# A three-dimensional coupled normal mode model for sound propagation in water in the presence of a conical seamount 

Philippos Saratsis

Master Thesis

Department of Mathematics and Applied Mathematics
University of Crete
July 2020
© Copyright 2020 by Philippos Saratsis
All Rights Reserved

## Acknowledgements

The biggest credits for the present thesis should go to my life companion, Dr. Marina Bitsaki. This work would - most probably - not be completed without her support and insistence.

I owe many thanks to my supervisor, Prof. Michael Taroudakis, for his help, understanding and patience.

Special thanks to my parents and in-laws, Nikos, Ourania, Manolis and Eleftheria for their love and support and my children Nicholas, Eleftheria and Christina for filling up my life.

Also, thanks to Dr. Costas Smaragdakis and Dr. Manos Kamarianakis for their help in various topics of my work.

Finally, I would like to dedicate this thesis in memory of my uncle George Vlachodimos who influenced my educational orientation.

# A three-dimensional coupled normal mode model for sound propagation in water in the presence of a conical seamount 

Philippos Saratsis<br>Master Thesis<br>Department of Mathematics and Applied Mathematics<br>University of Crete<br>Supervisor:<br>Michael I. Taroudakis,<br>Professor of Mathematics<br>Department of Mathematics and Applied Mathematics<br>University of Crete

## Contents

1 Introduction ..... 1
1.1 Previous Work ..... 2
1.2 My work ..... 3
1.3 Roadmap ..... 4
2 Formulation of the problem ..... 5
2.1 The environment ..... 5
2.2 The wave equation ..... 6
2.3 The Helmholtz equation ..... 7
2.4 Approximation with cylindrical rings ..... 10
3 The Normal Mode Solution to the Three-Dimensional Helmholtz Equation ..... 13
3.1 The homogeneous three-dimensional Helmholtz equation ..... 13
3.2 The inhomogeneous three-dimensional Helmholtz equation ..... 15
3.3 Source conditions of the inhomogeneous three-dimensional Helmholtz equation ..... 16
3.4 Representation of the field ..... 18
3.4.1 Convergence of the Model ..... 21
3.5 Interface conditions and two-way coupling ..... 22
3.5.1 Mode coupling in inward marching ..... 22
3.5.2 Mode coupling in outward marching ..... 26
3.5.3 Numerical Stability ..... 29
4 Numerical Results ..... 31
4.1 Range Independent Case ..... 31
4.2 Range Dependent Case: Single Cylinder ..... 38
4.3 Range Dependent Case:"Conical" Seamount ..... 50
5 Conclusions - Future Work ..... 57
A Properties of Bessel Functions ..... 59
A. 1 The Bessel Equation and its Solutions ..... 59
A. 2 Asymptotic Forms of Bessel Functions ..... 60
A.2.1 Asymptotic Expressions for Large Arguments ..... 60
A.2.2 Asymptotic Expressions for Small Arguments ..... 61
A.2.3 Asymptotic Expressions for Large Orders ..... 65
A. 3 Recursion Relations for Bessel Functions ..... 65
A. 4 Wronskian Relations for Bessel Functions ..... 66
A. 5 Linearly Independent Solutions of Bessel Equations for both Large and Small Ar- guments ..... 67
A.5.1 Linearly Independent Solutions for Large Arguments ..... 67
A.5.2 Linearly Independent Solutions for Small Arguments ..... 68

## List of Figures

2.1 Geometry of a conical seamount problem (side view) ..... 5
2.2 A conical seamount approximated by cylindrical sectors (side view) ..... 10
2.3 A conical seamount approximated by cylindrical sectors (top view) ..... 10
3.1 Use of the superposition method to obtain the pressure field outside the seamount ..... 21
3.2 A two-way coupled mode model ..... 22
3.3 Coupling between two neighboring rings in inward marching ..... 23
3.4 Coupling between two neighboring rings in outward marching ..... 27
4.1 Range Independent Case (seamount of zero elevation) ..... 32
4.2 Range Independent Case: Transmission Loss Contour, 50 Azimuthal Modes ..... 34
4.3 Range Independent Case: Transmission Loss vs Range, 50 Azimuthal Modes ..... 34
4.4 Range Independent Case: Transmission Loss Contour, 100 Azimuthal Modes ..... 35
4.5 Range Independent Case: Transmission Loss vs Range, 100 Azimuthal Modes ..... 35
4.6 Range Independent Case: Transmission Loss Contour, 150 Azimuthal Modes ..... 36
4.7 Range Independent Case: Transmission Loss vs Range, 150 Azimuthal Modes ..... 36
4.8 Range Independent Case: Transmission Loss Contour, 170 Azimuthal Modes ..... 37
4.9 Range Independent Case: Transmission Loss vs Range, 170 Azimuthal Modes ..... 37
4.10 Range Dependent Case 1: Cylinder of 100 m height ..... 38
4.11 Range Dependent Case 1: Cut-off angle ..... 39
4.12 Range Dependent Case: Transmission Loss Contour, 110 Azimuthal Modes ..... 40
4.13 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 0 deg. ..... 40
4.14 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 10 deg. ..... 41
4.15 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 19 deg. ..... 41
4.16 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 20 deg. ..... 42
4.17 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 30 deg. ..... 42
4.18 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 60 deg. ..... 43
4.19 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 90 deg. ..... 43
4.20 Range Dependent Case: Transmission Loss Contour, 110 Azimuthal Modes ..... 44
4.21 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 0 deg. ..... 44
4.22 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 1 deg. ..... 45
4.23 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 10 deg. ..... 45
4.24 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 19 deg. ..... 46
4.25 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 20 deg. ..... 46
4.26 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 30 deg. ..... 47
4.27 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 60 deg. ..... 47
4.28 Range Dependent Case: Single Cylinder, TL vs Range, Angle: 90 deg. ..... 48
4.29 Range Dependent Case: Single Cylinder, Inner Radius 200 m , TL vs Range ..... 49
4.30 Range Dependent Case: Single Cylinder, Inner Radius 500 m, TL vs Range ..... 49
4.31 Range Dependent Case: Six cylinders, Internal Radii 50 to 250 m ..... 50
4.32 Range Dependent Case: Frustum vs Single Cylinder (MODE4) $\mathrm{SD}=\mathrm{RD}=50 \mathrm{~m}$ ..... 51
4.33 Range Dependent Case: Frustum vs Single Cylinder (MODE4) $\mathrm{SD}=\mathrm{RD}=150 \mathrm{~m}$ ..... 51
4.34 Range Dependent Case: Frustum vs Single Cylinder (mycode) $\mathrm{SD}=\mathrm{RD}=50 \mathrm{~m}$ ..... 52
4.35 Range Dependent Case: Frustum vs Single Cylinder (mycode) $\mathrm{SD}=\mathrm{RD}=150 \mathrm{~m}$ ..... 52
4.36 Range Dependent Case: Six cylinders, Internal Radii 50 to 250 m ..... 53
4.37 Conical Frustum: Transmission Loss Contour, $\mathrm{SD}=\mathrm{RD}=50 \mathrm{~m}$ ..... 54
4.38 Conical Frustum: Transmission Loss Contour, $\mathrm{SD}=\mathrm{RD}=150 \mathrm{~m}$ ..... 54
4.39 Range Dependent Case: Conical Frustum, $\mathrm{SD}=\mathrm{RD}=50 \mathrm{~m}$ ..... 55
4.40 Range Dependent Case: Conical Frustum, $\mathrm{SD}=\mathrm{RD}=150 \mathrm{~m}$ ..... 55
A. 1 Bessel functions of orders $0,1,5$ and 10 for small arguments ..... 62
A. 2 Neumann functions of orders $0,1,5$ and 10 for small arguments ..... 64

## List of Tables

4.1 The environmental parameters ..... 33

## Chapter 1

## Introduction

During the past decades, many researchers have been studying the propagation of acoustic waves in the ocean and a large number of numerical models were developed for that purpose, given that the capability of modeling the sound propagation plays significant role in the majority of practical situations. Most of these models solve the problem in two dimensions (range and depth) and the solutions are satisfactory whenever the environmental dependence on the third dimension (azimuth) is negligible. There are, however, situations where three-dimensional effects cannot be overlooked. Problems of this form pose, for example, the existence of an underwater seamount or a strong eddy in the water column. For such cases, the 2D models, where there is no energy interaction between planes of constant azimuth with respect to the source, very often fail to provide realistic solutions, thus a 3D model is needed to depict with accuracy the pressure field in range, depth and azimuth.

The insurmountable obstacle for the introduction of a fully 3D model, which could give an accurate solution to the problem of acoustic propagation in an environment free of geometrical limitations, is the almost prohibitive computational cost required for the implementation of such an effort.

### 1.1 Previous Work

The first attempts for the expansion of the 2D models in order to be used in three-dimensional problems were applied in simplified oceanic environments, e.g. a two-dimensional axially symmetric model of a seamount or a planar 2D model for a continental slope (wedge). For such geometries, the solution of the fully 3D wave equation can be approximated by superposition of a finite number of 2 D solutions, with the use of integral transforms. The method of partitioning the space in N planes and subsequently the solution of N 2 D problems $(N \times 2 D)$, introduced by Perkins and Baer (1982), provides with sufficient accuracy a good approximation of the 3D approach in problems with weak transverse environmental variability, but is rather inadequate in the existence of strong environmental complexity.

Various models of the three-dimensional approach of the problem have been developed over the years, based in different methods for solving the wave equation. Some of them are:

- Adiabatic normal mode models
- Coupled normal mode models (one-way and two-way versions)
- Models based on the parabolic equation (PE) approximation (narrow or wide angle formulations and split-step or finite-difference implementations)
- Finite difference solutions to the full wave equation
- Finite element solutions to the full wave equation

All the aforementioned methods have advantages and disadvantages, the most considerable problem, however, remains the computational cost.

In 1996, M. Taroudakis published a paper titled: "A coupled-mode formulation for the solution of the Helmholtz equation in water in the presence of a conical sea-mount". The method applied in this paper produces an analytical solution of the 3D wave equation in an axially symmetric environment and the pressure of the acoustic field is presented as a series expansion in terms of normal modes and azimuthal Fourier series.

Although the formulation introduced was theoretically exact, the numerical implementation suffered from significant issues in the stability of the algorithms, a fact that prevented their use
in realistic conditions, limiting them in shallow water environments and sound sources of low frequencies. First, the numerical evaluation of the Hankel functions used in the applied expansion is unstable for high orders and small arguments. Furthermore, the number of azimuthal modes required for the convergence of the solution becomes very big when the source is far from the seamount.

More than a decade later Luo and Schmidt, based on the theoretical background of Taroudaki's work, introduce a numerical model in which they apply a number of modifications in the calculation of the acoustic pressure, inducing significant improvement both in the stability of the algorithm and in the computational cost of the method.

### 1.2 My work

The present work takes as starting point the work of M. Taroudakis. It extends in three dimensions the routines used in program MODE4, which was used to calculate the acoustic pressure in two dimensions and materializes the improvements introduced by Luo and Schmidt in the theory presented by Taroudakis, so that the new code becomes sufficient for the solution of more realistic scenarios in an underwater environment. The modifications are the following.

1. Instead of $H_{m}^{(1)}$ and $H_{m}^{(2)}$ (Hankel functions of order $m$ and the first and second kind, respectively), this model uses $J_{m}$ (the Bessel function of order $m$ ) and $H_{m}^{(1)}$ as the two linearly independent solutions of the Bessel equation. The advantage is that $J_{m}$ and $H_{m}^{(1)}$ remain linearly independent numerically for both large and small arguments, while the two Hankel functions become numerically indistinguishable at high orders.
2. Normalized Bessel and Hankel functions, as well as their asymptotic forms for small and large arguments, are used to avoid overflow and underflow problems. Moreover, the recurrence relations of Bessel and Hankel functions are used in evaluation of different orders, which improves efficiency.
3. The single-scatter approximation used in this model improves numerical efficiency. This is because for each azimuthal mode instead of solving one linear system of large dimension, this model solves multiple linear systems of small dimensions.
4. The efficiency is also improved with the implementation of the superposition representation of the external field with respect to the seamount. In the present model the number of azimuthal modes required for convergence depends only on the product of the wavenumber and the radius of the base of the seamount. In previous models this value depends on the product of the wavenumber and the distance between the source and the axis of the seamount.

### 1.3 Roadmap

Chapter Two, Formulation of the problem, describes the oceanic environment where the seamount is located.

Chapter Three, The Coupled Normal Mode Solution to the Three-Dimensional Helmholtz Equation, contains the analytical solution of the problem.

Chapter Four, Numerical Results, describes and discusses the results.
Chapter Five, Conclusions - Future Work, summarizes the results and offers suggestions for future work.

Appendix A, Properties of Bessel Functions, mentions some of the properties of Bessel functions used in the present work.

## Chapter 2

## Formulation of the problem

### 2.1 The environment



Figure 2.1: Geometry of a conical seamount problem (side view)

The geometry of the under consideration environment is illustrated in Figure 2.1. A cylindrical system of coordinates is introduced with the $z$-axis pointing downwards. The environment is defined by the sea surface at depth $z=0$, the water column of depth $h_{1}$ and the bottom consisted of a sedimentary layer of thickness $h_{2}$ and a semi-infinite half-space. The conical seamount of base radius $r^{I}$ and the same properties as the sediment is placed with its peak on the $z$-axis. Outside
the seamount the layers are horizontally stratified and range independent properties are assumed. A point harmonic source is located at range $r_{s}$, depth $z_{s}$ and angle $\phi_{s}=\pi$. The fact that the source is placed off the $z$-axis makes the problem three-dimensional.

