University of Crete
Department of Physics

Undergraduate Thesis

An Excursion Set Theory Approach to the End of Structure Formation in ΛCDM Cosmology

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to Sofia...

Complaining to Theocritus one day
the youthful poet Eumenes spoke thus:
“Two years have passed since I began to write,
and on one idyl my achievement rests:
it is the only perfect piece of mine.
Alas, the ladder of Poetic art
is lofty, very lofty, I perceive;
and from the first rung here I stand upon,
no moving upward for unhappy me”.
Answered Theocritus: “These words of yours
are unbecoming nay, are blasphemous.
Even though you are on the first rung only, still
hold your head high and know that you are blest.
So far as you have come is no small feat;
so much as you have done, uncommon praise.
Even this first rung lies distant from the range
of most, and has no fellowship therewith.
That you should come to stand upon this rung
needs must you be through your own blameless right
a citizen of the city of ideas.
And in that city never without care,
nor oft, do they award the citizenship:
for in its market-place are to be found
law-givers that no braggart can deceive.
So far as you have come is no small feat;
so much as you have done, uncommon praise”.

C.P.Cavafy, *The First Rung*

(Translation by John Cavafy)

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Abstract

One of the most important consequences of the existence of a dark energy component (in the form of a cosmological constant in the currently accepted ΛCDM cosmology) in the Universe, lies in the process of structure formation. In such a cosmology some initial overdensities are able to turnaround and collapse to virialized structures, while others are not. Here, after presenting the Excursion Set Theory of mass functions, we use it to obtain the mass function of the turnaround-ed structures in the distant future and we compare it with the mass functions of turnaround-ed and virialized structures in the present cosmological epoch and other cosmological epochs.

We also follow structures of various mass scales towards their final number densities. We find a "magical" mass scale, such that, structures with masses greater than that have always greater number density than their final while structures of lower mass have, at least at some time, greater number densities than their final.

We conclude that the process of structure formation is nearly completed in the present time. This work comes to complete and to extend similar results obtained from numerical simulations.

Contents

Chapter 1

An Overview of Cosmology

1.1 Introduction: Cosmology and Cosmological Ideas

Every civilization in the human history had its own myths about the creation of the world. These stories of cosmogony included usually fights between gods, giants and monsters. In other such stories the creation of the universe was nothing but the result of the mating of a male god with a female goddess.

For millennia cosmology had nothing in common with *reason* and *science*. It was only associated with religion, philosophy and metaphysics. Only very recently, for the first time in the history of mankind, the common efforts of theoretical/mathematical scientists, experimenters and observational scientists have provided us with a picture of the creation and evolution of the universe that we believe is more than a contemporary myth of cosmogony. This is the well known *Hot Big Bang Cosmology*, which we will briefly explore in this chapter.

Using Newton's words, to be here in this vantage point in history and being able to see a little bit further (and -for us- to understand the universe) is only because we are standing on the shoulders of giants. We will briefly discuss the evolution of the scientific and cosmological ideas.

The history of cosmology is a history of a gradual abandoning of the anthropocentric picture of the world. For the ancient Greeks, the first who tried to make a picture of the world based on reason, Earth was at the center of the cosmos. The Sun, the Moon and the planets rotated in circular orbits around it, while the stars were in "fixed" positions in much larger distances. Some astronomers, like Aristarchus of Samos, proposed that is more natural to think that is the Sun, and not the Earth, which is the center of the cosmos with the Earth and the other planets going around the Sun. This model solved various problems, as the phenomenon of retrogate motion of the planets, but it wasn't taken seriously at that time. The world had to wait for Nicolaus Copernicus (1473-1543), to re-invent the *heliocentric* view of the Universe.

Although Copernicus still believed that the Sun is the center of the Universe, is he who is credited with the *Copernican Principle*, that is the notion that we do not inhabit a special place in the Universe.

After Copernicus, the work of other scientists, as Galileo Galilei (1564-1642) who first *used* the telescope for astronomical observations, Johannes Kepler (1571-1630) who found his well- known laws for the movement of planets, and Isaac Newton (1642-1727) who set the basis of modern physics with the publication of his work, *Philosophiæ Naturalis Principia Mathematica*, changed the view of the world.

Newton's theory of gravitation was the first scientific theory, with the modern meaning of the term. With his theory he was able to explain, for example, the empirical laws of Kepler. He also formulated a cosmology based on the following way of thinking: since all massive bodies attract each other, a finite system of stars distributed over a finite system of stars over a finite region of space should collapse under their mutual attraction. But this was not observed, and in fact the stars were known to have had fixed positions since antiquity. Newton thought that this problem could be solved if the Universe extended over an infinite space and contained an infinite number of stars. He concluded that the attraction of a sufficient number of stars outside the system would counterbalance the self-gravitation of the finite system of stars. Although this theory is obviously false, it was the first scientific attempt to explain why the world looks like the way we see it.

After two centuries, the great astronomers William Herschel (1738-1822) and his son, John, conducted extensive observational research of the nearby stars and concluded ($\sim$1785) that the Milky Way was a disc-shaped star system. But they also, wrongly, concluded that our solar system was at the center of the Galaxy. Only in 1919 Harlow Shapley (1895-1972), with the observations of the distribution of the globular clusters found that our solar system lies at a distance of about two thirds of the galactic radius.

But, even then, Shapley believed that our Galaxy was at the center of the world - even that it consisted the whole Universe. Other astronomers, like Heber Curtis (1872-1932) thought that the objects then known as *spiral nebulae* were galaxies as our own. When Edwin Hubble (1889-1953) measured (1924) the distances of these nebulae found that they were extremely far away, indicating that they were indeed galaxies. Finally, in 1952, Walter Baade conclusively demonstrated that the Milky Way is an average size galaxy, surely no central or exceptional.

Modern Cosmology, as true science, was born at the first decades of the 20^{th} century. The publication of the *General Theory of Relativity* by Albert Einstein (1879-1955) and the observations of an expanding Universe by E. Hubble in 1929 set the theoretical and observational basis, respectively, of the science of Cosmology. Today the Big Bang Theory is the most widely accepted model of cosmology, although it faced rivals, with the *Steady State Theory* being the most known among them. But, even there are still some small problems with this model, a large amount of observational evidence suggests that *indeed* all started with a Big Bang!

1.2 Observational Properties of the Universe

In this section we will give some basic facts about the universe, its building blocks, the distances inside it and some very important observational properties, as the cosmological expansion, the tantalizing evidence for the existence of dark matter, and the unexpected discovery of the accelerating expansion of the Universe, which in turn indicates the existence of a dark energy component in the world .

1.2.1 Units and Typical Sizes

Although there are many units of distance used in astronomy, as the mean distance of the Earth to the Sun, the so-called *Astronomical Unit* (A.U.), which is about:

$$1\,\text{A.U.} \cong 1.49 \times 10^{11}\ m \tag{1.2.1}$$

or the very popular *light year, ly,* (the distance covered by the light, traveling with speed c for a time period of one year):

$$1\,light\,year \equiv c \times 1\,year \cong 0.951 \times 10^{16}\ m \tag{1.2.2}$$

the standard unit of distance used by astronomers is the *parsec*, (pc), which is defined to be the distance at which the semi-major axis of the Earth's orbit around the Sun would subtend an angle of one arc-second. So:

$$1\,\text{pc} \equiv \frac{1\,\text{A.U.}}{1''\text{in rad}} \cong 3.086 \times 10^{16}\,m \cong 3.26\,ly \tag{1.2.3}$$

Some examples of distances in the universe is the distance to the nearest star (α Centauri), which is about 1.3pc, our distance from the center of the Galaxy, about 8.5×10^3 pc $\equiv$ 8.5 kpc, the diameter of the disc of our Galaxy (13 kpc) and the distance to the nearest big galaxy (the Andromeda galaxy), which is about 770 kpc. In cosmology, where we consider distances much bigger than the distances between galaxies the most convenient unit is the *megaparsec* (Mpc), which is equal to 10^6 pc.

The most usual unit of mass is the *solar mass*:

$$1\,\text{M}_\odot \cong 1.99 \times 10^{30}\,kg \tag{1.2.4}$$

which is the typical mass scale of stars, while the spiral galaxies have typical masses of the order of 10^{11} $\text{M}_\odot$. For the luminosity the typical unit is the *solar luminosity*:

$$1\,\text{L}_\odot \cong 3.9 \times 10^{33}\ erg \cdot sec^{-1} \tag{1.2.5}$$

1.2.2 The Hierarchy of Structures in the Universe

Matter in the Universe is organized in structures, of gradually higher mass. We classify them as follows:

- **Stars:** Stars are the fundamental building blocks of the Universe, and the source of the visible light since they produce energy through fusion nuclear reactions in their cores. Our Sun is a typical star, of the mass and luminosity which we gave before.

- **Galaxies:** Galaxies are vast collections of stars, although in cosmology are thought as point-like objects emitting light. According to their morphology they come in three basic types: *spirals*, *ellipticals* and *irregular*. Our Galaxy is a typical spiral galaxy. It consists of a central bulge, plus a disc of radius 12.5 kpc and thickness of only 0.3 kpc. It has a mass of about 10^{11} $M_{\odot}$.

- **The Local group:** Our Galaxy is a member of a group of around 20 galaxies, called the *Local Group*. Exept of our Galaxy other members include the small irregular galaxies known as the Small Magellanic Cloud (SMC) and the Large Magellanic Cloud (LMC) and the M31, the great Andromeda galaxy, a spiral just as our own. A typical galaxy group occupies a volume of a few cubic megaparsecs.

- **Clusters of galaxies, Superclusters, Filaments and Voids:** In some places of the Universe galaxies grouped together to make gravitationally bound objects known as *clusters* of galaxies. Clusters may contain from approximately 50 galaxies to thousand of galaxies in a region of space about 6-7 Mpc in diameter. Two famous examples are Virgo cluster and Coma cluster (which contains about 10000 galaxies). Next in the hierarchy of the galactic clustering is the supercluster. This is the clustering of clusters on a grand scale (up to about 100 Mpc). Although many galaxies do not belong to clusters, every galaxy belongs to a supercluster. The mass of the Local Supercluster is about 10^{14} $M_{\odot}$. These superclusters are the largest bound structures in the Universe. The structures are distributed in such a way that they form filamentary forms over large voids.

1.2.3 The Cosmological Principle

The *Cosmological Principle* is the statement that at every moment in the cosmic history, the Universe (in large scales) is ***homogeneous*** and ***isotropic***. This is to say, that a privileged position or direction doesn't exist in the Universe. Homogeneity means that the Universe looks the same at each point, while isotropy means that it looks the same at every direction.

Observations confirm that the Universe is indeed homogeneous and isotropic at scales larger than about 100 Mpc. The homogeneity and isotropy of the Cosmic Background Radiation (CMB), which we will discuss briefly in this section and in more detail in a subsequent one, is also in favor of the validity of the Cosmological Principle. However in smaller scales there are important deviations from homogeneity and isotropy. These deviations allow (or indicated by) the existence of structures as stars, galaxies, clusters etc., the formation of which is the main topic of this thesis.

1.2.4 Hubble's Law and the Expansion of the Universe

As in the case of sound waves (Doppler effect), the frequency or, equivalently, the wavelength of light (electromagnetic radiation in general) *changes* if there is a relative motion between the source of the light and the observer. If we write

λ_{obs} for the observed wavelength and λ_{em} for the emitted wavelength, we then have from the Special Theory of Relativity, that:

$$\frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = \sqrt{\frac{1 \mp \upsilon/c}{1 \pm \upsilon/c}} \tag{1.2.6}$$

where υ is the relative speed between the source and the observer. The upper signs are for the case of an approaching source; in that case $\lambda_{\text{obs}} < \lambda_{\text{em}}$ and we say that we have *blueshift*. The lower signs are for the case of a receding source; in that case $\lambda_{\text{obs}} > \lambda_{\text{em}}$ and we say that we have *redshift*.

Astronomer Vesto Slipher was the first who observed that most galaxies tend to appear red-shifted rather than blue-shifted. Let us define the redshift as:

$$z \equiv \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} \tag{1.2.7}$$

then we can rewrite eq. (1.2.6) as (for the case of a receding source):

$$1 + z = \sqrt{\frac{1 + \upsilon/c}{1 - \upsilon/c}}$$

and, for small speed υ, we have that:

$$z \cong \frac{\upsilon}{c} \tag{1.2.8}$$

In the late 1920's Edwin Hubble measured the redshift and the distance (using the then-newly discovered period-luminosity relation of the *Cepheid variable* stars) of 18 nearby galaxies and he found a linear relation between the two. Using (1.2.8) he concluded that the speed of recession of a galaxy, υ, is proportional to its distance from us, r:

$$\boxed{\upsilon = H_0 \cdot r} \tag{1.2.9}$$

This is the famous ***Hubble's law*** and the constant of proportionality is called *Hubble's constant* or, better, *Hubble's parameter* since it's not constant but changes with time, as we will see. It's not so easy to measure with high accuracy the (present) value of Hubble's parameter. It's very common to write:

$$H_0 = 100h \;\; km \cdot s^{-1} \cdot Mpc^{-1} \tag{1.2.10}$$

and put the uncertainty into the constant h. The present value of h (taken from Particle Data Booklet) is:

$$h = 0.721(17) \tag{1.2.11}$$

Hubble's law, combined with the cosmological principle, gives us the image of an expanding universe. Every galaxy in the Universe (indeed, this is true for galaxies not very close to each other) is rushing away from any other galaxy with velocity which is proportional to their separation distance. Expansion looks the same from each point in the Universe. If this picture is true, we can conclude that in the distant past, everything in the Universe was much closer. Indeed there exist a time when the whole Universe was a single point. The "explosion" of this point is known as the ***Big Bang***.

From eq. (1.2.10) we see that Hubble's parameter has dimensions of inverse time. We define the *Hubble time* as:

$$t_{\rm H} \equiv H_0^{-1} = 9.8h^{-1} \times 10^9 \ yr \tag{1.2.12}$$

the Hubble time constitutes an upper limit to the age of the Universe in standard cosmological models.

1.2.5 The Matter with Dark Matter

Could there be invisible matter in the Universe, especially much more than the luminous matter we see in stars, nebulae, etc.? Observations based on the motions of astronomical objects seem to indicate such a thing.

Even back in 1933, the Swiss astronomer Franz Zwicky (1898-1974) mentioned that in order to remain gravitationally bound, some clusters of galaxies must have much more matter than the matter we can see.

One astonishing and easy-to-understand indication for the existence of invisible, dark matter comes from the *galactic rotation curves.* A galactic rotation curve is the graphic representation of the velocity of a particle/body orbiting around the center of the galaxy, as a function of radius from the center.

Let $M(r)$ to be the total mass of the galaxy inside radius r from the center. The centripetal force acting on a test particle, of mass m orbiting around the center of the galaxy at that a r, is the gravitational force from the the mass inside that radius. So:

$$\not{m}\frac{\upsilon(r)^2}{r} = G\frac{\not{m}M(r)}{r^2} \Rightarrow$$
$$\Rightarrow \upsilon(r) = \sqrt{G\frac{M(r)}{r}} \tag{1.2.13}$$

Now if the mass of the galaxy is only the luminous one, we expect at distances $r > R$, where R is the radius of the galaxy, to be $M(r) \cong M_{\rm lum,gal}$, where $M(r) \cong M_{\rm lum,gal}$ is the total luminous mass of the galaxy. So, combining this with the above eq. (1.2.13) we conclude that we expect :

$$\upsilon(r) \cong \sqrt{G\frac{M_{\rm lum,gal}}{r}} \Rightarrow \upsilon(r) \propto \frac{1}{\sqrt{r}} \ ,\text{for } \ r > R \tag{1.2.14}$$

But, with great surprise, observationally we see that:

$$\upsilon(r) \cong \text{const.} \ ,\text{for } \ r > R \tag{1.2.15}$$

What can we say about this deviation? One proposal, not very popular among physicists, is that the law of gravitation or the force law changes for very small accelerations of the order of $\approx cH_0 \cong 10^{-10} m/s^2$. This theory is known as Modified Newtonian Dynamics (MOND). But for most physicist this deviation indicates that there exists more mass than the mass we can see. This dark matter forms a halo with density $\rho(r) \propto 1/r^2$, and the total mass of it seems to be over 10 times the luminous mass.

What can the constituents of dark matter be? A part of it is, for sure, things *we know that exist*, as neutrinos (with very small mass < 2eV), black holes and objects of relative low mass as brown dwarfs, collectively known as MACHO's (Massive Compact Halo Objects). But the main part of the dark matter is still of unknown nature. There are some dark matter candidates, particles we *believe that may exist*. These candidates are coming from theories of particle physics, as *SuperSymmetry* (SUSY), and include particles with cute names, as photino, gravitino, neutralino etc., collectively known as WIMP's (Weakly Interacting Massive Particles). What's right and what's wrong? The time, the experiments and the observations will tell....

1.2.6 The Discovery of the Accelerating Expansion of the Universe and the Cosmological Constant

In 1990's two teams of cosmologists, one under Saul Perlmutter (Supernova Cosmology Project) of Lawrence Berkeley Laboratory and the other under Brian Schmidt (High-Z Supernova Search Team) of Mount Stromlo Observatory, used a very special astronomical object, known as Supernovae of Type Ia to trace the history of cosmic expansion.

Supernovae Type Ia are events/objects that can be used as ***standard candles***. A standard candle is an object of known intrinsic luminosity which can be used to measure astronomical or cosmological distances. As we have seen Hubble used as standard candle a particular type of variable stars, known as Cepheid Variables. Supernovae type Ia are events we believe that happen when a white dwarf which accretes material reaches the maximum mass for a white dwarf, the famous *Chandrasekhar limit* of $1.4M_{\odot}$. Then an explosion happens, and since the exploding star has the same properties in any case, we expect that all such events will have the same intrinsic luminosity. Observations of the *light curves* (evolution of the brightness of the events) confirm this hypothesis. The advantage of SN Type Ia over other standard candles is that they are extremely bright -so, we can observe them at great distances.

The two teams tried to use SN Ia to measure again the relation between the distance and the velocity of recession, as Hubble had done many decades ago. But now, the use of very distant objects (redshift $z \geq 1.0$) allowed cosmologists to measure the change of the rate of the expansion of the Universe.

In 1998, after years of efforts and overcoming of many difficulties, the first results came. To understand them, let us describe what they expected to find. Until then, it was believed that the main (the only, if we neglect radiation) constituent of the Universe is matter, luminous and dark. Now, according to the simple cosmological models which consider only matter, the expansion history of the Universe is determined completely by its mass density. The higher the density, the more the expansion is slowed by gravity. This implies that, in a Universe only with matter, the expansion rate would be greater in the past than today. This translates to the following observational fact: At a given redshift, in a high-density Universe which slows down, an object will be brighter than a an object of the same intrinsic brightness, in a Universe with constant rate of expansion.

They tried to measure the deceleration of the Universe. They discovered exactly the opposite. The Universe seems to be accelerating! They found that, at a given redshift, the SN Type Ia appeared much fainter than they expected.

Indeed they were less bright than predicted, even if they considered an empty Universe, with no gravitational self-attraction.

To explain the observed facts, the scientific community had to introduce a mysterious "dark energy", which acts as anti-gravity and causes the observed acceleration of the Universe. The nature of this dark energy is... dark: a possible explanation is that it is a *vacuum energy*, the zero-point energy of the quantum fields. But the calculations predict a value much much larger than the observed. In any case, the properties of this mysterious dark energy seem to can be incorporated to the old concept of the ***cosmological constant***, first introduced by A. Einstein, which is denoted by the Greek capital letter Λ. It has the bizarre property that its density remains constant as the Universe expands. Today, we believe that up to 70% of the matter/energy content of the Universe is dark energy, best described as a cosmological constant.

So, according to the currently accepted cosmological model, two are the main ingredients of the Universe: Matter, in the form of Cold Dark Matter (CDM) and dark energy, in the form of a cosmological constant. So, the cosmological model is known as ΛCDM Cosmology.

1.3 Geometry of the Universe, the Friedmann Equations and Simple Cosmological Models

In this section we will use the cosmological principle and the *General Theory of Relativity* (GR) to produce the equations that govern the evolution of the Universe, when we feed them with the content of it. Then we will apply them to simple cosmological models; simple but very useful models as approximate descriptions of the Universe at various phases of its evolution. We are not discussing here the GR in detail. The interested reader can see the bibliography for some nice books about GR

1.3.1 Three Possible Geometries of the Universe

The modern science of cosmology is based on the principle that at large distances the Universe is homogeneous and isotropic (the *Cosmological Principle* that we have discussed). How many different *geometries* exist which satisfy that principle? Let us try to find, for simplicity, 2D spaces which are homogeneous and isotropic. After a little bit of thought it's not hard to imagine that the familiar plane (the geometry on an *infinite* plane) or the surface of a sphere are 2D homogeneous and isotropic spaces. It's more difficult to imagine, but it also true, that the surface of a "saddle" (a *hyperboloid*) is also homogeneous and isotropic.

These spaces are the only 2D homogeneous and isotropic spaces (also called spaces with constant *curvature*). We are expecting that the 3D generalizations of those spaces to be the three *possible* geometries of the Universe. We will give now some properties of the three geometries:

Flat geometry: This is the the familiar Euclidean geometry on a plane surface. As we will see a flat space is a space with *zero curvature*. Two parallel lines on a plane not converge neither diverge: they always remain with a constant distance between them. In the Euclidean geometry, the geometry on a plane:

- The angles of a triangle add up to 180°
- The circumference of a circle of radius r is $c = 2\pi\, r$

If the universe has flat geometry is infinite and we talk about a *Flat Universe.*

Spherical geometry: In that case the space is still homogeneous and isotropic, as it must be, but now it remains finite, but without boundary (think the finite but without boundaries surface of a sphere). In the case of spherical geometry two parallel lines converge, and also:

- The angles of a triangle add up to *more* than 180°
- The circumference of a circle of radius r is *less* than $2\pi\, r$: $c < 2\pi\, r$

A Universe with spherical geometry is called *Closed Universe.*

Hyperbolic geometry: In that unfamiliar geometry, which is like the geometry on the surface of a saddle (in 2D), the space is infinite while two parallel lines diverge. Also:

- The angles of a triangle add up to *less* than 180°
- The circumference of a circle of radius r is *more* than $2\pi\, r$: $c > 2\pi\, r$

A Universe with such a geometry is an *Open Universe.*

The *line element*, that is the length of an infinitesimal line, for spaces with the three above types of geometries can be written, in a unified way, in spherical coordinates (r, θ, ϕ) as:

$$dl^2 = \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1.3.1}$$

with the introduction of the *curvature parameter* k, which can take only three values: $k = 0, \pm 1$. For $k = 0$ (*zero* curvature) we have *Flat geometry*, for $k = 1$ (*positive* curvature) we have *Spherical geometry* and for $k = -1$ (*negative* curvature) we have *Hyperbolic geometry.* We summarize all the above in the following table:

Table 1: **Properties of the Possible Geometries of the Universe**

Curvature	**Geometry**	**Angles of Triangle**	**Circumference of Circle**	**Type of Universe**
$k > 0$	Spherical	$> 180°$	$c < 2\pi\, r$	Closed
$k = 0$	Flat	180°	$c = 2\pi\, r$	Flat
$k < 0$	Hyperbolic	$< 180°$	$c > 2\pi\, r$	Open

1.3.2 The Robertson-Walker Metric and the Friedmann Equations

Now we want to find the four-dimensional (4D) metric which describes the space-time of the Universe. The metric must include the facts that:

- The (3D) space is homogeneous and isotropic at large scales.
- The Universe has the possibility to evolve with time (expansion or contraction of the Universe).

Using these two facts and eq. (1.3.1) for the expression for the line element of an isotropic and homogeneous space, we are coming to the most general form for the space-time metric of the Universe:

$$ds^2 = -dt^2 + R^2(t)\, dl^2 \Rightarrow$$

$$\Rightarrow \boxed{ds^2 = -dt^2 + R^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right]} \tag{1.3.2}$$

This is the well-known ***Robertson-Walker metric* (R-W)**. We have adopted here the signature $(- + ++)$ and that $c = 1$ (the speed of light in vacuum). Here (r, θ, ϕ) are called *co-moving coordinates* because we think the galaxies to be in "fixed" positions at that coordinate system, while the Universe expands or contracts. The t coordinate is called *cosmic time* and is the proper time of an observer fixed on a galaxy. $R(t)$ is a function known as *scale factor* and contains the information for the time evolution of the size of the Universe. As before, $k = 0$ corresponds to a *flat Universe*, $k = 1$ corresponds to a *closed Universe* and $k = -1$ corresponds to an *open Universe*.

We can immediately read the elements of the *metric tensor* of the R-W metric:

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & \frac{1}{1-kr^2} & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix} \Rightarrow$$

$$\Rightarrow g_{\mu\nu} = \mathrm{diag}\left(-1, \frac{1}{1-kr^2}, r^2, r^2 \sin^2\theta \right) \tag{1.3.3}$$

While the *inverse metric tensor* is:

$$g^{\mu\nu} = \mathrm{diag}\left(-1, 1-kr^2, r^{-2}, r^{-2} \sin^{-2}\theta \right) \tag{1.3.4}$$

Now we will use the *Einstein's equation* to determine the behavior of the scale factor $R(t)$. The Einstein's equation with the cosmological constant term, Λ, is written as:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G_N T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{1.3.5}$$

where $\mathcal{R}_{\mu\nu}$ is the (components of) the *Ricci tensor*, $\mathcal{R}$ is called the *curvature scalar* or the *Ricci scalar*, G_N is the Newton's gravitational constant, $T_{\mu\nu}$ is the *energy-momentum tensor*, Λ is the *cosmological constant* and $g_{\mu\nu}$ is the metric tensor. The Λ term can be interpreted as an effective energy-momentum tensor $\tilde{T}_{\mu\nu}$, for the vacuum, as:

$$\tilde{T}_{\mu\nu} = \frac{\Lambda g_{\mu\nu}}{8\pi G_N}$$

So we can finally rewrite the ***Einstein's equation*** as:

$$\boxed{\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G_N T_{\mu\nu}} \tag{1.3.6}$$

where the cosmological constant term is now included into the energy-momentum tensor.

