{47}$$

$$= \left(\frac{r}{L}\right)^2 \{\frac{1-h}{4b^2}dx^{+^2} - (1+h)dx^+(dx^- - \alpha^i dx^i) + (1-h)b^2(dx^- - \alpha^i dx^i)^2$$
$$+ (dy - \alpha_y dx^+)^2 + (dz - \alpha_z dx^+)^2\} + \left(\frac{L}{r}\right)^2 h^{-1}dr^2 \tag{48}$$

$$= \left(\frac{r}{L}\right)^2 \frac{1-h}{4b^2}dx^{+^2} + \left(\frac{l}{r}\right)^2 h^{-1}dr^2 + \left(\frac{r}{L}\right)^2 \{-(1+h)dx^+dx^- + (1+h)dx^+(\alpha_y dy + \alpha_z dz)$$
$$+ (1-h)b^2 dx^{-^2} + (1-h)b^2(\alpha_y dy + \alpha_z dz)^2 - 2(1-h)b^2(\alpha_y dy + \alpha_z dz)dx^-$$
$$+ dy^2 + (\alpha_y dx^+)^2 - 2\alpha_y dx^+ dy + dz^2 + (\alpha_z dx^+)^2 - 2\alpha_z dx^+ dz\} \tag{49}$$

$$= ds^2 + \left(\frac{r}{L}\right)^2 \{(1+h)dx^+(\alpha_y dy + \alpha_z dz) - 2(1-h)b^2(\alpha_y dy + \alpha_z dz)dx^- - 2dx^+(\alpha_y dy + \alpha_z dz)$$
$$+ (1-h)b^2(\alpha_y dy + \alpha_z dz)^2 + (dx^+)^2(\alpha_y^2 + \alpha_z^2)\} \tag{50}$$

$$= ds^2 + \left(\frac{r}{L}\right)^2 \{dx^+(\alpha_y dy + \alpha_z dz)(h-1) + 2(h-1)b^2(\alpha_y dy + \alpha_z dz)dx^-\} + \mathcal{O}(\alpha^2) \tag{51}$$

$$= ds^2 + \left(\frac{r}{L}\right)^2 \alpha^i dx^i (h-1)(dx^+ + 2b^2 dx^-) + \mathcal{O}(\alpha^2). \tag{52}$$

- The metric under the action of the dilatation operator in (44) takes the form:

$$d\tilde{s}^2 = \left(\frac{\tilde{r}}{L}\right)^2 \{\frac{1-h}{4b^2}d\tilde{x}^{+^2} - (1+h)d\tilde{x}^+ d\tilde{x}^- + (1-h)\tilde{b}^2 d\tilde{x}^{-^2} + d\tilde{y}^2 + d\tilde{z}^2\} + \left(\frac{L}{\tilde{r}}\right)^2 h^{-1}d\tilde{r}^2 \tag{53}$$

$$= (1+\alpha)^2 \left(\frac{r}{L}\right)^2 \{\frac{1-h}{4b^2}\frac{1}{(1-\alpha)^2}(1-\alpha)^4 dx^{+^2} - (1+h)(1-\alpha)^2 dx^{-^2}$$
$$+ (1-\alpha)^2(dy^2 + dz^2)\} + (1+\alpha)^{-2}\left(\frac{L}{r}\right)^2 h^{-1}(1+\alpha)^2 dr^2 \tag{54}$$

$$= (1+\alpha)^2(1-\alpha)^2 \left(\frac{r}{L}\right)^2 \{\frac{1-h}{4b^2}dx^{+^2} - (1+h)b^2 dx^{-^2} + dy^2 + dz^2\} + \left(\frac{L}{r}\right)^2 h^{-1}dr^2 \tag{55}$$

$$= (1-\alpha^2)^2 \left(\frac{r}{L}\right)^2 \{\frac{1-h}{4b^2}dx^{+^2} - (1+h)b^2 dx^{-^2} + dy^2 + dz^2\} + \left(\frac{L}{r}\right)^2 h^{-1}dr^2 \tag{56}$$

$$= ds^2 + \mathcal{O}(\alpha^2). \tag{57}$$

- Then, we implement the special conformal transformation ,C, in (45) and we end up with the following result:

$$d\tilde{s}^2 = \left(\frac{\tilde{r}}{L}\right)^2 \{\frac{1-h}{4b^2}d\tilde{x}^{+^2} - (1+h)d\tilde{x}^+ d\tilde{x}^- + (1-h)b^2 d\tilde{x}^{-^2} + d\tilde{y}^2 + d\tilde{z}^2\} + \left(\frac{L}{\tilde{r}}\right)^2 h^{-1} d\tilde{r}^2 \tag{58}$$

$$= (1+\alpha x^+)^2 \left(\frac{r}{L}\right)^2 \{\frac{1-h}{4b^2}dx^{+^2}(1-2\alpha x^+)^2 - (1+h)dx^+(1-2\alpha x^+)(dx^- - \alpha x^i dx^i + \frac{\alpha}{r^3}dr)$$
$$+ (1-h)b^2(dx^- - \alpha x^i dx^i + \frac{\alpha}{r^3}dr)^2 + [dy(1-\alpha x^+) - \alpha y dy]^2 + [dz(1-\alpha x^+) - \alpha z dz]^2\}$$
$$+ (\frac{L}{r})^2 \frac{1}{h(1+\alpha x^+)^2}[dr(1+\alpha x^+) + \alpha r dx^+]^2 \tag{59}$$

$$= \left(\frac{r}{l}\right)^2 [(1+\alpha x^+)(1-2\alpha x^+)]^2 \frac{1-h}{4b^2}dx^{+^2}$$
$$+ \left(\frac{r}{l}\right)^2 (1+h)[-(1-2\alpha x^+)dx^+ dx^- + \alpha(1-2\alpha x^+)x^i dx^i dx^+ - \alpha(1-2\alpha x^+)\frac{dr}{r^3}dx^+]$$
$$+ \left(\frac{r}{L}\right)^2 (1-h)b^2[dx^{-^2} - 2\alpha(x^i dx^i - \frac{dr}{r^3})dx^- + \alpha^2(x^i dx^i - \frac{dr}{r^3})^2]$$
$$+ \left(\frac{r}{l}\right)^2 [(1-\alpha x^+)^2 dx^{i^2} - 2\alpha(1-\alpha x^+)x^i dx^i dx^+ + (\alpha x^i dx^+)^2]$$
$$+ \left(\frac{l}{r}\right)^2 \frac{1}{h(1+\alpha x^+)^2}[(1+\alpha x^+)^2 dr^2 + 2\alpha(1+\alpha x^+) r dr dx^+ + (\alpha r dx^+)^2] \tag{60}$$

$$= \left(\frac{r}{L}\right)^2 (1-2\alpha x^+)\frac{1-h}{4b^2}dx^{+^2} + \left(\frac{r}{L}\right)^2 (1+h)[-(1-2\alpha x^+)dx^+ dx^- + \alpha x^i dx^i dx^+ - \alpha\frac{dr}{r^3}dx^+]$$
$$+ \left(\frac{r}{L}\right)^2 b^2(1-h)[dx^{-^2} - 2\alpha x^i dx^i dx^- + 2\alpha\frac{dr}{r^3}dx^-] + \left(\frac{r}{l}\right)^2 [(1-2\alpha x^+)dx^{i^2} - 2\alpha x^i dx^i dx^+]$$
$$+ \frac{1}{h}\left(\frac{l}{r}\right)^2 (dr^2 + 2\alpha r dr dx^+) + \mathcal{O}(\alpha^2) \tag{61}$$

$$= ds^2 - \left(\frac{r}{L}\right)^2 \frac{(1-h)\alpha x^+}{2b^2}dx^{+^2} + \left(\frac{r}{L}\right)^2 (1+h)[2\alpha x^+ dx^+ dx^- + \alpha x^i dx^i dx^+ - \alpha\frac{dr}{r^3}dx^+]$$
$$+ \left(\frac{r}{L}\right)^2 b^2(1-h)[-2\alpha x^i dx^i dx^- + 2\alpha\frac{dr}{r^3}dx^-] + \left(\frac{r}{L}\right)^2 [-2\alpha x^+ dx^{i^2} - 2\alpha x^i dx^i dx^+]$$
$$+ \frac{2}{h}\left(\frac{L}{r}\right)^2 \alpha r dr dx^+ + \mathcal{O}(\alpha^2) \tag{62}$$

$$= ds^2 + \left(\frac{r}{L}\right)^2 \alpha\{[\frac{h-1}{2b^2}x^+ dx^+ + 2(1+h)x^+ dx^- + (h-1)x^i dx^i]dx^+$$
$$+ 2[-x^+ dx^i + b^2(1-h)x^i dx^-]dx^i + [(-1-h+\frac{2l^4}{h})dx^+ + 2b^2(1-h)dx^-]\frac{dr}{r}\} + \mathcal{O}(\alpha^2) \tag{63}$$

Having imposed the set of transformations that correspond to the Schrödinger group, we can ascertain that the Schwartzschild metric does not preserve the Schrödinger symmetry.

More specifically, it omits the Galilean boosts and the special conformal transformation, while it retains the spacetime translations, the dilatations and the transformation that is generated by the Mass operator.

The above computations were performed regarding the Schwartzschild metric in the general case. Now, we are interested in applying the transformations that are implied by the Schrödinger algebra, to the AdS-Schwartzschild black hole solution in light-cone coordinates. To that extend, we need to fix the value of the parameter which is related to the location of the horizon, and more specifically we have to set h=1. After imposing this particular value, the metric in (38) takes the form:

$$ds^2 = \left(\frac{r}{L}\right)^2 [-2dx^+dx^- + dy^2 + dz^2] + \left(\frac{L}{r}\right)^2 dr^2 \tag{64}$$

Since the Schwartzschild metric in its general form violates the symmetries that are related to the Galilean boosts and the temporal component of the special conformal transformations, while preserving the rest of the symmetries, in order to examine the invariance of the AdS-Schwartzschild metric in (64), we just have to consider the transformation properties that apply to these particular cases.