### 2.2 The wave equation

The phenomenon of sound propagation produced by a point harmonic source is governed by the wave equation which represents a relation between the derivatives of sound pressure with respect to space and time. The wave equation in an ideal fluid can be derived from hydrodynamics and the adiabatic relation between pressure and density. The equation for conservation of mass, Euler's equation and the adiabatic equation of state are respectively

$$
\begin{gather*}
\frac{\partial \rho}{\partial t}=-\nabla \cdot \rho u  \tag{2.1}\\
\frac{\partial u}{\partial t}+(u \cdot \nabla) u=-\frac{1}{\rho} \nabla p(\rho)  \tag{2.2}\\
p=p_{0}+\rho_{1} \frac{\partial p}{\partial \rho}+\frac{1}{2} \rho_{1}^{2} \frac{\partial^{2} p}{\partial \rho^{2}}+\ldots \tag{2.3}
\end{gather*}
$$

In the above equations $\rho$ is the density, $u$ the particle velocity, $p$ the pressure and $\frac{\partial p}{\partial \rho} \equiv c^{2}$ is the sound speed in an ideal fluid. Using small perturbations for pressure and density, one can note that $u$ is also a small quantity. This means that the particle velocity which results from pressure and density perturbations is much smaller than the speed of sound. Retaining only first-order terms in the hydrodynamic equations and assuming that $\rho_{0}$ and $c^{2}$ are independent of time considering the fact that the time scale of oceanographic changes is much longer than the time scale of acoustic propagation - the combination of the above equations leads us to the "linearized" wave equation for pressure:

$$
\begin{equation*}
\rho \nabla \cdot\left(\frac{1}{\rho} \nabla p\right)-\frac{1}{c^{2}} \frac{\partial^{2} p}{\partial t^{2}}=0 \tag{2.4}
\end{equation*}
$$

### 2.3 The Helmholtz equation

If the density is constant in space, Eq.(2.4) can be replaced by the standard form of the wave equation:

$$
\begin{equation*}
\nabla^{2} P-\frac{1}{c^{2}} \frac{\partial^{2} P}{\partial t^{2}}=0 \tag{2.5}
\end{equation*}
$$

In Eq.(2.5) $P=P(\vec{x}, t)$ is the sound pressure and $\nabla^{2} P$ is the three dimensional Laplace operator. For a cartesian coordinate system $(x, y, z)$ the Laplace operator is

$$
\begin{equation*}
\nabla^{2} P=\frac{\partial^{2} P}{\partial x^{2}}+\frac{\partial^{2} P}{\partial y^{2}}+\frac{\partial^{2} P}{\partial z^{2}} \tag{2.6}
\end{equation*}
$$

For cylindrical coordinates $(r, z, \phi)$ the Laplace operator is given by

$$
\begin{equation*}
\nabla^{2} P=\frac{\partial^{2} P}{\partial r^{2}}+\frac{1}{r} \frac{\partial P}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2} P}{\partial \phi^{2}}+\frac{\partial^{2} P}{\partial z^{2}} \tag{2.7}
\end{equation*}
$$

A technique used to solve Eq.(2.5) is the method of separation of variables. We assume that the sound pressure can be written as

$$
\begin{equation*}
P(\vec{x}, t)=p(\vec{x}) T(t) \tag{2.8}
\end{equation*}
$$

Inserting Eq.(2.8) into Eq.(2.5) we get

$$
\begin{equation*}
T \nabla^{2} p=\frac{1}{c^{2}} p \frac{d^{2} T}{d t^{2}} \tag{2.9}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{c^{2}}{p} \nabla^{2} p=\frac{1}{T} \frac{d^{2} T}{d t^{2}} \tag{2.10}
\end{equation*}
$$

Since the left hand side term of Eq.(2.10) is a function of spatial variables only and the right hand side term is a function of time only, each term must be equal to a constant. Thus, we obtain:

$$
\begin{equation*}
\frac{c^{2}}{p} \nabla^{2} p=\frac{1}{T} \frac{d^{2} T}{d t^{2}}=-\omega^{2} \tag{2.11}
\end{equation*}
$$

where the negative sign of the constant $\omega^{2}$ is selected in order to facilitate the mathematical manipulations as well as the physical interpretation. In Eq.(2.11) one can recognize in constant $\omega$ the angular frequency of the sound wave ( $\omega=2 \pi f$, where $f$ is the frequency in Hz ).

Eq.(2.11) produces two equations:

$$
\begin{equation*}
\frac{d^{2} T}{d t^{2}}+\omega^{2} T=0 \tag{2.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\nabla^{2} p+\frac{\omega^{2}}{c^{2}} p=0 \tag{2.13}
\end{equation*}
$$

Eq.(2.12) gives us the time dependence of the acoustic pressure. It has two linearly independent solutions $T=A e^{ \pm i \omega t}$, where $A$ is a constant. Without loss of generality we can assume that $A=1$ and we will accept from the two possible solutions the one with the negative sign in the exponent. This means that we will study sources which emit energy with time dependence $e^{-i \omega t}$.

Eq.(2.13) is known as the Helmholtz equation. The three-dimensional character of our problem leads us to the inhomogeneous Helmholtz equation:

$$
\begin{equation*}
\nabla^{2} p+\frac{\omega^{2}}{c^{2}} p=-\delta\left(\vec{x}-\overrightarrow{x_{0}}\right) \tag{2.14}
\end{equation*}
$$

where $\delta\left(\vec{x}-\overrightarrow{x_{0}}\right)$ is the Dirac function. Introducing a cylindrical coordinates system Eq.(2.14) is written as:

$$
\begin{equation*}
\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial p}{\partial r}\right)+\frac{1}{r^{2}} \frac{\partial^{2} p}{\partial \phi^{2}}+\rho(z) \frac{\partial}{\partial z}\left(\frac{1}{\rho(z)} \frac{\partial p}{\partial z}\right)+\{k(r, z)\}^{2} p=-\frac{1}{r} \delta\left(r-r_{s}\right) \delta\left(z-z_{s}\right) \delta\left(\phi-\phi_{s}\right) \tag{2.15}
\end{equation*}
$$

Removing the left hand side parentheses the equation becomes:

$$
\begin{equation*}
\frac{\partial^{2} p}{\partial r^{2}}+\frac{1}{r} \frac{\partial p}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2} p}{\partial \phi^{2}}+\frac{\partial^{2} p}{\partial z^{2}}-\frac{1}{\rho(z)} \frac{\partial \rho}{\partial z} \frac{\partial p}{\partial z}+\{k(r, z)\}^{2} p=-\frac{1}{r} \delta\left(r-r_{s}\right) \delta\left(z-z_{s}\right) \delta\left(\phi-\phi_{s}\right) \tag{2.16}
\end{equation*}
$$

In this expression the density $\rho$ is generally considered as a function of $z$ only (in the present work only constant densities over the various layers will be considered) and $k$ is the wavenumber $\left(k=\frac{\omega}{c(z)}\right)$, where $c(z)$ is the sound speed profile.

Moreover, for the homogeneous Helmholtz equation we have:

$$
\begin{equation*}
\frac{\partial^{2} p}{\partial r^{2}}+\frac{1}{r} \frac{\partial p}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2} p}{\partial \phi^{2}}+\frac{\partial^{2} p}{\partial z^{2}}-\frac{1}{\rho(z)} \frac{\partial \rho}{\partial z} \frac{\partial p}{\partial z}+\{k(r, z)\}^{2} p=0 \tag{2.17}
\end{equation*}
$$

We define the pressure as $p^{(1)}$ in the water and $p^{(2)}$ in the bottom. The boundary conditions imposed in the problem require the pressure to vanish at the sea surface, to be continuous at the water-bottom interface, with a known discontinuity of its normal derivative at the same boundary. Finally, the normal derivative of the pressure must be zero at the sediment-subbottom boundary $h=h_{1}+h_{2}$ (Neumann boundary condition):

$$
\begin{gather*}
p^{(1)}(r, 0, \phi)=0  \tag{2.18}\\
p^{(1)}=p^{(2)} \text { on } S_{b}  \tag{2.19}\\
\frac{1}{\rho_{1}} \frac{\partial p^{(1)}}{\partial \vec{n}}=\frac{1}{\rho_{2}} \frac{\partial p^{(2)}}{\partial \vec{n}} \text { on } S_{b}  \tag{2.20}\\
\frac{\partial p^{(2)}}{\partial z}(r, h, \phi)=0 \tag{2.21}
\end{gather*}
$$

where $\vec{n}$ is the unit vector normal on the water-sediment interface $S_{b}$ and $\rho_{1}, \rho_{2}$ are the density values in the water and sediment respectively.

Further, a Sommerfeld radiation condition will be posed for the behavior of the field at infinity. In 1912, Sommerfeld stated a mathematically precise and easily applicable condition which, when added to boundary value problems for the Helmholtz equation, ensures that there are no waves originating at infinity and moving towards the source. This condition is applied at infinity and for three-dimensional problems requires that the solution $p$ of the Helmholtz equation (2.13) satisfies

$$
\begin{equation*}
\lim _{r \rightarrow \infty} r\left(\frac{\partial p}{\partial r}-i k p\right)=0 \tag{2.22}
\end{equation*}
$$

uniformly with respect to all directions in which the limit is approached.


Figure 2.2: A conical seamount approximated by cylindrical sectors (side view)

### 2.4 Approximation with cylindrical rings



Figure 2.3: A conical seamount approximated by cylindrical sectors (top view)

Two main regions can be determined in this geometry: The inner region defined by $0 \leq$ $r \leq r^{I}$, and the external region defined by $r \geq r^{I}$. In the coupled-mode approach a number of range-independent ring-shaped sectors are introduced to approximate the conical seamount. The notation $r^{j}$ is used to denote the range at the interface between ring $j$ and ring $j+1$, as illustrated
in Figure 2.2. Eq.(2.16) expresses the field produced by the source in the external region, while inside the inner region, which is a range-independent environment due to the absence of the source, the acoustic pressure is defined by Eq.(2.17). In each one of the rings the water-sediment interfaces lay horizontally at depth $h_{j}$ and the sound speed variates only with respect to depth. The pressure field satisfies the homogeneous Helmholtz equation and the boundary conditions are imposed in horizontal and vertical interfaces.

## Chapter 3

## The Normal Mode Solution to the Three-Dimensional Helmholtz <br> Equation

In this chapter we will show how to obtain normal mode solutions to the three-dimensional Helmholtz equation with a point source in cylindrical coordinates $r, z, \phi$.

### 3.1 The homogeneous three-dimensional Helmholtz equation

The homogeneous three-dimensional Helmholtz equation in cylindrical coordinates takes the form

$$
\begin{equation*}
\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial p}{\partial r}\right)+\frac{1}{r^{2}} \frac{\partial^{2} p}{\partial \phi^{2}}+\rho(z) \frac{\partial}{\partial z}\left(\frac{1}{\rho(z)} \frac{\partial p}{\partial z}\right)+\{k(r, z)\}^{2} p=0 \tag{3.1}
\end{equation*}
$$

Using the method of separation of variables we can write the acoustic pressure as

$$
\begin{equation*}
p(r, z, \phi)=R(r) U(z) \Phi(\phi) \tag{3.2}
\end{equation*}
$$

Substituting Eq.(3.2) to Eq.(3.1) we get

$$
\begin{equation*}
\frac{1}{r} \frac{d}{d r}\left(r \frac{d R}{d r}\right) U(z) \Phi(\phi)+\frac{1}{r^{2}} \frac{d^{2} \Phi}{d \phi^{2}} R(r) U(z)+\rho \frac{d}{d z}\left(\frac{1}{\rho} \frac{d U}{d z}\right) R(r) \Phi(\phi)+\frac{\omega^{2}}{c^{2}} R(r) U(z) \Phi(\phi)=0 \tag{3.3}
\end{equation*}
$$

Divided by $R(r) U(z) \Phi(\phi)$ Eq.(3.3) yields

$$
\begin{equation*}
\frac{1}{R} \frac{1}{r} \frac{d}{d r}\left(r \frac{d R}{d r}\right)+\frac{1}{\Phi} \frac{1}{r^{2}} \frac{d^{2} \Phi}{d \phi^{2}}+\frac{1}{U} \rho \frac{d}{d z}\left(\frac{1}{\rho} \frac{d U}{d z}\right)+\frac{\omega^{2}}{c^{2}}=0 \tag{3.4}
\end{equation*}
$$

By introducing the separation constants we can separate Eq.(3.4) into three ordinary differential equations. To get the depth-dependent equation let

$$
\begin{equation*}
\frac{1}{U_{n}} \rho \frac{d}{d z}\left(\frac{1}{\rho} \frac{d U_{n}}{d z}\right)+\frac{\omega^{2}}{c^{2}}=k_{n}^{2} \tag{3.5}
\end{equation*}
$$

or

$$
\begin{equation*}
\rho \frac{d}{d z}\left(\frac{1}{\rho} \frac{d U_{n}}{d z}\right)+\left(\frac{\omega^{2}}{c^{2}}-k_{n}^{2}\right) U_{n}=0 \tag{3.6}
\end{equation*}
$$

The solutions to the former equation $U_{n}(z)$ are depth-dependent eigenfunctions satisfying the orthonormal relation

$$
\begin{equation*}
\int_{0}^{\infty} \frac{1}{\rho(z)} U_{n}(z) U_{\nu}(z) d z=\delta_{n \nu} \tag{3.7}
\end{equation*}
$$

By substituting Eq.(3.6) into Eq.(3.4) and multiplying by $r^{2}$ we obtain the equation for $R(r)$ and $\Phi(\phi)$

$$
\begin{equation*}
\frac{r^{2}}{R} \frac{1}{r} \frac{d}{d r}\left(r \frac{d R}{d r}\right)+\frac{1}{\Phi} \frac{d^{2} \Phi}{d \phi^{2}}+k_{n}^{2} r^{2}=0 \tag{3.8}
\end{equation*}
$$

To obtain the angle-dependent equation let

$$
\begin{equation*}
\frac{1}{\Phi_{m}} \frac{d^{2} \Phi_{m}}{d \phi^{2}}=-m^{2}, \text { for } m=0,1,2, \ldots \tag{3.9}
\end{equation*}
$$

which leads to solutions

$$
\begin{equation*}
\Phi_{m}(\phi)=e_{m} \cos m \phi, \text { for } m=0,1,2, \ldots \tag{3.10}
\end{equation*}
$$

Using the orthonormal relation

$$
\begin{equation*}
\int_{-\pi}^{\pi} \Phi_{m}(\phi) \Phi_{\mu}(\phi) d \phi=\delta_{m \mu} \tag{3.11}
\end{equation*}
$$

we have

$$
e_{m}=\left\{\begin{array}{l}
\frac{1}{\sqrt{2 \pi}}, m=0  \tag{3.12}\\
\frac{1}{\sqrt{\pi}}, m \neq 0
\end{array}\right.
$$

If we substitute Eq.(3.9) into Eq.(3.8) then we get the equation for $R(r)$

$$
\begin{equation*}
\frac{r^{2}}{R_{m n}} \frac{1}{r} \frac{d}{d r}\left(r \frac{d R_{m n}}{d r}\right)+\left(k_{n}^{2}-\frac{m^{2}}{r^{2}}\right) r^{2}=0 \tag{3.13}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{1}{r} \frac{d}{d r}\left(r \frac{d R_{m n}}{d r}\right)+\left(k_{n}^{2}-\frac{m^{2}}{r^{2}}\right) R_{m n}=0 \tag{3.14}
\end{equation*}
$$

Eq.(3.14) is an $m$ th-order Bessel equation and its solution $R_{m n}(r)$ is any pair of $J_{m}\left(k_{n} r\right), Y_{m}\left(k_{n} r\right), H_{m}^{(1)}\left(k_{n} r\right)$ and $H_{m}^{(2)}\left(k_{n} r\right)$, where $J_{m}, Y_{m}$ are the Bessel functions of the first and second kind ( $Y_{m}$ is also called the Neumann function) and $H_{m}^{(1)}, H_{m}^{(2)}$ are the Hankel functions of the first and second kind $\left(H_{m}^{(1)}(x)=J_{m}(x)+i Y_{m}(x)\right.$ and $\left.H_{m}^{(2)}(x)=J_{m}(x)-i Y_{m}(x)\right)$.

