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Predicates of the 3D Apollonius Diagram

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"*A single idea, if it is right, saves us the labor of an infinity of experiences.*"

Jacques Maritain

Abstract

In this thesis we study one of the fundamental predicates required for the construction of the 3D Apollonius diagram (also known as the 3D Additively Weighted Voronoi diagram), namely the EdgeConflict predicate: given five sites S_i, S_j, S_k, S_l, S_m that define an edge e_{ijklm} in the 3D Apollonius diagram, and a sixth query site S_q, the predicate determines the portion of e_{ijklm} that will disappear in the Apollonius diagram of the six sites due to the insertion of S_q.

Our focus is on the algorithmic analysis of the predicate with the aim to minimize its algebraic degree. We decompose the main predicate into sub-predicates, which are then evaluated with the aid of additional primitive operations. We show that the maximum algebraic degree required to answer any of the sub-predicates and primitives, and, thus, our main predicate is 10 in non-degenerate configurations when the trisector is of Hausdorff dimension 1. We also prove that all subpredicates developed can be evaluated using 10 or 8-degree demanding operations for degenerate input for these trisector types, depending on whether they require the evaluation of an intermediate InSphere predicate or not.

Among the tools we use is the 3D inversion transformation and the so-called qualitative symbolic perturbation scheme. Most of our analysis is carried out in the inverted space, which is where our geometric observations and analysis is captured in algebraic terms.

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Key words and phrases. Computational geometry, algebraic computing, geometric predicates, Euclidean Apollonius diagram, EdgeConflict predicate, qualitative symbolic perturbation.

Περίληψη

Στην εργασία αυτή μελετάμε ένα από τα κεντρικότερα κατηγορήματα το οποίο απαιτείται για την κατασκευή του 3-Διάστατου Απολλώνιου Διαγράμματος (γνωστό και ως Voronoi διάγραμμα βεβαρυμένων σημείων), το επονομαζόμενο κατηγόρημα EDGE-CONFLICT: δεδομένων 5 σφαιρών S_i, S_j, S_k, S_l, S_m τα οποία ορίζουν μια ακμή e_{ijklm} στο Απολλώνιο διάγραμμα και μίας έκτης σφαίρας S_q, το κατηγόρημα αποφαίνεται ποιο υποσύνολο της e_{ijklm} θα πάψει να υπάρχει ως ακμή στο στο Απολλώνιο διάγραμμα των 6 σφαιρών.

Το κύριο μέλημά μας είναι η αλγοριθμική ανάλυση του κατηγορήματος αυτού, έχοντας ως βασικό στόχο την ελαχιστοποίηση του αλγεβρικού του βαθμού. Αρχικά αποσυνθέτουμε το βασικό κατηγόρημα σε υποκατηγορήματα, τα οποία βασίζονται με τη σειρά τους σε πιο βασικούς γεωμετρικοαλγεβρικούς ελέγχους. Αποδεικνύουμε ότι το σύνολο των υποκατηγορημάτων και άρα και το κεντρικό κατηγόρημα απαιτεί υπολογισμούς αλγεβρικού βαθμού το πολύ 10 για να απαντηθεί για μη εκφυλισμένες εισόδους. Επίσης, για του ίδιου τύπου τριχοτόμους, αποδεικνύουμε ότι όλα τα υποκατηγορήματα που έχουμε σχεδιάσει απαιτούν μέγιστο αλγεβρικό βαθμό 10 ή 8, εάν απαιτούν τον ενδιάμεσο υπολογισμό του κατηγορήματος InSphere ή όχι.

Ανάμεσα στα εργαλεία που χρησιμοποιούμε είναι ο μετασχηματισμός αντιστροφής και η συμβολική διαταραχή. Το μεγαλύτερο μέρος της ανάλυσής μας γίνεται στον αντεστραμμένο χώρο, όπου οι γεωμετρικές μας παρατηρήσεις μεταφράζονται σε αλγεβρικούς όρους.

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Λέξεις και φράσεις κλειδιά. Υπολογιστική Γεωμετρία, αλγεβρικοί υπολογισμοί, γεωμετρικά κατηγορήματα, Ευκλείδειο Απολλώνιο Διάγραμμα, κατηγόρημα EdgeConflict, συμβολική διαταραχή.

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Dedicated to those who believed in me
and, above all, Thanases

Chapter 1

Preliminaries

1.1 Introduction

Voronoi diagrams have been among the most studied structures in computational geometry since their inception [25, 10, 5, 26], due to their numerous applications, including motion planning and collision detection, communication networks, graphics, and growth of microorganisms in biology.

Despite being a central topic in research for many years, generalized Voronoi diagrams, and especially the Voronoi diagram of spheres (also known as the 3D Apollonius diagram) have not been explored sufficiently [34]; this is also pointed out by Aurenhammer et al. [6]. Moreover, due to recent scientific discoveries in biology and chemistry, 3D Apollonius diagrams are becoming increasingly important for representing and analysing the molecular 3D structure and surface [29] or the structure of the protein [33].

The methods used to calculate the Apollonius diagram usually rely on the construction of a different diagram altogether. Some methods include the intersection of cones [4] with the lifted power diagram and lower envelope calculations [42, 43, 27]. Boissonnat et al. use the convex hull to describe its construction [9, 8]. Aurenhammer's lifting method has also been implemented for two dimensions [2]. Karavelas and Yvinec [30] create the 2d Apollonius diagram from its dual, using the predicates developed in [22]. In [22], it is also reported that the Apollonius diagram can be obtained as a concrete case of the abstract Voronoi diagrams of Klein et al. [36].

Kim et al. made a major research contribution in the domain of the Voronoi diagrams of spheres. Their work provides many new algorithms related to the Voronoi diagrams including the computation of three-dimensional Voronoi diagrams [31, 35], Euclidean Voronoi diagram of 3D balls and its computation via tracing edges [32] and the Euclidean Voronoi diagrams of 3D spheres and applications to protein structure analysis [33, 14].

Hanniel and Elber [27] provide an algorithm for computation of the Voronoi diagrams for planes, spheres and cylinders in $\mathbb{R}^3$. Their algorithm relies on computing the lower envelope of the bisector surfaces similar to the algorithm of Will [43]. However, none of the current research efforts provide an exact method for computing the Apollonius diagram (or its dual Delaunay graph) of spheres. It is also true that very few works study the *exact* Voronoi diagram (or its dual Delaunay graph) for curved objects and relative predicates or invariants, such as [22] for circles, [23] for convex pseudo-circles and [11] for n-dimensional spheres.

Various algorithms and implementations that construct the Apollonius diagram of spheres (or its dual graph) can be found in the current bibliography [14, 39, 18, 38, 12, 13]. These approximative algorithms however are not exact as numerical approximation of roots of high degree polynomials is required throught their execution. The propagation of the errors can be critical, especially if the final output involves approximate intermediary computations or when degenerate cases arise. Regarding degeneracies, most of these implementations do not handle them separately or efficiently if not at all.

In this thesis, we are inspired by the the approach presented by Emiris and Karavelas in [22] for the exact evaluation of the 2D Apollonius diagram. In order to extend their work for the Apollonius diagram for 3D spheres, we develop equivalent predicates as the ones presented in their paper for the 2D case. Our ultimate goal is to analyze the most degree demanding predicate, called the EdgeConflict predicate: given five sites S_i, S_j, S_k, S_l, S_m that define a finite edge e_{ijklm} in the 3D Apollonius diagram, and a sixth query site S_q, the predicate determines the portion of e_{ijklm} that will disappear in the Apollonius diagram of the six sites due to the insertion of S_q.

A description of how these predicates could be used in the scope of a randomized incremental algorithm that constructs the Apollonius diagram is provided in [22, Section 3] and [30] for the 2D case. However, these procedure could easily be adapted for the 3D case, with minor modifications. The idea behind the algorithm is that one could construct the Voronoi diagram $\mathcal{VD}(\Sigma)$ of a set Σ by initially constructing the Voronoi diagram $\mathcal{VD}(\Sigma')$ of a subset $\Sigma' = \{S_i, S_j, S_k, S_l, S_m\}$ of Σ. Then, for a site $S_q \in \Sigma \backslash \Sigma'$, update $\mathcal{VD}(\Sigma')$ to $\mathcal{VD}(\Sigma' \cup \{S_q\})$ and set $\Sigma' \leftarrow \Sigma' \cup \{S_q\}$. If this process is repeated until $\Sigma \backslash \Sigma'$ is empty, i.e., $Sigma'$ becomes Σ, then we will have obtain $\mathcal{VD}(\Sigma)$. The difficult task of updating the Voronoi diagram requires, among other operations, to examine if a portion of a valid Voronoi edge e_{ijklm} in $\mathcal{VD}(\Sigma')$ will disappear in $\Sigma' \leftarrow \Sigma' \cup \{S_q\}$. The last task is accomplished via the call of the EdgeConflict predicate and it is evident that multiple calls of the predicate are demanded even after the insertion of a single site S_q. It is therefore crucial that an efficient design of this predicate would greatly impact the overall efficiency of the incremental algorithm that constructs $\mathcal{VD}(\Sigma')$.

Below, we provide a list of the major predicates that we need to incrementally construct the 3D Apollonius diagram, similarly with the 2D case (cf. [22, 30]). The indices of the input sites in the following predicates are indicative.

- NearestNeighbor(S_a), which computes the sphere of Σ' closest to S_a. This predicates involve finding the minimum of $d(S_q, S_n)$ for all $S_n \in \Sigma'$, which is a 4 degree-demanding operation.

- IsHidden(S_a, S_b), which computes if S_a lies inside S_b and therefore the Voronoi region of S_a lies inside the region of S_b. This predicate requires operations of degree 2 as it corresponds to the sign of $d(S_a, S_b) + 2r_a = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2} + r_a - r_b$.

- EdgeConflict($S_i, S_j, S_k, S_l, S_m, S_q$), which is the main predicate studied in this thesis and is fully analyzed for non-degenerate inputs in the case where the trisector τ_{ijk} is either hyperbolic or elliptic (see Sections 2 and 3). The subpredicates required to answer the EdgeConflict predicate for both of these trisector types, except the InSphere (also referred to as VertexConflict in the bibliography), are also studied for degenerate inputs in Chapter 4.

- InfiniteRightEdgeConflict(S_i, S_j, S_k, S_l, S_q), which computes the portion of a semi-finite edge e that no longer remains in the Voronoi diagram of the first four sites, after the insertion of the fifth. This edge e lies on the trisector τ_{ijk} and is bounded on the "left" by the Apollonius vertex v_{ijkl}. This predicate can be called only in the case where the trisector is infinite hence either of hyperbolic or parabolic type. It has been analyzed in Section 2.6 for non-degenerate configurations on hyperbolic trisectors.

- InfiniteLeftEdgeConflict(S_i, S_j, S_k, S_m, S_q), which is the symmetric predicate of the previous one and computes the portion of a semi-finite edge e that no longer remains in the Voronoi diagram of the first four sites, after the insertion of the fifth. This edge e lies on the trisector τ_{ijk} and is bounded on the "right" by the Apollonius vertex v_{ikjm}. The predicates analysis for non-degenerate configurations on hyperbolic trisectors can be found in Section 2.6.

- InCone(S_a, S_b, S_c), which returns the relative position of S_c against the semi-cone defined by S_a and S_b. This predicate's detailed description and analysis can be found in Sections 2.4.2 and 2.7.1. The predicate's analysis for degenerate input is provided in Section 4.1. Via the InCone predicate, we can determine the existence of disconnected components of the skeleton of the 3D Apollonius diagram, e.g. when a smaller sphere is inserted inside the convex hull of two others.

- FaceConflict(S_a, S_q) which computes if there are points of the Apollonius face of S_a in $\mathcal{VD}(\Sigma)$ that no longer are part of an Apollonius face in $\mathcal{VD}(\Sigma \cup \{S_q\})$ due to the insertion of S_q; we shall call that these points are *in conflict* with S_q. It holds that if S_q is not hidden, then a subset of the Apollonius face of it's nearest neighbor site, $NN(S_q) \in \Sigma$ will be in conflict with S_q. If this subset contains points that belong to Apollonius edges, then we can call the EdgeConflict predicate to detect the portion of the edge that is in conflict. However, it does not hold that there are always points of Apollonius edges that are in conflict, e.g., if a small sphere is inserted in the convex hull of two larger spheres. There has not been any references in the current bibliography on how the FaceConflict predicate could be designed and/or implemented in the 3D Apollonius diagram. It is however apparent that it will have to include calling the InCone and/or EdgeConflict as subpredicates.

In this thesis, we describe and analyze how all of the predicates described above can be answered, except for the FaceConflict. The development of these predicates along with the necessary subpredicates and primitives was made taking into consideration the modern shift of predicate design towards lower level algorithmic issues. Specifically, a critical factor that influenced our design was our goal to minimize the algebraic degree of the tested quantities (in terms of the input parameters) during a predicate evaluation. Such a minimization problem has become a main concern that influences algorithm design especially in geometric predicates, where zero tolerance in all intermediate computations is needed to obtain an exact result [17, 24, 7, 41, 44] .

Our main contribution in the research area is the development of a list of subpredicates that were not analyzed, either explicitly or implicitly, in the current bibliography and can be used within the scope of an incremental algorithm that constructs the 3D Apollonius diagram of a set of spheres. Our most outstanding result is the fact that all subpredicates presented in this thesis along with the EdgeConflict predicate require at most 10-fold degree demanding operation (with respect to the input quantities) in non-degenerate configurations. This is quite a low bound on the required degree since the equivalent EdgeConflict predicate in the 2D Apollonius diagram requires 6-fold operations [37] for non-degenerate inputs. Note that our approach of resolving the EdgeConflict predicate can also be applied for the 2D case to yield similar algebraic degrees.

Beside the implementation of these predicates under the no-degeneracies assumption, we also provide a way of resolving the predicates for degenerate inputs using the so-called qualitative symbolical perturbation technique (see Section 1.2 below). For every subpredicate appearing in the evaluation of the EdgeConflict predicate except for the InSphere predicate (also known as the VertexConflict predicate), we prove that the maximum algebraic degree required to evaluate it is 8, regardless of the existence of degeneracies. Therefore, we have proven that the algebraic degree of deciding the EdgeConflict predicate for degenerate inputs is $\max d, 8$, where d is the respective algebraic cost of VertexConflict. Moreover, the tools presented in this thesis and especially the observations made in the inverted space, suggest that the existing result $d = 28$ [15], provided by the author, can be as low as 10.

1.2 Introduction to Qualitative Symbolic Perturbation

In the field of computational geometry, the first non-primitive predicates that were designed usually ignored degenerate configurations. The sentences “no four points are cocircular” or “no four spheres are tagent to a common plane” are common in many papers; indeed, such assumptions can be found in almost all sections of Chapters 2 and 3. The non-degeneracy hypothesis allows simpler design and analysis of algorithms as special cases are not considered at all.

However, degenerate situations can not be ignored as they actually occur in practice [19, 46, 45]. Such degeneracies can be handled in various ways, usually by applying a *perturbation scheme*.

Before considering the evaluation of "perturbed" predicates, let us first describe what happens during the call of a predicate. Assume we have to evaluate the outcome of a predicate $G(\mathbf{x})$ for some input $\mathbf{x}_0$. In most such predicates, as the ones described in Chapters 2 and 3, the outcome is decided by looking at signs of various *geometric subpredicates* for the same input. Each of these subpredicates can be viewed as the sign of a polynomial $P(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^{\mu}$ includes the coordinates of all input sites. The original algorithm that evaluates $G(\mathbf{x})$ assumes that all these signs are never 0,, i.e., $\mathbf{x}_0$ is a no-degenerate input.

Otherwise, a degenerate configuration $G(\mathbf{x}_0)$ will result in some $P(\mathbf{x}_0)$ being 0. We handle these cases by exchanging the input $\mathbf{x}_0$ with a proper function $\pi(\mathbf{x}_0, \epsilon)$; this is called a *symbolic perturbation*. The required properties of $\pi(\mathbf{x}_0, \epsilon)$ is that it has to be a continuous function of the parameter ϵ, $\pi(\mathbf{x}_0, 0) = \mathbf{x}_0$ and that $x = \pi(\mathbf{x}_0, \epsilon)$ is not a degenerate input for G for sufficiently small positive values of ϵ. Under these assumptions, we can define the predicate $G(\mathbf{x}_0)$ as the limit of $G(\mathbf{x}_0, \epsilon)$ when $\epsilon \to 0^+$, denoted as $G^{\epsilon}(\mathbf{x}_0)$.

In some works [19, 40, 21, 1, 16, 20], the practical evaluation of the perturbed predicate $G^{\epsilon}(\mathbf{x}_0)$, involves expressing the sign of $P(\pi(\mathbf{x}_0, \epsilon))$ as a polynomial of ϵ and the coordinates of $\mathbf{x}_0$. The evaluation of the perturbed predicate $G^{\epsilon}(\mathbf{x}_0)$ ultimately amounts to determining the sign of the non-vanishing limit $\lim_{\epsilon\to 0^+} P(\pi(\mathbf{x}_0, \epsilon))$, for all predicates P such that $P(\mathbf{x}_0) = 0$. To decide the sign of this complex limit, the polynomial $P(\pi(\mathbf{x}_0, \epsilon))$ is rewritten as a polynomial in ϵ and its monomials are ordered in terms of increasing degree. The first degree is actually $P(\mathbf{x}_0)$ while the rest are evaluated in increasing degrees in ϵ until a non-vanishing coefficient is found. The sign of this coefficient is then returned as the sign of the limit.

A different approach to practically evaluate $G^{\epsilon}(\mathbf{x}_0)$ is presented in [15] by Devillers, Karavelas and Teillaud. Their approach, called *Qualitative Symbolic Perturbation*, is also the one we adopt in the analysis of Chapter 4. In their paper, perturbed predicates are not resolved algebraicly by considering non-vanishing terms as before. Instead, the evaluation of the sign of $P(\pi(\mathbf{x}_0, \epsilon))$ when $\epsilon \to 0^+$ is considered in geometric terms. Such strategy can be followed as $P(\mathbf{x}_0)$ itself expresses a geometric property of the input configuration $\mathbf{x}_0$; we simply have to determine how this property changes as the $\mathbf{x}_0$ infinitesimally changes to $\pi(\mathbf{x}_0, \epsilon)$.

As presented in [15], to resolve a degenerate geometric predicate P with input the spheres S_n, for $n \in I$, we first index the sites using the so-called *max-weight ordering* that assigns a larger index to the site with larger weight. To break ties between sites with the same weights, we use the lexicographic ordering of their centers: among two sites with the same weight, the site whose center is smaller lexicographically than the other is assigned a smaller max-weight order. If the site site S_n has larger index than S_m, we will denote it by $n > m$.

After the input sites are ordered, the predicate P for a degenerate input is resolved as follows. Initially, we symbolically perturbe the site with the largest index i.e. and consider the outcome of P after the site is infinitesimally inflated, while the others remain can be considered as fixed. If the configuration still remains degenerate, we enlarge the second largest index and so on until the resulting configuration is non-degenerate, in which case the predicate is resolved. Moreover, if more than one site has to be perturbed, each subsequent site is enlarged by an amount that is smaller than the previous ones; the benefits of this approach are presented in the respective paper [15].

This thesis is organised as follows. In Chapters 2 and 3, we present an algorithm that decides the EdgeConflict predicate assuming a non-degenerate input and that the supporting trisector of the Voronoi edge is *hyperbolic* or *elliptic* respectively. In Chapter 4, a description of the way we resolve degeneracies for all subpredicates is provided. Finally, in Section 6, we conclude the thesis.

1.3 Basic Definitions

Let $\mathcal{S}$ be a set of closed spheres S_n (also referred as *sites*) in $\mathbb{E}^3$, with centers $C_n = (x_n, y_n, z_n)$ and radii r_n. Define the Euclidean distance $\delta(p, S)$ between a point $p \in \mathbb{E}^3$ and a sphere $S = \{C, r\}$ as $\delta(p, S) = \|p - C\| - r$, where $\|\cdot\|$ stands for the Euclidean norm. The *Apollonius diagram* is then defined as the subdivision of the space induced by assigning each point $p \in \mathbb{E}^3$ to its nearest neighbor with respect to the distance function $\delta(\cdot, \cdot)$.

For each $i \neq j$, let $H_{ij} = \{y \in \mathbb{E}^3 : \delta(y, S_i) \leq \delta(y, S_j)\}$. Then the (closed) *Apollonius cell* V_i of S_i is defined to be $V_i = \cap_{i \neq j} H_{ij}$. The set of points that belong to exactly two Apollonius cells are called the *Apollonius faces*, whereas the 1-dimensional connected intersections of face closures are called *Apollonius edges*. Points that correspond to intersections of Apollonius edges and therefore belong to more than three Apollonius cells are called *Apollonius vertices*; the *Apollonius diagram* $\mathcal{VD}(\mathcal{S})$ of $\mathcal{S}$ is defined as the collection of the Apollonius cells, faces, edges and vertices.

An Apollonius vertex v is a point that is equidistant to 4 or more sites. If we denote 4 of these sites by S_i, S_j, S_k and S_l and let r be the common distance of v from each of them, then the sphere centered at v with radius r must be externally tagent to all four spheres. This cotagent sphere is called an *external Apollonius sphere* of the sites S_i, S_j, S_k and S_l. If T_n denotes the point of tangency of the Apollonius sphere and S_n, for $n \in \{i, j, k, l\}$, then the tetrahedron $T_iT_jT_kT_l$ can be either positively or negatively oriented, or even flat [15]. If $T_iT_jT_kT_l$ is positively oriented then the Apollonius vertex v will be denoted as v_{ijkl} whereas if $T_iT_jT_kT_l$ is negatively oriented it will be denoted as v_{ikjl}. Observe that a cyclic permutation of the indices does not alter the Apollonius vertex choice.

The *trisector* τ_{ijk} of three different sites S_i, S_j and S_k is the locus of points that are equidistant from the three sites. In the absense of degeneracies its Hausdorff dimension is 1, and it is either (a branch of) a hyperbola, a line, an ellipse, a circle, or a parabola [42]. We shall say that the trisector is

- *hyperbolic*, if it is a branch of a hyperbola or a line,
- *elliptic*, if it is an ellipse or a circle, and
- *parabolic*, if it is a parabola.

In this thesis, we mainly focus on the cases of hyperbolic and elliptic trisectors. The parabolic trisector can be viewed as a degenerate trisector as a symbolic perturbation of the sites S_i, S_j and S_k would result in τ_{ijk} becoming hyperbolic or elliptic. Therefore, analysis of all predicates on parabolic trisectors is reduced to the respective predicates for either hyperbolic or elliptic trisectors.

A finite Apollonius edge of a 3D Apollonius diagram is denoted by e_{ijklm} if the edge lies on the trisector τ_{ijk} of the sites S_i, S_j and S_k different sites it's endpoints are v_{ijkl} and v_{ikjm}.

1.4 Inversion

The 3-dimensional *inversion transformation* is a mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$ that maps a point $z \in \mathbb{R}^3$ to the point $W(z) = (z - z_0)/\|z - z_0\|^2$. The point z_0 is called the *pole of inversion*. Inversion maps spheres that do not pass through the pole to spheres, spheres that pass through the pole to planes and planes that pass through the pole to planes.

In the Apollonius diagram context we call $\mathcal{Z}$-space the space where the sites live. Since the Apollonius diagram does not change when we add to the radii of all spheres the same quantity, we will, most of the times, reduce the radii of the spheres S_i, S_j, S_k, S_l, S_m and S_q by the radius of a sphere S_I, where $I \in \{i, j, k, l, m, q\}$. The new spheres have obviously the same centers, whereas their radii become $r_n^\star = r_n - r_I$, $n \in \{i, j, k, l, m, q\}$. For convience, we call the image space of this radius-reducing transformation the $\mathcal{Z}^\star$-space. We may then apply inversion, with C_I as the pole, to get a new set of spheres or planes; we call $\mathcal{W}$-space the space where the radius-reduced, inverted sites live.

Note that we can safely assume that none of the sites $S_i, S_j, S_k, S_l, S_m, S_q$ are contained inside another. Indeed, in a Voronoi diagram $\mathcal{VD}(\mathcal{S})$, if $S_a, S_b \in \mathcal{S}$ and $S_a \subset S_b$ then S_a has an empty Voronoi cell, therefore the deletion of S_a from $\mathcal{S}$ does not alter the Voronoi diagram.

The non-inclusion assumption also implies that the image of the sphere S_n in $\mathcal{W}$-space, for $n \in \{i, j, k, l, m, n, q\}\backslash\{I\}$, is a sphere $S_n^\star$, centered at $C_n^\star = (u_n, v_n, w_n)$ with radius ρ_n, where

$$u_n = \frac{x_n^\star}{p_n^\star}, \qquad v_n = \frac{y_n^\star}{p_n^\star}, \qquad w_n = \frac{z_n^\star}{p_n^\star}, \qquad \rho_n = \frac{r_n^\star}{p_n^\star}, \tag{1.1}$$

$$x_n^\star = x_n - x_I, \quad y_n^\star = y_n - y_I, \quad z_n^\star = z_n - z_I, \quad r_n^\star = r_n - r_I, \tag{1.2}$$

and $p_n^\star = (x_n^\star)^2 + (y_n^\star)^2 + (z_n^\star)^2 - (r_n^\star)^2 > 0$.

Proof. Let's start with the equation of the sphere S_n in $\mathcal{Z}$-space:

$$(x - x_n)^2 + (y - y_n)^2 + (z - z_n)^2 = r_n^2 \tag{1.3}$$

The image of S_n in $\mathcal{Z}^\star$-space is the sphere S_n after its radius is reduced by r_I and therefore it's equation is:

$$(x - x_n)^2 + (y - y_n)^2 + (z - z_n)^2 = (r_n - r_I)^2 = (r_n^*)^2. \tag{1.4}$$

This can be rewritten as

$$(x - x_I - x_n^*)^2 + (y - y_I - y_n^*)^2 + (z - z_I - z_n^*)^2 = (r_n^*)^2. \tag{1.5}$$

Now, we apply the inversion transformation using the point (x_I, y_I, z_I) as the inversion pole and therefore a point (x, y, z) on S_n is mapped to (u, v, w) where

$$u = (x - x_I)/D,\ \ v = (y - y_I)/D,\ \ w = (z - z_I)/D,\ \text{ and } D = (x - x_I)^2 + (y - y_I)^2 + (z - z_I)^2. \tag{1.6}$$

Let $E := u^2 + v^2 + w^2 = 1/D$, it holds that

$$x = uD + x_I,\ \ y = vD + y_I,\ \text{ and } z = wD + z_I. \tag{1.7}$$

Substituting in (1.5), we get

$$(uD - x_n^*)^2 + (vD - y_n^*)^2 + (wD - z_n^*)^2 = (r_n^*)^2. \tag{1.8}$$

Multiplying by $E^2 = 1/D^2$ we get

$$(u - Ex_n^*)^2 + (v - Ey_n^*)^2 + (w - Ez_n^*)^2 = (Er_n^*)^2. \tag{1.9}$$

Now we expand the squares and regroup terms:

$$u^2 + v^2 + w^2 - 2uEx_n^* - 2vEy_n^* - 2wEz_n^* + E^2[(x_n^*)^2 + (y_n^*)^2 + (z_n^*)^2] = E^2(r_n^*)^2. \tag{1.10}$$

which can then be rewritten as (we use that $u^2 + v^2 + w^2 = E$)

$$E - 2uEx_n^* - 2vEy_n^* - 2wEz_n^* + E^2[(x_n^*)^2 + (y_n^*)^2 + (z_n^*)^2] = E^2(r_n^*)^2. \tag{1.11}$$

. Since E is not zero, we get the equation

$$1 - 2ux_n^* - 2vy_n^* - 2wz_n^* + E[(x_n^*)^2 + (y_n^*)^2 + (z_n^*)^2 - (r_n^*)^2] = 0 \tag{1.12}$$

or equivalently

$$1 - 2ux_n^* - 2vy_n^* - 2wz_n^* + Ep_n^* = 0. \tag{1.13}$$

Since p_n^* is strictrly positive due to the fact that S_I is not contained in S_n, we divide by p_n^* and substitute $E = u^2 + v^2 + w^2$:

$$u^2 - 2ux_n^*/p_n^* + v^2 - 2vy_n^*/p_n^* + w^2 - 2wz_n^*/p_n^* = -1/p_n^*. \tag{1.14}$$

After completing the squares, we get

$$(u - x_n^*/p_n^*)^2 + (v - y_n^*/p_n^*)^2 + (w - z_n^*/p_n^*)^2 = (x_n^*/p_n^*)^2 + (y_n^*/p_n^*)^2 + (z_n^*/p_n^*)^2 - 1/p_n^*. \tag{1.15}$$

The right part of the equation can be rewritten as

$$(x_n^*/p_n^*)^2 + (y_n^*/p_n^*)^2 + (z_n^*/p_n^*)^2 - 1/p_n^* = \frac{(x_n^*)^2 + (y_n^*)^2 + (z_n^*)^2 - p_n^*}{(p_n^*)^2} = \frac{(r_n^*)^2}{(p_n^*)^2}. \tag{1.16}$$

Therefore, (1.15) is written as

$$(u - u_n)^2 + (v - v_n)^2 + (z - z_n)^2 = \rho_n^2, \tag{1.17}$$

which is the equation of $S_n^\star$ in $\mathcal{W}$-space. □

We also define the quantities

$$D_{\lambda\mu\nu}^{\pi\theta} = \begin{vmatrix} \pi_\lambda & \theta_\lambda & 1 \\ \pi_\mu & \theta_\mu & 1 \\ \pi_\nu & \theta_\nu & 1 \end{vmatrix}, \quad D_{\lambda\mu\nu}^{\pi\theta\eta} = \begin{vmatrix} \pi_\lambda & \theta_\lambda & \eta_\lambda \\ \pi_\mu & \theta_\mu & \eta_\mu \\ \pi_\nu & \theta_\nu & \eta_\nu \end{vmatrix}, \quad D_{\lambda\mu\nu\xi}^{\pi\theta\eta} = \begin{vmatrix} \pi_\lambda & \theta_\lambda & \eta_\lambda & 1 \\ \pi_\mu & \theta_\mu & \eta_\mu & 1 \\ \pi_\nu & \theta_\nu & \eta_\nu & 1 \\ \pi_\xi & \theta_\xi & \eta_\xi & 1 \end{vmatrix},$$
$$D_{\lambda\mu\nu\xi}^{\pi\theta\eta\zeta} = \begin{vmatrix} \pi_\lambda & \theta_\lambda & \eta_\lambda & \zeta_\lambda \\ \pi_\mu & \theta_\mu & \eta_\mu & \zeta_\mu \\ \pi_\nu & \theta_\nu & \eta_\nu & \zeta_\nu \\ \pi_\xi & \theta_\xi & \eta_\xi & \zeta_\xi \end{vmatrix}, \tag{1.18}$$

and

$$E_{\lambda\mu\nu}^{\alpha\beta\gamma} = \begin{vmatrix} \alpha_\lambda^\star & \beta_\lambda^\star & \gamma_\lambda^\star \\ \alpha_\mu^\star & \beta_\mu^\star & \gamma_\mu^\star \\ \alpha_\nu^\star & \beta_\nu^\star & \gamma_\nu^\star \end{vmatrix}, \quad E_{\lambda\mu\nu\xi}^{\alpha\beta\gamma\delta} = \begin{vmatrix} \alpha_\lambda^\star & \beta_\lambda^\star & \gamma_\lambda^\star & \delta_\lambda^\star \\ \alpha_\mu^\star & \beta_\mu^\star & \gamma_\mu^\star & \delta_\mu^\star \\ \alpha_\nu^\star & \beta_\nu^\star & \gamma_\nu^\star & \delta_\nu^\star \\ \alpha_\xi^\star & \beta_\xi^\star & \gamma_\xi^\star & \delta_\xi^\star \end{vmatrix}, \tag{1.19}$$

for $\pi, \theta, \eta, \zeta \in \{x, y, z, r, u, v, w, \rho\}$, $\alpha, \beta, \gamma, \delta \in \{x, y, z, r\}$ and $\lambda, \mu, \nu, \xi \in \{i, j, k, l\}$.