- So, performing the galilean boosts, K^i, in (43), we get:

$$d\tilde{s}^2 = \left(\frac{r}{L}\right)^2 [-2dx^+d\tilde{x}^- + d\tilde{y}^2 + d\tilde{z}^2] + \left(\frac{L}{r}\right)^2 dr^2 \tag{65}$$

$$= \left(\frac{r}{L}\right)^2 [-2dx^+(dx^- - \alpha_y dy - \alpha_z dz) + (dy - \alpha_y dx^+)^2 + (dz - \alpha_z dx^+)^2] + \left(\frac{L}{r}\right)^2 dr^2 \tag{66}$$

$$= \left(\frac{r}{L}\right)^2 [-2dx^+dx^- + 2dx^+(\alpha_y dy + \alpha_z dz) + dy^2 + dz^2 - 2dx^+(\alpha_y dy + \alpha_z dz] + (\alpha_y dx^+)^2 + (\alpha_z dx^+)^2 + \left(\frac{L}{r}\right)^2 dr^2 \tag{67}$$

$$= ds^2 + \mathcal{O}(\alpha^2) \tag{68}$$

- While proceeding with the special conformal transformation ,C, in (45), we obtain:

$$d\tilde{s}^2 = \tilde{r}^2[-2d\tilde{x}^+ d\tilde{x}^- + d\tilde{y}^2 + d\tilde{z}^2] + \frac{d\tilde{r}^2}{\tilde{r}^2} \tag{69}$$

$$= -2(1+\alpha x^+)^2 r^2(1-2\alpha x^+)dx^+[dx^- - \alpha(x^i dx^i - \frac{dr}{r^3})] + r^2[dy^2 + dz^2 - 2\alpha(ydx^+dy + zdx^+dz)] + \frac{1}{r^2(1+\alpha x^+)^2}[dr(1+\alpha x^+) + ardx^+]^2 \tag{70}$$

$$= -2r^2dx^+dx^- + 2\alpha r^2(ydydx^+ + zdzdx^+ - \frac{drdx^+}{r^3}) + r^2(dy^2+dz^2) - 2\alpha r^2(ydydx^+ + zdzdx^+ - \frac{drdx^+}{r^3}) + \frac{dr^2}{r^2} + \mathcal{O}(\alpha^2) \tag{71}$$

$$= ds^2 + \mathcal{O}(\alpha^2) \tag{72}$$

where we have imposed that L=1.

It is now straightforward to confirm that the AdS-Schwartzschild metric in (64) exhibits a full Schrödinger symmetry.

5 Ohmic Conductivity of the AdS Black Hole

5.1 Theoretical Background

[7]

The conductivity tensor σ_{ij} measures the response of a conducting medium to externally applied fields. It is defined by

$$\langle J_i \rangle = \sigma_{ij} E_j \tag{73}$$

where E are externally applied electric fields and $\langle J \rangle$ are the currents induced in the medium.

Before the advent of the fluctuation-dissipation theorem [12], the most common method traditionally employed to calculate transport properties was to set up the kinetic equation for the molecular distribution functions, solved for stationary or periodic conditions. But the transport equation itself was an approximation that could not be derived without a rather strict and often unsatisfied certain condition, characteristics that would render the former method quite inferior compared to R.Kubo's method ; according to which physical quantities such as electric or magnetic susceptibility for alternating fields, electric conductivity and so on, are stringently expressed in terms of time-fluctuation of dynamical variables associated with such irreversible processes. In this way, the complete conductivity tensor for a given

[7] For a detailed approach you should resort to [19], [20].

frequency of an applied electric field can be rigorously expressed in terms of electric current components fluctuating spontaneously in equilibrium state.

We shall thus consider the corresponding Kubo formula [12] for the electric conductivity.

Consider a material in the presence of an electric field (in the long-wavelength limit)

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 e^{i(\mathbf{kx}-\omega t)} e^{\eta t}, \tag{74}$$

where η is a positive infinitesimal to turn on the field adiabatically from time $t = -\infty$. The Fourier transform of the ensemble average of the resulting current density is given by (to first order in $\mathbf{E_0}$)

$$\mathbf{J}_a(\mathbf{k}', \omega') = \delta_{\mathbf{k}',\mathbf{k}} \delta(\omega' - \omega) \sum \sigma_{ab}(\mathbf{k}, \omega) \mathbf{E}_{0b}, \tag{75}$$

where $\sigma_{ab}(\mathbf{k}, \omega)$ is the electrical-conductivity tensor. This tensor is then given by the well-known Kubo Formula

$$\sigma_{ab}(\mathbf{k}, \omega) = -\frac{1}{\hbar\omega} \int_{-\infty}^{0} d\tau e^{-i(\omega+i\eta)\tau} \langle [\mathbf{J}_b(\mathbf{k},\tau), \mathbf{J}_a(\mathbf{k},0)] \rangle, \tag{76}$$

where the square brackets denote a commutator, and the angular brackets denote an ensemble average in the absence of the electric field. Here $\mathbf{J}(\mathbf{k},\tau)$ is the Heisenberg operator

$$\mathbf{J}(\mathbf{k},\tau) = e^{i\frac{H_0}{\hbar}\tau} \mathbf{J}(\mathbf{k}) \mathbf{e}^{-\mathbf{i}\frac{\mathbf{H_0}}{\hbar},\tau} \tag{77}$$

with H_0 the Hamiltonian of the unperturbed system.

Our main target in the following subsections is to determine the conductivities associated with massive $\mathcal{N} = 2$ supersymmetric hypermultiplet flavour fields propagating in an $\mathcal{N}$ =4 supersymmetric SU(N_c) Yang-Mills (SYM) theory plasma at temperature T. Inserting a small number $N_f \ll N_c$ of them, the theory for massless hypermultiplets will remain approximately conformal to leading order in N_c. In the supergravity theory, this corresponds to introducing N_f D_7 - branes in the probe limit, so that the AdS background is left unmodified.

The method we follow in order to compute the conductivity is the Karch-O'Bannon probe brane method presented in detail in [19], [20] for which we only need a valid action for the probe brane, which will be the Dirac-Born-Infeld (DBI) action, and a probe brane worldvolume with a horizon. The desired transport property of the $\mathcal{N}$ =4 SYM theory plasma will be determined by restorting to the anti-de Sitter/conformal field theory (AdS/CFT) correspondence which equates the low-energy effective description of string theory, supergravity, on the background $AdS_5 \times S^5$ with $\mathcal{N}$ =4 supersymmetric SU(N_c) Yang-Mills (SYM) theory in the limits of large N_c and large 't Hooft coupling $\lambda = g_{YM} {N_c}^2$.

The hypermultiplet fields have a global U(N_f) vector symmetry and we may identify the U(1) subgroup of this as baryon number [21]. We work at finite U(1) density. In holography, a global symmetry of the boundary theory will be dual to some gauge invariance in the bulk theory. In our case, a finite baryon number density appears in the supergravity description as a nontrivial component $A_y(r)$ of the D_7 - brane gauge field, with r the AdS radial coordinate.

To accommodate the background electric field E and the induced current $\langle J^y \rangle$ in the field theory, we also need $A_+ = Ey + h_+(r)$, and $A_- = 2b^2 Ey + h_-(r)$ in the supergravity theory so that we have a nondynamical electric field, while the nontrivial r-dependence accounts for avoid getting a zero $\langle J^y \rangle$.

For an AdS-Schwartzschild background, with no gauge fields excited on the D_7 - brane worldvolume, two topologically distinct classes of D_7 - brane solutions are possible. The first are Minkowski embeddings that end outside the horizon, $r^{'} < r_H$. These do not possess a horizon on their worldvolume. The second are called black hole embeddings and they are D_7 - branes that intersect the horizon, thus developing a worldvolume horizon. Since we are aware of the fact that the embedding function is dual to the hypermultiplet mass operator, we note that the Minkowski solutions are dual to large mass in the field theory, while the black hole solutions are dual to small mass where "large" and "small" here are relative to the temperature.

In order to calculate the conductivity we will be based on two notable discoveries of [21]. The first is that when the D_7 - brane worldvolume gauge field corresponding to a finite charge density is turned on, only black hole embeddings are physically allowed. Once we have introduced the gauge field $A_y(r)$, the resulting radial field lines must have some place to end. For a Minkowski solution no such place exists. We may introduce point sources, strings stretching from the D_7 - brane to the horizon, to accommodate the radial field lines. As shown in [21], nevertheless, the force that the strings exert on the D_7 - brane will overcome the tension of the D_7 - brane, so the brane will be drawn into the horizon, producing a black hole embedding. This particular choice indicates that in a way the black hole solutions alone "perceive" holographically the whole range of hypermultiplet masses. It is then reasonable to wonder how can the black hole solutions incorporate large mass in the boundary theory, and the answer to that query brings us to the second considerable discovery of [21]. For large hypermultiplet mass in the boundary theory the D_7 - brane nearly resembles a Minkowski solution : it almost ends outside the horizon but develops a "spike" that extends all the way down to the horizon. Therefore, for our analysis of the boundary theory at finite baryon number density we need only consider black hole solutions in the supergravity description.

