Positive Solutions of Nonlinear Elliptic Equations Involving Critical Sobolev Exponents

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Preface

We are concerned with nonlinear elliptic equations admitting a variational structure. Their common feature is that we can not apply straightforward standard variational methods because of the lack of compactness of the Sobolev embedding. The goal of this master thesis is to study how in the cases of critical exponents we can overcome this lack of compactness and obtain results about the existence of positive solutions of nonlinear elliptic equations.

The structure of the thesis is as follows:

- In Chapter 1 we present some definitions and notations.
- Chapter 2 is about the existence of positive solutions of nonlinear elliptic equations with critical sobolev exponents in a bounded domain (The Brezis-Nirenberg Problem).
- In Chapter 3 we study the existence of positive solutions of nonlinear elliptic equation involving critical Sobolev exponents and Hardy potential in $\mathbb{R}^n$.
- Appendix.
- References.

Contents

CHAPTER 1

Definitions.

To begin with, we present some definitions and notations that will be used throughout this thesis.

• Let Ω be a bounded open subset of $\mathbb{R}^n$, the space

$$H^1(\Omega) := \left\{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\right\},$$

with inner product

$$\langle u, v\rangle_{H^1} = \int_\Omega (\nabla u \cdot \nabla v + uv)$$

and the corresponding norm

$$\|u\|_{H^1} = \left(\int_\Omega |\nabla u|^2 + \int_\Omega u^2\right)^{\frac{1}{2}}$$

is a Hilbert space.

• The space $H_0^1(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ functions in $H^1(\Omega)$.

Let $n \geq 3$ and $2^* := \frac{2n}{n-2}$.

• The space

$$D^{1,2}(\mathbb{R}^n) := \left\{u \in L^{2^*}(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n)\right\},$$

defined as the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{D^{1,2}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\nabla u|^2\right)^{\frac{1}{2}}$$

and inner product

$$\langle u, v\rangle_{D^{1,2}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \nabla u \cdot \nabla v$$

is a Hilbert space.

•$f(x) = O(g(x))$, when $x \to \alpha$, if $|f(x)| \leq C|g(x)|$ in a neighborhood of α.

•$f(x) = o(g(x))$, when $x \to \alpha$, if $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$.

• Let X be a real Banach space, I : $X \to R$ a map.

(i) I is Gateaux differentiable at $u \in X$ in the direction $w \in X$ if the map $\tau \to I(u + \tau w)$ is differentiable at $\tau = 0$.

(ii) I is Gateaux differentiable at $u \in X$ if I is Gateaux differentiable at u in all directions $w \in X$. In this case we denote by

$$I'(u)(w) := \lim_{\tau \to 0} \frac{I(u + \tau w) - I(u)}{\tau},$$

the Gateaux derivative at u in the direction w. Then, $I'(u) : X \to \mathbb{R}$ is the Gateaux derivative at u.

(iii) I is Fréchet differentiable at $u \in X$ if there exists $A_u \in X^*$ such that

$$\lim_{\|w\|_X \to 0} \frac{|I(u + w) - I(u) - A_u(w)|}{\|w\|_X} = 0.$$

Some known inequalities that will play a crucial role in our arguments are the following

•(*Poincaré*) Let $\Omega \subset \mathbb{R}^n$ be bounded, open

$$\int_\Omega |u|^q \leq C(q, \Omega) \int_\Omega |\nabla u|^q, \ 1 \leq q < \infty \ , \ \forall u \in W_0^{1,q}(\Omega).$$

Sobolev Inequalities .

•(Gagliardo-Nirenberg-Sobolev)

$$\left(\int_{\mathbb{R}^n} |u|^{q^*} \right)^{\frac{1}{q^*}} \leq C(n, q) (\int_{\mathbb{R}^n} |\nabla u|^q)^{\frac{1}{q}}, \ 1 \leq q < n \ , \forall u \in C_c^1(\mathbb{R}^n).$$

• Let Ω defined as above, $\partial\Omega \in C^1$

$$\left(\int_\Omega |u|^{q^*} \right)^{\frac{1}{q^*}} \leq C(n, q, \Omega) (\int_\Omega |u|^q + |\nabla u|^q)^{\frac{1}{q}}, \ 1 \leq q < n \ , \forall u \in W^{1,q}(\Omega).$$

• *Rellich–Kondrachov Compactness Theorem*

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with $\partial\Omega \in C^1$. Then $H^1(\Omega) \subset\subset L^q(\Omega)$, $1 \leq q < 2^*$.

So, regarding the last theorem the continuous embedding

$$H^1(\Omega) \subset L^{2^*}(\Omega)$$

is not compact and as a consequence we can not obtain the existence of solutions using standard variational methods.

CHAPTER 2

The Brezis-Nirenberg Problem

Let Ω be a bounded domain of $\mathbb{R}^n$ with $n \geq 3$. In this chapter we study the existence of u satisfying the nonlinear elliptic equation

$$\begin{cases} -\Delta u = u^{2^*-1} + \lambda u, & \Omega \\ u > 0, & \Omega \\ u = 0, & \partial\Omega, \end{cases} \tag{2.1}$$

where λ is a real constant.

The cases $n = 3$ and $n \geq 4$ turn out to be different.

- $n \geq 4$. Problem (2.1) has solution for every $\lambda \in (0, \lambda_1)$, where λ_1 denotes the first eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions. (Theorem 2.4.3(i)).
- $n = 3$. We can give a complete answer only when Ω is a ball. In this case, problem (2.1) has solution for $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$ (Theorem 2.4.3(ii)).
- Also, (2.1) has no solution for $\lambda \leq 0$ and Ω starshaped (Theorem 2.5.1.)

Our approach is variational so we observe solutions of (2.1) as nontrivial critical points of the functional

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*} - \frac{1}{2}\lambda\int_\Omega u^2.$$

Another viewpoint, which we shall use, is to seek for critical points of

$$\int_\Omega |\nabla u|^2 - \lambda\int_\Omega u^2$$

constrained on the sphere $\|u\|_{2^*} = 1$. In this way, such a critical point satisfies

$$-\Delta u - \lambda u = \mu u^{2^*-1},$$

where μ is a Lagrange multiplier and so after scaling we obtain a solution of (2.1).

Our goal is to prove that for suitable λ's the

$$\inf_{u\in H_0^1(\Omega)\,,\,\|u\|_{2^*}=1}\left(\int_\Omega |\nabla u|^2-\lambda\int_\Omega u^2\right)$$

is achieved. This minimization is not trivial, since as we mentioned before the embedding $H_0^1(\Omega)\subset L^{2^*}(\Omega)$ is not compact and as a result the mapping $u\mapsto \|u\|_{2^*}$ is not continuous under the weak convergence in the first space.

The first step in order to overcome the lack of compactness is to establish that for suitable λ's we have

$$\inf_{u\in H_0^1(\Omega)\,,\,\|u\|_{2^*}=1}\left(\int_\Omega |\nabla u|^2-\lambda\int_\Omega u^2\right)<\inf_{u\in H_0^1(\Omega)\,,\,\|u\|_{2^*}=1}\int|\nabla u|^2=S \tag{2.2}$$

and so for this matter we set

$$S_\lambda=\inf_{u\in H_0^1(\Omega)\,,\,\|u\|_{2^*}=1}\left(\|\nabla u\|_2^2-\lambda\,\|u\|_2^2\right)\,,\ \lambda\in\mathbb{R} \tag{2.3}$$

and

$$S=\inf_{u\in H_0^1(\Omega)\,,\,\|u\|_{2^*}=1}\|\nabla u\|_2^2\,, \tag{2.4}$$

where S corresponds to the best constant of the usual Sobolev embedding.
The arguments for estimating (2.3) were inspired by the work of Aubin [1] and the main point of the proof consists in evaluating the ratio

$$Q_\lambda(u)=\frac{\|\nabla u\|_2^2-\lambda\,\|u\|_2^2}{\|u\|_{2^*}^2},\ \text{for } u_\varepsilon(x)=\frac{\phi(x)}{(\varepsilon+|x|^2)^{\frac{n-2}{2}}},\ \text{with } \varepsilon>0.$$

The choice of these function is crucial since $(\varepsilon+|x|^2)^{-\frac{(n-2)}{2}}$ are extremal functions for the Sobolev inequality in $\mathbb{R}^n$ [15]. Proving (2.2) is essential, since as we will prove later (Lemma 2.4.1) this implies the existence of a minimizer of (2.3).

In the next sections, before we prove the main results about the Brezis-Nirenberg problem, we present some remarks concerning the best Sobolev constant. Next, we estimate S_λ, giving us a minimizer of (2.3). Regarding that, we can prove the existence of solutions of (2.1). Finally, we note some nonexistence results and we present some additional properties about the problem (2.1).

2.1 About the best Sobolev constant S

The best Sobolev constant S defined as in (2.4) will play an important role in our arguments, so in this section we summarize some facts about S.

It was shown by Talenti [15] that the best Sobolev constant is attained by a positive, radially symmetric function, besides all the functions obtained by rescaling it. Moreover, Gidas,Ni and Nirenberg [9,10] and Caffarelli, Gidas,Spuck [5] proved that any positive continuous solution of $-\Delta u=u^{2^*-1}$ is radially symmetric about a point which implies uniqueness among positive solutions that are regular at zero or infinity, in particular, of Talenti's minimizers.

We have that S depends only on the space dimension n, since the ratio

$$\frac{\|\nabla u\|_2}{\|u\|_{2^*}}$$

is invariant under scaling ;meaning that the ratio

$$\frac{\|\nabla u_k\|_2}{\|u_k\|_{2^*}}$$

is independent of k where $u_k(x) = u(kx)$.

For the next Lemma, we use a result of Pohozaev [11] that we will prove later (Theorem 2.5.1) and according to that as we mentioned in the beginning, we obtain a non existence result for (2.1), for $\lambda \leq 0$ and Ω smooth and starshaped.

Lemma 2.1.1. *The infimum in* (2.4) *is never achieved when Ω is a bounded domain.*

Proof. Suppose, by contradiction, that S is attained by some function $u \in H_0^1(\Omega)$. We may assume that $u \geq 0$ (otherwise we replace u by $|u|$).

Since Ω is bounded we can fix a ball $B_r(x) \supset \Omega$ and extending u we set

$$\breve{u} = \begin{cases} u, & \Omega \\ 0, & B_r(x) \setminus \Omega \end{cases}$$

Thus, S is also achieved on $B_r(x)$ and so $\breve{u}$ satisfies:

$$\Delta \breve{u} = \mu \breve{u}^p$$

for some constant $\mu > 0$ which contradicts Pohozaev's result, since there is no solution of (2.1) for $\lambda = 0$ on the ball $B_r(x)$ (starshaped). $\square$

Now, when $\Omega = \mathbb{R}^n$ the infimum is attained by

$$U(x) = C(1 + |x|^n)^{-\frac{(n-2)}{2}}. \tag{2.5}$$

or, after scaling of the type $\varepsilon^{-\frac{(n-2)}{2}} U(\frac{x}{\varepsilon})$, by any of the functions

$$U_\varepsilon(x) = C_\varepsilon(\varepsilon + |x|^2)^{-\frac{(n-2)}{2}}, \tag{2.6}$$

with C, C_ε normalization constants.[1,15]

So, comparing Pohozaev's negative result regarding the problem

$$\begin{cases} -\Delta u = u^{2^*-1}, & \Omega \\ u > 0, & \Omega \\ u = 0, & \partial\Omega, \end{cases}$$

with Ω being a starshaped domain, we observe how adding a lower-order term of $u^{2^*} - 1$ can change this nonexistence situation.

2.2 Proof of $S_\lambda < S$ for $n \geq 4$.

As we mentioned in the first section, the first step in order to establish the existence of solutions of (2.1) is to estimate (2.3). So, here we prove that for suitable λ's and space dimension $n \geq 4$, S_λ is strictly less than S.

Lemma 2.2.1.

$$\textit{We have that } S_\lambda < S \textit{ for all } \lambda > 0. \tag{2.7}$$

Proof. Without loss of generality we may assume $0 \in \Omega$.

We estimate the ratio

$$Q_\lambda(u_\varepsilon) = \frac{\|\nabla u_\varepsilon\|_2^2 - \lambda \|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2^*}^2},$$

with

$$u_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon + |x|^2)^{\frac{n-2}{2}}},\ \varepsilon > 0,$$

$\phi \in C_c^\infty(\Omega)$ positive, with $\phi(x) = 1$ in some neighborhood of 0.

We claim that for $\varepsilon \to 0$ we have

$$\|\nabla u_\varepsilon\|_2^2 = \frac{K_1}{\varepsilon^{\frac{n-2}{2}}} + O(1), \tag{2.8}$$

$$\|\nabla u_\varepsilon\|_{2^*}^2 = \frac{K_2}{\varepsilon^{\frac{n-2}{2}}} + O(\varepsilon), \tag{2.9}$$

$$\|u_\varepsilon\|_2^2 = \begin{cases} \frac{K_3}{\varepsilon^{\frac{n-4}{2}}} + O(1), & n \geq 5, \\ K_3 |log\ \varepsilon| + O(1), & n = 4, \end{cases} \tag{2.10}$$

where K_1, K_2, K_3 denote positive constants depending only on the space dimension n and such that $\frac{K_1}{K_2} = S$.

Indeed, for (2.8) we compute

$$\nabla u_\varepsilon(x) = \frac{\nabla\varphi(x)}{(\varepsilon + |x|^2)^{\frac{n-2}{2}}} - \frac{(n-2)\varphi(x)x}{(\varepsilon + |x|^2)^{\frac{n}{2}}}.$$

We have

$$|\nabla u_\varepsilon|^2 = \frac{|\nabla\varphi(x)|^2}{(\varepsilon + |x|^2)^{n-2}} + \frac{|x|^2(n-2)^2\varphi^2(x)}{(\varepsilon + |x|^2)^n} - \frac{(n-2)\varphi(x)2(x \cdot \nabla\phi(x))}{(\varepsilon + |x|^2)^{n-1}}.$$

Let $\delta > 0$ be small enough, using that

$$\int_\Omega |\nabla u_\varepsilon|^2 = \int_{B_\delta} |\nabla u_\varepsilon|^2 + \int_{\Omega \setminus B_\delta} |\nabla u_\varepsilon|^2,$$

we compute the middle term of $\int_\Omega |\nabla u_\varepsilon|^2$:

$$\begin{aligned}
\int_\Omega \frac{\phi^2(x)|x|^2}{(\varepsilon + |x|^2)^n} dx &= \int_{B_\delta} \frac{|x|^2}{(\varepsilon + |x|^2)^n} dx + \int_{\Omega \setminus B_\delta} \frac{\phi^2(x)|x|^2}{(\varepsilon + |x|^2)^n} dx \\
&= \int_{B_\delta} \frac{\phi^2(x)|x|^2}{(\varepsilon + |x|^2)^n} + \int_{\Omega \setminus B_\delta} \frac{(\phi^2(x) - 1)|x|^2}{(\varepsilon + |x|^2)^n} dx + \int_{\Omega \setminus B_\delta} \frac{|x|^2}{(\varepsilon + |x|^2)^n} dx \\
&= \int_\Omega \frac{|x|^2}{(\varepsilon + |x|^2)^n} dx + O(1),
\end{aligned}$$

using that, since $\phi = 1$ near 0, there exist C_1, C_2 constants such that

$$\left| \int_\Omega \frac{|\nabla\phi(x)|^2}{(\varepsilon + |x|^2)^{n-2}} \right| \leq C_1$$

and

$$\left| \int_\Omega \frac{(n-2)\varphi(x)2(x\cdot\nabla\phi(x))}{(\varepsilon+|x|^2)^{n-1}} \right| \leq C_2$$

for $\varepsilon \to 0$.

Thus, we conclude that

$$\int_\Omega |\nabla u_\varepsilon|^2 = (n-2)^2 \int_\Omega \frac{|x|^2}{(\varepsilon+|x|^2)^2} + O(1).$$

Now, we have

$$\int_\Omega \frac{|x|^2}{(\varepsilon+|x|^2)^n} = \int_{\mathbb{R}^n} \frac{|x|^2}{(\varepsilon+|x|^2)^n} - \int_{\mathbb{R}^n\setminus\Omega} \frac{|x|^2}{(\varepsilon+|x|^2)^n}$$

and by setting

$$I_1 = \int_{\mathbb{R}^n\setminus\Omega} \frac{|x|^2}{(\varepsilon+|x|^2)^n}$$

we have

$$B_\delta \subseteq \Omega \Rightarrow \mathbb{R}^n \setminus \Omega \subseteq \mathbb{R}^n \setminus B_\delta$$

$$|I_1| \leq \int_{\mathbb{R}^n\setminus B_\delta} \frac{|x|^2}{|x|^{2n}} < \infty$$

and so

$$\int_\Omega |\nabla u_\varepsilon|^2 = (n-2)^2 \int_{\mathbb{R}^n} \frac{|x|^2}{(\varepsilon+|x|^2)^n} + O(1) = \frac{K_1}{\varepsilon^{\frac{n-2}{2}}} + O(1),$$

where $K_1 = (n-2)^2 \int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^n} = \|\nabla U\|_2^2$, with $U(x) = (1+|x|^2)^{\frac{-(n-2)}{2}}$.

Moreover for (2.9) we have

$$\int_\Omega |u_\varepsilon|^{2^*} dx = \int_\Omega \frac{\phi^{2^*}(x)dx}{(\varepsilon+x^2)^n} = \int_\Omega \frac{\phi^{2^*}(x)-1}{(\varepsilon+|x|^2)^n}dx + \int_\Omega \frac{dx}{(\varepsilon+|x|^2)^n}$$
$$= O(1) + \int_{\mathbb{R}^n} \frac{dx}{(\varepsilon+|x|^2)^n} = \frac{K_2'}{\varepsilon^{\frac{n}{2}}} + O(1),$$

where

$$K_2' = \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^n}dx = \|U\|_{2^*}^{2^*}$$

and by Taylor expansion we have that (2.9) is obtained for $K_2 = \|U\|_{2^*}^2$ and $\frac{K_1}{K_2} = S$.

Indeed,

$$\|u_\varepsilon\|_{2^*}^2 = \left(\int_\Omega |u_\varepsilon|^{2^*} \right)^{\frac{n-2}{2}} = \frac{K_2}{\varepsilon^{\frac{n-2}{2}}} \left(1 + O(\varepsilon^{\frac{n}{2}})\right)^{\frac{n-2}{2}}$$

Setting

$$f(x) = (1+x)^{\frac{n-2}{2}}\ ,\ x = O(\varepsilon^{\frac{n}{2}})$$

we have

$$\|u_\varepsilon\|_{2^*}^2 = \frac{K_2}{\varepsilon^{\frac{n-2}{2}}} \left(1 + O(\varepsilon^{\frac{n}{2}})\right)$$
$$= \frac{K_2}{\varepsilon^{\frac{n-2}{2}}} + O(\varepsilon).$$

For (2.10) we have

$$\int_\Omega |u_\varepsilon|^2 = \int_\Omega \frac{[\varphi^2(x)-1]}{(\varepsilon+|x|^2)^{n-2}}dx + \int_\Omega \frac{dx}{(\varepsilon+|x|^2)^{n-2}} = O(1) + \int_\Omega \frac{dx}{(\varepsilon+|x|^2)^{n-2}}.$$

Now, we distinguish the cases where $n \geq 5$ and $n = 4$.

