The turnaround radius in Brane-World models

Stefanos Robert Kousvos*
ITCP and Department of Physics, University of Crete, 700 13 Heraklion, Greece

Abstract

In this thesis the maximum size predicted for cosmic structures (here onward called the maximum turnaround radius [32]) will be used to compare an alternative cosmological theory to experimental data. This theory proposed by Shtanov et al. [27, 14] treats our universe as a 4 dimensional brane embedded in a 5 dimensional bulk. Comparing the maximum turnaround radius predicted for this model to the value obtained experimentally we bound the parameter space of the theory. Specifically we discover experimentally disallowed regions for various values of the bulk cosmological constant Λ and the quasi density parameter of the theory Ω_l. A remarkable feature of our analysis is that as only input we use the value of Ω_m which is approximately equal to 0.3.

keywords : Braneworld model, large scale structures, maximum turn around radius

1 Introduction

The class of model we will be interested in is the so called Dvali-Gabadadze-Poratti class (DGP) [12]. We consider the matter fields to be constrained on the brane only. Specifically we will be interested in the 4 dimensional effective Einstein equations induced on the brane due to the presence of the 5 dimensional bulk. Do note also that the extra fifth dimension is space like. This model has two cosmological solutions, one that is self accelerated and one that requires a cosmological constant as to have an acceleration. The first was proved to have instabilities and thus is considered to be non physical, hence we will only be concerned with the second. The second cosmological solution is of interest since it can also be considered an alternative to ΛCDM.

The model we will be studying, originally proposed by [14], has the virtue that it is very general in the sense that it contains in the action a 5 dimensional Ricci scalar describing the intrinsic curvature in the bulk, a 5 dimensional cosmological constant, the extrinsic curvature induced on the brane due to its embedding in the bulk, as well as the usual terms of the brane Ricci scalar and the 4 dimensional brane cosmological constant which reproduce ΛCDM. Let us also note that the fifth dimension is took to be infinite thus gravity is modified at large scales, which also makes this case intuitively simple. Another virtue of this model is that it is a direct extension of ΛCDM thus one may actually do phenomenology with this model with relative ease. Braneworld models in general are also of interest as they can be thought as a bridge between string theory. It is of common acceptance that ΛCDM is believed to be incomplete

*skousvos@physics.uoc.gr

as a potential theory to describe our universe since various problems arise in it, such as the cosmological constant tuning problem, these also hint at a needed extension.

To achieve our goal we will be using the notion of the maximum turn around radius, which is the radius at which the attraction due to gravity is exactly balanced out by the effective repulsion due to the cosmic expansion, thus giving the maximum size of cosmic structures bound by gravity. This has been done already for ΛCDM and has been compared with experimental data with very satisfying results. The experimental maximum size was found to be approximately 0.9 times the theoretical prediction. Thus for the braneworld model we will demand for the values of the maximum turnaround radius to be no smaller than 0.9 times the maximum turn around value for ΛCDM or else that region of the theory is experimentally excluded. Hence our goal becomes to express the maximum turnaround radius of the braneworld model as a function of that of ΛCDM.

The structure of this thesis is as follows. We initially give a review of the model of interest and the equations governing it that will interest us. We then proceed to the calculation of the maximum turn around radius in the simplified case where the Weyl (see text for details) as well as the bulk cosmological constant terms are turned off. Then we turn the former on and lastly we also turn the bulk cosmological constant on. This way the method is demonstrated initially in a simplified form of the model and once a thorough understanding is achieved the full calculation is carried through.

We shall work with the mostly positive signature of the metric $(-,+,+,+,+)$ and we will set $c=1$.

2 The model and the field equations

We wish to give a brief review of ΛCDM cosmology in the notation we will use throughout the rest of the text before we get in to explicit details about the braneworld model. To do this we start from the 4 dimensional Einstein equation

$$G_{\mu\nu} + \frac{\sigma}{m^2} g_{\mu\nu} = \frac{1}{m^2} T_{\mu\nu} \tag{1}$$

it is obvious that in this notation the combination σ/m^2 in front of the metric must be the cosmological constant and $1/m^2$ must be equal to $8\pi G$. Now to solve this equation we must make an ansatz for the metric, we choose this to be the flat FLRW using conformal time (defined as $\tau = t/a$), with a being the scale factor

$$ds^2 = a^2(\tau)\left[-d\tau^2 + dx^2 + dy^2 + dz^2\right] \tag{2}$$

this ansatz if plugged in to Eq. (1), along with the use of the pressureless cold dark matter hypothesis (i.e. there is only an energy density term in the stress energy tensor) leads to the Friedmann equation

$$\mathcal{H}^2 = a^2\left(\frac{\rho+\sigma}{3m^2}\right) \tag{3}$$

it is clear with the identifications we made that this is the well known Friedmann equation.