The fact that the induced metric for a black hole embedding develops a horizon appears to be crucial for the behaviour of the brane in the presence of an electric field. A general attribute of the D-branes is that they exhibit an instability for sufficiently large worldvolume electric fields. For a string with both ends attached on the D-brane a worldvolume electric field will pull the endpoints of the string in opposite directions. At a critical value E_c this force grows large enough to overcome the string's tension and the string is ripped apart, hence the instability. For a black hole solution any electric field will trigger this instability since E_c goes to zero at a horizon where the vanishing of the time component of the metric makes the string basically tensionless.As we remarked before, at finite density we have only black hole solutions so at finite density this instability is always present. On the field theory side, this generic instability is ascribed to the fact that at any finite charge density any electric field will start to accelerate the charge carriers. Due to resistance from the $\mathcal{N}$ =4 SYM theory plasma the charge carriers will not accelerate forever but will reach a steady state. It is this steady state solution we are pursuing to find from the bulk point of view in our calculation.

5.2 The conductivity for the Schwartzschild metric

After having tested the invariance of the AdS-Schwartzschild metric in light-cone coordinates in subsection 4.0.2, we continue working with this background in order to compute holographically a DC conductivity related to transport of baryon number charge. The whole process relies on the work done by E. Kiritsis and B.S. Kim.

Once again we consider the AdS-Schwartzschild black hole solution in light-cone coordinates :

$$ds^2 = g_{++}dx^{+^2} + 2g_{+-}dx^+dx^- + g_{--}dx^{-^2} + g_{yy}dy^2 + g_{zz}dz^2 + g_{rr}dr^2 \tag{78}$$

where

$$g_{++} = \frac{(1-h)r^2}{4b^2Ł^2}, \quad g_{+-} = -\frac{(1+h)r^2}{2Ł^2}$$
$$g_{--} = \frac{(1-h)b^2r^2}{Ł^2}, \quad g_{yy} = g_{zz} = \frac{r^2}{Ł^2}, \quad g_{rr} = \frac{Ł^2}{hr^2} \tag{79}$$
$$h = 1 - \frac{r_H^4}{r^4}, \quad x^+ = b(t+x), \quad x^- = \frac{1}{2b}(t-x) \tag{80}$$

We require the number of the $D7$-branes filling AdS_5 to be $N_f \ll N_c$, so that we will be able to neglect their back-reaction on the geometry.(5.1) As a result, since we have chosen to probe the branes,the corresponding action will then be the Dirac-Born-Infeld (DBI) action which takes the form:

$$S_{D7} = -N_f T_{D7} \int d^8\sigma \sqrt{-\det(g_{ab} + F_{ab})}\ ,$$

where T_{D7}, σ and F_{ab} are the D-brane's tension, the world-volume coordinate and the $U(1)$ field strength, respectively.

Focusing on a $D7$-brane black hole embedding, we indicate the ansatz for a gauge field that contains a light-cone electric field E in the y-direction which is:

$$A_+ = Ey + h_+(r)\ , \quad A_- = 2b^2Ey + h_-(r)\ , \quad A_y = 2Eb^2x^- + h_y(r)\ . \tag{81}$$

Inserting the given expressions for the metric (78) and the gauge field (81) at this point, the DBI action becomes:

$$S_{D_7} = -\mathcal{N} \int dr \sqrt{-detM}, \tag{82}$$

where $\mathcal{N} = 2\pi^2 N_f T_{D_7}$ and,

$$detM = g_{zz}[g_{rr}(G_{+-}g_{yy} + \tilde{E}^2 g_{--}) + (g_{++}g_{yy} + \tilde{E}^2)\tilde{h_-}'^2 + G_{+-}\tilde{h_y}'^2 - 2g_{+-}g_{yy}\tilde{h_+}'\tilde{h_-}' + g_{--}g_{yy}\tilde{h_+}'^2] \tag{83}$$

where $G_{+-} = g_{++}g_{--} - g_{+-}^2$, while prime denotes a derivative with respect to r and the symbol ($\tilde{}$) the factor of $2\pi\alpha'$.

As a consequence of the fact that the action depends only on the first derivatives of the components of the gauge field, $h'_\mu(r)$ with $\mu = +, -, y$, the equations of motion correspond to constants of motion, in the sense that they are quantities that do not depend explicitly on the AdS radial coordinate. Denoting with $\mathcal{L}$ the Lagrangian of the DBI action, we write down the derived constants of motion:

$$\begin{aligned}
\langle J^+ \rangle &= \frac{\delta \mathcal{L}}{\delta h'_+} = -H \left(g_{--}\tilde{h}'_+ - g_{+-}\tilde{h}'_- \right) g_{yy} \ , \\
\langle J^- \rangle &= \frac{\delta \mathcal{L}}{\delta h'_-} = -H \left(g_{+-}g_{yy}\tilde{h}'_+ - [\tilde{E}^2 + g_{++}g_{yy}]\tilde{h}'_- \right) \ , \\
\langle J^y \rangle &= \frac{\delta \mathcal{L}}{\delta h'_y} = -H\, G_{+-}\tilde{h}'_y
\end{aligned} \tag{84}$$

where $H = -\frac{2\pi\alpha' \mathcal{N} g_{zz}}{\sqrt{-detM}}$.

Next we solve the equations of motion with respect to the terms h'_μ, and we obtain the following results:

$$h'_y = \frac{\mathcal{N}\langle J^y \rangle \varphi \sqrt{-g_{rr}g_{yy}}}{2\pi\alpha' \sqrt{\Xi}} \tag{85}$$

where

$$
\begin{aligned}
\Xi =\ & \langle J^+\rangle^2 g_{++}g_{+-}^4 g_{yy} + 2\langle J^+\rangle\langle J^-\rangle g_{+-}^5 g_{yy} - 2\langle J^+\rangle^2 g_{++}^2 g_{+-}^2 g_{--}g_{yy} - 4\langle J^+\rangle\langle J^-\rangle g_{++}g_{+-}^3 g_{--}g_{yy} \\
& + \langle J^-\rangle^2 g_{+-}^4 g_{--}g_{yy} + \langle J^+\rangle^2 g_{++}^3 g_{--}^2 g_{yy} + 2\langle J^+\rangle\langle J^-\rangle g_{++}^2 g_{+-}g_{--}^2 g_{yy} - 2\langle J^-\rangle^2 g_{++}g_{+-}^2 g_{--}^2 g_{yy} \\
& + \langle J^-\rangle^2 g_{++}^2 g_{--}^3 g_{yy} + (2\pi\alpha^{'})^2\mathcal{N}^2 g_{yy}^2 g_{zz}(g_{++}^3 g_{--}^3 - g_{+-}^6 + 3g_{++}g_{--}g_{+-}^4 - 3g_{++}^2 g_{--}^2 g_{+-}^2) \\
& + \tilde{E}^2\langle J^+\rangle^2 (g_{+-}^4 - 2g_{++}g_{--}g_{+-}^2 + g_{++}^2 g_{--}^2) \\
& + (2\pi\alpha^{'})^2\mathcal{N}^2\tilde{E}^2 g_{yy}g_{zz}(g_{++}^2 g_{--}^3 + g_{--}g_{+-}^4 - 2g{++}g_{--}^2 g_{+-}^2) \\
& + \mathcal{N}^2\langle J^y\rangle^2 (g_{+-}^4 g_{yy}^2 - 2g_{++}g_{--}g_{+-}^2 g_{yy}^2 + g_{++}^2 g_{--}^2 g_{yy}^2 - \tilde{E}^2 g_{--}g_{+-}^2 g_{yy} + \tilde{E}^2 g_{++}g_{--}^2 g_{yy}) \\
& + (\langle J^+\rangle^2 g_{++} + 2\langle J^+\rangle\langle J^-\rangle g_{+-} + \langle J^-\rangle^2 g_{--})G_{+-}^2 g_{yy} + (2\pi\alpha^{'})^2\mathcal{N}^2 G_{+-}^3 g_{yy}^2 g_{zz} + \tilde{E}^2\langle J^+\rangle^2 G_{+-}^2 \\
& + (2\pi\alpha^{'})^2\mathcal{N}^2\tilde{E}^2 G_{+-}^2 g_{--}g_{yy}g_{zz} + \mathcal{N}^2\langle J^y\rangle^2 (G_{+-}^2 g_{yy}^2 + \tilde{E}^2 G_{+-}g_{--}g_{yy}) \\
=\ & [\tilde{E}^2\langle J^+\rangle^2 + (\langle J^+\rangle^2 g_{++} + 2\langle J^+\rangle\langle J^-\rangle g_{+-} + \langle J^-\rangle^2 g_{--})g_{yy})]G_{+-}^2 \\
& + (2\pi\alpha^{'})^2\mathcal{N}^2 G_{+-}^2 \varphi g_{yy}g_{zz} + \mathcal{N}^2(2\pi\alpha^{'})^2\tilde{E}^2 G_{+-}^2 g_{--}g_{yy}g_{zz} + \mathcal{N}^2\langle J^y\rangle^2 G_{+-}\varphi g_{yy}. \qquad (86)
\end{aligned}
$$

In order to simplify the above equation and express it in a more elegant and convenient way we introduce the quantity $\varphi = G_{+-}g_{yy} + \tilde{E}^2 g_{--}$ and take advantage of the following relations:

$$G_{+-}^2 = (g_{++}g_{--})^2 + g_{+-}^4 - 2g_{++}g_{--}g_{+-}^2 \qquad (87)$$

$$G_{+-}^3 = (g_{++}g_{--})^3 - g_{+-}^6 + 3g_{++}g_{--}g_{+-}^4 - 3(g_{++}g_{--})^2 g_{+-}^2 \qquad (88)$$

$$
\begin{aligned}
\varphi^2 &= G_{+-}^2 g_{yy}^2 + \tilde{E}^4 g_{--}^2 + 2G_{+-}\tilde{E}^2 g_{--}g_{yy} \\
&= (g_{++}^2 g_{--}^2 + g_{+-}^4 - 2g_{++}g_{--}g_{+-}^2)g_{yy}^2 + \tilde{E}^4 g_{--}^2 + 2\tilde{E}^2 g_{--}g_{yy}(g_{++}g_{--} - g_{+-}^2) \qquad (89)
\end{aligned}
$$