For the Universe we can model it's matter and energy content by a perfect fluid of density $\rho(t)$ and pressure $p(t)$. For an isotropic fluid it's four-velocity is:

$$u_\mu = (1, 0, 0, 0) \tag{1.3.7}$$

so the energy-momentum tensor

$$T_{\mu\nu} = (\rho(t) + p(t))\, u_\mu u_\nu + p(t) g_{\mu\nu} \tag{1.3.8}$$

becomes

$$T_{\mu\nu} = \begin{pmatrix} \rho(t) & & & \\ & p(t)g_{11} & & \\ & & p(t)g_{22} & \\ & & & p(t)g_{33} \end{pmatrix} \Rightarrow$$

$$\Rightarrow T_{\mu\nu} = \mathrm{diag}\left(\rho(t), p(t)g_{11}, p(t)g_{22}, p(t)g_{33}\right) \tag{1.3.9}$$

Since we have the energy-momentum tensor what's left is to find the components of the Ricci tensor and the Ricci scalar, for the R-W metric. First we will compute the *Christoffel symbols* for the metric (1.3.3). The Christoffel symbols are given by:

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left(g_{\nu\rho,\lambda} + g_{\lambda\rho,\nu} - g_{\nu\lambda,\rho}\right) \tag{1.3.10}$$

where, for example, $g_{\nu\rho,\lambda} \equiv \frac{\partial g_{\nu\rho}}{\partial x^\lambda}$. Also note that, from the definition, $\Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\lambda\nu}$. Using the above relation, the non-vanishing Christoffel symbols for the Robertson-Walker metric are (we label our coordinates as $t \equiv x^0, r \equiv x^1, \theta \equiv x^2, \phi \equiv x^3$):

$$\begin{array}{lll} \Gamma^0_{11} = \dfrac{R\dot{R}}{1-kr^2}, & \Gamma^0_{22} = R\dot{R}r^2, & \Gamma^0_{33} = R\dot{R}r^2 \sin^2\theta \\ \Gamma^0_{01} = \dfrac{R}{\dot{R}}, & \Gamma^1_{11} = \dfrac{kr}{1-kr^2}, & \Gamma^1_{22} = -r(1-kr^2) \\ \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta & & \\ \Gamma^2_{02} = \dfrac{\dot{R}}{R}, & \Gamma^2_{12} = \dfrac{1}{r}, & \Gamma^2_{33} = -\sin\theta\cos\theta \\ \Gamma^3_{03} = \dfrac{R}{R}, & \Gamma^3_{13} = \dfrac{1}{r}, & \Gamma^3_{23} = \cot\theta \end{array} \tag{1.3.11}$$

Now we will compute the components of the Ricci tensor. Ricci tensor is the contracted form of the *Riemann tensor*:

$$\mathcal{R}^\rho{}_{\sigma\mu\nu} = \Gamma^\rho_{\nu\sigma,\mu} - \Gamma^\rho_{\mu\sigma,\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{1.3.12}$$

which means that:

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^{\lambda}{}_{\mu\lambda\nu} \tag{1.3.13}$$

Note that we have extensively used the *Einstein's summation convention*: the repeated upper and lower suffixes are summed. After a lot of calculations, one takes the non-vanishing components of the Ricci-tensor:

$$\begin{aligned}
\mathcal{R}_{00} &= -3\frac{\ddot{R}}{R} \\
\mathcal{R}_{11} &= \frac{R\ddot{R} + 2\dot{R}^2 + 2k}{1 - kr^2} \\
\mathcal{R}_{22} &= r^2\left(R\ddot{R} + 2\dot{R}^2 + 2k\right) \\
\mathcal{R}_{33} &= r^2\left(R\ddot{R} + 2\dot{R}^2 + 2k\right)\sin^2\theta
\end{aligned} \tag{1.3.14}$$

And finally we calculate the curvature (or Ricci) scalar, which is defined as:

$$\mathcal{R} \equiv g^{\mu\nu}\mathcal{R}_{\mu\nu} \tag{1.3.15}$$

and we find that:

$$\mathcal{R} = \left[\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2}\right] \tag{1.3.16}$$

Note that in all cases the dot means derivative with respect to the cosmic time, t. For example: $\dot{R} \equiv dR/dt$.

Now we are ready to derive the equations for the evolution of the scale factor. It's usual to work with a dimensionless scale factor, which we now introduce here:

$$a(t) \equiv \frac{R(t)}{R_0} \tag{1.3.17}$$

where R_0 is the present radius of the Universe, such that $a(t_0) = 1$ by definition. Now, working with this dimensionless scale factor, the $\mu\nu = 00$ Einstein equation is:

$$-3\frac{\ddot{a}}{a} = 4\pi G_N\left(\rho + 3p\right) \tag{1.3.18}$$

and the $\mu\nu = ij$ equations give:

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2R_0^2} = 4\pi G_N(\rho - p) \tag{1.3.19}$$

and rearranging we finally obtain the equations:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2R_0^2}} \tag{1.3.20}$$

and

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}\left(\rho + 3p\right)} \tag{1.3.21}$$

These are the famous ***Friedmann equations*** that govern the evolution of the scale factor. We can take a third, not independent, but very useful equation, the so-called *fluid equation*. This can be derived using the energy conservation equation:

$$T^{\mu\nu}{}_{;\mu} = 0 \tag{1.3.22}$$

where ; denotes covariant differentiation. Eq. (1.3.22) yields the continuity equation and the equations of motion of a perfect fluid particles. After some calculations, continuity equation reduces to:

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0} \tag{1.3.23}$$

Note that the above equation can be easily derived using the first law of thermodynamics, assuming that the universe is a perfect fluid under adiabatic expansion/contraction.

We have now *two* independent equations, but *three* unknown functions, namely $a(t), \rho(t)$ and $p(t)$. We need one more equation. This equation is the ***equation of state*** of the perfect fluid, that is a relation between the pressure $p(t)$ and the density $\rho(t)$ of the fluid. The perfect fluids relevant to cosmology obey the simple equation of state:

$$\boxed{p(t) = w\,\rho(t)} \tag{1.3.24}$$

or in the case where we don't have set $c = 1$:

$$p(t) = w\,\rho(t)\,c^2 \tag{1.3.25}$$

where w is a constant which depends of the nature of the fluid. We believe that three are the main ingredients of the Universe: *matter*, *radiation* and *dark* or *vacuum energy*. For each of these cases, w takes the value:

$$w = \begin{cases} 0 & \text{matter} \\ +\frac{1}{3} & \text{radiation} \\ -1 & \text{dark or vacuum energy} \end{cases} \tag{1.3.26}$$

Later we will study in brief the evolution of the scale factor in a Universe constituted of one or more of those ingredients.

1.3.3 Definition of Cosmological Parameters

Having derived the Friedmann equations we are ready to define some very important cosmological parameters. We can use these parameters to specify cosmological models and to compare them with observational data.

We can express the *rate of expansion* through the ***Hubble parameter***:

$$\boxed{H(t) \equiv \frac{\dot{a}}{a}} \tag{1.3.27}$$

The current value of Hubble parameter is Hubble's constant that we have seen before: $H(t_0) = \dot{a}(t_0) = H_0$, using the convention $a(t_0) = 1$. Using now the Hubble parameter, we can rewrite the first Friedmann equation as:

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2 R_0^2} \tag{1.3.28}$$

It's now obvious that, given the Hubble parameter, there exist a special value of the density which can ensure that the geometry of the universe is flat, which means $k = 0$. This value is known as the ***critical density*** and is given by:

$$\boxed{\rho_{\text{crit}}(t) \equiv \frac{3H^2(t)}{8\pi G_N}} \tag{1.3.29}$$

which is a function of time. We can compute the current critical density of the Universe, since we know the present value of the Hubble parameter (Hubble's constant):

$$\rho_{\text{crit},0} = 1.88 \times 10^{-26}\ h^2\ kg \cdot m^{-3} \tag{1.3.30}$$

Using the critical density, we can define now a very useful, dimensionless parameter, the ***density parameter*** Ω, by:

$$\boxed{\Omega(t) \equiv \frac{\rho}{\rho_{\text{crit}}}} \tag{1.3.31}$$

As usual, we denote the present value of the the density parameter as Ω_0. Note also, that we can define a density parameter for *each* constituent of the matter-density of the universe, for example Ω_{rad} the density parameter for the radiation content of the Universe, Ω_{DM} the density parameter for the dark matter content etc.

Now, after a little trivial algebra, we can use the density parameter to rewrite the Friedmann equation as:

$$\Omega - 1 = \frac{k}{H^2 a^2 R_0^2} \tag{1.3.32}$$

From the above equation, we can see that when $\Omega > 1$, $k = +1$ and the Universe is closed, when $\Omega < 1$, $k = -1$ and the Universe is open, and when $\Omega = 1$, $k = 0$ and the Universe is spatially flat.

Table 2: **Relation between ρ,Ω and the Geometry of the Universe**

Density	**Density Parameter**	**Curvature**	**Geometry**
$\rho < \rho_{\text{crit}}$	$\Omega < 1$	$k = -1$	Open
$\rho = \rho_{\text{crit}}$	$\Omega = 1$	$k = 0$	Flat
$\rho > \rho_{\text{crit}}$	$\Omega > 1$	$k = +1$	Closed

Another very useful cosmological parameter also exists. It's called ***deceleration parameter*** and is given by:

$$q(t) = -\frac{a\ddot{a}}{\dot{a}^2} \tag{1.3.33}$$

Deceleration parameter measures the *rate of change of the rate of expansion*. It's a very useful parameter, but since it's not important for the scope of this thesis we will neither prove it (it comes from the Taylor expansion of the scale factor about the present time) nor underling its observational applications.

1.3.4 Solution of the Friedmann Equations and Simple Cosmological Models

Since we have found the equations governing the evolution of the scale factor, given the matter-energy content of the Universe, we can solve them for some *simple* but *important* cases. We will give here solutions for the case when $k = 0$, that is the Universe is spatially flat. We believe that the Universe is indeed flat so the above is not a mere simplification. In a subsequent chapter we will need the solution of the Friedmann equation for a closed Universe. Because the solution is not so trivial and is given in a parametric form, we will give the solution there.

We are starting with the fluid equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p\right)$$

and the equation of state $p = w\rho$. Substituting this, back to the above, we have:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + w\rho\right) = 0 \Rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}\rho\left(1 + w\right) = 0 \Rightarrow$$

$$\Rightarrow \frac{\dot{\rho}}{\rho} = -3\left(1 + w\right)\frac{\dot{a}}{a} \tag{1.3.34}$$

Assuming that w is constant we can integrate the above equation to obtain:

$$\ln\rho = -3\left(1 + w\right)\ln a + const. \Rightarrow \rho \propto a^{-3(1+w)}$$

And finally, using that $a(t_0) \equiv a_0 = 1$, we have:

$$\boxed{\rho(t) = \frac{\rho_0}{a^{3(1+w)}}} \tag{1.3.35}$$

Where $\rho_0 \equiv \rho(t_0)$, as usual. So, we have now a relation between the evolution of the density and the scale factor. Together with the first Friedmann equation, we have two equations for two unknown functions, so we can solve the system of equations.

We will consider the cases where the Universe is flat and is fully dominated by each one of its three main ingredients:*matter*, *radiation* and *dark energy*. We will find the evolution of the scale factor and the Hubble parameter in each case.

A matter-dominated Universe: In this case we have only matter, $\Omega_{\rm M} = 1$, (dark matter and luminous matter content), and since for pressure-less matter $w = 0$, we have:

$$\rho_{\rm M}(t) = \frac{\rho_{\rm M,0}}{a^3(t)} \tag{1.3.36}$$

Now we can substitute this to the first Friedmann equation (with $k = 0$) and we have:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho(t) \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\frac{\rho_{\rm M,0}}{a^3} \Rightarrow$$

$$\Rightarrow a\,\dot{a}^2 = \frac{8\pi G_N}{3}\rho_{\rm M,0} \Rightarrow \sqrt{a}\,\dot{a} = \sqrt{\frac{8\pi G_N}{3}\rho_{\rm M,0}} \Rightarrow$$

$$\Rightarrow (\dot{a^{3/2}}) = \frac{3}{2}\sqrt{\frac{8\pi G_N}{3}\rho_{\rm M,0}}$$

Integrating (and remembering that $t \to 0 \Leftrightarrow a \to 0$) we have that:

$$a^{3/2} \propto t \Rightarrow a(t) \propto t^{2/3}$$

And finally, using the convention that $a(t_0) = 1$, we have:

$$\boxed{a(t) = \left(\frac{t}{t_0}\right)^{2/3}} \tag{1.3.37}$$

And the Hubble parameter takes the form:

$$H(t) = \frac{2}{3t} \tag{1.3.38}$$

A radiation-dominated Universe: We take now the case when we have only radiation, $\Omega_{\rm Rad} = 1$. As we have said, for radiation, $w = \frac{1}{3}$, so we have:

$$\rho_{\rm Rad}(t) = \frac{\rho_{\rm Rad,0}}{a^4(t)} \tag{1.3.39}$$

And substituting, as before, in the first Friedmann equation we have the time-dependence of the scale factor in a radiation-dominated Universe:

$$\boxed{a(t) = \left(\frac{t}{t_0}\right)^{1/2}} \tag{1.3.40}$$

And the Hubble parameter becomes:

$$H(t) = \frac{1}{2t} \tag{1.3.41}$$

A vacuum-dominated Universe: Finally we consider the case of a vacuum-dominated Universe, that is a Universe where the only ingredient is the dark energy (usually best described through the existence of a positive cosmological constant, Λ). As we have seen for the dark energy component we assume that $w = -1$, so we have:

$$\rho_\Lambda(t) = \rho_{\Lambda,0} \tag{1.3.42}$$

which means that the density of dark energy has the bizarre property to remain constant, even if the background Universe is expanding. Again we substitute this to the first Friedmann equation. We take:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho_\Lambda \Rightarrow \left(\frac{\dot{a}}{a}\right) = \sqrt{\frac{8\pi G_N}{3}\rho_\Lambda}$$

The density of the dark energy (in the form of a cosmological constant) is usually defined as:

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G_N} \tag{1.3.43}$$

Using this definition, we rewrite the above equation as:

$$\frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}}$$

And, after integration, we have that:

$$a(t) \propto e^{\sqrt{\frac{\Lambda}{3}}\,t}$$

And, using the usual constraint, $a(t_0) = 1$, we finally have that:

$$\boxed{a(t) = e^{\sqrt{\Lambda/3}\,(t-t_0)}} \tag{1.3.44}$$

while the Hubble parameter is:

$$H(t) = \sqrt{\frac{\Lambda}{3}} \tag{1.3.45}$$

which is not a great surprise, since we started from this equation to derive the time-dependence of the scale factor-see above!

We have considered the cases of Universes fully made of one constituent. Clearly this is not the case with our Universe. We know that our Universe contains radiation, matter and dark energy. So, why we spent time and effort to present models that seem to be irrelevant with the real Universe?

Let us write again the first Friedmann equation, but now let as write the total density ρ as the sum of the densities of the three main ingredients of the Universe. Also, let us assume from the beginning that the Universe is flat. So the Friedmann equation is written as:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\left(\rho_M + \rho_{Rad} + \rho_\Lambda\right)$$

and using eqs. (1.3.36),(1.3.39) & (1.3.42) we have that:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\left(\frac{\rho_{M,0}}{a^3} + \frac{\rho_{Rad,0}}{a^4} + \rho_{\Lambda,0}\right) \tag{1.3.46}$$

Since $a \to 0$ as $t \to 0$, at very early times in the cosmic history the dominant term is clearly that of the radiation, and we have, at a good level of accuracy, a *radiation-dominated Universe* as that we described before. On the other hand, at late times, $a \to \infty$ as $t \to \infty$, and we have a *vacuum-dominated Universe*. There also exists a time when the main constituent of the Universe is the pressure-less matter, and we have an *matter-dominated Universe*. Of course, there also exist times, as nowadays, when the densities of two ingredients are of the sameorder of magnitude and we cannot neglect one of them.

It's very common, in many books of cosmology, at this point, to present a few things about measuring distances in the Universe, and other elements of observational cosmology. Since this thesis isn't dedicated to observational work, we will avoid a long discussion about, for example, luminosity distance, angular diameter distance, particle horizons etc. We will only give, without proving it, a very useful relation, connecting the redshift of an object which is observed now, with the scale factor of the Universe when its light was emitted. This relation is:

$$\boxed{a_{\rm em} = \frac{1}{1 + z_{\rm em}}} \tag{1.3.47}$$

Thus, for example, a photon from an object with redshift 5 was emitted when the Universe was 6 times smaller than today.

1.4 The Hot Big Bang Cosmology

So far we have used GR and some vary basic observational properties (such as the homogeneity and isotropy of space, or the Hubble's law and the discovery of cosmic expansion) to derive general models that they can be applied in many different cases. Here we will describe the Standard Cosmological Model of Hot Big Bang, which is equally based in theory and observational facts. We will also review the main problems of the Model and we will discuss the most popular proposal which solves them, and also gives the necessary initial anisotropies to begin the structure formation.

1.4.1 Relics of the Big Bang I: Cosmic Background Radiation

Cosmic Microwave Background Radiation (CMB) is the isotropic microwave radiation coming to us from every part of the sky, which corresponds to a (perfect) black-body of temperature:

$$T_0 = 2.725 \pm 0.001 \text{ K}$$

CMB is a relic of the hot past of the Universe.The remaining glow of the Bang! Now, we have shown that the density of the radiation evolves as:

$$\rho_{\rm Rad}(t) \propto \frac{1}{a^4}$$

Now, the total energy density of the radiation of a black-body of temperature T is:

$$\epsilon_{\text{Rad}} \equiv \rho_{\text{Rad}} c^2 = \alpha T^4 \tag{1.4.1}$$

where α is the *radiation constant*. Combining the above we get:

$$(a \cdot T)^4 = const. \Rightarrow T(t) \propto \frac{1}{a(t)},$$

And using again the convention $a(t_0) = 1$, we have:

$$\boxed{T(t) = \frac{T_0}{a(t)}} \tag{1.4.2}$$

where $T_0 \equiv T(t_0)$ is the temperature of the black-body, today. Since now the temperature of the Universe is about 3 K, we conclude that in the Universe in the past was much hotter.

It's easy to see that as the Universe evolves, cools but the thermal distribution remains that of a black-body, of a lower temperature. The *black-body distribution* or *Planck's distribution* is given by:

$$\epsilon(f)\, df = \frac{8\pi h}{c^3} \frac{f^3}{exp\,(hf/k_B T) - 1}\, df \tag{1.4.3}$$

where h is the *Planck's constant*. As the Universe expands, the frequency f reduces as $f \propto 1/a$. So the denominator remains unchanged, while the numerator reduces as $1/a^3$, which is exactly what the energy density has to do as the Universe expands. So the form of the black-body distribution remains.

What's the origin of CMB? As we will see later, the Universe after about 10^{12} seconds was contained atomic nuclei, mainly free Hydrogen nuclei (protons) and a few Helium nuclei, free electrons and neutrinos, and a lot of photons. Hydrogen has a minimum ionization energy of 13.6 eV. That means that in order to ionize the atom (to kick off its electron) we must give to it energy of 13.6 eV or above.

When the Universe was very very hot, let's say 1 000 000 K, the product $k_B T$ was of the order of 100 eV, which means that almost all the photons had the necessary energy to ionize the hydrogen atoms immediately, as they tried to form. Then the Universe was a sea of free electrons and nuclei, and photons had a very short mean free path because they interacted very strongly with the electrons via Thompson scattering. But as the Universe expands, it cools and sometime the free electrons and nuclei unite to form neutral atoms. The Universe, then, becomes suddenly transparent and the photons are able to travel freely afterwards. This process is known as ***decoupling***.

To calculate the temperature of the Universe at decoupling we can make the crude estimation that decoupling happened when the mean photon energy (which for black-body distribution is $\cong 3k_B T$) is equal to the ionization energy of the Hydrogen, which gives:

$$T \cong \frac{13.6\text{eV}}{3k_B} \cong 50\,000\,\text{K} \tag{1.4.4}$$

This is an overestimation. A more detailed treatment, which -among other-takes into account that the number of photons is much greater than the number of electrons/nuclei gives:

$$T_{\text{dec}} \cong 3\,000\,\text{K} \tag{1.4.5}$$

which gives the scale factor the time of decoupling $a_{\text{dec}} \cong 1/1000$, which means that decoupling took place when the Universe was 1 000 times smaller. The age of the Universe at decoupling was about 350 000 yrs.

Observations of the last decades have shown that the Cosmic Microwave Background Radiation has indeed the form of a black-body spectrum (indeed is the most perfect black-body ever found), it's very isotropic with small anisotropies of the order of $\frac{\delta T}{T} \cong 10^{-5}$.

1.4.2 Relics of the Big Bang II: Primordial Nucleosynthesis

The abundance of the light elements (Deuterium, Helium, Lithium...) in the Universe, provides very strong evidence in favour of the Hot Big Bang theory. The heavier elements are produced inside the hot cores of stars. But for the lightest elements the hot past of the Universe, together combined with the rate of its cooling were enough to allow them to be formed.

Here we will consider the production of Helium which is the most stable element (and the most abundant, after Hydrogen). You may note that:

- $m_{\text{p}}c^2 = 938.3$ MeV and $m_{\text{n}}c^2 = 939.6$ MeV
- Free neutrons don't have infinite lifetime; they decay into protons instead. Half life: $t_{half} = 614$ sec.
- Neutrons bound into an atomic nucleus are stable.

We consider the case before nuclei form, but when the protons and neutron are non-relativistic, which means when $k_{\text{B}}T \ll m_{\text{p}}c^2$. Then, the number density of particles, N, is described by the Maxwell-Boltzmann distribution, so:

$$N \propto m^{3/2} \exp\left(-\frac{mc^2}{k_{\text{B}}T}\right) \tag{1.4.6}$$

The relative densities of neutrons and protons, will be then:

$$\frac{N_{\text{n}}}{N_{\text{p}}} = \left(\frac{m_{\text{n}}}{m_{\text{p}}}\right)^{3/2} \exp\left(-\frac{(m_{\text{n}} - m_{\text{p}})c^2}{k_{\text{B}}T}\right) \cong \exp\left(-\frac{(m_{\text{n}} - m_{\text{p}})c^2}{k_{\text{B}}T}\right) \tag{1.4.7}$$

The mass difference between the neutron and proton is 1.3 MeV. So, as long as $k_{\text{B}} \gg (m_{\text{n}} - m_{\text{p}})c^2$ we have that the number density of protons equals to that of neutrons: $N_{\text{n}} = N_{\text{p}}$.

Reactions converting neutrons to protons and vice versa (such $n + \nu_e \leftrightarrow p + e^-$) occur until $k_{\text{B}}T \cong 0.8$ MeV. Then, the relative densities of neutrons and protons are:

$$\frac{N_{\text{n}}}{N_{\text{p}}} \cong \exp\left(-\frac{1.3\text{MeV}}{0.8\text{MeV}}\right) \cong \frac{1}{5} \tag{1.4.8}$$

From that time only the decay of neutron changes the relative abundances. The process of nuclear fusion (when the energy/temperature of the Universe is low enough that cannot dissociate the formed nuclei) starts at an energy of 0.1 MeV. In that time interval (from the moment the Universe has energy 0.8 MeV until it has energy 0.1 MeV) the number density of neutrons falls. The (final) relative number density becomes:

$$\frac{N_{\rm n}}{N_{\rm p}} \cong \frac{1}{8} \tag{1.4.9}$$

Now, since every helium nucleus contains 2 neutrons (hydrogen has not neutrons), all the remaining neutrons end up in helium and the number density of He-4 is $N_{\text{He-4}} = N_{\rm n}/2$. Since each helium nucleus weights about four proton masses, the fraction of the total mass in the Universe in the form of helium-4, known as Y_4 is:

$$Y_4 \equiv \frac{2N_{\rm n}}{N_{\rm n}+N_{\rm p}} = \frac{2}{1+\frac{N_{\rm p}}{N_{\rm p}}} \cong 0.22 \tag{1.4.10}$$

We found that, according to this simple model, 22% of the (baryonic) matter in the Universe is in the form of helium-4. This is the mass fraction; since He-4 is four times heavier than the hydrogen, it means that there is one He-4 nucleus in the Universe for every 12 H nuclei. This prediction is consistent with the observations; indeed this ***primordial nucleosynthesis*** and the consistence between theory and observation provides a very convincing evidence for the Hot Big Bang.

1.4.3 Problems with the Hot Big Bang Model

Despite its success and the strong observational support, there are also some problems with the Hot Big Bang model. We will describe the two most important problems, namely the *flatness* and the *horizon* problem.

The Flatness Problem: We have given in eg. (1.3.32) a form of the Friedmann equation:

$$\Omega(t) - 1 = \frac{k}{R_0^2 a^2 H^2}$$

We know that Ω_0 is very close to 1 (1.002 ± 0.011). From the above equation we can deduce that in order to have today a density parameter so close to unity, in the past the density parameter should be even closer to unity. Let' s suppose that at a moment in the early Universe, we can neglect the curvature term. The evolution of the product $a(t)H^2(t)$ is:

$$a^2H^2 \propto t^{-1} \text{ in the case of radiation domination}$$

$$a^2H^2 \propto t^{-2/3} \text{ in the case of matter domination}$$

So we can easily see how the difference $\Omega(t) - 1$ evolves:

$$\Omega(t) - 1 \propto t \text{ in the case of radiation domination} \tag{1.4.11}$$

$$\Omega(t) - 1 \propto t^{2/3} \text{ in the case of matter domination} \tag{1.4.12}$$

You can see that in either case, as time goes by, the difference of the density parameter from unity increases. It's easy to prove that, in order to have the value of Ω_0 so close to 1 now, as given before, at the time of nucleosynthesis ($t \cong 1$ sec) should be:

$$|\Omega(t_{\rm nucl}) - 1| \leq 10^{-18} \tag{1.4.13}$$

which means that, at the time of nucleosynthesis:

$$0.999999999999999999 \leq \Omega(t_{\rm nucl}) \leq 1.000000000000000001 \text{ !!!!!!!}$$

So we face two possibilities: either the value of Ω is fixed to 1 from the beginning of the Universe or a mechanism exists, which is responsible for the apparent flatness of the Universe.

The Horizon Problem: We have said that the CMB is isotropic with very small anisotropies, of the order 1 to 100 000. Even when we observe opposite sides of the sky, we get electromagnetic radiation which corresponds to black-body radiation of temperature 2.725 K. We conclude that, in order to have the same temperature all the observed regions should have been casually connected before the time of decoupling.
But as we said CMB from opposite sides of the sky just reach us now, so they couldn't have interacted before. Even worse, since photons before the time of decoupling had very short mean free paths, the size of the "observable Universe" at decoupling was small, so there was no way for distant regions to had casually interacted. A possible explanation is that we had such initial conditions, a very isotropic temperature field, from the very first moments of the Universe. But such an explanation is unphysical and we need something better...

Apart from these two main problems, other problems arise too. Theories of particle physics predict the existence of monopoles in such an abundance that is inconsistent with observations (which haven't confirmed yet the existence of the monopole). We will see in the next subsection how a -then- young physicist tried to solve this problem and found a possible solution not only for this but also for the two main problems which we described before.

1.4.4 How Cosmic Inflation Solves the Problems

In 1981 Alan Guth, then a young physicist, proposed inflation as a *possible* solution to all these problems. By the term "inflation" we mean a period of time in the cosmic history, when the evolution of the scale factor is accelerating, so:

$$\text{Inflation} \Leftrightarrow \ddot{a}(t) > 0 \tag{1.4.14}$$

This, using the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{1.4.15}$$

means:

$$p < -\frac{\rho c^2}{3} \tag{1.4.16}$$

i.e. implies the existence of a constituent with negative pressure. We have seen the *cosmological constant* which has this property, and as expected the

Friedmann equation with a cosmological constant gives an exponential solution, eq. (1.3.44). The crucial thing, according to the inflationary model, is that this inflation must come to an end, somehow, after a very short time interval after its starting.

Let's see now how inflation solves the problems mentioned before. We start from the ***flatness problem***. Remember the form of the Friedmann equation:

$$\Omega(t) - 1 = \frac{k}{R_0^2 a^2 H^2}$$

We can see immediately that when the scale factor grows exponentially, we have:

$$\Omega(t) - 1 \to 0 \Rightarrow \Omega(t) \to 1 \tag{1.4.17}$$

So, the introduction of an era of exponential growth solves the flatness problem. It forces the density parameter of the Universe to go very close to unity during that era, end the subsequent evolution of the Universe change this only a little. In other words inflation predicts that (if we allow the present day Universe to have a cosmological constant) that: $\Omega_{\rm m} + \Omega_\Lambda = 1$.

Let's face now the ***horizon problem***. Inflation solves also this problem, because it predicts that an initially very small region, small enough to be every point in casual connection with every other point, expands -in a very short time- very much. The whole observable Universe -according to inflation- comes from a tiny region, small enough to achieve thermalization before inflation.

Finally, inflation solves the problem of the relics (monopoles) because in an exponentially expanding Universe the number density of such relics falls very very rapidly.