Thus, the solution to the homogeneous three-dimensional Helmholtz equation, Eq.(3.1) is

$$
\begin{equation*}
p(r, z, \phi)=\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} R_{m n}(r) U_{n}(z) \Phi_{m}(\phi) \tag{3.15}
\end{equation*}
$$

### 3.2 The inhomogeneous three-dimensional Helmholtz equation

The inhomogeneous three-dimensional Helmholtz equation with a point source takes the form

$$
\begin{equation*}
\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial p}{\partial r}\right)+\frac{1}{r^{2}} \frac{\partial^{2} p}{\partial \phi^{2}}+\rho(z) \frac{\partial}{\partial z}\left(\frac{1}{\rho(z)} \frac{\partial p}{\partial z}\right)+\{k(r, z)\}^{2} p=-\frac{\delta\left(r-r_{s}\right)}{r} \delta\left(z-z_{s}\right) \delta\left(\phi-\phi_{s}\right) \tag{3.16}
\end{equation*}
$$

As we described in Section 3.1 we may represent the solution to Eq.(3.16) as in Eq.(3.15), where $U_{n}(z)$ are the depth-dependent eigenfunctions satisfying Eq. $(3.6), \Phi_{m}(\phi)$ are the azimuthal eigenfunctions satisfying Eq.(3.9) and $\Phi_{m}(\phi)$ have the form

$$
\begin{equation*}
\Phi_{m}(\phi)=e_{m} \cos m\left(\phi-\phi_{s}\right), \text { for } m=0,1,2, \ldots \tag{3.17}
\end{equation*}
$$

where $\phi_{s}$ is the azimuthal angle of the point source.
By substituting Eq.(3.15) into Eq.(3.16) together with Eqs.(3.6) and (3.9), we obtain

$$
\begin{align*}
& \sum_{m=0}^{\infty} \sum_{n=1}^{\infty}\left[\frac{1}{r} \frac{d}{d r}\left(r \frac{d R_{m n}}{d r}\right) U_{n} \Phi_{m}+\frac{1}{r^{2}} \frac{d^{2} \Phi_{m}}{d \phi^{2}} R_{m n} U_{n}+\rho \frac{d}{d z}\left(\frac{1}{\rho} \frac{d U_{n}}{d z}\right) R_{m n} \Phi_{m}+k_{n}^{2} R_{m n} U_{n} \Phi_{m}\right] \\
& =\sum_{m=0}^{\infty} \sum_{n=1}^{\infty}\left[\frac{1}{r} \frac{d}{d r}\left(r \frac{d R_{m n}}{d r}\right) U_{n} \Phi_{m}+\frac{1}{r^{2}}\left(-m^{2} \Phi_{m}\right) R_{m n} U_{n}+k_{n}^{2} R_{m n} U_{n} \Phi_{m}\right] \\
& =\sum_{m=0}^{\infty} \sum_{n=1}^{\infty}\left[\frac{1}{r} \frac{d}{d r}\left(r \frac{d R_{m n}}{d r}\right)+\left(k_{n}^{2}-\frac{m^{2}}{r^{2}}\right) R_{m n}\right] U_{n} \Phi_{m} \\
& =-\frac{\delta\left(r-r_{s}\right)}{r} \delta\left(z-z_{s}\right) \delta\left(\phi-\phi_{s}\right) \tag{3.18}
\end{align*}
$$

Our next step is to eliminate $U_{n}(z)$ and $\Phi_{m}(\phi)$ by applying the orthonormal relations of $U_{n}$ and $\Phi_{m}$. First we apply the operator $\int_{-\pi}^{\pi} \Phi_{\mu}(\phi)(\cdot) d \phi$ to Eq.(3.18) and we get

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left[\frac{1}{r} \frac{d}{d r}\left(r \frac{d R_{\mu n}}{d r}\right)+\left(k_{n}^{2}-\frac{\mu^{2}}{r^{2}}\right) R_{\mu n}\right] U_{n}=-\frac{\delta\left(r-r_{s}\right)}{r} \delta\left(z-z_{s}\right) \Phi_{\mu}\left(\phi_{s}\right) \tag{3.19}
\end{equation*}
$$

Next we apply the operator $\int_{0}^{\infty} \frac{1}{\rho(z)} U_{\nu}(z)(\cdot) d z$ to Eq.(3.19) and obtain

$$
\begin{equation*}
\frac{1}{r} \frac{d}{d r}\left(r \frac{d R_{\mu \nu}}{d r}\right)+\left(k_{\nu}^{2}-\frac{\mu^{2}}{r^{2}}\right) R_{\mu \nu}=-\frac{\delta\left(r-r_{s}\right)}{r} \frac{U_{\nu}\left(z_{s}\right)}{\rho\left(z_{s}\right)} \Phi_{\mu}\left(\phi_{s}\right) \tag{3.20}
\end{equation*}
$$

Rewriting the last equation in terms of $m$ and $n$ we have

$$
\begin{equation*}
\frac{1}{r} \frac{d}{d r}\left(r \frac{d R_{m n}}{d r}\right)+\left(k_{n}^{2}-\frac{m^{2}}{r^{2}}\right) R_{m n}=-\frac{\delta\left(r-r_{s}\right)}{r} \frac{U_{n}\left(z_{s}\right)}{\rho\left(z_{s}\right)} \Phi_{m}\left(\phi_{s}\right) \tag{3.21}
\end{equation*}
$$

which is an $m$ th-order Bessel equation and, as we have seen, can be represented as a linear combination of any pair of $J_{m}\left(k_{n} r\right), Y_{m}\left(k_{n} r\right), H_{m}^{(1)}\left(k_{n} r\right)$ and $H_{m}^{(2)}\left(k_{n} r\right)$.

### 3.3 Source conditions of the inhomogeneous three-dimensional Helmholtz equation

The external region contains the point harmonic source, which appears as a forcing term in the Helmholtz equation. The solution $R_{m n}(r)$ of the range equation (3.21) takes the form:

$$
\begin{equation*}
R_{m n}(r)=A_{m n} H_{m}^{(1)}\left(k_{n} r\right)+B_{m n} H_{m}^{(2)}\left(k_{n} r\right) \tag{3.22}
\end{equation*}
$$

### 3.3 Source conditions of the inhomogeneous three-dimensional Helmholtz equation7

for $r^{I} \leq r<r_{s}$, and

$$
\begin{equation*}
R_{m n}(r)=C_{m n} H_{m}^{(1)}\left(k_{n} r\right) \tag{3.23}
\end{equation*}
$$

for $r \geq r_{s}$.
Applying the appropriate boundary conditions at $r=r_{s}$, we may obtain $B_{m n}$ and $C_{m n}$ in terms of $A_{m n}$. These conditions express: (a) the continuity of the pressure and (b) the jump of the normal particle velocity at $r=r_{s}$.
(a) Pressure continuity

$$
\begin{equation*}
p\left(r_{s}^{-}, z, \phi\right)=p\left(r_{s}^{+}, z, \phi\right) \tag{3.24}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
A_{m n} H_{m}^{(1)}\left(k_{n} r_{s}\right)+B_{m n} H_{m}^{(2)}\left(k_{n} r_{s}\right)=C_{m n} H_{m}^{(1)}\left(k_{n} r_{s}\right) \tag{3.25}
\end{equation*}
$$

(b) Jump of normal particle velocity

We rewrite Eq.(3.21) as follows:

$$
\begin{equation*}
\frac{d^{2} R_{m n}}{d r^{2}}+\frac{1}{r} \frac{d R_{m n}}{d r}+\left(k_{n}^{2}-\frac{m^{2}}{r^{2}}\right) R_{m n}=-\frac{\delta\left(r-r_{s}\right)}{r} \frac{U_{n}\left(z_{s}\right)}{\rho\left(z_{s}\right)} \Phi_{m}\left(\phi_{s}\right) \tag{3.26}
\end{equation*}
$$

By integrating the last expression over $\left[r_{s}^{-}, r_{s}^{+}\right]$, we obtain

$$
\begin{equation*}
\left.\frac{d R_{m n}}{d r}\right|_{r_{s}^{-}} ^{r_{s}^{+}}=-\frac{1}{r_{s}} \frac{U_{n}\left(z_{s}\right)}{\rho\left(z_{s}\right)} \Phi_{m}\left(\phi_{s}\right) \tag{3.27}
\end{equation*}
$$

By substituting Eqs.(3.22) and (3.23) to Eq.(3.27) we get

$$
\begin{equation*}
C_{m n} \frac{d H_{m}^{(1)}}{d r}-A_{m n} \frac{d H_{m}^{(1)}}{d r}-B_{m n} \frac{d H_{m}^{(2)}}{d r}=-S \tag{3.28}
\end{equation*}
$$

where

$$
\begin{equation*}
S \equiv \frac{1}{r_{s}} \frac{U_{n}\left(z_{s}\right)}{\rho\left(z_{s}\right)} \Phi_{m}\left(\phi_{s}\right) \tag{3.29}
\end{equation*}
$$

Now we can solve $B_{m n}$ and $C_{m n}$ in terms of $A_{m n}$. We rewrite Eqs.(3.25) and (3.28) as

$$
\begin{equation*}
B_{m n} H_{m}^{(2)}-C_{m n} H_{m}^{(1)}=-A_{m n} H_{m}^{(1)} \tag{3.30}
\end{equation*}
$$

and

$$
\begin{equation*}
B_{m n} \frac{d H_{m}^{(2)}}{d r}-C_{m n} \frac{d H_{m}^{(1)}}{d r}=-A_{m n} \frac{d H_{m}^{(1)}}{d r}+S \tag{3.31}
\end{equation*}
$$

and solve the deriving $2 \times 2$ linear system using Cramer's rule.
The determinants $D$ and $D_{B}$ are the following

$$
\begin{gather*}
D=\left|\begin{array}{cc}
H_{m}^{(2)} & -H_{m}^{(1)} \\
\frac{d H_{m}^{(2)}}{d r} & -\frac{d H_{m}^{(1)}}{d r}
\end{array}\right|=-\frac{4 i}{\pi r_{s}}  \tag{3.32}\\
D_{B}=\left|\begin{array}{cc}
-A_{m n} H_{m}^{(1)} & -H_{m}^{(1)} \\
-A_{m n} H_{m}^{(1)} \frac{d H_{m}^{(1)}}{d r}+S & -\frac{d H_{m}^{(1)}}{d r}
\end{array}\right|=S H_{m}^{(1)} \tag{3.33}
\end{gather*}
$$

Dividing $D_{B}$ by $D$ we find the solution for $B_{m n}$

$$
\begin{equation*}
B_{m n}=\frac{i \pi}{4} \frac{U_{n}\left(z_{s}\right)}{\rho\left(z_{s}\right)} \Phi_{m}\left(\phi_{s}\right) H_{m}^{(1)}\left(k_{n} r_{s}\right) \tag{3.34}
\end{equation*}
$$

Finally, we substitute Eq.(3.34) to Eq.(3.30) to find the expression for $C_{m n}$

$$
\begin{equation*}
C_{m n}=A_{m n}+B_{m n} \frac{H_{m}^{(2)}\left(k_{n} r_{s}\right)}{H_{m}^{(1)}\left(k_{n} r_{s}\right)}=A_{m n}+\frac{i \pi}{4} \frac{U_{n}\left(z_{s}\right)}{\rho\left(z_{s}\right)} \Phi_{m}\left(\phi_{s}\right) H_{m}^{(2)}\left(k_{n} r_{s}\right) \tag{3.35}
\end{equation*}
$$

### 3.4 Representation of the field

As we have seen in Eq.(3.15), the pressure field can be expressed as

$$
p(r, z, \phi)=\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} R_{m n}(r) U_{n}(z) \Phi_{m}(\phi)
$$

with $U_{n}(z)$ satisfying Eq.(3.6) and $\Phi_{m}(\phi)$ satisfying Eqs.(3.9) and (3.17).
We will use normalized Bessel and Hankel functions to represent $R_{m n}(r)$, as in [2]. The normalized Bessel and Hankel functions are defined as follows:

$$
\begin{equation*}
\hat{H}_{m n}^{j}(r) \equiv \frac{H_{m}^{(1)}\left(k_{n}^{j} r\right)}{H_{m}^{(1)}\left(k_{n}^{j} r^{j-1}\right)} \tag{3.36}
\end{equation*}
$$

$$
\begin{equation*}
\hat{J}_{m n}^{j}(r) \equiv J_{m}\left(k_{n}^{j} r\right) H_{m}^{(1)}\left(k_{n}^{j} r^{j}\right) \tag{3.37}
\end{equation*}
$$

With the use of the above functions, which are linearly independent both for high and low orders, the linear systems that we need to solve to obtain the coupling coefficients become stable and there are no overflow or underflow problems. Appendix A contains a proof about the linear independence of $J_{m}$ and $H_{m}^{(1)}$.