After these manipulations, the equation (85) takes the form:

$$
\begin{aligned}
h_y' &= \frac{\mathcal{N}\langle J^y\rangle\varphi(-g_{rr}g_{yy})^{1/2}}{(2\pi\alpha')G_{+-}(-Ug_{yy}\varphi + Vg_{yy}\varphi)^{1/2}} \\
&= \frac{\mathcal{N}\langle J^y\rangle(g_{rr})^{1/2}}{(2\pi\alpha')G_{+-}}\sqrt{\frac{\varphi}{U-V}} \qquad (90)
\end{aligned}
$$

where

$$U = -\mathcal{N}^2\frac{\langle J^y\rangle^2}{G_{+-}} - (2\pi\alpha^{'})^2\mathcal{N}^2 g_{zz} \qquad (91)$$

$$V = \frac{\tilde{E}^2\langle J^+\rangle^2 + (\langle J^+\rangle^2 g_{++} + 2\langle J^+\rangle\langle J^-\rangle g_{+-} + \langle J^-\rangle^2 g_{--})g_{yy})}{g_{yy}\varphi}. \qquad (92)$$

The other two terms that appear after solving the relevant system of equations are:

$$h'_+ = \frac{(\langle J^+\rangle g_{++}+\langle J^-\rangle g_{+-})g_{yy}+\tilde{E}^2\langle J^+\rangle}{2\pi(g_{yy})^{1/2}(\langle J^+\rangle g_{+-}+\langle J^-\rangle g_{--})}\sqrt{\frac{(\langle J^+\rangle g_{+-}+\langle J^-\rangle g_{--})^2(-g_{rr}\varphi-(2\pi\alpha')^2G_{+-}h_y'^2)}{\alpha'^2\varphi[2\langle J^+\rangle\langle J^-\rangle g_{+-}g_{yy}+\langle J^+\rangle^2(g_{++}g_{yy}+\tilde{E}^2)+\langle J^-\rangle^2 g_{--}g_{yy}+(2\pi\alpha')^2\mathcal{N}^2 g_{yy}g_{zz}\varphi]}}$$
$$= \frac{(\langle J^+\rangle g_{++}+\langle J^-\rangle g_{+-})g_{yy}+\tilde{E}^2\langle J^+\rangle}{(2\pi\alpha')(g_{yy}\varphi)^{1/2}}\sqrt{\frac{-g_{rr}\varphi-(2\pi\alpha')^2G_{+-}h_y'^2}{g_{yy}\varphi V+g_{yy}\varphi(2\pi\alpha')^2\mathcal{N}^2 g_{zz}}}$$
$$= \frac{(\langle J^+\rangle g_{++}+\langle J^-\rangle g_{+-})g_{yy}+\tilde{E}^2\langle J^+\rangle}{(2\pi\alpha')g_{yy}\varphi}\sqrt{\frac{-g_{rr}\varphi-(2\pi\alpha'^2)G_{+-}h_y'^2}{V+(2\pi\alpha')^2\mathcal{N}^2 g_{zz}}} \quad (93)$$

$$h'_- = \frac{(g_{yy})^{1/2}}{2\pi}\sqrt{\frac{(\langle J^+\rangle g_{+-}+\langle J^-\rangle g_{--})^2(-g_{rr}\varphi-(2\pi\alpha')^2G_{+-}h_y'^2)}{\alpha'^2\varphi[2\langle J^+\rangle\langle J^-\rangle g_{+-}g_{yy}+\langle J^+\rangle^2(g_{++}g_{yy}+\tilde{E}^2)+\langle J^-\rangle^2 g_{--}g_{yy}+(2\pi\alpha')^2\mathcal{N}^2 g_{yy}g_{zz}\varphi]}}$$
$$= \frac{(g_{yy})^{1/2}(\langle J^+\rangle g_{+-}+\langle J^-\rangle g_{--})}{(2\pi\alpha')\varphi^{1/2}}\sqrt{\frac{-g_{rr}\varphi-(2\pi\alpha')^2G_{+-}h_y'^2}{g_{yy}\varphi V+g_{yy}\varphi(2\pi\alpha')^2\mathcal{N}^2 g_{zz}}}$$
$$= \frac{\langle J^+\rangle g_{+-}+\langle J^-\rangle g_{--}}{(2\pi\alpha')\varphi}\sqrt{\frac{-g_{rr}\varphi-(2\pi\alpha')^2G_{+-}h_y'^2}{V+(2\pi\alpha')^2\mathcal{N}^2 g_{zz}}} \quad (94)$$

Plugging those solutions back into the action (82), we end up with the on-shell action:

$$S_{D_7} = -\mathcal{N}\int dr\ \sqrt{g_{zz}[-g_{rr}\varphi-(2\pi\alpha')^2[(g_{++}g_{yy}+\tilde{E}^2)h_-'^2+G_{+-}h_y'^2-2g_{+-}g_{yy}h'_+h'_-+g_{--}g_{yy}h_+'^2]]}$$
$$= -\mathcal{N}\int dr\ (g_{rr}g_{zz})^{1/2}\sqrt{\frac{[\mathcal{N}^2\langle J^y\rangle^2+G_{+-}(U-V)][\varphi g_{yy}\varphi V-g_{yy}\varphi^2(V+(2\pi\alpha')^2\mathcal{N}^2 g_{zz})]}{g_{yy}\varphi G_{+-}(U-V)(V+(2\pi\alpha')^2\mathcal{N}^2 g_{zz})}}$$
$$= -\mathcal{N}\int dr\ (g_{rr}g_{zz})^{1/2}\sqrt{\frac{[\mathcal{N}^2\langle J^y\rangle^2+G_{+-}(U-V)](V-V-(2\pi\alpha')^2\mathcal{N}^2 g_{zz})}{G_{+-}(V+(2\pi\alpha')^2\mathcal{N}^2 g_{zz})}}\sqrt{\frac{\varphi}{U-V}}$$
$$= -\mathcal{N}\int dr\ (g_{rr}g_{zz})^{1/2}\sqrt{\frac{G_{+-}(-U-(2\pi\alpha')^2\mathcal{N}^2 g_{zz}+U-V)(-(2\pi\alpha')^2\mathcal{N}^2 g_{zz})}{G_{+-}(V+(2\pi\alpha')^2\mathcal{N}^2 g_{zz})}}\sqrt{\frac{\varphi}{U-V}}$$
$$= -(2\pi\alpha')^2\mathcal{N}^2\int dr\ g_{zz}grr^{1/2}\sqrt{\frac{\varphi}{U-V}} \quad (95)$$

At this point, demanding reality of the expression written for the on-shell action (95), allows us to solve for the current $\langle J^y\rangle$. For that reason, we have to consider the fraction under the square root. Expanding the numerator of the action in powers of r, we observe that it vanishes at some r_* between the horizon and the boundary

$$\varphi = \tilde{E}^2 g_{--} + G_{+-}g_{yy} = -\frac{r^6}{L^6} + \frac{r^2 r_H^4}{L^6} + \frac{\tilde{E}^2 r_H^4 b^2}{r^2 L^2}. \quad (96)$$

In order to avoid having an imaginary DBI action, we demand the mutual vanishing of the denominator (get rid off the instability) and we postulate that both the numerator and the denominator change sign at the same place, r_*. Thus we demand

$$[\tilde{E}^2 g_{--} + G_{+-} g_{yy}]_{r=r_*} = 0 \; and \tag{97}$$

$$U(r_*) - V(r_*) = 0 \; . \tag{98}$$

Setting the numerator of V to be zero at $r = r_*$ and making use of the expression in (97), we get

$$\begin{aligned} &[\tilde{E}^2 \langle J^+ \rangle^2 + (\langle J^+ \rangle^2 g_{++} + 2\langle J^+ \rangle \langle J^- \rangle g_{+-} + \langle J^- \rangle^2 g_{--}) g_{yy})]_{r=r_*} = 0 \Longleftrightarrow \\ &(g_{+-}^2 - g_{++} g_{--}) g_{yy} \langle J^+ \rangle^2 + (\langle J^+ \rangle^2 g_{++} g_{--} + 2\langle J^+ \rangle \langle J^- \rangle g_{--} g_{+-} + \langle J^- \rangle^2 g_{--}^2) g_{yy} = 0 \Longleftrightarrow \\ &(\langle J^+ \rangle g_{+-})^2 + (\langle J^- \rangle g_{--})^2 + 2\langle J^+ \rangle \langle J^- \rangle g_{--} g_{+-} = 0 \Longleftrightarrow \\ &\langle J^- \rangle = -\frac{g_{+-}}{g_{--}}\Big|_{r=r_*} \langle J^+ \rangle \; . \end{aligned} \tag{99}$$

By plugging the condition (99) to the equation $V(r_*) = U(r_*)$, we obtain the expression of the current along $y-$direction as

$$\begin{aligned} &U(r_*) = V(r_*) \Longleftrightarrow \\ &\langle J^y \rangle^2 = -G_{+-}((2\pi\alpha^{'})^2 g_{zz} + \frac{V(r_*)}{\mathcal{N}^2}) \Longleftrightarrow \\ &\langle J^y \rangle^2 = \frac{\tilde{E}^2 g_{--}}{g_{yy}}((2\pi\alpha^{'})^2 g_{zz} + \frac{\tilde{E}^2 \langle J^+ \rangle^2 + \langle J^+ \rangle^2 (g_{++} - \frac{g_{+-}^2}{g_{--}}) g_{yy}}{\mathcal{N}^2 g_{yy} (\tilde{E}^2 g_{--} + G_{+-} g_{yy})}) \Longleftrightarrow \\ &\langle J^y \rangle^2 = \frac{\tilde{E}^2 g_{--}}{g_{yy}}[(2\pi\alpha^{'})^2 g_{zz} + \frac{\langle J^+ \rangle^2}{\mathcal{N}^2 g_{--} g_{yy}})]\Big|_{r=r_*} . \end{aligned} \tag{100}$$