We talked about inflation, but what is the physical motivation behind it? Are there any possible mechanism to produce it? This mechanism can be found in theories of particle physics known as Grand Unified Theories (because they try to describe electromagnetism, weak nuclear force and strong nuclear in a unified way) or, more recently, supersymmetry. It' is believed that *phase transitions*, controlled by scalar fields during the era of grand unification can provide the possible mechanism behind inflation. It's necessary to note that, even today, besides its successes, inflation is only a possibility (even a widely accepted possibility) and that there is not a *unique* inflationary model, but rather a class of such models.

Closing, let's note -it will become more clear later- that inflation -a quantum process- predicts the initial inhomogeneities in the density of the Universe, which lead to the cosmic structures we see today...

Chapter 2

The Inhomogeneous Universe: Linear Theory, Statistical Treatment and Mass Functions

2.1 Prelude: The Press-Schechter Mass Function

In the first chapter we described the Universe at its largest scales: at those scales where the assumptions of perfect homogeneity and isotropy are valid. But, as we know, at smaller scales the Universe is far from being homogeneous or isotropic: stars, galaxies, clusters and super-clusters of galaxies exist. We noted that quantum fluctuations during the inflationary era are probably responsible for the creation of initially small inhomogeneities, which, under the effect of the gravitational attraction grew and formed all those structures that fill the Universe.

Numerical simulations, using powerful supercomputers, are able to reproduce the process of structure formation by directly solving the equations of motion of a big sample of particles initially distributed almost -but not entirely-homogeneously. Despite the great value of those simulations, it's also useful -and one wants to have- an analytical or semi-analytical treatment of the structure formation.

One very frequently used concept in the theory of structure formation is the ***mass function*** of cosmic structures, defined by the relation:

$$dn = n(m,t)\,dm, \tag{2.1.1}$$

which is the number of structures in question with masses in the range $[m, m+dm]$ per (comoving) volume. Usually one speaks about *dark matter halos* and the relevant mass function of dark matter halos. This is because, as we have seen, it's believed that dark matter is the main ingredient of the matter content of the Universe. The mass function can be computed from the simulations, but for over forty years a lot of effort has been given in the direction of finding an analytical expression for the mass function based on a theoretical model. In 1974 two theoretical astrophysicists, William Press and Paul Schechter, using some crude assumptions and treating the Universe as a Gaussian random field of density fluctuations, presented such an expression for the mass function:

$$\boxed{\frac{dn}{dm}(m,a)\,dm = \sqrt{\frac{2}{\pi}}\frac{\rho_{\mathrm{m},0}}{m^2}\frac{\tilde{\delta}_{0,\mathrm{c}}(a)}{\sigma(m)}\left|\frac{d\,\ln\sigma(m)}{d\,\ln m}\right|\exp\left[-\frac{\tilde{\delta}^2_{0,\mathrm{c}}(a)}{2\sigma^2(m)}\right]\,dm.} \tag{2.1.2}$$

This is the well-known ***Press-Schechter mass function***, and it is a function of mass and cosmic time (through the scale factor, a). In this form, it gives the comoving number density of *virialized* objects dor every cosmic epoch. It has a good agreement with the results of numerical simulations, and although there are some improvements of it (such the Seth & Tormen mass function), it still remains valid today.

In what follows we will present the theory of structure formation, and especially those concepts which are necessary to understand and to derive the Press-Schechter (PS) mass function. We will give the main aspects of the *linear theory of perturbations* very briefly, the statistical treatment of the Universe as an overdensity field, quite in detail, and we will derive the PS mass function in two different ways: using the original assumptions of Press and Schechter and using the theory of Random Walks, also known as the *Excursion Set Theory* of halo mass functions.

The spherical collapse model is studied in detail in the next chapter, although we will use its results in what follows, especially the critical value for collapse, in the linear field of overdensities. Finally we will use the theory developed here and in the next chapter, together with some extensions for the concordance cosmology (i.e the $\Omega_\mathrm{m} + \Omega_\Lambda = 1$ cosmology), to derive interesting results about the end of the growth of structure in that cosmology.

2.2 The Linear Theory of Perturbations

2.2.1 Introduction

As we mentioned before, the Large Scale Structure we see today started with very small initial deviations from the homogeneous and isotropic model and grew by gravitational instability. These deviations can be treated as small perturbations around the smooth background, and we can keep only the terms of first order in perturbation quantities. The relevant theory is called the "linear theory" of perturbations and the regime where the linear theory is valid is called the "linear regime".

In other words, if $\rho(\tilde{\mathbf{x}})$ is the density distribution of matter in location $\tilde{\mathbf{x}}$, and if we define the ***overdensity*** field as:

$$\boxed{\delta(t,\tilde{\mathbf{x}}) = \frac{\rho(t,\tilde{\mathbf{x}}) - \overline{\rho}(t)}{\overline{\rho}(t)},} \tag{2.2.1}$$

then the linear theory is valid, when $|\delta| \ll 1$. Here $\tilde{\mathbf{x}}$ represents the *comoving coordinates* and $\overline{\rho}(t)$ denotes the mean density of the background Universe. Even if today the Universe is clearly in the non-linear regime, at least on small scales, we will see that using the linear theory, which is quite simple, we can extract useful information for the formation of the non-linear collapsed structures.

In what follows we give a very short, and simplistic, introduction to the linear theory of perturbations. We will present only the basic aspects of a much greater topic, only those aspects necessary to understand later the statistical theory of the overdensity field and its evolution, and to the derivation of the Press-Schechter mass function.

2.2.2 Newtonian Hydrodynamical Equations

Even if the theory should be worked out in the framework of general relativity, since the inhomogeneities are "small" (much smaller than the typical scale of the Universe) we can, in a very good approximation, neglect the effects of curvature and the finite speed of light and work within the Newtonian framework.

We can describe the matter content of the Universe (mainly Dark Matter) as an ideal fluid, with matter density $\rho(t, \vec{\mathbf{r}})$, velocity field $\tilde{v}(t, \vec{\mathbf{r}})$, pressure $p(t, \vec{\mathbf{r}})$ gravitational potential $\Phi(t, \vec{\mathbf{r}})$ and entropy per unit mass $S(t, \vec{\mathbf{r}})$, where $\vec{\mathbf{r}}$ are the physical coordinates. The basic hydrodynamical equations for these quantities are:

- **Continuity equation** (or mass conservation equation):

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}_r \cdot (\rho \vec{v}) = 0 \tag{2.2.2}$$

- **Euler equation** (or conservation of momentum equation):

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}_r)\vec{v} + \frac{1}{\rho}\vec{\nabla}_r p + \vec{\nabla}_r \Phi = 0 \tag{2.2.3}$$

- **Poisson equation** for the Newtonian gravitational potential Φ:

$$\nabla_r^2 \Phi = 4\pi G_N \rho \tag{2.2.4}$$

- **Conservation of entropy equation:**

$$\frac{\partial S}{\partial t} + (\vec{v} \cdot \vec{\nabla}_r) S = 0 \tag{2.2.5}$$

These equations, together with the *equation of state* $p = p(\rho, S)$, form a closed system of equations for the unknown functions $\rho, \vec{v}, p, \Phi$ and S. We explicitly expressed the symbol ∇ as ∇_r, to point the fact that the derivatives are derivatives in the physical coordinate system. For reasons we will not explain here, we can neglect the effects of the entropy in the process of structure formation, so we can neglect the last equation (conservation of entropy equation) and also we can simply write $p = p(\rho)$.

The above equations are non-linear and it's difficult to find a general solution of them. But we can solve them at first (or *linear*) order, by splitting each quantity into two parts: the quantity for the homogeneous background (these denoted by a bar over them) and a small inhomogeneous perturbation, indicated by a δ before it. So we set:

$$\begin{aligned}
&\rho(t,\vec{\mathbf{r}}) = \overline{\rho}(t) + \delta\rho(t,\vec{\mathbf{r}}), \quad \vec{v}(t,\vec{\mathbf{r}}) = \overline{\vec{v}}(t,\vec{\mathbf{r}}) + \delta\vec{v}(t,\vec{\mathbf{r}}) \\
&p(t,\vec{\mathbf{r}}) = \overline{p}(t) + \delta p(t,\vec{\mathbf{r}}), \quad \Phi(t,\vec{\mathbf{r}}) = \overline{\Phi}(t,\vec{\mathbf{r}}) + \delta\Phi(t,\vec{\mathbf{r}}) \\
&S(t,\vec{\mathbf{r}}) = \overline{S}(t) + \delta S(t,\vec{\mathbf{r}})
\end{aligned} \tag{2.2.6}$$

In what follows we still work at the physical coordinate system, but for convenience we will not indicate this fact explicitly. Substituting the first two equations of (2.2.6) to the continuity equation, and keeping only first order terms in the perturbed quantities, we get:

$$\begin{aligned}
\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \quad &\Rightarrow \quad \frac{\partial(\overline{\rho}+\delta\rho)}{\partial t} + \vec{\nabla}\cdot\big((\overline{\rho}+\delta\rho)(\overline{\vec{v}}+\delta\overline{\vec{v}})\big) = 0 \Rightarrow \\
&\Rightarrow \quad \frac{\partial\overline{\rho}}{\partial t} + \frac{\partial\delta\rho}{\partial t} + \vec{\nabla}\cdot\big(\overline{\rho}\overline{\vec{v}} + \overline{\rho}\delta\vec{v} + \delta\rho\overline{\vec{v}}\big) = 0 \Rightarrow \\
&\Rightarrow \quad \frac{\partial\overline{\rho}}{\partial t} + \frac{\partial\delta\rho}{\partial t} + \vec{\nabla}\cdot\big(\overline{\rho}\overline{\vec{v}}\big) + \vec{\nabla}\cdot(\overline{\rho}\delta\vec{v}) + \vec{\nabla}\cdot\big(\delta\rho\overline{\vec{v}}\big) = 0
\end{aligned}$$

But the continuity equation is also valid for the homogeneous background Universe, i.e:

$$\frac{\partial\overline{\rho}}{\partial t} + \vec{\nabla}\cdot\big(\overline{\rho}\overline{\vec{v}}\big) = 0$$

So we have:

$$\begin{aligned}
&\frac{\partial\delta\rho}{\partial t} + \vec{\nabla}\cdot(\overline{\rho}\delta\vec{v}) + \vec{\nabla}\cdot\big(\delta\rho\overline{\vec{v}}\big) = 0 \Rightarrow \\
\Rightarrow\ &\frac{\partial\delta\rho}{\partial t} + \overline{\rho}\vec{\nabla}\cdot\delta\vec{v} + \delta\rho\vec{\nabla}\cdot\overline{\vec{v}} + \overline{\vec{v}}\cdot\vec{\nabla}\delta\rho = 0
\end{aligned} \tag{2.2.7}$$

Now we can do the same for the Euler equation, using the expressions (2.2.6) for $\vec{v}$, ρ and Φ. So we write:

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} + \frac{1}{\rho}\vec{\nabla}p + \vec{\nabla}\Phi = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial(\overline{\vec{v}}+\delta\vec{v})}{\partial t} + \Big((\overline{\vec{v}}+\delta\vec{v})\cdot\vec{\nabla}\Big)(\overline{\vec{v}}+\delta\vec{v}) + \frac{1}{\overline{\rho}+\delta\rho}\vec{\nabla}(\overline{p}+\delta p) + \vec{\nabla}(\overline{\Phi}+\delta\Phi) = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial\overline{\vec{v}}}{\partial t} + \frac{\partial\delta\vec{v}}{\partial t} + \Big(\overline{\vec{v}}\cdot\vec{\nabla} + \delta\vec{v}\cdot\nabla\Big)(\overline{\vec{v}}+\delta\vec{v}) + \frac{1}{\overline{\rho}}\left(1-\frac{\delta\rho}{\overline{\rho}}\right)\Big(\vec{\nabla}\overline{p} + \vec{\nabla}\delta p\Big) + \vec{\nabla}\overline{\Phi} + \vec{\nabla}\delta\Phi = 0$$

Then we use the following:

- The homogeneous background Universe also obeys the Euler equation:

$$\frac{\partial\overline{\vec{v}}}{\partial t} + (\overline{\vec{v}}\cdot\vec{\nabla})\overline{\vec{v}} + \frac{1}{\overline{\rho}}\vec{\nabla}\overline{p} + \vec{\nabla}\overline{\Phi} = 0$$

- We keep only terms of first order in the perturbed quantities.

- We use that:
$$\delta p = c_s^2 \delta\rho,$$
with $c_s^2 = \partial\overline{p}/\partial\overline{\rho}$, the square of the speed of sound.

To obtain the linearized Euler equation:

$$\frac{\partial \delta\vec{v}}{\partial t} + \left(\overline{\vec{v}} \cdot \vec{\nabla}\right)\delta\vec{v} + \left(\delta\vec{v} \cdot \vec{\nabla}\right)\overline{\vec{v}} + \frac{c_s^2}{\overline{\rho}}\vec{\nabla}\delta\rho + \vec{\nabla}\delta\Phi = 0 \tag{2.2.8}$$

Finally we do the same for the Poisson equation:

$$\nabla^2\Phi = 4\pi G_N \rho = 0 \Rightarrow \nabla^2\left(\overline{\Phi} + \delta\Phi\right) = 4\pi G_N\left(\overline{\rho} + \delta\rho\right) = 0 \Rightarrow$$
$$\Rightarrow \nabla^2\delta\Phi = 4\pi G_N \delta\rho = 0 \tag{2.2.9}$$

where we have used that the background Universe also obeys the Poisson Equation.

Equations (2.2.7), (2.2.8) and (2.2.9) are the hydrodynamical equations for the perturbation quantities. We are going to transform them. First we will convert to ***comoving coordinates***: $\vec{\mathbf{x}} = \vec{\mathbf{r}}/a$, where a is the *scale factor*. Note that the background homogeneous velocity field is given by the *Hubble Law*:

$$\overline{\vec{v}}(t, \vec{\mathbf{r}}) = H(t)\vec{\mathbf{r}} \tag{2.2.10}$$

since:
$$\vec{v} \equiv \dot{\vec{\mathbf{r}}} = \dot{a}\vec{\mathbf{x}} + a\dot{\vec{\mathbf{x}}} = \overline{\vec{v}} + \delta\vec{v}$$

Remember that in eqs. (2.2.7)-(2.2.9) $\vec{\nabla} \equiv \vec{\nabla}_r$, i.e. we are in the physical coordinate system. To pass to the comoving coordinate system, we have to make the following transformations:

$$\vec{\nabla}_r = \frac{1}{a}\vec{\nabla}_x, \qquad \left.\frac{\partial}{\partial t}\right|_r = \left.\frac{\partial}{\partial t}\right|_x - \frac{1}{a}\overline{\vec{v}} \cdot \vec{\nabla}_x \tag{2.2.11}$$

This will be the first part of the transformation. We will also decompose the perturbation quantities into plane waves, i.e we will make a ***Fourier transform*** with respect to comoving coordinates. So, for any perturbation quantity δq:

$$\delta q\left(t, \vec{\mathbf{x}}\right) = \frac{1}{(2\pi)^3}\int \delta q(t, \vec{\mathbf{k}}) e^{+i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}\, d\vec{\mathbf{k}}, \quad \delta q(t, \vec{\mathbf{k}}) = \int \delta q(t, \vec{\mathbf{x}}) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}\, d\vec{\mathbf{x}} \tag{2.2.12}$$

Remember that the Fourier transform of the derivative of a function is simply the Fourier transform of the function, multiplied by ik or $i\vec{\mathbf{k}}$. So, when we will take the Fourier transforms of the equations we will replace:

$$\vec{\nabla}_x \to i\vec{\mathbf{k}}, \quad \nabla_x^2 \to -k^2 \tag{2.2.13}$$

with $k = |\vec{\mathbf{k}}|$.

Let's make a summary: we will take eqs. (2.2.7)-(2.2.9), we will, at first, pass to the comoving coordinate system, and then we will take the Fourier transform of the equations. For simplicity, from now on: $\vec{\nabla}_x \equiv \vec{\nabla}$.

We will present in detail the calculations for the *continuity equation*. First of all, we pass to the comoving coordinates:

$$\frac{\partial\delta\rho}{\partial t} + \overline{\rho}\vec{\nabla}_r \cdot \delta\vec{v} + \delta\rho\vec{\nabla}_r \cdot \overline{\vec{v}} + \overline{\vec{v}} \cdot \vec{\nabla}_r\delta\rho = 0 \longrightarrow \ \{\text{using (2.2.11) }\}$$

$$\longrightarrow \frac{\partial \delta\rho}{\partial t} - \frac{1}{a}\overline{\vec{v}}\cdot\vec{\nabla}\delta\rho + \frac{\overline{\rho}}{a}\vec{\nabla}\cdot\delta\vec{v} + \frac{\delta\rho}{a}\vec{\nabla}\cdot\overline{\vec{v}} + \frac{1}{a}\overline{\vec{v}}\cdot\vec{\nabla}\delta\rho = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial \delta\rho}{\partial t} + \frac{\overline{\rho}}{a}\vec{\nabla}\cdot\delta\vec{v} + \frac{\delta\rho}{a}\vec{\nabla}\cdot\overline{\vec{v}} = 0 \tag{2.2.14}$$

Now, we use that, from eq. (2.2.10):

$$\vec{\nabla}\cdot\overline{\vec{v}} = \vec{\nabla}\cdot H\,\vec{\mathbf{r}} = \vec{\nabla}\cdot(H a\,\vec{\mathbf{x}}) = (\vec{\nabla}\cdot\vec{\mathbf{x}})\,aH = 3a\,H, \tag{2.2.15}$$

to finally obtain:

$$\delta\dot{\rho} + 3H\delta\rho + \frac{\overline{\rho}}{a}\vec{\nabla}\cdot\delta\vec{v} = 0 \tag{2.2.16}$$

where, $d\delta\dot{\rho} = \delta\rho/dt$. And finally, making the Fourier transformation (using eq. (2.2.13)):

$$\delta\dot{\rho} + 3H\delta\rho + \frac{i\,\overline{\rho}\,\vec{\mathbf{k}}}{a}\cdot\delta\vec{v} = 0 \tag{2.2.17}$$

Similarly we work for the Euler and Poisson equations. Thus we finally obtain the *first order hydrodynamical equations* for a given mode $\vec{\mathbf{k}}$:

- **Continuity equation**:

$$\boxed{\delta\dot{\rho} + 3H\delta\rho + \frac{i\,\overline{\rho}\,\vec{\mathbf{k}}}{a}\cdot\delta\vec{v} = 0} \tag{2.2.18}$$

- **Euler equation**:

$$\boxed{\delta\dot{\vec{v}} + H\delta\vec{v} + \frac{i\,\vec{\mathbf{k}}}{a\,\overline{\rho}}c_s^2\delta\rho + \frac{i\,\vec{\mathbf{k}}}{a}\delta\Phi = 0} \tag{2.2.19}$$

- **Poisson equation**

$$\boxed{k^2\delta\Phi + 4\pi G_N a^2\delta\rho = 0} \tag{2.2.20}$$

2.2.3 Density Perturbations and the Growth Factor

We are going now to derive a single equation for the ***matter density contrast***, defined in the k-space as:

$$\delta(t,\vec{\mathbf{k}}) \equiv \frac{\delta\rho(t,\vec{\mathbf{k}})}{\overline{\rho}(t)} = \frac{\rho(t,\vec{\mathbf{k}}) - \overline{\rho}(t)}{\overline{\rho}(t)} \tag{2.2.21}$$

We will use the fact that the homogeneous background Universe obeys the continuity equation, which can be written as:

$$\dot{\overline{\rho}} = -3H\overline{\rho} \tag{2.2.22}$$

And also, we have:

$$\delta\rho = \overline{\rho}\delta, \;\; \delta\dot{\rho} = \overline{\rho}\dot{\delta} + \dot{\overline{\rho}}\delta \tag{2.2.23}$$

Using those equations, and also setting $\vec{\mathbf{k}}||\delta\vec{v}$,the continuity equation (2.2.18) becomes:

$$\delta\dot{\rho} + 3H\delta\rho + \frac{i\,\overline{\rho}\,k}{a}\delta\upsilon = 0 \Rightarrow \overline{\rho}\dot{\delta} + \dot{\overline{\rho}}\delta + 3H\overline{\rho}\delta + \frac{i\,\overline{\rho}\,k}{a}\delta\upsilon = 0 \Rightarrow$$

$$\Rightarrow \{\text{using eq. (22.22)}\} \quad \overline{\rho}\dot{\delta} - 3H\overline{\rho}\delta + 3H\overline{\rho}\delta + \frac{i\,\overline{\rho}\,k}{a}\delta\upsilon = 0 \Rightarrow$$

$$\dot{\delta} + \frac{i\,k}{a}\delta\upsilon = 0 \tag{2.2.24}$$

Now, we can differentiate this equation to obtain:

$$\ddot{\delta} + \frac{i\,k}{a}\delta\dot{\upsilon} - \frac{i\,k}{a^2}\dot{a}\delta\upsilon = 0 \Rightarrow \ddot{\delta} + \frac{i\,k}{a}\delta\dot{\upsilon} - \frac{i\,k}{a}H\,\delta\upsilon = 0, \tag{2.2.25}$$

and using again eq. (2.2.24): $\{\frac{i\,k}{a}\delta\upsilon = -\dot{\delta}\ \}$

$$\ddot{\delta} + \frac{i\,k}{a}\delta\dot{\upsilon} + H\dot{\delta} = 0 \tag{2.2.26}$$

We can use now the Euler equation, eq. (2.2.19) to eliminate $\delta\dot{\upsilon}$:

$$\delta\dot{\upsilon} + H\delta\upsilon + \frac{i\,k}{a\,\overline{\rho}}c_s^2\delta\rho + \frac{i\,k}{a}\delta\Phi = 0 \Rightarrow$$

$$\delta\dot{\upsilon} + H\delta\upsilon + \frac{i\,k}{a\,\overline{\rho}}c_s^2\overline{\rho}\delta + \frac{i\,k}{a}\delta\Phi = 0 \Rightarrow$$

$$\delta\dot{\upsilon} + H\delta\upsilon + \frac{i\,k}{a}\left(c_s^2\delta + \delta\Phi\right) = 0$$

with the aid of Poisson equation, we have:

$$\delta\Phi = -\frac{4\pi G_N a^2 \overline{\rho}\delta}{k^2}$$

So, substituting to the previous equation, we finally have for $\delta\dot{\upsilon}$ (also using eq. (2.2.24)):

$$\delta\dot{\upsilon} = \frac{Ha\dot{\delta}}{i\,k} + \frac{i\,k}{a}\left(\frac{4\pi G_N a^2\overline{\rho}\delta}{k^2} - c_s^2\right)\delta \tag{2.2.27}$$

We are ready to obtain the equation for the matter density contrast. Introducing eq.(2.2.27) into equation (2.2.26), we get:

$$\ddot{\delta} + H\dot{\delta} + H\dot{\delta} + \frac{i\,k}{a}\frac{i\,k}{a}\left(\frac{4\pi G_N a^2\overline{\rho}\delta}{k^2} - c_s^2\right)\delta = 0 \Rightarrow$$

$$\boxed{\Rightarrow \ddot{\delta} + 2H\dot{\delta} + \left(\frac{k^2 c_s^2}{a^2} - 4\pi G_N\overline{\rho}\right)\delta = 0} \tag{2.2.28}$$

This is a very basic equation, the equation for the ***linear theory*** of matter overdensities. In fact, this equation, since it is a *second order* ***linear*** *differential equation*, gives the name to the linear theory of perturbations.

We will explore the solutions of the previous equation. Consider the case of a static Universe, where $H = 0$, so eq. (2.2.28) becomes:

$$\ddot{\delta} + \left(\frac{k^2 c_s^2}{a^2} - 4\pi G_N \overline{\rho}\right)\delta = 0. \quad (2.2.29)$$

If the second term of the previous equation is negative, there are exponentially growing and decaying solutions, but if is positive we can write it as:

$$\ddot{\delta} + \omega_0^2 \delta, \qquad \omega_0 \equiv \sqrt{\frac{k^2 c_s^2}{a^2} - 4\pi G_N \overline{\rho}} \quad (2.2.30)$$

and it has oscillating solutions, as a result of the counteracting forces of gravity and pressure . Those oscillating solutions describe sound waves, traveling with sound speed c_s. To be positive, there must hold:

$$k \geq k_{\mathrm{J}} \equiv \frac{2\sqrt{\pi G_N \overline{\rho}}}{c_s} \quad (2.2.31)$$

k_{J} defines the ***Jeans length***, λ_{J}:

$$\lambda_{\mathrm{J}} \equiv \frac{2\pi}{k_{\mathrm{J}}} = c_s \sqrt{\frac{\pi}{G_N \overline{\rho}}} \quad (2.2.32)$$

perturbations smaller than the Jeans length oscillate, others grow or decay.

The behavior of perturbations are qualitatively the same in an expanding Universe. If we consider only modes of $k \ll k_{\mathrm{J}}$, eq. (2.2.28) becomes:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_N \overline{\rho}\delta = 0 \quad (2.2.33)$$

The general solution of the previous equation is:

$$\boxed{\delta(t, \vec{\mathbf{k}}) = \delta_+(\vec{\mathbf{k}})D_+(t) + \delta_-(\vec{\mathbf{k}})D_-(t)} \quad (2.2.34)$$

The {+} sign denotes growing modes, and the {-} sign denotes decaying modes. We are only interested in growing modes. $D_+(t)$ is called the ***linear growth factor***, and is usually normalized as $D_+(t_0) = 1$ .The linear growth factor can equally be expressed as a function of the scale factor, $D(a)$. The linear growth factor depends on the cosmic epoch and the cosmological model, i.e if we have a matter-dominated or a radiation dominated or a vacuum dominated Universe. We have:

- **Matter domination:** $D_+(a) \propto a$.
- **Radiation domination:** $D_+(a) \propto a^2$.
- **Concordance Model:** In a Universe with present day matter density parameter $\Omega_{\mathrm{m},0}$ and dark energy density parameter $\Omega_{\Lambda,0}$ the expression for the growth factor is more complicated. We have:

$$D_+(a) = A\left[\left(2\frac{\Omega_{\Lambda,0}}{\Omega_{\mathrm{m},0}}\right)^{1/3} a\right] \quad (2.2.35)$$

with:

$$A(x) = \frac{(x^3+2)^{1/2}}{x^{3/2}} \int_0^x \left(\frac{u}{u^3+2}\right)^{3/2} du \quad (2.2.36)$$

2.3 Statistics of the Overdensity Field

2.3.1 The Cosmological Density Field

We know that the overdensity field, $\delta(t, \vec{\mathbf{x}})$, contains all the information about the structure formation in the Universe. We also know that theoretical models (quantum fluctuations during inflation, for example) only allow to predict statistical properties of the cosmological fields. So we want to describe the cosmological overdensity field without having to specify the actual value of δ at each location in space-time $(t, \vec{\mathbf{x}})$.

As we have seen, we can decouple the spatial from the time dependence of the overdensity field. In what follows we focus on the spatial part $\delta(\vec{\mathbf{x}})$, at a specific time, t. We have told that $\delta(\vec{\mathbf{x}})$ is the outcome of some *stochastic process*, so it can normally be described in the language of ***random fields***. A random field $f(\vec{\mathbf{x}})$ is a field which at each point, $\vec{\mathbf{x}}$, $f(\vec{\mathbf{x}})$ is a random number. Our aim is to find the probability distribution:

$$\mathcal{P}(\delta_1, \delta_2, \ldots, \delta_N) d\delta_1 d\delta_2 \ldots d\delta_N \tag{2.3.1}$$

where $\delta_1 \equiv \delta(\vec{\mathbf{x}}_1)$, etc. A random field is specified by its *statistical moments*:

$$\langle \delta_1^{l_1}, \delta_2^{l_2}, \ldots, \delta_N^{l_n} \rangle \equiv \int \delta_1^{l_1} \delta_2^{l_2} \ldots \delta_N^{l_n} \, \mathcal{P}(\delta_1, \delta_2, \ldots, \delta_N) \, d\delta_1 d\delta_2 \ldots d\delta_N \tag{2.3.2}$$

where $\langle \ldots \rangle$ denotes the *ensemble average* of the stochastic process. That means that we take an average over an ensemble of universes with the same statistical properties. But if we introduce the mathematical property of ***ergodicity***, which says that for a random field the volume average goes to ensemble average as the survey volume goes to infinity, we can compute (and refer to) the statistical moments as a volume average in the overdensity field of our Universe (which a realization of the random process).