Our next step is to modify the expressions we found for $B_{m n}$ and $C_{m n}$ in order to depict the solution when $R_{m n}$ is represented as a combination of normalized Hankel and Bessel functions. In this case we have:

$$
\begin{equation*}
R_{m n}(r)=a_{m n} \frac{H_{m}^{(1)}\left(k_{n} r\right)}{H_{m}^{(1)}\left(k_{n} r^{I}\right)}+b_{m n} J_{m}\left(k_{n} r\right) H_{m}^{(1)}\left(k_{n} r_{s}\right) \tag{3.38}
\end{equation*}
$$

for $r^{I} \leq r<r_{s}$, and

$$
\begin{equation*}
R_{m n}(r)=c_{m n} \frac{H_{m}^{(1)}\left(k_{n} r\right)}{H_{m}^{(1)}\left(k_{n} r_{s}\right)} \tag{3.39}
\end{equation*}
$$

for $r \geq r_{s}$.
Now, instead of solving $b_{m n}$ and $c_{m n}$ as in the previous case, we can change $R_{m n}(r)$ in Eqs. (3.38) and (3.39) into the forms in Eqs.(3.22) and (3.23) and then make use of the results in Eqs. (3.34) and (3.35).

Using the equation

$$
\begin{equation*}
J_{m}\left(k_{n} r\right)=\frac{1}{2}\left[H_{m}^{(1)}\left(k_{n} r\right)+H_{m}^{(2)}\left(k_{n} r\right)\right] \tag{3.40}
\end{equation*}
$$

and combining Eqs.(3.22) and (3.38) we obtain

$$
\begin{equation*}
\frac{a_{m n}}{H_{m}^{(1)}\left(k_{n} r^{I}\right)}+\frac{1}{2} b_{m n} H_{m}^{(1)}\left(k_{n} r_{s}\right)=A_{m n} \tag{3.41}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{2} b_{m n} H_{m}^{(1)}\left(k_{n} r_{s}\right)=B_{m n} \tag{3.42}
\end{equation*}
$$

Substituting Eq.(3.34) into Eq.(3.42) we get

$$
\begin{equation*}
b_{m n}=\frac{2 B_{m n}}{H_{m}^{(1)}\left(k_{n} r_{s}\right)}=\frac{i \pi}{2} \frac{U_{n}\left(z_{s}\right)}{\rho\left(z_{s}\right)} \Phi_{m}\left(\phi_{s}\right) \tag{3.43}
\end{equation*}
$$

Working similarly we obtain $c_{m n}$ as

$$
\begin{equation*}
c_{m n}=a_{m n} \frac{H_{m}^{(1)}\left(k_{n} r_{s}\right)}{H_{m}^{(1)}\left(k_{n} r^{I}\right)}+\frac{i \pi}{2} \frac{U_{n}\left(z_{s}\right)}{\rho\left(z_{s}\right)} \Phi_{m}\left(\phi_{s}\right) J_{m}\left(k_{n} r_{s}\right) H_{m}^{(1)}\left(k_{n} r_{s}\right) \tag{3.44}
\end{equation*}
$$

Furthermore, we will apply the superposition method for the region outside the seamount. This method eliminates the virtual boundary at the source range and instead extends the outer region to the base of the seamount. The acoustic pressure is then obtained as a superposition of the unperturbed field produced by the source in the absence of the seamount and a scattered field produced by the seamount.

With this approach, the field in the three different regions is represented as:

1. Region I : $r \leq r^{1}$ (the innermost ring)

$$
\begin{equation*}
p^{1}(r, z, \phi)=\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} b_{m n}^{1} \hat{J}_{m n}^{1}(r) U_{n}^{1}(z) \Phi_{m}(\phi) \tag{3.45}
\end{equation*}
$$

2. Region II: (intermediate rings) $r^{1}<r \leq r^{I}$, where $r^{I}$ is the radius of the base of the seamount. In ring $j$, i.e. $r^{j-1}<r \leq r^{j}$

$$
\begin{equation*}
p^{j}(r, z, \phi)=\sum_{m=0}^{\infty} \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j} \hat{H}_{m n}^{j}(r)+b_{m n}^{j} \hat{J}_{m n}^{j}(r)\right] U_{n}^{j}(z) \Phi_{m}(\phi) \tag{3.46}
\end{equation*}
$$

3. Region III: $r>r^{I}$ (outside the base of the seamount)

$$
\begin{equation*}
p(r, z, \phi)=p_{i}\left(r^{\prime}, z\right)+\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \alpha_{m n}^{J} \hat{H}_{m n}^{J}(r) U_{n}^{J}(z) \Phi_{m}(\phi) \tag{3.47}
\end{equation*}
$$

where $r^{\prime}$ is the range of a field point with respect to the source (see Figure 3.1)

$$
\begin{equation*}
r^{\prime}(r, \phi)=\sqrt{r^{2}+r_{s}^{2}-2 r r_{s} \cos \left(\phi_{s}-\phi\right)} \tag{3.48}
\end{equation*}
$$

and $p_{i}\left(r^{\prime}, z\right)$ is the 2D normal mode solution

$$
\begin{equation*}
p_{i}\left(r^{\prime}, z\right)=\frac{i}{4 \rho\left(z_{s}\right)} \sum_{n=1}^{\infty} U_{n}^{J}\left(z_{s}\right) U_{n}^{J}(z) H_{0}^{(1)}\left(k_{n}^{J} r^{\prime}\right) \tag{3.49}
\end{equation*}
$$



Figure 3.1: Use of the superposition method to obtain the pressure field outside the seamount

### 3.4.1 Convergence of the Model

From Eq.(3.46), for $r^{j-1}<r<r^{j}$ the field is

$$
p^{j}(r, z, \phi)=\sum_{m=0}^{\infty} \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j} \hat{H}_{m n}^{j}(r)+b_{m n}^{j} \hat{J}_{m n}^{j}(r)\right] U_{n}^{j}(z) \Phi_{m}(\phi)
$$

We know that the summation of normal modes converges, given that for long range propagation problems we can take into account only the propagating modes. Below we will show the convergence of the azimuthal modes.

For $m \gg k_{n}^{j} r$ we may obtain the asymptotic forms of the normalized Bessel and Hankel functions (refer to Appendix A)

$$
\begin{equation*}
\hat{H}_{m n}^{j}(r)=\frac{H_{m}^{(1)}\left(k_{n}^{j} r\right)}{H_{m}^{(1)}\left(k_{n}^{j} r^{j-1}\right)} \sim\left(\frac{r^{j-1}}{r}\right)^{m} \rightarrow 0 \tag{3.50}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{J}_{m n}^{j}(r)=J_{m}\left(k_{n}^{j} r\right) H_{m}^{(1)}\left(k_{n}^{j} r^{j}\right) \sim-\frac{i}{m \pi}\left(\frac{r}{r^{j}}\right)^{m} \rightarrow 0 \tag{3.51}
\end{equation*}
$$

From the above we can see that when the order of azimuthal modes becomes much greater than the argument $k_{n}^{j} r$, the normalized Bessel and Hankel functions approach zero. Thus, we can use only a finite number of azimuthal modes to have convergent results. We set this number to


Figure 3.2: A two-way coupled mode model
be $M=\left[k_{0} r^{I}\right]$, where $k_{0}$ is the biggest possible eigenvalue in water, $r^{I}$ is the radius of the base of the seamount and $[x]$ rounds $x$ to the nearest integer towards infinity.

### 3.5 Interface conditions and two-way coupling

The interface conditions to be applied at the artificial cylindrical boundaries between each segment will ensure the continuity of the pressure and the radial component of the particle velocity. Subsequently, a two-way coupling method will be used for the calculation of the coupling coefficients. The first step contains the inward marching coupling, where the single-scatter approximation is used, and the second step contains the outward marching coupling, where the one-way approximation is used. The combination of the two steps leads to approximate two-way coupling. Figure 3.2 illustrates the marching two-way, single-scatter solution scheme.

### 3.5.1 Mode coupling in inward marching

The scheme for the inward marching technique is depicted in Figure 3.3. Here the single scatter approximation is applied. For two successive rings, i.e. ring $j+1$ and ring $j$, with $b_{m}^{j+1}$ known, we apply the interface conditions at $r=r^{j}$ to derive $a_{m}^{j+1}$ and $b_{m}^{j}$.

1. Continuity of pressure at $r=r^{j}$


Figure 3.3: Coupling between two neighboring rings in inward marching

The boundary condition at $r=r^{j}$ gives

$$
\begin{equation*}
p^{j}\left(r^{j}, z, \phi\right)=p^{j+1}\left(r^{j}, z, \phi\right) \tag{3.52}
\end{equation*}
$$

Substituting Eq.(3.46) into Eq.(3.52) we get

$$
\begin{align*}
& \sum_{m=0}^{\infty} \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j} \hat{H}_{m n}^{j}\left(r^{j}\right)+b_{m n}^{j} \hat{J}_{m n}^{j}\left(r^{j}\right)\right] U_{n}^{j}(z) \Phi_{m}(\phi) \\
= & \sum_{m=0}^{\infty} \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j+1} \hat{H}_{m n}^{j+1}\left(r^{j}\right)+b_{m n}^{j+1} \hat{J}_{m n}^{j+1}\left(r^{j}\right)\right] U_{n}^{j+1}(z) \Phi_{m}(\phi) \tag{3.53}
\end{align*}
$$

Restricting the last equation to the $m$-th azimuthal mode we have

$$
\begin{align*}
& \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j} \hat{H}_{m n}^{j}\left(r^{j}\right)+b_{m n}^{j} \hat{J}_{m n}^{j}\left(r^{j}\right)\right] U_{n}^{j}(z) \\
= & \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j+1} \hat{H}_{m n}^{j+1}\left(r^{j}\right)+b_{m n}^{j+1} \hat{J}_{m n}^{j+1}\left(r^{j}\right)\right] U_{n}^{j+1}(z) \tag{3.54}
\end{align*}
$$

We now apply the operator $\int_{0}^{\infty} \frac{1}{\rho^{j}(z)} U_{\nu}^{j}(z)(\cdot) d z$ to Eq. (3.54) and get

$$
\begin{align*}
& {\left[\alpha_{m \nu}^{j} \hat{H}_{m \nu}^{j}\left(r^{j}\right)+b_{m \nu}^{j} \hat{J}_{m \nu}^{j}\left(r^{j}\right)\right] } \\
= & \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j+1} \hat{H}_{m n}^{j+1}\left(r^{j}\right)+b_{m n}^{j+1} \hat{J}_{m n}^{j+1}\left(r^{j}\right)\right] \int_{0}^{\infty} \frac{1}{\rho^{j}(z)} U_{\nu}^{j}(z) U_{n}^{j+1}(z) d z \tag{3.55}
\end{align*}
$$

Using the notation

$$
\begin{equation*}
C_{\alpha \nu n}^{j+1} \equiv \int_{0}^{\infty} \frac{1}{\rho^{j}(z)} U_{\nu}^{j}(z) U_{n}^{j+1}(z) d z \tag{3.56}
\end{equation*}
$$

we can rewrite Eq.(3.55) in a matrix form

$$
\begin{equation*}
\hat{\mathbf{H}}_{m}^{j} \boldsymbol{\alpha}_{m}^{j}+\hat{\mathbf{J}}_{m}^{j} \mathbf{b}_{m}^{j}=\mathbf{C}_{\alpha}^{j+1}\left(\hat{\mathbf{H}}_{m}^{j+1} \boldsymbol{\alpha}_{m}^{j+1}+\hat{\mathbf{J}}_{m}^{j+1} \mathbf{b}_{m}^{j+1}\right) \tag{3.57}
\end{equation*}
$$

where $\hat{\mathbf{H}}_{m}^{j}, \hat{\mathbf{J}}_{m}^{j}, \hat{\mathbf{H}}_{m}^{j+1}$ and $\hat{\mathbf{J}}_{m}^{j+1}$ are diagonal matrices like

$$
\hat{\mathbf{H}}_{m}^{j}=\operatorname{diag}\left(\hat{H}_{m n}^{j}\left(r^{j}\right)\right)_{n=1,2, \ldots, N}=\operatorname{diag}\left(\frac{H_{m}^{(1)}\left(k_{n}^{j} r^{j}\right)}{H_{m}^{(1)}\left(k_{n}^{j} r^{j-1}\right)}\right)_{n=1,2, \ldots, N}
$$

and $\boldsymbol{\alpha}_{m}^{j}, \mathbf{b}_{m}^{j}, \boldsymbol{\alpha}_{m}^{j+1}$ and $\mathbf{b}_{m}^{j+1}$ are column vectors like

$$
\boldsymbol{\alpha}_{m}^{j}=\left[\begin{array}{c}
\alpha_{m 1}^{j} \\
\alpha_{m 2}^{j} \\
\vdots \\
\alpha_{m N}^{j}
\end{array}\right]
$$

where $N$ is the number of normal modes and $\mathbf{C}_{\alpha}^{j+1}=\left[C_{\alpha \nu n}^{j+1}\right], \nu=1, \ldots, N, n=1, \ldots, N$.
2. Continuity of the normal component of the particle velocity at $r=r^{j}$ This boundary condition gives

$$
\begin{equation*}
\left.\frac{1}{\rho^{j}} \frac{\partial p^{j}}{\partial r}\right|_{r^{j}}=\left.\frac{1}{\rho^{j+1}} \frac{\partial p^{j+1}}{\partial r}\right|_{r^{j}} \tag{3.58}
\end{equation*}
$$

Using the notations

$$
\begin{gather*}
D \hat{H}_{m n}^{j}(r) \equiv \frac{\frac{d H_{m}^{(1)}\left(k_{n}^{j} r\right)}{d\left(k_{n}^{j} r\right)}}{H_{m}^{(1)}\left(k_{n}^{j} r^{j-1}\right)}  \tag{3.59}\\
D \hat{J}_{m n}^{j}(r) \equiv \frac{d J_{m}\left(k_{n}^{j} r\right)}{d\left(k_{n}^{j} r\right)} H_{m}^{(1)}\left(k_{n}^{j} r^{j}\right) \tag{3.60}
\end{gather*}
$$

we have

$$
\begin{equation*}
\frac{d \hat{H}_{m n}^{j}(r)}{d r}=\frac{\frac{d H_{m}^{(1)}\left(k_{n}^{j} r\right)}{d r}}{H_{m}^{(1)}\left(k_{n}^{j} r^{j-1}\right)}=\frac{k_{n}^{j} \frac{d H_{m}^{(1)}\left(k_{n}^{j} r\right)}{d\left(k_{n}^{j} r\right)}}{H_{m}^{(1)}\left(k_{n}^{j} r^{j-1}\right)}=k_{n}^{j} D \hat{H}_{m n}^{j}(r) \tag{3.61}
\end{equation*}
$$

$$
\begin{equation*}
\frac{d \hat{J}_{m n}^{j}(r)}{d r}=\frac{d J_{m}\left(k_{n}^{j} r\right)}{d r} H_{m}^{(1)}\left(k_{n}^{j} r^{j}\right)=k_{n}^{j} \frac{d J_{m}\left(k_{n}^{j} r\right)}{d k_{n}^{j} r} H_{m}^{(1)}\left(k_{n}^{j} r^{j}\right)=k_{n}^{j} D \hat{J}_{m n}^{j}(r) \tag{3.62}
\end{equation*}
$$