When $n \geq 5$,

$$\int_\Omega \frac{dx}{(\varepsilon+|x|^2)^{n-2}} = \int_{\mathbb{R}^n} \frac{dx}{(\varepsilon+|x|^2)^{n-2}} + O(1)$$

and so (2.10) is valid for

$$K_3 = \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{n-2}}.$$

When $n = 4$, since Ω is bounded we have that for constants R_1 and R_2 with $R_1 \leq R_2$ such that

$$\int_{|x|\leq R_1} \frac{dx}{(\varepsilon+|x|^2)^2} \leq \int_\Omega \frac{dx}{(\varepsilon+|x|^2)^2} \leq \int_{|x|\leq R_2} \frac{dx}{(\varepsilon+|x|^2)^2}$$

and

$$\int_{|x|\leq R} \frac{dx}{(\varepsilon+|x|^2)^2} = \omega \int_0^R \frac{r^3}{(\varepsilon+r^2)^2} = \frac{1}{2}\omega|log\varepsilon| + O(1),$$

where ω is the area of S^3 and so (2.11) is obtained for $K_3 = \frac{1}{2}\omega$.

Combining the above results we conclude that

$$Q_\lambda(u_\varepsilon) = \begin{cases} S + O(\varepsilon^{\frac{n-2}{2}}) - \lambda\frac{K_3}{K_2}\varepsilon, & n \geq 5 \\ S + O(\varepsilon) - \lambda\frac{K_3}{K_2}\varepsilon|log\varepsilon|, & n = 4. \end{cases}$$

So, in all cases $Q_\lambda(u_\varepsilon) < S$ provided $\varepsilon > 0$ is small enough. □

2.3 Proof of $S_\lambda < S$ for $n = 3$.

For the case $n = 3$ we assume for simplicity that

$$\Omega = \{x \in \mathbb{R}^n; |x| < 1\},$$

so that $\lambda_1 = \pi^2$(with corresponding eigenfunction $|x|^{-1}\sin(\pi x)$).

The counterpart of Lemma 2.2.1 is the following.

Lemma 2.3.1. *We have that*

$$S_\lambda < S \textit{ for all } \lambda > \frac{1}{4}\lambda_1.$$

Proof. We shall estimate the ratio

$$Q_\lambda(u_\varepsilon) = \frac{\|\nabla u_\varepsilon\|_2^2 - \lambda\|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_6^2},$$

with

$$u_\varepsilon(x) = u_\varepsilon(r) = \frac{\phi(r)}{(\varepsilon+r^2)^{\frac{1}{2}}},\ r = |x|,\ \varepsilon > 0,$$

with ϕ fixed, smooth function such that

$$\phi(0) = 1, \ \phi(1) = 0, \ \phi'(0) = 0.$$

We claim that for $\varepsilon \to 0$ we have

$$\|\nabla u_\varepsilon\|_2^2 = \frac{K_1}{\varepsilon^{\frac{1}{2}}} + \omega \int_0^1 |\phi'(r)|^2 dr + O(\varepsilon^{\frac{1}{2}}), \tag{2.11}$$

$$\|u_\varepsilon\|_6^2 = \frac{K_2}{\varepsilon^{\frac{1}{2}}} + O(\varepsilon^{\frac{1}{2}}), \tag{2.12}$$

$$\|u_\varepsilon\|_2^2 = \omega \int_0^1 \varphi^2(r) dr + O(\varepsilon^{\frac{1}{2}}), \tag{2.13}$$

where K_1, K_2 are positive constants with $\frac{K_1}{K_2} = S$ and ω is the area of S^2.

For (2.11) we have

$$u_\varepsilon{}'(r) = \frac{\varphi'(r)}{(\varepsilon + r^2)^{\frac{1}{2}}} - \frac{r\phi(r)}{(\varepsilon + r^2)^{\frac{3}{2}}}$$

and so

$$\|\nabla u_\varepsilon\|_2^2 = \omega \int_0^1 [\frac{|\phi'(r)|^2}{(\varepsilon + r^2)} - \frac{2r\phi(r)\phi'(r)}{(\varepsilon + r^2)^2} + \frac{r^2\phi^2(r)}{(\varepsilon + r^2)^3}] r^2 dr.$$

For the middle term, integrating by parts we have

$$-2\int_0^1 \frac{\varphi(r)\varphi'(r)r^3}{(\varepsilon + r^2)^2} dr = \int_0^1 \phi^2(r)[\frac{3r^2}{(\varepsilon + r^2)^2} - \frac{4r^4}{(\varepsilon + r^2)^3}] dr$$

and so

$$\|\nabla u_\varepsilon\|_2^2 = \omega \int_0^1 \frac{|\phi'(r)|^2 r^2}{(\varepsilon + r^2)} dr + 3\omega\varepsilon \int_0^1 \frac{\phi^2(r) r^2}{(\varepsilon + r^2)^3} dr. \tag{2.14}$$

Now, for the first term we have

$$\begin{aligned} \int_0^1 \frac{|\phi'(r)|^2}{(\varepsilon + r^2)} r^2 dr &= \int_0^1 \frac{|\phi'(r)|^2(\varepsilon + r^2 - \varepsilon)}{(\varepsilon + r^2)} dr \\ &= \int_0^1 |\phi'(r)|^2 dr - \varepsilon \int_0^1 \frac{|\phi'(r)|^2}{(\varepsilon + r^2)} dr \\ &= \int_0^1 |\phi'(r)|^2 dr + O(\varepsilon). \end{aligned} \tag{2.15}$$

And for the last term we have

$$\int_0^1 \frac{\phi^2(r) r^2}{(\varepsilon + r^2)^3} dr = \int_0^1 \frac{r^2}{(\varepsilon + r^2)^3} dr + \int_0^1 \frac{r^2[\phi^2(r) - 1]}{(\varepsilon + r^2)^3} dr.$$

Using the fact that

$$\phi(0) = 1, \phi'(0) = 0$$

and applying the mean value theorem twice we get

$$|\phi^2(r) - 1| = |\phi^2(r) - \phi^2(0)| = |(\phi^2(\xi))' r| = |2\phi(\xi)\phi'(\xi) r|,$$

$$|\phi'(\xi)| = |\phi'(\xi) - \phi'(0)| = |\phi''(\theta)\xi| \leq |r\xi|$$

We have that for

$$I_1 = \int_0^1 \frac{r^2[\phi^2(r)-1]}{(\varepsilon+r^2)^3}dr$$

$$|I_1| \le C\int_0^1 \frac{r^4}{(\varepsilon+r^2)^3}dr,$$

where

$$\int_0^1 \frac{r^4}{(\varepsilon+r^2)^3}dr = \int_0^1 \frac{r^4}{\varepsilon^3(1+\frac{r^2}{\varepsilon})}dr = \int_0^{\varepsilon^{-\frac{1}{2}}} \frac{t^4\varepsilon^2}{\varepsilon^3(1+t^2)^3}\varepsilon^{\frac{1}{2}}dt = O(\varepsilon^{-\frac{1}{2}}).$$

So, we have

$$\int_0^1 \frac{\phi^2(r)r^2}{(\varepsilon+r^2)^3}dr = \int_0^1 \frac{r^2}{(\varepsilon+r^2)^3}dr + O(\varepsilon^{-\frac{1}{2}}) \tag{2.16}$$

and

$$\int_0^1 \frac{|\phi'(r)|^2r^2}{(\varepsilon+r^2)}dr = \int_0^1 |\phi'(r)|^2dr + O(\varepsilon). \tag{2.17}$$

Moreover,

$$\int_0^1 \frac{r^2}{(\varepsilon+r^2)^3}dr = \varepsilon^{-\frac{3}{2}}\int_0^{\varepsilon^{-\frac{1}{2}}} \frac{s^2}{(1+s^2)^3}ds = \varepsilon^{-\frac{3}{2}}\int_0^\infty \frac{s^2}{(1+s^2)^3}ds + O(1). \tag{2.18}$$

So, from (2.16), (2.17), (2.18) we have

$$\|\nabla u_\varepsilon\|_2^2 = \omega\int_0^1 |\phi'(r)|^2dr + 3\omega\varepsilon^{-\frac{1}{2}}\int_0^\infty \frac{s^2}{(1+s^2)^3}ds + O(\varepsilon^{-\frac{1}{2}}).$$

So, we obtain (2.11) with

$$K_1 = 3\omega\int_0^\infty \frac{s^2}{(1+s^2)}ds.$$

Also, since

$$\int_0^\infty \frac{s^2}{(1+s^2)^3}ds = \frac{1}{16}\pi \text{ and } \int_0^\infty \frac{s^4}{(1+s^2)^3}ds = \frac{3}{16}\pi$$

we have that

$$K_1 = \int_{\mathbb{R}^3} |\nabla U|^2dx, \text{ for } U(x) = \frac{1}{(1+|x|^2)^{\frac{1}{2}}},$$

since $\int_{\mathbb{R}^3}|\nabla U|^2 = \omega\int_0^\infty \frac{s^4}{(1+s^2)^3}ds$.

For (2.12)

$$\|u_\varepsilon\|_6^6 = \omega\int_0^1 \frac{\phi^6(r)r^2}{(\varepsilon+r^2)^3}dr = \omega\int_0^1 \frac{(\phi^6(r)-1)r^2}{(\varepsilon+r^2)^3}dr + \omega\int_0^1 \frac{r^2}{(\varepsilon+r^2)^3}dr = I_1 + I_2.$$

Since $\phi(0)=1, \phi'(0)=0$, applying again the mean value theorem twice we have

$$|I_1| \le C\int_0^1 \frac{r^4}{(\varepsilon+r^2)^3}dr = O(\varepsilon^{-\frac{1}{2}})$$

and

$$I_2 = \frac{\omega}{\varepsilon^{\frac{3}{2}}} \int_0^{\varepsilon^{-\frac{1}{2}}} \frac{s^2}{(1+s^2)^3} ds = \frac{\omega}{\varepsilon^{\frac{3}{2}}} \int_0^{\infty} \frac{s^2}{(1+s^2)^3} ds + O(1)$$

and so

$$\|u_\varepsilon\|_6^6 = \omega \int_0^1 \frac{(\phi^6(s)-1)s^2}{(\varepsilon+s^2)^3} ds + \frac{\omega}{\varepsilon^{\frac{3}{2}}} \int_0^{\infty} \frac{s^2}{(1+s^2)^3} ds + O(1) \Rightarrow$$

$$\|u_\varepsilon\|_6^6 = \frac{1}{\varepsilon^{\frac{3}{2}}}[\omega \varepsilon^{\frac{3}{2}} \int_0^1 \frac{(\phi^6(s)-1)s^2}{(\varepsilon+s^2)^3} ds + \omega \int_0^{\infty} \frac{s^2}{(1+s^2)^3} ds + O(1)]$$

that is,

$$\|u_\varepsilon\|_6^6 = \frac{1}{\varepsilon^{\frac{3}{2}}}[\omega \int_0^{\infty} \frac{s^2}{(1+s^2)^3} ds + O(\varepsilon)]$$

and by Taylor expansion again like we did in (2.9) for $K_2 = \left(\omega \int_0^\infty \frac{s^2}{(1+s^2)^3} ds\right)^{\frac{1}{3}} = \|U\|_6^2$ we obtain (2.12).

For (2.13)

$$\|u_\varepsilon\|_2^2 = \omega \int_0^1 \frac{\phi^2(r)}{(\varepsilon+r^2)} dr = \omega \int_0^1 \phi^2(r) dr + O(\varepsilon^{\frac{1}{2}}).$$

Combining (2.11), (2.12), (2.13) we get

$$Q_\lambda(u_\varepsilon) = S + \varepsilon^{\frac{1}{2}} \frac{\omega}{K_2} [\int_0^1 |\phi'(r)|^2 dr - \lambda \int_0^1 \phi^2(r) dr] + O(\varepsilon). \tag{2.19}$$

Choosing $\phi(r) = cos(\frac{\pi r}{2})$ we have

$$\int_0^1 |\phi'(r)|^2 dr = \frac{\pi^2}{4} \int_0^1 \phi^2(r) dr$$

and from (2.19) we have

$$Q_\lambda(u_\varepsilon) = S + (\frac{1}{4}\pi^2 - \lambda) C \varepsilon^{\frac{1}{2}} + O(\varepsilon),$$

for positive constant C. The conclusion of the lemma follows by choosing ε small enough. □

2.4 Existence of Positive solutions

Now, we shall move on to the next step regarding the proof of the existence of solutions of (2.1). This consists of obtaining a minimizer of (2.3). In the previous sections we proved that for suitable λ's in each case we have that

$$S_\lambda < S.$$

This result is crucial because of the next lemma.

Lemma 2.4.1. *(E.Lieb) If $S_\lambda < S$, the infimum in* (2.1) *is attained* .

We present two proofs. The first is the following

Proof. Suppose $S_\lambda < S$ and let $(u_j) \subset H_0^1(\Omega)$ be a minimizing sequence for (2.3) such that

$$\|\nabla u_j\|_2^2 - \lambda \|u_j\|_2^2 = S_\lambda + o(1)\ ,\ j \to \infty. \tag{2.20}$$

with

$$\|u_j\|_{2^*} = 1.$$

So, by Hölder's inequality we get

$$\int_\Omega |u_j|^2 \leq (\int_\Omega |u_j|^{2^*})^{\frac{n-2}{n}} |\Omega|^{\frac{2}{n}}$$

and so

$$\|u_j\|_2 \leq |\Omega|^{\frac{1}{n}},$$

i.e.

$$\sup_j \|u_j\|_2 < \infty.$$

Moreover, from (2.20) we obtain that $\sup_j \|\nabla u_j\|_2 < \infty$ and so u_j is bounded in $H_0^1(\Omega)$. Now, since H_0^1 is a reflexive space we may extract a subsequence u_j so that

$$u_j \rightharpoonup u \text{ in } H_0^1(\Omega).$$

.

From the Rellich-Kondrachov Theorem we have that

$$u_j \to u \text{ in } L^2(\Omega)$$

and

$$u_j \to u \text{ a.e. on } \Omega.$$

Also, we may assume $u_j \geq 0$ (otherwise we replace it by $|u_j|$) and from Fatou's Lemma we get $\|u\|_{2^*} \leq 1$.

Let $v_j = u_j - u$ and so

$$v_j \rightharpoonup 0 \text{ in } H_0^1(\Omega)$$

and

$$v_j \to 0 \text{ a.e. on } \Omega.$$

From (2.4) and since $\|u_j\|_{2^*} = 1$ we have

$$\|\nabla u_j\|_2 \geq S_0 = S.$$

So, from (2.20) passing to the limit we have

$$\lambda \|u\|_2^2 \geq S - S_\lambda > 0 \text{ and so } u \neq 0.$$

We have

$$\int_\Omega |\nabla v_j|^2 = \int_\Omega |\nabla(u_j - u)|^2 = \int_\Omega |\nabla u_j|^2 - 2\int_\Omega \nabla u_j \cdot \nabla u + \int_\Omega |\nabla u|^2$$

and so

$$\lim_j \int_\Omega |\nabla v_j|^2 = \lim_j \int_\Omega |\nabla u_j|^2 - \int_\Omega |\nabla u|^2. \tag{2.21}$$

So, using (2.20) we conclude that

$$\|\nabla u\|_2^2 + \|\nabla v_j\|_2^2 - \lambda \|u\|_2^2 = S_\lambda + o(1). \tag{2.22}$$

Moreover, we have that u_j is bounded in $L^{2^*}(\Omega)$ and $u_j \to u$ almost everywhere on Ω and so from Brezis-Lieb Lemma (proof in the Appendix) we have

$$\int_\Omega |u_j|^{2^*} - \int_\Omega |v_j|^{2^*} \to \int_\Omega |u|^{2^*},$$

that is

$$1 = \|v_j\|_{2^*}^{2^*} + \|u\|_{2^*}^{2^*} + o(1) \tag{2.23}$$

and by convexity we have

$$1 \le \left(\int_\Omega |u|^{2^*} + \int_\Omega |u_j|^{2^*} \right)^{\frac{2}{2^*}} + o(1) \le \|v_j\|_{2^*}^2 + \|u\|_{2^*}^2 + o(1).$$

and by Sobolev inequality we conclude

$$1 \le \|u\|_{2^*}^2 + \frac{1}{S} \|\nabla v_j\|_2^2 + o(1) \tag{2.24}$$

To finish the proof, we claim that

$$\|\nabla u\|_2^2 - \lambda \|u\|_2^2 \le S_\lambda \|u\|_{2^*}^2 . \tag{2.25}$$

In order to do that, we shall distinguish the cases where

(i) $S_\lambda > 0$, meaning $0 < \lambda < \lambda_1$,

(ii) $S_\lambda \le 0$, meaning $\lambda \ge \lambda_1$.

For the first case, myltiplying (2.24) by S_λ, since S_λ is positive, we have that

$$\begin{aligned} S_\lambda &\le S_\lambda \|u\|_{2^*}^2 + \frac{S_\lambda}{S} \|\nabla v_j\|_2^2 + o(1) \\ &\Rightarrow S_\lambda \le^{(2.22)} S_\lambda \|u\|_{2^*}^2 + S_\lambda + \lambda \|u\|_2^2 - \|\nabla u\|_2^2 \end{aligned}$$

So,

$$\|\nabla u\|_2^2 - \lambda \|u\|_2^2 \le S_\lambda \|u\|_{2^*}^2 .$$

Now, for the second case, since $\|u\|_{2^*}^2 \le 1$ and $S_\lambda < 0$, we have

$$S_\lambda \le S_\lambda \|u\|_{2^*}^2 .$$

Also, since the $\lim_j \int_\Omega |\nabla v_j|^2 \ge 0$, from (2.22) we have

$$\begin{aligned} 0 &\le S_\lambda + \lambda \|u\|_2^2 - \|\nabla u\|_2^2 \\ &\le S_\lambda \|u\|_{2^*}^2 + \lambda \|u\|_2^2 - \|\nabla u\|_2^2 \end{aligned}$$

So,

$$\|\nabla u\|_2^2 - \lambda \|u\|_2^2 \le S_\lambda \|u\|_{2^*}^2 .$$

So, we proved that in all cases (2.25) is true. Then, since $u \not\equiv 0$, the proof is complete.