If it holds for all $\tau \in \{\lambda, \mu, \nu\}$ that

$$\pi_\tau = \frac{\alpha_\tau^\star}{p_\tau^\star}, \quad \theta_\tau = \frac{\beta_\tau^\star}{p_\tau^\star}, \quad \eta_\tau = \frac{\gamma_\tau^\star}{p_\tau^\star}, \tag{1.20}$$

for some $\pi, \theta, \eta \in \{u, v, w, \rho\}$ and some $\alpha, \beta, \gamma \in \{x, y, z, r\}$, then it also holds that

$$D_{\lambda\mu\nu}^{\pi\theta} = \frac{1}{p_\lambda^\star p_\mu^\star p_\nu^\star} E_{\lambda\mu\nu}^{\alpha\beta p}, \quad D_{\lambda\mu\nu}^{\pi\theta\eta} = \frac{1}{p_\lambda^\star p_\mu^\star p_\nu^\star} E_{\lambda\mu\nu}^{\alpha\beta\gamma}. \tag{1.21}$$

1.5 Known tools

Let $\overrightarrow{v} = (v_1, v_2, v_3)$ and $\overrightarrow{w} = (w_1, w_2, w_3)$ be two vectors of $\mathbb{R}^3$. The inner product of these vectors is the quantity $\overrightarrow{v} \cdot \overrightarrow{w} := v_1 w_1 + v_2 w_2 + v_3 w_3$. It is known that the inner product of two vectors equals zero if and only if either vector is the zero vector or the vectors are perpendicular.

The cross product of the vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ is the vector $\overrightarrow{v} \times \overrightarrow{w} := (v_2 w_3 - v_3 w_2, -v_1 w_3 + v_3 w_1, v_1 w_2 - v_1 w_2)$. It is known that the cross product of two vectors is the zero vector if and only if either vector is the zero vector or the vectors are parallel.

Given four points K, L, M, N of $\mathbb{R}^3$, the outcome of the $\textsc{Orient3D}(K, L, M, N)$ predicate corresponds to the orientation of the tetrahedron $KLMN$. Specifically, if the predicate returns "+" or "-", then the tetrahedron $KLMN$ is positively or negatively oriented, respectively. Otherwise, if the predicate returns "0", the tetrahedron is flat, i.e., the four points are coplanar. Note that the $\textsc{Orient3D}$ predicate can be easily evaluated geometrically via the right-hand rule; if we orient our right hand with fingers curled to follow the circular sequence L, M, N then the predicate returns "+" (resp., "-") iff our thumb points towards (resp., away from) K. Algebraicly, the $\textsc{Orient3D}$ predicate can be evaluated as

$$\textsc{Orient3D}(K, L, M, N) = \mathsf{sign}\left(\begin{vmatrix} x_K & y_K & z_K & 1 \\ x_L & y_L & z_L & 1 \\ x_M & y_M & z_M & 1 \\ x_N & y_N & z_N & 1 \end{vmatrix}\right) = \mathsf{sign}(D^{xyz}_{KLMN}) \tag{1.22}$$

where $P = (x_P, y_P, z_P)$, for $P \in \{K, L, M, N\}$.

Chapter 2

Non-Degenerate Case Analysis for Hyperbolic Trisectors

In this chapter, we introduce the subpredicates and the algorithm that are used to answer the EdgeConflict predicate under the assumption that the trisector of the sites S_i, S_j and S_k is a branch of a hyperbola or a straight line and that no degeneracies arise. The section is organized as follows. In Section 2.1 we provide a way of orienting a *hyperbolic* trisector and in Section 2.2 we make useful remarks regarding the Voronoi edges that lie on such trisectors. In Section 2.3, we provide the list of possible outcomes for the EdgeConflict predicate as well as the assumptions that are considered to hold throughout this Chapter. The description of several useful subpredicates takes place in Sections 2.4, whereas in Section 2.5, we provide the algorithm that combines them and decides the EdgeConflict predicate. Lastly, a detailed analysis for each of the subpredicates can be found in Section 2.7 .

2.1 Orientation of a hyperbolic or linear trisector

Under the assumption that the trisector τ_{ijk} of the sites S_i, S_j, S_k is a line or a hyperbola, the three centers C_i, C_j, C_k cannot be collinear [42]. A natural way of orienting τ_{ijk} is accomplished via the well-known "right-hand rule"; if we fold our right hand to follow the centers C_i, C_j and C_k (in that order), our thumb will be showing the positive "end" of τ_{ijk} (see Figure 2.1).

By orienting τ_{ijk}, we clearly define an ordering on the points of τ_{ijk}, which we denote by $\prec$. Let o_{ijk} be the intersection of τ_{ijk} and the plane Π_{ijk} going through the centers C_i, C_j and C_k. We can now parametrize τ_{ijk} as follows: if $o_{ijk} \prec p$ then $\zeta(p) = \delta(p, S_i) - \delta(o_{ijk}, S_i)$; otherwise $\zeta(p) = -(\delta(p, S_i) - \delta(o_{ijk}, S_i))$. The function $\zeta(\cdot)$ is a 1-1 and onto mapping from τ_{ijk} to $\mathbb{R}$. Moreover, we define $\zeta(S)$, where S is an external tangent sphere to the sites S_i, S_j and S_k, to be $\zeta(c)$, where $c \in \tau_{ijk}$ is the center of S.

We also use τ^+_{ijk} (resp., τ^-_{ijk}) to denote the positive (resp., negative) semi-trisector, i.e., the set of points $p \in \tau_{ijk}$ such that $o_{ijk} \prec p$ (resp., $p \prec o_{ijk}$).

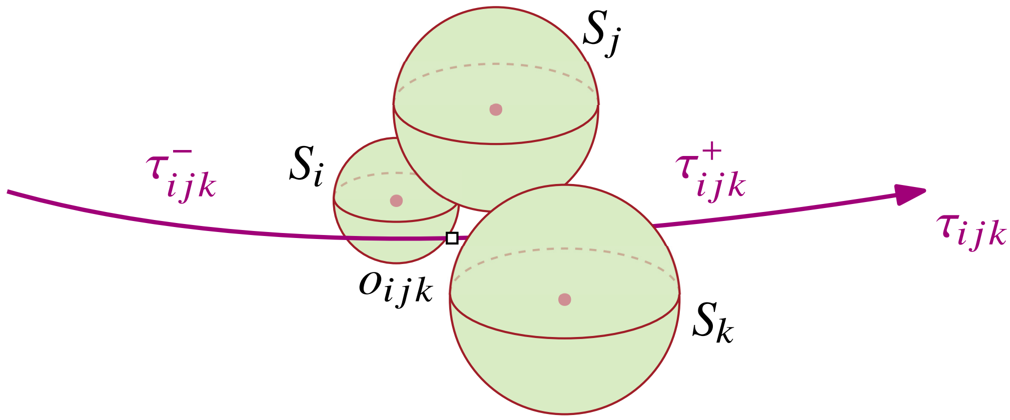

FIGURE 2.1: The case where the trisector τ_{ijk} of the spheres S_i, S_j and S_k is *hyperbolic*. Notice the orientation of τ_{ijk} based on the "right-hand rule". The point o_{ijk} of the trisector, which is coplanar with the centers C_i, C_j and C_k, separates the τ_{ijk} into two semi-trisectors, τ^-_{ijk} and τ^+_{ijk}.

2.2 Voronoi Edges on Hyperbolic Trisectors

In order to better understand our initial problem, more insight regarding the properties of a Voronoi diagram is required. Let us look closer at an edge e_{ijklm} (we drop the subscript for convenience) of $\mathcal{V}(\mathcal{S})$, where $\mathcal{S}$ is a set of given sites that includes S_n, for $n \in \{i, j, k, l, m\}$, and does not include S_q. This edge e lies on the trisector τ_{ijk}, the locus of points that are equidistant to the sites S_i, S_j and S_k. In the scope of this chapter, we assume that τ_{ijk} is of Hausdorff dimension 1 and is *hyperbolic*, i.e., either (a branch of) a hyperbola or a line. To ensure that the spheres S_i, S_j and S_k meet this criteria, the predicate TRISECTORTYPE (S_i, S_j, S_k), described in Section 2.4.2 and analyzed in Section 2.7.2, must return "hyperbolic".

We now focus on the edge e, i.e. the open continuous subset of τ_{ijk} whose closure is bounded by the Voronoi vertices v_{ijkl} and v_{ikjm}. If $\mathcal{T}(t)$ denotes the external Apollonius sphere of the sites S_i, S_j and S_k that is centered at t, the most crucial property of a point $t \in e$ is that $\mathcal{T}(t)$ does not intersect with any other site of $\mathcal{S}$. We call this the *Empty Sphere Principle* since it is a property that derives from the empty circle principle of a generic Voronoi diagram and its basic properties.

Notice that the original definition of the edge e included the fact that it was bounded by the Apollonius vertices v_{ijkl} and v_{ikjm}. Using the Empty Sphere Principle, we can show that the former and latter vertice correspond to the left and right endpoint of the edge, respectively (see Figure 2.2) ; this is equivalent to showing that $v_{ijkl} \prec v_{ijkm}$ on the oriented trisector.

We will now prove that the left endpoint is indeed v_{ijkl} and not v_{ikjm}. Assuming that v_{ikjm} is the left endpoint, we consider a point $t \in \tau_{ijk}$ such that initially $t \equiv v_{ikjm}$ and

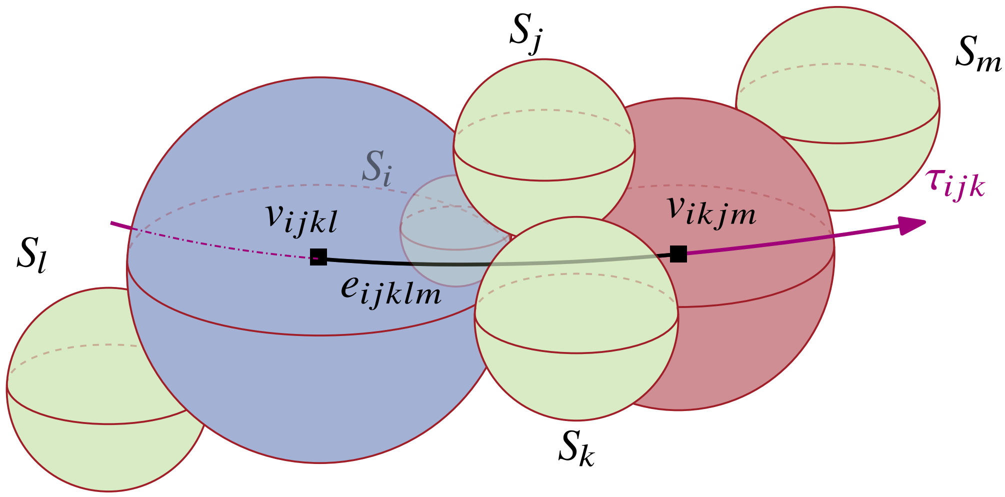

FIGURE 2.2: An edge e_{ijklm} that lies in a hyperbolic trisector τ_{ijk}. It's left and right endpoints are of the Apollonius vertices v_{ijkl} and v_{ikjm}, respectively. The blue and red sphere are the Apollonius spheres centered at v_{ijkl} and v_{ikjm}, respectively.

then move it infinitesimally on the trisector towards its positive direction. The Apollonius sphere $\mathcal{T}(t)$ was initially $\mathcal{T}(v_{ikjm})$ and therefore tangent to S_m. Since v_{ikjm} was assumed to be the left endpoint of e, t lies on e after being moved and therefore the sphere $\mathcal{T}(t)$ should not longer be tangent nor intersect S_m due to the Empty Sphere Principle. However, the tetrahedron $T_iT_jT_kT_m$, where T_n is the tangency point of the spheres $\mathcal{T}(v_{ikjm})$ and S_n, for $n\{i,j,k,l\}$, is negatively oriented by definition of the Apollonius vertex v_{ikjm} (see Section 1.3). Due to the tetrahedron's orientation and the movement of t towards the positive direction of τ_{ijk}, it must hold that the sphere $\mathcal{T}(t)$ contains T_m and therefore intersects S_m, yielding a contradiction. Consequently, we have proven that the left point of e is necessarily v_{ijkl} and the right endpoint is v_{ikjm}.

2.3 Problem Outline and Assertions

For clarity reasons, we restate the EDGECONFLICT predicate, highlighting its input, output as well as the assertions we are making for the rest of this paper.

The EDGECONFLICT predicate, one of the fundamental predicates required for the construction of the 3D Apollonius diagram (also known as the 3D Additively Weighted Voronoi diagram), takes as input five sites S_i, S_j, S_k, S_l and S_m that define an edge e_{ijklm} in the 3D Apollonius diagram as well as a sixth query site S_q. The predicate determines the portion of e_{ijklm} (we drop the subscripts for convenience) that will disappear in the Apollonius diagram of the six sites due to the insertion of S_q and therefore its output is one of the following

- NoConflict: no portion of e is destroyed by the insertion of S_q in the Apollonius diagram of the five sites.
- EntireEdge: the entire edge e is destroyed by the addition of S_q in the Apollonius diagram of the five sites.
- LeftVertex: a subsegment of e adjacent to its origin vertex (v_{ijkl}) disappears in the Apollonius diagram of the six sites.
- RightVertex: is the symmetric case of the LeftVertex case; a subsegment of e adjacent to the vertex v_{ikjm} disappears in the Apollonius diagram of the six sites.
- BothVertices: subsegments of e adjacent to its two vertices disappear in the Apollonius diagram of the five sites.
- Interior: a subsegment in the interior of e disappears in the Apollonius diagram of the five sites.

In Section 2.5, we prove that these are indeed the only possible answers to the studied predicate, under the assumption that no degeneracies occur. Specifically, all analysis presented in this section is done under the following two major assumptions:

- The trisector τ_{ijk} of the sites S_i, S_j and S_k is "hyperbolic", i.e., it is either a branch of a hyperbola or a straight line. Therefore, the spheres must lie in *convex position*; in other words, there must exist two distinct planes commonly tangent to all three spheres.
- None of the subpredicates called during the algorithm presented in Section 2.5 returns a degenerate answer. Mainly, this is equivalent to the statement: *All of the existing Apollonius vertices defined by the sites S_i, S_j, S_k and S_n, for $n \in \{l, m, q\}$, are distinct and the respective Apollonius spheres are all finite, i.e., they are not centered at infinity*. Such assertion dictates that the edge e is finite as none of its bounding vertices v_{ijkl} and v_{ikjm} can lie at infinity.

2.4 SubPredicates and Primitives

In this section, we describe the various subpredicates used throughout the evaluation of the EdgeConflict predicate via the main algorithm presented in Section 2.5. For convenience, only the input, output and specific geometric observations is provided in this section, whereas a detailed analysis along with an algebraic degree analysis of each subpredicate is found in Section 2.7.

2.4.1 The InSphere predicate

The InSphere(S_i, S_j, S_k, S_a, S_b) predicate returns $-, +$ or 0 if and only if the sphere S_b intersects, does not intersect or is tangent to the external Apollonius sphere of the sites S_i, S_j, S_k and S_b, centered at v_{ijka}. It is assumed that v_{ijka} exists and none of the first four inputed sites are contained inside one another. In [28], it is shown that the evaluation of the InSphere predicate requires operations of maximum algebraic degree 10, whereas in [3] an implicit InSphere predicate could be evaluated via the Delaunay graph, using 6-fold degree operations (although it is not clear if we could easily distinguish if we are testing against the Apollonius sphere centered at v_{ijka} or v_{ikja}).

Since degenerate configurations are beyond the scope of this chapter, the InSphere tests evaluated during the main algorithm (see Section 2.5) will always return + or −. We should also remark that, in bibliography, the InSphere predicate is also referred to as the VertexConflict predicate to reflect the fact that a negative (resp., positive) outcome of InSphere(S_i, S_j, S_k, S_a, S_b) amounts to the Apollonius vertex v_{ijka} in $\mathcal{VD}(\Sigma)$ vanishing (resp., remaining) in $\mathcal{VD}(\Sigma \cup \{S_b\})$, where Σ contains S_i, S_j, S_k and S_a but not S_b.

Lemma 1 *The* InSphere *predicate can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

2.4.2 The InCone and Trisector predicates

Given three spheres S_a, S_b and S_c, such that S_a and S_b are not contained one inside the other, we want to determine the relative geometric position of S_c with respect to the uniquely defined closed semi-cone $\mathcal{K}(S_a, S_b)$ that is tangent to both S_a and S_b and includes their centers (see Figure 2.3). We shall call this the InCone($S_a, S_b; S_c$) predicate.

In case the radii of S_a and S_b are equal, $\mathcal{K}(S_a, S_b)$ (we drop the parenthesis for convenience) degenerates into a cylinder without this having an impact to the predicate. If S_c° is used to denote the open sphere that corresponds to S_c, then all possible answers of the predicate InCone($S_a, S_b; S_c$) are

- Outside , if at least one point of S_c is outside $\mathcal{K}$,
- Inside, if S_c° lies inside $\mathcal{K}$ and $S_c \cap \vartheta\mathcal{K} = \emptyset$,
- OnePointTouch, if S_c° lies inside $\mathcal{K}$ and $S_c \cap \vartheta\mathcal{K}$ is a point,
- CircleTouch, if S_c° lies inside $\mathcal{K}$ and $S_c \cap \vartheta\mathcal{K}$ is a circle.

The last two answers are considered "degenerate" and therefore, we may consider that whenever InCone is called during the algorithm presented in Section 2.5, it will either return Outside or Inside.

This predicate is basic tool used in various other sub-predicates such as the TrisectorType, which returns the trisector type of a set of three spheres. It is known ([42])

that if the trisector τ_{abc} of S_a, S_b, S_c has Hausdorff dimension 1, it can either be a branch of a"hyperbola", a "line", an "ellipse", a "circle" or a "parabola"; these are the possible answers of the TrisectorType(S_a, S_b, S_c) predicate. However, since the "line" and the "circle" type are sub-case of the "hyperbolic" and "elliptic" trisector types respectively, we can characterize a trisector as either "hyperbolic", "elliptic" or "parabolic".

During the execution of the main algorithm of Section 2.5, the TrisectorType(S_i, S_j, S_k) has to be evaluated. Being able to distinguish the type of the trisector τ_{ijk} is essential since all the analysis presented in this section assumes that τ_{ijk} is hyperbolic.

The analysis followed to determine the outcome of the InCone or the TrisectorType predicate can be found in Sections 2.7.1 and 2.7.2 respectively, where the following lemma is proved.

Lemma 2 *The* InCone *and* TrisectorType *predicates can be evaluated by determining the sign of quantities of algebraic degree at most 4 (in the input quantities).*

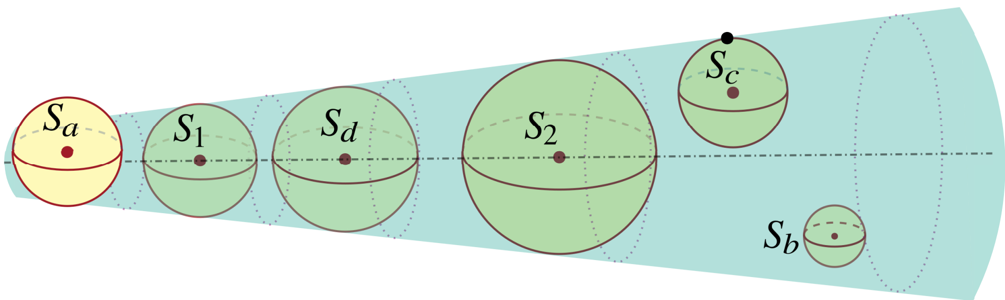

Figure 2.3: Some of the possible locations of a sphere S_n against the cone $\mathcal{K}$ defined by S_1 and S_2. The InCone($S_1, S_2; S_n$) returns Outside, Inside, OnePointTouch and CircleTouch for $n = a, b, c$ and d respectively.

2.4.3 The Distance predicate

When the trisector τ_{ijk} is a hyperbola or a line, there exist two distinct planes, denoted by Π^-_{ijk} and Π^+_{ijk}, such that each one is commonly tangent to the sites S_i, S_j, S_k and leave their centers on the same halfspace.

Observe that Π^-_{ijk} and Π^+_{ijk} correspond to the two Apollonius spheres *at infinity*, in the sense that they are centered at *infinity* and are cotangent to the spheres S_i, S_j and S_k. These planes are considered as oriented, and subdivide $\mathbb{R}^3$ into a positive and a negative halfspace, the positive being the halfspace containing the centers of the spheres.

If we consider a point p that moves on the trisector τ_{ijk} such that $\zeta(p)$ goes to $-\infty$ or $+\infty$, the sphere $\mathcal{T}(p)$ becomes the corresponding Apollonius sphere at infinity, i.e., the plane Π^-_{ijk} or Π^+_{ijk} (see Figure 2.4).

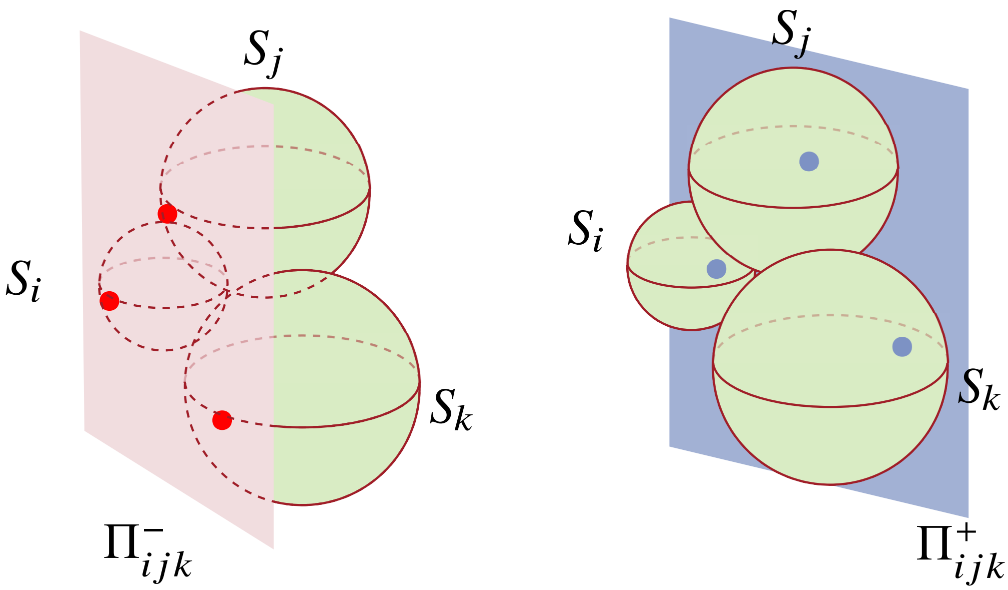

FIGURE 2.4: If the spheres S_i, S_j and S_k lie in a convex position there exist two distinct planes, Π^-_{ijk} and Π^+_{ijk}, cotangent to all spheres. These planes are considered as the Apollonius sphere of the sites S_i, S_j and S_k, centered at $p \in \tau_{ijk}$, as $\zeta(p)$ goes to $\pm\infty$ respectively.

Given the sites S_i, S_j, S_k and S_α, the DISTANCE $(S_i, S_j, S_k, S_\alpha)$ predicate determines whether S_α intersects, is tangent to, or does not intersect the (closed) negative halfspaces delimited by the two planes Π^-_{ijk} and Π^+_{ijk}. The "tangency" case is considered as degenerate and is beyond the scope of this paper. This predicate is used in the evaluation of the SHADOW predicate, and is equal to DISTANCE$(S_i, S_j, S_k, S_\alpha) = \left(\mathsf{sign}(\delta(S_\alpha, \Pi^-_{ijk})), \mathsf{sign}(\delta(S_\alpha, \Pi^+_{ijk}))\right)$, where $\delta(S, \Pi) = \delta(C, \Pi) - r$, and $\delta(C, \Pi)$ denotes the signed Euclidean of C from the plane Π and S is a sphere of radius r, centered at C. As for the EXISTENCE predicate, we reduce it to the computation of the signs of the two roots of a quadratic equation and prove the following lemma (see Section 2.7.4 for this analysis).

Lemma 3 *The* DISTANCE *predicate can be evaluated by determining the sign of quantities of algebraic degree at most 6 (in the input quantities).*

2.4.4 The EXISTENCE predicate

The next primitive operation we need for answering the EDGECONFLICT predicate is what we call the EXISTENCE predicate: given four sites S_a, S_b, S_c and S_n, we would like to determine the number of Apollonius spheres of the quadruple S_a, S_b, S_c, S_n. In general, given four sites there can be "0", "1", "2" or "infinite" Apollonius spheres (cf. [15]) including the Apollonius sphere(s) at infinity. The EXISTENCE(S_a, S_b, S_c, S_n) predicate only counts the Apollonius spheres that are *not* centered at infinity and since degenerate configurations of the input sites are beyond the scope of this paper, it is safe to assume that the outcome will always be "0","1" or "2". It is also clear that in case of a "1" outcome, the corresponding Apollonius center will either be v_{abcn} or v_{acbn} but not both; the case where $v_{abcn} \equiv v_{acbn}$ is ruled out by our initial no-degeneracies assumption.

The analysis of the EXISTENCE predicate can be found in Section 2.7.3 where we prove the following lemma.

Lemma 4 *The* EXISTENCE *predicate can be evaluated by determining the sign of quantities of algebraic degree at most 8 (in the input quantities).*

2.4.5 The SHADOW predicate

We now come to the second major subpredicate used by the EDGECONFLICT predicate: the SHADOW predicate.

Given three sites S_i, S_j and S_k, we define the *shadow region* $\mathcal{SR}(S_\alpha)$ of a site S_α, with respect to the trisector τ_{ijk}, to be the locus of points p on τ_{ijk} such that $\delta(C_\alpha, \mathcal{T}(p)) < r_\alpha$. The shadow region of the sites S_l, S_m and S_q play an important role when answering EDGECONFLICT$(S_i, S_j, S_k, S_l, S_m, S_q)$ (see Figure 2.5 and Section 2.5).

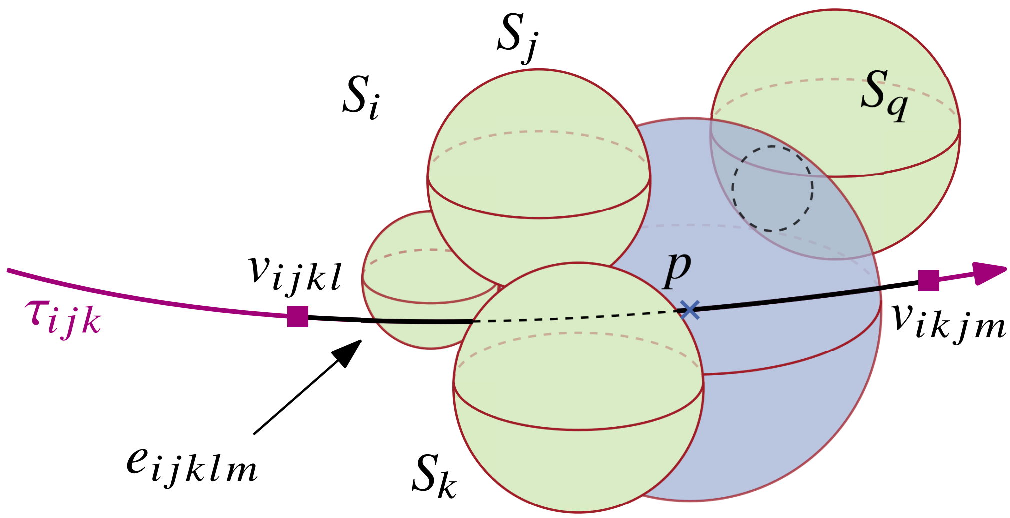

FIGURE 2.5: A finite edge e_{ijklm} of the Voronoi diagram $\mathcal{VD}(\mathcal{S})$ of the set $\mathcal{S} = \{S_n : n = i, j, k, l, m\}$ is the locus of points $p \in \tau_{ijk}$ such that $v_{ijkl} \prec p \prec v_{ikjm}$. Any sphere $\mathcal{T}(p)$, centered at p and cotangent to S_i, S_j and S_k does not intersect any sphere of $\mathcal{S}$ (*Empty Sphere Principle*). However, after inserting S_q in the existing Voronoi diagram, $\mathcal{T}(p)$ may intersect it. All points $p \in \mathcal{SR}(S_q)$ will no longer exist on in $\mathcal{VD}(\mathcal{S} \cup \{S_q\})$ as points on Voronoi edges.