Substituting Eq.(3.46) into Eq.(3.58) and using Eqs.(3.61) and (3.62) we get

$$
\begin{align*}
& \frac{1}{\rho^{j}} \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j} k_{n}^{j} D \hat{H}_{m n}^{j}\left(r^{j}\right)+b_{m n}^{j} k_{n}^{j} D \hat{J}_{m n}^{j}\left(r^{j}\right)\right] U_{n}^{j}(z)  \tag{3.63}\\
= & \frac{1}{\rho^{j+1}} \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j+1} k_{n}^{j+1} D \hat{H}_{m n}^{j+1}\left(r^{j}\right)+b_{m n}^{j+1} k_{n}^{j+1} D \hat{J}_{m n}^{j+1}\left(r^{j}\right)\right] U_{n}^{j+1}(z)
\end{align*}
$$

Applying the operator $\int_{0}^{\infty} U_{\nu}^{j}(z)(\cdot) d z$ to Eq.(3.63) we obtain

$$
\begin{aligned}
& k_{\nu}^{j}\left[\alpha_{m \nu}^{j} D \hat{H}_{m \nu}^{j}\left(r^{j}\right)+b_{m \nu}^{j} D \hat{J}_{m \nu}^{j}\left(r^{j}\right)\right] \\
= & \sum_{n=1}^{\infty} k_{n}^{j+1}\left[\alpha_{m n}^{j+1} D \hat{H}_{m n}^{j+1}\left(r^{j}\right)+b_{m n}^{j+1} D \hat{J}_{m n}^{j+1}\left(r^{j}\right)\right] \int_{0}^{\infty} \frac{1}{\rho^{j+1}(z)} U_{\nu}^{j}(z) U_{n}^{j+1}(z) d z
\end{aligned}
$$

or

$$
\begin{align*}
& \alpha_{m \nu}^{j} D \hat{H}_{m \nu}^{j}\left(r^{j}\right)+b_{m \nu}^{j} D \hat{J}_{m \nu}^{j}\left(r^{j}\right) \\
= & \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j+1} D \hat{H}_{m n}^{j+1}\left(r^{j}\right)+b_{m n}^{j+1} D \hat{J}_{m n}^{j+1}\left(r^{j}\right)\right] \frac{k_{n}^{j+1}}{k_{\nu}^{j}} \int_{0}^{\infty} \frac{1}{\rho^{j+1}(z)} U_{\nu}^{j}(z) U_{n}^{j+1}(z) d z \tag{3.64}
\end{align*}
$$

Using the notation

$$
\begin{equation*}
C_{b \nu n}^{j+1} \equiv \frac{k_{n}^{j+1}}{k_{\nu}^{j}} \int_{0}^{\infty} \frac{1}{\rho^{j+1}(z)} U_{\nu}^{j}(z) U_{n}^{j+1}(z) d z \tag{3.65}
\end{equation*}
$$

we can rewrite Eq.(3.64) in the matrix form

$$
\begin{equation*}
\mathbf{D} \hat{\mathbf{H}}_{m}^{j} \boldsymbol{\alpha}_{m}^{j}+\mathbf{D} \hat{\mathbf{J}}_{m}^{j} \mathbf{b}_{m}^{j}=\mathbf{C}_{b}^{j+1}\left(\mathbf{D} \hat{\mathbf{H}}_{m}^{j+1} \boldsymbol{\alpha}_{m}^{j+1}+\mathbf{D} \hat{\mathbf{J}}_{m}^{j+1} \mathbf{b}_{m}^{j+1}\right) \tag{3.66}
\end{equation*}
$$

where $\mathbf{D} \hat{\mathbf{H}}_{m}^{j}, \mathbf{D} \hat{\mathbf{J}}_{m}^{j}, \mathbf{D} \hat{\mathbf{H}}_{m}^{j+1}$ and $\mathbf{D} \hat{\mathbf{J}}_{m}^{j+1}$ are diagonal matrices and $\boldsymbol{\alpha}_{m}^{j}, \mathbf{b}_{m}^{j}, \boldsymbol{\alpha}_{m}^{j+1}$ and $\mathbf{b}_{m}^{j+1}$ are column vectors.

Equations (3.57) and (3.66) compose a linear system of equations. The solutions of this system are the coupling coefficient vectors $\boldsymbol{\alpha}_{m}^{j}$ and $\mathbf{b}_{m}^{j}$.

After some calculations we reach to the following equation

$$
\left[\begin{array}{c}
\mathbf{b}_{m}^{j}  \tag{3.67}\\
\boldsymbol{\alpha}_{m}^{j}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{R}_{m 4}^{j+1} & \mathbf{R}_{m 3}^{j+1} \\
\mathbf{R}_{m 2}^{j+1} & \mathbf{R}_{m 1}^{j+1}
\end{array}\right]\left[\begin{array}{c}
\mathbf{b}_{m}^{j+1} \\
\boldsymbol{\alpha}_{m}^{j+1}
\end{array}\right]
$$

where

$$
\begin{align*}
& \mathbf{R}_{m 1}^{j+1}=-\left(\mathbf{F}_{b}^{j}\right)^{-1}\left(\mathbf{D} \hat{\mathbf{J}}_{m}^{j} \mathbf{C}_{\alpha}^{j+1} \hat{\mathbf{H}}_{m}^{j+1}-\hat{\mathbf{J}}_{m}^{j} \mathbf{C}_{b}^{j+1} \mathbf{D} \hat{\mathbf{H}}_{m}^{j+1}\right)  \tag{3.68}\\
& \mathbf{R}_{m 2}^{j+1}=-\left(\mathbf{F}_{b}^{j}\right)^{-1}\left(\mathbf{D} \hat{\mathbf{J}}_{m}^{j} \mathbf{C}_{\alpha}^{j+1} \hat{\mathbf{J}}_{m}^{j+1}-\hat{\mathbf{J}}_{m}^{j} \mathbf{C}_{b}^{j+1} \mathbf{D} \hat{\mathbf{J}}_{m}^{j+1}\right)  \tag{3.69}\\
& \mathbf{R}_{m 3}^{j+1}=\left(\mathbf{F}_{b}^{j}\right)^{-1}\left(\mathbf{D} \hat{\mathbf{H}}_{m}^{j} \mathbf{C}_{\alpha}^{j+1} \hat{\mathbf{H}}_{m}^{j+1}-\hat{\mathbf{H}}_{m}^{j} \mathbf{C}_{b}^{j+1} \mathbf{D} \hat{\mathbf{H}}_{m}^{j+1}\right)  \tag{3.70}\\
& \mathbf{R}_{m 4}^{j+1}=\left(\mathbf{F}_{b}^{j}\right)^{-1}\left(\mathbf{D} \hat{\mathbf{H}}_{m}^{j} \mathbf{C}_{\alpha}^{j+1} \hat{\mathbf{J}}_{m}^{j+1}-\hat{\mathbf{H}}_{m}^{j} \mathbf{C}_{b}^{j+1} \mathbf{D} \hat{\mathbf{J}}_{m}^{j+1}\right) \tag{3.71}
\end{align*}
$$

and

$$
\begin{equation*}
\left(\mathbf{F}_{b}^{j}\right)^{-1}=\frac{\pi r^{j}}{2 i} \operatorname{diag}\left(k_{\nu}^{j} \frac{H_{m}^{(1)}\left(k_{\nu}^{j} r^{j-1}\right)}{H_{m}^{(1)}\left(k_{\nu}^{j} r^{j}\right)}\right), \nu=1,2, \ldots, N . \tag{3.72}
\end{equation*}
$$

To apply the single-scatter approximation we let $\boldsymbol{\alpha}_{m}^{j}=0$. Then, from Eq.(3.67) we have

$$
\begin{equation*}
\mathbf{R}_{m 2}^{j+1} \mathbf{b}_{m}^{j+1}+\mathbf{R}_{m 1}^{j+1} \boldsymbol{\alpha}_{m}^{j+1}=0 \tag{3.73}
\end{equation*}
$$

from which, solving for $\boldsymbol{\alpha}_{m}^{j+1}$ we get

$$
\begin{equation*}
\boldsymbol{\alpha}_{m}^{j+1}=-\left(\mathbf{R}_{m 1}^{j+1}\right)^{-1} \mathbf{R}_{m 2}^{j+1} \mathbf{b}_{m}^{j+1} \tag{3.74}
\end{equation*}
$$

Once we compute $\boldsymbol{\alpha}_{m}^{j+1}$, we find the solution for $\mathbf{b}_{m}^{j}$ which is

$$
\begin{equation*}
\mathbf{b}_{m}^{j}=\mathbf{R}_{m 4}^{j+1} \mathbf{b}_{m}^{j+1}+\mathbf{R}_{m 3}^{j+1} \boldsymbol{\alpha}_{m}^{j+1} \tag{3.75}
\end{equation*}
$$

### 3.5.2 Mode coupling in outward marching

The scheme for the outward marching technique is depicted in Figure 3.4. Here the one way approximation is applied. For two successive rings, i.e. ring $j$ and ring $j+1$, we need to obtain $\boldsymbol{\alpha}_{m}^{j+1}$, with $\boldsymbol{\alpha}_{m}^{j}$ known.

We now apply the boundary conditions

1. Continuity of pressure at $r=r^{j}$


Figure 3.4: Coupling between two neighboring rings in outward marching

The boundary condition at $r=r^{j}$ gives

$$
\begin{equation*}
p^{j+1}\left(r^{j}, z, \phi\right)=p^{j}\left(r^{j}, z, \phi\right) \tag{3.76}
\end{equation*}
$$

Substituting Eq.(3.46) into Eq.(3.76) we get (for the $m$-th azimuthal mode)

$$
\begin{align*}
& \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j+1} \hat{H}_{m n}^{j+1}\left(r^{j}\right)+b_{m n}^{j+1} \hat{J}_{m n}^{j+1}\left(r^{j}\right)\right] U_{n}^{j+1}(z) \\
= & \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j} \hat{H}_{m n}^{j}\left(r^{j}\right)+b_{m n}^{j} \hat{J}_{m n}^{j}\left(r^{j}\right)\right] U_{n}^{j}(z) \tag{3.77}
\end{align*}
$$

By applying $\int_{0}^{\infty} \frac{1}{\rho^{j+1}(z)} U_{\nu}^{j+1}(z)(\cdot) d z$ to the last equation we obtain

$$
\begin{align*}
& \alpha_{m \nu}^{j+1} \hat{H}_{m \nu}^{j+1}+b_{m \nu}^{j+1} \hat{J}_{m \nu}^{j+1} \\
= & \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j} \hat{H}_{m n}^{j}+b_{m n}^{j} \hat{J}_{m n}^{j}\right] \int_{0}^{\infty} \frac{1}{\rho^{j+1}(z)} U_{\nu}^{j+1}(z) U_{n}^{j}(z) d z \tag{3.78}
\end{align*}
$$

Using the notation

$$
\begin{equation*}
C_{c \nu n}^{j} \equiv \int_{0}^{\infty} \frac{1}{\rho^{j+1}(z)} U_{\nu}^{j+1}(z) U_{n}^{j}(z) d z \tag{3.79}
\end{equation*}
$$

we can rewrite Eq.(3.78) in the matrix form

$$
\begin{equation*}
\hat{\mathbf{H}}_{m}^{j+1} \boldsymbol{\alpha}_{m}^{j+1}+\hat{\mathbf{J}}_{m}^{j+1} \mathbf{b}_{m}^{j+1}=\mathbf{C}_{c}^{j}\left(\hat{\mathbf{H}}_{m}^{j} \boldsymbol{\alpha}_{m}^{j}+\hat{\mathbf{J}}_{m}^{j} \mathbf{b}_{m}^{j}\right) \tag{3.80}
\end{equation*}
$$

where $\hat{\mathbf{H}}_{m}^{j}, \hat{\mathbf{J}}_{m}^{j}, \hat{\mathbf{H}}_{m}^{j+1}$ and $\hat{\mathbf{J}}_{m}^{j+1}$ are diagonal matrices and $\boldsymbol{\alpha}_{m}^{j}, \mathbf{b}_{m}^{j}, \boldsymbol{\alpha}_{m}^{j+1}$ and $\mathbf{b}_{m}^{j+1}$ are column vectors.
2. Continuity of the normal component of the particle velocity at $r=r^{j}$

This boundary condition gives

$$
\begin{equation*}
\left.\frac{1}{\rho^{j+1}} \frac{\partial p^{j+1}}{\partial r}\right|_{r^{j}}=\left.\frac{1}{\rho^{j}} \frac{\partial p^{j}}{\partial r}\right|_{r^{j}} \tag{3.81}
\end{equation*}
$$

Inserting Eq.(3.46) into Eq.(3.81) and using the notations

$$
\begin{gather*}
D \hat{H}_{m n}^{j}(r) \equiv \frac{\frac{d H_{m}^{(1)}\left(k_{n}^{j} r\right)}{d\left(k_{n}^{j} r\right)}}{H_{m}^{(1)}\left(k_{n}^{j} r^{j-1}\right)}  \tag{3.82}\\
D \hat{J}_{m n}^{j}(r) \equiv \frac{d J_{m}\left(k_{n}^{j} r\right)}{d\left(k_{n}^{j} r\right)} H_{m}^{(1)}\left(k_{n}^{j} r^{j}\right) \tag{3.83}
\end{gather*}
$$

we have (for the $m$-th azimuthal mode)

$$
\begin{align*}
& \frac{1}{\rho^{j+1}(z)} \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j+1} k_{n}^{j+1} D \hat{H}_{m n}^{j+1}\left(r^{j}\right)+b_{m n}^{j+1} k_{n}^{j+1} D \hat{J}_{m n}^{j+1}\left(r^{j}\right)\right] U_{n}^{j+1}(z) \\
= & \frac{1}{\rho^{j}(z)} \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j} k_{n}^{j} D \hat{H}_{m n}^{j}\left(r^{j}\right)+b_{m n}^{j} k_{n}^{j} D \hat{J}_{m n}^{j}\left(r^{j}\right)\right] U_{n}^{j}(z) \tag{3.84}
\end{align*}
$$