□

Now, regarding the second proof we present the proof from [17] and the statement here is proved for $S_\lambda > 0$, i.e. $0 < \lambda < \lambda_1$.

Proof. We have that

$$\|u\|_\lambda^2 := \int_\Omega \left(|\nabla u|^2 - \lambda u^2\right)$$

defines an equivalent norm on $H_0^1(\Omega)$ thanks to Poincaré inequality.

Set $w_n = u_n - u$ and again from Brezis-Lieb Lemma we obtain that

$$1 = \|u\|_{2^*}^{2^*} + \lim_n \|w_n\|_{2^*}^{2^*}.$$

We have

$$\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2 + \lim \int_\Omega |\nabla(u_n - u)|^2 - \lambda \int_\Omega |u_n - u|^2 = \lim_n (\int_\Omega |\nabla u_n|^2 - \lambda \int_\Omega |u_n|^2)$$

and

$$\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2 + \lim_n \|\nabla w_n\|_2^2 = \lim_n (\int_\Omega |\nabla u_n|^2 - \lambda \int_\Omega |u_n|^2).$$

So combining the above results we have

$$\begin{aligned} S_\lambda = \lim_n \|u_n\|_\lambda^2 &= \|u\|_\lambda^2 + \lim_n \|w_n\|_\lambda^2 \\ &= \|u\|_\lambda^2 + \lim_n \|\nabla w_n\|_2^2 . \end{aligned}$$

Moreover, from Sobolev inequality and Brezis-Lieb Lemma we have

$$\lim_n \|\nabla w_n\|_2^2 \geq S \lim_n \|w_n\|_{2^*}^2 = S(1 - \|u\|_{2^*}^{2^*})^{\frac{2}{2^*}}.$$

So from the definition of S_λ we obtain

$$S_\lambda \geq S_\lambda \|u\|_{2^*}^2 + S(1 - \|u\|_{2^*}^{2^*})^{\frac{2}{2^*}}.$$

But

$$S_\lambda < S$$

and so

$$1 > \|u\|_{2^*}^2 + (1 - \|u\|_{2^*}^{2^*})^{\frac{2}{2^*}}. \tag{2.26}$$

Setting $a = \|u\|_{2^*}^{2^*}$ we have

$$1 > a^{\frac{2}{2^*}} + (1-a)^{\frac{2}{2^*}},$$

where $0 < a < 1$ and $\frac{2}{2^*} < 1$. But $f(x) = x^k$ for $k < 1$ is concave and so necessarily,

$$\|u\|_{2^*} = 1$$

and the proof is done, since from weak lower semicontinuity we have

$$\int_\Omega |\nabla u|^2 \leq \liminf_n \int_\Omega |\nabla u_n|^2$$

$$-\lambda \int_\Omega |u_n|^2 \to -\lambda \int_\Omega |u|^2 \Rightarrow$$

$$\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2 \leq \liminf_n \left(\int_\Omega |\nabla u_n|^2 - \lambda \int_\Omega |u_n|^2\right) = S_\lambda.$$

Remark 2.4.2. *We have, from the arguments of those proofs, that every minimizing sequence of* (2.3) *is relatively compact in* H_0^1*, with the strong topology.*

Proof. Indeed, we shall use the following. From (2.22) and (2.23) we have

$$\|\nabla u\|_2^2 - \lambda \|u\|_2^2 + \|\nabla v_j\|_2^2 \leq S_\lambda(\|u\|_{2^*}^2 + \|v_j\|_{2^*}^2) + o(1) \tag{2.27}$$

Moreover, from the definition of S_λ and from the usual Sobolev inequality we have

$$\int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2 \geq S_\lambda \|u\|_{2^*}^2 \tag{2.28}$$

and for $S_\lambda \leq S$ we have

$$S_\lambda \|v_j\|_{2^*}^2 \leq \frac{S_\lambda}{S} \int_\Omega |\nabla v_j|^2 \leq \int_\Omega |\nabla v_j|^2. \tag{2.29}$$

We claim that from (2.27), (2.28), (2.27) we conclude

$$\int_\Omega |\nabla v_j|^2 = \frac{S_\lambda}{S} \int_\Omega |\nabla v_j|^2 + o(1).$$

Indeed, we have that

$$\begin{aligned} \int_\Omega |\nabla v_j|^2 &\leq^{(2.27)} S_\lambda \|u\|_{2^*}^2 + S_\lambda \|v_j\|_{2^*}^2 - \int_\Omega |\nabla u|^2 + \lambda \int_\Omega u^2 + o(1) \\ (2.28), (2.29) &\leq \frac{S_\lambda}{S} \int_\Omega |\nabla v_j|^2 + o(1). \end{aligned} \tag{2.30}$$

So,

$$\int_\Omega |\nabla v_j|^2 = \frac{S_\lambda}{S} \int_\Omega |\nabla v_j|^2 + o(1)$$

and since from the hypothesis $S_\lambda < S$, we conclude that

$$\int_\Omega |\nabla v_j|^2 = o(1),$$

meaning that

$$u_j \to u, \text{ in } H_0^1(\Omega).$$

□

Now, we can proceed with the proof of the main results.

Theorem 2.4.3. *(i) Let $n \geq 4$. Then, for every $\lambda \in (0, \lambda_1)$ there exists a solution of* (2.1)*.*

(ii) Let $n = 3$ and assume that Ω is a ball. Then (2.1) *has a solution if $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$.*

Proof. Note here that the proof is the same for the cases $n = 3$ and $n \geq 4$, since from Lemma 2.4.1 we obtained a minimizer for (2.3). The only difference depends on the λ's for the estimate of (2.3).

So, let $u \in H_0^1(\Omega)$ be given by Lemma 2.4.1 such that

$$\|u\|_{2^*} = 1 \text{ and } \|\nabla u\|_2^2 - \lambda \|u\|_2^2 = S_\lambda.$$

We may assume $u \geq 0$ (otherwise we replace it by $|u|$). Since u is a minimizer of (2.3) we obtain a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$-\Delta u - \lambda u = \mu u^{2^*-1} \text{ in } \Omega. \tag{2.31}$$

In fact, due to (2.3) $\mu = S_\lambda$. Indeed, multiplying each side of (2.31) by u and integrating over Ω we have

$$S_\lambda = \int_\Omega |\nabla u|^2 - \lambda u^2 = \mu \int_\Omega u^{2^*} = \mu.$$

Moreover, $S_\lambda > 0$ since $\lambda < \lambda_1$.

So for $k > 0$,

$$-\Delta u - \lambda u = S_\lambda u^{2^*-1} \Rightarrow -k\Delta u - \lambda k u = S_\lambda k u^{2^*-1} = S_\lambda (ku)^{2^*-1} k^{2-2^*}$$

and let $\breve{v} = ku$, so

$$-\Delta \breve{u} - \lambda \breve{u} = S_\lambda k^{2-2^*} \breve{u}^{2^*-1}$$

and so for $k = S_\lambda^{\frac{1}{2-2^*}}$ we have that u after scaling is a solution of (2.1). Finally, $u > 0$ on Ω by the strong maximum principle . □

Remark 2.4.4. *The first proof of Theorem 2.4.3 did not involve Lemma 2.2.1. Instead, Brezis and Nirenberg denoted*

$$\mu_q = \inf_{u \in H_0^1(\Omega)\,,\, \|u\|_{q^*}=1} \left\{ \|\nabla u\|_2^2 - \lambda \|u\|_2^2 \right\} \text{ for } q < p. \tag{2.32}$$

We have that $\lim_{q \to p} \mu_q = S_\lambda$. Moreover the embedding $H_0^1(\Omega) \subset L^{q^*}(\Omega)$ is compact so the infimum in (2.32) is achieved by some $u_q \in H_0^1(\Omega)$ such that $u_q \geq 0$ on Ω, $\|u_q\|_{q^*} = 1$ and

$$-\Delta u_q - \lambda u_q = \mu_q u_q^{q^*-1}. \tag{2.33}$$

Thus, from Sobolev inequality and from (2.32) we get

$$S \|u_q\|_{2^*}^2 - \lambda \|u_q\|_2^2 \leq \|\nabla u_q\|_2^2 - \lambda \|u_q\|_2^2 = \mu_q. \tag{2.34}$$

For $q \to p$ passing to a subsequence u_q, we have $u_q \rightharpoonup u$ on $H_0^1(\Omega)$ and so passing to the limit in (2.34) we get

$$S - \lambda \|u\|_2^2 \leq S_\lambda$$

and $u \not\equiv 0$ (Lemma 2.2.1). Finally, from (2.33) we have that u satisfies

$$-\Delta u - \lambda u = S_\lambda u^{2^*-1}$$

and after scaling, as in the proof of the Theorem 2.4.3 we obtain a solution of (2.1). □

2.5 Some nonexistence results.

In order to have a more complete answer about the existence and nonexistence of solutions of (2.1) we shall prove the next nonexistence results.

Theorem 2.5.1. *(i) There is no solution of (2.1) when $\lambda \geq \lambda_1$.*

(ii) There is no solution of (2.1) for $\lambda \leq 0$ and Ω smooth and starshaped (Pohozaev [11]).

Proof. (i) Let ϕ_1 be the eigenfunction of $-\Delta$ corresponding to λ_1 with $\phi_1 > 0$ on Ω and let u solution of (2.1). We have

$$-\Delta\varphi_1 = \lambda_1\varphi_1.$$

Then, integrating by parts we have

$$-\int_\Omega (\Delta\phi_1)u = -\int_\Omega (\Delta u)\phi_1 = \lambda_1\int_\Omega u\phi_1 = \int_\Omega u^{2^*-1}\phi_1 + \lambda\int_\Omega u\phi_1 > \lambda\int_\Omega u\phi_1$$

and so $\lambda < \lambda_1$.

(ii) For the proof of this statement we need Pohozaev's identity [11] which is the following

Suppose u is a smooth function satisfying

$$\begin{cases} -\Delta u = g(u), & \Omega \\ u = 0, & \partial\Omega, \end{cases} \tag{2.35}$$

where g is continuous in $\mathbb{R}$. Then, we have

$$(1-\frac{n}{2})\int_\Omega g(u)u + n\int_\Omega G(u) = \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2, \tag{2.36}$$

with $G(u) = \int_0^u g(t)dt$ and ν denotes the outward normal to $\partial\Omega$.

In order to prove (2.36) we should use that for any $u \in C^2(\Omega)$ with $u = 0$ on $\partial\Omega$ we have that

$$\Delta(x\cdot\nabla u) = x\cdot\nabla\Delta u + 2\Delta u\ ,\ x\in\Omega$$

$$x\cdot\nabla u = (x\cdot\nu)\frac{\partial u}{\partial\nu}\ ,\ x\in\partial\Omega$$

and then integrating by parts 3 times we obtain

$$2\int_\Omega (x\cdot\nabla u)\Delta u dx = (2-n)\int_\Omega u\Delta u dx + \int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2 dS_x.$$

The proof is the following

$$\begin{aligned}\int_\Omega (x\cdot\nabla u)\Delta u dx &= -\int_\Omega \nabla(x\cdot\nabla u)dx + \int_{\partial\Omega}(x\cdot\nabla u)\frac{\partial u}{\partial\nu}dS_x \\ &= \int_\Omega \Delta(x\cdot\nabla u)dx + \int_\Omega (x\cdot\nabla u)\frac{\partial u}{\partial\nu}dS_x \\ &= \int_\Omega (x\cdot\nabla\Delta u + 2\Delta u)u dx + \int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2 dS_x.\end{aligned}$$

Setting $b = \Delta u$ we have

$$\int_\Omega (x\cdot\nabla b)u dx = \int_\Omega \sum_{i=1}^n x_i b_{x_i} u = -\sum_{i=1}^n \int_\Omega b(x_i u)_{x_i} = -\int_\Omega b\, div(xu)$$

thus, the last equality becomes

$$\int_\Omega (x\cdot\nabla\Delta u + 2\Delta u)u dx + \int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2 dS_x$$
$$= \int_\Omega \left(-div(xu)\Delta u + 2u\Delta u\right) dx + \int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2 dS_x,$$

so that

$$\int_\Omega (x\cdot\nabla\Delta u + 2\Delta u)u dx + \int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2 dS_x$$
$$= \int_\Omega [-(x\cdot\nabla u)\Delta u - (n-2)u\Delta u] dx + \int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2 dS_x.$$

So, we proved that

$$\int_\Omega (x\cdot\nabla u)\Delta u = (1-\frac{n}{2})\int_\Omega u\Delta u + \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)^2 (x\cdot\nu) dS_x.$$

Then, (2.36) is obtained by setting $-\Delta u = g(u)$ and noticing

$$-\int_\Omega x\cdot\nabla u g(u) dx = -\int_\Omega x\cdot\nabla G(u) dx = n\int_\Omega G(u) dx.$$

So, in order to move on with our proof, we choose $g(u) = u^{2^*-1} + \lambda u$ and so we get from (2.36) that

$$\lambda\int_\Omega u^2 = \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2. \tag{2.37}$$

But since Ω is starshaped $x\cdot\nu > 0$ almost everywhere on $\partial\Omega$. So, for $\lambda < 0$ using (2.37) we get $u = 0$. When $\lambda = 0$ again from (2.37) we conclude that $\frac{\partial u}{\partial\nu} = 0$ in $\partial\Omega$ and then from (2.1) we have

$$0 = -\int_\Omega \Delta u = \int_\Omega u^{2^*-1}$$

and so $u = 0$, using Green's identity.

□

Remark 2.5.2. *On the other hand, when Ω is annulus, Kazdan and Warner[12] proved that for $\lambda \in (-\infty, \lambda_1)$ there exists a radial solution of* (2.1).

Lemma 2.5.3. *Suppose $n = 3$. There is no solution of* (2.1) *for $\lambda \leq \frac{1}{4}\lambda_1$.*

Proof. Suppose that u is a solution for (2.1). From Gidas-Ni-Nirenberg result [9,10] we know that u must be spherically symmetric (since here Ω is a ball) and so u satisfies

$$\begin{cases} -u'' - \frac{2}{r}u' = u^5 + \lambda u, & (0,1) \\ u'(0) = u(1) = 0. \end{cases} \tag{2.38}$$

We claim that

$$\int_0^1 u^2(r)(\lambda\psi'(r)+\frac{1}{4}\psi'''(r))r^2dr = \frac{2}{3}\int_0^1 u^6(r)(r\psi(r)-r^2\psi'(r))dr+\frac{1}{2}|u'(1)|^2\psi(1), \quad (2.39)$$

for every smooth function ψ with $\psi(0)=0$.

Indeed, we should multiply (2.38) by $r^2\psi(r)u'(r)$ and integrate over $(0,1)$.

By integration by parts we have

$$-\frac{1}{2}\left[|u'|^2r^2\psi(r)\right]_0^1 - \int_0^1 |u'(r)|^2 r\psi(r)dr + \frac{1}{2}\int_0^1 |u'(r)|^2r^2\psi'(r)dr$$
$$= \left[\frac{1}{6}u^6 r\psi(r)\right]_0^1 - \frac{1}{6}\int_0^1 u^6(2r\psi(r)-r\psi(r))dr + [\lambda\frac{1}{2}u^2r^2\psi(r)]_0^1$$
$$-\frac{1}{2}\lambda\int_0^1 u^2(2r\psi(r)+r^2\psi'(r))dr$$

and so we obtain

$$\int_0^1 |u'|^2(\frac{1}{2}r^2\psi'(r)-r\psi(r))dr - \frac{1}{2}|u'(1)|^2\psi(1)$$
$$= -\frac{1}{6}\int_0^1 u^6(2r\psi(r)+r^2\psi'(r))dr - \frac{1}{2}\lambda\int_0^1 u^2(r)[2r\psi(r)+r^2\psi'(r)]dr. \quad (2.40)$$

Integrating and multiplying again (2.38) by $(\frac{1}{2}r^2\psi'(r)-r\psi(r))u$ we get

$$-\int_0^1 u''(r)r^2\psi(r)u'(r) - 2\int_0^1 |u'(r)|^2 r\psi(r)dr$$
$$= \int_0^1 u^5(r)r^2\psi(r)u'(r)dr + \lambda\int_0^1 r^2\psi(r)u'(r)dr$$

and so

$$-\int_0^1 (|u'(r)|^2)'r^2\psi(r)dr - 2\int_0^1 |u'(r)|^2 r\psi(r)dr$$
$$= \frac{1}{6}\int_0^1 (u^6(r))'r^2\psi(r)u'(r)dr + \lambda\int_0^1 u'(r)r^2\psi(r)dr.$$

Next, integrating by parts we obtain

$$\int_0^1 |u'(r)|^2(\frac{1}{2}r^2\psi'(r)-r\psi(r))dr - \frac{1}{4}\int_0^1 u^2(r)r^2\psi'''(r)dr$$
$$= \int_0^1 u^6(r)(\frac{1}{2}r^2\psi'(r)-r\psi(r))dr + \lambda\int_0^1 u^2(\frac{1}{2}r^2\psi'(r)-r\psi(r))dr. \quad (2.41)$$

Then, subtracting (2.41) from (2.40) we obtain (2.39).

We have seen, from Pohozaev's identity, that (2.1) has no solution for $\lambda \le 0$, thus we may assume that

$$0 < \lambda \le \frac{1}{4}\pi^2.$$

Choosing $\psi(r) = \sin\left((4\lambda)^{\frac{1}{2}} r\right)$ in (2.39) so

$$\psi(1) \geq 0, \ \lambda\psi'(r) + \frac{1}{4}\psi'''(r) = 0$$

and

$$r\psi(r) - r^2\psi'(r) = r\sin\left((4\lambda)^{\frac{1}{2}} r\right) - r^2(4\lambda)^{\frac{1}{2}}\cos\left((4\lambda)^{\frac{1}{2}} r\right) > 0 \text{ on } (0,1],$$

since $sin\theta - \theta cos\theta > 0$ for all $\theta \in (0, \pi]$ we obtain a contradiction. □

2.6 Additional properties.

• Regularity of solutions.

As we mentioned earlier, the solution u of (2.1) given by Theorem 2.4.3 belongs to $H_0^1(\Omega)$ (Remark 2.4.2). In particular, $u \in L^\infty(\Omega)$. Indeed, one should use the following Lemma.

Lemma 2.6.1. *(Brezis-Kato[4]) Let $u \in H_0^1(\Omega)$ such that*

$$-\Delta u = \alpha u \textit{ in } \Omega,$$

with $\alpha(x) \in L^{\frac{n}{2}}(\Omega)$ and $n \geq 3$. Then, $u \in L^t(\Omega)\ \forall\, t < \infty$.