Note that in order to be compatible with the properties of *homogeneity* and *isotropy*, the statistical moments of the overdensity field must be invariant under translations or rotations. Usually the overdensity field of the Universe is assumed to be a *Gaussian random field*, with zero mean. A random field $\delta(\vec{\mathbf{x}})$ is called Gaussian if the distribution of the field values at an arbitrary set of N points, is an N-variate Gaussian:

$$\mathcal{P}(\delta_1, \delta_2, \ldots, \delta_N) = \frac{\exp[-Q]}{[(2\pi)^{\mathrm{N}} \det(C)]^{1/2}}, \tag{2.3.3}$$

where:

$$Q \equiv \frac{1}{2} \sum_{i,j} \delta_i (C)^{-1}_{ij} \delta_j \ , \tag{2.3.4}$$

$$C_{ij} = \langle \delta_i \delta_j \rangle = \xi(r_i r_j). \tag{2.3.5}$$

The second moment, $\langle \delta_i \delta_j \rangle$, is called two-point correlation function. We will explore the correlation function of the overdensity field of the Universe in the following subsection. For the moment, note the Gaussian random field is completely specified by its second moment. Also note, if it's not clear to you, that the probability distribution for a specific point $\vec{\mathbf{x_i}}$ is also a Gaussian,i.e.:

$$\mathcal{P}(\delta_i) d\delta_i = \frac{1}{(2\pi\sigma_i^2)^{1/2}} exp\left(-\frac{\delta_i^2}{2\sigma_i^2}\right) d\delta_i \tag{2.3.6}$$

2.3.2 Correlation Function and the Power Spectrum

As before, we can decompose the *overdensity field*, or the *density contrast*, $\delta(\vec{\mathbf{x}})$ into plane waves (Fourier transform of the overdensity field).

$$\delta\left(\vec{\mathbf{x}}\right) = \frac{1}{(2\pi)^3}\int \delta(\vec{\mathbf{k}})e^{+i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}\, d\vec{\mathbf{k}}, \quad \delta(\vec{\mathbf{k}}) = \int \delta(\vec{\mathbf{x}})e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}\, d\vec{\mathbf{x}} \tag{2.3.7}$$

In what follows we will frequently pass from the *real space* description to the *Fourier space* description and vice versa. It's common practice to denote the overdensity with the letter δ in either space, and we can only distinguish them by their argument ($\vec{\mathbf{x}}$ or $\vec{\mathbf{k}}$).

The first moment of the overdensity field is the *mean*, which is zero by definition:

$$\langle\delta(\vec{\mathbf{x}})\rangle = \left\langle \frac{\rho - \overline{\rho}}{\overline{\rho}} \right\rangle = \frac{\overline{\rho} - \overline{\rho}}{\overline{\rho}} \equiv 0 \tag{2.3.8}$$

Next we define ***correlation function*** or the *two-point function* of the overdensity field, which is the second statistical moment of the field, as (in the real space):

$$\boxed{\xi\left(\vec{\mathbf{x}}, \vec{\mathbf{x}}'\right) \equiv \langle\delta(\vec{\mathbf{x}})\delta(\vec{\mathbf{x}}')\rangle} \tag{2.3.9}$$

The correlation function is widely used in cosmology. In this form the correlation function, given the value of the overdensity at one point in space, $\vec{\mathbf{x}}$, describes the probability that the same value will be found within a given distance, at a point $\vec{\mathbf{x}}'$. Since the overdensity field is homogeneous and isotropic, the correlation function must be only a function of the distance between two points. If $\vec{\mathbf{x}}' = \vec{\mathbf{x}} + \vec{\mathbf{r}}$, we write:

$$\xi\left(r\right) = \langle\delta(\vec{\mathbf{x}})\delta(\vec{\mathbf{x}} + \vec{\mathbf{r}})\rangle\,, \quad r \equiv |\vec{\mathbf{r}}| \tag{2.3.10}$$

We can decompose the correlation function into Fourier modes, $\delta(\vec{\mathbf{k}})$. The correlation function in *Fourier space* defines the ***power spectrum***, $P(k)$:

$$\langle\delta(\vec{\mathbf{k}})\delta^*(\vec{\mathbf{k}}')\rangle \equiv (2\pi)^3 P(k)\delta_{\mathrm{D}}(\vec{\mathbf{k}} - \vec{\mathbf{k}}') \tag{2.3.11}$$

where δ_{D} is Dirac's delta function, which ensures that modes of different wave vector $\vec{\mathbf{k}}$ are uncorrelated in Fourier space in order to ensure homogeneity. The power spectrum cannot depend on the direction of $\vec{\mathbf{k}}$ because of isotropy.

Now, we can insert eq. (2.3.7) into eq. (2.3.10), to get:

$$\begin{aligned}
\xi\left(r\right) &= \left\langle \frac{1}{(2\pi)^3}\int \delta(\vec{\mathbf{k}})e^{+i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}\, d\vec{\mathbf{k}}\, \frac{1}{(2\pi)^3}\int \delta^*(\vec{\mathbf{k}}')e^{-i\vec{\mathbf{k}}'\cdot(\vec{\mathbf{x}}+\vec{\mathbf{r}})}\, d\vec{\mathbf{k}}' \right\rangle \\
&= \left\langle \frac{1}{(2\pi)^3}\frac{1}{(2\pi)^3}\int d\vec{\mathbf{k}}\int d\vec{\mathbf{k}}'\, \delta(\vec{\mathbf{k}})\delta^*(\vec{\mathbf{k}}')\, e^{+i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}e^{-i\vec{\mathbf{k}}'\cdot(\vec{\mathbf{x}}+\vec{\mathbf{r}})} \right\rangle \\
&= \frac{1}{(2\pi)^3}\frac{1}{(2\pi)^3}\int d\vec{\mathbf{k}}\int d\vec{\mathbf{k}}'\, \left\langle \delta(\vec{\mathbf{k}})\delta^*(\vec{\mathbf{k}}') \right\rangle\, e^{+i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}e^{-i\vec{\mathbf{k}}'\cdot(\vec{\mathbf{x}}+\vec{\mathbf{r}})}
\end{aligned}$$

We use now the definition (2.3.11) and the properties of Dirac's delta function to get:

$$\begin{aligned}
\xi(r) &= \frac{1}{(2\pi)^3}\int d\vec{\mathbf{k}}\int d\vec{\mathbf{k}}'\,P(k)\delta_{\rm D}(\vec{\mathbf{k}}-\vec{\mathbf{k}}')e^{+i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}e^{-i\vec{\mathbf{k}}'\cdot(\vec{\mathbf{x}}+\vec{\mathbf{r}})} \\
&= \frac{1}{(2\pi)^3}\int d\vec{\mathbf{k}}\,P(k)\,e^{+i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}e^{-i\vec{\mathbf{k}}\cdot(\vec{\mathbf{x}}+\vec{\mathbf{r}})} \\
&= \frac{1}{(2\pi)^3}\int d\vec{\mathbf{k}}\,P(k)\,e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}
\end{aligned}$$

Next, we write the inner product $\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}$, as $\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} = k\,r\cos\theta$, with θ being the angle between the vectors $\vec{\mathbf{k}}$ and $\vec{\mathbf{r}}$. We also use that in spherical coordinates, in k-space: $d\vec{\mathbf{k}} = k^2\sin\theta\,dk\,d\theta\,d\phi$, so we have:

$$\begin{aligned}
\xi(r) &= \frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{\pi}\int P(k)e^{-ik\,r\cos\theta}k^2\sin\theta\,dk\,d\theta\,d\phi \\
&= \frac{2\pi}{(2\pi)^3}\int dk\,P(k)\,k^2\int_0^{\pi} e^{-ik\,r\cos\theta}\sin\theta\,d\theta
\end{aligned}$$

It's easy to calculate the second integral. The result is:

$$\int_0^{\pi} e^{-ik\,r\cos\theta}\sin\theta\,d\theta = 2\frac{\sin(kr)}{kr}$$

So, we finally have:

$$\boxed{\xi(r) = \frac{1}{2\pi^2}\int dk\,k^2 P(k)\frac{\sin(kr)}{kr}} \tag{2.3.12}$$

With a similar treatment we take for the Power spectrum:

$$\boxed{P(k) = 4\pi\int dr\,r^2\,\xi(r)\frac{\sin(kr)}{kr}} \tag{2.3.13}$$

As we have seen, a Gaussian random field is completely specified by the two-point correlation function $\xi(r)$. We can say now that, equivalently, it is completely specified by the power spectrum $P(k)$.

Another important statistical property is the ***variance*** of the overdensity field, defined -as you may expect- as (since the field has zero mean):

$$\sigma^2 \equiv \left\langle \delta^2(\vec{\mathbf{x}})\right\rangle = \left\langle |\delta(\vec{\mathbf{x}})|^2\right\rangle = \langle \delta(\vec{\mathbf{x}})\delta^*(\vec{\mathbf{x}})\rangle \tag{2.3.14}$$

From the above definition, you can understand that the variance of the field is simply the correlation function at $r = 0$. So, we can write for the variance of the overdensity field:

$$\boxed{\sigma^2 = \xi(0) = \frac{1}{2\pi^2}\int dk\,k^2 P(k)} \tag{2.3.15}$$

We will find this relation very useful later in order to compute the variance of the field, since one can have the power spectrum from the theory and fitting to simulations.

Note that the Power spectrum has dimensions of volume. Many people define the dimensionless quantity:

$$\Delta^2(k) \equiv \frac{1}{2\pi^2} P(k)\, k^3, \tag{2.3.16}$$

and write the variance as:

$$\sigma^2 = \int \Delta^2(k) \frac{dk}{k}. \tag{2.3.17}$$

We will need this definition later in the next subsection .

Note: In the above discussion we got the averages and integrals over all space. It's very common practice not to consider the whole Universe, but a volume $V = L^3$, which is much larger than any significant structure due to perturbations. V can be considered as a fair sample of the Universe. The Universe is considered periodic in the volume V. Then you can decompose the overdensity field in *discrete* modes, $\delta_{\vec{\mathbf{k}}}$. The previous results are regained in the limit $V \to \infty$. There is no need to investigate this approach further, and we will not consider it in what follows.

2.3.3 Evolution of the Power Spectrum and the Transfer Function

In the previous subsection we considered the power spectrum at some specific moment. Now we will find a way to describe the power spectrum at any moment in the cosmic history, i.e. to describe it's evolution. Remember that the power spectrum is related to the correlation function, which, in turn is related to the (square of) $\delta(\vec{\mathbf{k}})$. So, in principle, you may imagine that to calculate the power spectrum P(t,k), one has to take the linear theory equation (2.2.28) and solve for each independent mode.

If that was the case, we could write:

$$P(t,k) = P(t_i,k) \left[\frac{D_+(t)}{D_+(t_i)}\right]^2, \tag{2.3.18}$$

where $P(k,t_i)$ is the power spectrum at some *initial* time t_i, and D_+ the linear growth factor. Why IS this not entirely true? Remember that the linear equation for matter overdensities, eq.(2.2.28) obtained from the Newtonian hydrodynamical equations, under the assumption that Newtonian approximation is valid.

The Newtonian approximation is valid only for perturbations which are well *within the horizon.* As the Universe expands, the Hubble radius grows, and thus the scale of perturbations which can be in casual contact. A density perturbation mode is said to "enter the horizon" when its wave length, λ, equals the Hubble radius. For perturbations outside the horizon we need a general relativistic treatment which is outside the scope of this thesis.

The right equation which gives the evolution of the power spectrum is written as:

$$\boxed{P(t,k) = P(t_i,k)\, T^2(k) \left[\frac{D_+(t)}{D_+(t_i)}\right]^2} \tag{2.3.19}$$

and it's valid for times $t > t_{eq}$, where t_{eq} is the moment of *radiation-matter* equality. And:

- $P(t_i, k)$, as before, is the initial power spectrum, shortly after creation of the perturbations. The shape of the initial or *primordial* power spectrum will be given below.
- $T(k)$ is called the ***transfer function*** and it will be defined soon.
- $D_+(t)$ is the linear growth factor.

Let's explore now further the above parts of the equation of the evolution of the power spectrum.

Let's start with the *initial power spectrum*, $P(t_i, k)$. The theoretical model of inflation provides predictions for an initial spectrum of perturbations, responsible for the large scale structure. There are strong observational evidence (in the Cosmic Microwave Background) in favor of this scenario. Obtaining the initial power spectrum from the equations of the inflationary model is clearly outside the scope of this thesis.

Even before the existence of the idea of inflation, it was assumed that the initial power spectrum must have a power-law form:

$$P(t_i, k) \propto k^{n_s} \tag{2.3.20}$$

where n_s is called the ***spectral index***. The power spectra predicted by inflation have this form. The preference of this ansatz is because that it does not introduce any preferred length scale. If we choose for the spectral index $n_s = 1$ we have the ***Harrison -Zel'dovich spectrum***.

It's usually quoted that the Harrison-Zel'dovich spectrum is *scale invariant* but to understand what this means we have to do a little bit of work. Remember that the *dimensionless power spectrum* is defined as:

$$\Delta^2(k) \equiv \frac{1}{2\pi^2}\, k^3 P(k) \tag{2.3.21}$$

Similarly, we can define the dimensionless power spectrum for the *gravitational potential*, as:

$$\Delta^2_\Phi(k) \equiv \frac{1}{2\pi^2}\, k^3 P_\Phi(k) \tag{2.3.22}$$

Now consider again the Poisson equation in $k-$space:

$$k^2 \delta\Phi + 4\pi G_N a^2 \delta\rho = 0$$

which, can be written as:

$$k^2 \overline{\Phi} \delta_\Phi + 4\pi G_N a^2 \overline{\rho} \delta = 0 \tag{2.3.23}$$

where we have introduced the *potential overdensity*:

$$\delta_\Phi \equiv \frac{\delta\Phi}{\overline{\Phi}} \tag{2.3.24}$$

So we have:

$$k^2\delta_\Phi \propto \delta \Rightarrow \delta_\Phi \propto \delta/k^2 \tag{2.3.25}$$

And using the above:

$$P_\Phi(k) \propto k^{-4}P(k) \tag{2.3.26}$$

So, we finally have for the dimensionless power spectrum for the gravitational potential:

$$\Delta_\Phi(k) \propto k^3k^{-4}P(k) \Rightarrow \Delta_\Phi(k) \propto k^{n_s-1} \tag{2.3.27}$$

Now, if the spectral index is $n_s = 1$, as you can easily see:

$$\Delta_\Phi(k) \propto \text{const.} \tag{2.3.28}$$

and that's why is called scale-invariant spectrum. It has the desirable property that the gravitational potential is finite on both small and large scales.

Now we will consider the ***transfer function***. It's difficult to *define* what the transfer function *is*. One may say that the transfer function is a *filter*. A filter through which the initial power spectrum passes and takes another form. Between the time of its creation ant the time of complete matter domination various affect the growth of the initial perturbations. The transfer function tries to describe the combined effect of these processes that tend to change the form of the primordial power spectrum.

We will not explain in great detail all those processes. We will simply explain, mainly qualitatively, the basic effects that change the power spectrum and we will give a fitting-formula for the transfer function in the Cold Dark Matter model, which we will use later.

The critical point is ***when*** a mode enters the cosmological horizon, that is, as we have said, when its wave length equals the Hubble radius. Why is this so important? First consider ordinary, baryonic, matter. Modes entering the horizon while radiation dominates feel the radiation pressure, which almost completely stops the growth of the density perturbation until matter starts dominating and radiation pressure quickly becomes negligible. So, modes which are small enough to enter the horizon before a_{eq} (matter-radiation equality) are relatively suppressed compared to larger modes which enter the horizon afterwards.

The above are true, as we have said, for the baryonic matter. Remember that non-baryonic CDM does not have EM interactions. But there is another physical process that makes the stagnation of growth in pressureless matter perturbations during the radiation dominated era and is known as the ***Meszaros Effect***. It's very easy to understand this effect. The characteristic time for growth/collapse of perturbation of pressureless material is the free-fall time:

$$\tau_{\text{ff}} \propto (G\rho_{\text{m}})^{-1/2}$$

The characteristic time for the expansion of the Universe is the Hubble time $\tau_{\text{H}} = 1/H$. Using the Friedmann equation, we have that:

$$\tau_{\text{H}} \propto (G\rho_{\text{rad}})^{-1/2},$$

during radiation domination, and:

$$\tau_{\rm H} \propto (G\rho_{\rm m})^{-1/2}$$

So, as you can easily see that $\tau_{\rm H}/\tau_{\rm ff} \propto (\rho_{\rm rad}/\rho_{\rm m})^{-1/2}$ during the radiation-domination era and $\tau_{\rm H}/\tau_{\rm ff} \cong 1$ during the matter-domination era. So, as long as the Universe is dominated by the relativistic component, the fluctuations in the other component remain frozen. The perturbation can only grow after the time of matter-radiation equality.

We just saw how important is when a mode is entering the horizon. There are also other effects, beside the Meszaros effect that affect the growth of structure. These include Silk damping, free-streaming damping, acoustic oscillations etc. Unfortunately we cannot explain all those effects.

Now we will give a fitting formula for the transfer function. This is a fitting formula to numerical results obtained concerning all the above processes and effects. The formula, in a Cold Dark Matter (CDM) model, is the following:

$$\boxed{T(k) = \frac{\log(1+2.34q)}{2.34q}\left[1+3.89q+(16.1q)^2+(5.46q)^3+(6.71q)^4\right]^{-\frac{1}{4}}} \tag{2.3.29}$$

with $q = k\theta^{1/2}/(\Omega_{\rm CDM}h^2)$ and $\theta = 1$ for three neutrino flavours. Note that for small k (which means large wave length λ), $T(k) \to 1$, which is the expected behavior, since large modes enter the horizon during the matter-domination era.

2.3.4 The Smoothed Density Field and the Mass Variance

We will find useful, in what follows, to define the concept of the ***smoothed overdensity field***. Given an overdensity field, $\delta(\vec{\mathbf{x}})$, we can filter it using a filter or a ***window function***, $W(\vec{\mathbf{x}}; R).R$ is called the *size* of the filter. The smoothed field gives the average of the density fluctuations in a region of volume $V \sim R^3$, and so it's a quantity of great physical interest.

The smoothed or *filtered* field can be obtained by convolution of the "raw" density field with the window function:

$$\boxed{\delta(\vec{\mathbf{x}}; R) \equiv \int \delta(\vec{\mathbf{x}}')W(\vec{\mathbf{x}}+\vec{\mathbf{x}}'; R)\, d\vec{\mathbf{x}}} \tag{2.3.30}$$

The window function has some characteristics:

- Is normalized such that $\int W(\vec{\mathbf{x}}; R)d\vec{\mathbf{x}} = 1$.
- $W = \text{const} \cong R^{-3}$ if $|\vec{\mathbf{x}}' - \vec{\mathbf{x}}| \ll R$ and $W = 0$ if $|\vec{\mathbf{x}}' - \vec{\mathbf{x}}| \gg R$.

The filter function may have many shapes. For each shape we can define a mass:

$$M = \gamma_{\rm f}\,\overline{\rho}\,R^3, \tag{2.3.31}$$

where $\gamma_{\rm f}$ is a constant that depends on the size of the filter. So, a filter can be characterized by its size, R, or its mass, M.

As we told the smoothed density field is the convolution of the density field with the filter function. Now, according to the convolution theorem, convolution in *real-space* is equal to multiplication in *Fourier-space*. So, we have for the Fourier transform of the smoothed density field:

$$\boxed{\delta(\vec{\mathbf{k}};R) \equiv \int \delta(\vec{\mathbf{x}};R)e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}d\vec{\mathbf{x}} = \delta(\vec{\mathbf{k}})\widetilde{W}(kR)} \tag{2.3.32}$$

where $\widetilde{W}(kR) = \int W(\vec{\mathbf{x}};R)e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}d\vec{\mathbf{x}}$ is the Fourier transform of the window function. Note, as we denoted, that k and R only enter in the combination kR.

Three are the most frequently used window functions. We will present them. In what follows $r = |\vec{\mathbf{x}}|$.

- **Top Hat Filter**: $\left(\gamma_{\mathrm{f}} = \frac{4\pi}{3}\right)$

$$W(\vec{\mathbf{x}};R) = \begin{cases} \dfrac{3}{4\pi R^3} & r \leq R \\ 0 & r > R \end{cases} \tag{2.3.33}$$

$$\widetilde{W}(kR) = \frac{3}{(kR)^3}\left[\sin(kR) - (kR)\cos(kR)\right] \tag{2.3.34}$$

  The above is the most natural choice of window function, a simple sphere in real space. But this choice has the undesirable property that the sharp transition in real space leads to fringes in Fourier space.

- **Gaussian Filter**: $\left(\gamma_{\mathrm{f}} = (2\pi)^{3/2}\right)$

$$W(\vec{\mathbf{x}};R) = \frac{1}{(2\pi)^{3/2}R^3}\exp\left(-\frac{r^2}{2R}\right) \tag{2.3.35}$$

$$\widetilde{W}(kR) = \exp\left(-\frac{(kR)^2}{2}\right) \tag{2.3.36}$$

  As expected, the Gaussian window has the same form in real space *and* in Fourier space.

- **Sharp k-space Filter**: $\left(\gamma_{\mathrm{f}} = 6\pi^2\right)$

$$W(\vec{\mathbf{x}};R) = \frac{1}{2\pi^2 r^3}\left[\sin\left(\frac{r}{R}\right) - \left(\frac{r}{R}\right)\cos\left(\frac{r}{R}\right)\right] \tag{2.3.37}$$

$$\widetilde{W}(kR) = \begin{cases} 1 & k \leq 1/R \\ 0 & k > 1/R \end{cases} \tag{2.3.38}$$

  This filter is sharp in k-space but it has not well-defined boundaries in real space. We will find it very useful later, in the *excursion set theory* of the Press-Schechter mass function.

The smoothed density fluctuation field, $\delta(\vec{\mathbf{x}}; R)$, is also a Gaussian random field. The variance of the smoothed density field is:

$$\boxed{\sigma^2(R) = \left\langle \delta^2(\vec{\mathbf{x}}; R) \right\rangle = \frac{1}{2\pi^2} \int P(k) \widetilde{W}^2(kR) k^2 dk} \tag{2.3.39}$$

The probability of getting a value of $\delta(\vec{\mathbf{x}}; R)$ between δ and $\delta + d\delta$ is:

$$\mathcal{P}(\delta; R) d\delta = \frac{1}{\sqrt{2\pi\sigma^2(R)}} \exp\left(-\frac{\delta^2}{2\sigma^2(R)} \right) d\delta \tag{2.3.40}$$

Note that $\lim_{R\to 0} \widetilde{W}(kR) = 1$, so we have that $\lim_{R\to 0} \sigma^2(R) = \sigma^2$ as it should be.

We will define now an important cosmological parameter, the cosmological parameter σ_8. It is defined as the variance of the density field, linearly extrapolated to $z = 0$, when smoothed with top-hat filter of size $R = 8h^{-1}$ Mpc. So, it is:

$$\boxed{\sigma_8 = \left\langle \delta^2(\vec{\mathbf{x}}; R) \right\rangle^{1/2} \Big|_{R=8h^{-1}\,\mathrm{Mpc}} = \left[\frac{1}{2\pi^2} \int P(k) \widetilde{W}^2_{\mathrm{TH}}(kR) k^2 dk \right]} \tag{2.3.41}$$

The value of σ_8 given in the Particle Physics Booklet, July 2012, by Particle Data Group, is:

$$\sigma_8 = 0.800 \pm 0.003 \tag{2.3.42}$$

As we have said, we can define a mass, for each filter function. In other words, there IS a relationship between smoothing scale and mass, for every window function. So, we can label a filter by its size, R, or its mass M, and we can write, equally well:

$$\sigma^2(R) \quad \text{or} \quad \sigma^2(M) \tag{2.3.43}$$

Now let a filter function has corresponding volume V. Let $\langle M \rangle$ the mean mass inside the volume. We can define the ***mass variance*** inside the volume V as:

$$\boxed{\sigma^2(M) = \left\langle \left(\frac{M(\vec{\mathbf{x}}; R) - \langle M(R) \rangle}{\langle M(R) \rangle} \right)^2 \right\rangle} \tag{2.3.44}$$

and there is:

$$M(\vec{\mathbf{x}}; R) = \int \rho(\vec{\mathbf{x}}') W(\vec{\mathbf{x}} + \vec{\mathbf{x}}'; R) d\vec{x}$$

and, more specifically:

$$\langle M(R) \rangle \equiv \langle M(\vec{\mathbf{x}}; R) \rangle$$

Now we are finally ready. We have described the main ideas and theoretical concepts that are important to the theory of structure formation, and the statistical treatment of the inhomogeneous Universe. We have everything we need to make the next step: to derive and to understand the Press-Schechter Mass Function. . .

2.4 Derivation of the Press-Schechter Mass Function

Now we have almost achieved our goal. We we are ready to derive and to explain the Press-Schechter Mass Function. We have two ways to do this. The first using the original assumptions and postulates of William Press and Paul Schechter and the second using the theory of random walks or the so-called *excursion set formalism*. The first is simple and it comes directly from the previous discussion of random fields and their statistics. The second is more rigorous, solves some problems, but it demands new mathematics, not explored so far. We will give them both, for completeness.

2.4.1 Original Derivation

Press-Schechter theory, introduced by Press and Schechter in 1974 is a theory of non-linear evolution. In the second section of this chapter we developed the linear theory of the perturbed, inhomogeneous Universe. But as times goes by, this theory becomes invalid. The field soon becomes strongly non-linear, i.e overdensities can exist with $\delta \gg 1$.

As we said in the beginning of this chapter we want to find an expression for the ***mass function***, i.e. the number of structures in question with masses in the range $[m, m + dm]$ per (comoving) volume. How can we do this, which is to answer a fundamental question about the true, non-linear, overdensity field, using our knowledge about the evolution of the linear overdensity field and its statistics?

According to the Spherical Collapse model, presented in the next chapter, regions in the linear density field with $\delta > \delta_{\rm c}$ have collapsed to produce virialized dark matter halos. We want now to associate a mass to those halos and then use the statistics of the linear overdensity field to find the mass function.

Let as denote by δ_m the linear density field smoothed on a mass scale m, in other words $\delta_m = \delta(\vec{\mathbf{x}}; R)$ where $m = \gamma_{\rm f}\overline{\rho}R^3$. Then Press and Schechter postulated that:

> The probability that $\delta_m(\vec{\mathbf{x}}; R, t) > \delta_{\rm c}$ is the same as the mass fraction that at time t is contained in halos with mass greater than m.

The value of δ_c is related to the filter used. According to the spherical collapse model, which is most naturally associated with the top-hat filter, $\delta_c \cong 1.686$. Instead of evolving the whole overdensity field, we can consider the overdensity field ***linearly extrapolated*** *to the present epoch*, and change only the threshold, $\delta_{\rm c}$ which then will become time-dependent. Let us denote the linearly extrapolated field as $\tilde{\delta}_0(\vec{\mathbf{x}})$. It's the overdensity field that would result if all structures continued to grow according to the linear theory until the present epoch.