Next we find it useful to provide a brief recap of the Gauss-Codacci equations, since it is thanks to these we will be able to have a closed system of equations on the brane. To sketch how these equations are derived consider the following definition of the Riemann curvature tensor via a commutation relation of covariant derivatives

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma \tag{4}$$

where V^μ is a vector field. This holds true in arbitrary dimensions, as long as we use the appropriate dimensional covariant derivatives. The next step for us will be to write the metric of the 5 dimensional space as a sum of the induced 4 dimensional metric plus a term that will contain unit vectors normal to the 4 dimensional space (brane), it is not hard to convince thyself that this will look like (e.g. using the chain rule)

$$\mathcal{A}_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \tag{5}$$

where the symbol $\mathcal{A}$ was used to avoid confusion with symbols in the analysis later. Also g is the induced metric on the 4 dimensional brane and the vectors n_μ are perpendicular to the brane and have norm equal to unity. We rewrite this equation a little differently

$$g_{\mu\nu} = \mathcal{A}_{\mu\nu} - n_\mu n_\nu \tag{6}$$

doing this, the induced metric may now be viewed as a projection operator, and can be checked to satisfy all of the relations required by a projection operator. Thus if one uses this projection operator they can express the 4 dimensional covariant derivatives in Eq. (4) with respect to the 5 dimensional ones, and using the fact that the covariant derivative of the normal vector is defined as the external curvature ($K_{\mu\nu} = \nabla_\mu n_\nu$) as well as choosing the vector V^μ parallel to the 4 dimensional brane such that it is unaffected by the projection operators, one gets

$$R^\rho{}_{\sigma\mu\nu} = g^\rho{}_\alpha g^\beta{}_\rho g^\gamma{}_\mu g^\delta{}_\nu \mathcal{R}^\alpha{}_{\beta\gamma\delta} + K^\rho{}_\mu K_{\sigma\nu} - K^\rho{}_\nu K_{\sigma\mu} \tag{7}$$

where the $\mathcal{R}$ is used to label the 5 dimensional Ricci tensor and scalar. This is called Gauss's equation. Going one step further we get

$$R = g^{\sigma\nu} R^\lambda{}_{\sigma\lambda\nu} = \mathcal{R} - 2\mathcal{R}_{\mu\nu} n^\mu n^\nu + K^2 - K^{\mu\nu} K_{\mu\nu} \tag{8}$$

where $K = \mathcal{A}^{\mu\nu} K_{\mu\nu}$. Also we have

$$\nabla_{[\mu} K_{\nu]}{}^\mu = \frac{1}{2} g^\sigma{}_\nu \mathcal{R}_{\rho\sigma} n^\rho \tag{9}$$

which is called the Codacci equation (note that the covariant derivative here is 4 dimensional). This can be derived taking the covariant derivative of external curvature and then using the projection operator to rewrite the expression with respect to the higher dimensional covariant derivative.

We believe we are now ready to briefly review the action and the field equations of the braneworld model. The system is described by the action [14, 24, 27],

$$S = M^3 \left[\int_{\text{Bulk}} (\mathcal{R} - 2\Lambda) - 2 \int_{\text{Brane}} K \right] + \int_{\text{Brane}} (m^2 R - 2\sigma) + \int_{\text{Brane}} \mathcal{L}(g_{\mu\nu}, \phi) \tag{10}$$

Where M and m are respectively the 5 and 4 dimensional Planck masses whereas $\mathcal{R}$ and R are the corresponding Ricci scalars. Λ is the cosmological constant in the bulk, K is the extrinsic curvature on the brane, the combination of σ/m^2 plays the role of the cosmological constant on the brane with σ being the brane tension. $\mathcal{L}$ stands for the Lagrangian density of matter fields living on the brane, for our current purpose which would be the cold dark matter.

This leads to the Einstein equation in the bulk

$$\mathcal{G}_{\mu\nu} + \Lambda \mathcal{A}_{\mu\nu} = 0 \tag{11}$$

where $\mathcal{G}$ is the 5 dimensional Einstein tensor. The same action also leads to an equation for the fields on the brane

$$G_{\mu\nu} + \frac{\sigma}{m^2} g_{\mu\nu} + \frac{M^3}{m^2} S_{\mu\nu} = \frac{1}{m^2} T_{\mu\nu} \tag{12}$$

where

$$S_{\mu\nu} = K_{\mu\nu} - K g_{\mu\nu} \tag{13}$$

now Eq. (12) may be solved for $S_{\mu\nu}$ (notice that for m^2 vanishing this reduces to the Israel junction condition). Now one may use the Gauss identity Eq. (7) along with Eq. (12),Eq. (13) and Eq. (11) to construct the 4 dimensional Einstein tensor. This leads to the field equations projected onto the brane e.g. [14],

$$G_{\mu\nu} + \left(\frac{\Lambda_{\rm RS}}{b+1}\right) g_{\mu\nu} = \left(\frac{b}{b+1}\right) \frac{1}{m^2} T_{\mu\nu} + \left(\frac{1}{b+1}\right) \left[\frac{1}{M^6} Q_{\mu\nu} - \mathcal{C}_{\mu\nu}\right] \tag{14}$$

where the parameters entering this equation are defined as

$$b = \frac{\sigma l}{3M^3} \qquad l = \frac{2m^2}{M^3} \qquad \Lambda_{\rm RS} = \frac{\Lambda}{2} + \frac{\sigma^2}{3M^6} \tag{15}$$