At this point all we have to do in order to derive the desired result for the conductivity, is to consider Ohm's law, $\langle J^y \rangle = \sigma E$.

$$\begin{aligned} \sigma &= (2\pi\alpha^{'})(\frac{g_{--}(r_*)}{g_{yy}(r_*)})^{1/2} \sqrt{(2\pi\alpha^{'})^2 g_{zz}(r_*) + \frac{\langle J^+ \rangle^2}{\mathcal{N}^2 g_{--}(r_*) g_{yy}(r_*)}} \\ &= (2\pi\alpha^{'}) \sqrt{(2\pi\alpha^{'})^2 g_{--}(r_*) + \frac{\langle J^+ \rangle^2}{\mathcal{N}^2 g_{yy}(r_*) g_{zz}(r_*)}} \end{aligned}$$

where we can substitute the elements of the metric (79) and take

$$= (2\pi\alpha^{'}) \sqrt{(2\pi\alpha^{'})^2 (\frac{b}{L})^2 \frac{r_H^4}{r_*^2} + \frac{\langle J^+ \rangle^2 L^4}{\mathcal{N}^2 r_*^4}}. \tag{101}$$

In the next step we identify the location of the horizon as $r_H = \pi bTL^2$ [8] then we solve the equation (96) with respect to r_* and we obtain :

$$r_*^4 = \frac{(\pi bT)^4 L^8 + (\pi bT)^2 L^6 \sqrt{(\pi bTL)^4 + 4\tilde{E}^2 b^2}}{2} \tag{102}$$

so the conductivity in (101) appears as

$$\sigma = (2\pi\alpha^{'}) \sqrt{\frac{\sqrt{2}(2\pi\alpha')^2 \pi^3 b^5 T^3 L^3}{\sqrt{(\pi bTL)^2 + \sqrt{(\pi bTL)^4 + 4\tilde{E}^2 b^2}}} + \frac{2\langle J^+\rangle^2/(\pi bTL)^2}{(\pi bTL)^2 + \sqrt{(\pi bTL)^4 + 4\tilde{E}^2 b^2}}}. \tag{103}$$

The result in equation (103) consists of two terms adding in quadrature. This is obvious because even without the density $\langle J^+\rangle$, if we impose an electric field E, we will obtain the contribution arising from the first term in the sum. We will try to identify the physical origin of each addendum closely following the subsequent references [19],[20],[22],[23]. Two types of charge carriers contribute to the conductivity. The second term describes the contribution to the current from the charge carriers we inserted explicitly via a finite baryon number density, while we are going to explore the nature of the other term by examining its limiting behaviour.

Our intention to reveal the identity of this term leads us to study the zero-temperature limit of the conductivity, as expressed in equation (103). As shown explicitly in the Appendix B-(B.1), for sufficiently low temperatures, the Ohmic resistivity appears to be linear in temperature. This result is in agreement with experimental tests in unconventional superconductors at a critical point of a tuning parameter. ([24])

Suppose now that we have not introduced any baryon number density in our system, an assumption that corresponds to setting $\langle J^+\rangle = 0$ in (103). This choice, as it can be verified from the expansion performed in Appendix B-(B.2), leaves us with a term that depends on both the temperature and the electric field. Then, going to the zero-temperature limit suppresses this contribution, a result that is indicative of thermal pair production.

5.3 The conductivity for a general metric

Closely following the calculation performed by E. Kiritsis and B.S. Kim, we continue working in the spirit of the computation that was carried out in subsection 5.2, and thus we choose the following general metric in order to calculate the DC conductivity in this case.

The planar black hole solution is of the form:

$$ds^2 = \frac{e^{-2A}}{f(r)} dr^2 + e^{2A}(-f(r)dt^2 + dx^2 + dy^2 + dz^2). \tag{104}$$

[8] For a detailed derivation of this result see Appendix A , (A.11).

and having introduced the light-cone coordinates in (80) we express our metric in (104) as:

$$ds^2 = g_{++}dx^{+^2} + 2g_{+-}dx^+dx^- + g_{--}dx^{-^2} + g_{yy}dy^2 + g_{zz}dz^2 + g_{rr}dr^2 \qquad (105)$$

where

$$g_{++} = \frac{e^{2A}}{4b^2}(1-f(r)), \quad g_{+-} = -\frac{e^{2A}}{2}(1+f(r))$$
$$g_{--} = e^{2A}b^2(1-f(r)), \quad g_{yy} = g_{zz} = e^{2A}, \quad g_{rr} = \frac{e^{-2A}}{f(r)} \qquad (106)$$
$$f(r) \simeq 1 - \frac{{r_H}^\omega}{r^\omega}, \quad e^A \simeq \frac{r^c}{L^c}. \qquad (107)$$

Keeping in mind that we adopt here the same basic principles as we did in the case of the Schwartschild metric , the corresponding action will then be the Dirac-Born-Infeld (DBI) action which has the form:

$$S_{D7} = -N_f T_{D7} \int d^8\sigma \sqrt{-\det(g_{ab} + F_{ab})}\ ,$$

where T_{D7}, σ and F_{ab} are the D-brane's tension, the world-volume coordinate and the $U(1)$ field strength, respectively.

Focusing on a $D7$-brane black hole embedding, we indicate the ansatz for a gauge field that contains a light-cone electric field E in the y-direction which is:

$$A_+ = Ey + h_+(r)\ , \quad A_- = 2b^2Ey + h_-(r)\ , \quad A_y = 2Eb^2x^- + h_y(r)\ . \qquad (108)$$

Inserting the given expressions for the metric (105) and the gauge field (108) now, the DBI action becomes:

$$S_{D_7} = -\mathcal{N}\int dr\sqrt{-detM}, \qquad (109)$$

where $\mathcal{N} = 2\pi^2 N_f T_{D_7}$ and,

$$\begin{aligned} detM = g_{zz}[g_{rr}(G_{+-}g_{yy} + \tilde{E}^2 g_{--}) + (g_{++}g_{yy} + \tilde{E}^2)\tilde{h_-}'^2 \\ + G_{+-}\tilde{h_y}'^2 - 2g_{+-}g_{yy}\tilde{h_+}'\tilde{h_-}' + g_{--}g_{yy}\tilde{h_+}'^2] \end{aligned} \qquad (110)$$

where $G_{+-} = g_{++}g_{--} - g_{+-}^2$, while prime denotes a derivative with respect to r and the symbol (˜) the factor of $2\pi\alpha'$.

As a consequence of the fact that the action depends only on the first derivatives of the components of the gauge field, $h'_\mu(r)$ with $\mu = +, -, y$, the equations of motion correspond to constants of motion, in the sense that they are quantities that do not depend explicitly on the AdS radial coordinate. Denoting with $\mathcal{L}$ the Lagrangian of the DBI action, we write down the derived constants of motion:

$$
\begin{aligned}
\langle J^+\rangle &= \frac{\delta\mathcal{L}}{\delta h'_+} = -H\,\left(g_{--}\tilde{h}'_+ - g_{+-}\tilde{h}'_-\right) g_{yy}\ ,\\
\langle J^-\rangle &= \frac{\delta\mathcal{L}}{\delta h'_-} = -H\,\left(g_{+-}g_{yy}\tilde{h}'_+ - [\tilde{E}^2 + g_{++}g_{yy}]\tilde{h}'_-\right)\ ,\\
\langle J^y\rangle &= \frac{\delta\mathcal{L}}{\delta h'_y} = -H\,G_{+-}\tilde{h}'_y
\end{aligned}
\tag{111}
$$

where $H = -\frac{2\pi\alpha'\mathcal{N}g_{zz}}{\sqrt{-detM}}$.

Next we solve the equations of motion with respect to the terms h'_μ, and we obtain the following results:

$$
h'_y = \frac{\mathcal{N}\langle J^y\rangle\varphi\sqrt{-g_{rr}g_{yy}}}{2\pi\alpha'\sqrt{\Xi}}
\tag{112}
$$

where

$$
\begin{aligned}
\Xi = {} & \langle J^+\rangle^2 g_{++}g_{+-}^4 g_{yy} + 2\langle J^+\rangle\langle J^-\rangle g_{+-}^5 g_{yy} - 2\langle J^+\rangle^2 g_{++}^2 g_{+-}^2 g_{--}g_{yy} - 4\langle J^+\rangle\langle J^-\rangle g_{++}g_{+-}^3 g_{--}g_{yy}\\
& + \langle J^-\rangle^2 g_{+-}^4 g_{--}g_{yy} + \langle J^+\rangle^2 g_{++}^3 g_{--}^2 g_{yy} + 2\langle J^+\rangle\langle J^-\rangle g_{++}^2 g_{+-}g_{--}^2 g_{yy} - 2\langle J^-\rangle^2 g_{++}g_{+-}^2 g_{--}^2 g_{yy}\\
& + \langle J^-\rangle^2 g_{++}^2 g_{--}^3 g_{yy} + (2\pi\alpha')^2\mathcal{N}^2 g_{yy}^2 g_{zz}(g_{++}^3 g_{--}^3 - g_{+-}^6 + 3g_{++}g_{--}g_{+-}^4 - 3g_{++}^2 g_{--}^2 g_{+-}^2)\\
& + \tilde{E}^2\langle J^+\rangle^2(g_{+-}^4 - 2g_{++}g_{--}g_{+-}^2 + g_{++}^2 g_{--}^2)\\
& + (2\pi\alpha')^2\mathcal{N}^2\tilde{E}^2 g_{yy}g_{zz}(g_{++}^2 g_{--}^3 + g_{--}g_{+-}^4 - 2g{++}g_{--}^2 g_{+-}^2)\\
& + \mathcal{N}^2\langle J^y\rangle^2(g_{+-}^4 g_{yy}^2 - 2g_{++}g_{--}g_{+-}^2 g_{yy}^2 + g_{++}^2 g_{--}^2 g_{yy}^2 - \tilde{E}^2 g_{--}g_{+-}^2 g_{yy} + \tilde{E}^2 g_{++}g_{--}^2 g_{yy})\\
& + (\langle J^+\rangle^2 g_{++} + 2\langle J^+\rangle\langle J^-\rangle g_{+-} + \langle J^-\rangle^2 g_{--})G_{+-}^2 g_{yy} + (2\pi\alpha')^2\mathcal{N}^2 G_{+-}^3 g_{yy}^2 g_{zz} + \tilde{E}^2\langle J^+\rangle^2 G_{+-}^2\\
& + (2\pi\alpha')^2\mathcal{N}^2\tilde{E}^2 G_{+-}^2 g_{--}g_{yy}g_{zz} + \mathcal{N}^2\langle J^y\rangle^2(G_{+-}^2 g_{yy}^2 + \tilde{E}^2 G_{+-}g_{--}g_{yy})\\
= {} & [\tilde{E}^2\langle J^+\rangle^2 + (\langle J^+\rangle^2 g_{++} + 2\langle J^+\rangle\langle J^-\rangle g_{+-} + \langle J^-\rangle^2 g_{--})g_{yy})]G_{+-}^2\\
& + (2\pi\alpha')^2\mathcal{N}^2 G_{+-}^2\varphi g_{yy}g_{zz} + \mathcal{N}^2(2\pi\alpha')^2\tilde{E}^2 G_{+-}^2 g_{--}g_{yy}g_{zz} + \mathcal{N}^2\langle J^y\rangle^2 G_{+-}\varphi g_{yy}
\end{aligned}
\tag{113}
$$