Applying the operator $\int_{0}^{\infty} U_{\nu}^{j+1}(z)(\cdot) d z$ to Eq.(3.84) we obtain

$$
\begin{align*}
& \alpha_{m \nu}^{j+1} D \hat{H}_{m \nu}^{j+1}\left(r^{j}\right)+b_{m \nu}^{j+1} D \hat{J}_{m \nu}^{j+1}\left(r^{j}\right) \\
= & \sum_{n=1}^{\infty}\left[\alpha_{m n}^{j} D \hat{H}_{m n}^{j}\left(r^{j}\right)+b_{m n}^{j} D \hat{J}_{m n}^{j}\left(r^{j}\right)\right] \frac{k_{n}^{j}}{k_{\nu}^{j+1}} \int_{0}^{\infty} \frac{1}{\rho^{j}(z)} U_{\nu}^{j+1}(z) U_{n}^{j}(z) d z \tag{3.85}
\end{align*}
$$

With notation

$$
\begin{equation*}
C_{d \nu n}^{j} \equiv \frac{k_{n}^{j}}{k_{\nu}^{j+1}} \int_{0}^{\infty} \frac{1}{\rho^{j}(z)} U_{\nu}^{j+1}(z) U_{n}^{j}(z) d z \tag{3.86}
\end{equation*}
$$

we can rewrite Eq.(3.85) in the matrix form

$$
\begin{equation*}
\mathbf{D} \hat{\mathbf{H}}_{m}^{j+1} \boldsymbol{\alpha}_{m}^{j+1}+\mathbf{D} \hat{\mathbf{J}}_{m}^{j+1} \mathbf{b}_{m}^{j+1}=\mathbf{C}_{d}^{j}\left(\mathbf{D} \hat{\mathbf{H}}_{m}^{j} \boldsymbol{\alpha}_{m}^{j}+\mathbf{D} \hat{\mathbf{J}}_{m}^{j} \mathbf{b}_{m}^{j}\right) \tag{3.87}
\end{equation*}
$$

The solution of the system defined by Eqs.(3.80) and (3.87), together with the one-way approximation will give us an explicit expression for $\boldsymbol{\alpha}_{m}^{j+1}$ in terms of $\boldsymbol{\alpha}_{m}^{j}$.

Similar to the derivation in the inward marching, we reach to the following equation

$$
\left[\begin{array}{c}
\mathbf{b}_{m}^{j+1}  \tag{3.88}\\
\boldsymbol{\alpha}_{m}^{j+1}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{S}_{m 4}^{j} & \mathbf{S}_{m 3}^{j} \\
\mathbf{S}_{m 2}^{j} & \mathbf{S}_{m 1}^{j}
\end{array}\right]\left[\begin{array}{c}
\mathbf{b}_{m}^{j} \\
\boldsymbol{\alpha}_{m}^{j}
\end{array}\right]
$$

Using the one way approximation we let $\mathbf{b}_{m}^{j+1}=0$ and $\mathbf{b}_{m}^{j}=0$, so Eq.(3.88) reduces to

$$
\begin{equation*}
\boldsymbol{\alpha}_{m}^{j+1}=\mathbf{S}_{m 1}^{j} \boldsymbol{\alpha}_{m}^{j} \tag{3.89}
\end{equation*}
$$

where

$$
\begin{gather*}
\mathbf{S}_{m 1}^{j}=-\left(\mathbf{G}_{\alpha}^{j+1}\right)^{-1}\left(\mathbf{D} \hat{\mathbf{J}}_{m}^{j+1} \mathbf{C}_{c}^{j} \hat{\mathbf{H}}_{m}^{j}-\hat{\mathbf{J}}_{m}^{j+1} \mathbf{C}_{d}^{j} \mathbf{D} \hat{\mathbf{H}}_{m}^{j}\right)  \tag{3.90}\\
C_{c \nu n}^{j}=\int_{0}^{\infty} \frac{1}{\rho^{j+1}(z)} U_{\nu}^{j+1}(z) U_{n}^{j}(z) d z  \tag{3.91}\\
C_{d \nu n}^{j}=\frac{k_{n}^{j}}{k_{\nu}^{j+1}} \int_{0}^{\infty} \frac{1}{\rho^{j}(z)} U_{\nu}^{j+1}(z) U_{n}^{j}(z) d z  \tag{3.92}\\
\left(\mathbf{G}_{\alpha}^{j+1}\right)^{-1}=i \frac{\pi r^{j}}{2} \operatorname{diag}\left(k_{\nu}^{j+1} \frac{H_{m}^{(1)}\left(k_{\nu}^{j+1} r^{j}\right)}{H_{m}^{(1)}\left(k_{\nu}^{j+1} r^{j+1}\right)}\right), \nu=1,2, \ldots, N . \tag{3.93}
\end{gather*}
$$

Note that the coupling matrices $\mathbf{C}_{\alpha}, \mathbf{C}_{\mathbf{b}}, \mathbf{C}_{\mathbf{c}}$ and $\mathbf{C}_{\mathbf{d}}$, which express the exchange of energy between the propagating modes in neighboring rings, are independent of azimuthal orders so they can be calculated only once.

### 3.5.3 Numerical Stability

In the previous subsections we saw the formulas for inward and outward marching coupling. Combining them together we get the result depicted in Figure 3.2. However, we can observe from Eqs.(3.72) and (3.93) that $\left(\mathbf{F}_{\mathbf{b}}^{\mathbf{j}}\right)^{-\mathbf{1}}$ as well as $\left(\mathbf{G}_{\alpha}^{\mathbf{j}+\mathbf{1}}\right)^{\mathbf{- 1}}$ are not properly normalized both for very small and very large azimuthal orders with respect to $k_{n}^{j} r^{j}$, thus some more calculations have to be conducted in order to avoid unstable solutions.

1. Calculation of $\mathbf{R}_{\mathrm{m} 3}^{\mathrm{j}+1}$ and $\mathbf{R}_{\mathrm{m} 4}^{\mathrm{j}+1}$

The computation of the products $\left(\mathbf{F}_{\mathbf{b}}^{\mathbf{j}}\right)^{-\mathbf{1}} \mathbf{D} \hat{\mathbf{H}}_{\mathbf{m}}^{\mathbf{j}}$ and $\left(\mathbf{F}_{\mathbf{b}}^{\mathbf{j}}\right)^{-\mathbf{1}} \hat{\mathbf{H}}_{\mathbf{m}}^{\mathbf{j}}$ is sufficient to give us stable results for $\mathbf{R}_{\mathbf{m} \mathbf{3}}^{\mathbf{j + 1}}$ and $\mathbf{R}_{\mathbf{m} \mathbf{4}}^{\mathbf{j}+\mathbf{1}}$. Indeed, multiplying $\left(\mathbf{F}_{\mathbf{b}}^{\mathbf{j}}\right)^{\mathbf{- 1}}$ with the diagonal matrices $\mathbf{D} \hat{\mathbf{H}}_{\mathbf{m}}^{\mathbf{j}}$ and $\hat{\mathbf{H}}_{\mathbf{m}}^{\mathbf{j}}$ we get

$$
\begin{gather*}
\left(\mathbf{F}_{\mathbf{b}}^{\mathbf{j}}\right)^{-\mathbf{1}} \mathbf{D} \hat{\mathbf{H}}_{\mathbf{m}}^{\mathbf{j}}=\frac{\pi r^{j}}{2 i} \operatorname{diag}\left(k_{\nu}^{j} \frac{\frac{d H_{m}^{(1)}\left(k^{j} r^{j}\right)}{d\left(k_{\nu}^{\nu} r\right)}}{H_{m}^{(1)}\left(k_{\nu}^{j} r^{j}\right)}\right), \nu=1,2, \ldots, N .  \tag{3.94}\\
\left(\mathbf{F}_{\mathbf{b}}^{\mathbf{j}}\right)^{-\mathbf{1}} \hat{\mathbf{H}}_{\mathbf{m}}^{\mathbf{j}}=\frac{\pi r^{j}}{2 i} \operatorname{diag}\left(k_{\nu}^{j}\right), \nu=1,2, \ldots, N . \tag{3.95}
\end{gather*}
$$

The above matrices are stable, so by substituting them to Eqs.(3.70) and (3.71) we get stable $\mathbf{R}_{m 3}^{j+1}$ and $\mathbf{R}_{m 4}^{j+1}$.
2. Calculation of $\left(\mathbf{R}_{\mathbf{m} 1}^{\mathbf{j}+1}\right)^{-\mathbf{1}} \mathbf{R}_{\mathbf{m} 2}^{\mathbf{j}+1}$

It is easy to see that conducting the multiplication between $\left(\mathbf{R}_{m 1}^{j+1}\right)^{-1}$ and $\mathbf{R}_{m 2}^{j+1}$ the term $\left(\mathbf{F}_{\mathbf{b}}^{\mathbf{j}}\right)^{-\mathbf{1}}$ vanishes, so the result for $\boldsymbol{\alpha}_{m}^{j+1}$ is stable.

## 3. Calculation of $\mathbf{S}_{\mathrm{m} 1}^{\mathbf{j}}$

Similar to the "trick" used in the calculation of $\mathbf{R}_{m 3}^{j+1}$ and $\mathbf{R}_{m 4}^{j+1}$, we first compute the products of the diagonal matrix $\left(\mathbf{G}_{\alpha}^{\mathbf{j}+\mathbf{1}}\right)^{-\mathbf{1}}$ with $\mathbf{D} \hat{\mathbf{J}}_{\mathbf{m}}^{\mathbf{j}+\mathbf{1}}$ and $\hat{\mathbf{J}}_{\mathbf{m}}^{\mathbf{j}+\mathbf{1}}$. These multiplications give

$$
\begin{gather*}
\left(\mathbf{G}_{\alpha}^{\mathbf{j}+\mathbf{1}}\right)^{-\mathbf{1}} \mathbf{D} \hat{\mathbf{J}}_{\mathbf{m}}^{\mathbf{j}+\mathbf{1}}=i \frac{\pi r^{j}}{2} \operatorname{diag}\left(k_{\nu}^{j+1} \frac{d J_{m}\left(k_{\nu}^{j+1} r^{j}\right)}{d\left(k_{\nu}^{j+1} r\right)} H_{m}^{(1)}\left(k_{\nu}^{j+1} r^{j}\right)\right), \nu=1,2, \ldots, N .  \tag{3.96}\\
\left(\mathbf{G}_{\alpha}^{\mathbf{j}+\mathbf{1}}\right)^{\mathbf{- 1}} \hat{\mathbf{J}}_{\mathbf{m}}^{\mathbf{j}+\mathbf{1}}=i \frac{\pi r^{j}}{2} \operatorname{diag}\left(k_{\nu}^{j+1} J_{m}\left(k_{\nu}^{j+1} r^{j}\right) H_{m}^{(1)}\left(k_{\nu}^{j+1} r^{j}\right)\right), \nu=1,2, \ldots, N . \tag{3.97}
\end{gather*}
$$

Substituting the above to Eq.(3.90) we obtain stable results for $\mathbf{S}_{\mathbf{m} 1}^{\mathbf{j}}$.

## Chapter 4

## Numerical Results

Three test cases will be presented in this thesis. The first one is a range independent environment. With this, I check the validity of mycode comparing its results to the results obtained with program MODE4, a coupled normal mode $\mathrm{N} \times 2 \mathrm{D}$ model for calculating sound propagation in 3-D oceanic environments written by Taroudakis. In the second I use a single cylinder as a first, very simple approximation of a seamount. In the third, a conical frustum shaped seamount is approximated by cylindrical rings. In all the examples the seamount has an external radius of 1000 m and the distance between the source and the axis of the seamount is 3000 m . The source frequency used in the range independent case is 40 Hz , while the frequency used in both examples of the range dependent case is 25 Hz . The results will be expressed in terms of the transmission loss (TL), which is defined as:

$$
T L(r, z, \phi)=-20 \log \left|\frac{p(r, z, \phi)}{p_{0}}\right|
$$

where $p_{0}$ is the reference sound pressure, usually defined as the pressure at a distance of 1 m from a point harmonic source emitting in an unbounded region and equals to $\left|p_{0}\right|=\frac{1}{4 \pi}$. The transmission loss is expressed in dB (decibel) and describes the decrease in intensity of the acoustic field as the acoustic signal propagates through a waveguide.

### 4.1 Range Independent Case

The first example to be studied presents an environment with range independent properties, i.e. an environment where no seamount is present (in fact we assume a seamount of zero elevation) (Figure 4.1). Table 4.1 contains the parameters of the environment.


Figure 4.1: Range Independent Case (seamount of zero elevation)

In this example, for the source frequency of 40 Hz , the number of propagating modes is six $\left(k_{0}=0.1675516\right)$, while the number of azimuthal modes which are sufficient for convergence is 168 (in the graphs I used 170 azimuthal modes). Figures 4.2-4.9 show the progress of convergence for $50,100,150$ (not convergent results) and 170 azimuthal modes. Note that the results for 170 azimuthal modes are completely identical to those of program MODE4 for this environment.