and

$$Q_{\mu\nu} = \frac{1}{3} E E_{\mu\nu} - E_{\mu\lambda} E^{\lambda}_{\nu} + \frac{1}{2}\left(E_{\rho\lambda} E^{\rho\lambda} - \frac{E^2}{3}\right) g_{\mu\nu} \tag{16}$$

where

$$E_{\mu\nu} = m^2 G_{\mu\nu} - T_{\mu\nu} \quad \text{and} \quad E = E^{\mu}{}_{\mu} \tag{17}$$

and $\mathcal{C}_{\mu\nu}$ comes from the projection of the bulk Weyl tensor (i.e. the traceless part of the Riemann tensor) onto the brane. We take the Friedman-Robertson-Walker (FRW) ansatz for the metric with flat spatial sections and conformal time

$$ds^2 = a^2(\tau)\left[-d\tau^2 + dx^2 + dy^2 + dz^2\right] \tag{18}$$

and plug it into Eq. (14), with $T_{\mu\nu}$ corresponding to the cold dark matter; the contribution coming from the Weyl part is seen to behave like electromagnetic radiation density $\sim 1/a^4(\tau)$ and we shall ignore it from the homogeneous cosmological equations. However, we shall consider its inhomogeneous contribution later. One then obtains, for the so called normal branch,

$$\mathcal{H}^2 = \frac{a^2(\rho+\sigma)}{3m^2} + \frac{2a^2}{l^2}\left[1 - \sqrt{1 + l^2\left(\frac{\rho+\sigma}{3m^2} - \frac{\Lambda}{6}\right)}\right] \tag{19}$$

where ρ stands for the cold dark matter energy density, $\mathcal{H} = \dot{a}/a$ is the Hubble rate, where a 'dot' stands for derivative once with respect to τ. Initially we set the bulk cosmological constant Λ to zero, to get

$$\mathcal{H}^2 = \frac{a^2}{l^2}\left[\sqrt{1 + l^2\left(\frac{\rho+\sigma}{3m^2}\right)} - 1\right]^2 \tag{20}$$

It is customary then to define two useful quantities, β and γ, as

$$\beta = -2\sqrt{1 + l^2\left(\frac{\rho+\sigma}{3m^2}\right)} = -2\left(1 + \frac{l\mathcal{H}}{a}\right) \qquad 3\gamma - 1 = \frac{\dot{\beta}}{\mathcal{H}\beta} = \frac{\partial_\tau(\mathcal{H}/a)}{\mathcal{H}(1 + l\mathcal{H}/a)} \tag{21}$$

With the help of the cosmological density functions : $\Omega_m(a) = \rho a^2/(3m^2\mathcal{H}^2)$, $\Omega_\sigma = \sigma a^2/(3m^2\mathcal{H}^2)$ and $\Omega_l = a^2/(l^2\mathcal{H}^2)$, and recalling that $\rho \sim a^{-3}(\tau)$ for the cold dark matter, the above quantities can be re-expressed as

$$\beta = -\frac{2}{\sqrt{\Omega_l}}\sqrt{\Omega_m(1+z)^3+\Omega_\sigma+\Omega_l} \qquad 3\gamma - 1 = -\frac{3\Omega_m(1+z)^3}{2\left(\Omega_m(1+z)^3+\Omega_\sigma+\Omega_l\right)} \tag{22}$$

where z is the redshift : $1+z = 1/a(\tau)$, obtained by setting the current scale factor to unity and Ω is current observed value ($z=0$) of $\Omega(a)$. This completes the necessary review on the homogeneous cosmology front. It is clear that the ΛCDM limit corresponds to $\Omega_l \to 0$ or $l \to \infty$. Note also that in this limit we have $\beta \to -\infty$.

We are chiefly interested in the theory of spherical, scalar perturbations predicted by this model, pertaining the large scale cosmic structures. So we next take in the linear ansatz for the McVittie metric in Eq. (14),

$$ds^2 = a^2(\tau)\left[-(1+2\Phi(R,\tau))d\tau^2 + (1-2\Psi(R,\tau))\left(dx^2+dy^2+dz^2\right)\right] \tag{23}$$

where $R^2 = x^2+y^2+z^2$ and Φ and Ψ are the gravitational potentials. In the absence of anisotropic spatial stresses, we have $\Psi = \Phi$, which would not be the case here. The sources generating such spatial inhomogeneity are given by

$$\delta T^\mu{}_\nu = \begin{bmatrix} -\delta\rho & -\rho\nabla_i u \\ \frac{\rho\nabla^i u}{a^2} & \delta P\,\delta^i{}_j + \frac{\zeta^i{}_j}{a^2} \end{bmatrix} \tag{24}$$

which can be easily understood remembering the definition of the stress energy tensor for a perfect fluid

$$T^{\mu\nu} = (\rho+P)U^\mu U^\nu + Pg^{\mu\nu} \tag{25}$$

where $\zeta_{ij} = (\nabla_i\nabla_j - \delta_{ij}\nabla^2/3)\zeta$ $(i,\ j \equiv x,y,z)$ with ζ being a scalar used to parametrize the anisotropic strength tensor ζ_{ij}. $u(R,\tau)$ is the velocity potential function ignoring any vorticity, $\delta\rho(R,\tau)$ is the perturbation representing the central overdensity and δP is the pressure perturbation. We also have for the Weyl fluid perturbation (in analogy to the previous case but this time keeping in mind we are talking about a radiation like energy density),