Then, we can restate the argument of Press and Schechter as:

> The probability that $\tilde{\delta}_{m,0}(\vec{\mathbf{x}}; R) > \tilde{\delta}_{0,{\rm c}}(a)$ is the same as the mass fraction that at time a (since we can characterize a cosmic epoch/ time using the value of the scale factor) is contained in halos with mass greater than m.

The threshold value $\tilde{\delta}_{0,\mathrm{c}}(a)$ is defined using the *linear growth factor*, as:

$$\tilde{\delta}_{0,\mathrm{c}}(a) = \tilde{\delta}_c \frac{D_+(a_0)}{D_+(a)}, \tag{2.4.1}$$

For a Gaussian random field, we have that:

$$\mathcal{P}(\tilde{\delta}_{m,0} > \tilde{\delta}_{0,\mathrm{c}}(a)) = \frac{1}{\sqrt{2\pi}\sigma(m)} \int_{\tilde{\delta}_{0,\mathrm{c}}(a)}^{\infty} \exp\left[-\frac{\tilde{\delta}_{m,0}}{2\sigma^2(m)}\right] d\tilde{\delta}^2_{m,0} = \frac{1}{2}\mathrm{erfc}\left[\frac{\tilde{\delta}_{0,\mathrm{c}}(a)}{\sqrt{2}\sigma(m)}\right] \tag{2.4.2}$$

where $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ is the ***complementary error function***, and is defined as:

$$\boxed{\mathrm{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-u^2)\ du} \tag{2.4.3}$$

Now, according to the postulate by Press and Schechter, we have that:

$$F(> m, a) = \frac{1}{2}\mathrm{erfc}\left[\frac{\tilde{\delta}_{0,\mathrm{c}}(a)}{\sqrt{2}\sigma(m)}\right] \tag{2.4.4}$$

The quantity $F(> m, a)$ is called the *mass fraction* and is the fraction of the matter in the Universe belonging to collapsed structures with mass $> m$.

BUT there is a problem with the above postulate. For a Gaussian field, half the volume of the Universe is necessarily underdense, and will never exceed the threshold regardless of how much the density field evolves. To see this better, consider that $\lim_{m\to 0} \sigma(m) = \infty$ and $\mathrm{erfc}(x) = 1$. So, the postulate by Press and Schechter predicts that only the 1/2 of all matter in the Universe is locked-up in collapsed halos. Clearly this is not the case in the real Universe. Underdense regions can be enclosed within larger overdense regions, so they can be included in some larger collapsed object. We can "solve" this problem by simply introducing a *fudge factor two*:

$$F(> m, a) = 2\mathcal{P}(\tilde{\delta}_{m,0} > \tilde{\delta}_{0,\mathrm{c}}(a)) = \mathrm{erfc}\left[\frac{\tilde{\delta}_{0,\mathrm{c}}(a)}{\sqrt{2}\sigma(m)}\right] \tag{2.4.5}$$

Now we are ready to derive the Press-Schechter mass function. Since we know the mass fraction $F(> m, a)$, we can calculate the mass function $dn/dm = n(m, a)$ from the relation:

$$\frac{dn}{dm}(m, a)dm \equiv n(m, a)dm = \frac{\rho_{\mathrm{m},0}}{m}\left|\frac{d}{dm}F(> m, a)\right| dm, \tag{2.4.6}$$

where $\rho_{\mathrm{m},0}$ is the mean matter density of the present-day Universe. $\rho_{\mathrm{m},0}/m$ is the number density of halos with mass m if all of the mass in the Universe were composed of such halos. What follows now is only an amount of dirty algebra.

Let as differentiate the complementary error function, at first:

$$\frac{d}{dx}\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \frac{d}{dx} \int_x^{\infty} \exp(-u^2)\, du \Rightarrow$$

$$\Rightarrow \frac{d}{dx}\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \frac{d}{dx} \left(- \int_{\infty}^{x} \exp(-u^2)\, du\right) \Rightarrow$$

$$\Rightarrow \frac{d}{dx}\text{erfc}(x) = -\frac{2}{\sqrt{\pi}}\exp(-x^2) \tag{2.4.7}$$

We want to calculate $\frac{d}{dm}F(>m,a) = \frac{d}{dm}\text{erfc}\left[\frac{\tilde{\delta}_{0,\mathrm{c}}(a)}{\sqrt{2}\sigma(m)}\right]$. Using the above derivative of the complementary error function, we have ($x = \frac{\tilde{\delta}_{0,\mathrm{c}}(a)}{\sqrt{2}\sigma(m)}$):

$$\frac{d}{d\left(\frac{\tilde{\delta}_{0,\mathrm{c}}(a)}{\sqrt{2}\sigma(m)}\right)}F(>m,a) = -\frac{2}{\sqrt{\pi}}\exp\left[\frac{\tilde{\delta}^2_{0,\mathrm{c}}(a)}{2\sigma^2(m)}\right] \tag{2.4.8}$$

But also:

$$\begin{aligned}
\frac{d}{dm}F(>m,a) &= \frac{d\left(\frac{\tilde{\delta}_{0,\mathrm{c}}(a)}{\sqrt{2}\sigma(m)}\right)}{dm}\frac{d}{d\left(\frac{\tilde{\delta}_{0,\mathrm{c}}(a)}{\sqrt{2}\sigma(m)}\right)}F(>m,a) \\
&= \frac{\tilde{\delta}_{0,\mathrm{c}}}{\sqrt{2}}\frac{d}{dm}\left(\frac{1}{\sigma(m)}\right)\frac{d}{d\left(\frac{\tilde{\delta}_{0,\mathrm{c}}(a)}{\sqrt{2}\sigma(m)}\right)}F(>m,a) \;\; \{\text{using } (2.4.5)\} \\
&= \frac{\tilde{\delta}_{0,\mathrm{c}}}{\sqrt{2}}\frac{d}{dm}\left(\frac{1}{\sigma(m)}\right)\left(-\frac{2}{\sqrt{\pi}}\exp\left[\frac{\tilde{\delta}^2_{0,\mathrm{c}}(a)}{2\sigma^2(m)}\right]\right) \\
&= \frac{\tilde{\delta}_{0,\mathrm{c}}}{\sqrt{2}}\left(-\frac{1}{\sigma^2(m)}\frac{d\sigma(m)}{dm}\right)\left(-\frac{2}{\sqrt{\pi}}\exp\left[\frac{\tilde{\delta}^2_{0,\mathrm{c}}(a)}{2\sigma^2(m)}\right]\right) \Rightarrow
\end{aligned}$$

$$\Rightarrow \frac{d}{dm}F(>m,a) = \sqrt{\frac{2}{\pi}}\frac{\tilde{\delta}_{0,\mathrm{c}}(a)}{\sigma(m)}\left(\frac{1}{\sigma(m)}\frac{d\sigma}{dm}\right)\exp\left[\frac{\tilde{\delta}^2_{0,\mathrm{c}}(a)}{2\sigma^2(m)}\right] \tag{2.4.9}$$

Now, note that:

$$\frac{d\ln\sigma(m)}{d\ln m} = \frac{dm}{d\ln m}\frac{d\ln\sigma(m)}{dm} = \frac{1}{\frac{dm}{d\ln m}}\frac{d\ln\sigma(m)}{dm} \Rightarrow$$

$$\Rightarrow \frac{d\ln\sigma(m)}{d\ln m} = m\frac{d\ln\sigma(m)}{dm} = m\frac{1}{\sigma(m)}\frac{d\sigma(m)}{dm} \Rightarrow$$

$$\Rightarrow \frac{1}{\sigma(m)}\frac{d\sigma(m)}{dm} = \frac{1}{m}\frac{d\ln\sigma(m)}{d\ln m} \tag{2.4.10}$$

So, combining eqs. (2.4.6) and (2.4.7), we have that:

$$\frac{d}{dm}F(>m,a) = \sqrt{\frac{2}{\pi}}\frac{1}{m}\frac{\tilde{\delta}_{0,\mathrm{c}}(a)}{\sigma(m)}\left(\frac{d\ln\sigma(m)}{d\ln m}\right)\exp\left[\frac{\tilde{\delta}^2_{0,\mathrm{c}}(a)}{2\sigma^2(m)}\right] \tag{2.4.11}$$

And finally, using eq. (2.4.3), we have the mass function:

$$\frac{dn}{dm}(m,a)dm = \frac{\rho_{\mathrm{m},0}}{m}\left|\frac{d}{dm}F(>m,a)\right|dm \Rightarrow$$

$$\boxed{\frac{dn}{dm}(m,a)\,dm = \sqrt{\frac{2}{\pi}}\frac{\rho_{\mathrm{m},0}}{m^2}\frac{\tilde{\delta}_{0,\mathrm{c}}(a)}{\sigma(m)}\left|\frac{d\,\ln\sigma(m)}{d\,\ln m}\right|\exp\left[-\frac{\tilde{\delta}^2_{0,\mathrm{c}}(a)}{2\sigma^2(m)}\right]\,dm.} \tag{2.4.12}$$

which is the well-known ***Press-Schechter mass function***.

Now some comments for the mass function. We can define a characteristic mass, $m^\star$ by $\sigma(m^\star) = \tilde{\delta}_{0,\mathrm{c}}(a)$. Then we have that:

- For $m \ll m^\star$ we have that $\frac{dn}{dm}(m,a) \propto m^{\beta-2}$ where $\beta = \frac{d\ln\sigma(m)}{d\ln m}$. For CDM cosmology $\beta \to 0$ at low masses so that we have $\frac{dn}{dm}(m,a) \propto m^{-2}$.
- For $m \gg lm^\star$ the abundance of halos is exponentially suppressed.
- From the above we can conclude that $m^\star$ gives us the mass scale of structures that start to form at epoch a.

2.4.2 Excursion Set Formalism

Remember that in the previous derivation of the mass fraction (and consequently the mass function), using the original postulate by Press and Schechter, we had to introduce a fudge factor of two to get the correct normalization. It may seem of no great importance, but an explanation must be given for the introduction of this factor. This problem is closely connected with the *cloud-in-cloud* problem. This problem comes from the fact that underdense regions can be embedded inside larger overdense regions; so there is a miscounting of the number of low-mass clumps many of which would have been subsumed into larger objects.

Here we will give another derivation of the Press-Schechter mass function, which solves the above problems, and also opens new roads to the exploration of the structure formation, halo growth and halo mergers. The solution, first proposed in 1990 by *Bond et. al.*, uses the language of ***random walks***, and interprets the statistics of halo formation as a random walk. This approach is usually called the *excursion set theory*.

So, first of all, we must give the main points of the theory of random walks. Here we make only a review, without derivations and proofs. The theory of random walks and many other aspects of stochastic processes and their applications, are uniquely explored in the article by S. Chandrasekhar, *"Stochastic Problems in Physics and Astronomy"* (see the bibliography).

- The simplest problem of random flights (as generally called) is the one-dimensional random walk. Consider a particle which executes a series of displacements of equal length, ℓ, along a straight line. Each step can be taken in the forward or in the backward direction, with equal probability 1/2. Let the particle executes n steps per unit time. Then we have that the probability $\mathcal{W}(x,t)dx$ for the particle to be found between x and $x+dx$ after a time t is:

$$\mathcal{W}(x,t)dx = \frac{1}{2\sqrt{\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right)dx, \tag{2.4.13}$$

where, we have written:

$$D = \frac{1}{2}n\ell^2 \tag{2.4.14}$$

The mean (average) displacement and the mean square displacement are, respectively:

$$\langle x \rangle = 0, \qquad \langle x^2 \rangle = 2Dt \tag{2.4.15}$$

So, finally, the root mean square displacement is:

$$x_{rms} = \sqrt{2Dt} \tag{2.4.16}$$

- Next we consider the case of the a random walk with an absorbing barrier at $x = x_1$. The existence of the perfect absorber at x_1 means that whenever the particle arrives at x_1 it immediately becomes incapable to undergone any more displacements. We want to find, again, the probability $\mathcal{W}(x, t; x_1)dx$ to find the particle between x and $x + dx$ after a time t, with $x \leq x_1$. To do this, we can first count all possible sequences of steps which lead to x in the absence of the absorbing wall we should then exclude a certain number of "forbidden" sequences.
  There is a smart way to do this. A trajectory which meets the absorbing wall has an equal probability for moving above or below it. For any *forbidden* trajectory continuing above the absorbing wall, there is an *allowed* mirror trajectory continuing below it, and conversely. So, for any path reaching a point $x < x_1$ along only *allowed* trajectories, there is a path reaching its mirror point on the line $x = x_1$, $x_1 + (x_1 - x) = 2x_1 - x$, along only *forbidden* trajectories. So, finally, the probability to reach the point $x < x_1$ along only *allowed* trajectories is the probability to reach x along *any* trajectory, minus the probability to reach its mirror point $(2x_1 - x)$, along forbidden trajectories (*all* trajectories leading to $2x_1 - x$ are forbidden). So:

  $$\mathcal{W}(x, t; x_1) = \mathcal{W}(x, t)dx - \mathcal{W}(2x_1 - x, t)dx \tag{2.4.17}$$

  And finally:

  $$\mathcal{W}(x, t; x_1)dx = \frac{1}{2\sqrt{\pi D t}} \left[\exp\left(-\frac{x^2}{4Dt} \right) - \exp\left(-\frac{(2x_1 - x)^2}{4Dt} \right) \right] dx \tag{2.4.18}$$

We are now ready to apply our knowledge of the theory of random walks to get Press-Schechter mass function. Indeed, what we can calculate using the above theory is the mass fraction, $F(> m, a)$. We can calculate this, by assigning every mass element dm in the Universe to a collapsed structure of some mass m.

Consider again the overdensity field, linearly extrapolated to the present epoch, as described in the previous subsection. It's convenient to work in the linearly extrapolated overdensity field, since this remains Gaussian, which is a very useful property, as we will see. Then, a region is considered to have "collapsed" in cosmic epoch a, if its (linearly extrapolated) overdensity exceeds a critical value, $\tilde{\delta}_{0,\mathrm{c}}(a)$.

Now consider a point $\vec{\mathbf{x}}$ in space. We smooth the overdensity field $\tilde{\delta}_0(a, \vec{\mathbf{x}})$ with a spherically symmetric filter function of varying mass scale. We start from $m \to \infty$, and we proceed to smaller scales. Then, if in some mass scale m, $\tilde{\delta}_0(a, \vec{\mathbf{x}}) = \tilde{\delta}_{0,\mathrm{c}}(a)$, then we think that the mass element dm in $\vec{\mathbf{x}}$ belongs to an object of mass m. Since, in this way we assign each mass element to the biggest possible collapsed object which includes it, we immediately solve the cloud-in-cloud problem.

The previous process is similar to the one-dimensional random walk described before. More specifically, it is very similar to a random walk with an absorbing barrier at $\tilde{\delta}_{0,\mathrm{c}}(a)$. The "space-like" variable (analogous to x) is now the overdensity $\tilde{\delta}_0$. What is the "time-like" variable (analogous to t)? From the theory of one-dimensional random walks we have that:

$$\left\langle x^2 \right\rangle = 2Dt$$

But remember that we also have:

$$\sigma^2(m) = \left\langle \tilde{\delta}_0^2(\vec{\mathbf{x}}; m) \right\rangle \tag{2.4.19}$$

So we can identify $Dt \to \sigma^2(m)/2$, and if we write $S(m) \equiv \sigma^2(m)$, then $Dt \to S(m)/2$. For a hierarchical cosmology (as CDM), $S(m)$ is a monotonically declining function of halo mass, so there is an one-to-one relation between $S(m)$ and m.

So we describe the random walk as follows: We consider a point $\vec{\mathbf{x}}$, with overdensity linearly extrapolated to the present day, $\tilde{\delta}_0(\vec{\mathbf{x}})$. We choose a filter function (we will see in the next paragraph which is the appropriate), and we filter the overdensity field. For each value of filtering mass, m (which means, for each value $S(m)$, the smoothed overdensity $\tilde{\delta}_{0,\mathrm{m}}(\vec{\mathbf{x}})$ will have a different value. So, with each point $\vec{\mathbf{x}}$ corresponds a trajectory $\tilde{\delta}_{0,\mathrm{m}}(\vec{\mathbf{x}})$. The trajectory starts at $m \to \infty$, which means $S(m) \to 0$, and continues until meeting the absorbing barrier at $\tilde{\delta}_{0,\mathrm{c}}(a)$.

To be the above trajectory *Markovian*, i.e the walk to be indeed a random walk, each step of the walk must be independent from the previous (the walk must have no "memory" of its prior path). In order to ensure this, we must chose a filter such that, when changing $S(m)$, new and independent modes are added. If we choose the *sharp k-space filter*, which we rewrite here as:

$$\widetilde{W}_m(k) = \begin{cases} 1 & k \le k_c(m) \\ \\ 0 & k > k_c(m) \end{cases} \tag{2.4.20}$$

where:

$$k_c(m) = \left(\frac{6\pi^2 \rho_{\mathrm{m},0}}{m} \right)^{1/3} \tag{2.4.21}$$

(this result is obtained if we consider the mass scale associated with the sharp k-space filter), then we satisfy the necessary condition. Indeed, the filtered overdensity field becomes:

$$\tilde{\delta}_{0,\mathrm{m}}(\vec{\mathbf{x}}) = \int d^3\,\vec{\mathbf{k}} W_m(k) \tilde{\delta}_{0,\mathrm{m}}(\vec{\mathbf{k}}) e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}} = \int_{k \le k_c(m)} d^3\,\vec{\mathbf{k}} \tilde{\delta}_{0,\mathrm{m}}(\vec{\mathbf{k}}) e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}} \tag{2.4.22}$$

and a change of the mass scale from m to $m - dm$ adds only modes between $k_c(m)$ and $k_c(m - dm)$. Also using the linearly extrapolated overdensity field which is Gaussian random field, we ensure that there is an equal probability for the system to move towards any one of the two available directions.

Using the correspondence $x \to \tilde{\delta}_0$, $Dt \to S/2$ and $x_1 \to \tilde{\delta}_{0,\mathrm{c}}$ we have from the the theory of random walks with an absorbing barrier, that the probability that a point in space will have an average extrapolated overdensity between $\tilde{\delta}_0$ and $\tilde{\delta}_0 + d\tilde{\delta}_0$ when filtered at a scale m corresponding to a variance of $S(m)$ is:

$$\mathcal{W}(\tilde{\delta}_0, S; \tilde{\delta}_{0,\mathrm{c}})d\tilde{\delta}_0 = \frac{1}{\sqrt{2\pi S}}\left[\exp\left(-\frac{\tilde{\delta}_0^2}{2S}\right) - \exp\left(-\frac{(2\tilde{\delta}_{0,\mathrm{c}} - \tilde{\delta}_0)^2}{2S}\right)\right] d\tilde{\delta}_0 \tag{2.4.23}$$

with $\tilde{\delta}_0 \leq \tilde{\delta}_{0,\mathrm{c}}$. Now the Press-Schechter ansatz becomes:

> The fraction of trajectories with a first up-crossing of the barrier $\tilde{\delta}_{0,\mathrm{c}}(a)$ at $S(m) > S(m_1) = \sigma^2(m_1)$ is equal to the mass fraction that at time a resides in halos with masses $m < m_1$.

Thus the mass fraction is $F(> m, a) = F(> \tilde{\delta}_{0,\mathrm{c}}(a))$, the fraction of points which are lost of the walk when filtering at higher mass scales. But we have $F(> m, a) = 1 - F(< m, a) = 1 - F(< \tilde{\delta}_{0,\mathrm{c}}(a))$. So, we finally have:

$$\begin{aligned} F(> m, a) &= 1 - \int_{-\infty}^{\tilde{\delta}_{0,\mathrm{c}}(a)} \mathcal{W}(\tilde{\delta}_0, S; \tilde{\delta}_{0,\mathrm{c}})d\tilde{\delta}_0 \\ &= 1 - \frac{1}{\sqrt{2\pi S}}\left(\int_{-\infty}^{\tilde{\delta}_{0,\mathrm{c}}(a)} \exp\left(-\frac{\tilde{\delta}_0^2}{2S}\right) d\tilde{\delta}_0 - \int_{-\infty}^{\tilde{\delta}_{0,\mathrm{c}}(a)} \exp\left(-\frac{(2\tilde{\delta}_{0,\mathrm{c}} - \tilde{\delta}_0)^2}{2S}\right) d\tilde{\delta}_0\right) \end{aligned}$$

Using that:

$$\int_{-\infty}^{\tilde{\delta}_{0,\mathrm{c}}(a)} \exp\left(-\frac{\tilde{\delta}_0^2}{2S}\right) d\tilde{\delta}_0 = \sqrt{2S}\int_{-\infty}^{\tilde{\delta}_{0,\mathrm{c}}(a)/\sqrt{2S}} \exp(-u^2)\, du, \tag{2.4.24}$$

with the substitution $u = \frac{\tilde{\delta}_0}{\sqrt{2S}}$, and also that:

$$\int_{-\infty}^{\tilde{\delta}_{0,\mathrm{c}}(a)} \exp\left(-\frac{(2\tilde{\delta}_{0,\mathrm{c}} - \tilde{\delta}_0)^2}{2S}\right) d\tilde{\delta}_0 = -\sqrt{2S}\int_{+\infty}^{\tilde{\delta}_{0,\mathrm{c}}(a)/\sqrt{2S}} \exp(-u^2)\, du, \tag{2.4.25}$$

with the substitution $u = \frac{2\tilde{\delta}_{0,\mathrm{c}} - \tilde{\delta}_0}{\sqrt{2S}}$, we get:

$$\begin{aligned} F(> m, a) &= 1 - \frac{\sqrt{2S}}{\sqrt{2\pi S}}\left(\int_{-\infty}^{\tilde{\delta}_{0,\mathrm{c}}(a)/\sqrt{2S}} \exp(-u^2)\, du + \int_{+\infty}^{\tilde{\delta}_{0,\mathrm{c}}(a)/\sqrt{2S}} \exp(-u^2)\, du\right) \\ &= 1 - \frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^{0} \exp(-u^2)\, du + 2\int_{0}^{\tilde{\delta}_{0,\mathrm{c}}(a)/\sqrt{2S}} \exp(-u^2)\, du + \int_{+\infty}^{0} \exp(-u^2)\, du\right) \\ &= 1 - \frac{2}{\sqrt{\pi}}\int_{0}^{\tilde{\delta}_{0,\mathrm{c}}(a)/\sqrt{2S}} \exp(-u^2)\, du \end{aligned}$$

The definition of the *error function* is:

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x \exp(-u^2)\, du, \tag{2.4.26}$$

while the *complementary error function* is defined through the previous function, as:

$$\mathrm{erfc}(x) \equiv 1 - \mathrm{erf}(x) \tag{2.4.27}$$

So, finally we have the result for the mass fraction, from the excursion set theory, to be:

$$\boxed{F(> m, a) = \mathrm{erfc}\left[\frac{\tilde{\delta}_{0,\mathrm{c}}(a)}{\sqrt{2S(m)}}\right]} \tag{2.4.28}$$

Since $S(m) = \sigma^2(m)$, the above expression for the mass fraction is identical to the expression we got using the original Press-Schechter postulate, and also contains the -then- fudge factor of two!!

Then, we can derive the Press-Schechter mass function in the same way as in the previous subsection. The method described here is powerful. Not only it solves the cloud-in-cloud problem and explains the introduction of the fudge factor of two, but allows us to expand the theory. For example to describe merger rates, to consider ellipsoidal rather than spherical collapse and to study the environment of the collapsed structures.

Chapter 3

The Spherical Collapse Model

3.1 Introduction

Here we will consider the simplest case for the growth and collapse of a structure in an *Einstein-deSitter* (matter-dominated and flat) universe. The model is simplistic, but gives us the necessary physical insight into the processes of structure formation. Imagine that in an otherwise critical-matter density universe ($\Omega_{\mathrm{m},0} = 1$) there is a spherical region with matter density above the critical (that means $\Omega_{\mathrm{m},0} > 1$ or $\rho > \rho_{crit}$) . In order to ensure that the background universe is still flat we can imagine that the extra mass inside the spherical region is taken from a thin shell just outside of it. According to the General Theory of Relativity the overdense region will grow as an independent non-flat (here closed) sub-universe, obeying its own Friedmann equations. According to these, the region will grow, will reach a maximum size and then it will start to collapse. According to Mathematics the final destiny of this overdense region is to be a nice, mathematical, point but Physics dictate that the final destiny of it is to be an (even nicer) bound structure.

Using the spherical collapse we can get the threshold value of overdensity which defines what is a structure and we use it in the Press-Schechter formalism. The spherical collapse model for the most complicated case of a Universe with a cosmological constant is given in the next chapter.

3.2 The Friedmann Equation for the Overdense Region

We are now coming to describe more formally the above concepts. Introducing the *scale factor* for the overdense region (which from now on we will call "structure"), a_{s}, the first Friedmann equation is written as:

$$\left(\frac{\dot{a}_{\mathrm{s}}}{a_{\mathrm{s}}}\right)^2 = \frac{8\pi G}{3}\rho_{\mathrm{m}} - \frac{kc^2}{a_{\mathrm{s}}^2} \tag{3.2.1}$$

where $\rho_{\rm m}$ is the matter density of the structure, G the *Newton's* gravitational constant, c is the speed of light in vacuum and k gives us the curvature. As usual, dot denotes differentiation with respect to time. Using the second Friedmann equation we have that the matter density, $\rho_{\rm m}$, evolves as $\rho_{\rm m} \propto a^{-3}$, and after a little bit of algebra we can write eq. (3.2.1) as:

$$\frac{\dot{a}_{\rm s}}{a_{\rm s}} = H_0 \left(\Omega_{\rm m,0} a_{\rm s}^{-3} + (1 - \Omega_{\rm m,0}) a_{\rm s}^{-2}\right)^{1/2} \tag{3.2.2}$$

where H_0 is the "present" value of Hubble's constant for the structure (for convenience we do not put an extra s here and in $\Omega_{\rm m,0}$ but they are also referred to the structure). For a closed universe $1 - \Omega_{\rm m,0} < 0$.

The solution of eq. (3.2.1) or -equivalently- eq.(3.2.2) is given in parametric form:

$$a_{\rm s}(\theta) = A(1 - \cos\theta) \tag{3.2.3}$$

$$t(\theta) = B(\theta - \sin\theta) \tag{3.2.4}$$

with $0 \leq \theta \leq 2\pi$, and the constants A and B are:

$$A = \frac{\Omega_{\rm m,0}}{2(\Omega_{\rm m,0} - 1)} \quad , \quad B = \frac{1}{H_0} \frac{\Omega_{\rm m,0}}{2(\Omega_{\rm m,0} - 1)^{3/2}} \tag{3.2.5}$$

There is also a relation between A and B:

$$A^3 = GMB^2 \tag{3.2.6}$$

where M is the *enclosed mass* of the spherical region. The proof of all those is not so trivial and is given at the end of this chapter.