For our purpose, we use Lemma 2.6.1 with $\alpha(x) = \lambda + u^{2^*-2}$. So, since $u \in L^{2^*}$ we have $a \in L^{\frac{n}{2}}$. Thus, by elliptic regularity we obtain that $u \in C^\infty(\Omega)$.

• When the exponent in (2.1), name it p, is greater than $2^* - 1$ and Ω is starshaped, problem (2.1) has no solution if $\lambda \leq \lambda^*$, where λ^* is a positive constant depending on Ω and the exponent p.

Suppose u satisfies (2.1). From Pohozaev's identity we have

$$(1 - \frac{n}{2})\int_\Omega (u^{p+1} + \lambda u^2) + n\int_\Omega (\frac{u^{p+1}}{p+1} + \frac{\lambda u^2}{2}) = \frac{1}{2}\int_{\partial\Omega} (x \cdot \nu)\left(\frac{\partial u}{\partial \nu}\right)^2 > 0$$

and so

$$(-1 + \frac{n}{2} - \frac{n}{p+1})\int_\Omega u^{p+1} < \lambda \int_\Omega u^2. \tag{2.42}$$

Then, combining (2.1), (2.42) and using Poincaré inequality we obtain

$$\begin{aligned}\lambda_1 \int_\Omega u^2 \leq \int_\Omega |\nabla u|^2 &= \int_\Omega u^{p+1} + \lambda \int_\Omega u^2 \\ &< \lambda(-1 + \frac{n}{2} - \frac{n}{p+1})^{-1}\int_\Omega u^2 + \lambda \int_\Omega u^2,\end{aligned}$$

so

$$\lambda > \lambda_1 \frac{n-2}{n}\frac{p - \frac{n+2}{n-2}}{p-1}.$$

• Uniqueness-nonuniqueness. When Ω is a ball Gidas,Ni and Nirenberg proved that every solution of (2.1) is spherically symmetric [9,10]. On the other hand, if Ω is an annulus with

$n \geq 4$ then (2.1) admits both radial and nonradial solutions for all $\lambda > 0$ sufficiently small. For this matter, regarding the radial solutions, we define

$$\Sigma_\lambda = \inf_{u \in H_r \, , \, \|u\|_{2^*}=1} \left\{ \|\nabla u\|_2^2 - \lambda \|u\|_2^2 \right\}, \tag{2.43}$$

where $H_r = \{u \in H_0^1(\Omega), \text{ u is radial } \}$. In this case, it is known [14] that the embedding

$$H_r(\Omega) \subset L^{2^*}(\Omega)$$

is compact and so the infimum in (2.43) is achieved by some $u_\lambda \in H_r$ such that assuming $u_\lambda \geq 0$ on Ω, with $\|u_\lambda\|_{2^*} = 1$, satisfying

$$-\Delta u_\lambda - \lambda u_\lambda = \Sigma_\lambda u_\lambda^p.$$

If $\lambda < \lambda_1$ then $\Sigma_\lambda > 0$ and so by scaling we obtain a solution of (2.1).

Now, for the nonradial case we consider S_λ defined by (2.3). We have that $\lambda \mapsto S_\lambda$ and $\lambda \mapsto \Sigma_\lambda$ are continuous. Moreover, $S = S_0 < \Sigma_0$, since otherwise we would have that the best Sobolev constant would be achieved, which is a contradiction, since we are in a bounded domain (Lemma 2.1.1). Thus, from continuity, $S_\lambda < \Sigma_\lambda$ and for $\lambda > 0$ sufficiently small from Lemma 2.2.1 the infimum in (2.3) is achieved by some nonradial function. So, we conclude that in the case where Ω is an annulus and $n \geq 4$ we obtain both radial and nonradial solutions.

• **Equations with variable coefficients.** Let $\Omega \subset \mathbb{R}^n, \; n \geq 4$ and $a(x) \in C^\infty(\Omega)$ such that

$$a(x) \geq \delta \text{ on some open subset of } \Omega, \tag{2.44}$$

with

$$\int_\Omega \left(|\nabla v|^2 - a(x)v^2\right) \geq \delta \int_\Omega v^2 \text{ for all } v \in H_0^1, \text{ and } \delta > 0. \tag{2.45}$$

Then, there exists a solution for the following

$$\begin{cases} -\Delta u = u^{2^*-1} + a(x)u, & \Omega \\ \quad\quad u > 0, & \Omega \\ \quad\quad u = 0, & \partial\Omega. \end{cases} \tag{2.46}$$

Assumption (2.45) is essential since $L = -\Delta - a$ needs to be positive.
Indeed, let

$$\mu_1 = \min_{v \in H_0^1(\Omega)} \left\{ \frac{\int_\Omega \left(|\nabla v|^2 - av^2\right)}{\|v\|_{2^*}^2} \right\},$$

denote the first eigenvalue of L and $\phi_1 > 0$ the corresponding eigenfunction we have that multiplying (2.46) by ϕ_1 we conclude

$$\mu_1 \int_\Omega u\phi_1 = \int_\Omega u^{2^*-1}\phi_1$$

so that μ_1 is necessarily positive. In order to solve (2.46) one should consider

$$J = \inf_{u \in H_0^1(\Omega) \, , \, \|u\|_{2^*}=1} \left(\int_\Omega (|\nabla u|^2 - au^2) \right) \tag{2.47}$$

and from (2.45) we have that $J > 0$. Without loss of generality we assume that $0 \in \Omega$ and

$$a(0) > 0.$$

By expansion as $\varepsilon \to 0$ as in Lemma 2.2.1 we conclude

$$Q(u_\varepsilon) = \begin{cases} S - a(0)\frac{K_3}{K_2}\varepsilon + O(\varepsilon^{\frac{n-2}{2}}), & n \geq 5 \\ S - a(0)\frac{K_3}{K_2}\varepsilon|\log\varepsilon| + O(\varepsilon), & n = 4, \end{cases}$$

where K_2, K_3 defined as in Lemma 2.2.1. Thus, we have that for $\varepsilon > 0$ small enough $J < S$ and as a result the infimum in (2.47) is achieved. Indeed, the proof that estimating J gives a minimizer is the same as in Lemma 2.4.1. So, suppose u_j is a minimizing sequence for J. Then, u_j is bounded in $H_0^1(\Omega)$ and so passing to a subsequence, u_j we obtain

$$u_j \rightharpoonup u, \text{ in } H_0^1(\Omega).$$

Again, setting $v_j = u_j - u$ we have that

$$v_j \rightharpoonup 0, \text{ in } H_0^1(\Omega)$$

,

$$v_j \to 0, \ L^2(\Omega),$$

and

$$v_j \to 0, \text{ a.e on } \Omega.$$

Moreover, since

$$\int_\Omega |\nabla u_j|^2 - a(x)u_j^2 = J + o(1), \text{ for } j \to \infty$$

we have that

$$\int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v_j|^2 - \int_\Omega a(x)u^2 = J + o(1) \tag{2.48}$$

and by Brezis-Lieb Lemma

$$1 = \int_\Omega |u|^{2^*} + \int_\Omega |v_j|^{2^*} + o(1). \tag{2.49}$$

Now, by convexity, (2.48) and (2.49) we conclude

$$\int_\Omega |\nabla u|^2 - \int_\Omega a(x)u^2 + \int_\Omega |\nabla v_j|^2 \leq J\,\|u\|_{2^*}^2 + J\,\|v_j\|_{2^*}^2 + o(1).$$

Then, by the definition of J and from Sobolev inequality we have

$$(1 - \frac{J}{S})\int_\Omega |\nabla v_j|^2 = o(1)$$

and since $J < S$ we obtain the result.

So, we obtain some $u \in H_0^1(\Omega)$ such that

$$-\Delta u - a(x)u = Ju^{2^*-1}$$

and since $J > 0$ by scaling we obtain a solution of (2.46).

• Improved Sobolev Inequalities.

Corollary 2.6.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Then there exists a constant*

$$\lambda^* \textit{ with } 0 < \lambda^* < \lambda_1, \textit{ where } \lambda^* \textit{ depends on } \Omega$$

such that

$$\|\nabla u\|_2^2 \geq S\,\|u\|_6^2 + \lambda^*\,\|u\|_2^2 \textit{ for all } u \in H_0^1(\Omega). \tag{2.50}$$

Proof. Let Ω^* be the ball such that $|\Omega^*| = |\Omega|$. Let u^* denote the symmetric decreasing rearrangement of u. It is known from [15] that if $u \in H_0^1(\Omega)$ then $u^* \in H_0^1(\Omega^*)$ and

$$\|\nabla u^*\|_{L^2(\Omega^*)}^2 \le \|\nabla u\|_{L^2(\Omega)}^2 \tag{2.51}$$

Also, for every $u^* \in H_0^1(\Omega^*)$

$$\|\nabla u^*\|_{L^2(\Omega^*)}^2 \ge S\,\|u^*\|_{L^6(\Omega^*)} + \frac{1}{4}\lambda_1(\Omega^*)\,\|u^*\|_{L^2(\Omega^*)}\,. \tag{2.52}$$

Indeed, from (2.52) we have that $S_\lambda \ge S$, when $\lambda = \frac{1}{4}\lambda_1(\Omega^*)$, which is true, because if it wasn't, we would have that $S_\lambda < S$ and this would imply that there exists a solution of (2.1) on Ω^* with $\lambda = \frac{1}{4}\lambda_1(\Omega^*)$ which contradicts Theorem (2.4.3). Thus, since $\lambda_1(\Omega^*) = \frac{\pi^2}{R^2}$, where $\frac{3}{4}\pi R^3 = |\Omega|$ and combining (2.51), (2.52) and the fact that

$$\|u^*\|_{L^q(\Omega^*)} = \|u\|_{L^q(\Omega)}$$

we obtain (2.50). □

So, from Corollary 2.6.2 we have that for a bounded domain $\Omega \subset \mathbb{R}^3$ there exists a number λ^* depending on the domain with $0 < \lambda^* < \lambda_1$ such that

$$\begin{cases} S_\lambda < S, & \lambda > \lambda^* \\ S_\lambda = S, & 0 \le \lambda \le \lambda^*. \end{cases} \tag{2.53}$$

When Ω is a ball then $\lambda^* = \frac{1}{4}\lambda_1$ (Lemma 2.3.1.)

Remark 2.6.3. *When $n \ge 4$, there is no inequality of the type*

$$\|\nabla u\|_2^2 \ge S\,\|u\|_{2^*}^2 + \lambda^*\,\|u\|_2^2 \textit{ for every } u \in H_0^1(\Omega) \textit{ and } \lambda^* > 0. \tag{2.54}$$

Indeed, this would imply $S_{\lambda^*} \ge S$ contradicting Lemma 2.2.1.

On the other hand, the following inequality holds

$$\|\nabla u\|_2^2 \ge S\,\|u\|_{2^*}^2 + \lambda_q\,\|u\|_q^2\,, \tag{2.55}$$

for every $u \in H_0^1(\Omega)$ $n \ge 3,\ q < \frac{n}{n-2},\ \lambda_q > 0$ a constant depending on q and Ω.

Proof. By symmetrization as we did before we may assume that Ω is a ball. Let

$$\breve{S}_\lambda = \inf_{u \in H_0^1(\Omega),\, \|u\|_{2^*}=1} \left(\int_\Omega |\nabla u|^2 - \lambda\,\|u\|_q^2 \right).$$

Inequality (2.55) implies

$$\breve{S}_\lambda \ge S \text{ for some positive } \lambda = \lambda_q.$$

This is true. Suppose not, i.e. suppose that

$$\breve{S}_\lambda < S \text{ for all } \lambda > 0.$$

Then Lemma 2.4.1 would imply that $\breve{S}_\lambda$ is achieved by u and as a consequence we would get a solution of

$$\begin{cases} -\Delta u = u^{2^*-1} + \lambda \frac{u^{q-1}}{\|u\|_q^{q-2}}, & \Omega \\ u > 0, & \Omega \\ u = 0, & \partial\Omega. \end{cases}$$

Then, from Pohozaev's identity (2.36), we get

$$\begin{aligned}\lambda(\frac{n}{q}+1-\frac{n}{2})\,\|u\|_q^2 &= \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)(\frac{\partial u}{\partial \nu})^2 \geq C\left(\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\right)^2\\ &= C\left(\int_\Omega \Delta u\right)^2 = \left(\int_\Omega |\Delta u|\right)^2 \geq C\,\|u\|_q^2\,,\end{aligned}$$

where the last inequality is obtained since Δ^{-1} is a bounded operator from $L^1(\Omega)$ into $L^2(\Omega)$. So, $\lambda \geq \lambda_0 > 0$ a contradiction, since we have assumed that the hypothesis was true for all $\lambda > 0$. $\square$

Remark 2.6.4. *Assume* $\Omega \subset \mathbb{R}^n,\ n\geq 3$ *and* $\lambda \leq 0$. *Then* $S_\lambda = S$ *and the infimum* (2.3) *is not attained.*

Indeed, combining Lemma 2.2.1 and 2.3.1 we have

$$Q_\lambda(u_\varepsilon) = \begin{cases} S + O(\varepsilon), & n\geq 5\\ S+O(\varepsilon|log\varepsilon|), & n=4\\ S+O(\varepsilon^{\frac{1}{2}}), & n=3.\end{cases}$$

Thus, $S_\lambda \leq S$. Moreover, for $\lambda \leq 0$ we have $S_\lambda \geq S$. We already know from Lemma 2.1.1 that the infimum in (2.3) is not attained for $\lambda = 0$ and so it cannot be attained for $\lambda < 0$, since $S_\lambda = S$.

• Let $\Omega \subset \mathbb{R}^3$ strictly starshaped (i.e. $x\cdot\nu \geq a > 0$ on $\partial\Omega$.) Then, if (2.1) has a solution then

$$\lambda \geq \lambda_0 > 0.$$

Indeed, using again Pohozaev's identity we conclude

$$\begin{aligned}\lambda\int_\Omega u^2 &= \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2\\ &\geq a\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)^2 \geq b\left(\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\right)^2\\ &= \left(\int_\Omega\Delta u\right)^2 = b\left(\int_\Omega|\Delta u|\right)^2 \geq c\int_\Omega u^2,\ c>0.\end{aligned} \tag{2.56}$$

Thus, $\lambda \geq \lambda_0$ and the proof is done.

CHAPTER 3

An equation involving critical exponent and Hardy potential

In this chapter we are concerned with the existence of positive solutions of the form

$$-\Delta u = a(x/|x|)\frac{u}{|x|^2} + f(x,u)\ , \mathbb{R}^n \setminus \{0\}\,, \tag{3.1}$$

depending on the behaviour of the function a where $a \in C^1(S^{n-1})$ and $n \geq 3$. In particular, the existence results presented in this thesis are about $f(s) = s^{2^*-1}$. Equations of this type arise in the study of nonlinear Schrödinger operators when the field presents a nonisotropic singularity at the origin. Among the nonlinear functions f, the case

$$-\Delta u = a(x/|x|)\frac{u}{|x|^2} + u^{2^*-1},\ \mathbb{R}^n \setminus \{0\} \tag{3.2}$$

is of special interest. Indeed, this equation is invariant under the scaling of the type $u(x) \to R^{\frac{(n-2)}{2}}u(xR)$ and under particular conditions the critical exponent $2^* - 1$ is the more likely to admit solutions having nice behavior near the origin and at infinity (Theorem 3.1.4).

To approach the problem we associate to the linear part in (3.1) the quadratic form

$$Q(u) := \int_{\mathbb{R}^n} \left(|\nabla u|^2 dx - a(x/|x|)\frac{u^2}{|x|^2} \right) dx$$

and its first eigenvalue

$$\lambda_1(a) = \inf_{u \in D^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{Q(u)}{\int_{\mathbb{R}^n} \frac{u^2}{|x|^2}}. \tag{3.3}$$

When $a \equiv 0$ Hardy's inequality yields $\lambda_1(0) = \frac{(n-2)^2}{4}$.

3.1 The quadratic form and the first eigenvalue problem.

For simpler notations we will identify $a(x/|x|)$ with its positively homogeneous extension of degree 0.

So, associating with a the bilinear and quadratic forms:

$$Q(u,v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v - a(x)\frac{uv}{|x|^2}$$

and

$$Q(u) := Q(u,u) = \int_{\mathbb{R}^n} |\nabla u|^2 - a(x)\frac{u^2}{|x|^2},$$

we obtain from Hardy's inequality

$$\frac{(n-2)^2}{4}\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \leq \int_{\mathbb{R}^n} |\nabla u|^2,$$

that the above forms are continuous in $D^{1,2}(\mathbb{R}^n) \times D^{1,2}(\mathbb{R}^n)$ and $D^{1,2}(\mathbb{R}^n)$ respectively. In this way, we conclude that there exists a unique bounded symmetric operator $L_Q \in D^{1,2}(\mathbb{R}^n)$ such that

$$\langle L_Q u, v\rangle_{D^{1,2}(\mathbb{R}^n)} = Q(u,v).$$

For the arguments of the next propositions we shall prove the next lemma

Lemma 3.1.1. *We have that*

$$\lambda_1(a) = \inf_{\phi\in H^1(S^{n-1})\setminus\{0\}} \frac{\int_{S^{n-1}} |\nabla\phi|^2 + (\frac{(n-2)^2}{4} - a(x))\phi^2}{\int_{S^{n-1}} \phi^2}.$$

Proof. Let

$$l = \inf_{\phi\in H^1(S^{n-1})\setminus\{0\}} \frac{\int_{S^{n-1}} |\nabla\phi|^2 + (\frac{(n-2)^2}{4} - a(x))\phi^2}{\int_{S^{n-1}} \phi^2}.$$

We first show that $\lambda_1(a) \leq l$.

Let $\xi : \mathbb{R}^+ \to [0,1]$ a smooth cut-off function such that $\xi(s) = 0,\ \forall\, s \in [0,\frac{1}{2}]$ and $\xi(s) = 1,\ \forall s \geq 1$. Moreover, for $\varepsilon \in (0,1)$ we define

$$\xi_\varepsilon(s) = \begin{cases} \xi(s/\varepsilon), & s \leq 1 \\ \xi(1/\varepsilon s), & s \geq 1. \end{cases}$$

Let $w_\varepsilon \in D^{1,2}(\mathbb{R}^n)$ such that

$$w_\varepsilon(x) := |x|^{\frac{2-n}{2}}\xi_\varepsilon(|x|)\phi_1(x/|x|),$$

where $\phi_1 \in H^1(S^{n-1})$ is a positive eigenfunction associated to l.