The SHADOW$(S_i, S_j, S_k, S_\alpha)$ predicate returns the type of $\mathcal{SR}(S_\alpha)$ seen as an interval, or union of intervals, in $\mathbb{R}$. More precisely, the SHADOW predicate returns the topological structure of the set $\zeta(\mathcal{SR}(S_\alpha)) = \zeta(\{p \in \tau_{ijk} \mid \delta(C_\alpha, \mathcal{T}(p)) < r_\alpha\})$, which we denote by $SRT(S_\alpha)$.

Clearly, the boundary points of the closure of Shadow(S_i, S_j, S_k, S_α) are the points p on τ_{ijk} for which $\delta(C_\alpha, \mathcal{T}(p)) = r_\alpha$. These points are nothing but the centers of the Apollonius spheres of the four sites S_i, S_j, S_k and S_α, and, as such, there can only be 0, 1 or 2 (assuming no degeneracies). Since τ_{ijk} is assumed to be hyperbolic, this immediately suggests that $SRT(S_\alpha)$ can have one of the following 6 types: $\emptyset$, $(-\infty, \infty) = \mathbb{R}$, $(-\infty, \phi)$, $(\chi, +\infty)$, (χ, ϕ), or $(-\infty, \phi) \cup (\chi, +\infty)$, where $\phi, \chi \neq \pm\infty$.

For convenience, we will use the $\mathcal{SR}(S_a)$ notation instead of $SRT(S_\alpha)$; for example, the statement "$\mathcal{SR}(S_a) = (-\infty, \phi)$" will be often used instead of "$\mathcal{SR}(S_a)$'s type is $(-\infty, \phi)$" or "$SRT(S_\alpha) = (-\infty, \phi)$" (see Figure 2.6 for an example). This notation change further highlights the fact that we are only interested in the topological structure of $\mathcal{SR}(S_a)$ rather than the actual set itself.

In Section 2.7.5, we prove that the evaluation of the Shadow predicate only requires the call of the respective Distance and Existence predicate, yielding the following lemma.

Lemma 5 *The* Shadow *predicate can be evaluated by determining the sign of quantities of algebraic degree at most 8 (in the input quantities).*

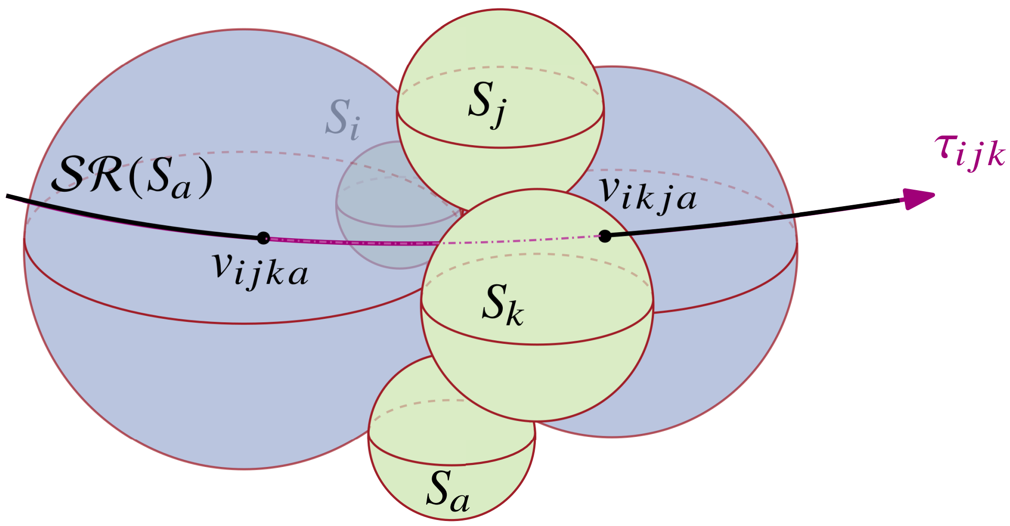

Figure 2.6: Since there are two Apollonius spheres of the sites S_n, for $n \in \{i, j, k, a\}$, centered at v_{ijka} and v_{ikja}, the $\mathcal{SR}(S_a)$ on τ_{ijk} must have two endpoints. In this specific configuration notice that, for every point p on the segments of τ_{ijk} painted black, the sphere $\mathcal{T}(p)$ will intersect S_a. Therefore, the black segments are indeed the shadow region $\mathcal{SR}(S_a)$ of the sphere S_a on the trisector τ_{ijk}.

2.4.6 The Order predicate

The most important sub-predicate used to evaluate the EdgeConflict predicate is what we call the Order predicate. When Order(S_i, S_j, S_k, S_a, S_b) is called, it returns the order of appearance of any of the existing Apollonius vertices $v_{ijka}, v_{ikja}, v_{ijkb}$ and v_{ikjb} on the oriented trisector τ_{ijk}.

This sub-predicate is called during the main algorithm that answers the EdgeConflict($S_i, S_j, S_k, S_l, S_m, S_q$), for $(a, b) \in \{(l, q), (m, q)\}$, only in the case that either v_{ijkq}, v_{ikjq} or both exist. Let us also recall that, in this chapter, the trisector τ_{ijk} is "hyperbolic" and that e_{ijklm} is a valid finite Apollonius edge; the Apollonius vertices v_{ijkl} and v_{ikjm} both exist on (the oriented) τ_{ijk} and $v_{ijkl} \prec v_{ikjm}$.

In order to answer the Order predicate, we first call the Shadow(S_i, S_j, S_k, S_n) predicate, for $n \in \{a, b\}$, to obtain the type of $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$. From the shadow region types, two pieces of information is easily obtained; firstly, we determine which of the Apollonius vertices v_{ijka}, v_{ikja} and v_{ijkb}, v_{ikjb} actually exist and secondarily, if both v_{ijkn} and v_{ikjn} exist for some $n \in \{a, b\}$, then their ordering on the oriented trisector is also retrieved. Such deductions derive from the study of the shadow region, as shown in Section 2.7.6 (see Lemma 15). For example, if $\mathcal{SR}(S_a) = (\chi, \phi)$, then both v_{ijka} and v_{ikja} exist and appear on the oriented trisector in this order: $v_{ikja} \prec v_{ijka}$.

Now that the existence and partial ordering of the Apollonius vertices v_{ijka} and v_{ikja} (resp., v_{ijkb} and v_{ikjb}) is known, we must provide a way of "merging" them into a complete ordering. For this reason, we examine all possible complete orderings of the Apollonius vertices on the oriented trisector τ_{ijk}. The study of these orderings is seen in the inverted plane $\mathcal{W}$-space. In Section 2.7.6, we present the strong geometric relationship that holds between the spheres of the original $\mathcal{Z}$-space and their images in the inverted plane. The observations we make regarding the connection of the two spaces allow us to interpret geometric configurations on one space to equivalent ones on the other. A full analysis of how we tackle all possible configurations is presented in Section 2.7.6.

In our analysis, we prove that the Order predicate, in the worst case, amounts to evaluate up to 4 InSphere predicates plus some auxiliary tests of lesser algebraic cost. We have therefore proven the following lemma.

Lemma 6 *The* Order *predicate can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

2.5 The main algorithm

In this section, we describe in detail how the predicate EdgeConflict ($S_i, S_j, S_k, S_l, S_m, S_q$) is resolved with the use of the subpredicates InCone, TrisectorType, Existence, Shadow and Order.

We begin by determining the type of the trisector τ_{ijk}; this is done via the call of the TrisectorType (S_i, S_j, S_k) predicate. Recall that in the scope of this chapter, it is assumed that the τ_{ijk} is a hyperbola (or a line) and that none of the subpredicates called return a degenerate answer.

To answer the EdgeConflict predicate, one must determine which "part" of the edge e_{ijklm} remains in the Voronoi diagram after the insertion of the site S_q. This is plausible by identifying the set of points of e_{ijklm} that still remain in the updated Voronoi Diagram; each of these points must satisfy the "empty-sphere property": a sphere, centered at that point and tangent to the spheres S_i, S_j, S_k, must not intersect any other sites of the Voronoi Diagram. As an immediate result, a point p of the edge e_{ijklm} in $\mathcal{VD}(\mathcal{S})$ remains in $\mathcal{VD}(\mathcal{S} \cup \{S_q\})$ if and only if $\mathcal{T}(p)$ does not intersect S_q. Since the shadow region of the sphere S_q with respect to the trisector τ_{ijk} consists of all points p such that $\mathcal{T}(p)$ intersects S_q, it must hold that the part of the edge e_{ijklm} that no longer remains in $\mathcal{VD}(\mathcal{S} \cup \{S_q\})$ is actually $e_{ijklm} \cap \mathcal{SR}(S_q)$ (see Figure 2.5) . In conclusion, the result of the EdgeConflict predicate is exactly the set $e_{ijklm} \cap \mathcal{SR}(S_q)$ seen as an interval or union of intervals of $\mathbb{R}$.

To determine the intersection type of $e_{ijklm} \cap \mathcal{SR}(S_q)$, we first take into account that the finite edge e_{ijklm} consists of all points p on the oriented trisector τ_{ijk} bounded by the points v_{ijkl} and v_{ikjm} from left and right respectively (see Section 2.2). Next, we consider the type of $\mathcal{SR}(S_q)$ which can be evaluated as shown in Section 2.7.5 and is one of the following: $(-\infty, \phi)$, $(\chi, +\infty)$, (χ, ϕ), $(-\infty, \phi) \cup (\chi, +\infty)$, $\emptyset$ or $\mathbb{R}$.

If the edge e_{ijklm} is seen as the interval (λ_1, μ_2), evidently the intersection type of $E' = e_{ijklm} \cap \mathcal{SR}(S_q)$ must be one of the following 6 types, each corresponding to a different answers of the EdgeConflict predicate.

- If E' is of type $\emptyset$, the predicate returns NoConflict.
- If E' is of type e_{ijklm}, the predicate returns EntireEdge.
- If E' is of type (λ_1, ϕ), the predicate returns LeftVertex.
- If E' is of type (χ, μ_2), the predicate returns RightVertex.
- If E' is of type $(\lambda_1, \phi) \cup (\chi, \mu_2)$, the predicate returns BothVertices.
- If E' is of type (χ, ϕ), the predicate returns Interior.

This observation suggests that, if we provide a way to identify the type of E', we can answer the EdgeConflict predicate. Taking into consideration that

- λ_1 and μ_2 correspond to v_{ijkl} and v_{ikjm}, respectively as shown in Section 2.2, and
- χ, ϕ correspond to v_{ikjq} and v_{ijkq}, respectively as stated in Lemma 15 that we prove in Section 2.7.6,

it becomes apparent that if we order all Apollonius vertices v_{ijkl}, v_{ikjm} and any of the existing among v_{ijkq}, v_{ikjq}, bearing in mind the type of $\mathcal{SR}(S_q)$, we can deduce the type of E'.

For example, let us assume that $\mathcal{SR}(S_q)$ type is $(-\infty, \chi) \cup (\phi, +\infty)$. If $v_{ijkl} \prec v_{ikjq} \prec v_{ikjm} \prec v_{ijkq}$ on the oriented trisector τ_{ijk}, or equivalently $\lambda_1 < \chi < \mu_2 < \phi$, we can conclude that E' is of type (λ_1, ϕ) and the EdgeConflict predicate would return LeftVertex.

Therefore, it is essential that we are able to provide an ordering of the Apollonius vertices v_{ijkl}, v_{ikjm} and any of the existing among v_{ijkq}, v_{ikjq}. Such a task is accomplished via the call of the Order $(S_i, S_j, S_k, S_a, S_b)$ predicate $(a, b) = (l, q)$ and (m, q). The outcomes of these predicates consist of the orderings of all possible Apollonius vertices of the sites S_i, S_j, S_k, S_a and S_i, S_j, S_k, S_b on the trisector τ_{ijk}. These partial orderings can then be merged into a complete ordering, which contains the desired one. The results' combination principle is identical to the one used when we have to order a set of numbers but we can only compare two at a time.

A detailed algorithm that summarizes the analysis of this Section and can be followed to answer the EdgeConflict$(S_i, S_j, S_k, S_l, S_m; S_q)$ is described in the following steps. Note that the trisector τ_{ijk} is assumed to be a branch of a hyperbola or a line, i.e., the outcome of TrisectorType (S_i, S_j, S_k) is "hyperbolic".

Step 1 We evaluate $SRT(q) = \text{Shadow}(S_i, S_j, S_k, S_q)$. If $SRT(q) = \emptyset$ or $\mathbb{R}$, we return NoConflict or EntireEdge respectively. Otherwise, if $SRT(q)$ is $(-\infty, \phi)$, $(\chi, +\infty)$, (χ, ϕ) or $(-\infty, \phi) \cup (\chi, +\infty)$, we go to Step 2a, 2b, 2c or 2d, respectively. Note that ϕ and χ correspond to the Apollonius vertices v_{ijkq} and v_{ikjq} respectively.

Step 2a We evaluate $I_1 = \text{InSphere}(S_i, S_j, S_k, S_l, S_q)$. If $I_1 = +$ then $v_{ijkl} \notin \mathcal{SR}(S_q)$ or equivalently $v_{ijkq} \prec v_{ijkl} \prec v_{ikjm}$; in this case the predicate's outcome is NoConflict. Otherwise, if $I_1 = -$, we know that $v_{ijkl} \in \mathcal{SR}(S_q)$ and we have to evaluate $I_2 = \text{InSphere}(S_i, S_k, S_j, S_m, S_q)$. If $I_2 = +$ then $v_{ikjm} \notin \mathcal{SR}(S_q)$ or equivalently $v_{ijkq} \prec v_{ikjm}$; since $v_{ijkl} \prec v_{ijkq} \prec v_{ikjm}$ the predicate's outcome is LeftVertex. If $I_2 = -$ then $v_{ikjm} \in \mathcal{SR}(S_q)$ and subsequently $v_{ikjm} \prec v_{ijkq}$; since $v_{ijkl} \prec v_{ikjm} \prec v_{ijkq}$ the predicate's outcome is EntireEdge.

Step 2b We evaluate $I_1 = \text{InSphere}(S_i, S_j, S_k, S_l, S_q)$. If $I_1 = -$ then $v_{ijkl} \in \mathcal{SR}(S_q)$ or equivalently $v_{ikjq} \prec v_{ijkl} \prec v_{ikjm}$; in this case the predicate's outcome is EntireEdge. Otherwise, if $I_1 = -$, we know that $v_{ijkl} \notin \mathcal{SR}(S_q)$ and we have to evaluate $I_2 = \text{InSphere}(S_i, S_k, S_j, S_m, S_q)$. If $I_2 = -$ then $v_{ikjm} \in \mathcal{SR}(S_q)$ or equivalently $v_{ikjq} \prec v_{ikjm}$; since $v_{ijkl} \prec v_{ikjq} \prec v_{ikjm}$ the predicate's outcome is RightVertex. If $I_2 = +$ then $v_{ikjm} \notin \mathcal{SR}(S_q)$ and subsequently $v_{ikjm} \prec v_{ikjq}$; since $v_{ijkl} \prec v_{ikjm} \prec v_{ikjq}$ the predicate's outcome is NoConflict.

Step 2c We evaluate $I_1 = \text{InSphere}(S_i, S_j, S_k, S_l, S_q)$ and $I_2 = \text{InSphere}(S_i, S_k, S_j, S_m, S_q)$.

- If $(I_1, I_2) = (-, +)$ then $v_{ijkl} \in \mathcal{SR}(S_q)$ or equivalently $v_{ikjq} \prec v_{ijkl} \prec v_{ijkq}$ and $v_{ikjm} \notin \mathcal{SR}(S_q)$. Since $v_{ijkl} \prec v_{ikjm}$ it must hold that $v_{ikjq} \prec v_{ijkl} \prec v_{ijkq} \prec v_{ikjm}$ and therefore the predicate's outcome is LeftVertex.

- If $(I_1, I_2) = (+, -)$ then $v_{ikjm} \in \mathcal{SR}(S_q)$ or equivalently $v_{ikjq} \prec v_{ikjm} \prec v_{ijkq}$ and $v_{ijkl} \notin \mathcal{SR}(S_q)$. Since $v_{ijkl} \prec v_{ikjm}$ it must hold that $v_{ijkl} \prec v_{ikjq} \prec v_{ikjm} \prec v_{ijkq}$ and therefore the predicate's outcome is RightVertex.
- If $(I_1, I_2) = (-, -)$ then $v_{ijkl}, v_{ikjm} \in \mathcal{SR}(S_q)$. Since $v_{ijkl} \prec v_{ikjm}$ it must hold that $v_{ikjq} \prec v_{ijkl} \prec v_{ikjm} \prec v_{ijkq}$; in this case the predicate's outcome is EntireEdge.
- If $(I_1, I_2) = (+, +)$ then there are three possible cases, due to $v_{ijkl}, v_{ikjm} \notin \mathcal{SR}(S_q)$: either $v_{ikjq} \prec v_{ijkq} \prec v_{ijkl} \prec v_{ikjm}$, $v_{ijkl} \prec v_{ikjq} \prec v_{ijkq} \prec v_{ikjm}$ or $v_{ijkl} \prec v_{ikjm} \prec v_{ikjq} \prec v_{ijkq}$. To distinguish among the cases, we first call the ORDER$(S_i, S_j, S_k, S_l, S_q)$ predicate. If it returns that $v_{ikjq} \prec v_{ijkq} \prec v_{ijkl}$ then we are in the first case and the predicate returns NoConflict. Otherwise, it will necessarily return that $v_{ijkl} \prec v_{ikjq} \prec v_{ijkq}$ and we also have to call the ORDER$(S_i, S_j, S_k, S_m, S_q)$ predicate. If it returns that $v_{ikjq} \prec v_{ijkq} \prec v_{ikjm}$ then we are in the second case and the predicate returns Interior. Otherwise, it will return $v_{ikjm} \prec v_{ikjq} \prec v_{ijkq}$ and therefore we are in the third case thus the predicate's outcome is NoConflict.

Step 2d We evaluate $I_1 = \text{INSPHERE}(S_i, S_j, S_k, S_l, S_q)$ and $I_2 = \text{INSPHERE}(S_i, S_k, S_j, S_m, S_q)$.

- If $(I_1, I_2) = (-, +)$ then $v_{ikjm} \notin \mathcal{SR}(S_q)$ or equivalently $v_{ikjq} \prec v_{ikjm} \prec v_{ijkq}$ and $v_{ijkl} \in \mathcal{SR}(S_q)$. Since $v_{ijkl} \prec v_{ikjm}$ it must hold that $v_{ijkl} \prec v_{ijkq} \prec v_{ikjm} \prec v_{ikjq}$ and therefore the predicate's outcome is LeftVertex.
- If $(I_1, I_2) = (+, -)$ then $v_{ijkl} \notin \mathcal{SR}(S_q)$ or equivalently $v_{ikjq} \prec v_{ijkl} \prec v_{ijkq}$ and $v_{ikjm} \in \mathcal{SR}(S_q)$. Since $v_{ijkl} \prec v_{ikjm}$ it must hold that $v_{ijkq} \prec v_{ijkl} \prec v_{ikjq} \prec v_{ikjm}$ and therefore the predicate's outcome is RightVertex.
- If $(I_1, I_2) = (+, +)$ then $v_{ijkl}, v_{ikjm} \notin \mathcal{SR}(S_q)$. Since $v_{ijkl} \prec v_{ikjm}$ it must hold that $v_{ijkq} \prec v_{ijkl} \prec v_{ikjm} \prec v_{ikjq}$; in this case the predicate's outcome is NoConflict.
- If $(I_1, I_2) = (-, -)$ then there are three possible cases, due to $v_{ijkl}, v_{ikjm} \in \mathcal{SR}(S_q)$: either $v_{ijkq} \prec v_{ikjq} \prec v_{ijkl} \prec v_{ikjm}$, $v_{ijkl} \prec v_{ijkq} \prec v_{ikjq} \prec v_{ikjm}$ or $v_{ijkl} \prec v_{ikjm} \prec v_{ijkq} \prec v_{ikjq}$. To distinguish among the cases, we first call the ORDER$(S_i, S_j, S_k, S_l, S_q)$ predicate. If it returns that $v_{ijkq} \prec v_{ikjq} \prec v_{ijkl}$ then we are in the first case and the predicate returns EntireEdge. Otherwise, it will necessarily return that $v_{ijkl} \prec v_{ijkq} \prec v_{ikjq}$ and we also have to call the ORDER$(S_i, S_j, S_k, S_m, S_q)$ predicate. If it returns that $v_{ijkq} \prec v_{ikjq} \prec v_{ikjm}$ then we are in the second case and the predicate returns BothVertices. Otherwise, it will return $v_{ikjm} \prec v_{ijkq} \prec v_{ikjq}$ and therefore we are in the third case thus the predicate's outcome is EntireEdge.

A sketch of the subpredicates used when answering the EDGECONFLICT predicate is shown in Figure 2.7. Since the highest algebraic degree needed in the evaluation of the subpredicates used is 10, we have proven the following theorem.

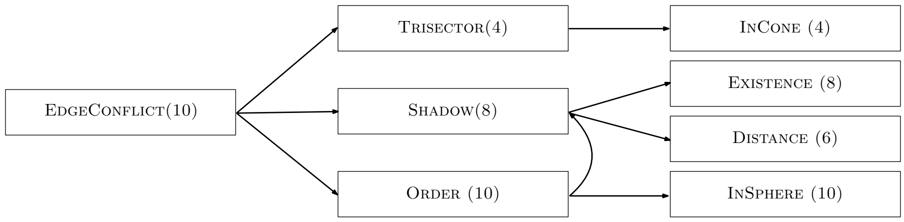

FIGURE 2.7: The layout of predicates and their subpredicates used to answer the EDGECONFLICT predicate in the case of a hyperbolic trisector. The number next to each predicate corresponds to its algebraic degree. It is assumed that every subpredicate returns a non-degenerate answer.

Theorem 1 *The* EDGECONFLICT *predicate for hyperbolic trisectors can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

2.6 The EdgeConflict Predicate for infinite hyperbolic edges

A variation of the algorithm presented in the previous section can be used to answer the following predicates:

- INFINITERIGHTEDGECONFLICT(S_i, S_j, S_k, S_l, S_q): describes the intersection type of an infinite Voronoi edge e that lies on the hyperbolic trisector τ_{ijk} and is bounded on the left by v_{ijkl}. The existence of such a Voronoi edge is equivalent to the fact that either only v_{ijkl} exist among $\{v_{ijkl}, v_{ikjl}\}$ or both exist and $v_{ikjl} \prec v_{ijkl}$, based on the remarks of previous sections.

- INFINITELEFTEDGECONFLICT(S_i, S_j, S_k, S_m, S_q): describes the intersection type of an infinite Voronoi edge e that lies on the hyperbolic trisector τ_{ijk} and is bounded on the right by v_{ikjm}. The existence of such a Voronoi edge is equivalent to the fact that either only v_{ikjm} exist among $\{v_{ijkm}, v_{ikjm}\}$ or both exist and $v_{ikjm} \prec v_{ijkm}$, based on the remarks of previous sections.

These two predicates are also called in the scope of a randomized incremental algorithm that constructs the 3D Apollonius diagram, similarly with the EDGECONFLICT predicate. The algorithms that decides them have minor differences than the algorithm presented in Section 2.5 to reflect the fact that the Voronoi edge e has only one finite bound. The possible non-degenerate outcomes of these predicates are the same with the EDGECONFLICT predicate with the following differences.

- In INFINITERIGHTEDGECONFLICT, the edge e is only bounded on the left and therefore the outcome RightVertex denotes that only a part of e adjacent to v_{ijkl} remains on the updated Voronoi diagram after the insertion of S_q. Moreover, BothVertices denotes that only a finite part of e that is not adjacent to v_{ijkl} remains on the updated Voronoi diagram.

- In InfiniteLeftEdgeConflict, the edge e is only bounded on the right and subsequently the outcome LeftVertex denotes that only a part of e adjacent to v_{ikjm} remains on the updated Voronoi diagram after the insertion of S_q. Furthermore, BothVertices denotes that only a finite part of e that is not adjacent to v_{ikjm} remains on the updated Voronoi diagram.

The main idea behind the following algorithms is the same with the one presented in Section 2.5. We consider the topological form of the edge e as an interval $(\lambda_1, +\infty)$ or $(-\infty, \mu_2)$ and the outcome of the predicate reflects the intersection of this interval with the respective interval that represents the shadow region of S_q.

Algorithm for InfiniteRightEdgeConflict$(S_i, S_j, S_k, S_l; S_q)$.

Step 1 We evaluate $SRT(q) = \textsc{Shadow}(S_i, S_j, S_k, S_q)$. If $SRT(q) = \emptyset$ or $\mathbb{R}$, we return NoConflict or EntireEdge respectively. Otherwise, if $SRT(q)$ has the form $(-\infty, \phi)$, $(\chi, +\infty)$, (χ, ϕ) or $(-\infty, \phi) \cup (\chi, +\infty)$, then we go to Step 2a, 2b, 2c or 2d respectively. Note that ϕ and χ correspond to the Apollonius vertices v_{ijkq} and v_{ikjq} respectively.

Step 2a We evaluate $I = \textsc{InSphere}(S_i, S_j, S_k, S_l, S_q)$. If $I = +$ then $v_{ijkl} \notin \mathcal{SR}(S_q)$ or equivalently $v_{ijkq} \prec v_{ijkl}$; in this the predicate's outcome is NoConflict. Otherwise, if $I = -$ then $v_{ijkl} \in \mathcal{SR}(S_q)$ or equivalently $v_{ijkl} \prec v_{ijkq}$ hence the predicate's outcome is LeftVertex.

Step 2b We evaluate $I = \textsc{InSphere}(S_i, S_j, S_k, S_l, S_q)$. If $I = +$ then $v_{ijkl} \notin \mathcal{SR}(S_q)$ or equivalently $v_{ijkl} \prec v_{ikjq}$; in this the predicate's outcome is RightVertex. Otherwise, if $I = -$ then $v_{ijkl} \in \mathcal{SR}(S_q)$ or equivalently $v_{ikjq} \prec v_{ijkl}$ hence the predicate's outcome is EntireEdge.

Step 2c We evaluate $I = \textsc{InSphere}(S_i, S_j, S_k, S_l, S_q)$. If $I = -$ then $v_{ijkl} \in \mathcal{SR}(S_q)$ or equivalently $v_{ikjq} \prec v_{ijkl} \prec v_{ijkq}$; in this the predicate's outcome is LeftVertex. Otherwise, if $I = +$ we call the $\textsc{Order}(S_i, S_j, S_k, S_l, S_q)$ and we can determine whether $v_{ijkl} \prec v_{ikjq} \prec v_{ijkq}$ or $v_{ikjq} \prec v_{ijkq} \prec v_{ijkl}$. In the former case, the predicate returns Interior otherwise, in the latter case it returns NoConflict.

Step 2d We evaluate $I = \textsc{InSphere}(S_i, S_j, S_k, S_l, S_q)$. If $I = +$ then $v_{ijkl} \notin \mathcal{SR}(S_q)$ or equivalently $v_{ikjq} \prec v_{ijkl} \prec v_{ijkq}$; in this the predicate's outcome is RightVertex. Otherwise, if $I = -$ we call the $\textsc{Order}(S_i, S_j, S_k, S_l, S_q)$ and we can determine whether $v_{ijkl} \prec v_{ikjq} \prec v_{ijkq}$ or $v_{ikjq} \prec v_{ijkq} \prec v_{ijkl}$. In the former case, the predicate returns BothVertices otherwise, in the latter case it returns EntireEdge.

Algorithm for InfiniteLeftEdgeConflict($S_i, S_j, S_k, S_m; S_q$).

Step 1 We evaluate $SRT(q) = \text{Shadow}(S_i, S_j, S_k, S_q)$. If $SRT(q) = \emptyset$ or $\mathbb{R}$, we return NoConflict or EntireEdge respectively. Otherwise, if $SRT(q)$ has the form $(-\infty, \phi)$, $(\chi, +\infty)$, (χ, ϕ) or $(-\infty, \phi) \cup (\chi, +\infty)$, then we go to Step 2a, 2b, 2c or 2d respectively. Note that ϕ and χ correspond to the Apollonius vertices v_{ijkq} and v_{ikjq} respectively.

Step 2a We evaluate $I = \text{InSphere}(S_i, S_k, S_j, S_m, S_q)$. If $I = +$ then $v_{ikjm} \notin \mathcal{SR}(S_q)$ or equivalently $v_{ijkq} \prec v_{ikjm}$; in this the predicate's outcome is LeftVertex. Otherwise, if $I = -$ then $v_{ikjm} \in \mathcal{SR}(S_q)$ or equivalently $v_{ikjm} \prec v_{ijkq}$ hence the predicate's outcome is EntireEdge.

Step 2b We evaluate $I = \text{InSphere}(S_i, S_k, S_j, S_m, S_q)$. If $I = +$ then $v_{ikjm} \notin \mathcal{SR}(S_q)$ or equivalently $v_{ikjm} \prec v_{ikjq}$; in this the predicate's outcome is NoConflict. Otherwise, if $I = -$ then $v_{ikjm} \in \mathcal{SR}(S_q)$ or equivalently $v_{ikjq} \prec v_{ikjm}$ hence the predicate's outcome is RightVertex.

Step 2c We evaluate $I = \text{InSphere}(S_i, S_k, S_j, S_m, S_q)$. If $I = -$ then $v_{ikjm} \in \mathcal{SR}(S_q)$ or equivalently $v_{ikjq} \prec v_{ikjm} \prec v_{ijkq}$; in this the predicate's outcome is RightVertex. Otherwise, if $I = +$ we call the $\text{Order}(S_i, S_j, S_k, S_m, S_q)$ and we can determine whether $v_{ikjm} \prec v_{ikjq} \prec v_{ijkq}$ or $v_{ikjq} \prec v_{ijkq} \prec v_{ikjm}$. In the former case, the predicate returns NoConflict otherwise, in the latter case it returns BothVertices.