$$m^2\delta\mathcal{C}^\mu{}_\nu = \begin{bmatrix} -\delta\rho_\mathcal{C} & -(\rho_r+P_r)\nabla_i u_\mathcal{C} \\ \frac{(\rho_r+P_r)\nabla^i u_\mathcal{C}}{a^2} & \frac{\delta\rho_c\delta^i{}_j}{3} + \frac{\delta\pi^i{}_j}{a^2} \end{bmatrix} \tag{26}$$

where ρ_r and P_r are the homogeneous density and pressure of radiation, $\delta_C = \frac{3\delta\rho_C}{4\rho_r}$ and u_C is the velocity potential for the Weyl Fluid, also $\delta\pi_{ij} = \left(\nabla_i\nabla_j - \delta_{ij}\nabla^2/3\right)\delta\pi_\mathcal{C}$ and $\delta\pi_\mathcal{C}$ is again a scalar used to parametrize $\delta\pi_{ij}$. We have

$$\frac{1}{a}\dot{\delta}_C = \nabla^2 u_C \tag{27}$$

from which one gets

$$\frac{1}{a^2}\ddot{\delta}_C + \left(\frac{2\beta}{2+\beta} - 3\gamma\right)\frac{\mathcal{H}}{a^2}\dot{\delta}_C - \frac{1}{3}\left(2+3\gamma\right)\nabla^2\delta_C = \frac{1+3\gamma}{4\rho_r}\nabla^2\left(\rho\Delta\right) \tag{28}$$

where

$$\Delta = \delta + \frac{3\mathcal{H}u}{a} \tag{29}$$

with $\delta = \delta\rho/\rho$. Finally we have the differential equations determining the two potentials

$$\nabla^2 \Psi = \frac{2+\beta}{2m^2\beta}\rho\Delta + \frac{4\rho_r}{3m^2\beta}\left(\delta_C + \frac{3\mathcal{H}u_C}{a}\right) \tag{30}$$

and

$$\nabla^2(\Psi - \Phi) = \frac{8\rho_r}{3m^2\beta}\left[\delta_C + \frac{6\mathcal{H}u_C}{a(2+\beta)} + \frac{3\rho\Delta}{4\rho_r}\right] \tag{31}$$

In the limit $\beta \to -\infty$, the right hand side of the above equation vanishes and we recover ΛCDM, $\Psi = \Phi$. We shall not be requiring all the perturbation sources described above, as will be clear from the discussions below.

With all this equipment, we are now ready to go into the maximum turn around calculations.

3 Calculation of the maximum turnaround radius

We shall first demonstrate the calculation of the maximum turn around radius for a spherical cosmic structure ignoring the Weyl term to demonstrate the method. Our starting point will be (see [33]) to consider the proper or physical spatial coordinate corresponding to the cold dark matter's perturbation, $\delta\rho(R,\tau)$,

$$\vec{r} = a(\tau)\vec{x} \tag{32}$$

The velocity and acceleration of this element with respect to the proper or physical time $dt = a(\tau)d\tau$ reads

$$\frac{dr^i}{dt} = \frac{1}{a(\tau)}\frac{d\vec{r}}{d\tau} = \delta u^i + \mathcal{H}x^i \tag{33}$$

where δu^i is the peculiar velocity. Using Eq. (23) we can derive the conservation equation for the perturbation

$$\delta\dot{\vec{u}} + \mathcal{H}\delta\vec{u} = -\vec{\nabla}\Phi$$

Differentiating Eq. (33) with respect to the proper time once again and using the above equation, we obtain the acceleration

$$\frac{d^2\vec{r}}{dt^2} = \left(\frac{\ddot{a}}{a^3} - \frac{\dot{a}^2}{a^4}\right)\vec{r} - \frac{1}{a}\vec{\nabla}\Phi \equiv \frac{\dot{\mathcal{H}}}{a^2}\vec{r} - \frac{1}{a}\vec{\nabla}\Phi \tag{34}$$

Note that the above relation is model independent. The explicit model dependence will enter via $\mathcal{H}$ and Φ. Since length scales pertaining the structures are essentially sub-Hubble, the velocity potential for matter can be ignored and spatial derivatives of the perturbations will be favoured over time derivatives. Subtracting Eq. (31) from Eq. (30) and dropping all the Weyl terms gives the Poisson equation for the potential Φ,

$$\nabla^2\Phi = \frac{2+\beta}{2m^2\beta}\delta\rho - \frac{2}{m^2\beta}\delta\rho \tag{35}$$

where we have ignored the velocity perturbation, $\rho\Delta \sim \delta\rho$ in Eq. (29). Thus we get

$$\nabla^2\Phi = 4\pi G_{\text{eff}}\delta\rho \tag{36}$$

where

$$G_{\text{eff}} = G\left(1 - \frac{2}{\beta}\right) = G\left(1 + \frac{1}{1 + \frac{l\mathcal{H}}{a}}\right) \qquad \text{and} \qquad G \equiv \frac{1}{8\pi m^2} \tag{37}$$