Table 4.1: The environmental parameters

| Sound speed profile in water |  |
| :---: | :---: |
| Depth (m) | Sound speed (m/sec) |
| 0 | 1500 |
| 200 | 1500 |
| Water density $=1 \mathrm{gr} / \mathrm{cm}^{3}$ |  |
| Sediment thickness $=50 \mathrm{~m}$ |  |
| Sound speed profile in sediment |  |
| Depth (m) | Sound speed (m/sec) |
| 200 | 1650 |
| 250 | 1650 |
| Sediment density $=1.3 \mathrm{gr} / \mathrm{cm}^{3}$ |  |
| Substrate density $=1.8 \mathrm{gr} / \mathrm{cm}^{3}$ |  |
| Substrate sound speed $=1750 \mathrm{~m} / \mathrm{sec}$ |  |

Range Independent Case


Figure 4.2: Range Independent Case: Transmission Loss Contour, 50 Azimuthal Modes


Figure 4.3: Range Independent Case: Transmission Loss vs Range, 50 Azimuthal Modes

Range Independent Case


Figure 4.4: Range Independent Case: Transmission Loss Contour, 100 Azimuthal Modes


Figure 4.5: Range Independent Case: Transmission Loss vs Range, 100 Azimuthal Modes

Range Independent Case
Frequency 40 Hz , 150 Azimuthal Modes, Inner 100m, SD=RD=50m


Figure 4.6: Range Independent Case: Transmission Loss Contour, 150 Azimuthal Modes


Figure 4.7: Range Independent Case: Transmission Loss vs Range, 150 Azimuthal Modes

Range Independent Case
Frequency $40 \mathrm{~Hz}, 170$ Azimuthal Modes, Inner 100m, SD=RD=50m


Figure 4.8: Range Independent Case: Transmission Loss Contour, 170 Azimuthal Modes


Figure 4.9: Range Independent Case: Transmission Loss vs Range, 170 Azimuthal Modes

### 4.2 Range Dependent Case: Single Cylinder

In this case a cylindrical seamount of 100 m height and properties identical to the sediment arises from the bottom of the sea (Figure 4.10). The source frequency of 25 Hz produces 4 propagating modes in each of the segments. Note that an artificial inner segment has to be introduced so that the geometry of the environment is in compliance with the model which needs at least three segments to work properly.


Figure 4.10: Range Dependent Case 1: Cylinder of 100m height

Conducting some elementary calculations we see that for $r_{s}=3000 \mathrm{~m}$ and $r=1000 \mathrm{~m}$ the angles (with respect to the source) for which there is "contact" between an acoustic signal leaving from the source and the seamount are amid $-19,47^{\circ}$ and $19,47^{\circ}$. A major improvement in the results originating from the 3-D model comes from the fact that it takes into consideration the effects from the presence of the seamount, in antithesis with MODE4 which considers the environment outside the seamount as completely range independent.

The following figures provide a comparison between the results produced by mycode with respect to program MODE4, for a number of angles. In the first set (Figures 4.12-4.19) both the source and the receiver are above the seamount ( $\mathrm{SD}=\mathrm{RD}=50 \mathrm{~m}$ ), while in the second (Figures 4.20-4.28) they are below the seamount's peak ( $\mathrm{SD}=\mathrm{RD}=150 \mathrm{~m}$ ).

It is obvious that the radius of the artificial inner interface should have no impact to the results, given that there is no change in the seamount properties. However, numerical issues concerning


Figure 4.11: Range Dependent Case 1: Cut-off angle
the Bessel and Hankel functions evolve when this radius increases, altering the results, as we can see in Figures 4.29 and 4.30. This is a fact that motivates further examination.

Cylinder Height 100m


Figure 4.12: Range Dependent Case: Transmission Loss Contour, 110 Azimuthal Modes


Figure 4.13: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 0 deg.


Figure 4.14: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 10 deg.


Figure 4.15: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 19 deg.


Figure 4.16: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 20 deg.


Figure 4.17: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 30 deg.


Figure 4.18: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 60 deg.


Figure 4.19: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 90 deg.

## Cylinder Height 100m



Figure 4.20: Range Dependent Case: Transmission Loss Contour, 110 Azimuthal Modes


Figure 4.21: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 0 deg.


Figure 4.22: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 1 deg.


Figure 4.23: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 10 deg.


Figure 4.24: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 19 deg.


Figure 4.25: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 20 deg.


Figure 4.26: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 30 deg.


Figure 4.27: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 60 deg.


Figure 4.28: Range Dependent Case: Single Cylinder, TL vs Range, Angle: 90 deg.


Figure 4.29: Range Dependent Case: Single Cylinder, Inner Radius 200 m, TL vs Range


Figure 4.30: Range Dependent Case: Single Cylinder, Inner Radius 500 m, TL vs Range

### 4.3 Range Dependent Case: "Conical" Seamount

## Test Case 1

In the following test case the seamount is slightly different than the previous one. Instead of a single cylinder, in this case we examine a seamount that is approximated by six cylinders. The height of the inner one is 105 m , its radius is 50 m and for the next four we have a 1 m decrease in height every 50 m in range, as we can see in Figure 4.31. The sixth cylinder has a height of 100 m , extending from 250 to 1000 m .


Figure 4.31: Range Dependent Case: Six cylinders, Internal Radii 50 to 250 m

Comparing the environment of this test case to the previous one we can see that the differences are not essential, so it is not unreasonable to expect similar results. Indeed, as we can see in Figures 4.32 and 4.33, program MODE4 gives us almost identical results for both cases.

Exactly the same happens in mycode with the difference that the numerical issues arising from the placement of the artificial cylinder radius at 250 m in the case of a single cylinder are depicted in the case of the frustum as well, as we can see in Figures 4.34 and 4.35.


Figure 4.32: Range Dependent Case: Frustum vs Single Cylinder (MODE4) SD=RD=50m


Figure 4.33: Range Dependent Case: Frustum vs Single Cylinder (MODE4) $\mathrm{SD}=\mathrm{RD}=150 \mathrm{~m}$


Figure 4.34: Range Dependent Case: Frustum vs Single Cylinder (mycode) $\mathrm{SD}=\mathrm{RD}=50 \mathrm{~m}$


Figure 4.35: Range Dependent Case: Frustum vs Single Cylinder (mycode) $\mathrm{SD}=\mathrm{RD}=150 \mathrm{~m}$

## Test Case 2

In the last example the seamount has the same partitioning in range (i.e. the segment boundaries are positioned at $50,100,150,200,250$ and 1000 m ) but the difference in height per segment is 5 m starting from 125 m in the inner cylinder down to 100 m in the external one (see Figure 4.36).


Figure 4.36: Range Dependent Case: Six cylinders, Internal Radii 50 to 250 m

Figures 4.37 and 4.38 present the transmission loss contours for source and receiver depth at 50 and 150 meters respectively, while Figures 4.39 and 4.40 provide a comparison between the results generated by mycode and program MODE4.


Figure 4.37: Conical Frustum: Transmission Loss Contour, $\mathrm{SD}=\mathrm{RD}=50 \mathrm{~m}$


Figure 4.38: Conical Frustum: Transmission Loss Contour, $\mathrm{SD}=\mathrm{RD}=150 \mathrm{~m}$


Figure 4.39: Range Dependent Case: Conical Frustum, $\mathrm{SD}=\mathrm{RD}=50 \mathrm{~m}$


Figure 4.40: Range Dependent Case: Conical Frustum, $\mathrm{SD}=\mathrm{RD}=150 \mathrm{~m}$

## Chapter 5

## Conclusions - Future Work

In my thesis I extended the normal mode program MODE4 in order to compute acoustic pressure in a 3 -dimensional environment in the presence of an obstacle of axially symmetric characteristics (conical seamount). The code is written in Fortran, using the majority of the routines of program MODE4, although with serious transmutations, plus some subroutines of LAPACK. The graphs are drawn in MATLAB and the representations of the field in the figures are drawn with GEOGEBRA.

One major difference between the two programs is that in mycode z -axis is the axis of the conical seamount, while in MODE4 z-axis contains the source. I used some of the improvements introduced by Luo to avoid the numerical limitations (as regards underflows and overflows) generated by the use of Bessel and Hankel functions.

To test the validity of mycode I first used a range independent environment. The representation of the solution as a double series in terms of vertical and azimuthal coefficients is identical to the solution produced by a 2-D program, like MODE1.

In the case of a cylindrical seamount, the results for various angles are very much alike those obtained by MODE4, with a significant improvement around the seamount, where a $\mathrm{N} \times 2 \mathrm{D}$ model cannot depict precisely the scattering effects in sound propagation because of the existence of the seamount in the nearby region.

However, as shown in some figures of the previous chapter, numerical issues arising due to the use of Hankel and Bessel functions still exist, especially when the radii of the inner cylindrical rings increase in range. The implementation of the asymptotic representations of those functions
should solve many of these problems, but although they have been included in mycode, the "starting point" for their use has yet to be determined.

Another issue that needs detailed examination is the existence of different number of propagating modes between the segments because of the difference in the heights of the cylinders that approximate the seamount. The numerical requirements for rectangular matrices impose the use of the same number of propagating modes in each segment. To succeed in this, we either need to "cut" propagating modes from the segments with more than the minimum number, thus losing a lot of useful "information", or - a technique that Prof. Taroudakis has used in his work to increase the thickness of the sedimentary layer above the semi-infinite half-space. With this manipulation we can increase the number of propagating modes so that each segment reaches at least the number of propagating modes of the range independent environment. The restriction to this number for all segments afterwards has a small impact to the results. Implementing this technique, I found that, in order to obtain 4 propagating modes in each segment of the range dependent test cases, the thickness of the sediment layer had to be 80 m for the single cylinder and the "almost cylindrical" frustum, while in the last example, where the inner cylinder was 125 m high, I had to increase this thickness to 110 m .

## Appendix A

## Properties of Bessel Functions

Bessel functions are solutions to the Bessel equations. Below we present some of their properties.

## A. 1 The Bessel Equation and its Solutions

The standard form of a Bessel equation of order $\nu$ is

$$
\begin{equation*}
z^{2} \frac{d^{2} w}{d z^{2}}+z \frac{d w}{d z}+\left(z^{2}-\nu^{2}\right) w=0 \tag{A.1}
\end{equation*}
$$

where $z=x+i y$ and $\nu$ is real. Its solutions are Bessel functions of the first kind $J_{\nu}(z)$, of the second kind $Y_{\nu}(z)$ (also called Weber or Neumann functions) and of the third kind $H_{\nu}^{(1)}(z), H_{\nu}^{(2)}(z)$ (also called the Hankel functions). Some important features of the various solutions are the following [4], [5]: $J_{\nu}(z)$ is bounded as $z \rightarrow 0$ in any bounded range of $\arg z . H_{\nu}^{(1)}(z)$ tends to zero as $|z| \rightarrow \infty$ in the sector $0<\arg z<\pi . H_{\nu}^{(2)}(z)$ tends to zero as $|z| \rightarrow \infty$ in the sector $-\pi<\arg z<0$. $J_{\nu}(z)$ and $Y_{\nu}(z)$ are linearly independent for all values of $\nu . H_{\nu}^{(1)}(z)$ and $H_{\nu}^{(2)}(z)$ are linearly independent for all values of $\nu$. The general solution to Eq.(A.1) is represented as a linear combination of two linearly independent functions with arbitrary constants $C_{1}$ and $C_{2}$, such as

$$
\begin{equation*}
w(z)=C_{1} H_{\nu}^{(1)}(z)+C_{2} H_{\nu}^{(2)}(z) \tag{A.2}
\end{equation*}
$$

## A. 2 Asymptotic Forms of Bessel Functions

## A.2.1 Asymptotic Expressions for Large Arguments

Bessel functions of the first and second kind, $J_{n}(x)$ and $Y_{n}(x),(x \in \mathbb{R}$ and $n \in \mathbb{N})$, are called standing wave solutions of Eq.(A.1), because their asymptotic behaviors (as $x \rightarrow \infty$ ) are given by

$$
\begin{equation*}
J_{n}(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left(x-\frac{n \pi}{2}-\frac{\pi}{4}\right) \tag{A.3}
\end{equation*}
$$

and

$$
\begin{equation*}
Y_{n}(x) \sim \sqrt{\frac{2}{\pi x}} \sin \left(x-\frac{n \pi}{2}-\frac{\pi}{4}\right) \tag{A.4}
\end{equation*}
$$

The asymptotic forms of their derivatives $J_{n}^{\prime}(x)$ and $Y_{n}^{\prime}(x)$, are as following

$$
\begin{equation*}
J_{n}^{\prime}(x) \sim-\sqrt{\frac{2}{\pi x}} \sin \left(x-\frac{n \pi}{2}-\frac{\pi}{4}\right) \tag{A.5}
\end{equation*}
$$

and

$$
\begin{equation*}
Y_{n}^{\prime}(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left(x-\frac{n \pi}{2}-\frac{\pi}{4}\right) \tag{A.6}
\end{equation*}
$$

A linear combination of the Bessel and Neumann functions gives us the Hankel functions of the first and second kind $H_{n}^{(1)}(x)$ and $H_{n}^{(2)}(x)$.

$$
\begin{equation*}
H_{n}^{(1)}(x)=J_{n}(x)+i Y_{n}(x) \tag{A.7}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{n}^{(2)}(x)=J_{n}(x)-i Y_{n}(x) \tag{A.8}
\end{equation*}
$$

Hankel functions are also called traveling wave solutions of Eq.(A.1) because with time dependence $e^{-i \omega t}, H_{n}^{(1)}(x)$ corresponds to a diverging outgoing wave while $H_{n}^{(2)}(x)$ stands for an incoming and converging wave, if we examine their asymptotic behaviors (as $x \rightarrow \infty$ )

$$
\begin{equation*}
H_{n}^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{i\left(x-\frac{n \pi}{2}-\frac{\pi}{4}\right)} \tag{A.9}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{n}^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{-i\left(x-\frac{n \pi}{2}-\frac{\pi}{4}\right)} \tag{A.10}
\end{equation*}
$$

The asymptotic forms of their derivatives are as below:

$$
\begin{equation*}
H_{n}^{(1)^{\prime}}(x) \sim i \sqrt{\frac{2}{\pi x}} e^{i\left(x-\frac{n \pi}{2}-\frac{\pi}{4}\right)}=i H_{n}^{(1)}(x) \tag{A.11}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{n}^{(2)^{\prime}}(x) \sim-i \sqrt{\frac{2}{\pi x}} e^{i\left(x-\frac{n \pi}{2}-\frac{\pi}{4}\right)}=-i H_{n}^{(2)}(x) \tag{A.12}
\end{equation*}
$$

## A.2.2 Asymptotic Expressions for Small Arguments

## Bessel Function $J_{n}(x)$

The series representation of $J_{n}(z)$ is [5] [4]

$$
\begin{equation*}
J_{n}(z)=\left(\frac{z}{2}\right)^{n} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!\Gamma(n+k+1)}\left(\frac{z}{2}\right)^{2 k}, \text { where }|\arg z|<\pi \tag{A.13}
\end{equation*}
$$

From this representation we can conclude that for small arguments we have

$$
\begin{equation*}
J_{n}(x)=\left(\frac{x}{2}\right)^{n}\left[\frac{1}{n!}-\frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{2}+\cdots\right]=\frac{1}{n!}\left(\frac{x}{2}\right)^{n}+O\left(x^{n+2}\right) \tag{A.14}
\end{equation*}
$$

from which we obtain

$$
\begin{equation*}
J_{n}(x) \sim \frac{1}{n!}\left(\frac{x}{2}\right)^{n}, \text { for }|x| \rightarrow 0 \tag{A.15}
\end{equation*}
$$

Thus, when $|x| \rightarrow 0$, we get $J_{0}(x) \sim 1$ for $n=0$ and $J_{n}(x) \sim 0$ for $n>0$. Fig.(A.1) shows the graphs of $J_{n}(x)$ for $n=0,1,5$ and 10.