Step 2d We evaluate $I = \text{InSphere}(S_i, S_k, S_j, S_m, S_q)$. If $I = +$ then $v_{ikjm} \notin \mathcal{SR}(S_q)$ or equivalently $v_{ikjq} \prec v_{ikjm} \prec v_{ijkq}$; in this the predicate's outcome is LeftVertex. Otherwise, if $I = -$ we call the $\text{Order}(S_i, S_j, S_k, S_m, S_q)$ and we can determine whether $v_{ikjm} \prec v_{ikjq} \prec v_{ijkq}$ or $v_{ikjq} \prec v_{ijkq} \prec v_{ikjm}$. In the former case, the predicate returns EntireEdge otherwise, in the latter case it returns BothVertices.

Since the evaluation of InfiniteRightEdgeConflict and InfiniteLeftEdgeConflict only require the call of the Shadow, InSphere and Order predicates we have proven the following theorem.

Theorem 2 *The* InfiniteLeftEdgeConflict *and* InfiniteRightEdgeConflict *predicates for hyperbolic trisectors can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

2.7 Design and Analysis of SubPredicates

In this Section, we provide a detailed description on how to answer every subpredicate involved in the algorithm presented in Section 2.5. For each primitive, besides analyzing how we derive the outcome, we also compute its algebraic degree, i.e., the maximum algebraic degree of all quantities that have to be evaluated to obtain the subpredicate's result.

2.7.1 The InCone predicate

To answer the InCone (S_a, S_b, S_c) predicate we first determine the number of possible tangent planes to the sites S_a, S_b, and S_c that leave them all on the same side; there can be either 0, 1, 2 or ∞ such planes. Bear in mind that the InCone predicate can be called only if no one of the spheres S_a and S_b are contained inside another.

If S_a and S_b have different radii, then $\mathcal{K}$ will denote the cone that contains and is tangent to these spheres, whereas $\mathcal{K}^-$ will symbolize the symmetric cone with the same axis and apex. Let us no consider each of the four possible cases regarding the number of cotagent planes, since it is indicative of the relative position of the three spheres.

1. If no such plane exists, there are four cases to consider;
   - S_c lies strictly inside the cone $\mathcal{K}$; the predicate returns Inside.
   - S_c lies strictly inside the cone $\mathcal{K}^-$; the predicate returns Outside.
   - S_c fully intersects the cone $\mathcal{K}$ in the sense that there is a circle on S_c that is outside $\mathcal{K}$. In this case, the predicate returns Outside.
   - There is a circle on S_c that is outside $\mathcal{K}$; S_c could lie strictly inside the cone $\mathcal{K}^-$ or fully intersect the cone $\mathcal{K}$, $\mathcal{K}^-$ or both. In all these cases, the predicate returns Outside.
2. If there is only one such plane, then S_c touches $\mathcal{K}$ in a single point. There are three cases to consider;
   - S_c lies strictly inside the cone $\mathcal{K}$; the predicate returns OnePointTouch.
   - S_c fully intersects the cone $\mathcal{K}$ (there is a circle in S_c that is outside $\mathcal{K}$). In this case, the predicate returns Outside.
   - S_c is tangent to the cone $\mathcal{K}^-$ at a single point. In this case, the predicate returns Outside.
3. If there are two such planes, the spheres must lie in convex position, hence the predicate returns Outside.
4. If there are infinite such planes, the spheres S_a, S_b and S_c have collinear centers and the points of tangency of each sphere with the cone is a single circle. The predicate returns CircleTouch in this scenario.

In the case no cotagent plane to all sites S_a, S_b and S_c exist, we must be able to tell if S_c lies inside the cone $\mathcal{K}^-$. However, this check is only needed in the case $r_a \neq r_b$; if $r_a = r_b$, the cone $\mathcal{K}$ degenerates into a cylinder and $\mathcal{K}^-$ does not exist.

If $r_a = r_b = r$ we can immediately answer the predicate if $r_c > r$; S_c has a point outside the cylinder and the outcome is Outside. Otherwise, we initially consider the case where C_a, C_b and C_c are collinear; this is equivalent to the cross product of the vectors

$\overrightarrow{C_aC_c}$ and $\overrightarrow{C_aC_b}$ being zero. Since $\overrightarrow{C_aC_c} \times \overrightarrow{C_aC_b} = (-D^{yz}_{abc}, D^{xz}_{abc}, -D^{xy}_{abc})$, it holds that $\overrightarrow{C_aC_c} \times \overrightarrow{C_aC_b} = \overrightarrow{0}$ if and only if $D^{xz}_{abc} = D^{xz}_{abc} = D^{xz}_{abc} = 0$, which is a 2-degree demanding operation. In the collinear scenario, we can easily answer the InCone predicate based on the sign of $r_c - r$; if it is negative or zero, the answer is Inside or CircleTouch, respectively.

If $r_a = r_b = r$ but the centers of C_a, C_b and C_c are not collinear, then if $r_c = r$ we immediately answer Outside. Otherwise, if $r_c - r < 0$, we consider the sign of the quantity $A = d(C_c, \ell) + r_c - r$, where ℓ denotes the line going through C_a and C_b and $d(C_c, \ell)$ denotes the Euclidean distance of C_c from ℓ. If A is positive or negative, then the outcome of InCone predicate is Outside or Inside, respectively. Otherwise, if A equals zero, the answer is OnePointTouch. Notice that $d(C_c, \ell) = |\overrightarrow{C_aC_c} \times \overrightarrow{C_aC_b}|/|\overrightarrow{C_aC_b}|$ and therefore

$$A = \mathsf{sign}(d(C_c, \ell) + r_c - r) = \mathsf{sign}(|\overrightarrow{C_aC_c} \times \overrightarrow{C_aC_b}|/|\overrightarrow{C_aC_b}| + r_c - r) \tag{2.1}$$

$$= \mathsf{sign}(|\overrightarrow{C_aC_c} \times \overrightarrow{C_aC_b}| + (r_c - r)|\overrightarrow{C_aC_b}|) = \mathsf{sign}(|\overrightarrow{C_aC_c} \times \overrightarrow{C_aC_b}|^2 - (r_c - r)^2|\overrightarrow{C_aC_b}|^2) \tag{2.2}$$

$$= \mathsf{sign}(((D^{xz}_{abc})^2 + (D^{xz}_{abc})^2 + (D^{xz}_{abc})^2) - (r_c - r)^2((x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2)) \tag{2.3}$$

since $r_c - r < 0$.

Let us now consider the case $r_a \neq r_b$ in detail. First, observe that we can assume without loss of generality that $r_a < r_b$ as this follows from the definition of the InCone predicate. Indeed, since InCone (S_a, S_b, S_c) and InCone (S_b, S_a, S_c) represent the same geometric inquiry, we can exchange the notation of the spheres S_a and S_b in case $r_a > r_b$.

Taking this into consideration, we denote K to be the apex of the cone $\mathcal{K}$ and Π_K to be the plane that goes through K and perpendicular to axis ℓ of the cone. We also denote by Π^+_K the half-plane defined by the plane Π_K and the centers C_a and C_b, whereas its compliment half-plane is denoted by Π^-_K.

It is obvious that if the center of the sphere S_c does not lie in Π^+_K the predicate must return Outside since S_c has at least one point outside the cone $\mathcal{K}$. Moreover, if the center C_c lies in Π^+_K then S_c cannot lie inside $\mathcal{K}^-$ and this case is ruled out. To check if C_c lies on Π^+_K, we first observe that C_a and C_b define the line ℓ hence for every point P of ℓ stands that $\overrightarrow{OP} = \overrightarrow{OC_a} + t\overrightarrow{C_aC_b}$ for some $t \in \mathbb{R}$. It is clear that, for a sphere with center $P(t)$ to be tangent to the cone it must have radius $r(t)$ that is linearly dependent with t, i.e., $r(t) = k_1t + k_0$. To evaluate k_1 and k_0, we observe that for $t = 0$, $P(0) \equiv C_a$ hence $r(0) = r_a$ and respectively for $t = 1$, $P(1) \equiv C_b$ hence $r(1) = r_b$. We conclude that $r(t) = t \cdot (r_b - r_a) + r_a$, $t \in \mathbb{R}$. The cone apex lies on ℓ so $\overrightarrow{OK} = \overrightarrow{OC_a} + t_c\overrightarrow{C_aC_b}$ for $t_c \in \mathbb{R}$ such that $r(t_c) = 0$ or equivalently $t_c = r_a/(r_a - r_b)$. In this way we have evaluated the cone apex coordinates which derive from the relation $\overrightarrow{OK} = \overrightarrow{OC_a} + \overrightarrow{C_bC_a} \cdot r_a/(r_b - r_a)$. Since Π_K is perpendicular to ℓ and therefore to $\overrightarrow{C_aC_b}$, and $\overrightarrow{C_aC_b}$ points towards the positive side of Π_K, the point C_c lies on the positive half plane Π_K^+ iff the quantity $M = \overrightarrow{C_aC_b} \cdot \overrightarrow{KC_c}$ is strictly positive.

To evaluate the sign of M we have

$$\begin{aligned}
\mathsf{sign}(M) &= \mathsf{sign}(\overrightarrow{C_aC_b} \cdot \overrightarrow{KC_c}) = \mathsf{sign}(\overrightarrow{C_aC_b} \cdot (\overrightarrow{OC_c} - \overrightarrow{OK})) && (2.4)\\
&= \mathsf{sign}(\overrightarrow{C_aC_b} \cdot \overrightarrow{OC_c} - \overrightarrow{C_aC_b} \cdot \overrightarrow{OK}) && (2.5)\\
&= \mathsf{sign}(\overrightarrow{C_aC_b} \cdot (\overrightarrow{OC_c} - \overrightarrow{OC_a} - \frac{r_a}{r_b - r_a}\overrightarrow{C_bC_a})) && (2.6)\\
&= \mathsf{sign}(\overrightarrow{C_aC_b} \cdot ((r_b - r_a)\overrightarrow{C_aC_c} + r_a\overrightarrow{C_aC_b}))\mathsf{sign}(r_b - r_a) && (2.7)\\
&= \mathsf{sign}((r_b - r_a)\overrightarrow{C_aC_c} \cdot \overrightarrow{C_aC_b} + r_a\overrightarrow{C_aC_b} \cdot \overrightarrow{C_aC_b}) && (2.8)
\end{aligned}$$

so determining $\mathsf{sign}(M)$ requires operations of degree 3, since $r_b - r_a$ is strictly positive.

If $M < 0$ or $M = 0$ and $r_c > 0$ the predicate immediately returns Outside. In the special case where $M = 0$ and $r_c = 0$, the sphere S_k is essentially the apex of the cone $\mathcal{K}$ and the predicate returns CircleTouch.

Lastly and for the rest of this section, we consider the case $M > 0$ in detail, as further analysis is required to answer the InCone predicate. We break down our analysis depending on the collinearity of the centers C_a, C_b and C_c.

The Centers C_a, C_b, C_c are Collinear

If C_a, C_b and C_c are collinear, they all lie on the line ℓ, and therefore $\overrightarrow{OC_c} = \overrightarrow{OC_a} + t_o\overrightarrow{C_aC_b}$ for some $t_o \in \mathbb{R}$. Equivalently, we get that $t_o\overrightarrow{C_aC_b} = \overrightarrow{C_aC_c}$ and since C_a and C_b cannot be identical we can evaluate $t_o = X/Y$ where $(X, Y) = (x_c - x_a, x_b - x_a)$ or $(y_c - y_a, y_b - y_a)$ or $(z_c - z_a, z_b - z_a)$, if $x_b - x_a \neq 0$ or $y_b - y_a \neq 0$ or $z_b - z_a \neq 0$ respectively.

Denote $r(t)$ as before, we evaluate the sign S of $r_c - r(t_o)$,

$$\begin{aligned}
S &= \mathsf{sign}(r_c - r(t_o)) = \mathsf{sign}(r_c - r_a - (r_b - r_a)t_o) && (2.9)\\
&= \mathsf{sign}(r_c - r_a - (r_b - r_a)X/Y) && (2.10)\\
&= \mathsf{sign}(Y)\mathsf{sign}((r_c - r_a)Y - (r_b - r_a)X) && (2.11)\\
&= -\mathsf{sign}(Y)\mathsf{sign}((r_a - r_c)Y + (r_b - r_a)X) && (2.12)
\end{aligned}$$

which requires operations of degree at most 2.

We can now answer the predicate because if $r_c < r(t_o)$, i.e., S is negative, then S_c lies strictly inside the cone; otherwise, if $r_c > r(t_o)$, i.e., S is positive, then S_c intersects the cone. If $r_c = r(t_c)$ or equivalently S is zero, then S_c touches $\mathcal{K}$ in a circle. In conclusion, we get that

$$\textsc{InCone}(S_a, S_b, S_c) = \begin{cases} \mathsf{Inside}, & \text{if } S < 0,\\ \mathsf{CircleTouch}, & \text{if } S = 0,\\ \mathsf{Outside}, & \text{if } S > 0, \end{cases}$$

Non-Collinear Centers

If C_a, C_b and C_c are not collinear and must examine the number of possible tritangent planes to the sites S_n for $n \in \{a, b, c\}$. Denote $\Pi : \kappa x + \lambda y + \mu z + \nu = 0$ a plane tangent to S_a, S_b and S_c that leaves the spheres on the same half-plane, and assume without loss of generality that $\kappa^2 + \lambda^2 + \mu^2 = 1$. Since the sphere S_n for $n \in \{i, j, k\}$ touches the plane Π, we get that $\delta(S_n, \Pi) = \kappa x_n + \lambda y_n + \mu z_n + \nu = r_n$.

We examine the resulting system of equations

$$\kappa x_a + \lambda y_a + \mu z_a = r_a - \nu \tag{2.13}$$
$$\kappa x_b + \lambda y_b + \mu z_b = r_b - \nu \tag{2.14}$$
$$\kappa x_c + \lambda y_c + \mu z_c = r_c - \nu \tag{2.15}$$
$$\kappa^2 + \lambda^2 + \mu^2 = 1 \tag{2.16}$$

and distinguish the following cases

- if $D^{xyz}_{abc} \neq 0$, we can express κ, λ and μ linearly in terms of ν. Substituting these expressions in the last equation, we get a quadratic equation that vanishes at d; the sign of the discriminant Δ' of this quadratic reflects the number of possible ν's and therefore tangent planes to the spheres S_a, S_b and S_c.

- if $D^{xyz}_{abc} = 0$ and since the centers of the spheres S_a, S_b and S_c are not collinear, one of the quantities $D^{xy}_{abc}, D^{xz}_{abc}$ or D^{yz}_{abc} is non-zero, without loss of generality assume $D^{xy}_{abc} \neq 0$. In this case, we can express κ and λ linearly in terms of μ, whereas $\nu = D^{xyr}_{abc}/D^{xy}_{abc}$. From the last equation, we get a quadratic equation that vanishes at μ, and the sign of the discriminant Δ'' again reflects the number of possible μ's and therefore tangent planes to the spheres S_a, S_b and S_c.

Writing down the expressions of the discriminants, we finally evaluate that $\Delta' = 4(D^{xyz}_{abc})^2\Delta$ and $\Delta'' = 4(D^{xy}_{abc})^2\Delta$, where

$$\Delta = (D^{xy}_{abc})^2 + (D^{xz}_{abc})^2 + (D^{yz}_{abc})^2 - (D^{xr}_{abc})^2 - (D^{yr}_{abc})^2 - (D^{zr}_{abc})^2. \tag{2.17}$$

Since the signs of the discriminants Δ', Δ'' and Δ are identical, we evaluate $\mathsf{sign}(\Delta)$ and proceed as follows:

1. If $\Delta > 0$, there are two planes tangent to all three spheres S_a, S_b and S_c; the predicate returns Outside.

2. If $\Delta = 0$, there is a single plane tangent to the spheres S_a, S_b and S_c and we have to determine if S^o_c lies strictly inside $\mathcal{K}$ or not. In the former case, the predicate returns OnePointTouch whereas in the latter, S^o_c intersects the exterior of $\mathcal{K}$ and the predicate returns Outside.

Notice that, only if we are in the former case, there exists a proper $\epsilon > 0$ such that the inflated sphere $\tilde{S_c}$, with radius $\tilde{r_c} = r_c + \epsilon$, will point-touch the cone $\mathcal{K}$. (Note that the tangency points of the cone $\mathcal{K}$ and the spheres S_c and $\tilde{S_c}$ are not the same!) Therefore, if we consider the analysis of the predicate InCone $(S_a, S_b, \tilde{S_c})$, we will conclude that the "perturbed" discriminant $\tilde{\Delta}$ vanishes for this $\epsilon > 0$.

For the evaluation of $\tilde{\Delta}$, we simply substitute r_c in Δ with $\tilde{r_c} = r_c + \epsilon$ and rewrite $\tilde{\Delta} = \tilde{\Delta}(\epsilon)$ as a polynomial in terms of ϵ: $\tilde{\Delta}(\epsilon) = \Delta_2\epsilon^2 + \Delta_1\epsilon + \Delta_0$, where $\Delta_0 = \Delta = 0$ and

$$\Delta_2 = -[(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2](< 0), \tag{2.18}$$

$$\Delta_1 = -2\big((x_b - x_a)D^{xr}_{abc} + (y_b - y_a)D^{yr}_{abc} + (z_b - z_a)D^{zr}_{abc}\big). \tag{2.19}$$

Since $\epsilon = 0$ is a root of the quadratic (in terms of ϵ) $\tilde{\Delta}(\epsilon)$, a simple use of Vieta's formula shows that $\tilde{\Delta}(\epsilon)$ has a positive root, if and only if $\mathsf{sign}(\Delta_1)$ is strictly positive. In conjunction with our previous remarks, the predicate should return OnePointTouch if $\Delta_1 > 0$; otherwise it should return Outside.

3. If $\Delta < 0$, there is no plane tangent to the spheres S_a, S_b and S_c. Since S_c lying inside the cone $\mathcal{K}^-$ is ruled out (due to the sign of M being positive), we have to distinguish between the two possible cases; S_c lies strictly inside $\mathcal{K}$ or S_c intersects $\vartheta\mathcal{K}$.

   It follows that, either S_c lies strictly within the cone and the predicate must return Inside, or S_c fully intersects the cone and the predicate must return Outside. Using the same analysis as in case $\Delta = 0$, we observe that if we inflate (or deflate) S_c, the perturbed sphere $\tilde{S_c}$ with radius $\tilde{r_c} = r_c + \epsilon$ will touch the cone $\mathcal{K}$ for two different values ϵ_1 and ϵ_2. The predicate must return Inside if we must inflate S_c to touch $\mathcal{K}$, i.e., if $\epsilon_1, \epsilon_2 > 0$ whereas the predicate must return Outside if we must deflate S_c, to point-touch $\mathcal{K}$, i.e., if $\epsilon_1, \epsilon_2 < 0$. As shown in the case $\Delta = 0$, the perturbed discriminant that will appear during the evaluation of InCone $(S_a, S_b, \tilde{S_c})$ is $\tilde{\Delta}(\epsilon) = \Delta_2\epsilon^2 + \Delta_1\epsilon + \Delta_0$ where

$$\Delta_2 = -[(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2](< 0), \tag{2.20}$$

$$\Delta_1 = -2\big((x_b - x_a)D^{xr}_{abc} + (y_b - y_a)D^{yr}_{abc} + (z_b - z_a)D^{zr}_{abc}\big), \tag{2.21}$$

$$\Delta_0 = \Delta < 0. \tag{2.22}$$

   Since the sphere $\tilde{S_c}$) point-touches the cone $\mathcal{K}$ for $\epsilon = \epsilon_1, \epsilon_2$, the discriminant $\tilde{\Delta}(\epsilon)$ must vanish for these epsilons, as mentioned in the previous case. Therefore, ϵ_1 and ϵ_2 are the roots of the quadratic (in terms of ϵ) $\Delta_2\epsilon^2 + \Delta_1\epsilon + \Delta_0$ and we know, using Vieta's rule and the fact that $\Delta_0 < 0$, that ϵ_1, ϵ_2 are both negative (resp., positive) if and only if Δ_1 is negative (resp., positive). In summary, if Δ_1 is positive or negative, the predicate returns Inside or Outside, respectively.

The analysis of this section summarizes to the following algorithm that answers the InCone(S_a, S_b, S_c) predicate.

Step 1 If $C_c = C_n$ for $n \in \{a, b\}$ return Outside, Inside or CircleTouch if the quantity $r_c - r_n$ is positive, negative or zero, respectively. Otherwise, go to step 2

Step 2 If $r_a = r_b$ go to Step 3, otherwise go to Step 5.

Step 3 If $r_c > r_a$ return Outside. Otherwise, evaluate the quantities $D^{xz}_{abc}, D^{xz}_{abc}$ and D^{xz}_{abc}. If all equal zero go to Step 4a, otherwise go to Step 4b.

Step 4a If $r_c = r_a$ return CircleTouch otherwise, if $r_c < r_a$ return Inside.

Step 4b If $r_c = r_a$ return Outside, otherwise evaluate $A = \mathsf{sign}(((D^{xz}_{abc})^2 + (D^{xz}_{abc})^2 + (D^{xz}_{abc})^2) - (r_c - r)^2((x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2))$. If A is positive, negative or zero, return Outside, Inside or OnePointTouch, respectively.

Step 5 If $r_b < r_a$ exchange the notation of S_a and S_b. Evaluate $M = \mathsf{sign}(\overrightarrow{C_aC_c} \cdot \overrightarrow{C_aC_b}(r_b - r_a) + r_a\overrightarrow{C_aC_b} \cdot \overrightarrow{C_aC_b})$ and if $M < 0$ or $M = 0$ and $r_c > 0$ return Outside. Otherwise, if $M = r_c = 0$ return CircleTouch. In any other case, go to Step 6.

Step 6 Evaluate the quantities $D^{xz}_{abc}, D^{xz}_{abc}$ and D^{xz}_{abc}. If all equal zero, go to Step 7, otherwise go to step 8.

Step 7 Evaluate $S = -\mathsf{sign}(Y)\mathsf{sign}((r_a - r_c)Y + (r_b - r_a)X)$ where $(X, Y) = (x_c - x_a, x_b - x_a)$ or $(y_c - y_a, y_b - y_a)$ or $(z_c - z_a, z_b - z_a)$, if $x_b - x_a \neq 0$ or $y_b - y_a \neq 0$ or $z_b - z_a \neq 0$, respectively. If S is positive, negative or zero, we return Outside, Inside or CircleTouch, respectively.

Step 8 Evaluate $\Delta = (D^{xy}_{abc})^2 + (D^{xz}_{abc})^2 + (D^{yz}_{abc})^2 - (D^{xr}_{abc})^2 - (D^{yr}_{abc})^2 - (D^{zr}_{abc})^2$. If Δ is positive, we return Outside. Otherwise, if Δ is zero or negative, we go to Step 9a or 9b, respectively.

Step 9a Evaluate $\Delta_1 = -2\big((x_b - x_a)D^{xr}_{abc} + (y_b - y_a)D^{yr}_{abc} + (z_b - z_a)D^{zr}_{abc}\big)$. If $\Delta_1 > 0$ we return OnePointTouch, otherwise we return Outside.

Step 9b Evaluate $\Delta_1 = -2\big((x_b - x_a)D^{xr}_{abc} + (y_b - y_a)D^{yr}_{abc} + (z_b - z_a)D^{zr}_{abc}\big)$. If $\Delta_1 > 0$ we return Inside, otherwise we return Outside.

Note that, since the algorithm that decides the InCone predicate demands the evaluation of quantities of degree at most 4 (in the input quantities), we have proven the following lemma.

Lemma 7 *The* InCone *predicate can be evaluated by determining the sign of quantities of algebraic degree at most 4 (in the input quantities).*

2.7.2 The TrisectorType predicate

Assuming the trisector τ_{ijk} exists and has Hausdorff dimension 1, observe that the trisector's type and the relative position of the spheres are closely related. Specifically,

- If the spheres are in *convex position*, i.e. there exist two distinct commonly tangent planes that leave them on the same side, then their trisector can either be a hyperbola or a line, in the special case $r_i = r_j = r_k$. The predicate returns "hyperbolic" in this scenario.

- If the spheres are in in *strictly non-convex position*, i.e. one of them lies strictly inside the cone defined by the other two, then their trisector can either be an ellipse or a circle, in the special case where C_i, C_j and C_k are collinear. The predicate returns "elliptic" in this scenario (see Figure 2.8, Left).

- If the spheres are in *degenerate non-convex position*, i.e. they are in non-convex position and the closure of all three touch their convex hull, then their trisector is a parabola and the predicate returns "parabolic"(see Figure 2.8, Right).

Therefore, one can answer the TrisectorType predicate if the relative position of the spheres three input spheres is identified. We accomplish such task by combining the outcomes of the three InCone predicates with inputs (S_i, S_j, S_k), (S_i, S_k, S_j) and (S_j, S_k, S_i).

- If at least one outcome is Inside, then the spheres are in *strictly non-convex position* and the trisector's type is "elliptic".

- If at least one outcome is OnePointTouch, then the spheres are in *degenerate non-convex position* and the TrisectorType predicate returns "parabolic".

- Finally, if all three outcomes are Outside, then the sites are in *convex position* and the trisector's type is "hyperbolic".

We have argued that the TrisectorType predicate can be resolved by calling the InCone predicate at most three times (for example, if the first InCone returns Inside the trisector must be "elliptic"). Since the InCone predicate is a 4-degree demanding operation in the input quantities, we have proven the following lemma.

Lemma 8 *The* TrisectorType *predicate can be evaluated by determining the sign of quantities of algebraic degree at most 4 (in the input quantities).*

2.7.3 The Existence predicate

To answer the Existence (S_i, S_j, S_k, S_a) predicate, we first break up our analysis depending on whether the radii of all sites S_n, for $n \in \{i, j, k, a\}$, are equal; if r is indeed their common radius, we deflate them by r. The existence of a sphere tangent to the original spheres

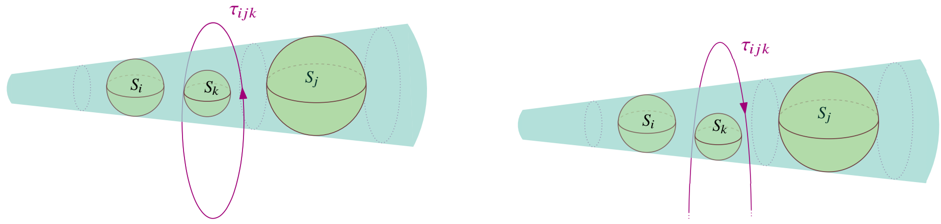

FIGURE 2.8: The case of an *elliptic* (Left) and a *parabolic* trisector (Right).

amounts to the existence of a sphere tangent to the centers of the sites where the latter cotangent sphere is the former inflated by r. Regarding the existence of the Apollonius sphere of the centers C_n for $n \in \{i, j, k, a\}$, we consider the tetrahedron $C_iC_jC_kC_a$ formed and check it for flatness; this is a 3-degree demanding operation. If it is not flat, there are is a single sphere S passing through all the centers, hence there is only one external Apollonius sphere and therefore the EXISTENCE predicate returns 1. If the tetrahedron is flat, i.e., the centers lie on the same plane Π_{ijka}, we check them for co-circularity (this is a 5-degree demanding operation); if co-circular, there are infinite number of spheres tangent to the sites (either internal or external), hence the EXISTENCE predicate returns ∞ (degenerate answer). If they are not co-circular, there can not be a sphere tangent to all sites and the EXISTENCE predicate returns 0. Note that if the EXISTENCE predicate would return ∞ we apply a QSP scheme to resolve the degeneracy.

Let us now consider the case where the radii of the sites S_n for $n \in \{i, j, k, a\}$ are not all equal. Since a reordering of the sites does not affect the EXISTENCE predicate, we reorder them such that $r_a = \min\{r_i, r_j, r_k, r_a\}$. We now deflate all spheres by r_a and then invert all sites with respect to the point C_a; we call this "inversion through the sphere S_a".

In the inverted $\mathcal{W}$-space, a plane tritangent to the inverted spheres $S_i^\star, S_j^\star, S_k^\star$ amounts to a sphere tangent to all sites S_i, S_j, S_k and S_a in the original $\mathcal{Z}$-space. For the corresponding sphere to be an external Apollonius sphere of the sites, the following conditions must stand:

1. The plane must leave all inverted spheres on one side, called the positive side of the plane, and
2. The origin $O = (0, 0, 0)$ of the $\mathcal{W}$-space, that corresponds to the "point at infinity" in the $\mathcal{Z}$-space, must also lie on the positive side of the plane. Geometrically, this means that we are looking for an external Apollonius sphere.

Considering Condition 1, we denote $\Pi_{ijk}^\star : au + bv + cw + d = 0$ a plane tangent to $S_i^\star, S_j^\star$ and $S_k^\star$ in the inverted space and assume without loss of generality that $a^2 + b^2 + c^2 = 1$. Since the signed Euclidean distance of a point $P(u_p, v_p, w_p)$ from the plane $\Pi_{ijk}^\star$ is $\delta(P, \Pi_{ijk}) = au_p + bv_p + cw_p + d$ and the sphere $S_n^\star$, for $n \in \{i, j, k\}$, touches $\Pi_{ijk}^\star$ and lies on its positive side, we get that $\delta(C_n^\star, \Pi_{ijk}) = au_n + bv_n + cw_n + d = \rho_n$.

A tuple (a, b, c, d) that satisfies the resulting system of equations, amounts to a tangent plane in $\mathcal{W}$-space and an Apollonius sphere in $\mathcal{Z}$-space. In order for condition 2 to be valid, the point $O = (0,0,0)$ of the $\mathcal{W}$-space must lie on the positive side of Π_{ijk}, i.e., the signed distance of O from the plane Π_{ijk} must be positive. Equivalently, we want $\delta(O, \Pi_{ijk}) = d$ to be positive, hence we are only interested in the solutions (a, b, c, d) that satisfy $d > 0$.