where we have also used Eq. (21). $G_{\rm eff}$ approaches G as $l \to \infty$, i.e. the ΛCDM limit. Thus in this theory the effective Newton's 'constant' is larger than G, indicating the increase of gravitational attraction. We next approximate the whole structure as a point mass located at $\vec{r} = 0 : \ \delta\rho = M\delta^3(\vec{r})$ as the perturbation,

$$\nabla^2 \Phi = 4\pi G_{\rm eff} M \delta^3(\vec{R}a(\tau)) \tag{38}$$

giving

$$\Phi = -\frac{G_{\rm eff} M}{R} \tag{39}$$

The maximum turn around radius $R_{\rm TA,max}$ is by definition the point of vanishing acceleration. Thus setting $d^2\vec{r}/dt^2 = 0$ in Eq. (34) and noting that from the spherical symmetry of the problem we have $\vec{r} \equiv r$, we obtain

$$\frac{\dot{\mathcal{H}}\, R_{\rm TA,max}}{a} - \frac{G_{\rm eff} M}{R^2_{\rm TA,max}} = 0 \tag{40}$$

Using now Eq. (20) along with the homogeneous conservation equation, $\dot{\rho} + 3\mathcal{H}\rho = 0$, and

$$\dot{\mathcal{H}} = \mathcal{H}^2 + \frac{1}{6m^2} \frac{al\dot{\rho}}{\sqrt{1 + l^2 \left(\frac{\rho+\sigma}{3m^2}\right)}} \tag{41}$$

we finally arrive at

$$R_{\rm TA,max} = \left(\frac{G_{\rm eff} M}{\frac{1}{l^2}\left(-1 + \sqrt{1 + l^2 \frac{\rho+\sigma}{3m^2}}\right)^2 - \frac{\rho}{2m^2}\left(1 - \frac{1}{\sqrt{1+l^2\frac{\rho+\sigma}{3m^2}}}\right)} \right)^{1/3} \tag{42}$$

We recover the ΛCDM result by setting $l \to \infty$ above,

$$R_{\rm TA,max} = \left(\frac{GM}{\frac{\Lambda_\sigma}{3} - \frac{\rho}{6m^2}} \right)^{1/3} \tag{43}$$

where we have written $\Lambda_\sigma = \sigma/m^2$ for the brane cosmological constant. We can rewrite the above equation as

$$R_{\rm TA,max} = \left(\frac{3GM}{\Lambda_\sigma}\right)^{1/3} \left(1 - \frac{\rho}{2m^2\Lambda_\sigma}\right)^{-1/3} \tag{44}$$

We may just include the background density ρ in the mass term via the redefinition $M' = M\left(1 + \rho/2m^2\Lambda_\sigma\right)$, and arrive at

$$R_{\rm TA,max} = \left(\frac{3GM'}{\Lambda_\sigma}\right)^{1/3} \tag{45}$$

The mass function M' clearly should be regarded as a total or effective mass function, taking into account the effect of the homogeneous matter density as well. For nearby cosmic structures, which is our main focus, we may take $z \sim 0$. Then recalling $1/2m^2 \equiv 4\pi G$, using $\Omega_m \simeq 0.3$ and $\Omega_{\Lambda_\sigma} \simeq 0.7$ for ΛCDM, it is easy to see that $M' \approx 1.214M$. One then uses the observed mass versus actual size data to do phenomenology in this context [33].

Next we shall include the effect of the inhomogeneous Weyl fluid to investigate how it modifies Eq. (42), while keeping still $\Lambda = 0$. This would simply correspond to modifying G_{eff} as follows. First, we recall that we already have ignored the homogeneous cosmological part of it (Sec. 2). The velocity potential of the perturbation of the fluid satisfies $\nabla^2 u_C = 0$, as we may ignore the temporal variation with respect to the spatial ones in Eq. (27), in the subhorizon length scale we are concerned with. Next note that in Eq. (30) and Eq. (31), u_C comes multiplied with the homogeneous radiation density, ρ_r. Thus we may ignore any term containing the Weyl fluid's velocity perturbation. Ignoring the temporal variations of the perturbation with respect to the spatial variation in Eq. (28), we have

$$\delta\rho_C = -\frac{1+3\gamma}{2+3\gamma}\delta\rho \tag{46}$$

where we have set an additive integration constant to zero, as the inhomogeneity is by definition sourced by the central overdensity. Subtracting now Eq. (31) from Eq. (30), we obtain

$$\nabla^2\Phi = 4\pi G\left(1 - \frac{2}{\beta\,(3\gamma+2)}\right)\delta\rho \tag{47}$$

We have from Eq. (22) $\beta \leq 0$ and

$$3\gamma + 2 = 3 - \frac{3\Omega_m(1+z)^3}{2\left[\Omega_m(1+z)^3 + \Omega_\sigma + \Omega_l\right]} > 0, \quad \text{always}$$

In other words, the effective Newton's 'constant'

$$G_{\text{eff}} = G\left(1 - \frac{2}{\beta\,(3\gamma+2)}\right) \tag{48}$$

appearing in Eq. (47) is always larger than that of ΛCDM, as earlier. Also, G_{eff} reduces to G in the limit $l \to \infty$ (or $\beta \to -\infty$).