Now, we can easily find the *maximum scale factor* that the initial overdensity will have (which can also give us the *maximum radius of the overdensity*), by setting equal to zero the derivative of $a_{\rm s}$, as given in eq. (3.2.3), with respect to θ:

$$\frac{da_{\rm s}}{d\theta} = 0 \quad \Rightarrow \quad A \sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \tag{3.2.7}$$

$$\Rightarrow \quad \theta = 0, \pi, 2\pi \tag{3.2.8}$$

The solution $\theta = 0$ corresponds to $t = 0$ (when the expansion started for the perturbation/structure). But the solution $\theta = \pi$ corresponds (from eq. (3.2.4)) to time $t \equiv t_{\rm max} = B\pi$, which is the time of *turn-around*, that is the time when the overdensity reaches its maximum radius and begins to collapse. The scale factor of the overdensity at this time is (from eqs. (3.2.3) and (3.2.5)):

$$a_{\rm s}(\theta = \pi) \equiv a_{\rm s,max} = A(1 - \cos\pi) = 2A \tag{3.2.9}$$

or:

$$a_{\rm s,max} = \frac{\not{2}\Omega_{\rm m,0}}{\not{2}(\Omega_{\rm m,0} - 1)} \Rightarrow a_{\rm s,max} = \frac{\Omega_{\rm m,0}}{(\Omega_{\rm m,0} - 1)} \tag{3.2.10}$$

As we have said, the time of turnaround is (similarly from eqs. (3.2.4) and (3.2.5)):

$$t(\theta = \pi) \equiv t_{\rm max} = B\pi = \frac{\pi}{2H_0} \frac{\Omega_{\rm m,0}}{(\Omega_{\rm m,0} - 1)^{3/2}} \tag{3.2.11}$$

Finally the solution $\theta = 2\pi$ corresponds to the time of the (theoretical) complete collapse of the structure, when:

$$t = t(\theta = 2\pi) = 2\pi B \ \ , \ \ a_{\rm s} \to 0 \tag{3.2.12}$$

As we will see later, the final fate of the overdensity isn't to collapse at one point of infinite density, but to form a so-called *virialized structure*.

We have completed the mathematical description of what we described in the introduction in words. An initial overdensity will grow, reach a maximum size and recollapse, as a independent sub-universe. In the subsequent sections we will explore with more details some aspects of this model. Before closing this section we combine eqs. (3.2.3) & (3.2.9) and eqs. (3.2.4) & (3.2.10) (we divide them) to produce two very useful formulae:

$$\frac{a_{\rm s}}{a_{\rm s,max}} = \frac{1}{2}\left(1 - \cos\theta\right) \tag{3.2.13}$$

$$\frac{t}{t_{\rm max}} = \frac{1}{\pi}\left(\theta - \sin\theta\right) \tag{3.2.14}$$

We will find them very useful when studying the linear theory for the growth of perturbations just in the next section.

3.3 The Linear Theory for the Growth of the Overdense Region

Although it seems bizarre, it is very useful and in use in various theoretical concepts in cosmology (as in the Press-Schechter formalism), to formulate the *linear theory* for the growth of perturbations. That is, to take as the equations for the evolution of the overdensity/structure only the expansions to the second order (to the parameter θ) of the equations (3.2.13) and (3.2.14)).
The first terms of the Maclaurin series for $cos\theta$ and $sin\theta$ are:

- $\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$
- $\sin\theta \approx \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5$

Using these, we can rewrite eqs. (3.2.13) and (3.2.14):

$$\frac{a_{\rm s}}{a_{\rm s,max}} \cong \frac{1}{4}\left(\theta^2 - \frac{\theta^4}{12}\right) \tag{3.3.1}$$

$$\frac{t}{t_{\rm max}} \cong \frac{1}{6\pi}\left(\theta^3 - \frac{\theta^5}{20}\right) \tag{3.3.2}$$

Our structure will grow obeying these equations, according to the *linear theory*. Note (we will use this later) that if we had kept only the leading order of eqs. (3.3.1) and (3.3.2) that would give us the expansion of the background universe. Now, we want to express $\frac{a_{\rm s}}{a_{\rm s,max}}$ as a function of $\frac{t}{t_{\rm max}}$. In doing this we will use a kind of "*perturbation theory*". We start from eq. (3.3.2), and we have that:

$$\frac{t}{t_{\rm max}} \cong \frac{1}{6\pi}\left(\theta^3 - \frac{\theta^5}{20}\right) \Rightarrow \theta^3 - \frac{\theta^5}{20} \cong 6\pi\frac{t}{t_{\rm max}} \Rightarrow \theta^3 \cong 6\pi\frac{t}{t_{\rm max}} + \frac{\theta^5}{20} \tag{3.3.3}$$

For convenience we define:

$$x \equiv 6\pi \frac{t}{t_{\max}} \tag{3.3.4}$$

So eq. (3.3.3) takes the form:

$$\theta^3 \cong x + \frac{\theta^5}{20} \tag{3.3.5}$$

We are trying to express θ as a function of x. Since we can't have an exact solution, we will find an approximate one. In zeroth order approximation, we can say that:

$$\theta^3_{(0)} = x \Rightarrow \theta_{(0)} = x^{1/3} \tag{3.3.6}$$

Substituting this zeroth approximation back to eq. (3.3.5) we find the first-order approximation:

$$\theta^3_{(1)} = x + \frac{\theta^5_{(0)}}{20} \Rightarrow \theta^3_{(1)} = x + \frac{x^{5/3}}{20} \Rightarrow \theta^3_{(1)} = x\left(1 + \frac{x^{2/3}}{20}\right) \Rightarrow \theta_{(1)} = x^{1/3}\left(1 + \frac{x^{2/3}}{20}\right)^{1/3} \tag{3.3.7}$$

And, finally, remembering that $x \ll 1$, we have:

$$\theta_{(1)} \cong x^{1/3}\left(1 + \frac{1}{60}x^{2/3}\right) \tag{3.3.8}$$

We stop here, and we take this first-order approximation as our $\theta(x)$. Substitunging this back to eq. (3.3.1), we express $\frac{a_{\rm s}}{a_{\rm s,max}}$ as a function of x:

$$\begin{aligned}
\frac{a_{\rm s}}{a_{\rm s,max}} &\cong \frac{1}{4}\left(\theta^2_{(1)} - \frac{\theta^4_{(1)}}{12}\right) \Rightarrow \\
\frac{a_{\rm s}}{a_{\rm s,max}} &\cong \frac{1}{4}\left[x^{2/3}\left(1 + \frac{1}{60}x^{2/3}\right)^2 - \frac{1}{12}x^{4/3}\left(1 + \frac{1}{60}x^{2/3}\right)^4\right] \Rightarrow \\
\frac{a_{\rm s}}{a_{\rm s,max}} &\cong \frac{1}{4}\left[x^{2/3}\left(1 + \frac{1}{30}x^{2/3}\right) - \frac{1}{12}x^{2/3}x^{2/3}\right] \Rightarrow \\
\frac{a_{\rm s}}{a_{\rm s,max}} &\cong \frac{1}{4}x^{2/3}\left[1 + \left(\frac{1}{30} - \frac{1}{12}\right)x^{2/3}\right] \Rightarrow
\end{aligned}$$

$$\Rightarrow \frac{a_{\rm s}}{a_{\rm s,max}} \cong \frac{1}{4}x^{2/3}\left[1 - \frac{1}{20}x^{2/3}\right] \tag{3.3.9}$$

Where, we have used the fact that $x \ll 1$, in the third line, above. Finally, we substitute x from eq. (3.3.4) and we find that:

$$\boxed{\frac{a_{\rm s,lin}}{a_{\rm s,max}} \cong \frac{1}{4}\left(6\pi\frac{t}{t_{\max}}\right)^{2/3}\left[1 - \frac{1}{20}\left(6\pi\frac{t}{t_{\max}}\right)^{2/3}\right]} \tag{3.3.10}$$

This equation is the ***linear theory expression*** for the growth of perturbations. It gives us the evolution of the scale factor of the perturbation/structure in the linear regime.

We have said before that the leading terms of equations (3.3.1) and (3.3.2) give us the expansion of the *background universe* (remember again that our overdensity /perturbation/ structure evolves completely independently from the background universe) . Denoting as $a_{\rm bu}$ the *scale factor of the background universe* we have:

$$\frac{a_{\rm bu}}{a_{\rm s,max}} = \frac{1}{4}\theta^2 \Rightarrow$$

$$\Rightarrow \frac{a_{\rm bu}}{a_{\rm s,max}} = \frac{1}{4}\left(6\pi\frac{t}{t_{\rm max}}\right)^{2/3} \tag{3.3.11}$$

which has the familiar $a \propto t^{2/3}$ dependence of the time, that has the scale factor of a flat universe. Defining the *overdensity* δ of the structure as:

$$\delta \equiv \frac{\rho_{\rm s} - \langle\rho_{\rm bu}\rangle}{\langle\rho_{\rm bu}\rangle} \tag{3.3.12}$$

we will use eqs. (3.3.10) and (3.3.11) to find the *linearly extrapolated overdensity,* $\delta_{\rm lin}$, at the time of turnaround and at the time of the collapse of the structure. That is, the overdensity which the structure would have if it continued to grow according to the linear theory, at the time of turnaround and at the time of collapse. Even that sounds really odd, it's of great importance and in use in structure formation formalisms, like Press-Schechter formalism, which we will explore in the next chapter.

Since the "universes" are matter dominated (and remain matter-dominated trough their evolution), their densities evolve as: $\rho \propto a^{-3}$. So we can write:

$$\frac{\rho_{\rm s,\ lin}}{\rho_{\rm bu}} = \left(\frac{a_{\rm bu}}{a_{\rm s,lin}}\right)^3$$

Using eq. (3.3.12), we have: $\rho_{\rm s,lin} = (\delta + 1)\rho_{\rm bu}$, and substituting this back to the previous we finally have that:

$$\delta_{\rm lin} + 1 = \left(\frac{a_{\rm bu}}{a_{\rm lin}}\right)^3 \tag{3.3.13}$$

Also, for $\delta_{\rm lin} \ll 1$ we can write:

$$\frac{a_{\rm lin}}{a_{\rm bu}} = (1 + \delta_{\rm lin})^{-1/3} \Rightarrow \frac{a_{\rm lin}}{a_{\rm bu}} \cong \left(1 - \frac{1}{3}\delta_{\rm lin}\right) \tag{3.3.14}$$

Since, from eq.(3.3.11) we have:

$$\frac{a_{\rm bu}}{a_{\rm s,max}} = \frac{1}{4}\left(6\pi\frac{t}{t_{\rm max}}\right)^{2/3}$$

we can substitute this back to eq. (3.3.10) to take:

$$\frac{a_{\rm s,lin}}{\cancel{a}_{\rm s,max}} \cong \frac{a_{\rm bu}}{\cancel{a}_{\rm s,max}}\left[1 - \frac{1}{20}\left(6\pi\frac{t}{t_{\rm max}}\right)^{2/3}\right] \Rightarrow$$

$$\Rightarrow \frac{a_{\rm s,lin}}{a_{\rm bu}} \cong 1 - \frac{1}{20}\left(6\pi\frac{t}{t_{\rm max}}\right)^{2/3}$$

and finally using eq. (3.3.14):

$$1 - \frac{1}{3}\delta_{\rm lin} \cong 1 - \frac{1}{20}\left(6\pi\frac{t}{t_{\rm max}}\right)^{2/3} \Rightarrow$$

$$\Rightarrow \boxed{\delta_{\rm lin} = \frac{3}{20}\left(6\pi\frac{t}{t_{\rm max}}\right)^{2/3}} \tag{3.3.15}$$

This equation gives the *linearly extrapolated overdensity* of the structure. In the last line we converted the symbol of approximate equality ($\cong$) into the symbol of equality ($=$), beqause eq. (3.3.15) is a kind of definition of the linear extrapolated overdensity.

We are now ready to find the overdensity that our initial perturbation/structure will have at the time of turnaround and complete collapse, in this simplified and linearly extrapolated model. Remembering that the time for turnaround is $t_{\rm turn} \equiv t_{\rm max}$, we have that the linear density contrast, at turnaround, is (using eq.(3.3.15)):

$$\delta_{\rm lin}^{\rm turn} = \frac{3}{20}(6\pi)^{2/3} \Rightarrow$$

$$\Rightarrow \boxed{\delta_{\rm lin}^{\rm turn} \cong 1.062} \tag{3.3.16}$$

This result tells us that the end of the linear regime, when $\delta_{\rm lin}$ reaches unity, corresponds roughly to the time that structures break away from the general expansion, but at that time, gravitationally bound structures have yet to form. After turnaround, collapse continues symmetrically to the expansion phase, and the object collapses to a point at $t_{\rm coll} = 2t_{\rm max}$. Using again eq. (29), the linear density contrast at this time is:

$$\delta_{\rm lin}^{\rm coll} = \frac{3}{20}(12\pi)^{2/3} \Rightarrow$$

$$\Rightarrow \boxed{\delta_{\rm lin}^{\rm coll} \cong 1.686} \tag{3.3.17}$$

This value, as we will see, is usually used in analytical models of structure formation (Press-Schecter formalism) to identify an overdensity as a gravitationally collapsed structure.

We can also find the actual *nonlinear* density contrast at turnaround. That is:

$$1 + \delta_{\rm nonlin}^{\rm turn} = \left(\frac{a_{\rm bu}}{a_{\rm s,max}}\right)^3 = \left[\frac{1}{4}\left(6\pi\frac{t}{t_{\rm max}}\right)^{2/3}\right]^3 = \frac{(6\pi)^2}{4^3} \cong 5.55 \tag{3.3.18}$$

obtained considering only the first term of eq. (3.3.10)

3.4 Virialization

Until now we have talked a lot of times about the final destiny of our initial perturbation. The equations in previous sections show that the fate of this perturbation is to become a (mathematical) point of infinite density, in a time $t = 2t_{\max}$. Clearly this is an ideal and not physical situation. Since the initial perturbation is neither spherical nor homogeneous, the collapse will not proceed until the end, but our overdensity will stop its collapse and it will form a bound structure in *virial equlibrium*.

Virial theorem states that for a structure or for a body in equilibrium, the mean potential energy is egual to minus two times the mean kinetic energy:

$$\langle V_{\rm vir}^{\rm pot}\rangle = -2\langle T_{\rm vir}^{\rm kin}\rangle \tag{3.4.1}$$

At the time of turnaround, when the structure reaches its maximum size, $T_{\rm turn}^{\rm kin} = 0$. Using conservation of energy, at the time of turnaround and at the time of virialization, we have:

$$V_{\rm turn}^{\rm pot} + \not{T}_{\rm turn}^{\rm kin} = V_{\rm vir}^{\rm pot} + T_{\rm vir}^{\rm kin} \Rightarrow V_{\rm turn}^{\rm pot} = V_{\rm vir}^{\rm pot} + T_{\rm vir}^{\rm kin} \tag{3.4.2}$$

and using Virial theorem, eq.(3.4.1), eq.(3.4.2) gives:

$$V_{\rm turn}^{\rm pot} = V_{\rm vir}^{\rm pot} - \frac{1}{2}V_{\rm vir}^{\rm pot} \Rightarrow V_{\rm turn}^{\rm pot} = \frac{1}{2}V_{\rm vir}^{\rm pot} \tag{3.4.3}$$

Since the gravitational potential energy of a mass M inside a spherical volume of radius r is $V^{\rm pot} \propto 1/r$, we conclude from eq. (35) that:

$$\frac{1}{r_{\rm turn}} \equiv \frac{1}{r_{\max}} = \frac{1}{2r_{\rm vir}} \Rightarrow r_{\rm vir} = \frac{1}{2}r_{\max} \tag{3.4.4}$$

So the radius of the final, stabilized, virialized structure would be half its maximum size at turnaround.

Summarizing, all the above analysis shows us that an initial spherical perturbation/overdensity will grow as an over-dense sub-universe. It will reach a maximum size (the *turnaround radius*), which is also the point when it will break away from the expansion of the background universe, then it will begin to collapse, reaching finally a state of virial equilibrium, when it will have half the size it had at turnaround.

The condition for virialization is achieved when $\theta = 2\pi$. Now we will explore some general characteristics of the virialized structures. First we will consider the actual, non-linear, overdensity of the virialized structure. We have found in eq. (3.3.18) that the actual, nonlinear density contrast at turnaround is: $1 + \delta_{\rm nonlin}^{\rm turn} \cong 5.55$. Since the final virialized structure has the half radius that it had at turnaround, and since we are dealing with mass only, the density inside the spherical volume has increased by a factor of 2^3. Meanwhile, for matter $\rho \propto a^-3$ and $a \propto t^{2/3}$ for a flat and matter-dominated universe, so the density of the background universe has decreased by a factor of 2^2 that time. Combining all these, we take for the actual, *non-linear* overdensity of the structure, at virialization:

$$1 + \delta_{\rm nonlin}^{\rm vir} = (1 + \delta_{\rm nonlin}^{\rm turn}) \times 2^3 \times 2^3 \Rightarrow$$
$$\Rightarrow 1 + \delta_{\rm nonlin}^{\rm vir} \cong 5.55 \times 8 \times 4 \Rightarrow$$
$$\Rightarrow \boxed{1 + \delta_{\rm nonlin}^{\rm vir} \cong 178} \tag{3.4.5}$$

This value is confirmed by simulations. You must remember that we have considered a case of a universe with $\Omega_{\rm m,0} = 1$. We know that this is not the case in our universe, which has $\Omega_{\rm m,0} \cong 0.3$ and $\Omega_{\Lambda,0} \cong 0.7$. But, even then, the linear density contrast at collapse, $\delta_{\rm lin}^{\rm coll}$, has a value close to $\delta_{\rm lin}^{\rm coll} \cong 1.7$. On the other hand, in such a universe, the true non-linear overdensity is about 2 times that we have found for the matter-dominated one.

The threshold $\delta_{\rm nonlin}^{\rm vir} \cong 200$ is often used to define a collapsed object. The *virial radius* is the radius around a structure within which the density is about 200 times higher than the average background density.

Another thing we want to know for a collapsed structure is a relation connecting the velocity dispersion and the mass of the structure. We will give the answer without proving it, since we will not need it in the rest of our work. The relation is:

$$\left(\frac{\upsilon^2}{127\mathrm{km\ s^{-1}}}\right)^2 = \left(\frac{M}{10^{12}\mathrm{h^{-1}M_\odot}}\right)^{2/3}(1+z_{\rm vir}) \tag{3.4.6}$$

From the above relation, we see that perturbations which collapse at earlier times have higher velocity dispersions for the same enclosed mass. To prove eq. (3.4.5) one has to use that for bound objects the virial theorem says that:

$$\upsilon^2 = \frac{GM}{r_g} \tag{3.4.7}$$

where M is the mass of the system and r_g is the radius within which the gravitational energy is $U = -GM^2/r_g$, and the fact that the mass within an initial comoving radius $r_{\rm i,com}$ is:

$$M = \frac{4\pi}{3}\rho_{\rm m,0} r_{\rm i,com}^3 \tag{3.4.8}$$

Finally, we will say a few words about the time of virialization of a structure, and the mass dependence of it. We will see in the appendix that the parameter θ, which we have used,is related to the cosmic time and the initial overdensity. This relation says to us that higher overdensities turn around and collapse at the earlier times, when the background universe was smaller and denser.

3.5 Chapter Appendix A: Solution of the Friedmann Equations for a Closed, Matter-Dominated Universe

We have given in eqs. (3.2.3),(3.2.4) and (3.2.5) a parametric solution of the Friedmann equations for our closed, matter-dominated sub-universe. Here we will prove them, for the general case of a closed matter-dominated universe. The first Friedmann equation is:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\mathrm{m}} - \frac{kc^2}{a^2} \tag{3.5.1}$$

while the second, the so-called *fluid equation* is:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{3.5.2}$$

For pressure-less matter $p = 0$, and combining (3.5.1),(3.5.2) we have for a matter-dominated universe:

$$\frac{\dot{a}}{a} = H_0\left(\Omega_{\mathrm{m},0}a^{-3} + (1-\Omega_{\mathrm{m},0})a^{-2}\right)^{1/2} \tag{3.5.3}$$

which is the same as equation (3.2.2). For a *closed* matter-dominated universe, $(1-\Omega_{\mathrm{m},0}) < 0$, or $(\Omega_{\mathrm{m},0} - 1) > 0$. We will give the derivation of eq. (3.5.3) from eqs. (3.5.1) and (3.5.2) in the next appendix.

We will define now a useful consept, the *conformal time* η, also called the "arc parameter measure of time". During the interval of time dt, a photon travelling on a hypersphere of radius $a(t)$ covers an arc measured in radians, equal to:

$$d\eta = \frac{dt}{a(t)} \tag{3.5.4}$$

The "arc parameter" is defined by the integral of $d\eta$ from the start of the expansion:

$$\eta = \int_0^{\mathrm{t}} \frac{dt'}{a(t')} \tag{3.5.5}$$

As you will see, we will need conformal time in our derivation of the parametric solution.

We are starting now from eq. (3.5.3) and multiplying both sides with a, we have that:

$$\frac{\dot{a}}{a} = H_0(\Omega_{\mathrm{m},0}a^{-3} + (1-\Omega_{\mathrm{m},0})a^{-2})^{1/2} \Rightarrow \dot{a} = H_0(\Omega_{\mathrm{m},0}a^{-1} + (1-\Omega_{\mathrm{m},0}))^{1/2}$$

or, after a little bit of "massaging":

$$\frac{da}{dt} = H_0\sqrt{\Omega_{\mathrm{m},0}a^{-1} + (1-\Omega_{\mathrm{m},0})} \Rightarrow \frac{da}{dt} = H_0\sqrt{\Omega_{\mathrm{m},0}a^{-1} - (\Omega_{\mathrm{m},0}-1)} \Rightarrow$$

$$\Rightarrow \frac{da}{dt} = H_0\sqrt{\Omega_{\mathrm{m},0}-1}\sqrt{\frac{\Omega_{\mathrm{m},0}}{\Omega_{\mathrm{m},0}-1}a^{-1} - 1} \Rightarrow$$

$$\Rightarrow H_0\sqrt{\Omega_{\mathrm{m},0}-1}\,dt = \frac{da}{\sqrt{\frac{\Omega_{\mathrm{m},0}}{\Omega_{\mathrm{m},0}-1}a^{-1}-1}} \tag{3.5.6}$$

Now, we will use the conformal time, defined in eq. (44) to rewrite eq. (46), using also that $dt = a\,d\eta$:

$$\Rightarrow H_0\sqrt{\Omega_{\mathrm{m},0}-1}\,a\,d\eta = \frac{da}{\sqrt{\frac{\Omega_{\mathrm{m},0}}{\Omega_{\mathrm{m},0}-1}a^{-1}-1}} \Rightarrow$$

$$\Rightarrow H_0\sqrt{\Omega_{\mathrm{m},0}-1}\,d\eta = \frac{da}{a\sqrt{\frac{\Omega_{\mathrm{m},0}}{\Omega_{\mathrm{m},0}-1}a^{-1}-1}} \Rightarrow$$

$$\Rightarrow H_0\sqrt{\Omega_{\mathrm{m},0}-1}\,d\eta = \frac{da}{\sqrt{\frac{\Omega_{\mathrm{m},0}}{\Omega_{\mathrm{m},0}-1}a-a^2}} \tag{3.5.7}$$

Finally we define the parameter θ as:

$$\theta \equiv \left(H_0\sqrt{\Omega_{\mathrm{m},0}-1}\right)\eta \tag{3.5.8}$$

Also, for future convenience, we set:

$$A = \frac{\Omega_{\mathrm{m},0}}{2(\Omega_{\mathrm{m},0}-1)} \tag{3.5.9}$$

Using these, eq. (3.5.7) becomes:

$$d\theta = \frac{da}{\sqrt{2Aa-a^2}} \tag{3.5.10}$$

Now, we can integrate eq. (3.5.10):

$$\int_{\theta_0}^{\theta} d\theta' = \int_0^a \frac{da'}{\sqrt{2Aa'-a'^2}} \tag{3.5.11}$$

Looking at a table of integrals we find the following integral:

$$\int \frac{dx}{\sqrt{2Ax-x^2}} = \sin^{-1}\left(\frac{x-A}{A}\right) + C \tag{3.5.12}$$

which is the same as that on the right side of eq. (3.5.11). So using (3.5.12) we have from eq.(3.5.11):

$$\theta - \theta_0 = \sin^{-1}\left(\frac{a-A}{A}\right) + \frac{1}{2}\pi \tag{3.5.13}$$

But, the requirement $\eta = 0$ at $a = 0$ or, equivalently, $\theta = 0$ at $a = 0$, sets $\theta_0 = 0$, so we have:

$$\frac{a-A}{A} = \sin\left(\theta - \frac{1}{2}\pi\right) = -\cos\theta \Rightarrow a\,(\theta) = A(1-\cos\theta) \tag{3.5.14}$$

with A as defined in eq. (3.5.9). Now, from eq. (3.5.4) $dt = a d\eta$ or, using eqs. (3.5.10) and then (3.5.14) :

$$dt = a d\eta \Rightarrow dt = \frac{a}{H_0\sqrt{\Omega_{\mathrm{m},0} - 1}} d\theta \Rightarrow dt = \frac{A}{H_0\sqrt{\Omega_{\mathrm{m},0} - 1}}(1 - \cos\theta) d\theta \quad (3.5.15)$$

We define now B as:

$$B \equiv \frac{A}{H_0\sqrt{\Omega_{\mathrm{m},0} - 1}} = \frac{1}{H_0} \frac{\Omega_{\mathrm{m},0}}{2(\Omega_{\mathrm{m},0} - 1)^{2/3}} \quad (3.5.16)$$

So, we have from eq.(3.5.15):

$$dt = B(1 - \cos\theta)\, d\theta \Rightarrow \int_{t_0}^{t} dt' = B \int_0^{\theta} (1 - \cos\theta')\, d\theta'$$

$$\Rightarrow t - t_0 = B(\theta - \sin\theta)$$

But, the requirement $\theta = 0$ at $t = 0$ sets $t_0 = 0$, so we finally have:

$$t = B(\theta - \sin\theta) \quad (3.5.17)$$

with B as defined in eq. (3.5.15). So we found the parametric solution of the Friedmann equations for a closed matter-dominated universe:

$$a(\theta) = A(1 - \cos\theta)$$

$$t(\theta) = B(\theta - \sin\theta)$$

with

$$A = \frac{\Omega_{\mathrm{m},0}}{2(\Omega_{\mathrm{m},0} - 1)}$$

and

$$B = \frac{1}{H_0} \frac{\Omega_{\mathrm{m},0}}{2(\Omega_{\mathrm{m},0} - 1)^{2/3}}$$

Q.E.D!

3.6 Chapter Appendix B: Proof of eq. (3.5.3)

In order to derive the parametric solution of the Friedmann equations, in the previous appendix, we used that for a matter-dominated universe, the first Friedmann equation (using the second) can take the form:

$$\frac{\dot{a}}{a} = H_0(\Omega_{\mathrm{m},0}a^{-3} + (1 - \Omega_{\mathrm{m},0})a^{-2})^{1/2}$$

Here we will prove this statement. Hubble's parameter is defined as:

$$H \equiv \frac{\dot{a}}{a} \tag{3.6.1}$$

Using this, the critical density is defined as:

$$\rho_{\mathrm{crit}} = \frac{3H^2}{8\pi G} \tag{3.6.2}$$

From eqs. (3.6.1) and (3.6.2) we conclude that:

$$\left(\frac{H}{H_0}\right)^2 = \frac{\rho_{\mathrm{crit}}}{\rho_{\mathrm{crit},0}} \tag{3.6.3}$$

Where, the subscript "0" stands always for the quantity at the present time. The first Friedmann equation can also take another well-known form:

$$H^2a^2(1 - \Omega_i) = -kc^2 \tag{3.6.4}$$

where "i" stands for every constituent of the universe. Here we consider only mass. Using this equation, and adopting the convention that $a_0 = 1$ we have:

$$H_0^2(1 - \Omega_{\mathrm{m},0}) = -kc^2 \tag{3.6.5}$$

also for arbitrary time:

$$H^2a^2(1 - \Omega_{\mathrm{m}}) = -kc^2 \tag{3.6.6}$$

The right-hand sides of eqs. (3.6.5) and (3.6.6) are equal, so we have:

$$H^2a^2(1 - \Omega_{\mathrm{m}}) = H_0^2(1 - \Omega_{\mathrm{m},0}) \tag{3.6.7}$$

From the definition of Ω and using eq.(3.6.3) we have that:

$$\Omega_{\mathrm{m}} = \Omega_{\mathrm{m},0}\left(\frac{H}{H_0}\right)^2 a^{-3} \tag{3.6.8}$$

We put that back to (3.6.7), and after some trivial algebra we have:

$$H = H_0(\Omega_{\mathrm{m},0}a^{-3} + (1 - \Omega_{\mathrm{m},0})a^{-2})^{1/2} \tag{3.6.9}$$

or, using again eq.(3.6.1):

$$\frac{\dot{a}}{a} = H_0\left(\Omega_{\mathrm{m},0}a^{-3} + (1 - \Omega_{\mathrm{m},0})\, a^{-2}\right)^{1/2} \tag{3.6.10}$$

Q.E.D

Chapter 4

The End of Structure Formation in ΛCDM Cosmology

4.1 Introduction

In the first chapter we described the discovery that our Universe is accelerating. The acceleration of the Universe indicates the existence of a "dark" energy also called vacuum energy (because a possible explanation of its nature is that it is the "zero-point" energy of the quantum fields, an energy of the quantum vacuum). If the density of this dark energy component of the Universe remains constant with time, it can be described as a *Cosmological Constant*, denoted by the Greek letter Λ.