We have that

$$w_\varepsilon(x) = \varepsilon^{\frac{(2-n)}{2}} w_1(x/\varepsilon),\ |x| \leq \varepsilon$$

and

$$w_\varepsilon(x) = \varepsilon^{\frac{(n-2)}{2}} w_1(\varepsilon x),\ |x| \geq \varepsilon^{-1}.$$

So we obtain

$$\int_{\{|x|\leq\varepsilon\}\cup\{|x|\geq\varepsilon^{-1}\}} |\nabla w_\varepsilon|^2 + \frac{1}{|x|^2}w_\varepsilon^2 = \int_{\mathbb{R}^n} |\nabla w_1|^2 + \frac{1}{|x|^2}w_1^2. \tag{3.4}$$

Indeed, for

$$|x| \le \varepsilon \text{ we have } \nabla w_\varepsilon(x) = \varepsilon^{-\frac{n}{2}} \nabla w_1(x/\varepsilon)$$

and so

$$\int_{|x|\le\varepsilon} |\nabla w_\varepsilon|^2 = \int_{|x|\le\varepsilon} \varepsilon^{-n} |\nabla w_1(x/\varepsilon)|^2 = \int_{|y|\le 1} |\nabla w_1(y)|^2 dy.$$

Doing the same for $|x| \ge \varepsilon^{-1}$ we obtain (3.4).

So,

$$\int_{\{|x|\le\varepsilon\}\cup\{|x|\ge\varepsilon^{-1}\}} |\nabla w_\varepsilon|^2 + \frac{1}{|x|^2} w_\varepsilon^2 \le C, \tag{3.5}$$

with C constant independent of ε.

From the definition of $\lambda_1(a)$ and from (3.5) we have

$$\lambda_1(a) \le \frac{\int_{\mathbb{R}^n} |\nabla w_\varepsilon|^2 - a(x)\frac{w_\varepsilon^2}{|x|^2}}{\int_{\mathbb{R}^n} \frac{w_\varepsilon{}^2}{|x|^2}} \le \frac{C_1 + \int_{\varepsilon\le|x|\le\varepsilon^{-1}} |\nabla w_\varepsilon|^2 - a(x)\frac{w_\varepsilon^2}{|x|^2}}{\int_{\varepsilon\le|x|\le\varepsilon^{-1}} \frac{w_\varepsilon{}^2}{|x|^2}}.$$

We have that

$$\nabla w_\varepsilon(x) = \frac{2-n}{2} x|x|^{\frac{-n-2}{2}} \phi_1(x/|x|) + |x|^{-\frac{n}{2}} \nabla\phi_1(x/|x|)$$

So,

$$\int_{\varepsilon\le|x|\le\varepsilon^{-1}} |\nabla w_\varepsilon|^2 - a(x)\frac{w_\varepsilon^2}{|x|^2} = \int_\varepsilon^{\varepsilon^{-1}} \frac{(2-n)^2}{4} r^{-n} \int_{S^{n-1}} \phi_1^2(\theta) r^{n-1} + r^{-n}|\nabla\phi_1|^2 r^{n-1} - a(x) r^{-1}\phi_1^2(\theta)$$

Thus,

$$\lambda_1(a) \le \frac{C_1 + 2\log\varepsilon^{-1} \int_{S^{n-1}} |\nabla\phi_1|^2 + (\frac{(n-2)^2}{4} - a(x))\phi_1^2}{2\log\varepsilon^{-1} \int_{S^{n-1}} \phi_1^2}$$

and finally as $\varepsilon \to 0$ we conclude $\lambda_1(a) \le l$.

For the inverse inequality, we associate to each $w \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ its transfrom $\breve{w}$ homogeneous of degree $\frac{2-n}{2}$ defined

$$\breve{w}(x) := \left(\int_0^\infty \frac{1}{R^{n-1}} w^2(x/R) dR\right)^{\frac{1}{2}}, \tag{3.6}$$

$\breve{w}$ is indeed homogeneous of degree $\frac{2-n}{2}$, since

$$\breve{w}(\lambda x) = \left(\int_0^\infty \frac{1}{R^{n-1}} w^2(\lambda x/R) dR\right)^{\frac{1}{2}}$$

and setting $\frac{\lambda}{R} = \frac{1}{R'}$, we have

$$\begin{aligned}\left(\int_0^\infty \frac{1}{R^{n-1}} w^2(\lambda x/R) dR)\right)^{\frac{1}{2}} &= \left(\int_0^\infty \frac{1}{R'^{n-1}\lambda^{n-1}} w^2(x/R')\lambda dR\right)^{\frac{1}{2}} \\ &= \lambda^{\frac{n-2}{2}} \breve{w}(x).\end{aligned}$$

Moreover, we have

$$\int_{\mathbb{R}^n} \frac{w^2(x)}{|x|^2} dx = \int_0^\infty \int_{S^{n-1}} \frac{w^2(r\theta)}{r^2} r^{n-1} d\theta dr$$

and setting $r = \frac{1}{R}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{w^2(x)}{|x|^2} dx &= \int_{S^{n-1}} \int_0^R \frac{w^2(\frac{\theta}{R})}{R^{n-1}} dR d\theta \\ &= \int_{S^{n-1}} \breve{w}^2 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} a(x) \frac{w^2}{|x|^2} &= \int_0^\infty \int_{S^{n-1}} a(\theta) \frac{w^2(r\theta)}{r^2} \\ &= \int_{S^{n-1}} a(x) \breve{w}^2, \end{aligned} \tag{3.8}$$

where the last equality is obtained by setting $r = \frac{1}{R}$.
Differentiating (3.6) we have

$$\breve{w}_{x_i}(x) = \breve{w}^{-1}(x) \int_0^\infty \frac{1}{R^{n-1}} w(x/R) \left(w(x/R)\right)_{x_i} dR$$

and so

$$|\nabla \breve{w}(x)| \leq \breve{w}^{-1} \int_0^\infty \frac{1}{R^{\frac{n-1}{2}}} w(x/R) \frac{1}{R^{\frac{n+1}{2}}} \nabla w(x/R) dr$$

and using Cauchy-Schwarz inequality we obtain

$$|\nabla \breve{w}(x)| \leq \left(\int_0^\infty \frac{1}{R^{n+1}} |\nabla w(x/R)|^2 dR \right)^{\frac{1}{2}}$$

and so

$$\int_{S^{n-1}} |\nabla \breve{w}|^2 \leq \int_{\mathbb{R}^n} |\nabla w|^2. \tag{3.9}$$

Let $\phi(x) = \breve{w}(\frac{x}{|x|})$.
Since $\breve{w}$ is homogeneous of degree $\frac{2-n}{2}$ we have

$$\phi(x) = \frac{1}{|x|^{\frac{2-n}{2}}} \breve{w}(x) \text{ and so } \breve{w}(x) = |x|^{\frac{2-n}{2}} \phi(x).$$

Thus,

$$\nabla \breve{w}(x) = \nabla \phi(x) |x|^{\frac{2-n}{2}} + \frac{(2-n)}{2} x |x|^{\frac{-n-2}{2}} \phi(x)$$

and so

$$\int_{S^{n-1}} |\nabla \breve{w}|^2 - a(x) \breve{w}^2 = \int_{S^{n-1}} |\nabla \phi|^2 + \frac{(2-n)^2}{4} \phi^2 - a(x) \phi^2.$$

Therefore, using (3.7), (3.8), (3.9) we have

$$\begin{aligned} l &\leq \frac{\int_{S^{n-1}} |\nabla \phi|^2 + (\frac{(n-2)^2}{4} - a(x))\phi^2}{\int_{S^{n-1}} \phi^2} = \frac{\int_{S^{n-1}} |\nabla \breve{w}|^2 - a(x)\breve{w}^2}{\int_{S^{n-1}} \breve{w}^2} \\ \text{(3.7), (3.8), (3.9)} &\leq \frac{\int_{\mathbb{R}^n} |\nabla w|^2 - a(x) \frac{w^2}{|x|^2}}{\int_{\mathbb{R}^n} \frac{w^2}{|x|^2}}. \end{aligned}$$

Finally, by the density of $C_c^\infty(\mathbb{R}^n \setminus \{0\})$ the proof is complete. □

Remark 3.1.2. *Since S^{n-1} is a compact manifold, $H^1(S^{n-1})$ is compactly embedded in $L^2(S^{n-1})$ and so $\lambda_1(a)$ defined as in Lemma 3.1.1 is achieved by a positive function ϕ_1 which satisfies*

$$-\Delta_{S^{n-1}}\phi + \left(\frac{(n-2)^2}{4} - a(x)\right)\phi = \lambda_1(a)\phi.$$

Indeed, let $\phi_k \in H^1(S^{n-1}),\ \|\phi_k\|_{L^2(S^{n-1})} = 1$, with

$$\int_{S^{n-1}} |\nabla\phi_k|^2 + \left(\frac{(n-2)^2}{4} - a(x)\right)\phi_k^2 \to \lambda_1(a).$$

In particular,

$$\int_{S^{n-1}} |\nabla\phi_k|^2 = \lambda_1(a) - \frac{(2-n)^2}{4} - \int_{S^{n-1}} a(x)\phi_k^2 + o(1)\ ,\ k\to\infty.$$

But $a \in C^1(S^{n-1})$ and so it is bounded, let M be the upper bound, and so

$$\sup_k \|\nabla\phi_k\|_{L^2(S^{n-1})} < \infty.$$

Passing to a subsequence, let ϕ_k we have

$$\phi_k \rightharpoonup \phi,\ H^1(S^{n-1})$$
$$\phi_k \to \phi,\ L^2(S^{n-1})$$

thus, by weak lower semicontinuity ϕ is a minimizer.

Theorem 3.1.3. *If $\lambda_1(a) > 0$ then $(Q(u))^{\frac{1}{2}}$ defines equivalent norm in $D^{1,2}(\mathbb{R}^n)$.*

Proof. Thanks to Hardy inequality we have that

$$|Q(u)| \le \int |\nabla u|^2 + \int_{\mathbb{R}^n} |a(x)\frac{u^2}{|x|^2}| \le \int_{\mathbb{R}^n} |\nabla u|^2 + sup_{S^{n-1}}|a(x)|\frac{4}{(n-2)^2}\int_{\mathbb{R}^n} |\nabla u|^2$$

and so there exists $C > 0$ such that

$$Q(u) \le C\,\|u\|^2_{D^{1,2}(\mathbb{R}^n)}\,.$$

In order to complete the proof we shall prove that there exists a constant C such that

$$C\,\|u\|^2_{D^{1,2}(\mathbb{R}^n)} \le Q(u).$$

We argue by contradiction and assume that corresponding to every $\varepsilon > 0$ there exists $u_\varepsilon \in D^{1,2}(\mathbb{R}^n)$ such that

$$Q(u_\varepsilon) < \varepsilon\,\|u_\varepsilon\|^2_{D^{1,2}(\mathbb{R}^n)}\,.$$

We then deduce that $\lambda_1((1-\varepsilon)^{-1}a) < 0$, but from Lemma 3.1.1 we have that

$$a \mapsto \lambda_1(a)$$

is continuous, so as $\varepsilon \to 0$ we obtain $\lambda_1(a) < 0$, a contradiction. $\square$

Theorem 3.1.4. *(i) If $\lambda_1(a) \leq 0$ and $f > 0$ then (3.1) has no positive solution in $D^{1,2}(\mathbb{R}^n)$.*

(ii) When $f(s) = s^\theta$, with $\theta \neq 2^* - 1$, then there is no positive solution $u \in D^{1,2}(\mathbb{R}^n) \cap L^{\theta+1}(\mathbb{R}^n)$.

Proof. (i) Suppose, by contradiction, that for $\lambda_1(a) \leq 0$ and $f > 0$ there exists a positive solution u of (3.1). To begin with, let ϕ_1 be the minimizer of Remark 3.1.2. Then, since

$$-\Delta_{S^{n-1}}\phi_1 + \left(\frac{(n-2)^2}{4} - a(x)\right)\phi_1 = \lambda_1(a)$$

using polar coordinates

$$\Delta w = \frac{\partial^2 w}{\partial r^2} + \frac{n-1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}}w$$

we conclude that

$$w_1(x) = |x|^{\frac{2-n}{2}}\phi_1(x/|x|)$$

solves

$$-\Delta w = (a(x) + \lambda_1(a))\frac{w}{|x|^2},\ \mathbb{R}^n \setminus \{0\}\,. \tag{3.10}$$

Now, multipling (3.1) by w_1 and (3.10) by u, subtracting and integrating over

$$A_{r,R} = \{x \in \mathbb{R}^n : r \leq |x| \leq R\}\,,$$

we obtain

$$\int_{A_{r,R}} f(u)w_1 - \lambda_1(a)\frac{uw_1}{|x|^2} = \int_{\partial A_{r,R}} w_1\frac{\partial u}{\partial \nu} - u\frac{\partial w_1}{\partial \nu}.$$

Since the left hand term is always positive, for $\lambda_1(a) \leq 0$, the proof will be done when we show that we can choose sequences of $r_\nu \to 0$, $R_\nu \to \infty$ such that, when $r_\nu = \rho_\nu$ and $R_\nu = \rho_\nu$ the integral tends to zero as $\nu \to \infty$.

For every $\rho > 0$ from Hölder's inequality we obtain

$$\int_{|x|=\rho} u|\frac{\partial w_1}{\partial \nu}| + w_1|\frac{\partial u}{\partial \nu}| \leq \|u\|_{L^{2^*}(\rho S^{n-1})}\|\nabla w_1\|_{L^{(2^*)'}(\rho S^{n-1})} + \|w_1\|_{L^2(\rho S^{n-1})}\|\nabla u\|_{L^2(\rho S^{n-1})}\,.$$

We have $w_1(x) = |x|^{\frac{2-n}{2}}\phi_1(\frac{x}{|x|})$ and so

$$\begin{aligned}
\left(\int_{|x|=\rho}|\nabla w|^{(2^*)'}\right)^{\frac{1}{(2^*)'}} &= \left(\int_{|x|=\rho}|\nabla w|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}}\\
&= \left(\int_{|y|=1}(|y|^{-n}|\nabla\phi|^2\rho^{-n} + \frac{(2-n)^2}{4}\rho^{-n}\phi^2)^{\frac{n}{n+2}}\rho^{n-1}d\rho\right)^{\frac{2+n}{2n}}\\
&= C_1(\rho^{-n(\frac{n}{n+2})}\rho^{n-1})^{\frac{2+n}{2n}} = C_1\rho^{\frac{1}{2^*}},
\end{aligned}$$

with C_1 independent of ρ.

Also,

$$\int_{|x|=\rho} w_1^2 = \int_{|x|=\rho}|x|^{2-n}\phi_1^2(\frac{x}{|x|}) = \rho\int_{|y|=1} y^{2-n}\phi_1^2(y)dy = C_2\rho,$$

with C_2 independent of ρ.

So,

$$\int_{|x|=\rho} u|\frac{\partial w_1}{\partial\nu}| + w_1|\frac{\partial u}{\partial\nu}| \le C_1(\rho\int_{|x|=\rho} u^{2^*})^{\frac{1}{2^*}} + C_2(\rho\int_{|x|=\rho}|\nabla u|^2)^{\frac{1}{2}}.$$

Since $u \in D^{1,2}(\mathbb{R}^n)$ the integrals

$$\int_0^\infty d\rho\int_{|x|=\rho} u^{2^*} \text{ and } \int_0^\infty d\rho\int_{|x|=\rho}|\nabla u|^2$$

converge .Thus, there exists $r_\nu \to 0, R_\nu \to \infty$ such that when $\rho_\nu = r_\nu$ and $\rho_\nu = R_\nu$

$$\rho_\nu\int_{|x|=\rho_\nu} u^{2^*} + \rho_\nu\int_{|x|=\rho_\nu}|\nabla u|^2 \to 0.$$

Indeed, if this wasn't true there would exists a constant $C > 0$ such that

$$\rho\int_{|x|=\rho} u^{2^*} + \rho\int_{|x|=\rho}|\nabla u|^2 \ge C$$

i.e.

$$\int_{|x|=\rho} u^{2^*} + \int_{|x|=\rho}|\nabla u|^2 \ge \frac{C}{\rho} \Rightarrow \int_0^\infty\int_{|x|=\rho} u^{2^*} + |\nabla u|^2 \ge \int_0^\infty \frac{C}{\rho},$$

a contradiction, since ρ^{-1} is not integrable at the origin and at infinity.

(ii) We will work as above using a Pohozaev's type identity. Suppose that there exists a solution $u \in D^{1,2}(\mathbb{R}^n) \cap L^{\theta+1}(\mathbb{R}^n)$, when $\theta \ne 2^* - 1$.

Multiplying (3.1) by $(x\cdot\nabla u + \frac{n-2}{2}u)$ we have

$$0 = (\Delta u + a(x)\frac{u}{|x|^2} + u^\theta)(x\cdot\nabla u + \frac{n-2}{2}u),$$

where by calculations we have

$$\Delta u(x\cdot\nabla u) = div(\nabla u(x\cdot\nabla u) - \frac{x}{2}|\nabla u|^2) + \frac{n-2}{2}|\nabla u|^2.$$

Moreover, we have

$$\begin{aligned} div(\frac{1}{\theta+1}u^{\theta+1}x) &= \sum_{i=1}^n (\frac{1}{\theta+1}u^{\theta+1}x_i)_{x_i} \\ &= \frac{n}{\theta+1}u^{\theta+1} + u^\theta(x\cdot\nabla u). \end{aligned}$$

Thus, integrating over $A_{r,R} = \{x \in \mathbb{R}^n : r \le |x| \le R\}$ and from the Divergence Theorem we obtain

$$\begin{aligned} 0 = &\int_{\partial A_{r,R}}\left(\nabla u\cdot x\frac{\partial u}{\partial\nu} - \frac{|\nabla u|^2}{2}x\cdot\nu + \frac{n-2}{2}u\frac{\partial u}{\partial\nu}\right) \\ &+ \int_{A_{r,R}}\left(a(x)\frac{u}{|x|^2}(x\cdot\nabla u) + \frac{n-2}{2}a(x)\frac{u^2}{|x|^2}\right) \\ &+ \int_{\partial A_{r,R}}\frac{1}{\theta+1}u^{\theta+1}x\cdot\nu - \int_{A_{r,R}} u^{\theta+1}\left(\frac{n}{\theta+1} + \frac{n-2}{2}\right) \end{aligned}$$

and so we conclude

$$\frac{2n-(n-2)(\theta+1)}{2(\theta+1)}\int_{A_{r,R}} u^{\theta+1} = \int_{\partial A_{r,R}}\left(\frac{n-2}{2}u\frac{\partial u}{\partial \nu} - \frac{|\nabla u|^2}{2}x\cdot\nu + x\cdot\nabla u\frac{\partial u}{\partial \nu}\right)$$
$$+\int_{\partial A_{r,R}}\left(\frac{1}{\theta+1}u^{\theta+1}x\cdot\nu\right) + \int_{A_{r,R}}\left(a(x)\frac{u}{|x|^2}(x\cdot\nabla u) + \frac{n-2}{2}a(x)\frac{u^2}{|x|^2}\right)$$

Now for the last terms, we have

$$div(\frac{1}{2}a(x)\frac{u^2}{|x|^2}x) = \frac{n}{2}a(x)\frac{u^2}{|x|^2} + (x\cdot\nabla a(x))\frac{1}{2}\frac{u^2}{|x|^2}$$
$$+a(x)\frac{u^2}{|x|^2}(x\cdot\nabla u) - a(x)\frac{u^2}{|x|^2}.$$

So,

$$div(\frac{1}{2}a(x)\frac{u^2}{|x|^2}x) = \frac{n-2}{2}a(x)\frac{u^2}{|x|^2} + a(x)\frac{u^2}{|x|^2}(x\cdot\nabla u) + \frac{u^2}{2|x|^2}(x\cdot\nabla a(x)).$$

So by the Divergence Theorem

$$\int_{A_{r,R}} a(x)\frac{u^2}{|x|^2}(x\cdot\nabla u) = \int_{\partial A_{r,R}}\frac{u^2}{2|x|^2}a(x)x\cdot\nu - \frac{1}{2}\int_{A_{r,R}}\frac{u^2}{|x|^2}x\cdot\nabla a(x),$$

where the last integral vanishes since the vectors are vertical.