Let us now compute $R_{\text{TA,max}}$ for this case and compare the result with ΛCDM. We rewrite Eq. (20) as

$$\left(\frac{\mathcal{H}}{a}\right)^2 = \frac{\rho+\sigma}{3m^2} + \frac{2}{l^2}\left[1 - \sqrt{1 + l^2\left(\frac{\rho+\sigma}{3m^2}\right)}\right] \tag{49}$$

Using Eq. (21) and the density parameter corresponding to l, we get

$$\left(\frac{\mathcal{H}}{a}\right)^2 = \frac{\rho+\sigma}{3m^2} - \frac{2\mathcal{H}}{la} = \frac{\rho+\sigma}{3m^2}\frac{1}{1+2\sqrt{\Omega_l}} \tag{50}$$

We also note that

$$1 - \frac{1}{\sqrt{1 + l^2(\frac{\rho+\sigma}{3m^2})}} = 1 - \frac{1}{1 + \frac{l\mathcal{H}}{a}} = 1 - \frac{\sqrt{\Omega_l}}{\sqrt{\Omega_l}+1} \tag{51}$$

Since $\sqrt{\Omega_l}$ is expectedly a 'small' number, we shall now proceed perturbatively in it. Using the above equations and Eq. (34), Eq. (41), we obtain after a lengthy but straightforward computations, up to the leading order in Ω_l,

$$R_{\mathrm{TA,max}} = \left[\frac{G_{\mathrm{eff}}M}{\frac{\sigma}{3m^2} - \frac{\rho}{6m^2} - \frac{2\sigma\sqrt{\Omega_l}}{3m^2} - \frac{\rho\sqrt{\Omega_l}}{2m^2}}\right]^{1/3} \tag{52}$$

Recalling $\Lambda_\sigma = \sigma/m^2$, we now compare the above expression with Eq. (43) corresponding to ΛCDM. We have already proven that $G_{\mathrm{eff}} \geq G$ and the denominator of the above equation is obviously smaller than that of Eq. (43). Since $\sqrt{\Omega_l}$ is a small number and $\beta \sim \Omega_l^{-1/2}$, Eq. (22), we can express the leading corrections to $R_{\mathrm{TA,max}}$ as

$$R_{\mathrm{TA,max}} \approx \left(\frac{3GM'}{\Lambda_\sigma}\right)^{1/3}\left[1 + \frac{2\sqrt{\Omega_l}}{3} + \frac{\sqrt{\Omega_l}\rho}{2m^2\Lambda_\sigma} - \frac{2}{3\beta(3\gamma+2)}\right] \tag{53}$$

where $M' = M(1 + \rho/2m^2\Lambda_\sigma)$ as earlier. Thus the maximum turn around radius or the maximum possible size of a cosmic structure predicted by this theory is larger than ΛCDM. Even though this conclusion is based upon perturbative arguments, we shall see in the next section that it holds true fully non-perturbatively, as well. Thus the phantom braneworld model with the bulk cosmological constant Λ set to zero, passes the test of stability of non-virial structures via $R_{\mathrm{TA,max}}$ with flying colours. We believe it is a result interesting in its own right.

However, we shall show below that once we 'turn on' Λ we can indeed have values of $R_{\mathrm{TA,max}}$ *smaller* than ΛCDM. In particular, we shall demonstrate that owing to the complicated structures of the field equations, all the other sources would 'interact' with Λ to make such decrement. As a consequence, we shall be able to obtain clear constraints to rule out certain region of the parameter space of the theory.

3.1 Inclusion of the bulk cosmological constant

We shall now be needing Eq. (19) which reads out

$$\left(\frac{\mathcal{H}}{a}\right)^2 = \frac{\rho+\sigma}{3m^2} + \frac{2}{l^2}\left(1 - \sqrt{1 + l^2\left(\frac{\rho+\sigma}{3m^2} - \frac{\Lambda}{6}\right)}\right) \tag{54}$$

along with the general form of Eq. (21),

$$\beta = -2\sqrt{1 + l^2\left(\frac{\rho+\sigma}{3m^2} - \frac{\Lambda}{6}\right)} \qquad 3\gamma - 1 = -\frac{\rho}{2m^2\left(\frac{\rho+\sigma}{3m^2} + \frac{1}{l^2} - \frac{\Lambda}{6}\right)} \tag{55}$$

The generalization to the effective Newton's constant comes readily substituting β and γ from Eq. (55) into Eq. (47) or Eq. (48) and our $R_{\mathrm{TA,max}}$ will be given by

$$R_{\mathrm{TA,max}} = \left(\frac{G_{\mathrm{eff}}M}{\dot{\mathcal{H}}/a^2}\right)^{1/3} \tag{56}$$