The rest of this section is devoted to the algebraic analysis of the aforementioned system of equations to determine the number of such solutions with the minimum algebraic cost. The conclusion of our analysis is that such a task is possible by evaluating expressions of algebraic degree at most 8 (in the input quantities) yielding the lemma at the end of this section.

Our main tool is Crammer's rule and therefore two major cases rise during the analysis of the system

$$au_i + bv_i + cw_i = \rho_i - d, \tag{2.23}$$
$$au_j + bv_j + cw_j = \rho_j - d, \tag{2.24}$$
$$au_k + bv_k + cw_k = \rho_k - d, \tag{2.25}$$
$$a^2 + b^2 + c^2 = 1. \tag{2.26}$$

If $D_{ijk}^{uvw} \neq 0$, we can express a, b and c in terms of d as follows

$$a = \frac{D_{ijk}^{vw\rho} - dD_{ijk}^{vw}}{D_{ijk}^{uvw}}, \quad b = -\frac{D_{ijk}^{uw\rho} - dD_{ijk}^{uw}}{D_{ijk}^{uvw}}, \quad c = \frac{D_{ijk}^{uv\rho} - dD_{ijk}^{uv}}{D_{ijk}^{uvw}}. \tag{2.27}$$

We will then substitute the expressions of a, b and c to the equation $a^2 + b^2 + c^2 = 1$ and conclude that d is a root of $M(d) = M_2 d^2 + M_1 d + M_0$, where

$$M_2 = (D_{ijk}^{uv})^2 + (D_{ijk}^{uw})^2 + (D_{ijk}^{vw})^2, \tag{2.28}$$
$$M_1 = D_{ijk}^{vw\rho} D_{ijk}^{vw} + D_{ijk}^{uw\rho} D_{ijk}^{uw} + D_{ijk}^{uv\rho} D_{ijk}^{uv}, \tag{2.29}$$
$$M_0 = (D_{ijk}^{vw\rho})^2 + (D_{ijk}^{uw\rho})^2 + (D_{ijk}^{uv\rho})^2 - (D_{ijk}^{uvw})^2. \tag{2.30}$$

The signs of M_1 and M_0 are determined using the the following equalities

$$\mathsf{sign}(M_1) = -\mathsf{sign}(D_{ijk}^{vw\rho} D_{ijk}^{vw} + D_{ijk}^{uw\rho} D_{ijk}^{uw} + D_{ijk}^{uv\rho} D_{ijk}^{uv}) \tag{2.31}$$
$$= -\mathsf{sign}(E_{ijk}^{yzr} E_{ijk}^{yzp} + E_{ijk}^{xzr} E_{ijk}^{xzp} + E_{ijk}^{xyr} E_{ijk}^{xyp}), \tag{2.32}$$
$$\mathsf{sign}(M_0) = \mathsf{sign}((D_{ijk}^{vw\rho})^2 + (D_{ijk}^{uw\rho})^2 + (D_{ijk}^{uv\rho})^2 - (D_{ijk}^{uvw})^2) \tag{2.33}$$
$$= \mathsf{sign}((E_{ijk}^{yzr})^2 + (E_{ijk}^{xzr})^2 + (E_{ijk}^{xyr})^2 - (E_{ijk}^{xyz})^2), \tag{2.34}$$

where M_2 is always positive unless $E_{ijk}^{xyp}, E_{ijk}^{xzp}$ and E_{ijk}^{yzp} are all zero; in this case $M_2 = 0$. The expressions that appear in the evaluation of M_2, M_1 and M_0 have maximum algebraic degree 7 in the input quantities.

First, we shall consider the case $M_2 \neq 0$; this is geometrically equivalent to the non-collinearity of the inverted centers $C_n^\star$ for $n \in \{i, j, k\}$. Since M_2 is assumed to be strictly positive, the quadratic polynomial $M(d)$ has 0, 1 or 2 real roots depending on whether the sign of the discriminant $\Delta_M = M_1^2 - 4M_2M_0$ is negative, zero or positive respectively. We evaluate the discriminant Δ_M of $M(d)$ to be

$$\Delta_M = 4(D_{ijk}^{uvw})^2\big((D_{ijk}^{uv})^2 + (D_{ijk}^{uw})^2 + (D_{ijk}^{vw})^2 - (D_{ijk}^{u\rho})^2 - (D_{ijk}^{v\rho})^2 - (D_{ijk}^{w\rho})^2\big), \tag{2.35}$$

hence determining $\mathsf{sign}(\Delta_M)$ requires 8-fold algebraic operations since

$$\mathsf{sign}(\Delta_M) = \mathsf{sign}\left((E_{ijk}^{xyp})^2 + (E_{ijk}^{xzp})^2 + (E_{ijk}^{yzp})^2 - (E_{ijk}^{xrp})^2 - (E_{ijk}^{yrp})^2 - (E_{ijk}^{zrp})^2\right). \tag{2.36}$$

If $\Delta_M < 0$, the predicate returns "0", whereas if $\Delta_M = 0$ we get that $M(d)$ has a double root $d = -M_1/(2M_2)$; the predicate returns "1 double" if $\mathsf{sign}(d) = -\mathsf{sign}(M_1)$ is positive, otherwise it returns "0". Finally, if $\Delta_M > 0$, $M(d)$ has two distinct roots $d_1 < d_2$, whose sign we check for positiveness. Using Vieta's rules and since $\mathsf{sign}(M_1) = -\mathsf{sign}(d_1 + d_2)$ and $\mathsf{sign}(M_0) = \mathsf{sign}(d_1)\mathsf{sign}(d_2)$ we conclude that, if M_0 is negative the predicate returns "1". Otherwise, the predicate's outcome is "0" if $\mathsf{sign}(M_1)$ is positive or "2" if $\mathsf{sign}(M_1)$ is negative.

We now analyse the case where $M_2 = 0$. Since the quantities $D_{ijk}^{uv}, D_{ijk}^{uw}$ and D_{ijk}^{vw} are all zero, we get that $a = D_{ijk}^{vw\rho}/D_{ijk}^{uvw}$ $b = D_{ijk}^{uw\rho}/D_{ijk}^{uvw}$ and $c = D_{ijk}^{uv\rho}/D_{ijk}^{uvw}$. Therefore, since $d = \rho_i - (au_i + bv_i + cw_i)$ we can evaluate the sign of d as

$$\mathsf{sign}(d) = \mathsf{sign}(D_{ijk}^{uvw})\mathsf{sign}(D_{ijk}^{uvw}\rho_i - D_{ijk}^{vw\rho}u_i - D_{ijk}^{uw\rho}v_i - D_{ijk}^{uv\rho}w_i) \tag{2.37}$$

$$= \mathsf{sign}(E_{ijk}^{xyz})\mathsf{sign}(E_{ijk}^{xyz}r_i^\star - E_{ijk}^{yzr}x_i^\star - E_{ijk}^{xzr}y_i^\star - E_{ijk}^{xyr}z_i^\star) \tag{2.38}$$

The evaluation of the sign of d therefore demands operations of algebraic degree 4 (in the input quantities). If $d > 0$ the predicate returns "1" otherwise it returns "0".

Last, we consider the case $D_{ijk}^{uvw} = 0$. We can safely assume that at least one of the quantities $D_{ijk}^{uv}, D_{ijk}^{uw}$ and D_{ijk}^{vw} is non-zero since otherwise the centers $C_n^\star$ for $n \in \{i, j, k\}$ would be collinear [1] ,yielding a contradiction. Assume without loss of generality that $D^{uv} \neq 0$, we can solve the system of equations in terms of c and we get that

$$a = \frac{-D_{ijk}^{v\rho} + cD_{ijk}^{vw}}{D_{ijk}^{uv}}, \quad b = \frac{D_{ijk}^{u\rho} - cD_{ijk}^{uw}}{D_{ijk}^{uv}}, \quad d = \frac{D_{ijk}^{uv\rho}}{D_{ijk}^{uv}}. \tag{2.39}$$

If we substitute a and b in the equation $a^2 + b^2 + c^2 = 1$, we get that c is a root of a quadratic polynomial $L(c) = L_2c^2 + L_1c + L_0$, where

[1] If $D_{ijk}^{uv} = D_{ijk}^{uw} = D_{ijk}^{vw} = 0$, the projections of the points $C_n^\star$, $n \in \{i, j, k\}$, on all three planes $w = 1$, $v = 1$ and $u = 1$ would form a flat triangle. For each projection, this is equivalent to either some of the projection points coinciding or all three being collinear. Since the original centers $C_n^\star$ are distinct points for $n \in \{i, j, k\}$, they must be collinear for such a geometric property to hold.

$$L_2 = (D_{ijk}^{uv})^2 + (D_{ijk}^{uw})^2 + (D_{ijk}^{vw})^2, \tag{2.40}$$
$$L_1 = -2(D_{ijk}^{v\rho} D_{ijk}^{vw} + D_{ijk}^{u\rho} D_{ijk}^{uw}), \tag{2.41}$$
$$L_0 = (D_{ijk}^{u\rho})^2 + (D_{ijk}^{v\rho})^2 - (D_{ijk}^{uv})^2. \tag{2.42}$$

We evaluate the discriminant of $L(c)$ to be

$$\Delta_L = 4(D_{ijk}^{xy})^2 \left[(D_{ijk}^{xy})^2 + (D_{ijk}^{xz})^2 + (D_{ijk}^{yz})^2 - (D_{ijk}^{xr})^2 - (D_{ijk}^{yr})^2 - (D_{ijk}^{zy})^2\right], \tag{2.43}$$

and therefore $\mathsf{sign}(\Delta_L) = \mathsf{sign}(\Delta_M)$. The evaluation of $\mathsf{sign}(\Delta_M)$ is known to require 8-fold algebraic operations as shown in a previous case.

If $\Delta_M < 0$, the predicate returns "0"; there is no tangent plane in the inverted space. Otherwise, we determine the sign of d to be

$$\mathsf{sign}(d) = \mathsf{sign}\big(D_{ijk}^{uv\rho}/D_{ijk}^{uv}\big) = \mathsf{sign}\big(E_{ijk}^{xyr}/E_{ijk}^{xyp}\big) = \mathsf{sign}(E_{ijk}^{xyr})\mathsf{sign}(E_{ijk}^{xyp}) \tag{2.44}$$

and the following cases arise

1. If $\Delta_M = 0$, the predicate returns "0" if $d < 0$ or "1" if $d > 0$.
2. If $\Delta_M > 0$, the predicate returns "0" if $d < 0$ or "2" if $d > 0$.

Note that in the last case, the algebraic degrees of Δ_M and d are 8 and 4 respectively and that, in every possible scenario, 8 was the maximum algebraic degree of any quantity we had to evaluate.

Lemma 9 *The* EXISTENCE *predicate can be evaluated by determining the sign of quantities of algebraic degree at most 8 (in the input quantities).*

2.7.4 The DISTANCE predicate

In this section, we provide a detailed analysis regarding the evaluation of the DISTANCE(S_i, S_j, S_k, S_a) predicate, as this was defined in Section 2.4.3. As stated there, the outcome of this primitive is the tuple $(\mathsf{sign}(\delta(S_a, \Pi_{ijk}^-)), \mathsf{sign}(\delta(S_a, \Pi_{ijk}^+)))$, where the planes Π_{ijk}^- and Π_{ijk}^+ are commonly tangent to the sites S_i, S_j and S_k. The existence of these planes is guaranteed since the trisector τ_{ijk} is assumed to be "hyperbolic" (see Section 2.4.2). Also take into consideration that in the scope of this chapter, this subpredicate never returns a degenerate answer, i.e., neither $(\mathsf{sign}(\delta(S_a, \Pi_{ijk}^-))$ nor $\mathsf{sign}(\delta(S_a, \Pi_{ijk}^+)))$ can equal zero.

We shall now consider such a plane $\Pi : ax + by + cz + d = 0$ tangent to all sites S_i, S_j and S_k, that leaves them all on the same side (this site is denoted as the *positive* side). If we assume without loss of generality that $a^2 + b^2 + c^2 = 1$, it must stand that $\delta(C_n, \Pi) = ax_n + by_n + cz_n + d = r_n$ for $n \in \{i, j, k\}$, where $\delta(C_n, \Pi)$ denotes the signed Euclidean of the center C_n from the plane Π. If we consider the distance $\epsilon = \delta(S_a, \Pi) = \delta(C_a, \Pi) - r_a$ of the sphere S_a from this plane, then the following system of equations must hold.

$$ax_i + by_i + cz_i + d = r_i, \tag{2.45}$$
$$ax_j + by_j + cz_j + d = r_j, \tag{2.46}$$
$$ax_k + by_k + cz_k + d = r_k, \tag{2.47}$$
$$ax_a + by_a + cz_a + d = r_a + \epsilon, \tag{2.48}$$
$$a^2 + b^2 + c^2 = 1 \tag{2.49}$$

Due to the initial assumption of a hyperbolic trisector, only two such planes, Π^-_{ijk} and Π^+_{ijk}, are cotangent to the spheres S_i, S_j and S_k and therefore algebraicly satisfy the system of equations above. In other words, there exist only two distinct algebraic solutions $(a_\nu, b_\nu, c_\nu, d_\nu, \epsilon_\nu)$ for $\nu \in \{1, 2\}$, and apparently $\{\epsilon_1, \epsilon_2\} = \{\delta^+_a, \delta^-_a\}$, where $\delta^+_a = \delta(S_a, \Pi^+_{ijk})$ and $\delta^-_a = \delta(S_a, \Pi^-_{ijk})$.

Bearing in mind that the answer to the Distance predicate is actually the tuple (δ^+_a, δ^-_a), we want to determine the signs of ϵ_1 and ϵ_2 and correspond them to δ^+_a and δ^-_a. Regarding the signs of $\{\epsilon_1, \epsilon_2\}$ we algebraicly study the system of equations above.

First, we consider the case $D^{xyz}_{ijka} \neq 0$ in detail. Under this assumption and with the use of Crammer's rule, we may express a, b and c with respect to ϵ as

$$a = \frac{D^{yzr}_{ijka} - \epsilon D^{yz}_{ijk}}{D^{xyz}_{ijka}}, \quad b = \frac{-D^{xzr}_{ijka} + \epsilon D^{xz}_{ijk}}{D^{xyz}_{ijka}}, \quad c = \frac{D^{xyr}_{ijka} + \epsilon D^{xy}_{ijk}}{D^{xyz}_{ijka}}. \tag{2.50}$$

After substituting these expressions in the equation $a^2 + b^2 + c^2 = 1$, we obtain that $\Lambda(\epsilon) = \Lambda_2\epsilon^2 + \Lambda_1\epsilon + \Lambda_0 = 0$,

$$\Lambda_2 = (D^{xy}_{ijk})^2 + (D^{yz}_{ijk})^2 + (D^{xz}_{ijk})^2, \tag{2.51}$$
$$\Lambda_1 = -2(D^{yzr}_{ijka}D^{yz}_{ijk} + D^{xzr}_{ijka}D^{xz}_{ijk} - D^{xyr}_{ijka}D^{xy}_{ijk}), \tag{2.52}$$
$$\Lambda_0 = (D^{xyr}_{ijka})^2 + (D^{xzr}_{ijka})^2 + (D^{yzr}_{ijka})^2 - (D^{xyz}_{ijka})^2. \tag{2.53}$$

Take into consideration that Λ_2 cannot be zero since otherwise the centers C_i, C_j and C_k would be collinear, yielding a contradiction; we have assumed that τ_{ijk} is hyperbolic. Since $\Lambda(\epsilon)$, which is definitely a quadratic in terms of ϵ, has the aforementioned ϵ_1 and ϵ_1 as roots, we may use Vieta's formula to determine the signs of ϵ_1, ϵ_2. All we need is the signs of Λ_1 and Λ_0, quantities of algebraic degree 5 and 6 respectively (we already proved Λ_2 is positive).

If both roots are positive, negative or zero, the predicate returns $(+, +)$, $(-, -)$ or $(0, 0)$ respectively. If the sign of the roots differ, we must consider a way of distinguishing which of them corresponds to δ_a^+ and δ_a^-. Since the ϵ_1 and ϵ_2 have different signs, the set $\{\epsilon_1, \epsilon_2\} = \{\delta_a^-, \delta_a^+\}$ is one of the following: $\{+, -\}, \{0, -\}$ or $\{0, +\}$.

The cases we now consider are products of geometric observations based on the three possible configurations of the centers C_i, C_j, C_k and C_a:

- If D_{ijka}^{xyz} is positive, then C_a lies on the positive side of the plane Π_{ijk}. In this case, only the geometric configurations $(\delta_a^-, \delta_a^+) = (+, -), (0, -)$ or $(+, 0)$ are possible.
- If D_{ijka}^{xyz} is negative, then C_a lies on the negative side of the plane Π_{ijk}. In this case, only the geometric configurations $(\delta_a^-, \delta_a^+) = (-, +), (-, 0)$ or $(0, +)$ are possible.

For example, if the roots of $\Lambda(\epsilon)$ turn out to be one positive and one negative, the predicate will return $(+, -)$ if D_{ijka}^{xyz} is positive or $(-, +)$ if D_{ijka}^{xyz} is negative.

Lastly, we consider the case $D_{ijka}^{xyz} = 0$, where the centers C_n, for $n \in \{i, j, k, a\}$, are coplanar. In this context, it is both algebraicly and geometricaly apparent that $\epsilon_1 = \epsilon_2 = \epsilon$; the sphere S_a either intersects, is tangent or does not intersect both planes Π_{ijk}^- and Π_{ijk}^+. Specifically, if $D_{ijk}^{xyz} \neq 0$, then $\epsilon = -D_{ijka}^{xyzr}/D_{ijk}^{xyz}$ and we immediately evaluate $\mathsf{sign}(\epsilon) = -\mathsf{sign}(D_{ijka}^{xyzr}) \cdot \mathsf{sign}(D_{ijk}^{xyz})$. If $D_{ijk}^{xyz} = 0$ then at least one of the quantities $D_{ijk}^{xy}, D_{ijk}^{xz}, D_{ijk}^{yz}$ does not equal zero, since the centers C_n for $n \in \{i, j, k\}$ are not collinear. Assume without loss of generality that $D_{ijk}^{xy} \neq 0$, then $\epsilon = D_{ijka}^{xyr}/D_{ijk}^{xy}$ and we evaluate $\mathsf{sign}(\epsilon) = \mathsf{sign}(D_{ijka}^{xyr})\mathsf{sign}(D_{ijk}^{xy})$. In every case the predicate returns $(\mathsf{sign}(\epsilon), \mathsf{sign}(\epsilon))$.

Taking into consideration that the evaluation of Λ_1 is the most degree-demanding operation to resolve the Distance predicate, we have proven the following lemma.

Lemma 10 *The* Distance *predicate can be evaluated by determining the sign of quantities of algebraic degree at most 6 (in the input quantities).*

2.7.5 The Shadow Predicate

In this section, we provide a way of resolving the Shadow $(S_i, S_j, S_k, S_\alpha)$ predicate as this was described in Section 2.4.5. Assuming that the trisector τ_{ijk} is "hyperbolic" and no degeneracies occur, we have shown that the outcome, which is the topological structure of $\mathcal{SR}(S_a)$ on τ_{ijk}, is one of the following: $\emptyset$, $(-\infty, \infty) = \mathbb{R}$, $(-\infty, \phi)$, $(\chi, +\infty)$, (χ, ϕ), or $(-\infty, \phi) \cup (\chi, +\infty)$, where $\phi, \chi \neq \pm\infty$.

Initially, the predicates Existence$(S_i, S_j, S_k, S_\alpha)$ and Distance$(S_i, S_j, S_k, S_\alpha)$ are called; let us denote by E and (σ_1, σ_2) their respective outcomes. The geometric interpretation of these two quantities leads to the resolution of the shadow region $\mathcal{SR}(S_\alpha)$ with respect to τ_{ijk}.

Regarding the meaning of the signs σ_1 and σ_2, assume that $\sigma_1 = +$. In this case, the sphere S_α does not intersect the plane Π_{ijk}^- which is in fact the Apollonius sphere $\mathcal{T}(\zeta^{-1}(-\infty))$. In other words, $\zeta^{-1}(-\infty)$ does not belong to $\mathcal{SR}(S_\alpha)$, i.e., $-\infty$ does not "show up" in the predicate's outcome. Using similar arguments, if $\sigma_2 = +$ then $+\infty$ does not show up in the outcome whereas, and if σ_1 or σ_2 are negative then $-\infty$ or $+\infty$ shows up, respectively. Note that $\sigma_1, \sigma_2 \in \{+, -\}$ under the assumption of no degeneracies.

Regarding the geometric interpretation of E, we have mentioned that the boundary points of the closure of $\mathcal{SR}(S_\alpha)$ correspond to centers of finite Apollonius spheres of the sites S_n, for $n \in \{i, j, k, \alpha\}$ that are not centered at infinity. We have shown in Section 2.7.3 that the cardinality of these Apollonius spheres is in fact E and assuming no degeneracies there are either 0, 1 or 2.

The combined information of σ_1, σ_2 and E is used to determine the type of $\mathcal{SR}(S_\alpha)$, as follows:

- If $E = 0$, the shadow region $\mathcal{SR}(S_\alpha)$ has no boundary points, hence it's type is either $(-\infty, +\infty)$ or $\emptyset$. If $\sigma_1 = -$ (or $\sigma_2 = -$), the predicate returns $(-\infty, +\infty)$ otherwise, if $\sigma_1 = +$ (or $\sigma_2 = +$), it returns $\emptyset$.

- If $E = 1$, then the closure of $\mathcal{SR}(S_\alpha)$ has one finite boundary point, hence it's type is either $(-\infty, \phi)$ or $(\chi, +\infty)$. If $\sigma_1 = -$ (or $\sigma_2 = +$), the predicate returns $(-\infty, \phi)$ otherwise, if $\sigma_1 = +$ (or $\sigma_2 = -$) it returns $(\chi, +\infty)$).

- Finally, if $E = 2$ then the closure of $\mathcal{SR}(S_\alpha)$ has two finite boundary points, hence it's type is either (χ, ϕ) or $(-\infty, \phi) \cup (\chi, +\infty)$. If $\sigma_1 = +$ (or $\sigma_2 = +$) the predicate returns (χ, ϕ) otherwise, if $\sigma_1 = -$ (or $\sigma_2 = -$), it returns $(-\infty, \phi) \cup (\chi, +\infty)$.

Not that all other combinations of the outcomes of the Existence and Distance predicates correspond to degenerate shadow regions; these are handled in detail in Section 4.3.

Since the evaluation of the Shadow predicate only requires the call of the Distance and the Existence predicates, which demand operations of maximum algebraic degree 6 and 8 respectively, we proved the following lemma.

Lemma 11 *The* Shadow *predicate can be evaluated by determining the sign of quantities of algebraic degree at most 8 (in the input quantities).*

2.7.6 The Order predicate

The major subpredicate called during the evaluation of the EdgeConflict predicate via the algorithm described in Section 2.5 is the so called Order predicate. As already described in Section 2.4.6, the Order $(S_i, S_j, S_k, S_a, S_b)$ returns the order of appearance on the oriented trisector τ_{ijk} of any Apollonius vertices defined by the sites S_i, S_j, S_k and S_a or S_b. This primitive needs to be evaluated only if both $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ are not of the form $\emptyset$ or $(-\infty, +\infty)$, with $\{a, b\} \in \{l, m, q\}$. Indeed, for $a \in \{l, m\}$ we know that v_{ijkl} and v_{ikjm} exist and therefore $\mathcal{SR}(S_l)$ and $\mathcal{SR}(S_m)$ can not be of that type. Moreover, $\mathcal{SR}(S_q)$ cannot be of the form $\emptyset$ or $(-\infty, +\infty)$; the Order predicate is not necessary if this is the case. To conclude, the predicates Shadow (S_i, S_j, S_k, S_n) for $n \in \{a, b\}$ are called in advance and their outcomes are considered to be known for the rest of the analysis. As already mentioned, τ_{ijk} is also assumed to be a “hyperbolic” trisector.

The analysis of the Order predicate is the major contribution of this thesis. We demonstrate the usefulness of the inversion technique by proving the strong connection between the original and the inverted space. By exploiting this relation, we are able to create useful tools which can also be used in both 2D and 3D Apollonius diagrams to improve existing results.

In the rest of this Section, we introduce the inverted space and make some initial observations on how it is connected with the original space. Afterwards, we define a 2-dimensional sub-space of the inverted space, to make useful geometric observations effortless. Lastly, we break up our analysis of the Order predicate according to the shadow region types $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$. Ultimately, we prove the following lemma.

Lemma 12 *The* Order *predicate can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

The $\mathcal{W}$-space

The original space where the sites S_i, S_j, S_k, S_a and S_b lie is called the $\mathcal{Z}$-space. However, most of our analysis is carried in the inverted $\mathcal{W}$-space, as defined in Section 1.4 with a slight modification.

Specifically, observe that the definition of the $\mathcal{W}$-space depends on the choice of the sphere S_I. Since a cyclic permutation of the sites S_i, S_j and S_k does not alter the outcome of the Order predicate, we assume that $k < i, j$; otherwise we reorder the sites so it does, hence r_k is not larger than r_i and r_j. We now select to invert $\mathcal{Z}$-space "through the sphere S_k", i.e., we reduce the radii of all initial sites by r_k (and obtain $\mathcal{Z}^\star$-space) and then invert all points with C_k as the pole.

Notice that when we reduce the sites S_a or S_b by r_k we may end up with a sphere of negative radius if $r_a < r_k$ or $r_b < r_k$. Although the existence (or not) of spheres with negative radius in $\mathcal{W}$-space makes the geometric configurations quite different to handle, the algebraic methods we present here can handle both cases without modifications. For example, if a sphere with a negative radius is tangent to the positive side of a plane, then its center must lie on the negative side of the plane; geometrically, this is confusing but algebraicly there is no difference with the respective positive-radius scenario. For this reason, we shall assume for the rest of the Section that all sites lying in $\mathcal{W}$-space have positive radii.

The analysis that follows is based on the strong relation that holds between the geometric configuration of the sites S_i, S_j, S_k, S_a and S_b in $\mathcal{Z}$-space and the corresponding configuration of the inverted sites $S_i^\star, S_j^\star, S_a^\star$ and $S_b^\star$ in $\mathcal{W}$-space.

In $\mathcal{W}$-space, O denotes the point $(0, 0, 0)$ which is the image of the "point at infinity" of $\mathcal{Z}$-space. Given a point p that lies on τ_{ijk}, $\mathcal{T}(p)$ denotes the external Apollonius sphere tangent to the sites S_i, S_j and S_k, centered at p. Such a sphere $\mathcal{T}(p)$ in $\mathcal{Z}$-space corresponds to a plane in $\mathcal{W}$-space, denoted $\Pi^\star(p)$, that is tangent to the inverted sites $S_i^\star$ and $S_j^\star$ and therefore tangent to the cone defined by them. Notice that, since $S_i^\star$ and $S_j^\star$ are distinct spheres of $\mathcal{W}$-space due to their pre-images S_i and S_j also being distinct in $\mathcal{Z}$-space, the cone $\mathcal{K}(S_i^\star, S_j^\star)$ is well defined. For the rest of this Section, we define $\mathcal{K}^\star$ to be the semi cone (or cylinder if $\rho_i = \rho_j$) that contains $S_i^\star$ and $S_j^\star$.

Let us observe what happens in $\mathcal{W}$-space when we consider this point p moving on τ_{ijk} such that $\zeta(p)$ goes from $-\infty$ towards $+\infty$. The corresponding plane $\Pi^\star(p)$ rotates while remaining tangent to $\mathcal{K}^\star$, with starting and ending positions the planes denoted by $\Pi^\star(-\infty)$ and $\Pi^\star(+\infty)$ respectively.

It is obvious that these two planes correspond to the two Apollonius spheres of S_i, S_j and S_k "at infinity" , i.e., the planes $\{\Pi^-_{ijk}, \Pi^+_{ijk}\}$. These planes of $\mathcal{Z}$-space must be distinct in the case of a "hyperbolic" trisector τ_{ijk} as shown in Sections 2.7.1 and 2.7.4, and, as a result, their images in $\mathcal{W}$-space must also be distinct.

Moreover, each of $\Pi^\star(-\infty)$ and $\Pi^\star(+\infty)$ must go through O because the their pre-images are planes that go through the "point at infinity" in $\mathcal{Z}$-space. Combining these last two remarks, we conclude that the points $C_i^\star, C_j^\star$ and O are not collinear and O lies strictly outside the semi cone $\mathcal{K}^\star$. It is of great importance to understand that the last fact holds only because τ_{ijk} is "hyperbolic"; if we were studying the "elliptic" trisector type, O would lie strictly inside the semi cone $\mathcal{K}^\star$ and in the degenerate case of a "parabolic" trisector, O would lie on the boundary of $\mathcal{K}^\star$.

For every point $p \in \tau_{ijk} \backslash \{\pm\infty\}$, the sphere $\mathcal{T}(p)$ is an external Apollonius sphere and therefore does not contain the "point at infinity" in $\mathcal{Z}$-space. Correspondingly, its image in $\mathcal{W}$-space, i.e., the plane $\Pi^\star(p)$, must leave the point O and the centers of the spheres $S_i^\star$ and $S_j^\star$ on the same side. The side of the plane $\Pi^\star(p)$ that contains O is called *positive* whereas the other is referred to as *negative*.

Let us now consider the plane $\Pi^\star$ that goes through the points $C_i^\star, C_j^\star$ and the point O of $\mathcal{W}$-space. The well-definition of $\Pi^\star$ follows from the non-collinearity of the three points we proved earlier. This plane turns out to be the image of the plane Π_{ijk} that goes through the centers C_i, C_j and C_k (and apparently the point at infinity) in $\mathcal{Z}$-space.