Since we believe that today, about 70% of the matter/energy content of the Universe is in the form of this dark energy (in the form of a cosmological constant) while the remaining 30% is mostly dark matter, the currently accepted cosmological model is called the Λ (dark energy) CDM (Cold Dark Matter) Cosmology.

In such a Cosmology, the existence of the dark energy has a tremendous effect: the creation, the creation of new structures in the Universe cannot last forever. Since we can imagine the effect of the dark energy as an "anti-gravitational" effect, there are two counteracting "forces" which fight each other in the process of structure formation. Here we will use the Press-Schechter formalism we developed before, to probe the way towards the end of structure formation in the currently accepted cosmological model.

Since the acceleration has already started, we expect that the process of structure formation has already almost finished. Numerical simulations have shown that this is true; in the near future the larger structures will grow a little and then further growth will stop. The *comoving* number density of structures will remain nearly constant thereafter.

The use of an analytical tool, as the Press-Schechter formalism allows as to answer, even in a statistical sense many questions concerning the end of structure formation in ΛCDM Cosmology. Our results, also agree with the general trend shown in numerical simulations. But here, we go a step further from this general trend. We compare the mass function of the present cosmological epoch and the mass function at an infinite time in the future; we answer how different mass scales approach their final comoving number density; we show different trends between structures of different masses: high-mass structures have always less number density than their final while low-mass structures have now greater number density than their final.

While answering the above, we point the importance of the ***turnaround*** radius of a structure and the corresponding *overdensity* for turnaround. We note that in some cases the turnaround radius/overdensity may provide a better identification of a structure than the frequently used ***virial*** radius/overdensity.

In what follows we briefly describe the spherical collapse model in a Universe with matter and cosmological constant/dark energy. Through this we find a minimum overdensity a region must have today to be able to turnaround even at infinite time. Using this critical minimum overdensity and the PS formalism we can get the information we need about the end of structure formation in ΛCDM model.

4.2 The Spherical Collapse Model in a $\Omega_m + \Omega_\Lambda = 1$ Universe

As we said before, now we will describe the spherical collapse model in a flat Universe with matter (mainly dark matter) and dark energy in the form of a cosmological constant. Since now the solution of this problem is much more complicated than in the case of a flat, matter-dominated Universe, here we give only the necessary main points that allows us to understand why structure formation will not last forever in such a cosmological model.

4.2.1 The Criterion for Turnaround

In this paragraph we show quantitatively what we described before: that only regions where the average matter density exceeds some certain value are able to turnaround and collapse, even at infinite time.

Consider a small homogeneous spherical density perturbation in an otherwise homogeneous universe, with $\Omega_m + \Omega_\Lambda = 1$. This density perturbation evolves as an independent, non-flat sub-universe, which obeys the following Friedmann equation:

$$\left(\frac{da_p}{dt}\right)^2 = H_0^2 \Omega_m a_p^2 \left(a_p^{-3} + \omega - \kappa a_p^{-2}\right) \tag{4.2.1}$$

where $\omega = \Omega_\Lambda/\Omega_m = \Omega_m^{-1} - 1$ and a_p is the scale factor/radius of the overdensity (Ω_m and Ω_Λ are the matter and cosmological constant/vacuum energy density parameters of the background Universe). The constant κ (*curvature constant*) is a constant which characterizes the magnitude and the sign of the perturbation. If κ is positive it corresponds to an overdensity, while if it is negative corresponds to an underdensity. The magnitude, $|\kappa|$ shows the deviation from the homogeneous background.

The background Universe is flat and evolves according to the Friedmann equation:

$$\left(\frac{da}{dt}\right)^2 = H_0^2\Omega_\mathrm{m}a^2\left(a^{-3}+\omega\right) \tag{4.2.2}$$

where a is the scale factor of the Universe. Dividing eqs. (4.2.1),(4.2.2) and rearranging we get:

$$\left(\frac{da_\mathrm{p}}{da}\right)^2 = \frac{a}{a_\mathrm{p}}\frac{\omega a_\mathrm{p}^3 - \kappa a_\mathrm{p} + 1}{\omega a^3 + 1} \tag{4.2.3}$$

The previous equation describes the behavior of the scale factor of the perturbation (overdensity in our case) as a function of the scale factor of the Universe.

So, to be able to turnaround and collapse an overdensity, the following equation must have a real and positive solution:

$$\omega a_\mathrm{p}^3 - \kappa a_\mathrm{p} + 1 = 0 \tag{4.2.4}$$

This gives a minimum value of the curvature parameter, κ:

$$\kappa_\mathrm{min,coll} = \frac{3\omega^{1/3}}{2^{2/3}} \tag{4.2.5}$$

Regions with The corresponding *maximum* turnaround radius is:

$$a_\mathrm{p,ta,max} = (2\omega)^{-1/3} \tag{4.2.6}$$

A perturbation with curvature parameter $\kappa_\mathrm{min,coll}$ will approach its maximum turnaround radius $a_\mathrm{p,ta,max}$ as $t \to \infty$. We will talk later for the importance of the maximum turnaround radius.

4.2.2 Translating the Criterion to a Critical Overdensity in the Linearly Extrapolated Overdensity Field

We have a minimum parameter, $\kappa_\mathrm{min,coll}$, which identifies perturbations that are able to have turnaround. To use the Press-Schechter formalism to take the mass function of the final distribution of the turnaround-ed structures we must translate this to a minimum critical overdensity in the field of overdensities linearly extrapolated to the present epoch ($a_0 = 1$).

To do this, we need to find a relation between κ and $\tilde{\delta}_0$, i.e.to correspond *every* value of the curvature parameter κ, to a unique value of overdensity in the overdensity field linearly extrapolated to the present epoch, $\tilde{\delta}_0$.

We have seen that the linear growth factor in $\Omega_\mathrm{m} + \Omega_\Lambda = 1$ cosmology is given by:

$$D(a) = A[(2\omega)^{1/3}a] \tag{4.2.7}$$

where:

$$A(x) = \frac{(x^3+2)^{1/2}}{x^{3/2}}\int_0^x\left(\frac{u}{u^3+2}\right)^{3/2}du \tag{4.2.8}$$

So the linear theory evolution of an overdensity is given by:

$$\tilde{\delta} = \tilde{\delta}_0\frac{D(a)}{D(a_0)} \tag{4.2.9}$$

where $\tilde{\delta}_0$ is the value of the overdensity linearly extrapolated to the present epoch. To find the relation we want, between $\tilde{\delta}_0$ and κ we expand the exact:

$$\delta = \left(\frac{a}{a_{\rm p}}\right)^3 - 1 \tag{4.2.10}$$

to first order in a, which gives:

$$\delta = \frac{3\kappa}{5} a + \mathcal{O}(a^2) \tag{4.2.11}$$

and also the linear relation, which gives:

$$\tilde{\delta} = \tilde{\delta}_0 \frac{(2\omega)^{1/3}}{5A\left[a_0(2\omega)^{1/3}\right]} a + \mathcal{O}(a^2) \tag{4.2.12}$$

By demanding the coefficients to be equal, we get (using also the fact that $a_0 = 1$):

$$\boxed{\kappa = \frac{(2\omega)^{1/3}}{3A\left[(2\omega)^{1/3}\right]}\tilde{\delta}_0} \tag{4.2.13}$$

Now, substituting to this equation the minimum κ for collapse given by eq. (4.2.5) we get the critical (minimum) overdensity, linearly extrapolated to the present epoch, that a perturbation must have to be able to reach turnaround, at some time (in the past or in the future, even at infinite time):

$$\boxed{\tilde{\delta}_{\rm min,coll,0} = \tfrac{9}{2}\, A\left[(2\omega)^{1/3}\right]} \tag{4.2.14}$$

which, using the values $\Omega_{\rm m,0} \cong 0.27$ and $\Omega_{\Lambda,0} \cong 0.73$ gives the numerical value:

$$\tilde{\delta}_{\rm min,coll,0} \cong 1.2006 \tag{4.2.15}$$

So, having the value of the minimum critical overdensity for turnaround we can plot (and we will plot, later) the *final* mass function of turnaround-ed structures.

4.2.3 The Evolution Equation and its Solutions for Eventually Collapsing Overdensities

Using eq. (4.2.3) we can describe the evolution of the scale factor of the perturbation as a function of the scale factor of the Universe. The so-called *evolution equation* is:

$$\frac{da_{\rm p}}{da} = \begin{cases} \left(\frac{a_{\rm p}^{-1}+\omega a_{\rm p}^2-\kappa}{a^{-1}+\omega a^2}\right)^{1/2}, & \kappa < \kappa_{\rm min,coll} \ \textbf{or} \\ & \kappa \geq \kappa_{\rm min,coll}, a < a_{\rm ta} \\ -\left(\frac{a_{\rm p}^{-1}+\omega a_{\rm p}^2-\kappa}{a^{-1}+\omega a^2}\right)^{1/2}, & \kappa \geq \kappa_{\rm min,coll},\ a > a_{\rm ta} \end{cases} \tag{4.2.16}$$

where the right sign was chosen using also eq. (4.2.4). $a_{\rm ta}$ is the scale factor of the Universe when the perturbation reaches the turnaround radius, which is the maximum size of the perturbation. The turnaround radius can be obtained solving eq. (4.2.4), which gives:

$$a_{\rm p,ta} = \omega^{-1/3}\sqrt{\frac{4}{3}\frac{\kappa}{\omega^{\frac{1}{3}}}}\cos\frac{1}{3}\left(\cos^{-1}\sqrt{\frac{27}{4}\left(\frac{\kappa}{\omega^{\frac{1}{3}}}\right)^{-3}}+\pi\right) \tag{4.2.17}$$

The *maximum* turnaround radius is taken if we put $\kappa = \kappa_{\rm min,coll}$ into the previous equation. What we get is eq. (4.2.6) : $a_{\rm p,ta,max} = (2\omega)^{-1/3}$.

Note again the great difference between the $\Omega_{\rm m} = 1$ cosmology and the $\Omega_{\rm m} + \Omega_\Lambda = 1$ cosmology. In $\Omega_{\rm m} = 1$ cosmology *all* overdensities turn around and collapse while in $\Omega_{\rm m} + \Omega_\Lambda = 1$ cosmology some overdensities are forever expanding. In such a cosmology the process of structure formation becomes a battle between the gravitational self attraction of the overdensity and the anti-gravitational effect of the dark energy. And in this battle the winner takes it all!

For $\kappa \geq \kappa_{\rm min,coll}$ we have the solution of the evolution equation (4.2.16):

$$\int_0^a \frac{\sqrt{y}\,dy}{\sqrt{\omega y^3+1}} = \begin{cases} \int_0^{a_{\rm p}} \frac{\sqrt{x}dx}{\sqrt{\omega x^3-\kappa x+1}}, & a < a_{\rm ta} \\ \\ 2\int_0^{a_{\rm p,ta}} \frac{\sqrt{x}dx}{\sqrt{\omega x^3-\kappa x+1}} - \int_0^{a_{\rm p}} \frac{\sqrt{x}dx}{\sqrt{\omega x^3-\kappa x+1}}, & a \geq a_{\rm ta} \end{cases} \tag{4.2.18}$$

The integral on the left hand side of the equation (4.2.18) has the solution:

$$\int \frac{\sqrt{y}\,dy}{\sqrt{\omega y^3+1}} = \frac{2}{3}\omega^{-1/2}\sinh^{-1}\sqrt{\omega y^3} \tag{4.2.19}$$

If we *define* (for more details see the paper by V. Pavlidou & B. Fields mentioned in the bibliography) the *incomplete vacuum integral of the first kind* as:

$$\mathcal{V}_1(r,\mu) \equiv \frac{3}{2}\int_0^r \frac{\sqrt{x}\,dx}{\sqrt{(1-x)(-x^2-x+\mu)}} \tag{4.2.20}$$

with $0 \leq r \leq 1$ and $\mu \geq 2$, where the parameters r and μ, are defined as:

$$r \equiv a_{\rm p}/a_{\rm p,ta}\,, \quad \mu \equiv (\omega a_{\rm p,ta}^3)^{-1} \tag{4.2.21}$$

then we can write:

$$\int_0^{a_{\rm p}} \frac{\sqrt{x}dx}{\sqrt{\omega x^3-\kappa x+1}} = \frac{2}{3}\omega^{-1/2}\mathcal{V}_1(r,\mu) \tag{4.2.22}$$

So finally we can write equation (4.2.18) as:

$$a = \begin{cases} \omega^{-1/3}\left\{\sinh\left[\mathcal{V}_1(r,\mu)\right]\right\}^{2/3}, & a \leq a_{\rm ta} \\ \\ \omega^{-1/3}\left\{\sinh\left[2\mathcal{V}_1(1,\mu)-\mathcal{V}_1(r,\mu)\right]\right\}^{2/3}, & a > a_{\rm ta} \end{cases} \tag{4.2.23}$$

The previous equation is the solution of the evolution equation in ΛCDM cosmology. It gives the value of r (also the value of $a_{\rm p,ta}$) as a function of the scale factor of the Universe.

Using (4.2.23) we can calculate the scale factor of the Universe when the perturbation reaches the turnaround point and when it collapses. When the perturbation reaches turnaround we have that $a_{\rm p} = a_{\rm p,ta} \Rightarrow r = 1$. So we have:

$$a_{\rm ta} = \omega^{-1/3} \left[\sinh \mathcal{V}_1(1,\mu)\right]^{2/3} \tag{4.2.24}$$

while for the scale factor of the Universe when the perturbation collapses (formally $a_{\rm p} = 0 \Rightarrow r = 1$) we have (since $\mathcal{V}_1(0,\mu) = 0$):

$$a_{\rm coll} = \omega^{-1/3} \left[\sinh 2\mathcal{V}_1(1,\mu)\right]^{2/3} \tag{4.2.25}$$

Using equations (4.2.24) and (4.2.25) we can find the value of the overdensity in the linearly extrapolated to the present epoch field of overdensities, which corresponds to turnaround or to collapse of a perturbation/structure at a certain cosmological epoch. Indeed, what we can calculate from those two equations are the parameters μ that correspond to turnaround or collapse at a cosmological epoch, i.e. $\mu_{\rm ta}(a)$ and $\mu_{\rm coll}(a)$.

The parameter μ is connected to the curvature parameter κ of an overdensity. We have from the definition (4.2.21) that:

$$\omega a_{\rm p,ta}^3 = \mu^{-1} \Rightarrow a_{\rm p,ta} = (\omega\mu)^{-1/3} \tag{4.2.26}$$

So, using now eq. (4.2.4):

$$\omega a_{\rm p,ta}^3 - \kappa a_{\rm p,ta} + 1 = 0$$

we get:

$$\omega(\omega\mu)^{-1} - \kappa(\omega\mu)^{-1/3} + 1 = 0 \Rightarrow \mu^{-1} + 1 = \kappa(\omega\mu)^{-1/3} \Rightarrow$$

$$\Rightarrow \frac{1+\mu}{\mu} = \frac{\kappa}{\omega^{1/3}\mu^{1/3}} \Rightarrow$$

$$\Rightarrow \kappa = \omega^{1/3}\frac{1+\mu}{\mu^{2/3}} \tag{4.2.27}$$

Combining equations (4.2.13) and (4.2.27) we get a very important result:

$$\boxed{\tilde{\delta}_{0,\rm c}(a) = \frac{3A\left[(2\omega)^{1/3}\right]}{2^{1/3}} \frac{1+\mu_{\rm c}(a)}{\left[\mu_{\rm c}(a)\right]^{2/3}}} \tag{4.2.28}$$

This result is extremely important. It gives us a recipe to calculate the overdensity in the linearly etrapolated to the present epoch density field which characterizes the turnaround or the collapse ($\mu_{\rm c}$ can be $\mu_{\rm ta}$ or $\mu_{\rm coll}$). The parameter $\mu_{\rm c}(a)$ can be calculated from (4.2.24) or (4.2.25). So, we can use this result to the Press-Schechter formalism (which wants as an input a value of the overdensity which characterizes what is a structure).

4.3 The Importance of the (Maximum) Turnaround Radius

Here we will point out the importance of the *maximum* turnaround radius of a structure/perturbation as a test of the ΛCDM cosmology. What follows is based on the work of V. Pavlidou and T. Tomaras. We also point out the probability the turnaround radius of a structure to be, in certain cases, better in defining what is the size of the structure than the usually used *virial* radius.

We have seen a way to obtain the *maximum turnaround radius* of a perturbation in the previous section. Here we will present one more way to obtain this, and then we will discuss what it really means and how this can be used as an observational test of the currently accepted cosmological model.

Let us consider a mass M in a flat background with a cosmological constant, $\Lambda > 0$. The space-time outside the mass is described by the Scharzschild-de Sitter (SdS) metric (units $G_N = c = 1$):

$$d\tau^2 = A(r)dt^2 - \frac{dr^2}{A(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.3.1}$$

with:

$$A(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} \tag{4.3.2}$$

where τ is the proper time, t is the cosmic time and r is (in the case of a weak gravitational field, as in the case of cosmic structures) the physical distance from M. A test particles follows the geodesics of the SdS metric. The trajectory, parametrized with the proper time, satisfies the equations:

$$A(r)\,\dot{t}^2 - \frac{\dot{r}^2}{A(r)} - r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2 = 1 \tag{4.3.3}$$

$$A(r)\,\dot{t}^2 = \eta,\ \frac{d}{d\tau}(r^2\dot{\theta}) = r^2\sin\theta\cos\theta\,\dot{\phi}^2,\ r^2\sin^2\theta\dot{\phi} = \lambda \tag{4.3.4}$$

where η and λ are parameters defined from the initial conditions and $\dot{} \equiv \frac{d}{d\tau}$. We can choose $\theta(\tau) = \pi/2$ and also we can consider only radial motion. Then, after a little bit of massaging, we can get the following equation for the radial coordinate (with initial conditions $r(\tau_0) = r_0$ and $\dot{r}(\tau_0) = u_0$):

$$\frac{1}{2}\dot{r}^2 + U_0(r) = \mathcal{E}_0 \tag{4.3.5}$$

with:

$$U_0(r) = -\frac{M}{r} - \frac{\Lambda r^2}{6}, \quad \mathcal{E}_0 = \frac{u_0^2}{2} + U_0(r_0) \tag{4.3.6}$$

We can obtain the force acting on a test particle, from the effective potential:

$$F = -\frac{dU_0}{dr} = -\frac{M}{r^2} + \frac{\Lambda r}{3} \tag{4.3.7}$$

If we set now the force equal to zero, and after restoring the physical units (using dimensional arguments), we get:

$$\boxed{r_c = \left(\frac{3G_N M}{\Lambda c^2}\right)^{1/3}} \tag{4.3.8}$$

This is the point where the gravitational force and the "anti-gravitational" "force" originated from the existence of the cosmological constant cancel each other. A point particle left from rest at $r < r_c$ will fall to the central mass, while a point particle left from rest at $r > R_c$ will follow the expansion of the Universe. Note that, after some substitutions, the result (4.3.8) is identical to the result (4.2.6): it describes the *maximum* turnaround radius of a perturbation of mass M.

This maximum turnaround radius can be used as a strong test of ΛCDM cosmology. What's the previous discussion and calculations tell us is that in such a cosmology a ***nonexpanding*** structure cannot have radius greater than the maximum turnaround radius. Accurate measurements of mass and radius for a great amount of structures (which is not an easy work to do) will test our cosmological model, searching for the "non-black raven" as Karl Popper would put it. Our work, beside its self-standing interest, aims to help us to design an experiment to test the upper bound of structures.

Finally we want to mention the following thing: we argue that the turnaround radius of a structure may characterize better the structure than the usually used virial radius. Virial radius is a quite fudge definition of the size of the structure. Nothing special happens to the structure at this very point. But the turnaround radius out of a structure gives a boundary: material inside this radius falls to the center of the structure, while material outside the turnaround radius follows the general expansion of the Universe. Note that, at late times we expect that the number density of turnaround-ed structures will be almost the same than the number of virialized structures. We will tell more about that in the next section.

4.4 Results

We have developed all the necessary tools to face our problem: the end of structure formation in ΛCDM cosmology. We will follow an approach based on the Press-Schechter formalism developed in the second chapter. The values of physical constants and cosmological parameters, where needed were obtained from the *Review of Particle Physics*, 2012, from Particle Data Group. The most frequently used are:

Newtonian gravitational constant	G_N	$6.6738(8) \times 10^{-11}$ m^3 kg^{-1} s^{-2}
Solar mass	$M_\odot$	$1.9885(2) \times 10^{30}$ kg
Parsec (1 *au*/ 1 arc sec)	pc	$3.0856776 \times 10^1 6$ m
Present day Hubble expansion rate	H_0	$100\,h\,\text{km}\;s^{-1}$ Mpc^{-1}
Scale factor for Hubble expansion rate	h	0.721(17)
Critical density of the Universe	ρ_c	$2.775366 \times 10^{11} h^2 M_\odot$
Dark energy density of the ΛCDM Universe	Ω_Λ	0.73(3)
Baryon density of the Universe	Ω_b	0.045(3)
Cold Dark Matter density of the Universe	Ω_cdm	0.22(3)
Pressureless matter density of the Universe	$\Omega_\text{m} = \Omega_\text{cdm} + \Omega_\text{b}$	0.27 ± 0.03
Fluctuation amplitude at $8h^{-1}$ Mpc scale	σ_8	0.80(3)

The definitions of the cosmological parameters can be found in the first and the second chapter. In the second chapter we have also given the necessary formulas to calculate the "ingredients" of the PS mass function, as the mass variance. The threshold value of the overdensity, in the linearly extrapolated to the present day field of overdensities, is given by eq. (4.2.14) if we want to define in that field regions that will have turnaround at infinite time in the future, and by eq. (4.2.28) if we want to define in the same field regions that had/will have turnaround (or virialization)at some cosmological epoch α.

4.4.1 Comparison of the Present Day PS Mass Function with the Ultimate PS Mass Function

In the first figure we present the PS mass function of the *turnaround-ed* structures (overdense regions in space that have passed their turnaround point) today (at the present cosmological epoch, $a_0 = 1$) and at infinite time in the future.

Remember that the mass function gives us the (comoving, which means that we do not take into account the expansion of the Universe)number density of structures (structures per comoving volume) per mass interval, as a function of mass (and cosmic time, generally, but usually we plot the mass function at a certain cosmic epoch). Here we have chosen to plot the mass function in a mass range starting from $M = 10^{11} M_{\odot}$ (the mass scale of a small galaxy) to $M = 10^{16}$ (the mass scale of a supercluster).

The ultimate mass function (the mass function of turnaround-ed structures at infinite time in the future) is obtained using as the critical threshold value (which defines a turnaround-ed structure at infinity) the one taken by eq. (4.2.14). The two mass functions have the expected characteristics: a power-law part (far low-mass scales) followed by a part of exponential suppression. The exponential suppression starts at a mass scale of about $M = 10^{14} M_{\odot}$ (group of galaxies) for both mass functions.

This first figure shows us something very important! The two mass functions are almost identical. Especially for low-mass scales the two coincide and only for very high-mass scale there is a small deviation between them. The deviation starts after the point of exponential suppression, so what this plot tells us is that, if we define a structure by its turnaround overdensity, is that structure formation is ***almost completed*** in the present epoch! There will be a very small change in the number density of structures of high mass, but they are -anyway- much less abundant than the structures of low mass.

For completeness we also present, in the second figure, the PS mass function of virialized structures, for the present cosmological epoch. As we can see, now the divergence between the ultimate mass function and the present day mass function for virialized structures is greater. But, even then, the deviation is quite small and starts for large mass scales. What that figure shows us, combined with the first, is that almost all structures have reached their turnaround point; but among them structures with mass $> 5 \times 10^{13} M_{\odot}$ haven't all of them reached their virial point -thy are not all of them virialized structures. In the future, what remains in their evolution, is to condensate into denser virialized structures.

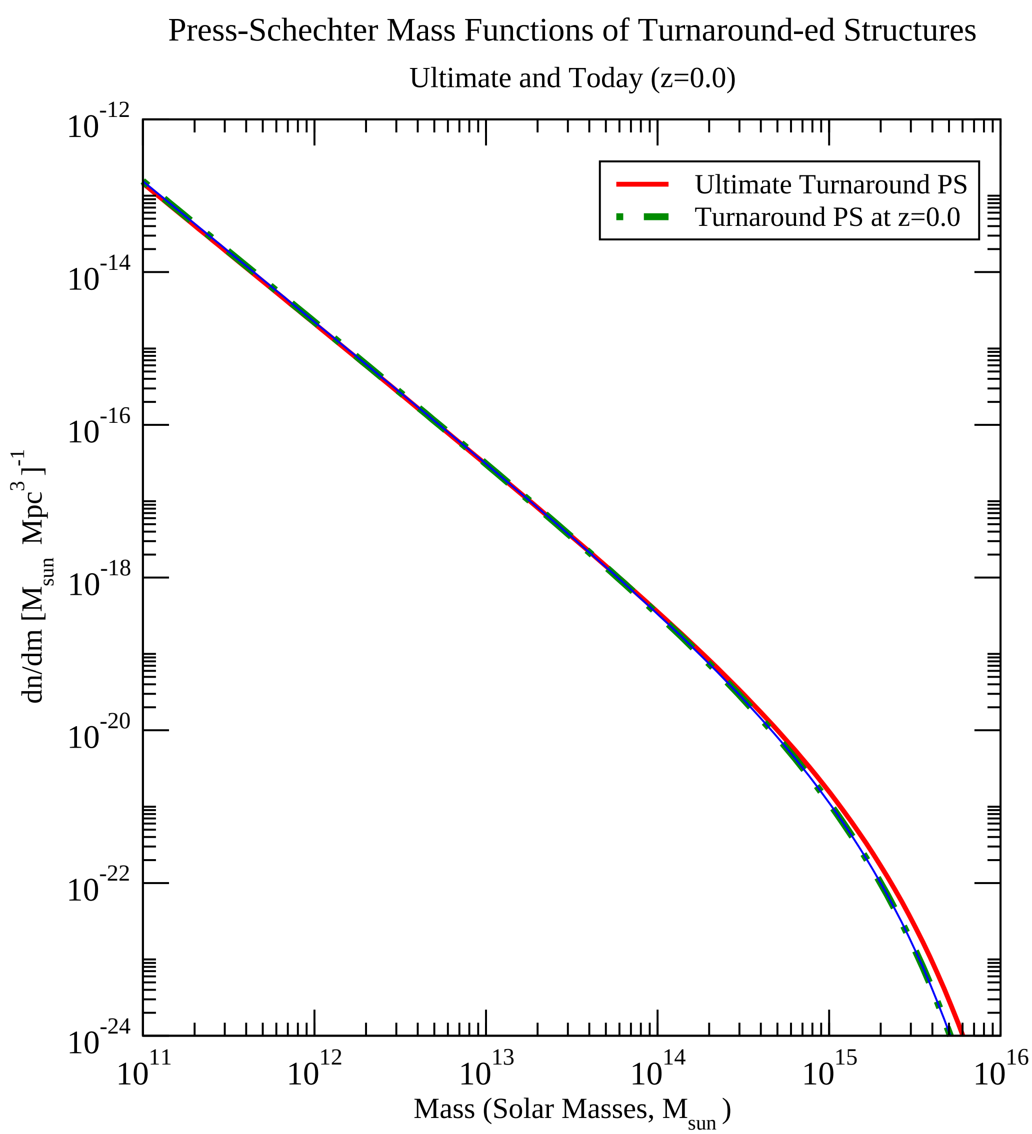

Figure 4.1: Comparison of PS mass function for turnaround-ed structures -today and at infinity.