Following similar steps as earlier, we now obtain a modified expression for $R_{\mathrm{TA,max}}$, incorporating the effect off the bulk cosmological constant,

$$R_{\mathrm{TA,max}} = \left(\frac{3GM}{\Lambda_\sigma} \frac{1 - \frac{2}{\beta(3\gamma+2)}}{1 - \frac{\rho}{2m^2\Lambda_\sigma} + \frac{6}{l^2\Lambda_\sigma}\left[1 - \sqrt{1 + l^2\left(\frac{\rho+m^2\Lambda_\sigma}{3m^2} - \frac{\Lambda}{6}\right)}\right] + \frac{3\rho}{2m^2\Lambda_\sigma}\frac{1}{\sqrt{1+l^2\left(\frac{\rho+m^2\Lambda_\sigma}{3m^2} - \frac{\Lambda}{6}\right)}}} \right)^{1/3} \tag{57}$$

Expressing this with respect to the primed mass $M' = M\left(1 + \rho/2m^2\Lambda_\sigma\right)$, we finally obtain

$$R_{\mathrm{TA,max}} = \left(\frac{3GM'}{\Lambda_\sigma} \frac{1 - \frac{2}{\beta(3\gamma+2)}}{1 - \frac{\rho}{2m^2\Lambda_\sigma} + \frac{6}{l^2\Lambda_\sigma}\left[1 - \sqrt{1 + l^2\left(\frac{\rho+m^2\Lambda_\sigma}{3m^2} - \frac{\Lambda}{6}\right)}\right] + \frac{3\rho}{2m^2\Lambda_\sigma}\frac{1}{\sqrt{1+l^2\left(\frac{\rho+m^2\Lambda_\sigma}{3m^2} - \frac{\Lambda}{6}\right)}}} \right)^{1/3} \left(\frac{1}{1+\rho/2m^2\Lambda_\sigma}\right)^{1/3} \tag{58}$$

Squaring both sides of Eq. (54) and using $\Omega_l = \frac{a^2}{l^2\mathcal{H}^2}$, we have

$$\sqrt{1 + l^2\left(\frac{\rho+\sigma}{3m^2} - \frac{\Lambda}{6}\right)} = \sqrt{\frac{1}{\Omega_l} - \frac{l^2\Lambda}{6}} + 1 \tag{59}$$

Factoring out now the ΛCDM maximum turnaround radius, we get

$$\frac{R_{\mathrm{TA,max}}}{R_{\mathrm{TA,max0}}} = \left(\frac{1 - \frac{2}{\beta(3\gamma+2)}}{1 - \frac{\rho}{2m^2\Lambda_\sigma} - \frac{6\Omega_l\mathcal{H}^2}{a^2\Lambda_\sigma}\sqrt{\frac{1}{\Omega_l} - \frac{\Lambda a^2}{6\Omega_l\mathcal{H}^2}} + \frac{3\rho}{2m^2\Lambda_\sigma}\frac{1}{\sqrt{\frac{1}{\Omega_l} - \frac{a^2\Lambda}{6\Omega_l\mathcal{H}^2}+1}}} \right)^{1/3} \left(\frac{1}{1+\rho/2m^2\Lambda_\sigma}\right)^{1/3} \tag{60}$$

where the suffix '0' in the denominator of the left hand side stands for ΛCDM. Note in the above expression that the ratio $\rho/m^2\Lambda_\sigma$ could be replaced with Ω_m/Ω_σ.

We now wish to make a plot of Eq. (60) subject to the variation of the bulk cosmological constant and the parameter Ω_l, by taking $\Omega_m \simeq 0.3$ as the only input, supported by observation. Let us divide both sides of Eq. (54) with $(\mathcal{H}/a)^2$ and use Eq. (59) to get

$$1 - \Omega_m \simeq 0.7 = \Omega_\sigma - 2\Omega_l\sqrt{\frac{1}{\Omega_l} - \frac{a^2\Lambda}{6\Omega_l\mathcal{H}^2}} \tag{61}$$

This equation helps us to replace Ω_σ (or Λ_σ) in Eq. (60) in the favour of Ω_l.

Fig. 1 then depicts the variation of the ratio on the left hand side of Eq. (60), with respect to the variation in the parameter space of the bulk cosmological constant and Ω_l. As we have discussed earlier, for some nearby non-virial cosmic structures with $M \gtrsim 10^{14}M_\odot$, the theoretical prediction of ΛCDM on $R_{\mathrm{TA,max}}$ is only roughly about 10% larger than their actual observed sizes [32, 33]. Thus *any* alternative dark energy/gravity model predicting an $R_{\mathrm{TA,max}}$ lesser than about 10% compared to ΛCDM, gets ruled out.

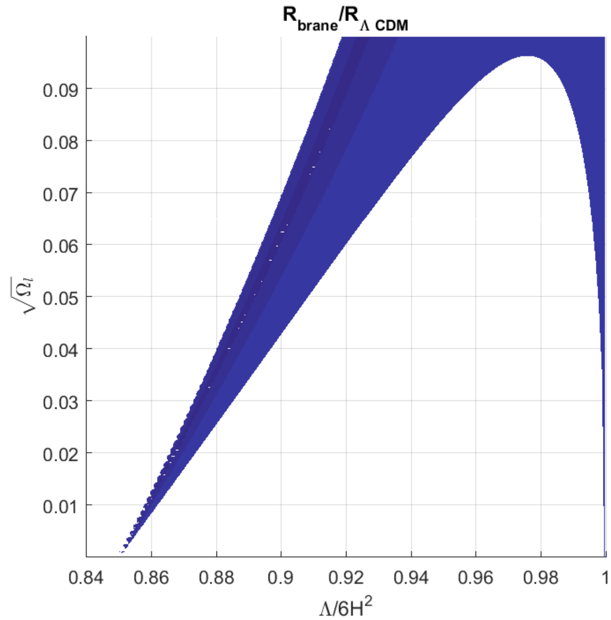

Figure 1: Plot of Eq. (60) to demonstrate the region of parameters excluded by mass versus actual size observations (the coloured region). The current scale factor is set to unity and we have taken $\Omega_m \simeq 0.3$ as the only input. Ω_σ or Λ_σ at each point is given via Eq. (61).