The latter plane separates $\mathcal{Z}$-space into two half-spaces, $\mathcal{H}_+$ and $\mathcal{H}_-$, where $\mathcal{H}_+$ (resp., $\mathcal{H}_-$) denotes the set of points N such that Orient3D (N, C_i, C_j, C_k) is positive (resp., negative). The plane $\Pi^\star$ also separates $\mathcal{W}$-space into two half-spaces, $\mathcal{H}_+^\star$ and $\mathcal{H}_-^\star$, where $\mathcal{H}_+^\star$ (resp., $\mathcal{H}_-^\star$) denotes the set of points M such that Orient3D $(M, C_i^\star, C_j^\star, O)$ is positive (resp., negative).

If we now consider a point $C_n = (x_n, y_n, z_n)$ of $\mathcal{Z}$-space and its inversion image $C_n^\star$ in $\mathcal{Z}$-space,we can easily prove that

$$\textsc{Orient3D}(C_n^\star, C_i^\star, C_j^\star, O) = \mathsf{sign}(p_i^\star p_j^\star p_n^\star D^{uvw}_{nij}) = \mathsf{sign}(D^{xyz}_{nijk}) \tag{2.54}$$

$$= \textsc{Orient3D}(C_n, C_i, C_j, C_k). \tag{2.55}$$

since $C_n^\star = (u_n, v_n, w_n)$, where $(u_n, v_n, w_n) = (x_n^\star, y_n^\star, z_n^\star)(p_n^\star)^{-1}$ and

$$x_n^\star = x_n - x_k, \quad y_n^\star = y_n - y_k, \quad z_n^\star = z_n - z_k, \quad p_n^\star = (x_n^\star)^2 + (y_n^\star)^2 + (z_n^\star)^2. \tag{2.56}$$

This result indicates that $\mathcal{H}_+^\star$ and $\mathcal{H}_-^\star$ are in fact the inversion images of the open semi-spaces $\mathcal{H}_+$ and $\mathcal{H}_-$.

This simple correspondence of semi-spaces yields a remarkable result. If we consider a point $p \in \tau_{ijk}^+$ (resp., τ_{ijk}^-), then the tangency points of the Apollonius sphere $\mathcal{T}(p)$ with the spheres S_i and S_j must lie on $\mathcal{H}_+$ (resp., $\mathcal{H}_-$). If this fact is considered through inversion, the tangency points of the plane $\Pi^\star(p)$ with the semi cone $\mathcal{K}^\star$ must lie on $\mathcal{H}_+^\star$ (resp., $\mathcal{H}_-^\star$). Furthermore, if we let p move on τ_{ijk}^+ (resp., τ_{ijk}^-) such that $\zeta(p)$ goes to $+\infty$ (resp., $-\infty$), we can deduce that the plane $\Pi^\star(+\infty)$ (resp., $\Pi^\star(-\infty)$) goes through the point O and its tangency points with the spheres $S_i^\star$ and $S_j^\star$ lie on $\mathcal{H}_+^\star$ (resp., $\mathcal{H}_-^\star$).

The $\mathcal{Y}$-space

All these observations are indicative of the strong connection of the original and and the inverted space. However, since a three-dimensional space such as the $\mathcal{W}$-space makes observations and case breakdowns too complex, we will now consider a sub-space to carry out our analysis. For this reason, we consider a (random) plane $\Pi^\perp$ in $\mathcal{W}$-space that is perpendicular to the axis of the semi-cone $\mathcal{K}^\star$ at point $\hat{\mathcal{A}}$ and intersects it at a full circle C'; the intersection of $\mathcal{W}$-space and $\Pi^\perp$ is called the $\mathcal{Y}$-space. Such a plane $\Pi^\perp$ exist both when $\rho_i^\star \leq \rho_j^\star$ and $\rho_i^\star > \rho_j^\star$ and therefore $\mathcal{Y}$-space is well-defined in both cases. Notice that in every figure representing the $\mathcal{Y}$-space, we always depict the $\Pi^\perp$ plane such that the vector $\overrightarrow{C_i^\star C_j^\star}$ points "towards" the reader (see Figure 2.9).

In $\mathcal{Y}$-space, we will use the following notation:

- $\hat{\ell}(p), \hat{\ell}(\pm\infty)$ and $\hat{\ell}(o_{ijk})$ denote the intersection of the plane $\Pi^\perp$ with the planes $\Pi^\star(p), \Pi^\star(\pm\infty)$ and $\Pi^\star(o_{ijk})$ respectively.

- $\hat{\eta}, \hat{\theta}, \hat{o}$ and $\hat{p}$ denote the points of tangency of C' and the lines $\hat{\ell}(-\infty)$, $\hat{\ell}(+\infty)$, $\hat{\ell}(o_{ijk})$ and $\hat{\ell}(p)$ respectively (for $p \in \tau_{ijk}$).

- The intersection of $\mathcal{H}_+^\star$ (resp., $\mathcal{H}_-^\star$) with the $\Pi^\perp$ plane is called the positive (resp., negative) half-plane $\hat{\mathcal{H}}_+$ (resp., $\hat{\mathcal{H}}_-$).

- The positive (resp., negative) side of the line $\hat{\ell}(p)$ for a point $p \in \tau_{ijk}$ to be the side that contains (resp., does not contain) the point $\hat{\mathcal{A}}$.

- $\hat{O}$ denotes the point of intersection of the lines $\hat{\ell}(-\infty)$ and $\hat{\ell}(+\infty)$.

We shall now define an equivalency relation between the trisector τ_{ijk} and an arc of C', which is the biggest idea upon which the rest of our analysis is based. If a point p moves on τ_{ijk} such that $\zeta(p)$ goes from $-\infty$ to $+\infty$ then, in $\mathcal{Y}$-space, the corresponding point $\hat{p}$ moves on C' from the point $\hat{\eta}$ to the point $\hat{\theta}$, going through the point $\hat{o}$. Observe that there is a 1-1 correspondence between the oriented trisector τ_{ijk} and the oriented arc $(\hat{\eta}\hat{o}\hat{\theta})$. We denote this 1-1 and onto mapping from τ_{ijk} to the arc $(\hat{\eta}\hat{o}\hat{\theta})$ by $\psi(\cdot)$, such that $\psi(p) = \hat{p}$.

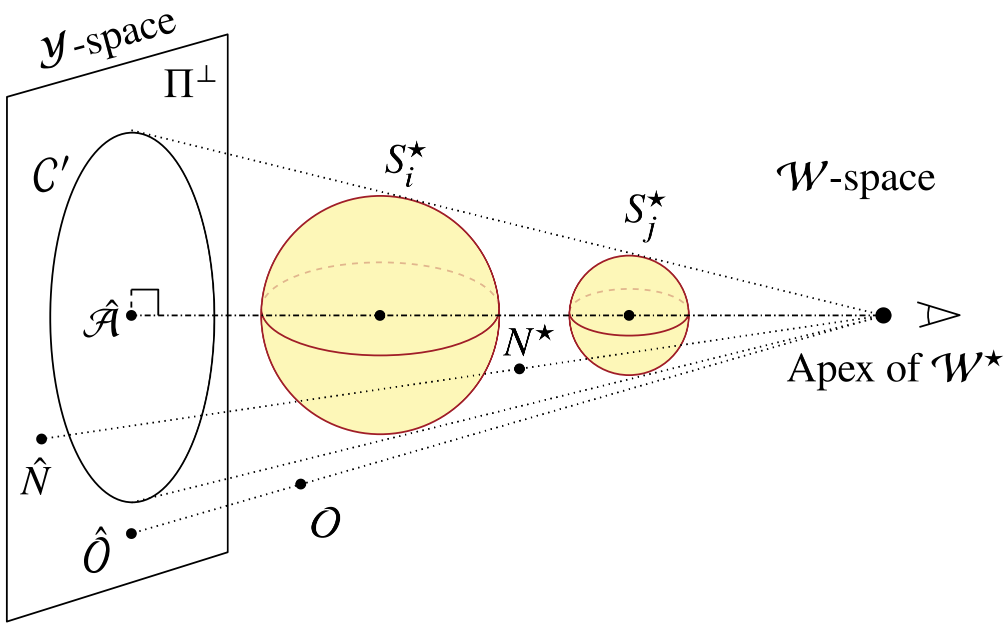

FIGURE 2.9: The $\mathcal{Y}$-space, where most of the analysis of the ORDER predicate is carried on, is in essence a projection of $\mathcal{W}$-space to a plane $\Pi^{\perp}$ via the apex of the cone $\mathcal{K}^\star$. Since $\mathcal{Y}$-space is a 2-dimensional space, an observation regarding a $\mathcal{W}$-space geometric configuration is made easier if we consider the corresponding configuration in $\mathcal{Y}$-space.

What naturally follows is that the order of appearance of the vertices v_{ijka}, v_{ikja}, v_{ijkb} and v_{ikjb} on the oriented trisector amounts to the order of appearance of the points $\psi(v_{ijka})$, $\psi(v_{ikja})$, $\psi(v_{ijkb})$ and $\psi(v_{ikjb})$ on the oriented arc $(\psi(\eta), \psi(o), \psi(\theta))$ (see Figure 2.10). Consider, however, that we only need to order the Apollonius vertices that actually exist among $v_{ijka}, v_{ikja}, v_{ijkb}$ and v_{ikjb}.

Lemma 13 *There is a 1-1 correspondence between the order of appearance of the existing vertices among $v_{ijka}, v_{ikja}, v_{ijkb}$ and v_{ikjb} on the oriented hyperbolic trisector τ_{ijk} and the order of appearance of the existing points $\psi(v_{ijka})$, $\psi(v_{ikja})$, $\psi(v_{ijkb})$ and $\psi(v_{ikjb})$ on the oriented arc $(\psi(\eta), \psi(o), \psi(\theta))$.*

The lemma suggests that the outcome of the Order predicate could return the order of appearance of the images of the aforementioned Apollonius vertices on the arc $(\psi(\eta), \psi(o), \psi(\theta))$ instead of the order of the original vertices on the trisector τ_{ijk}.

Towards our goal of obtaining the ordering of the inverted Apollonius vertices, we denote the circle $\hat{S_n} \in \mathcal{Y}$-space for $n \in \{a, b\}$ which will be considered as the image of $S_n^\star$ of $\mathcal{W}$-space. We need to define the image of these spheres in a proper way such that they carry their geometric properties from $\mathcal{W}$-space to $\mathcal{Y}$-space. For this reason, we consider the center $\hat{C_n}$ of $\hat{S_n}$ to be the intersection of $\Pi^\perp$ with the line that goes through the apex of the cone $\mathcal{K}^\star$ and the center $\hat{C_n}$. Observe that such a line is well defined since the latter two points cannot coincide; if they did then $\mathcal{SR}(S_n)$ would be $\emptyset$ or $\mathbb{R}$ yielding a contradiction [2]. Finally, the radius of $\hat{C_n}$ is such that $\hat{C_n}$ is tangent to each of the existing lines $\hat{\ell}(v_{ijkn})$ and $\hat{\ell}(v_{ikjn})$ (at least one of them exists due to $\mathcal{SR}(S_n)$ not being $\emptyset$ or $\mathbb{R}$).

Another crucial property we want to point out derives from the inversion mapping we used to go from $\mathcal{Z}^\star$-space to $\mathcal{W}$-space. The mapping $W(z)$ we used is known to be *inclusion preserving*, i.e., the relative position of two spheres in the original space is preserved in the inverted one. For example, consider a sphere S_μ that intersects the (existing) Apollonius sphere $\mathcal{T}(v_{ikjn})$ (resp., $\mathcal{T}(v_{ijkn})$), for $n \in \{a, b\}$ in $\mathcal{Z}$-space. After reducing both spheres by r_k, their images in $\mathcal{Z}^\star$ retain the same relative position, i.e., they intersect. Applying the inversion mapping, it must stand that the sphere $S_\mu^\star$ must intersect the negative side of $\Pi^\star(v_{ikjn})$ (resp., $\Pi^\star(v_{ijkn})$) since this half space is precisely the inversion image of the interior of $\mathcal{T}(v_{ikjn})$ (resp., $\mathcal{T}(v_{ijkn})$). Finally, if we consider this configuration in $\mathcal{Y}$-space, we deduce that $\hat{S_\mu}$ must intersect the negative side of $\hat{\ell}(v_{ikjn})$ (resp., $\hat{\ell}(v_{ijkn})$).

In a similar way we can show that if S_μ is tangent or does not intersect the Apollonius sphere $\mathcal{T}(v_{ikjn})$ (resp., $\mathcal{T}(v_{ijkn})$) then, in $\mathcal{Y}$-space, $\hat{S_\mu}$ is tangent to $\hat{\ell}(v_{ikjn})$ (resp., $\hat{\ell}(v_{ijkn})$) or does not intersect its negative side (see Figure 2.11). A fact tightly connected with these observations is that the relative position of S_μ and $\mathcal{T}(v_{ikjn})$ (resp., $\mathcal{T}(v_{ijkn})$) is provided by the InSphere predicate. Specifically,

[2] In such a geometric configuration, there would not exist a plane in $\mathcal{W}$-space co-tangent to all spheres $S_i^\star$,$S_j^\star$ and $S_n^\star$. Equivalently, in $\mathcal{Z}$-space there would not exist an Apollonius sphere of the sites S_i,S_j,S_k and S_n hence $\mathcal{SR}(S_n)$ would be either $\emptyset$ or $\mathbb{R}$ based on the analysis of Section 2.7.5. An equivalent arguements is that, if the apex of $\mathcal{K}^\star$ and $\hat{C_n}$ were coinciding then this would mean that S_n either contains or is contained in S_k, which is ruled out.

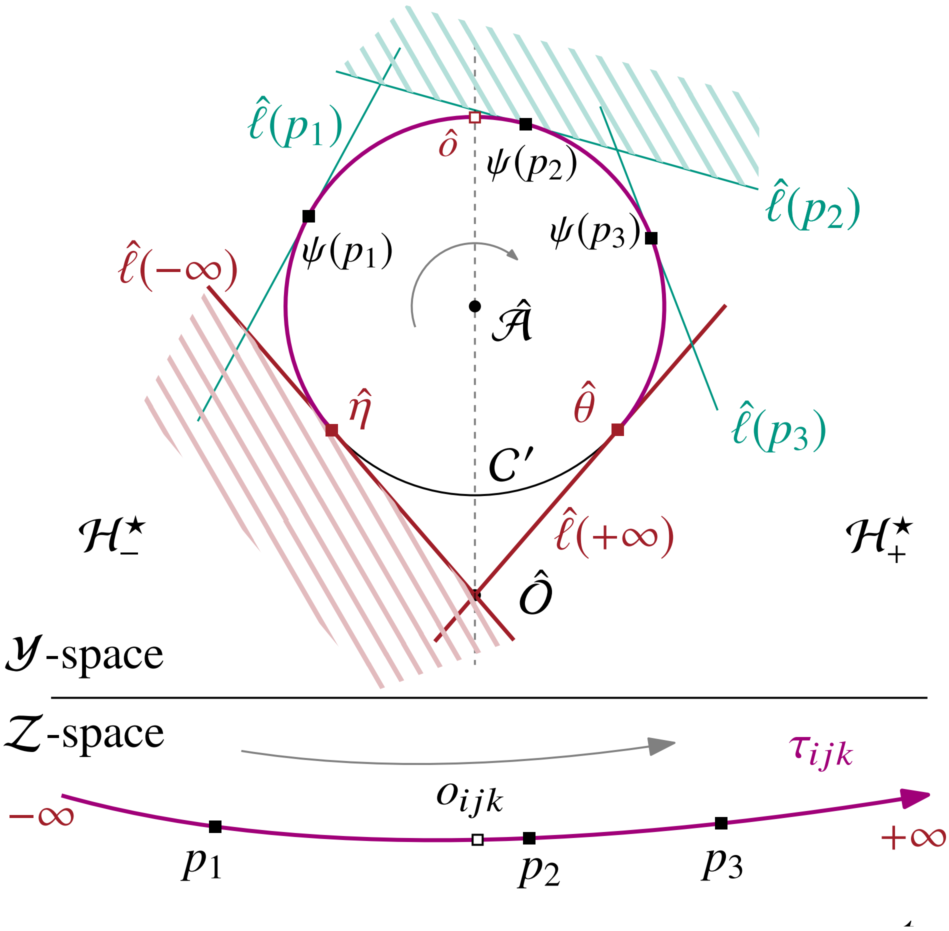

FIGURE 2.10: There is a strong relation between the $\mathcal{Z}$-space and the $\mathcal{Y}$-space. When the point p_i of $\mathcal{Z}$-space traverses the trisector τ_{ijk} towards its positive direction, the corresponding point $\psi(p_i)$ of $\mathcal{Z}$-space traverses the arc $(\psi(\eta), \psi(o), \psi(\theta))$. At the same time, $\hat{\ell}(p_i)$ rotates in $\mathcal{Y}$-space, remaining tangent to C', with starting and ending positions the lines $\hat{\ell}(-\infty)$ and $\hat{\ell}(+\infty)$ respectively. The red and green area symbolize the negative side of the lines $\hat{\ell}(-\infty)$ and $\hat{\ell}(p_2)$, respectively.

- if InSphere $(S_i, S_k, S_j, S_n, S_m)$ is $-$, 0 or $+$ then S_m intersects, is tangent to or does not intersect the Apollonius sphere $\mathcal{T}(v_{ikjn})$ and,
- if InSphere $(S_i, S_j, S_k, S_n, S_m)$ is $-$, 0 or $+$ then S_m intersects, is tangent to or does not intersect the Apollonius sphere $\mathcal{T}(v_{ijkn})$.

Since these InSphere predicates can be evaluated as shown in Section 2.4.1, the relative position of $\hat{S_\mu}$ with respect to any of the existing lines $\hat{\ell}(v_{ikjn})$ and $\hat{\ell}(v_{ijkn})$ can be determined in $\mathcal{Y}$-space, for $n \in \{a, b\}$.

Lemma 14 *The circle $\hat{S_m}$ intersects, is tangent to or does not intersect the negative side of $\hat{\ell}(v_{ikjn})$ (resp., $\hat{\ell}(v_{ijkn})$) if and only if the* InSphere *predicate with input (S_i,S_k,S_j,S_n,S_m) (resp., (S_i,S_j,S_k,S_n, S_m)) is negative, zero or positive respectively.*

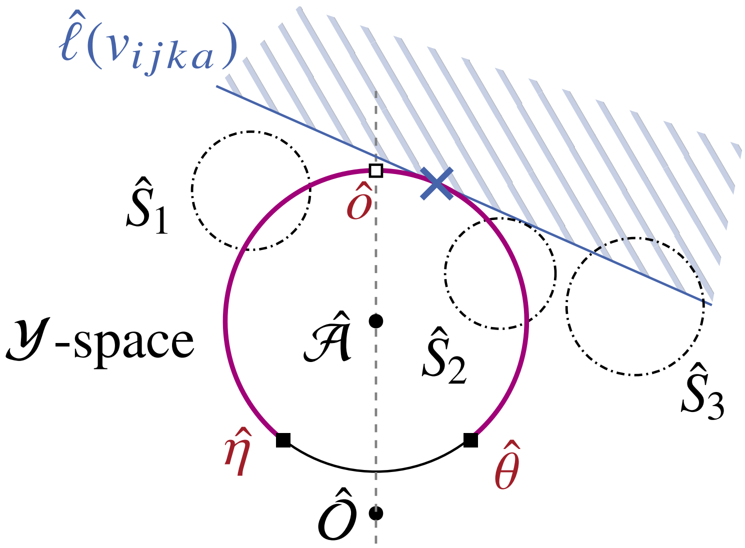

Figure 2.11: As Lemma 14 suggests, the InSphere predicate with input $(S_i, S_j, S_k, S_a, S_n)$ is positive, zero or negative for $n = 0, 1$ or 2 respectively.

The classic configuration

When the Order$(S_i, S_j, S_k, S_a, S_b)$ predicate is called, we initially determine the shadow region types of $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ via the appropriate Shadow predicates. If the type of each shadow region is (χ, ϕ) or $(\chi, +\infty)$ (not necessary the same), we say that we are in *classic configuration*. In such a setup, we can distinguish simpler cases regarding the ordering the images of the Apollonius vertices on the oriented arc $(\hat{\eta}\hat{o}\hat{\theta})$.

We therefore break up the analysis the Order predicate depending on whether we are in a classic (Section 2.7.6) or non-classic configuration (Section 2.7.6), the latter being reduced to the former using various observations. Let us now study in more detail what kind of information derives from the fact that the sites S_a and S_b satisfy the conditions of a classic configuration.

Suppose S_n, for $n = a$ or b, is (χ, ϕ) and therefore, the endpoints $\{\chi, \phi\}$ must correspond to the two Apollonius vertices $\{\zeta(v_{ijkn}), \zeta(v_{ikjn})\}$ on the trisector τ_{ijk} based on the remarks of Section 2.4.5. Let us consider χ and ϕ as $\zeta(v_n)$ and $\zeta(v'_n)$ respectively, with $\{v_n, v'_n\} = \{v_{ijkn}, v_{ikjn}\}$. Then, for every $p \in \tau_{ijk}$ such that $v_n \prec p \prec v'_n$, the sphere $\mathcal{T}(p)$ must intersect S_n as this follows from the definition of $\mathcal{SR}(S_n)$.

If we consider this point p moving on τ_{ijk}, initialy starting from the left-endpoint position v_n, then we observe that the Apollonius sphere $\mathcal{T}(v_n)$ intersects S_n if we move its center infinitesimally towards the positive direction of τ_{ijk}. Taking a closer look at the tangency points T_i, T_j, T_k and T_n of $\mathcal{T}(v_n)$ with the spheres S_i, S_j, S_k and S_n respectively, and since the orientation of τ_{ijk} is based in such a way on the orientation of C_i, C_j and C_k, it must hold that T_n must lie with respect to the plane formed by T_i, T_j and T_k such that $T_iT_jT_kT_n$ be negative oriented. For that reason, v_n is in fact v_{ikjn} and subsequently v'_n is v_{ijkn}.

The same argument can be used to prove that if $\mathcal{SR}(S_n)$ is $(\chi, +\infty)$ then χ corresponds to $\zeta(v_{ikjn})$. If we apply a similar analysis in all shadow region types that contain a finite endpoint, it will lead to the following lemma.

Lemma 15 *If the type of the shadow region $\mathcal{SR}(S_n)$ of a sphere S_n on a hyperbolic trisector is one of the following: $(-\infty, \phi)$, $(\chi, +\infty)$, (χ, ϕ), or $(-\infty, \phi) \cup (\chi, +\infty)$ where $\phi, \chi \neq \pm\infty$, then $\chi \equiv \zeta(v_{ikjn})$ and $\phi \equiv \zeta(v_{ijkn})$.*

As the lemma suggest, in a classic configuration, for $n \in \{a, b\}$,

- if $\mathcal{SR}(S_n) = (\chi, \phi)$, then both v_{ijkn} and v_{ikjn} exist and $v_{ikjn} \prec v_{ijkn}$, whereas
- if $\mathcal{SR}(S_n) = (\chi, +\infty)$, then v_{ikjn} exists while v_{ijkn} does not.

An equally important result arises when pondering of the possible positions of the circle $\hat{S_n}$, for $n \in \{a, b\}$ with any of the existing lines $\hat{\ell}(v_{ikjn})$ and $\hat{\ell}(v_{ijkn})$. Firstly, let us consider the scenario where $\mathcal{SR}(S_n) = (\chi, \phi)$ and in consequence, both lines exist. In this case, both points $\psi(v_{ikjn})$ and $\psi(v_{ijkn})$ exist on the oriented arc such that $\psi(v_{ikjn}) \prec \psi(v_{ijkn})$, as this follows from all previous remarks. From the definition of $\mathcal{SR}(S_n) = (\chi, \phi)$, it derives as a result that for a point p on the trisector τ_{ijk} such that $v_{ikjn} \prec pv_{ijkn}$, the sphere $\mathcal{T}(p)$ intersects with S_n. Using the “inclusion preserving” argument, it must stand that, in $\mathcal{Y}$-space, $\hat{S_n}$ intersects with the negative side of $\hat{\ell}(p)$. Therefore, if $\hat{M}$ is the midpoint $\psi(v_{ikjn})$ and $\psi(v_{ijkn})$ on the arc $(\eta o \theta)$ and $\mathbb{V}$ denotes the open ray from $\hat{\mathcal{A}}$ towards $\hat{M}$, then the circle $\hat{S_n}$ must be centered at a point on $\mathbb{V}$, i.e., $\hat{C_n} \in \mathbb{V}$ (see Figure 2.12).

Lastly, let us examine the case where $\mathcal{SR}(S_n) = (\chi, +\infty)$. We begin by observing that S_n must intersect with Π^+_{ijk} due to the definition of the shadow region and this amounts, in $\mathcal{Y}$-space, to $\hat{S_n}$ intersecting the negative side of the line $\hat{\ell}(+infty)$. Moreover, following a similar analysis as in the case of $\mathcal{SR}(S_n) = (\chi, \phi)$, we come to the conclusion that $\hat{S_n}$ must

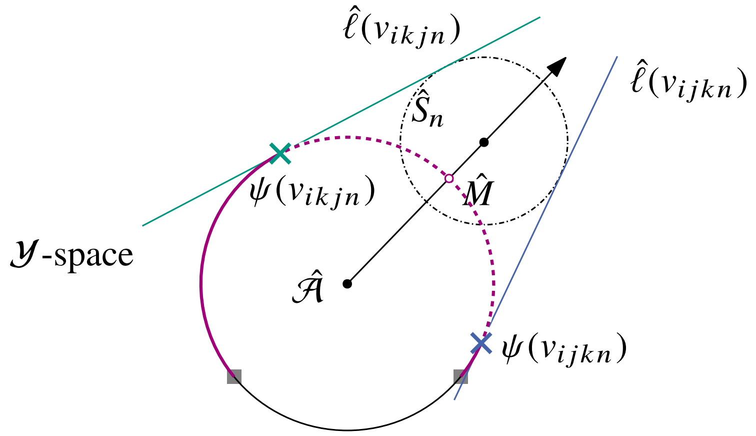

FIGURE 2.12: If both v_{ikjn}, v_{ijkn} exist and $\mathcal{SR}(S_a)$ is (χ, ϕ) with respect to the trisector τ_{ijk}, $\hat{C}_n$ must lie on the ray $(\hat{\mathcal{A}}, \hat{M})$. The dotted arc represent the image of $\mathcal{SR}(S_n)$ in $\mathcal{Y}$-space; it is obvious that, for any $\psi(p)$ in this arc, $\hat{S_n}$ intersects the negative side of the line $\hat{\ell}(p)$. As of Lemma 14, S_n must intersect the sphere $\mathcal{T}(p)$ which is equivalent to $p \in \mathcal{SR}(S_n)$.

be tangent to $\hat{\ell}(v_{ikjn})$ at a point $\hat{T}$ such that the (counterclowise) angle $(\hat{\mathcal{A}}, \psi(v_{ikjn}), \hat{T})$ is 90° (not 270°, see Figure 2.13). This fact must hold for the line $\hat{\ell}(p)$ to intersect $\hat{S_n}$, for every point $p \in \tau_{ijk}$ with $\psi(v_{ikjn}) \prec \psi(p) \prec \hat{\theta}$.

Ordering the Apollonius vertices in a classic configuration

In a classic configuration, we take for granted that, for $n \in \{a, b\}$, either $\mathcal{SR}(S_n) = (\chi, \phi)$ and therefore $v_{ikjn} \prec v_{ijkn}$ on the trisector τ_{ijk} or $\mathcal{SR}(S_n) = (\chi, +\infty)$ and only v_{ikjn} exists on τ_{ijk}. To order all of these existing Apollonius vertices, we break down our analysis into four sub-configurations.

Case A. All vertices $v_{ikja}, v_{ijka}, v_{ikjb}$ and v_{ijkb} exist, i.e., both $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ are of type (χ, ϕ).

Case B. Only the vertices v_{ikja} and v_{ikjb} exist, i.e., both $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ are of type $(\chi, +\infty)$.

Case C. Only the vertices v_{ikja}, v_{ijka} and v_{ikjb} exist, i.e., the type of $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ are (χ, ϕ) and $(\chi, +\infty)$ respectively.

Case D. Only the vertices v_{ikjb}, v_{ijkb} and v_{ikja} exist, i.e., the type of $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ are $(\chi, +\infty)$ and (χ, ϕ) respectively.

The last Case D is identical with the Case C if we name exchange the spheres S_a and S_b. Therefore, if Case D arises, we evaluate ORDER $(S_i, S_j, S_k, S_b, S_a)$ instead, which falls

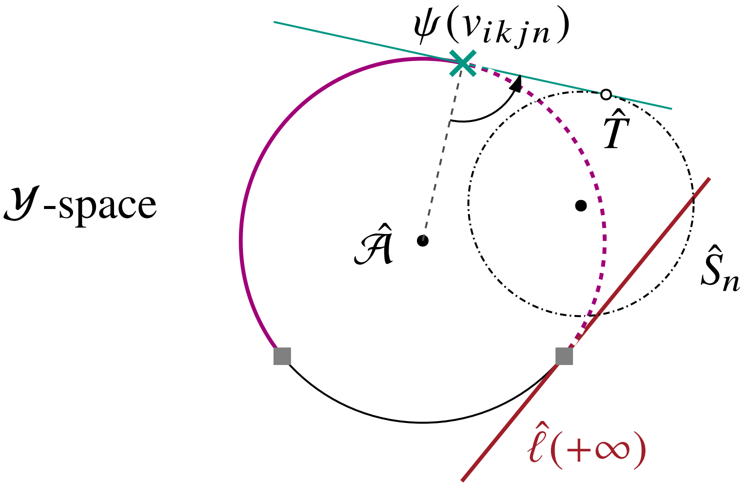

FIGURE 2.13: If v_{ikjn} exists and v_{ijkn} does not, the shadow region of S_n is known to be $(\chi, +\infty)$. This means that the tangency point $\hat{T}$ of $\hat{S_n}$ with the line $\hat{\ell}(v_{ikjn})$ is on the side of the line that forms a 90° angle with the vector $(\psi(v_{ikjn}), \hat{\mathcal{A}})$. The image of $\mathcal{SR}(S_n)$ in $\mathcal{Y}$-space in this case is the dotted arc which is indeed of the form $(\chi, +\infty)$.

in Case C, and return the resulting ordering of v_{ikjb}, v_{ijkb} and v_{ikja}. Consequently, we only need to consider the Cases A, B and C; the analysis of each configuration is deployed separately in the following sections.