So,

$$\frac{2n-(n-2)(\theta+1)}{2(\theta+1)}\int_{A_{r,R}} u^{\theta+1} = \int_{\partial A_{r,R}}[\frac{n-2}{2}u\frac{\partial u}{\partial \nu} - \frac{1}{2}|\nabla u|^2(x\cdot\nu)$$
$$+(x\cdot\nabla u)\frac{\partial u}{\partial \nu} + \frac{1}{\theta+1}u^{\theta+1}(x\cdot\nu) + a(x)(x\cdot\nu)\frac{u^2}{2|x|^2}].$$

From the hypothesis we have that $\theta \neq 2^*-1 = \frac{n+2}{n-2}$ and so the left handed term does not vanish. Now it remains to show, that the boundary integrals tend to zero, at least for suitable sequences r_ν, R_ν.

For this matter, we will show that there exists $r_\nu \to 0, R_n \to \infty$ such that when $\rho_\nu = r_\nu$ and $\rho_\nu = R_\nu$ then

$$\int_{|x|=\rho_\nu}\left(\frac{n-2}{2\rho_\nu}|u|^2 + \frac{(n-2)\rho_\nu}{2}|\nabla u|^2 + \frac{3\rho_\nu}{2}|\nabla u|^2 + \frac{1}{2\rho_\nu}a(x)u^2 + \frac{\rho_\nu}{\theta+1}|u|^{\theta+1}\right) \to 0. \tag{3.11}$$

Let $r_\nu \to 0$ and $\rho_\nu = r_\nu$ then since $u \in D^{1,2}(\mathbb{R}^n) \cup L^{\theta+1}(\mathbb{R}^n)$ from the continuity of the Lebesgue integral we have

$$\int_{\rho_\nu}|\nabla u|^2 \to 0,$$
$$\int_{\rho_\nu}|u|^{2^*} \to 0,$$
$$\int_{\rho_\nu}|u|^{\theta+1} \to 0.$$

Moreover from Hölder's inequality we obtain

$$\int_{|x|=\rho_\nu} u^2 \le \left(\int_{|x|=\rho_\nu} |u|^{2^*}\right)^{\frac{n-2}{n}} \rho_\nu^{\frac{2(n-1)}{n}}$$

so that

$$\frac{1}{\rho_\nu}\int_{|x|=\rho_\nu} u^2 \le \left(\int_{|x|=\rho_\nu} |u|^{2^*}\rho_\nu\right)^{\frac{n-2}{n}} \overset{\rho_\nu\to 0}{\to} 0$$

and so (3.11) is valid for $r_\nu \to 0$ and $\rho_\nu = r_\nu$.

Now, suppose $R_\nu \to \infty$ and $\rho_\nu = R_\nu$. There exists ρ_ν such that

$$\rho_\nu \int_{|x|=\rho_\nu|} |\nabla u|^2 + \rho_\nu \int_{|x|=\rho_\nu} |u|^{2^*} + \frac{\rho_\nu}{\theta+1}\int_{|x|=\rho_\nu} |u|^{\theta+1} \to 0\,. \tag{3.12}$$

Indeed, otherwise as in the proof of (i) we would get a contradiction.

So, by Young's Inequality we have

$$\begin{aligned}\int_{|x|=\rho_\nu} &\left(\frac{(n-2)}{2}|u||\nabla u| + \frac{3}{2}\rho_\nu|\nabla u|^2 + \frac{\rho_\nu}{\theta+1}|u|^{\theta+1} + \frac{1}{2\rho_\nu}u^2\right)\\ &\le \frac{n-2}{2\rho_\nu}\int_{|x|=\rho_\nu} u^2 + \frac{(n-2)\rho_\nu}{2}\int_{|x|=\rho_\nu}|\nabla u|^2 + \frac{3\rho_\nu}{2}\int_{|x|=\rho_\nu}|\nabla u|^2\\ &\quad + \frac{1}{2\rho_\nu}\int_{|x|=\rho_\nu} u^2 + \frac{\rho_\nu}{\theta+1}\int_{|x|=\rho_\nu}|u|^{\theta+1},\end{aligned}$$

but as above from Hölder's inequality we have

$$\frac{1}{\rho_\nu}\int_{|x|=\rho_\nu} u^2 \le \left(\int_{|x|=\rho_\nu} \rho_\nu|u|^{2^*}\right)^{\frac{n-2}{n}}$$

and so from (3.12) we have

$$\frac{1}{\rho_\nu}\int_{|x|=\rho_\nu} u^2 \to 0.$$

So, every part of (3.11) tends to 0 over ρ_ν and we obtain a contradiction.

□

Theorem 3.1.5. *There exists positive homogeneous solution u of degree* $\frac{2-n}{2}$ *with* $u \notin D^{1,2}(\mathbb{R}^n)$ *of*

$$-\Delta u = a(x)\frac{u}{|x|^2} + u^{2^*-1},\ \mathbb{R}^n\setminus\{0\}\,, \tag{3.13}$$

if and only if $\lambda_1(a) > 0$.

Proof. As we did before, using polar coordinates, $u(x) = |x|^{\frac{2-n}{2}}\phi(\frac{x}{|x|})$ is a solution of (3.13) if and only if ϕ is a solution of

$$-\Delta_{S^{n-1}}\phi + \frac{(2-n)^2}{4}\phi = a(x)\phi + \phi^{2^*-1}. \tag{3.14}$$

To solve (3.14) we consider the minimization problem

$$\inf_{\phi\in H^1(S^{n-1})\setminus\{0\}} \frac{\int_{S^{n-1}} |\nabla\phi|^2 + (\frac{(2-n)^2}{4} - a(x))\phi^2}{(\int_{S^{n-1}} |\phi|^{2^*})^{\frac{2}{2^*}}}.$$

Suppose $\lambda_1(a) > 0$. Then from Lemma 3.1.1 we have that the infimum will be positive .We have that for $p < \frac{2(n-1)}{(n-3)}$, $H^1(S^{n-1})$ is compactly embedded in $L^p(S^{n-1})$. Since $2^* = \frac{2n}{n-2} < \frac{2(n-1)}{(n-3)}$ we have that the infimum is achieved and so (3.14) admits a solution.

Now, suppose $\lambda_1(a) \leq 0$. Then (3.14) has no positive solution. Indeed, we argue by contradiction and let ϕ be a positive solution of (3.14) and w_1 the first positive eigenfunction associated to $\lambda_1(a)$. From (3.14) we obtain

$$-\int_{S^{n-1}} \Delta\phi w_1 + \int_{S^{n-1}} \left(\frac{(2-n)^2}{4} - a(x)\right) w_1\phi = \int_{S^{n-1}} \phi^{2^*-1}w_1$$

and integrating by parts we have

$$-\int_{S^{n-1}} \Delta w_1\phi + \int_{S^{n-1}} \left(\frac{(2-n)^2}{4} - a(x)\right) w_1\phi = \int_{S^{n-1}} \phi^{2^*-1}w_1$$

and since w_1 is eigenfunction associated to $\lambda_1(a)$ we have

$$\lambda_1(a)\int_{S^{n-1}} \phi w_1 = \int_{S^{n-1}} \phi^{2^*-1}w_1$$

and since $\lambda_1(a) \leq 0$ the solution ϕ can not be positive. □

3.2 Symmetry properties of positive solutions.

Now, we use some symmetry results obtained by the moving planes method ([9],[10]). The main result that we use is the following.

Let u be a positive solution of

$$-\Delta u = A\frac{u}{|x|^2} + f(|x|, u),\ \mathbb{R}^n \setminus \{0\}\,. \tag{3.15}$$

and suppose the following are true

(A_1) $A \in [0, \frac{(n-2)^2}{4})$,

(f_1) $f : \mathbb{R}^+ \setminus \{0\} \times \mathbb{R}^+ \to \mathbb{R}$ is locally Lipschitz in s and nonincreasing in $|x|$,

(f_2) $\exists\rho(x) \geq 0$, $\rho \in L^\infty_{loc}(\mathbb{R}^n \setminus \{0\})$ such that $\forall\ 0 < s < t$, $\frac{f(|x|,s)-f(|x|,t)}{s-t} \leq \rho(x)t^\mu$.

(u_1) $\rho u^\mu \in L^{\frac{n}{2}}(\mathbb{R}^n \setminus B_r)$,$\forall r > 0$,

(u_2) $u \in W^{1,2}_{loc}(\mathbb{R}^n \setminus \{0\}) \cap L^\infty_{loc}(\mathbb{R}^n \setminus \{0\})$,

(u_3) $u \in L^{2^*}(\mathbb{R}^n \setminus B_1)$.

Then, the following holds

Theorem 3.2.1. *Under the above assumptions, every positive solution of* (3.15) *is radially symmetric about a point.*

Here, we use the above theorem in order to obtain symmetry results about the family of equations

$$-\Delta u = A\frac{u}{|x|^2} + u^\theta,\ u \in W^{1,2}_{loc}(\mathbb{R}^n \setminus \{0\}) \cap L^\infty_{loc}(\mathbb{R}^n \setminus \{0\}), \tag{3.16}$$

and $\theta > 1$.

Theorem 3.2.2. *Suppose $A \in (0, \frac{(n-2)^2}{4})$ and $\theta > 1$. A positive solution u of (3.16) is radially symmetric about the origin in the following cases*
(i) $\theta = 2^* - 1$ and $u \in L^{2^*}(B_1)$ or $u \in L^{2^*}(\mathbb{R}^n \setminus B_1)$.
(ii) $\theta < 2^* - 1$ and $u \in L^{2^*}(B_1)$ or $u \in L^{(\theta-1)\frac{n}{2}}(\mathbb{R}^n \setminus B_1) \cap L^{2^*}(\mathbb{R}^n \setminus B_1)$.
(iii) $\theta > 2^* - 1$ and $u \in L^{2^*}(\mathbb{R}^n \setminus B_1) \cap L^{(\theta-1)\frac{n}{2}}(\mathbb{R}^n \setminus B_1)$.

Proof. We need to confirm that the assumptions of Theorem 3.2.1 are fulfilled. We first prove the assertion under the integrability conditions at infinity. We have that in all cases assumptions $(A_1), (f_1), (u_2)$ are satisfied from the hypothesis. Moreover, regarding (f_2) we shall use the convexity of f and we obtain

$$\frac{f(s) - f(t)}{s - t} \leq \theta t^{\theta-1}, \ \forall\, 0 < s < t$$

and when $\theta = 2^* - 1$, we have $\frac{(\theta-1)n}{2} = 2^*$ we conclude $u^{\theta-1} \in L^{\frac{n}{2}}(\mathbb{R}^n \setminus B_1) \cap L^{\frac{n}{2}}_{loc}(\mathbb{R}^n \setminus B_1)$. Now, for the L^{2^*} assumption at the origin we use the conformal equivalence of the Laplacian and we define

$$v(x) = \frac{1}{|x|^{n-2}} u(\frac{x}{|x|^2})$$

and since $\Delta v = |x|^{-n-2} \Delta u(x/|x|^2)$ it is a solution of equation

$$-\Delta v = A\frac{v}{|x|^2} + |x|^{\theta(n-2)-n-2} v^{\theta}.$$

In order to secure (f_1) we want $n + 2 - \theta(n-2) \geq 0$ and this happens for $\theta \leq 2^* - 1$. Finally, calculating the norms and by Hölder's inequality we have

$$\|v\|_{L^{2^*}(\mathbb{R}^n \setminus B_1)} = \|u\|_{L^{2^*}(B_1)}$$

and

$$\left\| |x|^{\theta(n-2)-n-2} v^{\theta-1} \right\|_{L^{\frac{n}{2}}(\mathbb{R}^n \setminus B_1)} = \left\| u^{\theta-1} \right\|_{L^{\frac{n}{2}}(B_1)} \leq C \|u\|_{L^{2^*}(B_1)}.$$

Thus, the proof is finished since all the assumptions of Theorem 3.2.1 are satisfied. □

3.3 Analysis of the radially symmetric case.

In this section we present, without proofs, the classification of positive radially solutions of (3.16).

Writing $u(x) = \phi(|x|), \ |x| = r, \ \phi$ satisfies

$$\phi'' + \frac{n-1}{r}\phi' + \frac{A}{r^2}\phi + \phi^{\theta} = 0. \tag{3.17}$$

Actually, this class of equations is equivalent to that with $A = 0$. For this purpose, one should set $\psi(s) = e^{as}\phi(e^s)$. So, (3.17) is equivalent to

$$\ddot{\psi} + h\ddot{\psi} + k\psi + \psi^{\theta} = 0 \tag{3.18}$$

with $a = \frac{2}{(\theta-1)}$, $2a + 1 + h = n - 1$ and $a^2 + ha + k = A$. Then, setting

$$\bar{\theta} = \frac{\theta(n-2) - n - 2}{2(\theta - 1)}$$

we have that in the case of $\theta = 2^* - 1$ since $\bar{\theta} = 0$ (3.18) becomes

$$\ddot{\psi} - (\frac{(n-2)^2}{4} - A)\psi + \psi^{2^*-1} = 0. \tag{3.19}$$

When $\theta = 2^* - 1$ one can classify all the positive entire solutions of (3.17) as follows

• one solution homogeneous of degree $\frac{(2-n)}{2}$, corresponding to the nonzero constant solution of (3.19).

• a two parameter family of solutions behaving like $O(|x|^{\frac{(2-n)}{2}})$ near the origin and at infinity , correspoding to the periodic solutions of (3.19).

• the solution

$$u_A(x) = \frac{(n(n-2)\eta_A^2)^{\frac{(n-2)}{4}}}{(|x|^{1-\eta_A}(1+|x|^{2\eta_A}))^{\frac{(n-2)}{2}}}$$

with

$$\eta_A = (1 - \frac{4A}{(n-2)^2})^{\frac{1}{2}}$$

3.4 The minimization problem.

In this chapter we study the minimization problem

$$S(a) = \inf_{u \in D^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{Q(u)}{\|u\|_{2^*}^2}, \tag{3.20}$$

with

$$S = S(0)$$

denoting the best constant in the usual Sobolev inequality in $\mathbb{R}^n$.

• As $\lambda_1(a) > 0$ the quadratic form $(Q(u))^{\frac{1}{2}}$ defines equivalent norm and inner product in $D^{1,2}(\mathbb{R}^n)$. Therefore $S(a) > 0$, since thanks to norm equivalence we have

$$S(a) \geq \inf_{u \in D^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{C\int_{\mathbb{R}^n} |\nabla u|^2}{\|u\|_{2^*}^2} > 0.$$

Proposition 3.4.1. *Let $(u_\nu)_\nu$ be a minimizing sequence ,weakly converging to $u_0 \not\equiv 0$. Then u_0 is a minimum and the convergence holds in the strong $D^{1,2}(\mathbb{R}^n)$ topology.*

Proof. The arguments here are similar to Lemma (2.4.1), as again «breaking» the minimizing sequence regarding to the weak limit gives an estimate below the level $S(a)$.