Based on that, we get the constraint on the parameter space of the theory depicted in the figure. It is also evident that for $\Lambda = 0$, there is no constraint whatsoever, on the parameter space of the theory. Since the analysis in this section in essentially non-perturbative, it proves our earlier claim made at the end of Sec. 3, regarding the consistency of this model with the bound of $R_{\mathrm{TA,max}}$ with a vanishing bulk cosmological constant.

The above analysis is essentially with $\Lambda > 0$. What happens when we flip the sign of the bulk Λ? Quite remarkably, it turns out that for this case, the theory once again passes the maximum turn around test with flying colours. In Fig. 2, we have numerically depicted Eq. (60) with respect to independent parameters, with $\Lambda/6\mathcal{H}^2$ as high as up to ~ -1000. Clearly, owing to the complicated nature of Eq. (60), this result was far from obvious *a priori*. We believe this result is also interesting in its own right.

4 Discussion

In this thesis the notion of the maximum turn around radius was used to bound the parameter space of braneworld cosmology. It is remarkable that this was done using as sole input the value of Ω_m which is considered concrete experimentally, unless we are talking about something like MOND. We saw that at zero bulk cosmological constant the theory receives no constraints whatsoever. Once the cosmological constant is turned on, if it is positive experimentally disallowed regions appear in the parameter space of the theory. If it is negative there is no bound up to the values tested.

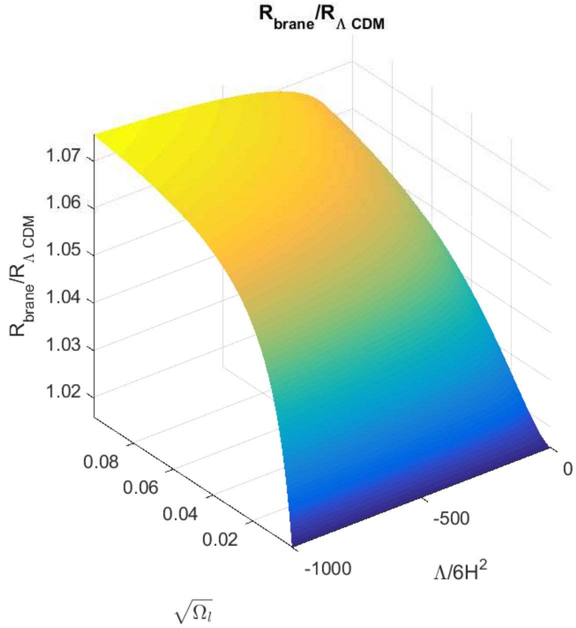

Figure 2: Plot of Eq. (60) for a negative bulk Λ. We see that the ratio of the maximum turnaround radii does not dip below 0.9.

We intend to extend the results of this thesis further to the case where we do not consider any sub-Hubble approximations. This work is already underway. It is of interest since firstly it reduces the number of approximations made thus strengthening the results and secondly it will possibly allow for a broader data pool yet again strengthening our results.

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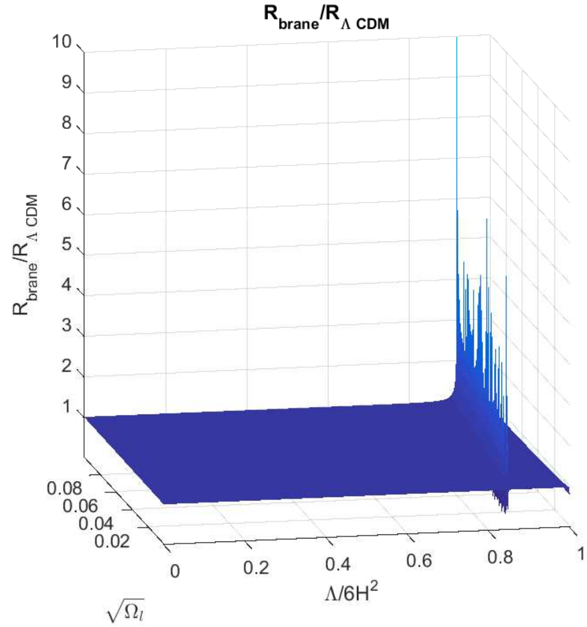

Figure 3: Plot of Eq. (60), but this time we demonstrate the 3d plot to show the values of the ratio of the maximum turnaround radii, what one sees is that the values of the ratio become interesting as we approach the experimentally excluded regions demonstrated in Figure 1.

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