In order to simplify the above equation and express it in a more elegant and convenient way we introduce the quantity $\varphi = G_{+-}g_{yy} + \tilde{E}^2 g_{--}$ and take advantage of the following relations:

$$
G_{+-}^2 = (g_{++}g_{--})^2 + g_{+-}^4 - 2g_{++}g_{--}g_{+-}^2
\tag{114}
$$

$$
G_{+-}^3 = (g_{++}g_{--})^3 - g_{+-}^6 + 3g_{++}g_{--}g_{+-}^4 - 3(g_{++}g_{--})^2 g_{+-}^2
\tag{115}
$$

$$
\begin{aligned}
\varphi^2 &= G_{+-}^2 g_{yy}^2 + \tilde{E}^4 g_{--}^2 + 2G_{+-}\tilde{E}^2 g_{--}g_{yy} \\
&= (g_{++}^2 g_{--}^2 + g_{+-}^4 - 2g_{++}g_{--}g_{+-}^2)g_{yy}^2 + \tilde{E}^4 g_{--}^2 + 2\tilde{E}^2 g_{--}g_{yy}(g_{++}g_{--} - g_{+-}^2)
\end{aligned} \tag{116}
$$

After these manipulations, the equation (112) takes the form:

$$
\begin{aligned}
h'_y &= \frac{\mathcal{N}\langle J^y\rangle\varphi(-g_{rr}g_{yy})^{1/2}}{(2\pi\alpha')G_{+-}(-Ug_{yy}\varphi + Vg_{yy}\varphi)^{1/2}} \\
&= \frac{\mathcal{N}\langle J^y\rangle(g_{rr})^{1/2}}{(2\pi\alpha')G_{+-}}\sqrt{\frac{\varphi}{U-V}}
\end{aligned} \tag{117}
$$

where

$$
U = -\mathcal{N}^2\frac{\langle J^y\rangle^2}{G_{+-}} - (2\pi\alpha')^2\mathcal{N}^2 g_{zz} \tag{118}
$$

$$
V = \frac{\tilde{E}^2\langle J^+\rangle^2 + (\langle J^+\rangle^2 g_{++} + 2\langle J^+\rangle\langle J^-\rangle g_{+-} + \langle J^-\rangle^2 g_{--})g_{yy})}{g_{yy}\varphi} \tag{119}
$$

.

The other two terms that appear after solving the relevant system of equations are:

$$
\begin{aligned}
h'_+ &= \frac{(\langle J^+\rangle g_{++}+\langle J^-\rangle g_{+-})g_{yy}+\tilde{E}^2\langle J^+\rangle}{2\pi(g_{yy})^{1/2}(\langle J^+\rangle g_{+-}+\langle J^-\rangle g_{--})}\sqrt{\frac{(\langle J^+\rangle g_{+-}+\langle J^-\rangle g_{--})^2(-g_{rr}\varphi-(2\pi\alpha')^2G_{+-}h'^2_y)}{\alpha'^2\varphi[2\langle J^+\rangle\langle J^-\rangle g_{+-}g_{yy}+\langle J^+\rangle^2(g_{++}g_{yy}+\tilde{E}^2)+\langle J^-\rangle^2 g_{--}g_{yy}+(2\pi\alpha')^2\mathcal{N}^2 g_{yy}g_{zz}\varphi]}} \\
&= \frac{(\langle J^+\rangle g_{++} + \langle J^-\rangle g_{+-})g_{yy} + \tilde{E}^2\langle J^+\rangle}{(2\pi\alpha')(g_{yy}\varphi)^{1/2}}\sqrt{\frac{-g_{rr}\varphi - (2\pi\alpha')^2 G_{+-}h'^2_y}{g_{yy}\varphi V + g_{yy}\varphi(2\pi\alpha')^2\mathcal{N}^2 g_{zz}}} \\
&= \frac{(\langle J^+\rangle g_{++} + \langle J^-\rangle g_{+-})g_{yy} + \tilde{E}^2\langle J^+\rangle}{(2\pi\alpha')g_{yy}\varphi}\sqrt{\frac{-g_{rr}\varphi - (2\pi\alpha'^2)G_{+-}h'^2_y}{V + (2\pi\alpha')^2\mathcal{N}^2 g_{zz}}}
\end{aligned} \tag{120}
$$

$$
\begin{aligned}
h'_- &= \frac{(g_{yy})^{1/2}}{2\pi}\sqrt{\frac{(\langle J^+\rangle g_{+-} + \langle J^-\rangle g_{--})^2(-g_{rr}\varphi - (2\pi\alpha')^2 G_{+-}h'^2_y)}{\alpha'^2\varphi[2\langle J^+\rangle\langle J^-\rangle g_{+-}g_{yy} + \langle J^+\rangle^2(g_{++}g_{yy} + \tilde{E}^2) + \langle J^-\rangle^2 g_{--}g_{yy} + (2\pi\alpha')^2\mathcal{N}^2 g_{yy}g_{zz}\varphi]}} \\
&= \frac{(g_{yy})^{1/2}(\langle J^+\rangle g_{+-} + \langle J^-\rangle g_{--})}{(2\pi\alpha')\varphi^{1/2}}\sqrt{\frac{-g_{rr}\varphi - (2\pi\alpha')^2 G_{+-}h'^2_y}{g_{yy}\varphi V + g_{yy}\varphi(2\pi\alpha')^2\mathcal{N}^2 g_{zz}}} \\
&= \frac{\langle J^+\rangle g_{+-} + \langle J^-\rangle g_{--}}{(2\pi\alpha')\varphi}\sqrt{\frac{-g_{rr}\varphi - (2\pi\alpha')^2 G_{+-}h'^2_y}{V + (2\pi\alpha')^2\mathcal{N}^2 g_{zz}}}
\end{aligned} \tag{121}
$$

Plugging those solutions back into the action (109), we end up with the on-shell action:

$$
\begin{aligned}
S_{D_7} &= -\mathcal{N}\int dr\ \sqrt{g_{zz}[-g_{rr}\varphi-(2\pi\alpha')^2[(g_{++}g_{yy}+\tilde{E}^2)h_-'^2+G_{+-}h_y'^2-2g_{+-}g_{yy}h_+'h_-'+g_{--}g_{yy}h_+'^2]]}\\
&= -\mathcal{N}\int dr\ (g_{rr}g_{zz})^{1/2}\sqrt{\frac{[\mathcal{N}^2\langle J^y\rangle^2+G_{+-}(U-V)][\varphi g_{yy}\varphi V-g_{yy}\varphi^2(V+(2\pi\alpha')^2\mathcal{N}^2g_{zz})]}{g_{yy}\varphi G_{+-}(U-V)(V+(2\pi\alpha')^2\mathcal{N}^2g_{zz})}}\\
&= -\mathcal{N}\int dr\ (g_{rr}g_{zz})^{1/2}\sqrt{\frac{[\mathcal{N}^2\langle J^y\rangle^2+G_{+-}(U-V)](V-V-(2\pi\alpha')^2\mathcal{N}^2g_{zz})}{G_{+-}(V+(2\pi\alpha')^2\mathcal{N}^2g_{zz})}}\sqrt{\frac{\varphi}{U-V}}\\
&= -\mathcal{N}\int dr\ (g_{rr}g_{zz})^{1/2}\sqrt{\frac{G_{+-}(-U-(2\pi\alpha')^2\mathcal{N}^2g_{zz}+U-V)(-(2\pi\alpha')^2\mathcal{N}^2g_{zz})}{G_{+-}(V+(2\pi\alpha')^2\mathcal{N}^2g_{zz})}}\sqrt{\frac{\varphi}{U-V}}\\
&= -(2\pi\alpha^{'})^2\mathcal{N}^2\int dr\ g_{zz}grr^{1/2}\sqrt{\frac{\varphi}{U-V}} \qquad (122)
\end{aligned}
$$