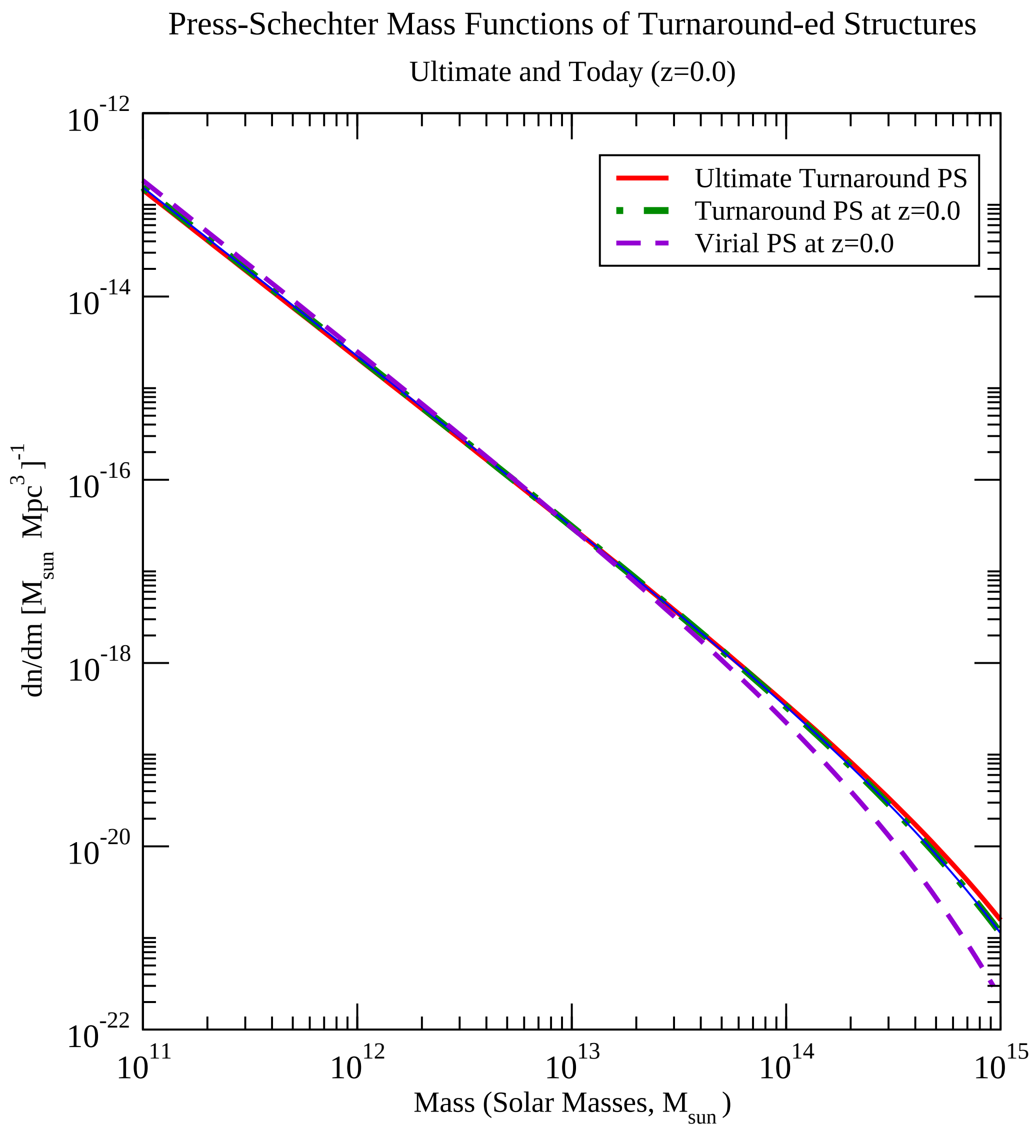

Figure 4.2: Comparison of PS mass functions: For the present cosmological epoch using as a threshold the virial overdenity and the turnaround overdensity, and for infinite time using the threshold value for turnaround at that time.

4.4.2 Comparison of the PS mass function at z=1.0 with the Ultimate PS Mass Function

We compare now (see the third figure) the Ultimate PS mass function with that at the cosmological epoch $z = 1.0$, which means (using eq. (1.3.47)) when the scale factor of the Universe was $a = 1/2$, i.e. when the Universe had half its present size. We have chosen to present this cosmological epoch, because it a distant cosmological epoch, but accessible observationally. The distant supernovae which helped cosmologists to made the discovery of the accelerating Universe were at $z \cong 1$.

Again we have plotted the mass functions in a mass range $10^{11} - 10^{16} M_{\odot}$. We can see now a much greater divergence between the Ultimate mass function and the mass functions at $z = 1.0$ (even we consider the mass function of turnaround-ed or virialized structures, than the case with the present cosmological epoch. This, simply tells us that the process of structure formation was still ongoing that epoch. That epoch is the epoch where dark energy starts to play an important role in the mass/energy density of the Universe, so it is expected that after its domination (at that epoch) the structure formation will come to an end.

4.4.3 The Road to the Final Number Density of Structures of Various Mass Scales

As we have seen, there is a final distribution of structures among different masses (the Ultimate mass function). Now we will investigate how different mass scales approach their final number density. To do so, we take (for some characteristic mass scales) the ratio of the PS mass function (for turnarounded structures) at cosmic epoch a over the Ultimate PS mass function, as a function of the cosmic epoch (function of the scale factor, a).

As you can see in the fourth figure, we plot the mentioned ratio starting from $a = 0.1$ to $a = 15$. We choose mass scales between $10^9 M_{\odot}$ (dwarf galaxies) and $10^{16} M_{\odot}$ (superclusters). There many interesting things to notice in this plot:

- Low-mass scales reach their final number density (ratio=1) at about $a = 1$, which means in the present cosmological epoch. It's very interesting to mention that these low-mass structures seem to reach their final number density at the same time (present) ***independently*** of their mass scale. Structures of mass $10^{10} M_{\odot}$ or $10^{12} M_{\odot}$, for example, have ratio=1 at $a \cong 1$.

- For greater mass scales there is another trend. Very high-mass structures reach their final number density (ratio=1) later than high-mass structures. For example, structures of $M = 10^1 6 M_{\odot}$ have ratio=1 later than structures with $M = 10^{15} M_{\odot}$.

- There is another great difference between low-mass structures and high-mass structures. High mass structures have ***always*** lower number density than their final. At the other hand, low-mass structures start with number density lower than their final, they reach a maximum number (density) -greater than their final- and then their number density will fall reaching

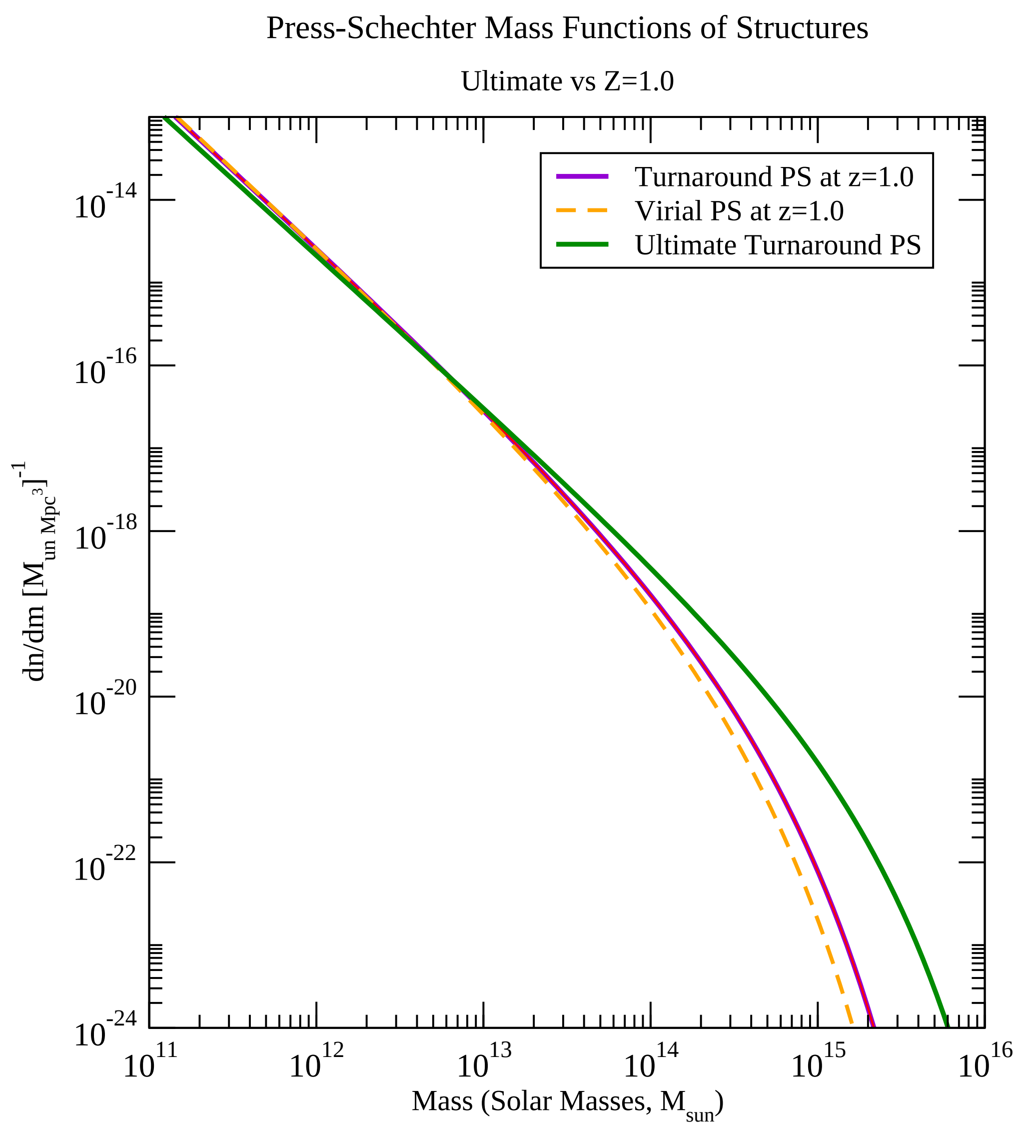

Figure 4.3: Comparison of PS mass functions: Ultimate PS mass function and mass functions (turnaround and virial) at z=1.0.

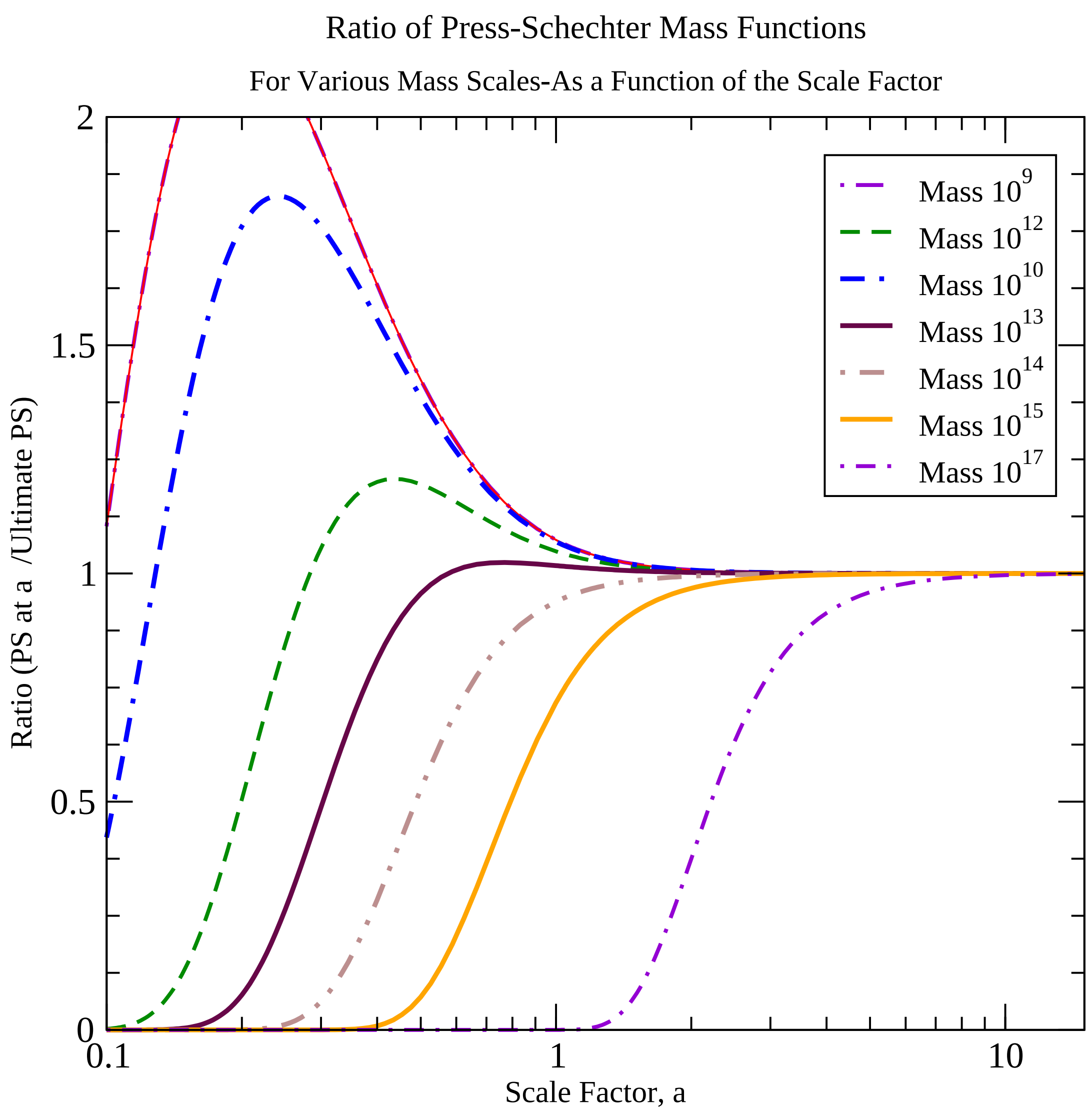

Figure 4.4: Evolution of the ratio of number density over the final number density, for various mass scales.

the final (ratio=1) at about $a = 1$, as we have seen. We will find the "magical" mass scale which divides the two regions (high-mass / low-mass) in a while.

We must make clear why the number density of low-mass structures falls. Low mass structures are inside overdensities of higher total mass. As the time progresses, these higher mass regions collapse to structures, and the initial low-mass structures inside them are not counted any more as structures by the PS formalism.

So, what figure 4 also tells us is that high-mass structures (larger than a certain mass scale) are -statistically- not inside even larger structures (or, more modestly, they are not *all* of them inside larger structures.

We will find now the "magical" scale we mentioned before. The procedure we follow the next procedure: if you see again figs. (1), (2) and (3) for low-mass the Ultimate mass function and the mass functions an epoch are seem to coincide; indeed the PS mass function at a (for low-masses) is a little bit *higher* than the Ultimate. This is rational, since some low-mass structures will be parts of greater structures, as we told. Exactly the opposite happens for higher-mass scales. The Ultimate number density is higher than the number density at a. What we do now, is to find (as a function of the scale factor, a) the mass scale of intersection of the PS mass functions.

We find (see fig. 5) that the mass of intersection of the PS mass functions approaches a constant value, of about $M \cong 2.6 \times 10^{13} M_{\odot}$ (group of galaxies -small cluster). This is the magical scale which divides the two regions/trends we saw. So:

$$M_{\text{magical}} \cong 2.6 \times 10^{13} M_{\odot} \tag{4.4.1}$$

As we have seen, structures with masses greater than that have always lower number density than the final.

4.4.4 Further Comparison of the PS Mass Functions: Ultimate vs Today

We will compare again the Ultimate mass function with the mass function of the present epoch. What we do now is to take the ratio of the two mass functions as a function of mass, at a mass range $10^{11} M_{\odot} - 10^{17} M_{\odot}$.

We see that for $M < 10^{14} M_{\odot}$ the ratio is $\cong 1$. For larger masses there is a greater divergence of the ratio from unity, but this mass scale ($M = 10^{14} M_{\odot}$) indicates the beginning of the exponential suppression of the mass functions, so this divergence is not important. One more time, we have shown that structure formation is, today, almost completed.

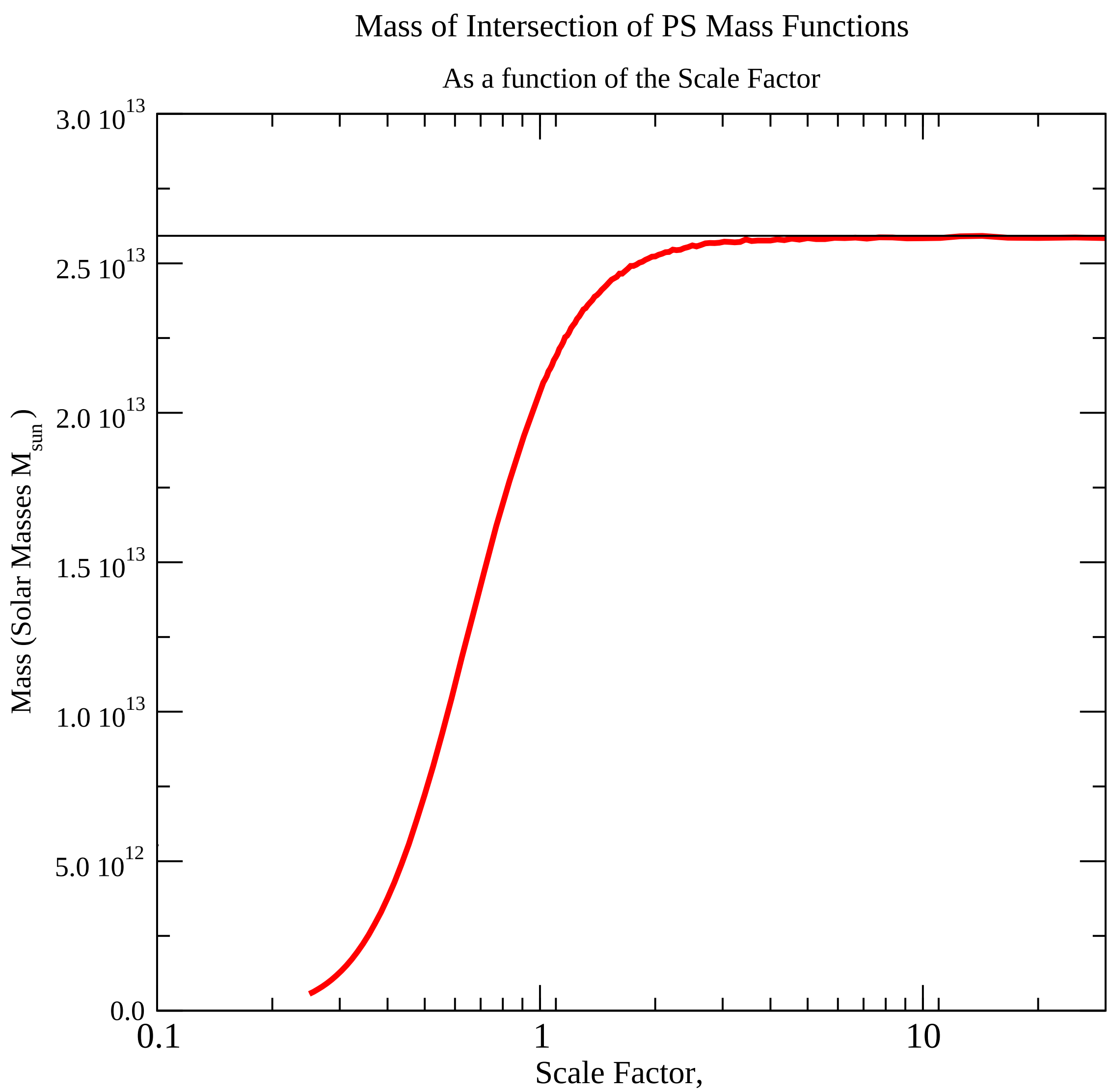

Figure 4.5: The mass scale of intersection of PS mass functions as a function of the scale factor, a.

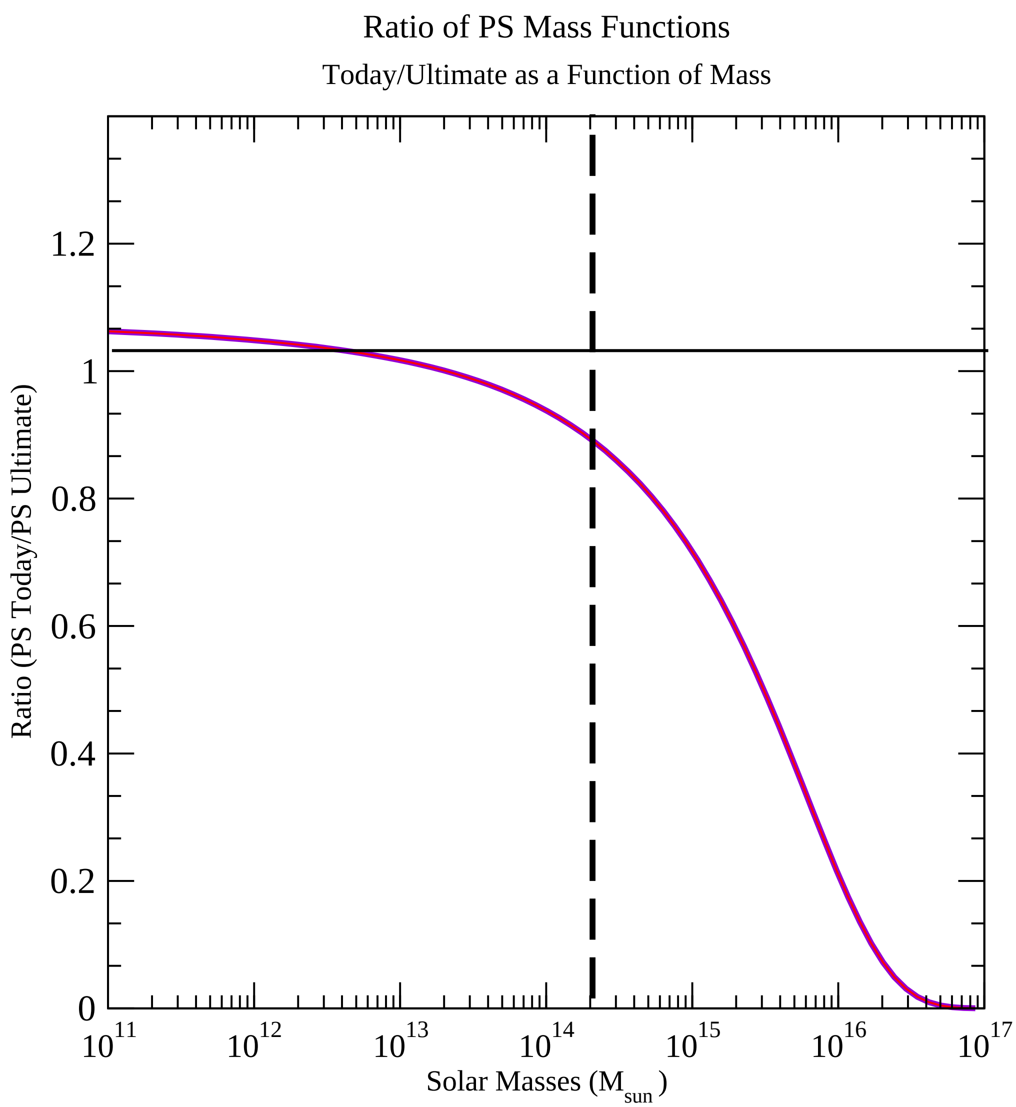

Figure 4.6: Ratio of the PS mass functions (Today/Ultimate) as a function of mass

Chapter 5

Discussion

In the present thesis we presented the Press-Schechter formalism (based on excursion sets) of cosmological mass functions. We used this analytical tool to explore how structure formation comes to an end in the currently accepted ΛCDM cosmological model, which contains a dark energy component.

Numerical simulations have been contacted (see the article by Busha et. al. mentioned in the bibliography) and show that the structure formation, already today, has come to an end. Here, we have confirmed this result, using the analytical formalism. But, the aim of the thesis was not a simple confirmation of something already known.

Studying how different mass scales behave, we shown that low-mass structures had in the near past greater (comoving) number density than today (and today they have number density equal to their final) while high-mass structures have always less number density than the final. A "magical" scale of about $2.6 \times 10^{13} M_{\odot}$ exists which divides the two trends. In a subsequent work we will try to find an observation implication of this result, and probably a connection with the cosmological test proposed by Pavlidou & Tomaras.

We have used mainly the turnaround radius (and the corresponding overdensity) to characterize structures rather than the usually used virial radius. We argue that the turnaround radius defines better what is a structure. Virial radius is calculated based on some crude assumptions and, if we have a real structure in front of us we will nothing special happening in that radius. The turnaround radius is the region around a structure which defines a region where the gravitational attraction of the mass is greater than the general trend of expansion in the Universe. It divides the Universe around a structure to a "local environment" where mass falls to the structure from the rest of the Universe where the Hubble flow is in action.

Let us now describe the future evolution of the Universe, if it really contains dark energy with properties similar to that of a cosmological constant. We have proven that the *comoving* number density of structures will remain nearly constant. This does not mean that in the world will be as we now know it! A dark energy-dominated Universe expands exponentially (see eq. (1.3.44)).The *proper* mass function is related to the comoving mass function as: $dn/dm|_{\rm proper} = a^{-3} dn/dm|_{\rm comoving}$. So the true number density also falls exponentially. Every overdensity/dark matter halo which has value of overdensity that exceeds (4.2.14) will grow (slowly and slowly) and will become the center of its own sub-universe. An observer that time will not be able to see other structures, outside the sphere of influence.

Cosmology, as any other natural science is an experimental/observational science. This work, a theoretical work, intends to help us design observations to test our cosmological model, to made *local* tests (remember that we conclude that dark energy exists from observations on cosmological scales-supernovae) of the existence of the cosmological constant based on the structure formation.

This work is devoted to the exploration and understanding of the architecture of the Universe. We are only ***trying*** to understand it. It's not sure that we will ever be able to understand it completely. But, in any case, -as with the ordinary architecture- we can only make a step back, look at the big picture, the great design and feel the beauty of it. At last, the Architecture of the Universe is the most beautiful of all...

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