For the derivatives of $J_{n}(x)$ we have

$$
\begin{gather*}
J_{0}^{\prime}(x) \sim 0  \tag{A.16}\\
J_{n}^{\prime}(x) \sim \frac{1}{2(n-1)!}\left(\frac{x}{2}\right)^{n-1}, \text { for } n>0 \tag{A.17}
\end{gather*}
$$



Figure A.1: Bessel functions of orders $0,1,5$ and 10 for small arguments

## Neumann Function $Y_{n}(x)$

The small argument approximations of $Y_{n}(x)$ are

$$
\begin{gather*}
Y_{0}(x) \sim \frac{2}{\pi}\left[\ln \frac{x}{2}+\gamma\right]  \tag{A.18}\\
Y_{n}(x) \sim-\frac{(n-1)!}{\pi}\left(\frac{2}{x}\right)^{n}, \text { for } n>0 \tag{A.19}
\end{gather*}
$$

In Eq.(A.18) $\gamma$ is Euler's constant $\gamma=0.57721 \ldots$.... From the last two equations we can see that $x=0$ is a singular point of $Y_{n}(x)$. Fig.(A.2) shows the graphs of $Y_{n}(x)$ for $n=0,1,5$ and 10 . We can observe that as $x \rightarrow 0, Y_{n}(x) \rightarrow-\infty$.

The small argument approximations of their derivatives are

$$
\begin{gather*}
Y_{0}^{\prime}(x) \sim \frac{1}{\pi} \frac{2}{x}  \tag{A.20}\\
Y_{n}^{\prime}(x) \sim \frac{n!}{2 \pi}\left(\frac{2}{x}\right)^{n+1}, \text { for } n>0 \tag{A.21}
\end{gather*}
$$

Hankel Function of the first kind $H_{n}^{(1)}(x)$
Since $H_{n}^{(1)}(x)=J_{n}(x)+i Y_{n}(x)$, for small arguments we have

$$
\begin{gather*}
H_{0}^{(1)}(x) \sim 1+i \frac{2}{\pi}\left[\ln \frac{x}{2}+\gamma\right]  \tag{A.22}\\
H_{0}^{(1)^{\prime}}(x) \sim \frac{i}{\pi} \frac{2}{x}  \tag{A.23}\\
H_{n}^{(1)}(x) \sim-i \frac{(n-1)!}{\pi}\left(\frac{2}{x}\right)^{n}, n \neq 0  \tag{A.24}\\
H_{n}^{(1)^{\prime}}(x) \sim \frac{i n!}{2 \pi}\left(\frac{2}{x}\right)^{n+1}, n \neq 0 \tag{A.25}
\end{gather*}
$$



Figure A.2: Neumann functions of orders $0,1,5$ and 10 for small arguments

Hankel Function of the second kind $H_{n}^{(2)}(x)$
The small argument approximations for $H_{n}^{(2)}(x)=J_{n}(x)-i Y_{n}(x)$ are

$$
\begin{gather*}
H_{0}^{(2)}(x) \sim 1-i \frac{2}{\pi}\left[\ln \frac{x}{2}+\gamma\right]  \tag{A.26}\\
H_{0}^{(2)^{\prime}}(x) \sim-\frac{i}{\pi} \frac{2}{x}  \tag{A.27}\\
H_{n}^{(2)}(x) \sim i \frac{(n-1)!}{\pi}\left(\frac{2}{x}\right)^{n}, n \neq 0  \tag{A.28}\\
H_{n}^{(2)^{\prime}}(x) \sim-\frac{i n!}{2 \pi}\left(\frac{2}{x}\right)^{n+1}, n \neq 0 \tag{A.29}
\end{gather*}
$$

## A.2.3 Asymptotic Expressions for Large Orders

In the following equations it is supposed that $n \rightarrow \infty$ through real positive values, the other variables being fixed [8]

$$
\begin{gather*}
J_{n}(x) \sim \frac{1}{\sqrt{2 \pi n}}\left(\frac{e x}{2 n}\right)^{n}  \tag{A.30}\\
Y_{n}(x) \sim-\sqrt{\frac{2}{\pi n}}\left(\frac{e x}{2 n}\right)^{-n}  \tag{A.31}\\
H_{n}^{(1)}(x) \sim-i Y_{n}(x) \sim-i \sqrt{\frac{2}{\pi n}}\left(\frac{e x}{2 n}\right)^{-n}  \tag{A.32}\\
H_{n}^{(2)}(x) \sim i Y_{n}(x) \sim i \sqrt{\frac{2}{\pi n}}\left(\frac{e x}{2 n}\right)^{-n} \tag{A.33}
\end{gather*}
$$

## A. 3 Recursion Relations for Bessel Functions

Some useful recursion relations for Bessel functions are [5]

$$
\begin{equation*}
\mathcal{Z}_{n-1}(z)+\mathcal{Z}_{n+1}(z)=\frac{2 n}{z} \mathcal{Z}_{n}(z) \tag{A.34}
\end{equation*}
$$

$$
\begin{equation*}
\mathcal{Z}_{n-1}(z)-\mathcal{Z}_{n+1}(z)=2 \mathcal{Z}_{n}^{\prime}(z) \tag{A.35}
\end{equation*}
$$

Adding eqs.(A.34) and (A.35) we get

$$
\begin{equation*}
\mathcal{Z}_{n}^{\prime}(z)=\mathcal{Z}_{n-1}(z)-\frac{n}{z} \mathcal{Z}_{n}(z) \tag{A.36}
\end{equation*}
$$

Subtracting eqs.(A.34) and (A.35) we get

$$
\begin{equation*}
\mathcal{Z}_{n}^{\prime}(z)=-\mathcal{Z}_{n+1}(z)+\frac{n}{z} \mathcal{Z}_{n}(z) \tag{A.37}
\end{equation*}
$$

## A. 4 Wronskian Relations for Bessel Functions

Some useful Wronskian relations for Bessel functions are

$$
\begin{gather*}
\mathcal{W}\left[J_{n}(z), Y_{n}(z)\right]=\frac{2}{\pi z}  \tag{A.38}\\
\mathcal{W}\left[H_{n}^{(1)}(z), H_{n}^{(2)}(z)\right]=-\frac{4 i}{\pi z} \tag{A.39}
\end{gather*}
$$

where the Wronskian of functions $f(z)$ and $g(z)$ is defined as

$$
\mathcal{W}[f(z), g(z)] \equiv\left|\begin{array}{ll}
f(z) & g(z)  \tag{A.40}\\
f^{\prime}(z) & g^{\prime}(z)
\end{array}\right|=f(z) g^{\prime}(z)-f^{\prime}(z) g(z)
$$

We notice that the Wronskian of $J_{n}(z)$ and $Y_{n}(z)$, or that of $H_{n}^{(1)}(z)$ and $H_{n}^{(2)}(z)$ are independent of the order $n$. Furthermore, we can see that eq.(A.38) is the basic Wronskian relation and may be used to derive other Wronskian relations for $J_{n}(z), Y_{n}(z), H_{n}^{(1)}(z)$ and $H_{n}^{(2)}(z)$. For example, we may obtain $\mathcal{W}\left[J_{n}(z), H_{n}^{(1)}(z)\right]$ as follows

$$
\begin{align*}
\mathcal{W}\left[J_{n}(z), H_{n}^{(1)}(z)\right] & =\mathcal{W}\left[J_{n}(z), J_{n}(z)+i Y_{n}(z)\right] \\
& =\mathcal{W}\left[J_{n}(z), J_{n}(z)\right]+\mathcal{W}\left[J_{n}(z), i Y_{n}(z)\right]  \tag{A.41}\\
& =i \mathcal{W}\left[J_{n}(z), Y_{n}(z)\right] \\
& =\frac{2 i}{\pi z}
\end{align*}
$$

## A. 5 Linearly Independent Solutions of Bessel Equations for both Large and Small

 Arguments
## A. 5 Linearly Independent Solutions of Bessel Equations for both Large and Small Arguments

Theoretically, solutions to the Bessel equation (A.1) can be any pair of two of the functions $J_{\nu}(z)$, $Y_{\nu}(z), H_{\nu}^{(1)}(z)$ and $H_{\nu}^{(2)}(z)$. However, as shown below, only one pair remains independent for both large and small arguments.

## A.5.1 Linearly Independent Solutions for Large Arguments

Let $|x| \gg \nu$ or $|x| \rightarrow \infty$ with $\nu$ fixed. Denote $x=\alpha+i \beta,(\beta>0)$ and assume that $\beta \rightarrow \infty$. The asymptotic forms of Hankel functions are

$$
\begin{align*}
H_{\nu}^{(1)}(x) & \sim \sqrt{\frac{2}{\pi x}} e^{i\left(x-\frac{\nu \pi}{2}-\frac{\pi}{4}\right)} \\
& =\sqrt{\frac{2}{\pi x}} e^{i\left(\alpha+i \beta-\frac{\nu \pi}{2}-\frac{\pi}{4}\right)}  \tag{A.42}\\
& =\sqrt{\frac{2}{\pi x}} e^{i\left(\alpha-\frac{\nu \pi}{2}-\frac{\pi}{4}\right) e^{-\beta}} \\
& =0, \text { as } \beta \rightarrow \infty
\end{align*}
$$

and

$$
\begin{align*}
H_{\nu}^{(2)}(x) & \sim \sqrt{\frac{2}{\pi x}} e^{-i\left(x-\frac{\nu \pi}{2}-\frac{\pi}{4}\right)} \\
& =\sqrt{\frac{2}{\pi x}} e^{-i\left(\alpha+i \beta-\frac{\nu \pi}{2}-\frac{\pi}{4}\right)}  \tag{A.43}\\
& =\sqrt{\frac{2}{\pi x}} e^{-i\left(\alpha-\frac{\nu \pi}{2}-\frac{\pi}{4}\right) e^{\beta}} \\
& =\infty, \text { as } \beta \rightarrow \infty
\end{align*}
$$

In wave theory, $H_{\nu}^{(1)}(x) \rightarrow 0$ means that an outgoing wave decays exponentially with range and $H_{\nu}^{(2)}(x) \rightarrow \infty$ means that an incoming wave increases exponentially with range. Using eqs.(A.42) and (A.43) we have

$$
\begin{equation*}
J_{\nu}(x)=\frac{1}{2}\left[H_{\nu}^{(1)}(x)+H_{\nu}^{(2)}(x)\right] \sim \frac{1}{2} H_{\nu}^{(2)}(x), \text { as } \Im(x) \rightarrow \infty \tag{A.44}
\end{equation*}
$$

and

$$
\begin{equation*}
Y_{\nu}(x)=\frac{1}{2}\left[H_{\nu}^{(1)}(x)-H_{\nu}^{(2)}(x)\right] \sim-\frac{1}{2} H_{\nu}^{(2)}(x), \text { as } \Im(x) \rightarrow \infty \tag{A.45}
\end{equation*}
$$

Thus, we can see that for $|x| \gg \nu, J_{\nu}(z), Y_{\nu}(z)$ and $H_{\nu}^{(2)}(z)$ are linearly dependent and all of them are linearly independent of $H_{\nu}^{(1)}(x)$.

## A.5.2 Linearly Independent Solutions for Small Arguments

Let $|x| \ll \nu$ or $|x| \rightarrow 0$ with $\nu$ fixed. We have seen in A.2.2 that as $|x| \rightarrow 0, J_{\nu}(x)$ approaches a finite value, which is either 1 (when $\nu=0$ ) or 0 (when $\nu \neq 0$ ), while $Y_{\nu}(x) \rightarrow-\infty$. For the Hankel functions we have

$$
\begin{gather*}
H_{\nu}^{(1)}(x)=J_{\nu}(x)+i Y_{\nu}(x) \sim i Y_{\nu}(x)  \tag{A.46}\\
H_{\nu}^{(2)}(x)=J_{\nu}(x)-i Y_{\nu}(x) \sim-i Y_{\nu}(x) \tag{A.47}
\end{gather*}
$$

This means that when $|x| \ll \nu, Y_{\nu}(z), H_{\nu}^{(1)}(x)$ and $H_{\nu}^{(2)}(z)$ are linearly dependent and all of them are linearly independent of $J_{\nu}(x)$.

Combining all the above we come to the conclusion that the only pair of functions that preserves linear independence for both large and small arguments is $J_{\nu}(x)$ and $H_{\nu}^{(1)}(x)$.

## Bibliography

[1] Michael I. Taroudakis: A coupled-mode formulation for the solution of the Helmholtz equation in water in the presence of a conical sea-mount, Journal of Computational Acoustics, Vol.4, No. 1 (1996), pp.101-121.
[2] Wenyu Luo: Three-Dimensional Propagation and Scattering around a Conical Seamount, PhD Thesis, MIT 2007.
[3] Wenyu Luo and Henrik Schmidt: Three-dimensional propagation and scattering around a conical seamount, Journal of the Acoustical Society of America, Vol. 125, No. 1 (2009), pp.5265.
[4] Earl G. Williams: Fourier Acoustics. Academic Press, 1999.
[5] Milton Abramowitz and Irene A. Stegun: Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Applied Mathematics Series No.55, National Bureau of Standards, Washington D.C., 1972.
[6] John S. Perkins and Ralph N. Baer: An approximation to the three-dimensional parabolicequation method for acoustic propagation, Journal of the Acoustical Society of America, Vol.72, No.2, 1982, pp.515-522.
[7] I.S. Gradshteyn and I.M. Ryzhik: Table of Integrals, Series and Products, Fifth Edition, Academic Press, 1994.
[8] Digital Library of Mathematical Functions, National Institute of Standards and Technology, http://dlfm.nist.gov
[9] Jérémie Eskenazi: A computer model for sound propagation around conical seamounts, Master Thesis, MIT 2001.