Analysis of Case A Given that all Apollonius vertices $v_{ijka}, v_{ikja}, v_{ijkb}, v_{ikjb}$ exist on τ_{ijk} and $v_{ikja} \prec v_{ijka}$ as well as $v_{ikjb} \prec v_{ijkb}$, the list of all possible orderings (and thus ouctomes of the ORDER predicate) is the following

OrderCase 1. $v_{ikja} \prec v_{ijka} \prec v_{ikjb} \prec v_{ijkb}$,

OrderCase 2. $v_{ikja} \prec v_{ikjb} \prec v_{ijka} \prec v_{ijkb}$,

OrderCase 3. $v_{ikjb} \prec v_{ikja} \prec v_{ijka} \prec v_{ijkb}$,

OrderCase 4. $v_{ikjb} \prec v_{ikja} \prec v_{ijka} \prec v_{ijka}$,

OrderCase 5. $v_{ikjb} \prec v_{ijkb} \prec v_{ikja} \prec v_{ijka}$,

OrderCase 6. $v_{ikja} \prec v_{ikjb} \prec v_{ijkb} \prec v_{ijka}$.

Any of these ordering on the trisector is equivalent to the corresponding ordering of $\psi(v_{ikja}), \psi(v_{ikjb}), \psi(v_{ijkb})$ and $\psi(v_{ijka})$ on the arc $(\hat{\eta}, \hat{o}, \hat{\theta})$ as stated in Lemma 13.

We now study separately all these possible cases in $\mathcal{Y}$-space following the same approach. Firstly, we place the images of the Apollonius vertices on the arc according to the

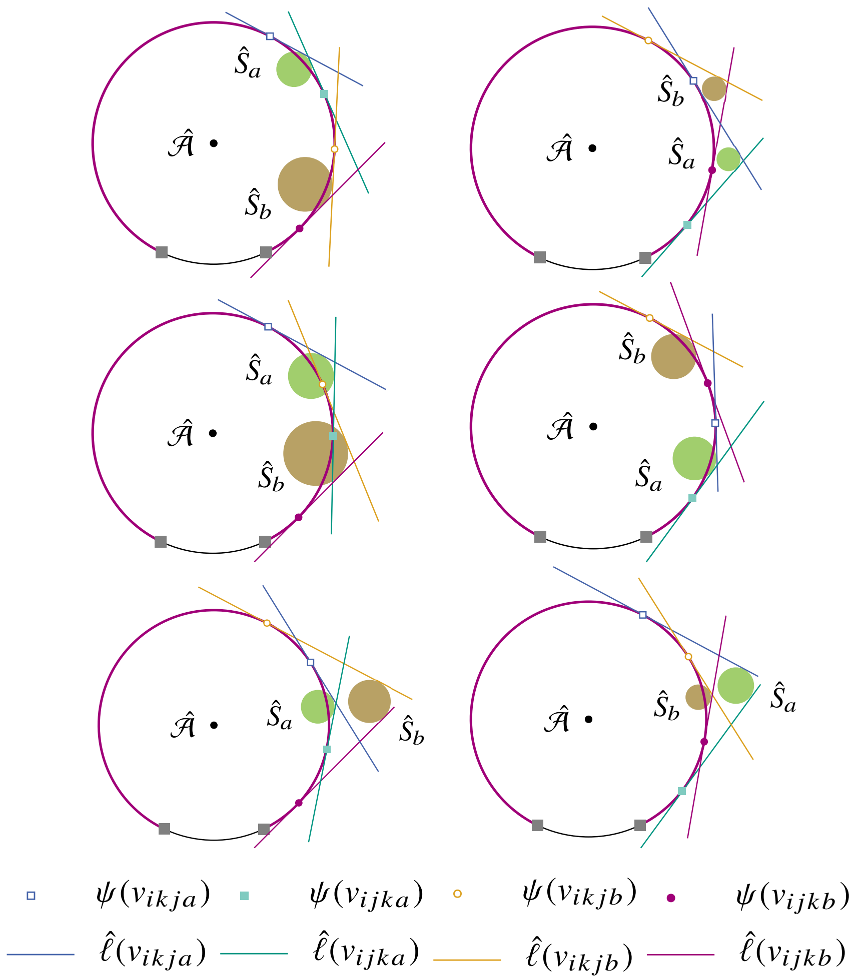

FIGURE 2.14: Under the assumption that all Apollonius vertices $v_{ikja}, v_{ijka}, v_{ikjb}$ and v_{ijkb} exist on the trisector τ_{ijk}, we consider all possible orderings of these vertices. As of Lemma 13, each of these orderings is equivalent to a respective ordering of the points $\psi(v_{ikja}), \psi(v_{ijka}), \psi(v_{ikjb})$ and $\psi(v_{ijkb})$ on the oriented arc $(\psi(\eta), \psi(o), \psi(\theta))$ of $\mathcal{Y}$-space. For every possible ordering a possible location of $\hat{S_a}$ and $\hat{S_b}$ is considered, such that the shadow regions $\mathcal{SR}(S_n)$ for $n \in \{a, b\}$ is of type (χ, ϕ) since we examine a classic configuration. From top to bottom, Left: OrderCase 1, 2, 3. From top to bottom, Right: OrderCase 4, 5, 6.

OrderCase we are examining. Then, we consider a possible location for each of the circles $\hat{S_a}$ and $\hat{S_b}$ taking into consideration the remarks made in the Section 2.7.6. Lastly, we draw some conclusions regarding the relative position of $\hat{S_a}$ and $\hat{S_b}$ with the lines $\psi(v_{ikjb}), \psi(v_{ijkb})$ and $\psi(v_{ijka}), \psi(v_{ikja})$ respectively. The later observations are then translated as InSphere test's results based on Lemma 14.

Let us consider one case in detail, for example the OrderCase 2 configuration; a similar approach will be applied to each OrderCase. In Figure 2.14 (Left Column, 2nd Row), we consider a random[3] layout of the points $\psi(v_{ikja})$, $\psi(v_{ikjb})$, $\psi(v_{ijkb})$ and $\psi(v_{ijka})$ (and the respective tangent planes at these points) that appear in the order OrderCase 2 dictates. In the same figure, we provide a possible location of $\hat{S_n}$, for $n \in \{a, b\}$ with respect to the selected layout; $\hat{S_n}$ must be tangent to both $\hat{\ell}(v_{ikjn})$ and $\hat{\ell}(v_{ijkn})$, and centered according to the analysis of Section 2.7.6.

Finally, we inspect the relative position of $\hat{S_a}$ (resp., $\hat{S_b}$) with the lines $\psi(v_{ikjb})$ and $\psi(v_{ikjb})$ (resp., $\psi(v_{ikja})$ and $\psi(v_{ikja})$). In any such random layout, it must hold that

- $\hat{S_a}$ intersects the negative side of $\psi(v_{ikjb})$ but does not intersect the negative side of $\psi(v_{ijkb})$ and,
- $\hat{S_b}$ intersects the negative side of $\psi(v_{ijka})$ but does not intersect the negative side of $\psi(v_{ijka})$.

Another way of proving this, is by looking at the shadow regions of S_a and S_b on the arc. For example, in a OrderCase 2 configuration, $v_{ikja} \prec v_{ikjb} \prec v_{ijka}$ and subsequently $v_{ikjb} \in \mathcal{SR}(S_a)$, since $\mathcal{SR}(S_a)$ consists of all points $p \in \tau_{ijk}$ with $v_{ikja} \prec p \prec v_{ijka}$. As a result the sphere $\mathcal{T}(v_{ikjb})$ must intersect the sphere S_a, i.e., $\hat{S_a}$ intersects the negative side of $\hat{\ell}(v_{ikjb})$.

Lastly, we translate the obtained relative positions of circles and lines of $\mathcal{Y}$-space to InSphere tests outcomes. For example, if $\hat{S_a}$ intersects the negative side of $\hat{\ell}(v_{ikjb})$, we conclude that InSphere $(S_i, S_k, S_j, S_b, S_a)$ is negative, as an immediate result of Lemma 2.10. In conclusion, we get that if the Apollonius vertices we seek to order appear as in OrderCase 2, then

- InSphere $(S_i, S_k, S_j, S_b, S_a) = -$ and InSphere $(S_i, S_j, S_k, S_b, S_a) = +$,
- InSphere $(S_i, S_k, S_j, S_a, S_b) = +$ and InSphere $(S_i, S_j, S_k, S_a, S_b) = -$.

Ultimately, we create a table of the four possible InSphere outcomes that hold in each of the OrderCase's 1 to 6 (see Table 2.1). A simple way of distinguishing the ordering of the Apollonius vertices becomes clear now, due to tuple of outcomes being so different in most

[3]In Figure 2.14, the circles $\hat{S_a}$ and $\hat{S_b}$ always appear to be centered on the same side of the line going through $\hat{\mathcal{A}}$ and $\hat{O}$. This was done for reasons of consistency and does not always correspond to reality, since it would be equivalent to $C^\star_a$ and $C^\star_b$ always lying on the same side of the plane going through the points $C^\star_i, C^\star_j$ and $C^\star_k$.

OrderCase:	1	2	3	4	5	6
InSphere($S_i, S_k, S_j, S_b; S_a$)	+	−	+	+	+	−
InSphere($S_i, S_j, S_k, S_b; S_a$)	+	+	+	−	+	−
InSphere($S_i, S_k, S_j, S_a; S_b$)	+	+	−	−	+	+
InSphere($S_i, S_j, S_k, S_a; S_b$)	+	−	−	+	+	+

Table 2.1: Case A: Signs of all possible InSphere tests that follow from the analysis of each OrderCase. Notice that only the rows that correspond to OrderCase 1 and OrderCase 5 are identical and therefore we only need the signs of these InSphere predicates to determine the OrderCase most of the time. If all predicates return positive, we require some auxiliary tests to distinguish between the two cases.

OrderCase's. Indeed, if $Q = (Q_1, Q_2, Q_2, Q_4)$ denotes the ordered tuple of the InSphere predicate outcomes, where

$$Q_1 = \text{InSphere}(S_i, S_k, S_j, S_b, S_a), \quad Q_2 = \text{InSphere}(S_i, S_j, S_k, S_b, S_a),$$
$$Q_3 = \text{InSphere}(S_i, S_k, S_j, S_a, S_b), \quad Q_4 = \text{InSphere}(S_i, S_j, S_k, S_a, S_b),$$

then the Order predicate returns :

- the ordering of OrderCase 2, if $Q = (-, +, +, -)$ or,
- the ordering of OrderCase 3, if $Q = (+, +, -, -)$ or,
- the ordering of OrderCase 4, if $Q = (+, -, -, +)$ or,
- the ordering of OrderCase 6, if $Q = (-, -, +, +)$.

Finally, if $Q = (+, +, +, +)$ then either OrderCase 1 or OrderCase 5 is the correct ordering of the vertices (see Figure 2.15). To resolve this dilemma, we distinguish cases depending on the ordering of the midpoints M_a and M_b of the arcs $(\psi(v_{ikja}), \psi(v_{ijka}))$ and $(\psi(v_{ikjb}), \psi(v_{ijkb}))$ respectively. Since it must either hold that $\{v_{ikja} \prec v_{ijka}\} \prec \{v_{ikjb} \prec v_{ijkb}\}$ (OrderCase 1) or $\{v_{ikjb} \prec v_{ijkb}\} \prec \{v_{ikja} \prec v_{ijka}\}$ (OrderCase 5), then we are obviously in the former case if $M_a \prec M_b$ or in the latter if $M_b \prec M_a$.

To determine the ordering of M_a and M_b on the arc $(\hat{\eta}, \hat{o}, \hat{\theta})$, we shall use the auxiliary point $\hat{o}$. Initially, we reflect on the fact that, for $n \in \{a, b\}$, $\hat{C}_n$ is known to lie on the open ray from $\mathcal{A}$ towards M_n. It is also apparent that the points O, $\mathcal{A}$ and $\hat{o}$ are collinear and appear in this order on the line $\hat{\ell}$ they define.

Based on the definition of $\mathcal{Y}$-space and the remarks of Section 2.7.6, the midpoint M_n, for $n \in \{a, b\}$, satisfies

- $M_n \prec \hat{o}$ if and only if $\text{Orient3D}(C_n^\star, C_i^\star, C_j^\star, O) < 0$,

- $\hat{o} \prec M_n$ if and only if $\textsc{Orient3D}(C_n^\star, C_i^\star, C_j^\star, O) > 0$,
- $M_n \equiv \hat{o}$ if and only if $\textsc{Orient3D}(C_n^\star, C_i^\star, C_j^\star, O) = 0$.

Lastly, we notice that $\textsc{Orient3D}(C_b^\star, C_i^\star, C_j^\star, C_a^\star) < 0$ is equivalent to $\hat{C}_b$ lying on the "right side" of the oriented line going from $\mathcal{A}$ to $\hat{C}_a$.

Ultimately, we determine the relative position of M_a and M_b by combining all the information extracted of the $\textsc{Orient3D}$ predicates mentioned, using the following algorithm.

Step 1. We evaluate $o_1 = \textsc{Orient3D}(C_a^\star, C_i^\star, C_j^\star, O)$, $o_2 = \textsc{Orient3D}(C_b^\star, C_i^\star, C_j^\star, O)$ and $\Pi = \mathsf{sign}(o_1)\mathsf{sign}(o_2)$. If $\Pi > 0$ go to Step 2a, otherwise go to Step 2b.

Step 2a. Either $M_a, M_b \prec \hat{o}$ or $\hat{o} \prec M_a, M_b$. In either case, we evaluate $o_3 = \textsc{Orient3D}(C_b^\star, C_i^\star, C_j^\star, C_a^\star)$. If $o_3 < 0$ then $M_a \prec M_b$, and the $\textsc{Order}$ predicate returns the ordering of OrderCase 1. Otherwise, $M_b \prec M_a$ and the ordering of OrderCase 5 is returned. (see Figure 2.16).

Step 2b. Either $\hat{o}$ lies in-between M_a and M_b or is identical with one of them. In both cases, if $o_1 < o_2$ then $M_a \prec M_b$ and the $\textsc{Order}$ predicate return the ordering of OrderCase 1, otherwise, $M_b \prec M_a$ and the ordering of OrderCase 5 is returned. (see Figure 2.17).

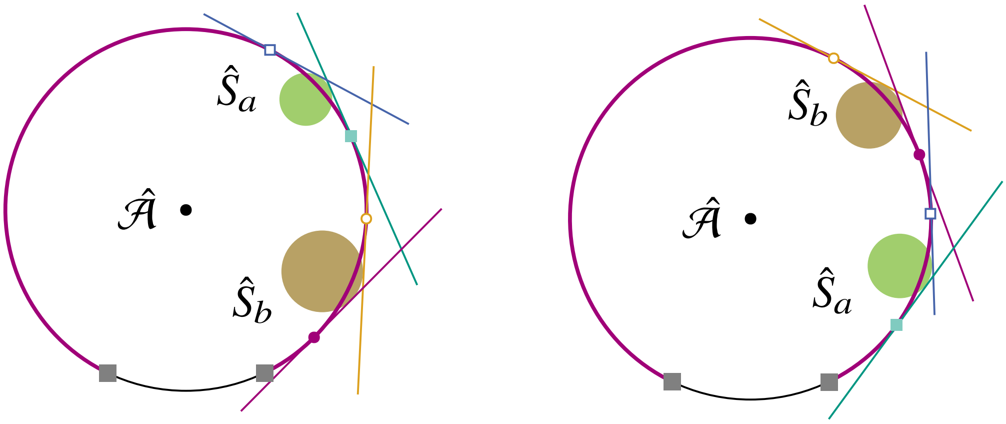

FIGURE 2.15: If $Q = (+,+,+,+)$ then we must determine if the ordering of the Apollonius vertices correspond to OrderCase 1 (Left) or 5 (Right). It is apparent that we are in the first case if and only if the ray $(\mathcal{A}, t)$ "meets" $\hat{C}_a$ first as t traverses the arc $(\psi(\eta), \psi(o), \psi(\theta))$.

Analysis of Case B Given that $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ are both of the form $(\chi, +\infty)$ and therefore only the Apollonius vertices v_{ikja} and v_{ikjb} exist on τ_{ijk}, the ordering of these vertices on $(\hat{\eta}, \hat{o}, \hat{\theta})$ is either

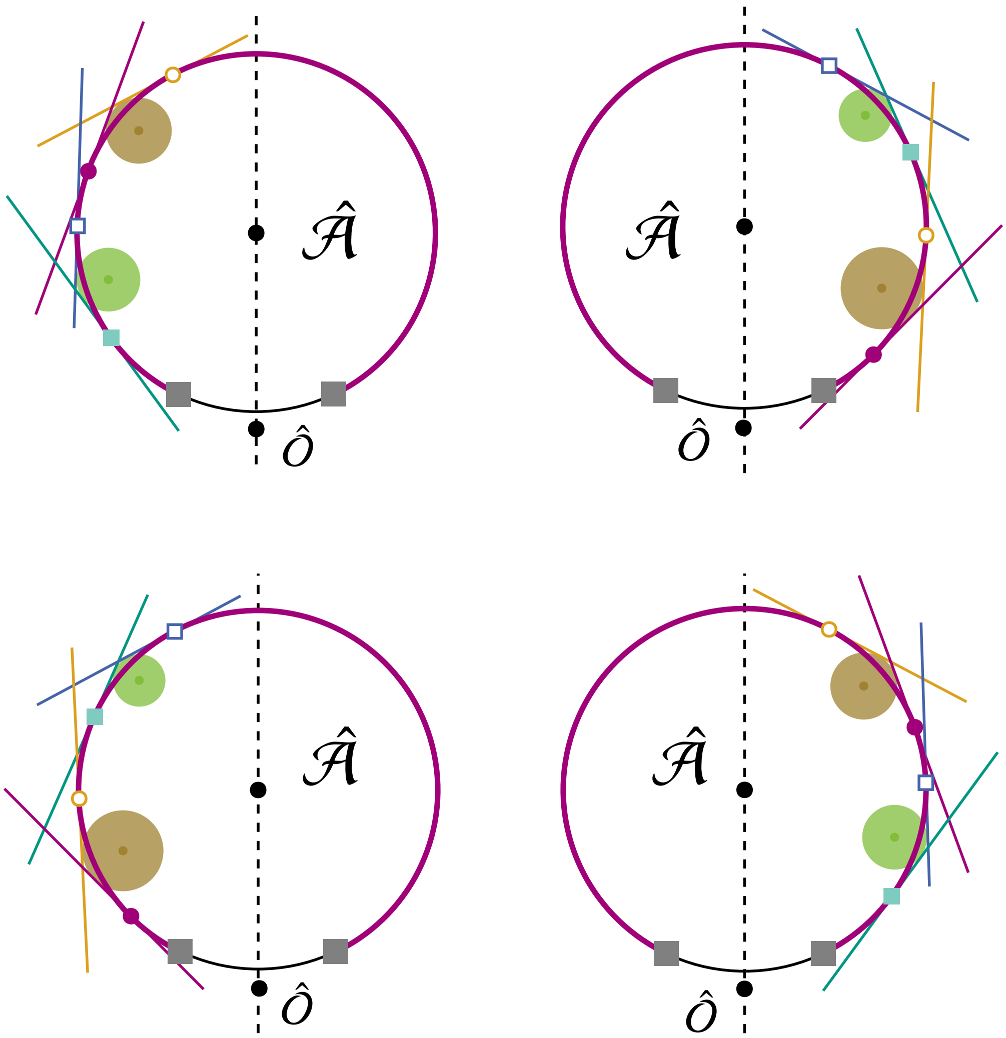

FIGURE 2.16: If $o_1 \cdot o_2 > 0$, the centers $\hat{C}_a$ and $\hat{C}_b$ must lie on the same side of the line that goes through $\hat{\mathcal{A}}$ and $\hat{O}$. No matter which side the centers lie on, if $o_3 < 0$ or equivalently $\hat{C}_a$ lies on the left side of the oriented line that goes from $\hat{\mathcal{A}}$ to $\hat{C}_b$, (Top 2 Figures) then we obtain the ordering described in OrderCase 1. Otherwise, we obtain the ordering described in OrderCase 5 (Bottom 2 Figures).

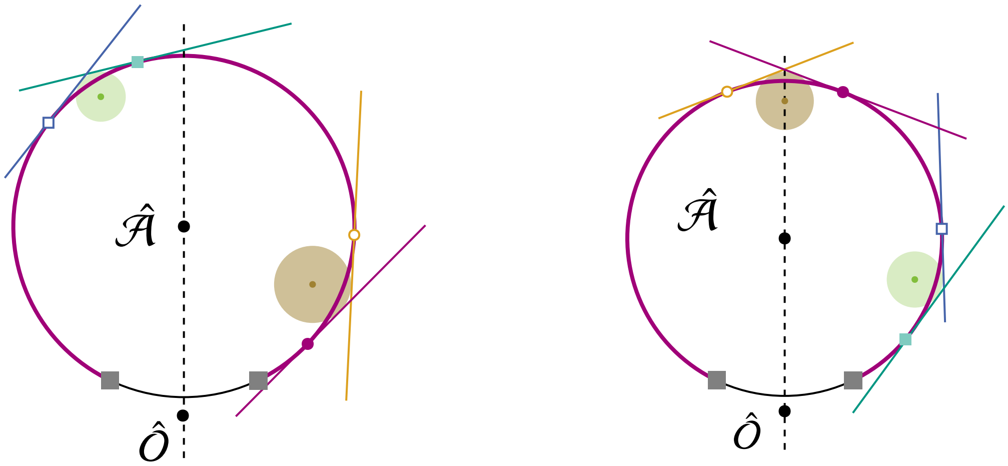

FIGURE 2.17: If $o_1 \cdot o_2 \leq 0$, the centers $\hat{C}_a$ and $\hat{C}_b$ lie on different sides of the line that goes through $\hat{\mathcal{A}}$ and $\hat{O}$ (Left) or only one of them lies on the line (since we are in either OrderCase 1 or 5). No matter which side the centers lie on, if $o_1 < o_2$ we obtain the ordering described in OrderCase 1. Otherwise, we obtain the ordering described in OrderCase 5 (Bottom 2 Figures).

OrderCase 1. $v_{ikja} \prec v_{ikjb}$ or,

OrderCase 2. $v_{ikjb} \prec v_{ikja}$.

A similar analysis with the Case A is used to resolve the predicate in Case B; we create a table regarding the possible outcomes of the INSPHERE tests with inputs (S_i,S_k,S_j,S_a,S_b) and (S_i,S_k,S_j,S_b,S_a). Recall that the outcome of Q_1 = INSPHERE(S_i, S_k, S_j, S_a, S_b) (resp., Q_2 = INSPHERE(S_i, S_k, S_j, S_a, S_b)) is $-$, 0 or + if the circle $\hat{S_b}$ (resp., $\hat{S_a}$) intersects, is tangent to or does not intersect the negative side of $\hat{\ell}(v_{ikja})$ (resp., $\hat{\ell}(v_{ikjb})$).

Using a simpler approach, we observe that

- in OrderCase 1, v_{ikja} does not belong to the shadow region of the sphere S_b on τ_{ijk} and therefore $\mathcal{T}(v_{ikja})$ does not intersect S_b or equivalently $Q_1 = +$. Moreover, in this case, v_{ikjb} belongs to the shadow region of the sphere S_a on τ_{ijk} and therefore $\mathcal{T}(v_{ikjb})$ intersects S_a or equivalently $Q_2 = -$.

- In OrderCase 2, v_{ikja} belongs to the shadow region of the sphere S_b on τ_{ijk} and therefore $\mathcal{T}(v_{ikja})$ intersects S_b or equivalently $Q_1 = -$. Furthermore, v_{ikjb} does not belong to the shadow region of the sphere S_a on τ_{ijk} and therefore $\mathcal{T}(v_{ikjb})$ does not intersects S_a or equivalently $Q_2 = +$.

In conclusion we can answer the ORDER(S_i, S_j, S_k, S_a, S_b) predicate in case B by evaluating Q_1; if $Q_1 = +$ then return OrderCase 1 otherwise, if $Q_1 = -$ return OrderCase

	OrderCase 1	OrderCase 2
InSphere($S_i, S_k, S_j, S_b; S_a$)	+	−
InSphere($S_i, S_k, S_j, S_a; S_b$)	−	+

Table 2.2: Case B: Signs of all possible InSphere tests that follow from the analysis of each OrderCase. Notice that each column is distinct and therefore we can determine the OrderCase after the outcomes of the InSphere predicates.

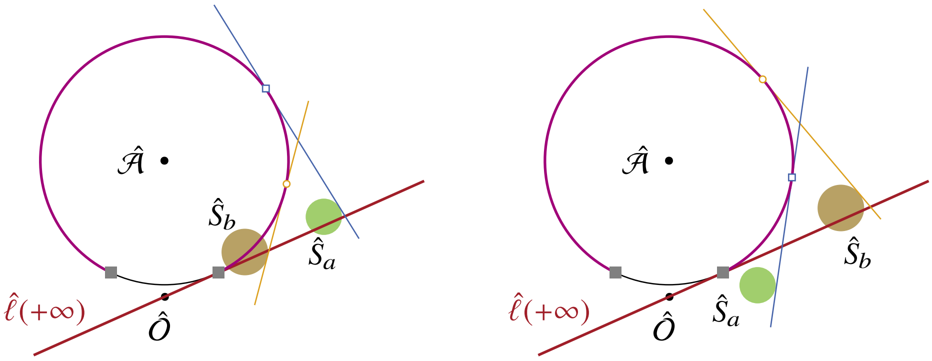

Figure 2.18: In Case B, it is assumed that only the Apollonius vertices v_{ikja} and v_{ijkb} exist on the trisector τ_{ijk}. We consider the two possible orderings of these vertices: OrderCase 1 (Left) and OrderCase 2 (Right). Similar with Case A, we consider the corresponding ordering of the points $\psi(v_{ikja})$ and $\psi(v_{ijkb})$ on the arc $(\psi(\eta), \psi(o), \psi(\theta))$. A possible location for the circles $\hat{S_a}$ and $\hat{S_b}$ is drawn based on the analysis of Section 2.5.

2. Equivalently, we could evaluate Q_2 instead of Q_1; if $Q_2 = -$ then return OrderCase 1 otherwise if $Q_2 = -$ return OrderCase 2. The following equivalencies are depicted in Table 2.2 and this concludes the analysis of Case B.

Analysis of Case C In Case C, it is assumed that $\mathcal{SR}(S_a) = (\chi, \phi)$ hence $v_{ikja} \prec v_{ijka}$ while $\mathcal{SR}(S_b) = (\chi, +\infty)$ and consequently only v_{ikjb} exists on the arc $(\hat{\eta}, \hat{o}, \hat{\theta})$. All three possible orderings of these three Apollonius vertices on the arc are

OrderCase 1. $v_{ikja} \prec v_{ijka} \prec v_{ikjb}$,

OrderCase 2. $v_{ikja} \prec v_{ikjb} \prec v_{ijka}$ and

OrderCase 3. $v_{ikjb} \prec v_{ikja} \prec v_{ijka}$.

The analysis of this Case uses the same tools and analysis presented in the previous two cases with small adjustments, since $\mathcal{SR}(S_a) = (\chi, \phi)$ and $\mathcal{SR}(S_b) = (\chi, +\infty)$ in the case

	OrderCase 1	OrderCase 2	OrderCase 3
InSphere($S_i, S_k, S_j, S_b; S_a$)	$+$	$-$	$+$
InSphere($S_i, S_k, S_j, S_a; S_b$)	$+$	$+$	$-$
InSphere($S_i, S_j, S_k, S_a; S_b$)	$+$	$-$	$-$

Table 2.3: Case C: Signs of all possible InSphere tests that follow from the analysis of each OrderCase. Notice that each column is distinct and therefore we can determine the OrderCase after the outcomes of the InSphere predicates, as in Case B.

studied. Let us denote by Q_1, Q_2 and Q_3 the results of the InSphere predicates with inputs $(S_i, S_k, S_j, S_b, S_a)$, $(S_i, S_k, S_j, S_a, S_b)$ and $(S_i, S_j, S_k, S_b, S_a)$ respectively.

Notice now that

- in OrderCase 1, v_{ikjb},v_{ikja} and v_{ijka} do not belong to the shadow region of S_a, S_b and S_b respectively and therefore it must stand that $Q_1 = Q_2 = Q_3 = +$.
- in OrderCase 2, v_{ikjb} and v_{ijka} belong to the shadow region of S_a and S_b respectively and for this reason $Q_1 = -$ and $Q_3 = -$. On the other hand, v_{ikja} does not belong to the shadow region of S_b and therefore $Q_2 = +$. Finally,
- in OrderCase 3, both v_{ikja} and v_{ijka} belong to the shadow region of S_b and consequently $Q_2 = -$ and $Q_3 = -$ whereas, v_{ikjb} does not belong to the shadow region of S_a and therefore $Q_1 = +$.

Since the tuple $Q = (Q_1, Q_2, Q_3)$ is different in each OrderCase 1 to 3, we can answer the predicate by evaluating the three InSphere predicates hence Q and correspond it the respective ordering (also see Table 2.3):

- if $Q = (+, +, +)$, return the ordering of OrderCase 1 or,
- if $Q = (-, +, -)$, return the ordering of OrderCase 2 otherwise,
- if $Q = (+, -, -)$, return the ordering of OrderCase 3.

Algebraic Cost to resolve the Cases A, B or C The analysis of the Cases A, B and C showed that the answer of the Order(S_i, S_j, S_k, S_a, S_b) predicate in a classic configuration ultimately amounts to determining the outcomes of up to four InSphere predicates and, if needed, some auxiliary Orient3D tests.

To answer any of the InSphere predicates that may require evaluation, we must perform operations of maximum algebraic degree 10 (in the input quantities), as mentioned in Section 2.4.1.