Setting $v_\nu = u_\nu - u_o$ we have

$$v_\nu \rightharpoonup 0, \text{ in } D^{1,2}(\mathbb{R}^n)$$
$$v_\nu \to 0, \text{ a.e. on } \mathbb{R}^n.$$

So, to begin with, we remark that by calculations, weak convergence implies

$$Q(u_o + v_\nu) = Q(u_o) + Q(v_\nu) + 2Q(u_o, v_\nu),$$

but from the definition of the inner product due to weak convergence we get

$$Q(u_o, v_\nu) = \langle L_Q u_o, v_\nu \rangle_{D^{1,2}(\mathbb{R}^n)} \to 0,$$

therefore

$$Q(u_o + v_\nu) = Q(u_o) + Q(v_\nu) + o(1). \tag{3.21}$$

Moreover, using Brezis-Lieb Lemma, we have

$$\|u_o + v_\nu\|_{2^*}^{2^*} = \|u_o\|_{2^*}^{2^*} + \|v_\nu\|_{2^*}^{2^*} + o(1).$$

Also, from hypothesis we have

$$Q(u_o + v_\nu) = S(a)\,\|u_o + v_\nu\|_{2^*}^2 + o(1) \Rightarrow Q^{\frac{2^*}{2}}(u_o + v_\nu) = S(a)^{\frac{2^*}{2}}\,\|u_o + v_\nu\|_{2^*}^{2^*} + o(1)$$

and by defintion of $S(a)$

$$Q(v_\nu) \geq S(a)\,\|v_\nu\|_{2^*}^2 \Rightarrow Q^{\frac{2^*}{2}}(v_\nu) \geq S(a)^{\frac{2^*}{2}}\,\|v_\nu\|_{2^*}^{2^*}$$

Thus

$$Q^{\frac{2^*}{2}}(u_o + v_\nu) - Q^{\frac{2^*}{2}}(v_\nu) \leq S(a)^{\frac{2^*}{2}}\,\|u_o + v_\nu\|_{2^*}^{2^*} - S(a)^{\frac{2^*}{2}}\,\|v_\nu\|_{2^*}^{2^*} + o(1)$$

and so

$$Q^{\frac{2^*}{2}}(u_o + v_\nu) - Q^{\frac{2^*}{2}}(v_\nu) \leq S(a)^{\frac{2^*}{2}} \|u_o\|_{2^*}^{2^*} + o(1)$$

Thus, we conclude

$$\frac{Q(u_o)}{\|u_o\|_{2^*}^2} \leq S(a) \frac{Q(u_o + v_\nu) - Q(v_\nu) + o(1)}{(Q^{\frac{2^*}{2}}(u_o + v_\nu) - Q^{\frac{2^*}{2}}(v_\nu) + o(1))^{\frac{2}{2^*}}}.$$

Here, since $S(a) > 0$ we have

$$0 \leq \limsup_n Q(v_\nu) =^{(3.21)} \limsup_\nu Q(u_o + v_\nu) - Q(u_o) < \limsup_n Q(u_o + v_\nu).$$

Passing to a subsequence of u_ν (still forming a minimizing sequence) so that $\lim_\nu Q(u_\nu)$ exists, we conclude from convexity that the right hand side converges to a limit less than $S(a)$, contradicting the definition of $S(a)$, unless $Q(v_\nu)$ converges to 0. Thus, we conclude that necessarily $Q(v_\nu)$ converges to zero. So, u_o is a minimizer and since $Q^{\frac{1}{2}}$ defines an equivalent norm we obtain the strong convergence in $D^{1,2}(\mathbb{R}^n)$ and the proof is complete. □

Proposition 3.4.2. *Let $(u_\nu)_\nu$ be a minimizing sequence ,weakly converging to zero. Then, for every ball B_r and for every $\varepsilon \in (0, r)$(or $\varepsilon \in (-r, 0)$) there exists $\rho \in (0, \varepsilon)$(or $\rho \in (\varepsilon, 0)$) such that for a subsequence*

$$\int_{B_{r+\rho}} |\nabla u_\nu|^2 \to 0,$$

or

$$\int_{\mathbb{R}^n \setminus B_{r+\rho}} |\nabla u_\nu|^2 \to 0.$$

Proof. We are going to prove the assertion for $\varepsilon > 0$ and for a minimizing sequence such that $\|u_\nu\|_{2^*} = 1$. Moreover, suppose that $(u_\nu)_\nu$ has the Palais-Smale property, namely in this case setting

$$J(u_\nu) = \frac{Q(u_\nu)}{\|u_\nu\|_{2^*}^2},$$

we have

$$Q(u_\nu) \to S(a)$$

and

$$\|DJ(u_\nu)\|_{(D^{1,2}(\mathbb{R}^n))^*} = \varepsilon_\nu, \text{ when } \nu \to \infty,$$

with

$$\|DJ(u_\nu)\|_{(D^{1,2}(\mathbb{R}^n))^*} = \lim_{\|w\|_{D^{1,2}(\mathbb{R}^n)} \to 0} \frac{J(u_\nu + w) - J(u_\nu)}{\|w\|_{D^{1,2}(\mathbb{R}^n)}}.$$

For fixed v, setting $w = \varepsilon v$, with $\varepsilon \to 0$ we conclude

$$\frac{J(u_\nu + \varepsilon v) - J(u_\nu)}{\varepsilon} = \varepsilon_\nu \|v\|_{D^{1,2}(\mathbb{R}^n)}.$$

That is

$$\int_{\mathbb{R}^n} \nabla u_\nu \cdot \nabla v - a(x)\frac{u_\nu v}{|x|^2} = S(a) \int_{\mathbb{R}^n} |u_\nu|^{2^*-2} u_\nu v + o(\|v\|_{D^{1,2}(\mathbb{R}^n)}), \ \forall v \in D^{1,2}(\mathbb{R}^n), \quad (3.22)$$

since

$$\int_{\mathbb{R}^n}\left(\nabla u_\nu\cdot\nabla v - a(x)\frac{u_\nu v}{|x|^2}\right) = \int_{\mathbb{R}^n}|u_\nu|^{2^*-2}u_\nu v(\int_{\mathbb{R}^n}|\nabla u_\nu|^2 - a(x)\frac{u_\nu^2}{|x|^2})$$
$$= \int_{\mathbb{R}^n}|u_\nu|^{2^*-2}u_\nu v(S(a)+o(1)),$$

with

$$\int_{\mathbb{R}^n}|u_\nu|^{2^*-1}|v| \le \|v\|_{D^{1,2}(\mathbb{R}^n)}$$

using the normalization of the L^{2^*}'s norm. Then by virtue of Ekeland's principle [7,14] this will be true for any weakly convergent subsequence, since regarding that, close to any minimizing sequence there exists another one having the Palais-Smale property.

The following argument is borrowed from [13] (proof in the Appendix). Using the fact that

$$\int_{B_{r+\varepsilon}\setminus B_r}|\nabla u_\nu|^2 < \infty$$

and

$$\int_{\rho=r}^{\rho=r+\varepsilon}d\rho\int_{\rho S^{n-1}}|\nabla u_\nu|^2 = \int_{B_{r+\varepsilon}\setminus B_r}|\nabla u_\nu|^2$$

passing to a subsequence and by selection Lemma [6, Appendix] there exists $\rho\in(0,\varepsilon)$

$$\int_{(\rho+r)S^{n-1}}|\nabla u_\nu|^2 \le C\int_{B_{r+\varepsilon}\setminus B_r}|\nabla u_\nu|^2,$$

for infinitely many ν's and therefore u_ν is bounded in $H^1((r+\rho)S^{n-1})$.

Since $H^1((r+\rho)S^{n-1})$ is compactly embedded in $H^{\frac{1}{2}}((r+\rho)S^{n-1})$, passing to a subsequence we obtain the strong convergence

$$u_\nu \to u \text{ in } H^{\frac{1}{2}}((r+\rho)S^{n-1}).$$

Moreover, we have that there exists $C_1>0$ such that

$$\|u_\nu\|_{H^{\frac{1}{2}}((r+\rho)S^{n-1})} \le C_1\|u_\nu\|_{H^1(B_{r+\rho})}$$

and also, from Sobolev inequality, we obtain a constant $C_2>0$ such that

$$\|u_\nu\|_{H^1(B_{r+\rho})} \le C_2\|u_\nu\|_{D^{1,2}(\mathbb{R}^n)}.$$

We shall show that the weak convergence of the u_ν's to zero in $D^{1,2}(\mathbb{R}^n)$ forces the limit of u_ν's in $H^{\frac{1}{2}}((\rho+r)S^{n-1})$ to be zero. To do this, we prove that the weak convergence to 0 in $D^{1,2}(\mathbb{R}^n)$ implies the weak convergence to 0 in $H^1(B_{r+\rho})$. Then, we will be done, thanks to the continuity of the embedding of $H^1(B_{r+\rho})$ into $H^{\frac{1}{2}}((r+\rho)S^{n-1})$.

From the definition of the weak convergence we have that for every $l\in(D^{1,2}(\mathbb{R}^n))^*$

$$l(u_\nu)\to 0.$$

Let $\bar{l}\in(H^1(B_{r+\rho}))^*$, i.e.

$$|\bar{l}(u_\nu)| \le C_1\|u_\nu\|_{H^1(B_{r+\rho})} \le C_2\|u_\nu\|_{D^{1,2}(\mathbb{R}^n)}.$$

Thus, $\bar{l} \in (D^{1,2}(\mathbb{R}^n))^*$ and so $\bar{l}(u_\nu) \to 0$ for every $\bar{l} \in (H^1(B_{r+\rho}))^*$ meaning that

$$u_\nu \rightharpoonup 0 \text{ in } H^1(B_{r+\rho}).$$

So, since $u_\nu \rightharpoonup 0$ in $H^1(B_{r+\rho})$, thanks to the continuity of the embedding in $H^{\frac{1}{2}}((r+\rho)S^{n-1})$ by the same argument we conclude that

$$u_\nu \rightharpoonup 0 \text{ in } H^{\frac{1}{2}}((r+\rho)S^{n-1}).$$

So, invoking the strong convergence in this space we conclude

$$u_\nu \to 0 \text{ in } H^{\frac{1}{2}}((r+\rho)S^{n-1}).$$

Now, let $w_{1,\nu}$ harmonic functions in $B_{r+\varepsilon} \setminus B_{r+\rho}$ such that

$$w_{1,\nu} = u_\nu,\ (r+\rho)S^{n-1},$$

$$w_{1,\nu} = 0,\ (r+\varepsilon)S^{n-1}$$

and $w_{2,\nu}$ harmonic functions in $B_{r-\varepsilon} \setminus B_{r+\rho}$ such that

$$w_{2,\nu} = u_\nu,\ (r+\rho)S^{n-1},$$

$$w_{2,\nu} = 0,\ (r-\varepsilon)S^{n-1}.$$

So,

$$\|w_{1,\nu}\|_{H^{\frac{1}{2}}(\partial\Omega_1 \cup \partial\Omega_2)} = \|u_\nu\|_{H^{\frac{1}{2}}((r+\rho)S^{n-1})},$$

with $\partial\Omega_1 = (r+\rho)S^{n-1}$ and $\partial\Omega_2 = (r+\varepsilon)S^{n-1}$.

Similarly

$$\|w_{2,\nu}\|_{H^{\frac{1}{2}}(\partial\Omega_1 \cup \partial\Omega_3)} = \|u_\nu\|_{H^{\frac{1}{2}}((r+\rho)S^{n-1})},$$

where $\partial\Omega_3 = (r-\varepsilon)S^{n-1}$.

The inverse Laplace operator

$$\Delta^{-1} : H^{\frac{1}{2}}((r+\rho)S^{n-1} \cup (r+\varepsilon)S^{n-1}) \to H^1(B_{r+\varepsilon} \setminus B_{r+\rho})$$

is continuous. Therefore, we have

$$\|w_{1,\nu}\|_{H^1(B_{r+\varepsilon}\setminus B_{r+\rho})} \le \|w_{1,\nu}\|_{H^{\frac{1}{2}}(\partial\Omega_1 \cup \partial\Omega_2)} = \|u_\nu\|_{H^{\frac{1}{2}}((r+\rho)S^{n-1})} \to 0$$

and also

$$\|w_{2,\nu}\|_{H^1(B_{r+\rho}\setminus B_{r-\varepsilon})} \le \|w_{2,\nu}\|_{H^{\frac{1}{2}}(\partial\Omega_1 \cup \partial\Omega_3)} = \|u_\nu\|_{H^{\frac{1}{2}}((r+\rho)S^{n-1})} \to 0$$

i.e.

$$w_{1,\nu} \to 0, \text{ on } H^1(B_{r+\varepsilon} \setminus B_{r+\rho})$$

and

$$w_{2,\nu} \to 0 \text{ on } H^1(B_{r+\rho} \setminus B_{r-\varepsilon}).$$

Now we define two auxiliary sequences as follows

$$u_{1,\nu} = \begin{cases} u_\nu, & x \in B_{r+\rho} \\ w_{1,\nu}, & x \in B_{r+\varepsilon} \setminus B_{r+\rho} \\ 0, & \text{elsewhere} \end{cases} \tag{3.23}$$

$$u_{2,\nu} = \begin{cases} 0, & x \in B_{r-\varepsilon} \\ w_{2,\nu}, & x \in B_{r+\rho} \setminus B_{r-\varepsilon} \\ u_\nu, & \text{elsewhere.} \end{cases} \tag{3.24}$$

Since $\|u_{1,\nu}\|_{D^{1,2}(\mathbb{R}^n)}$ and $\|u_{2,\nu}\|_{D^{1,2}(\mathbb{R}^n)}$ are uniformly bounded we can test (3.22) with $v = u_{1,\nu}$ and $v = u_{2,\nu}$.

So,

$$Q(u_{1,\nu}) = \int_{\mathbb{R}^n} |\nabla u_{1,\nu}|^2 - a(x)\frac{u_{1,\nu}^2}{|x|^2} =^{(3.22)} S(a)\int_{\mathbb{R}^n} |u_{1,\nu}|^{2^*} + o(1),$$

thus,

$$\begin{aligned} Q(u_{1,\nu}) &= \int_{B_{r+\rho}} |\nabla u_\nu|^2 - a(x)\frac{u_\nu^2}{|x|^2} + \int_{B_{r+\varepsilon}\setminus B_{r+\rho}} |w_{1,\nu}|^2 - a(x)\frac{w_{1,\nu}^2}{|x|^2} \\ &= S(a)\left(\int_{B_{r+\rho}} u_\nu^{2^*} + \int_{B_{r+\varepsilon}\setminus B_{r+\rho}} w_{1,\nu}^{2^*}\right). \end{aligned}$$

But since $w_{1,\nu} \to 0$ on $H^1(B_{r+\varepsilon} \setminus B_{r+\rho})$ and

$$\|w_{1,\nu}\|_{L^{2^*}(B_{r+\varepsilon}\setminus B_{r+\rho})} \le C\,\|w_{1,\nu}\|_{H^1(B_{r+\varepsilon}\setminus B_{r+\rho})}$$

we have

$$Q(u_{1,\nu}) = S(a)\int_{B_{r+\rho}} u_\nu^{2^*} + o(1) = S(a)\int_{B_{r+\rho}} u_{1,\nu}^{2^*} + o(1).$$

So combining the above results we come up with

$$\begin{aligned} Q(u_{1,\nu}) &= \int_{B_{r+\rho}} |\nabla u_\nu|^2 - a(x)\frac{u_\nu^2}{|x|^2} + o(1) \\ &= S(a)\int_{B_{r+\rho}} u_\nu^{2^*} + o(1) = S(a)\int_{B_{r+\rho}} u_{1,\nu}^{2^*} + o(1). \end{aligned}$$

In the same way, for $u_{2,\nu}$ due to $w_{2,\nu} \to 0$ on $H^1(B_{r+\rho} \setminus B_{r-\varepsilon})$ we obtain

$$\begin{aligned} Q(u_{2,\nu}) &= \int_{\mathbb{R}^n\setminus B_{r+\rho}} |\nabla u_\nu|^2 - a(x)\frac{u_\nu^2}{|x|^2} = S(a)\int_{\mathbb{R}^n\setminus B_{r+\rho}} u_\nu^{2^*} + o(1) \\ &= S(a)\int_{\mathbb{R}^n\setminus B_{r+\rho}} u_{2,\nu}^{2^*} + o(1). \end{aligned}$$

Thus,

$$\|u_\nu\|_{2^*}^{2^*} = \|u_{1,\nu}\|_{2^*}^{2^*} + \|u_{2,\nu}\|_{2^*}^{2^*} + o(1)$$

and

$$Q(u_\nu) = Q(u_{1,\nu}) + Q(u_{2,\nu}) + o(1).$$

Suppose $u_{1,\nu}$ does not converge to 0. So

$$\begin{aligned} \frac{Q(u_{1,\nu})}{\|u_{1,\nu}\|_{2^*}^2} &= \frac{Q(u_\nu) - Q(u_{2,\nu}) + o(1)}{(\|u_\nu\|_{2^*}^{2^*} - \|u_{2,\nu}\|_{2^*}^{2^*} + o(1))^{\frac{2}{2^*}}} \\ &\le S(a)\frac{Q(u_\nu) - Q(u_{2,\nu}) + o(1)}{(Q^{\frac{2^*}{2}}(u_\nu) - Q^{\frac{2^*}{2}}(u_{2,\nu}) + o(1))^{\frac{2}{2^*}}}. \end{aligned}$$

Now, passing to a subsequence of $u_{2,\nu}$ so that the $\lim_\nu Q(u_{2,\nu})$ exists, since

$$\lim_\nu Q(u_\nu) = S(a) > 0,$$

we conclude that

$$\limsup_\nu \frac{Q(u_{1,\nu})}{\|u_{1,\nu}\|_{2^*}^2} < S(a),$$

which is a contradiction unless $Q(u_{2,\nu}) \to 0$. Thus, since Q defines equivalent norm we obtain

$$\int_{\mathbb{R}^n} |\nabla u_{2,\nu}|^2 = \int_{\mathbb{R}^n \setminus B_{r+\rho}} |\nabla u_\nu|^2 + o(1) \to 0.$$

Alternatively, we may suppose that $u_{2,\nu}$ does not converge to 0 and conclude $Q(u_{1,\nu}) \to 0$, so that

$$\int_{\mathbb{R}^n} |\nabla u_{1,\nu}|^2 = \int_{B_{r+\rho}} |\nabla u_\nu|^2 + o(1) \to 0.$$

$\square$

Proposition 3.4.3. *Let $\lambda_1(a) > 0$. Moreover, suppose for $n \geq 4$ that $\max_{S^{n-1}} a > 0$ and for $n = 3$, $\int_{S^{n-1}} a \geq 0$. Then $S(a) < S$ unless $a \equiv 0$.*

Proof. Let $n \geq 4$. We use the result of Lemma 2.2.1. Let $x_o \in S^{n-1}$ such that $a(x_o) > 0$. Then, there exists a ball $B_r(x_o)$ such that $\bar{a} = \inf_{B_r(x_o)} a > 0$. We define $\lambda = \bar{a}(1+r)^{-2}$.

We have

$$H_0^1(B_r(x_o)) \subset D^{1,2}(\mathbb{R}^n),$$

since

$$\int_{\mathbb{R}^n} |\nabla u|^2 = \int_{B_r(x_o)} |\nabla u|^2 < \infty$$

and from Sobolev inequality we have $u \in L^{2^*}(\mathbb{R}^n)$.

So,

$$S(a) \leq \inf_{u \in H_0^1(B_r(x_o)) \setminus \{0\}} \frac{Q(u)}{\|u\|_{2^*}^2} \leq \inf_{u \in H_0^1(B_r(x_o)) \setminus \{0\}} \frac{\int_{B_r(x_o)} |\nabla u|^2 - \lambda u^2}{\|u\|_{2^*}^2} = S_\lambda \stackrel{2.2.1}{<} S,$$

where for the last inequality we use the fact that $\lambda > 0$.

Now, for the case $n = 3$, let u_o be one of the minimizers of the usual Sobolev quotient. Since

$$\int_{\mathbb{R}^n} a(x) \frac{u^2}{|x|^2} \geq 0,$$

we have

$$Q(u_o) \leq \|\nabla u_o\|_2^2 \Rightarrow \frac{Q(u_o)}{\|u_o\|_{2^*}^2} \leq \frac{\|\nabla u_o\|_2^2}{\|u_o\|_{2^*}^2} = S.$$

Thus $S(a) \leq S$. But u_o cannot be a minimizer of $S(a)$, since it does not solve the associated Euler-Lagrange equation ($-\Delta u = a(x)\frac{u}{|x|^2} + u^{2^*-1}$) unless $a \equiv 0$. So, we obtain

$$S(a) < S,$$

unless $a \equiv 0$. $\square$

Proposition 3.4.4. *Using the assumptions of Proposition 3.4.3*

$$S(a) = \inf_{u \in D^{1,2}(\mathbb{R}^n) \setminus \{0\}} \frac{Q(u)}{\|u\|_{2^*}^2}$$

is achieved. Therefore,

$$-\Delta u = a(x) \frac{u}{|x|^2} + u^{2^*-1} \tag{3.25}$$

admits a positive solution in $D^{1,2}(\mathbb{R}^n)$.