At this point, demanding reality of the expression written for the on-shell action (122), allows us to solve for the current $\langle J^y\rangle$. For that reason, we have to consider the fraction under the square root. Expanding the numerator of the action in powers of r, we observe that it vanishes at some r_* between the horizon and the boundary

$$
\varphi = \tilde{E}^2g_{--}+G_{+-}g_{yy} = \frac{r^{2c}}{L^{2c}}[\tilde{E}^2b^2\frac{{r_H}^\omega}{r^\omega}-\frac{r^{4c}}{L^{4c}}(1-\frac{{r_H}^\omega}{r^\omega})]. \qquad (123)
$$

In order to avoid having an imaginary DBI action, we demand the mutual vanishing of the denominator (get rid off the instability) and we postulate that both the numerator and the denominator change sign at the same place, r_*. Thus we demand

$$
[\tilde{E}^2g_{--}+G_{+-}g_{yy}]_{r=r_*} = 0\ and \qquad (124)
$$

$$
U(r_*)-V(r_*) = 0\ . \qquad (125)
$$

Setting the numerator of V to be zero at $r=r_*$ and making use of the expression in (124), we get

$$
\begin{aligned}
&[\tilde{E}^2\langle J^+\rangle^2+(\langle J^+\rangle^2g_{++}+2\langle J^+\rangle\langle J^-\rangle g_{+-}+\langle J^-\rangle^2g_{--})g_{yy})]_{r=r_*}=0 \Longleftrightarrow\\
&(g_{+-}^2-g_{++}g_{--})g_{yy}\langle J^+\rangle^2+(\langle J^+\rangle^2g_{++}g_{--}+2\langle J^+\rangle\langle J^-\rangle g_{--}g_{+-}+\langle J^-\rangle^2g_{--}^2)g_{yy}=0 \Longleftrightarrow\\
&(\langle J^+\rangle g_{+-})^2+(\langle J^-\rangle g_{--})^2+2\langle J^+\rangle\langle J^-\rangle g_{--}g_{+-}=0 \Longleftrightarrow\\
&\langle J^-\rangle = -\frac{g_{+-}}{g_{--}}\bigg|_{r=r_*}\langle J^+\rangle\ . \qquad (126)
\end{aligned}
$$

By plugging the condition (126) to the equation $V(r_*)=U(r_*)$, we obtain the expression of the current along $y-$direction as

$$
\begin{aligned}
&U(r_*) = V(r_*) \Longleftrightarrow \\
&\langle J^y\rangle^2 = -G_{+-}((2\pi\alpha^{'})^2 g_{zz} + \frac{V(r_*)}{\mathcal{N}^2}) \Longleftrightarrow \\
&\langle J^y\rangle^2 = \frac{\tilde{E}^2 g_{--}}{g_{yy}}((2\pi\alpha^{'})^2 g_{zz} + \frac{\tilde{E}^2\langle J^+\rangle^2 + \langle J^+\rangle^2(g_{++} - \frac{g_{+-}^2}{g_{--}})g_{yy}}{\mathcal{N}^2 g_{yy}(\tilde{E}^2 g_{--} + G_{+-}g_{yy})}) \Longleftrightarrow \\
&\langle J^y\rangle^2 = \frac{\tilde{E}^2 g_{--}}{g_{yy}}[(2\pi\alpha^{'})^2 g_{zz} + \frac{\langle J^+\rangle^2}{\mathcal{N}^2 g_{--}g_{yy}})]\bigg|_{r=r_*}.
\end{aligned}
\tag{127}
$$

At this point, after considering Ohm's law, $\langle J^y\rangle = \sigma E$, the conductivity appears as:

$$
\begin{aligned}
\sigma &= (2\pi\alpha^{'})(\frac{g_{--}(r_*)}{g_{yy}(r_*)})^{1/2}\sqrt{(2\pi\alpha')^2 g_{zz}(r_*) + \frac{\langle J^+\rangle^2}{\mathcal{N}^2 g_{--}(r_*)g_{yy}(r_*)}} \\
&= (2\pi\alpha^{'})\sqrt{(2\pi\alpha')^2 g_{--}(r_*) + \frac{\langle J^+\rangle^2}{\mathcal{N}^2 g_{yy}(r_*)g_{zz}(r_*)}}
\end{aligned}
$$

where we can substitute the elements of the metric (106) and take

$$
= (2\pi\alpha^{'})\sqrt{(2\pi\alpha')^2\frac{b^2}{L^{2c}}{r_H}^{\omega}{r_*}^{2c-\omega} + \frac{\langle J^+\rangle^2 L^{4c}}{\mathcal{N}^2{r_*}^{4c}}}. \tag{128}
$$

In the next step we identify the location of the horizon as $r_H = (\frac{4\pi}{\omega}bTL^{2c})^{\frac{1}{2c-1}}$ [9] and then we solve the equation (123) with respect to r_* :

To obtain a justifiable result, we Taylor-expand our expression near $r = r_H$, and thus we get:

$$
r_* = r_H + \frac{\tilde{E}^2 b^2}{\omega {r_H}^{4c-1}L^{-4c} + \frac{\omega}{r_H}\tilde{E}^2 b^2} \tag{129}
$$

so the conductivity in (128) appears as:

[9]For further information on the derivation of this result see Appendix A,(A.13).

$$\sigma = (2\pi\alpha') \sqrt{(2\pi\alpha')^2 \frac{b^2}{L^{2c}} r_H^{\omega} \left(r_H + \frac{\tilde{E}^2 b^2}{\omega {r_H}^{4c-1} L^{-4c} + \frac{\omega}{r_H} \tilde{E}^2 b^2}\right)^{2c-\omega} + \frac{\langle J^+ \rangle^2 L^{4c}}{\mathcal{N}^2 \left(r_H + \frac{\tilde{E}^2 b^2}{\omega {r_H}^{4c-1} L^{-4c} + \frac{\omega}{r_H} \tilde{E}^2 b^2}\right)^{4c}}}$$

while setting c = 1 and $\omega = 4$ brings us to the familiar case of the Schwartzschild metric (78)

$$= (2\pi\alpha') \frac{1}{1 + \frac{\tilde{E}^2 b^2 {r_H}^2}{4L^{-4} + 4{r_H}^2 \tilde{E}^2 b^2}} \sqrt{(2\pi\alpha')^2 \frac{b^2}{L^2} {r_H}^2 + \frac{\langle J^+ \rangle^2 L^4}{\mathcal{N}^2 {r_H}^4}}$$

$$= (2\pi\alpha') \frac{1}{1 + \frac{\tilde{E}^2 b^2 (\pi b T L^2)^2}{4L^{-4} + 4(\pi b T L^2)^2 \tilde{E}^2 b^2}} \sqrt{(2\pi\alpha')^2 \frac{b^2}{L^2} (\pi b T L^2)^2 + \frac{\langle J^+ \rangle^2 L^4}{\mathcal{N}^2 (\pi b T L^2)^4}}. \tag{130}$$

Choosing to perform the same expansions as we did in the case of the conductivity for the Schwartzschild metric, i.e. considering the limits $T = 0$, and then $\langle J^+ \rangle = 0$, we are lead to the same arguments regarding the behaviour of the conductivity. [10]

.

A Appendix: Hawking temperature of a general black brane metric

Here we compute the Hawking temperature for a general class of black brane metrics closely following the procedure described in [18]. The metrics to which we apply this method are of the form:

$$ds^2 = k(r)(-l(r)dt^2 + dx^i dx^i) + \frac{1}{w(r)} dr^2. \tag{A.1}$$

where we assume that l(r) and w(r) have a first-order zero at the horizon, whereas k(r) is non-vanishing there.

The metric in (A.1) possesses a regular, finite-area horizon at $r = r_H$. The Hawking temperature of this horizon is interpreted as the temperature of the dual CFT. A straightforward way to calculate it is to demand that the Euclidean continuation of the metric (A.1),

$$ds^2 = k(r)(l(r)dt_E^2 + dx^i dx^i) + \frac{1}{w(r)} dr^2, \tag{A.2}$$

obtained as usual by the replacement t $\rightarrow it_E$, be regular at the horizon. Since the Euclidean time direction shrinks to zero size at $r = r_H$, we must require that t_E is periodically identified with an appropriate period β. This period is then interpreted as the inverse temperature, $\beta = \frac{1}{T}$. The reason for this is that, at a finite temperature T, one is interested in calculating the partition function $Tre^{-\beta H}$ where H is the Hamiltonian of the theory. In a path

[10] For details you should resort to Appendix B-(B.3), (B.4).

integral formulation, the trace may be implemented by periodically identifying the Euclidean time with period β .

If we Taylor-expand (A.2) near $r = r_H$ now, we find:

$$\begin{aligned}
ds^2 &\simeq (k(r_H) + (r - r_H)k'(r_H))(l(r_H) + (r - r_H)l'(r_H))dt_E^2 \\
&+ (k(r_H) + (r - r_H)k'(r_H))dx^i dx^i + \frac{dr^2}{w(r_H) + (r - r_H)w'(r_H)} \\
&\simeq (k(r_H)l(r_H) + (r - r_H)k(r_H)l'(r_H) + (r - r_H)k'(r_H)l(r_H) + (r - r_H)^2 k'(r_H)l'(r_H))dt_E^2 \\
&+ ((k(r_H) + (r - r_H)k(r_H))dx^i dx^i + \frac{dr^2}{w(r_H) + (r - r_H)w'(r_H)} \\
&\simeq (r - r_H)k(r_H)l'(r_H)dt_E^2 + k(r_H)dx^i dx^i + \frac{dr^2}{(r - r_H)w'(r_H)}
\end{aligned} \tag{A.3}$$

where we have taken into consideration that $l(r_H) = w(r_H) = 0$.

At this point we are about to introduce new variables ρ, θ and bring our metric in the form :

$$ds^2 \simeq d\rho^2 + \rho^2 d\theta^2 + k(r_H)dx^i dx^i \tag{A.4}$$

The new coordinates are defined as follows :

$$\begin{aligned}
d\rho^2 = \frac{dr^2}{(r - r_H)w'(r_H)} &\Longleftrightarrow d\rho = \frac{dr}{\sqrt{(r - r_H)w'(r_H)}} \\
\text{so} \quad \rho = \frac{1}{\sqrt{\mathrm{w'(r_H)}}} &\int_0^{\mathrm{r}} \frac{\mathrm{dr'}}{\sqrt{\mathrm{r' - r_H}}} = 2\sqrt{\frac{\mathrm{r - r_H}}{\mathrm{w'(r_H)}}}
\end{aligned} \tag{A.5}$$

and

$$\rho^2 d\theta^2 = (r - r_H)k(r_H)l'(r_H)dt_E^2 \Longleftrightarrow d\theta = \frac{\sqrt{(r - r_H)k(r_H)l'(r_H)}}{\rho}dt_E = \frac{1}{2}\sqrt{k(r_H)l'(r_H)w'(r_H)}dt_E \tag{A.6}$$

where in order to avoid having a conical singularity at $\rho = 0$ we must require θ to have a period 2π so

$$dt_E = \frac{2d\vartheta}{\sqrt{k(r_H)l'(r_H)w'(r_H)}} \text{ and } \mathrm{t_E} = \frac{2}{\sqrt{\mathrm{k(r_H)l'(r_H)w'(r_H)}}} \int_0^{2\pi} \mathrm{d}\theta = \frac{4\pi}{\sqrt{\mathrm{k(r_H)l'(r_H)w'(r_H)}}} \tag{A.7}$$

From (A.7) we then see that the period of the Euclidean time must be:

$$\frac{1}{T} = \frac{4\pi}{\sqrt{k(r_H)l'(r_H)w'(r_H)}}. \tag{A.8}$$

After the description of the derivation procedure, we are able to apply the method to various cases, as long as the conditions for the involved functions hold.

Let's now use the results of the above method in order to compute the location of the horizon in the case of the metric given in (78). As we know, before introducing the light-cone coordinates, the planar black hole solution can be written as:

$$ds^2 = (\frac{r}{L})^2(-hdt^2 + dx^i dx^i) + (\frac{L}{r})^2 h^{-1} dr^2 \quad \text{with} \quad \mathrm{h} = 1 - \frac{\mathrm{r_H^4}}{\mathrm{r^4}}. \tag{A.9}$$

Since the above metric is in the desired form which appears in (A.1), we can directly apply the described standard method of evaluating the Hawking temperature in the expression given in (A.9). Once we identify $k(r_H) = (\frac{r_H}{L})^2, l'(r_H) = h'(r_H) = \frac{4}{r_H}, \quad \text{and} \quad \mathrm{w'(r_H)} = \frac{\mathrm{4r_H}}{\mathrm{L^2}}$ we get:

$$\frac{1}{T} = \frac{\pi L^2}{r_H} \tag{A.10}$$

and since we have expressed our spacetime (78) in the light-cone coordinates with a specific normalization (80), we proceed in rescaling our final result and obtain:

$$r_H = \pi b T L^2. \tag{A.11}$$

At this point we want to adapt the standard method to a general metric of the form:

$$ds^2 = e^{2A}(-f(r)dt^2 + dx^i dx^i) + e^{-2A}\frac{1}{f(r)}dr^2 \tag{A.12}$$

where for our application, $e^A, f(r)$ vary as $\frac{r^c}{L^c}, 1 - (\frac{r_H}{r})^\omega$ respectfully, while the considered identifications are: $k(r_H) = \frac{{r_H}^{2c}}{L^{2c}}, l'(r_H) = f'(r_H) = \frac{\omega}{r_H}, \quad and \quad w'(r_H) = \frac{\omega r_H^{2c-1}}{L^{2c}}$.

So, the corresponding rescaled location of the horizon comes of as:

$$r_H = (\frac{4\pi}{\omega} b T L^{2c})^{\frac{1}{2c-1}}. \tag{A.13}$$

B Appendix : Behaviour of the derived conductivities at certain limits

As we have shown in subsection 5.3 our main result is (103). Here we want to check our answer in the limit with $T = 0$, so we will Taylor-expand our expression around this value. What we get after having performed the expansion up to third order - having neglected the multiplicative constants-is:

$$\frac{\sqrt{\frac{\langle J^+\rangle^2}{E}}}{T} - \frac{\sqrt{\langle J^+\rangle^2 E}}{2E^2}T + \frac{\sqrt{\frac{\langle J^+\rangle^2}{E}}}{8E^2}T^3 + \mathcal{O}[T]^4 \,. \tag{B.1}$$

Now we want to examine what happens when we set $\langle J^+\rangle = 0$ in (103), when the above considerations - regarding the approximation's order and the multiplicative factors - still apply:

$$\sqrt{\frac{T^3}{\sqrt{T^2+\sqrt{T^4+E^2}}}} + \frac{\sqrt{\frac{T^3}{\sqrt{T^2+\sqrt{T^4+E^2}}}}\langle J^+\rangle^2}{2T^5\sqrt{T^2+\sqrt{T^4+E^2}}} + \mathcal{O}[\langle J^+\rangle]^4 \,. \tag{B.2}$$

Willing to consider the same limits as far as the conductivity appearing in (130) is concerned, we first take $T = 0$, and what we obtain is:

$$\frac{\sqrt{\langle J^+\rangle^2}}{T^2} - E^2\sqrt{\langle J^+\rangle^2} + 2E^4\sqrt{\langle J^+\rangle^2}T^2 + \mathcal{O}[T]^4 \,, \tag{B.3}$$

while in the second case which corresponds to $\langle J^+\rangle = 0$, we have:

$$\frac{T+E^2T^3}{1+2E^2T^2} + \frac{1+E^2T^2}{2T^5(1+2E^2T^2)}\langle J^+\rangle^2 + \mathcal{O}[\langle J^+\rangle]^4 \,. \tag{B.4}$$

References

[1] E. Kiritsis, *"String theory in a nutshell,"* Princeton, USA: Univ. Pr. (2007) 588 p.

[2] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," Nucl. Phys. B **72** (1974) 461.

[3] J. D. Bekenstein, "Black holes and entropy," Phys. Rev. D **7** (1973) 2333.

[4] H. -W. Hammer, "Universality in few-body systems with large scattering length," AIP Conf. Proc. **777** (2005) 1 [nucl-th/0502080].

[5] D. T. Son, *"Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrödinger symmetry,"* Phys. Rev. **D78** (2008) 0460 [ArXiv:0804.3972][hep-th].

[6] K. Balasubramanian and J. McGreevy, *"Gravity duals for non-relativistic CFTs,"* Phys. Rev. Lett. **101** (2008) 061601 [ArXiv:0804.4053][hep-th].

[7] S. A. Hartnoll, *"Lectures on holographic methods for condensed matter physics,"* Class. Quant. Grav. **26** (2009) 224002. [ArXiv:0903.3246][hep-th].

[8] M. Rangamani,"Holography for non-relativistic CFTs," *"Holography for non-relativistic CFTs,"* Acta Phys. Polon. **B40** (2009) 3745-3770.

[9] A. Bagchi, R. Gopakumar, *"Galilean Conformal Algebras and AdS/CFT,"* JHEP **0907** (2009) 037. [ArXiv:0902.1385]hep-th].

[10] J. Maldacena, D. Martelli, Y. Tachikawa, *"Comments on string theory backgrounds with non-relativistic conformal symmetry,"* JHEP **0810** (2008) 072. [ArXiv:0807.1100][hep-th].

[11] D. Mateos, *"String Theory and Quantum Chromodynamics,"* Class. Quant. Grav. **24** (2007) S713-S740. [ArXiv:0709.1523][hep-th].

[12] R. Kubo, "Statistical mechanical theory of irreversible processes. 1. General theory and simple applications in magnetic and conduction problems," J. Phys. Soc. Jap. **12** (1957) 570.

[13] B. S. Kim, D. Yamada, *"Properties of Schroedinger Black Holes from AdS Space,"* JHEP **1107** (2011) 120. [ArXiv:1008.3286][hep-th].

[14] A. Zaffaroni, *"Introduction to the AdS-CFT correspondence,"* Class. Quant. Grav. **17** (2000) 3571-3597.

[15] S. Sachdev, " Quantum Phase Transitions," *" Quantum Phase Transitions,"* (Cambridge University Press, Cambridge, England, 1999).

[16] A. V. Manohar, *"Large N QCD,"*
[ArXiv:hep-ph/9802419].

[17] J. M. Maldacena, *"Large N field theories, string theory and gravity,"* [ArXiv:hep-th/9905111].

[18] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal and U. A. Wiedemann, [ArXiv:1101.0618][hep-th]

[19] A. Karch and A. O'Bannon, JHEP **0709** (2007) 024 [arXiv:0705.3870 [hep-th]]. [ArXiv:0705.3870][hep-th]

[20] A. O'Bannon, Phys. Rev. D **76** (2007) 086007 [arXiv:0708.1994 [hep-th]]. [ArXiv:0708.1994][hep-th]

[21] S. Kobayashi, D. Mateos, S. Matsuura, R. C. Myers and R. M. Thomson, JHEP **0702** (2007) 016 [hep-th/0611099]. [ArXiv:hep-th/0611099].

[22] A. Karch, A. O'Bannon and E. Thompson, JHEP **0904** (2009) 021 [arXiv:0812.3629 [hep-th]]. [ArXiv:0812.3629][hep-th]

[23] M. Ammon, C. Hoyos, A. O'Bannon and J. M. S. Wu, JHEP **1006** (2010) 012 [arXiv:1003.5913 [hep-th]]. [ArXiv:1003.5913][hep-th]

[24] S.H. Naqib, J.R. Cooper, J.L. Tallon, and C. Panagopoulos,"Temperature dependence of electrical resistivity of high-Tc cuprates- from pseudogap to overdoped regions," [arXiv:cond-mat/0209457]