Proof. Let $(u_\nu)_\nu$ such that $u_\nu \in D^{1,2}(\mathbb{R}^n)$ and $\|u_\nu\|_{L^{2^*}(\mathbb{R}^n)} = 1$ with

$$Q(u_\nu) \to S(a).$$

Suppose $u_\nu \rightharpoonup u_o$. If $u_o \not\equiv 0$ the proof is done from the result of Proposition 3.4.1. If not, we are going to show that a sequence of suitably rescaled u_ν has a nonzero weak limit.

For every ν, let $R_\nu > 0$ such that

$$\int_{B_{R_\nu}} |\nabla u_\nu|^2 - a(x) \frac{u_\nu^2}{|x|^2} = \int_{\mathbb{R}^n \setminus B_{R_\nu}} |\nabla u_\nu|^2 - a(x) \frac{u_\nu^2}{|x|^2} = \frac{1}{2} Q(u_\nu) = \frac{S(a)}{2} + o(1).$$

We define

$$v_\nu(x) = R_\nu^{\frac{n-2}{2}} u_\nu(x R_\nu).$$

So, we obtain

$$\begin{aligned} \int_{B_1} |\nabla v_\nu|^2 - a(x) \frac{v^2}{|x|^2} dx &= \int_{B_1} R_\nu^n |\nabla u(x R_\nu)|^2 - a(x) R_\nu^{n-2} \frac{u^2(x R_\nu)}{|x|^2} dx \\ &= \int_{B_{R_\nu}} |\nabla u_\nu|^2 - a(y) \frac{u^2(y)}{|y|^2} dy. \end{aligned}$$

and from the above form we can conclude that

$$\int_{B_1} |\nabla v_\nu|^2 - a(x) \frac{v_\nu^2}{|x|^2} = \int_{\mathbb{R}^n \setminus B_1} |\nabla v_\nu|^2 - a(x) \frac{v_\nu^2}{|x|^2} = \frac{1}{2} Q(u_\nu). \tag{3.26}$$

But, the ratio $\frac{Q(u_\nu)}{\|u_\nu\|_{2^*}^2}$ is invariant under scaling of the type $R^{\frac{n-2}{2}} u_\nu(xR)$ so v_ν still forms a minimizing sequence which, being $D^{1,2}(\mathbb{R}^n)$ bounded, admits a weakly convergent subsequence.

We will argue by contradiction applying the Proposition 3.4.2 twice. Then for $r = 1$ and $\varepsilon = \frac{1}{4}$ and $\varepsilon = -\frac{1}{4}$ we obtain that there exists $\rho^+ \in (0, \frac{1}{4})$ and $\rho^- \in (-\frac{1}{4}, 0)$ such that

$$\int_{B_{1+\rho-}} |\nabla v_\nu|^2 \to 0 \text{ or } \int_{\mathbb{R}^n \setminus B_{1+\rho-}} |\nabla v_\nu|^2 \to 0$$

and

$$\int_{B_{1+\rho+}} |\nabla v_\nu|^2 \to 0 \text{ or } \int_{\mathbb{R}^n \setminus B_{1+\rho+}} |\nabla v_\nu|^2 \to 0.$$

We shall rule out all possibilities other than

$$\int_{B_{1+\rho-}} |\nabla v_\nu|^2 \to 0$$

and

$$\int_{\mathbb{R}^n \setminus B_{1+\rho+}} |\nabla v_\nu|^2 \to 0$$

Indeed, suppose $\int_{B_{1+\rho+}} |\nabla v_\nu|^2 \to 0$ then $\int_{B_1} |\nabla v_\nu|^2 \to 0$. Using Hardy's Inequality [8]

$$\int_{B_r} \frac{u^2}{|x|^2} dx \leq C \int_{B_r} |\nabla u|^2 dx + \frac{1}{r^2} \int_{B_r} u^2 dx$$

and using the L^2_{loc} convergence of the v_ν's we obtain that

$$\int_{B_1} a(x) \frac{v_\nu^2}{|x|^2} \to 0,$$

which contradicts (3.26) since $S(a) > 0$.

So, we conclude that

$$\int_{\mathbb{R}^n \setminus B_{1+\rho+}} |\nabla v_\nu|^2 \to 0.$$

Moreover, since

$$B_1 \subset B_{1+\rho+} \text{ we have } \int_{\mathbb{R}^n \setminus B_{1+\rho+}} |\nabla v_\nu|^2 < \int_{\mathbb{R}^n \setminus B_1} |\nabla v_\nu|^2,$$

we have from (3.26) that

$$\int_{\mathbb{R}^n \setminus B_1} a(x) \frac{v_\nu^2}{|x|^2} + \frac{S(a)}{2} + o(1) > \int_{\mathbb{R}^n \setminus B_{1+\rho+}} |\nabla v_\nu|^2$$

and using the L^2_{loc} convergence of the v_ν's we have

$$-\int_{\mathbb{R}^n} a(x) \frac{v_\nu^2}{|x|^2} + o(1) < \frac{S(a)}{2} + o(1). \tag{3.27}$$

Suppose

$$\int_{\mathbb{R}^n \setminus B_{1+\rho-}} |\nabla v_\nu|^2 \to 0 \text{ and so } \int_{\mathbb{R}^n \setminus B_1} |\nabla v_\nu|^2 \to 0.$$

Then from (3.26) we have

$$o(1) - \int_{\mathbb{R}^n \setminus B_{1+\rho+}} a(x) \frac{v_\nu^2}{|x|^2} = \frac{S(a)}{2} + o(1),$$

contradicting (3.27). So, necessarily

$$\int_{B_{1+\rho-}} |\nabla v_\nu|^2 \to 0.$$

Moreover, from the L^2_{loc} convergence of the v_ν's we have that

$$\int_{B_{\frac{3}{2}} \setminus B_{\frac{1}{2}}} a(x) \frac{v_\nu^2}{|x|^2} \to 0.$$

Now, let η be smooth cut-off function such that $0 \leq \eta \leq 1$, $\eta(x) = 1,$ for $|x| \in [\frac{3}{4}, \frac{5}{4}]$ and $\eta(x) = 0,$ if $|x| \notin [\frac{1}{2}, \frac{3}{2}]$.

We have

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla v_\nu|^2 &= \int_{B_{1+\rho-}} |\nabla v_\nu|^2 + \int_{B_{1+\rho+}\setminus B_{1+\rho-}} |\nabla v_\nu|^2 + \int_{\mathbb{R}^n\setminus B_{1+\rho+}} |\nabla v_\nu|^2 \\ &= o(1) + \int_{B_{1+\rho+}\setminus B_{1+\rho-}} |\nabla v_\nu|^2.\end{aligned}$$

Thus,

$$\begin{aligned}0 \le \|(\eta v_\nu) - v_\nu\|^2_{D^{1,2}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\nabla(\eta v_\nu)|^2 - 2\int_{\mathbb{R}^n} \nabla(\eta v_\nu)\cdot\nabla v_\nu + \int_{\mathbb{R}^n} |\nabla v_\nu|^2 \\ &\le \int_{\frac{1}{2}\le|x|\le\frac{3}{2}} |\nabla v_\nu|^2 - \int_{\frac{1}{2}\le|x|\le\frac{3}{2}} |\nabla v_\nu|^2\eta \\ &= \int_{B_{1+\rho+}\setminus B_{1+\rho-}} |\nabla v_\nu|^2 - \int_{B_{1+\rho+}\setminus B_{1+\rho-}} |\nabla v_\nu|^2 + o(1) = o(1).\end{aligned}$$

i.e.

$$\|(\eta v_\nu) - v_\nu\|_{D^{1,2}(\mathbb{R}^n)} \to 0.$$

Also, using the equivalence of $Q^{\frac{1}{2}}$ norm we obtain

$$Q((\eta v_\nu) - v_\nu) = o(1),$$

i.e.

$$\int_{\mathbb{R}^n} |\nabla((\eta v_\nu) - v_\nu)|^2 - \int_{\mathbb{R}^n} a(x)\frac{|\eta v_\nu - v_\nu|^2}{|x|^2} = o(1).$$

Now, since $\|(\eta v_\nu - v_\nu)\|_{D^{1,2}(\mathbb{R}^n)} \to 0$ we have

$$-\int_{\mathbb{R}^n} a(x)\frac{\eta^2 v_\nu^2}{|x|^2} + 1\int_{\mathbb{R}^n} a(x)\frac{\eta v_\nu^2}{|x|^2} - \int_{\mathbb{R}^n} a(x)\frac{v_\nu^2}{|x|^2} = o(1),$$

and using that

$$v_\nu \to 0 \text{ in } L^2_{loc}(\mathbb{R}^n)$$

, we conclude that

$$\int_{\mathbb{R}^n} a(x)\frac{v_\nu^2}{|x|^2} = o(1).$$

Therefore,

$$Q(v_\nu) = \int_{B_{1+\rho+}\setminus B_{1+\rho-}} |\nabla v_\nu|^2 + o(1). \tag{3.28}$$

Moreover, since $\|\eta v_\nu - v_\nu\|_{D^{1,2}(\mathbb{R}^n)} \to 0$ from Sobolev inequality we conclude that

$$\int_{\mathbb{R}^n} |\eta v_\nu - v_\nu|^{2^*} \to 0,$$

i.e.

$$\int_{\mathbb{R}^n} |\eta v_\nu - v_\nu|^{2^*} = o(1) \text{ for } \nu \to \infty.$$

From Minkowski's inequality we have

$$\|\eta v_\nu - v_\nu\|_{2^*} \geq |\|\eta v_\nu\|_{2^*} - \|v_\nu\|_{2^*}|$$

and so

$$\|\eta v_\nu\|_{2^*} = \|v_\nu\|_{2^*} + o(1), \text{ for } \nu \to \infty.$$

Thus,

$$\|\eta v_\nu\|_{2^*}^2 = \|v_\nu\|_{2^*}^2 + o(1) \tag{3.29}$$

Combining (3.28) and (3.29) we obtain

$$S = S(0) \leq \frac{\int_{\mathbb{R}^n} |\nabla(\eta v_\nu)|^2}{\|\eta v_\nu\|_{2^*}^2} = \frac{Q(v_\nu) + o(1)}{\|v_\nu\|_{2^*}^2 + o(1)}$$

and passing to the limit we have $S \leq S(a)$ which is a contradiction from Proposition 3.4.3. Finally, we have that in each case we obtain a non zero weak limit, obtaining a minimizer from Proposition 3.4.1. So, let u be a minimizer of $S(a)$. We may suppose that $u \geq 0$ (since, otherwise we replace with $|u|$) such that

$$-\Delta u - a(x)\frac{u}{|x|^2} = S(a)u^{2^*-1}.$$

After scaling, we obtain a positive solution of (3.25). □

CHAPTER 4

Appendix.

Lemma 4.0.1. *(Brezis-Lieb Lemma) Let (Ω, Σ, μ) be a measure space and suppose $f_n \subset L^p$ with $0 < p < \infty$. If*

(i) f_n is bounded in L^p,

(ii) $f_n \to f$ almost everywhere, then

$$\lim_n \left(\|f_n\|_p^p - \|f_n - f\|_p^p \right) = \|f\|_p^p .$$

Proof. Regarding the case $0 < p \leq 1$ we do not need the uniform boundedness of $\|f_n\|_p$ and the result follows from the inequality

$$||f_n|^p - |f_n - f|^p| \leq |f|^p$$

and by dominated convergence theorem. In particular, for the case $p = 1$ from Fatou's Lemma we have $f \in L^1$. Setting $g_n = f_n - f$ so that g_n converges to 0 almost everywhere.

Define

$$G_n := |g_n| + |f| - |g_n + f|$$

and then since $|G_n| \leq 2|f|$ by dominated convergence theorem we have

$$\int_\Omega G_n \to 0.$$

Then for $p > 1$ we consider

$$G_n := |g_n|^p + |f|^p - |g_n + f|^p.$$

There is a constant C_ε such that

$$|G_n| - \varepsilon |g_n|^p \leq C_\varepsilon |f|^p.$$

Indeed, using the convexity of $h(t) = |t|^p$, with $t \in [0, 1]$ we get

$$G_n \geq |g_n|^p + |f|^p - t^{1-p}|g_n|^p - (1-t)^{1-p}|f|^p.$$

Now, choosing $t = t_\varepsilon = (1+\varepsilon)^{\frac{1}{1-p}}$ we get

$$G_n \geq -\varepsilon|g_n|^p - C_{1,\varepsilon}|f|^p,$$

where $C_{1,\varepsilon} = (1-t_\varepsilon)^{1-p} - 1$.

Moreover, we have

$$\begin{aligned} G_n &= (1-\varepsilon)|g_n|^p + |f|^p - |f+g_n|^p + \varepsilon|g_n|^p \\ &\leq \left((1-\varepsilon)s^{1-p} - 1\right)|f+g_n|^p + \left(1 + (1-s)^{1-p}\right)|f|^p + \varepsilon|g_n|^p. \end{aligned}$$

Now, setting $s = s_\varepsilon = (1-\varepsilon)^{\frac{1}{p-1}}$ we conclude

$$G_n \leq \varepsilon|g_n|^p + \left(1 + (1-s_\varepsilon)^{1-p}\right)|f|^p$$

and we obtain the result for $C_\varepsilon = 1 + (1-s_\varepsilon)^{1-p}$. Thus, by dominated convergence theorem we have

$$\lim_{n\to\infty} \int_\Omega |G_n| - \varepsilon|g_n|^p = 0.$$

Now we use the assumption of uniform boundedness. Since t^p is convex for $p \geq 1$ we have

$$\int_\Omega |g_n|^p = \int_\Omega |f_n - f|^p \leq 2^{p-1} \int_\Omega |f_n|^p + |f|^p \leq 2^p C$$

Thus,

$$\int_\Omega |G_n| \leq \varepsilon 2^p C + \int_\Omega (|G_n| - \varepsilon|g_n|^p)$$

and consequently

$$\limsup_n \int_\Omega |G_n| \leq \varepsilon 2^p C$$

and since $\varepsilon > 0$ arbitrary small, the proof is done. □

Theorem [13]. Let $u_m \in H^1(\Omega)$ and define

$$A_1 = \sup_m \int_\Omega |\nabla u_m|^2.$$

Then from Fubini's Theorem we have $u_m \in H^1(rS^{n-1})$ for almost every $r > 0$. Moreover, for $\delta \in [0,1]$ and due to

$$\int_{1-\delta}^{1+\delta} \int_{rS^{n-1}} |\nabla u_m|^2 dS dr \leq A_1$$

for every m there exists a set Λ_m with radii in $[1-\delta, 1+\delta]$ with measure $|\Lambda_m| \geq \frac{3\delta}{2}$ and

$$\int_{rS^{n-1}} |\nabla u_m|^2 \leq \frac{2A_1}{\delta} = c_1$$

for every $r \in \Lambda_m$.

Proof. Indeed , suppose for contradiction that $\exists m$ such that

$$|\Lambda_m| \geq \frac{3\delta}{2} \text{ and } \int_{rS^{n-1}} |\nabla u_m|^2 > \frac{2A_1}{\delta}.$$

Let

$$f_m(r) = \int_{rS^{n-1}} |\nabla u_m|^2.$$

Then,

$$\int_{1-\delta}^{1+\delta} f_m(r)dr \geq \int_{\Lambda_m} f_m(r)dr > 3A_1,$$

which is a contradiction. □

Selection Lemma. Let $G \subset \mathbb{R}^n$ measurable with $\mu(G) < \infty$ and M_k a sequence of measurable sets in G such that for some $\varepsilon > 0$ we have $\mu(M_k) > \varepsilon\ \forall k \in \mathbb{N}$. We can choose by passing to a subsequence that $\{M_{k_l}\}$ such that $\bigcap_{l=1}^{\infty} M_{k_l} \neq \varnothing$.

Proof. Let

$$\phi(x) = \lim_m \sum_{m=1}^{\infty} \chi_{M_k}(x).$$

i.e. $\phi(x)$ indicates the number of M_k's containing x.

Suppose

$$\bigcap_{l=1}^{\infty} M_{k_l} \not\equiv \varnothing$$

and so $\phi(x) < \infty$.

From Lusin's Theorem, there exists compact set $K \subset G$ with $\mu(G \setminus K) < \frac{\varepsilon}{2}$ and $\phi_{|K}$ continuous,thus $\phi(x) \leq n_0\ \forall x \in K$.

Let $M_k' = M_k \cap K$ so that $\mu(M_k') > \frac{\varepsilon}{2}$. We have

$$\infty > n_0\mu(K) > \int_K \phi(x) = \int_K \lim_m \sum_{k=1}^{\infty} \chi_{M_k'}$$

$$=^{Beppo-Levi} \lim_m \sum_{k=1}^{\infty} \int_K \chi_{M_k'}(x) = \lim_m \mu(M_k') > \lim_m (m\frac{\varepsilon}{2}) = \infty,$$

which is a contradiction. □

CHAPTER 5

References.

[1] Aubin, Th., Equations différentielles non lineaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures et Appl. 55, (1976), pp. 269-293.
[2] H. Brezis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc., 88 (1983), 486–490.
[3] H. Brezis and L. Nirenberg, Positive solutions of Nonlinear elliptic equations involving Critical exponents, Comm. Pure Appl. Math., 36 (1983), 437–477.
[4] H. Brézis and T. Kato, Remarks on the Schrödinger operator with singular complex potentials, J. Math. Pures Appl. (9) 58 (1979), no. 2, 137–151.
[5] L. Cafarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math., 42 (1989) 271–297.
[6] G.Eisen, A selection lemma for sequences of measurable sets , and lower semicontinuity of multiple integrals , manuscripta mathematica volume 27, pages73–79 (1979).
[7] Ekeland, I., On the variational principle, J. Math. Anal. Appl. 47, 1974, pp. 324-353.
[8] L.C. Evans, Partial differential equations, Amer. Math. Soc., 1998 (2nd edition 2010).
[9] B. Gidas, W.M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979), 209–243.
[10] B. Gidas, W.M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in Rn, Math. Anal. and Applic., Part A, Advances in Math. Suppl. Studies 7A, ed. L. Nachbin, Academic Press (1981), 209–243.
[11] S.I. Pohozaev, Eigenfunctions of the equation u + f(u) = 0, Sov. Math. Doklady, 5 (1965),1408–1411.
[12] Kazdan, J., and Warner, F., Remarks on some quasilinear elliptic equations, Comm. Pure Appl.Math. 28, 1975, pp. 567-597.
[13] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, Math. Z., 187 (1984), 511–517.
[14] M. Struwe, “Variational Methods and Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems,” Springer-Verlag, Berlin/New York (1990).
[15] G. Talenti, Best constants in Sobolev inequality, Annali di Mat., 110 (1976), 353–372.
[16] S. Terracini, On positive entire solutions to a class of equations with singular coefficient and critical exponent, Adv. Diff. Equa., 1 (1996), no. 2, 241–264.
[17] M.Willem, Minimization problems with lack of compactness , Banach Center Publications 35 (1996), 97-107.