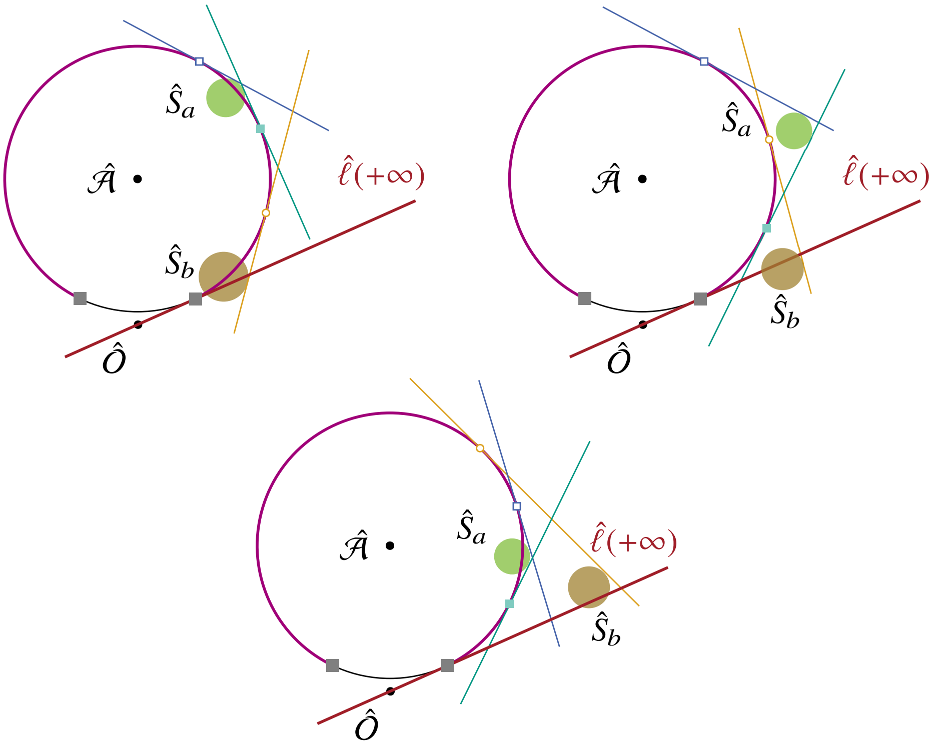

FIGURE 2.19: In Case C, it is assumed that only the Apollonius vertices v_{ikja}, v_{ijka} and v_{ijkb} exist on the trisector τ_{ijk}. We consider the three possible orderings of these vertices: OrderCase 1 (Top Left), OrderCase 2 (Top Right) and OrderCase 3 (Bottom). Similar with Case A and B, we consider the corresponding ordering of the points $\psi(v_{ikja})$ and $\psi(v_{ijkb})$ on the arc $(\psi(\eta), \psi(o), \psi(\theta))$. A possible location for the circles $\hat{S_a}$ and $\hat{S_b}$ is drawn based on the analysis of Section 2.5.

Regarding the auxiliary ORIENT3D primitives, we observe that

$$\textsc{Orient3D}(C_b^\star, C_i^\star, C_j^\star, C_a^\star) = \mathsf{sign}(D_{bija}^{uvw}) = \mathsf{sign}(p_i^\star p_j^\star p_a^\star p_b^\star)\mathsf{sign}(E_{bija}^{xyzp}) \tag{2.57}$$
$$= \mathsf{sign}(E_{bija}^{xyzp}), \tag{2.58}$$

where the quantity E_{bija}^{xyzp} and is an expression of algebraic degree 5 on the input quantities. The expression $\textsc{Orient3D}(C_n^\star, C_i^\star, C_j^\star, O)$, for $n \in \{a, b\}$ can be evaluated as shown in Section 2.7.6,

$$\textsc{Orient3D}(C_n^\star, C_i^\star, C_j^\star, O) = \mathsf{sign}(p_i^\star p_i^\star p_n^\star D_{nij}^{uvw}) = \mathsf{sign}(D_{nijk}^{xyz}) \tag{2.59}$$
$$= \textsc{Orient3D}(C_n, C_i, C_j, C_k) \tag{2.60}$$

and therefore its evaluation requires operations of algebraic degree 4 (in the input quantities).

In conclusion, since the evaluation of the INSPHERE predicates is the most degree-demanding operation throughout the evaluation of the ORDER predicate in a classic configuration, we have proven the following lemma.

Lemma 16 *The* ORDER *predicate in a classic configuration can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

Ordering the Apollonius vertices in a non-classic configuration

In the previous section we presented a way to resolve the $\textsc{Order}(S_i, S_j, S_k, S_a, S_b)$ predicate under the assumption that $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ were either $(\chi, +\infty)$ or (χ, ϕ) (not necessary the same); we called this *a classic configuration*. In this section, we will assume we are in a non-classic configuration, i.e., at least one of $\mathcal{SR}(S_a)$ or $\mathcal{SR}(S_b)$ is $(-\infty, \phi)$ or $(-\infty, \phi) \cup (\chi, \phi)$. For convenience, these last two forms of a shadow region are labelled as non-classic whereas the classic forms are $(\chi, +\infty)$ and (χ, ϕ).

If $\mathcal{SR}(S_n)$ has a non-classic type, for $n = a$ or b, then we claim that there exist a sphere S_N, for $N = A$ or B respectively, such that:

- if $\mathcal{SR}(S_n) = (-\infty, \phi)$ then $\mathcal{SR}(S_N) = (\chi, +\infty)$ and $v_{ijkn} \equiv v_{ikjN}$ or,
- if $(-\infty, \phi) \cup (\chi, \phi)$ then $\mathcal{SR}(S_N) = (\chi, \phi)$ and $v_{ijkn} \equiv v_{ikjN}$ as well as $v_{ikjn} \equiv v_{ijkN}$.

If these conditions hold, we will say that S_n and S_N are *equivalent spheres*. Notice that if $\mathcal{SR}(S_n)$ has a non-classic type then the shadow region of its equivalent sphere has a classic type and vice versa. The utility of this equivalency is that it enable us to make a connection between a classic and a non-classic configuration in the following way.

When the predicate $\textsc{Order}(S_i, S_j, S_k, S_a, S_b)$ is called then

1. if $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ have a classic type, we are in a classic and therefore, we resolve the predicate based on the analysis of Section 2.7.6.

2. If $\mathcal{SR}(S_a)$ has a classic type and $\mathcal{SR}(S_b)$ does not, then we call ORDER $(S_i, S_j, S_k, S_a, S_B)$. Since both $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_B)$ have a classic type, this predicate can be evaluated using analysis of Section 2.7.6 with some adjustments. The predicate's outcome would be the ordering of v_{ikja}, v_{ikjB} and any of the existing v_{ijka} or v_{ijkB}. Using the property of equivalent spheres, we could answer the initial predicate by substituting v_{ikjB} with v_{ijkb} and, if it exists, v_{ijkB} with v_{ikjb}.

3. If $\mathcal{SR}(S_b)$ has a classic type and $\mathcal{SR}(S_a)$ does not, then we follow a similar analysis with the previous case. We evaluate ORDER $(S_i,\ S_j, S_k, S_A, S_b)$ and in the resulting ordering of the Apollonius vertices v_{ikjA}, v_{ikjb} and any of the existing v_{ijkA} or v_{ijkb}, we will substitute $v_{ikjA} \equiv v_{ijka}$ and if necessary, $v_{ijkA} \equiv v_{ikja}$, to obtain the answer to the initial ORDER predicate.

4. Finally, if both $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ do not have a classic type we evaluate ORDER$(S_i, S_j, S_k, S_A, S_B)$. As before, we substitute $v_{ikjA} \equiv v_{ijka}$, $v_{ikjB} \equiv v_{ijkb}$ and if necessary, $v_{ijkA} \equiv v_{ikja}$ and/or $v_{ijkB} \equiv v_{ikjb}$, and the acquired ordering is the answer of the initial ORDER predicate.

The evaluation of the ORDER predicate called in any of these 4 cases will eventually require determining INSPHERE or ORIENT3D predicates with inputs that involve the sites S_i, S_j, S_k, S_A (or S_a) and S_B(or S_b). The list of all possible predicates that must be evaluated, in the worst case scenario and assuming a classic configuration, would be:

- INSPHERE $(S_i, S_k, S_k, S_a, S_b)$,
- INSPHERE $(S_i, S_j, S_k, S_a, S_b)$,
- INSPHERE $(S_i, S_k, S_j, S_b, S_a)$,
- INSPHERE $(S_i, S_j, S_k, S_b, S_a)$,
- ORIENT3D (C_a, C_i, C_j, C_k),
- ORIENT3D (C_b, C_i, C_j, C_k) and
- ORIENT3D $(C_a^\star, C_i^\star, C_j^\star, C_b^\star)$.

It is apparent that we must be able to answer these predicates when either one or both of S_a and S_b are substituted by S_A and S respectively.

Firstly, we present a way of defining an equivalent sphere S_N when $\mathcal{SR}(S_n)$ has a non-classic type, for $N = A$ or B and $n = a$ or b respectively. Since $\hat{C_n}$ cannot coincide with $\hat{\mathcal{A}}$ (because there are either 1 or 2 cotangent lines to $\mathcal{K}^\star$ and $\hat{S_n}$), these points define a line $\hat{\ell_n}$. If a random point $\hat{C_N}$ is selected on $\hat{\ell_n}$ such that $\hat{\mathcal{A}}$ lies in-between $\hat{C_N}$ and $\hat{C_n}$, then we may choose an appropriate radius such that a circle $\hat{S_n}$, centered at $\hat{C_N}$, is tangent to any of the existing lines $\hat{\ell}(v_{ikjn})$ and $\hat{\ell}(v_ijkn)$.

Notice that any sphere S_N of $\mathcal{W}$-space whose corresponding image in $\mathcal{Y}$-space is the circle $\hat{S_N}$ has the desired properties of an equivalent sphere of S_n. Indeed, if $\mathcal{SR}(S_n)$ is (χ, ϕ) then it must stand that $\mathcal{SR}(S_N) = (-\infty, \phi) \cup (\chi, +\infty)$ and specifically the actual endpoints of these shadow regions on the trisector τ_{ijk} coincide. To prove this argument, we only need observe in $\mathcal{Y}$-space that the circle $\hat{C_n}$ intersects the negative side of a line $\hat{\ell}(p)$ only for $v_{ikjn} \prec v_{ijkn}$ whereas, these are the only family of lines $\hat{\ell}(p)$ for $p \in \tau_{ijk}$ that do not intersect $\hat{S_N}$. As a conclusion the shadow region of S_n and S_N must be complementary, i.e., $\mathcal{SR}(S_N) = (-\infty, \phi) \cup (\chi, +\infty)$. From Lemma 15, we deduce that $v_{ijkN} \prec v_{ikjN}$ and since these endpoints coincide with the endpoints of $\mathcal{SR}(S_n)$ it must hold that $v_{ijkN} \equiv v_{ikjn}$ and $v_{ikjN} \equiv v_{ijkn}$, since $v_{ikjn} \prec v_{ijkn}$ (see Figure 2.20).

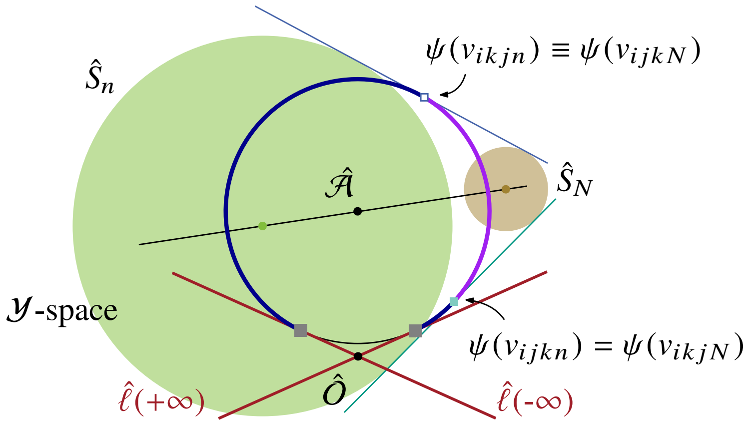

FIGURE 2.20: The shadow region of S_n is $(-\infty, \phi) \cup (\chi, +\infty)$ as its image in $\mathcal{Y}$-space is the blue area of the arc. Notice that the respective image of $\mathcal{SR}(S_N)$ is the purple area and therefore $\mathcal{SR}(S_n)$ must equal (χ, ϕ). Since the endpoints of the two shadow regions coincide and based on Lemma 15, it must hold that $v_{ijkN} \equiv v_{ikjn}$ and $v_{ikjN} \equiv v_{ijkn}$. Therefore, $\hat{S_N}$ and $\hat{S_n}$ are equivalent.

Using a similar analysis, one can consider an equivalent sphere S_N of S_n, when $\mathcal{SR}(S_n)$ is assumed to be $(-\infty, \phi)$. The center of the respective circle $\hat{C_N}$ is selected in the same way as above, and the radius of $\hat{S_N}$ is chosen such that the circle is tangent to $\hat{\ell}(v_{ijkn})$. Again, we can conclude that $\mathcal{SR}(S_N)$ and $\mathcal{SR}(S_n)$ are complementary since the family of lines $\hat{\ell}(p)$ for $\hat{\eta} \prec p$ are the locus of lines $\hat{\ell}(p)$, with $\hat{p} \in (\hat{\eta}, \hat{o}, \hat{\theta})$, whose negative side is intersected by $\hat{S_n}$ and simultaneously, whose negative side is not intersected by $\hat{S_N}$ (see Figure 2.21).

An interesting observation is that S_N is not uniquely defined in the sense that we do not provide its exact coordinates expressed as a function of the input quantities. This is a consequence of the fact that there are infinite spheres S_n that all share the same Apollonius vertices v_{ikjn} and v_{ijkn}.

Resuming the analysis of the properties of the equivalent sphere, we notice that if a point $p \in \tau_{ijk}$ lies on the shadow region of S_n then it must not lie on the shadow region of S_N and vice versa. An equivalent statement would be that a sphere $\mathcal{T}(p)$, for $p \in \tau_{ijk}$, intersects S_n

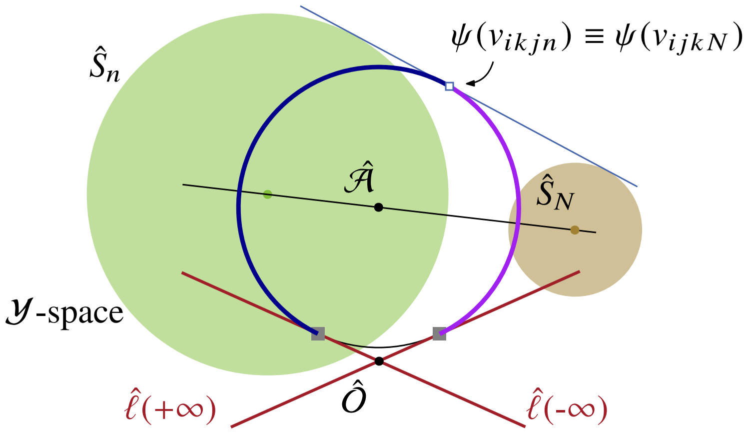

FIGURE 2.21: The shadow region of S_n is $(-\infty, \phi)$ as its image in $\mathcal{Y}$-space is the blue area of the arc. Notice that the respective image of $\mathcal{SR}(S_N)$ is the purple area and therefore $\mathcal{SR}(S_n)$ must equal $(\chi, +\infty)$. Since the endpoints of the two shadow regions coincide and based on Lemma 15, it must hold that $v_{ijkN} \equiv v_{ikjn}$. Therefore, $\hat{S_N}$ and $\hat{S_n}$ are equivalent.

if and only if it does not intersect S_N (see Figure 2.22). If p is chosen to be either v_{ikjm} or v_{ijkm}, where $m \in \{a,b\} \backslash \{n\}$, we get the following relations

$$\text{InSphere}(S_i, S_j, S_k, S_m, S_N) = -\text{InSphere}(S_i, S_j, S_k, S_m, S_n), \tag{2.61}$$
$$\text{InSphere}(S_i, S_k, S_j, S_m, S_N) = -\text{InSphere}(S_i, S_k, S_j, S_m, S_n). \tag{2.62}$$

Moreover, if S_m has a non-classic type and S_M is an equivalent sphere, where $M = B$ if $m = b$ or $M = A$ if $m = a$, it is known that $v_{ikjM} \equiv v_{ijkm}$ and, if v_{ikjm} also exists, then $v_{ijkM} \equiv v_{ikjm}$. Therefore, using the previous observation for $p = v_{ijkM}$ or $v_{ikjM}\}$, we obtain the following expressions,

$$\text{InSphere}(S_i, S_j, S_k, S_M, S_N) = -\text{InSphere}(S_i, S_j, S_k, S_M, S_n) \tag{2.63}$$
$$= -\text{InSphere}(S_i, S_k, S_j, S_m, S_n), \tag{2.64}$$
$$\text{InSphere}(S_i, S_k, S_j, S_M, S_N) = -\text{InSphere}(S_i, S_k, S_j, S_M, S_n) \tag{2.65}$$
$$= -\text{InSphere}(S_i, S_j, S_k, S_m, S_n). \tag{2.66}$$

These last four equalities can be used to evaluate any InSphere predicate that arises during the evaluation of the Order predicate in the case of a non-classic configuration.

Regarding the respective Orient3D predicates that may have to be evaluated, we consider the fact that $\hat{\mathcal{A}}$, $\hat{C}_n$ and $\hat{C_N}$ are collinear and the latter two lie on opposite sides of $\hat{\mathcal{A}}$. Subsequently, it is also true that $\hat{C}_n$ and $\hat{C_N}$ must lie on opposite sides with respect to any line $\hat{\lambda}$ that goes through $\hat{\mathcal{A}}$ (see Figure 2.22).

If we choose $\hat{\lambda}$ to be the line $\hat{\ell}$ that goes through $\hat{O}$ and bear in mind that the position of a point of $\mathcal{Y}$-space with respect to this line corresponds to the position of its pre-image in $\mathcal{Z}$-space against the plane Π_{ijk}, we infer that C_n and C_N lie on different sides of Π_{ijk} and therefore

$$\textsc{Orient3D}(C_N, C_i, C_j, C_k) = -\textsc{Orient3D}(C_n, C_i, C_j, C_k).$$

If $\hat{\lambda}$ is chosen to be the line that goes through $\hat{C_m}$ for $m \in \{a, b\}\backslash\{n\}$ then it must hold in $\mathcal{Y}$-space that $C_n^\star$ and $C_N^\star$ lie on different sides with respect to the plane that goes through $C_i^\star$, $C_j^\star$ and $C_m^\star$, which is equivalent to

$$\textsc{Orient3D}(C_N^\star, C_i^\star, C_j^\star, C_m^\star) = -\textsc{Orient3D}(C_n^\star, C_i^\star, C_j^\star, C_m^\star), \tag{2.67}$$

$$\textsc{Orient3D}(C_m^\star, C_i^\star, C_j^\star, C_N^\star) = -\textsc{Orient3D}(C_m^\star, C_i^\star, C_j^\star, C_n^\star). \tag{2.68}$$

Finally, combining the last two equations, we obtain that

$$\textsc{Orient3D}(C_N^\star, C_i^\star, C_j^\star, C_M^\star) = -\textsc{Orient3D}(C_n^\star, C_i^\star, C_j^\star, C_M^\star) \tag{2.69}$$

$$= \textsc{Orient3D}(C_n^\star, C_i^\star, C_j^\star, C_m^\star). \tag{2.70}$$

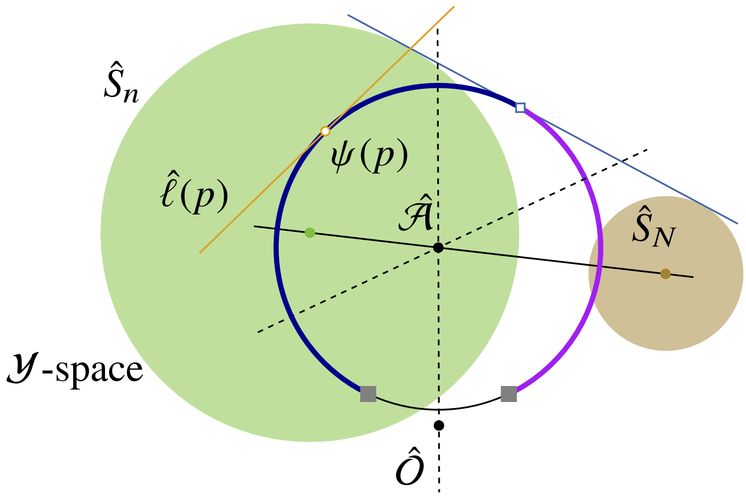

Figure 2.22: If S_N is an equivalent sphere of S_n, then it must hold that the centers $\hat{C}_n$, $\hat{C}_N$ and $\hat{\mathcal{A}}$ are collinear and the former two points lie on opposite sides with respect to the latter. Observe that they also lie on opposite sides with respect to any line that goes through $\hat{\mathcal{A}}$. Lastly, it is apparent that a point $\psi(p)$ on the arc $(\psi(\eta), \psi(o), \psi(\theta))$ must lie on the image of the shadow region of either S_n or S_N.

In conclusion, we have shown that the evaluation of all 7 InSphere or Orient3D predicates, that may involve one or two equivalent spheres, can be amounted to the evaluation of respective predicates that contain only the original spheres S_a and S_b instead. Ultimately, we proved that the algebraic cost of the Order predicate in a non-classic configuration is the same as in a classic configuration, yielding the following lemmas.

Lemma 17 *The* Order *predicate in a non-classic configuration can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

Lemma 18 *The* Order *predicate can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

Chapter 3

Non-Degenerate Case Analysis for Elliptic Trisectors

In this chapter, our goal is to answer the EDGECONFLICT($S_i, S_j, S_k, S_l, S_m, S_q$) predicate under the assumption that the trisector of the first three input sites is either an ellipse or a circle and no degeneracies occur. If the trisector τ_{ijk} has one of the aforementioned types, we will say that we are in an *elliptic trisector* case. Note that for the respective non-degenerate *hyperbolic case*, where τ_{ijk} is a branch of a hyperbola or a line, the EDGECONFLICT predicate is analysed thoroughly in Chapter 2.

These two cases share a lot of similarities in the sense that, most of the analysis and the subpredicates presented for the hyperbolic trisector type can be used with little or no modifications for the elliptic trisector type. In Section 3.1, we present the major differences that are found among the two cases, with the most important being spotted when trying to order Voronoi vertices on the oriented trisector. The main algorithm that answers the EDGECONFLICT predicate in an elliptic-trisector scenario is presented in Section 3.2.

3.1 Differences between the elliptic and the hyperbolic case

Orienting an elliptic trisector

If the trisector of the sites S_i, S_j and S_k is an ellipse or a circle, it necessarily holds that one of the three sphere lies strictly inside the convex hull defined by the other two [42]. Since a cyclic permutation of the sites S_i, S_j and S_k does not alter the outcome of the EDGECONFLICT predicate, we may assume that S_k has the minimum radius and therefore lies in the convex hull of S_i and S_j.

It is apparent that a naive way of ordering the trisector τ_{ijk} would be via the use of the "right-hand rule": if our thumb points from C_j towards C_i, the direction pointed by our hand, if it "wraps" τ_{ijk}, is the positive direction. Variations of this rule would include pointing our thumb from C_i towards C_j or even considering the radii of the spheres S_i and S_j in order to choose among the two possible orientations.

However, there is a more clever way of defining the orientation of this elliptic trisector within the context of what we are trying to achieve, i.e., answer the EDGECONFLICT predicate.

Firstly, we must understand that orienting the trisector τ_{ijk} was initially needed to properly define a Voronoi edge e_{ijklm}. In the case of a hyperbolic trisector type, we were able to use a simple ordering $\prec$ that derived by the orientation of τ_{ivk} via the "right-hand rule" to observe that $e_{ijklm} := \{\tau \in \tau_{ijk} : v_{ijkl} \prec \tau \prec v_{ikjm}\}$. Therefore, in order to be consistent with the analysis and notation of Chapter 2, we must define an orientation that will preserve this fact.

To accomplish this, we begin by considering the shadow region of S_l and S_m on the trisector τ_{ijk}. It is clear that both these shadow regions must have at least one boundary point, corresponding to the center of the respective Apollonius sphere of S_i, S_j, S_k and either S_l or S_m respectively. Assuming no degeneracies and according to the analysis that we will present in the subsection "Shadow Regions" below, if at least one of the Apollonius vertices $\{v_{ikjn}, v_{ijkn}\}$ exist, for some $n \in \{l, m, q\}$, then both must exist. As a consequence, it must hold that all vertices $v_{ijkl}, v_{ikjl}, v_{ijkm}$ and v_{ikjm} exist on the trisector τ_{ijk} since e_{ijklm} is assumed to exist.

We now orient τ_{ijk} such that, when traversing the shadow region of S_l on τ_{ijk} following the positive orientation, we "start" from v_{ikjl} and "end" at v_{ijkl}. Since e_{ijklm} is a valid Voronoi edge, it is now certain that, while traversing the rest of the trisector from v_{ijkl} to v_{ikjl}, we will encounter v_{ikjm} first among $\{v_{ijkm}, v_{ikjm}\}$. Furthermore, for all points $\tau \in e_{ijklm}$, it holds that $v_{ijkl} \prec \tau \prec v_{ikjm}$, where $a \prec b \prec c$ denotes that we will encounter the points a, b and c on the trisector in this order while traversing it in its positive direction.

Although, this way of orienting τ_{ijk} helps us preserve most of the analysis of Section 2.7.5 as well as Lemma 15, it is quite complicated. We may simplify this task as follows. If $\mathcal{A} = \mathcal{A}_{ijk}(t)$ is the Apollonius sphere of the sites S_i, S_j and S_k, centered at $t \in \tau_{ijk}$, let T_n be the tangency point $\mathcal{A}$ and S_n, for $n \in \{i, j, k\}$. If we fold our right hand to follow the points T_i, T_j and T_k (in that order), our thumb will be showing the positive direction of τ_{ijk}. However, since we can safely assume that S_k lies in the convex hull of S_i and S_j after a proper name exchange, the above right-hand rule is equivalent to an even simpler rule. The orientation of τ_{ijk} is the one provided by the original "naive" right-hand rule: if our thumb points from C_j towards C_i, the direction pointed by our hand, if it "wraps" τ_{ijk}, is the positive direction (see Figure 3.1).

Differences in $\mathcal{W}$-space and $\mathcal{Y}$-space

Following the same analysis presented in Section 2.7.6, $\mathcal{W}$-space is the plane we will end up after reducing all original sites S_n, for $n \in \{i, j, k, l, m, q\}$, by radius $r_k = \min\{r_i, r_j, r_k\}$ and inverting (see Section 1.4) with C_k as the pole. As before, we can also define the 2-dimensional $\mathcal{Y}$-space, where geometric remarks are easier to take place.

However, there is a crucial difference between the observations of Section 2.7.6 that hold for hyperbolic trisector types and the respective elliptic case we want to study. In the former case, there exist two Apollonius spheres-planes of the sites S_i, S_j and S_k, centered at infinity, whereas, in the latter case, no such sphere exist. The equivalency of this fact in $\mathcal{W}$-space is that, the image of all these Apollonius spheres are planes, commonly tagent to the spheres

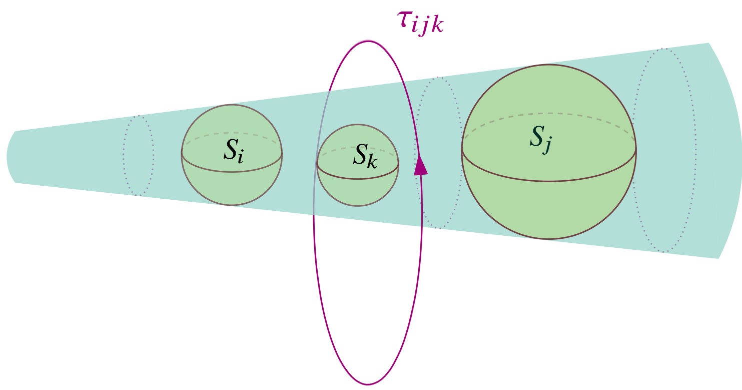

FIGURE 3.1: Orienting an elliptic trisector, using the "right-hand rule"; if our thumb points from C_j towards C_i, the direction pointed by our hand, if it "wraps" τ_{ijk}, is the positive direction.

$S_i^\star$ and $S_j^\star$, that leave the point O on the same side as the latter two spheres. Moreover, there can not exist a plane cotagent to the spheres $S_i^\star$ and $S_j^\star$ such that $S_i^\star$ and $S_j^\star$ lie on the one half-plane and O on the other. Subsequently, the point O must lie strictly inside the semi-cone $\mathcal{K}^\star$ defined by $S_i^\star$ and $S_j^\star$ (or the respective cylinder if these two spheres have equal radii).

Respectively, the point $\hat{O}$ in $\mathcal{Y}$-space must lie inside the circle C'. Using the same arguements as in Section 2.7.6, one can show that the image of $\mathcal{A}_{ijk}(t)$, i.e., the Apollonius sphere of S_i, S_j and S_k centered at t, is a plane in $\mathcal{W}$-space. As the point t moves on the oriented trisector τ_{ijk}, the plane rotates remaining tangent to $\hat{S}_i$ and $\hat{S}_j$. In $\mathcal{Y}$-space, the corresponding image is that of a line $\hat{\ell}(t)$ that rotates remaining tangent to the circle C'. The main difference with the hyperbolic case is that there are no maximal positions in these rotations; the rotating plane in $\mathcal{W}$-space (resp., line in $\mathcal{Y}$-space) never goes through the point O (resp., $\hat{O}$).

Note that we can appropriately orient the circle C' using the "right-hand rule" again: if our thumb points from $\hat{C}_j$ towards $\hat{C}_i$, the direction pointed by our hand, if it "wraps" C', is the positive direction. (see Figure 3.2) Observe that there is a 1-1 correspondence between the oriented trisector τ_{ijk} and the oriented circle C'. We denote this 1-1 and onto mapping from τ_{ijk} to the circle C' by $\psi(\cdot)$, such that $\mathcal{A}_{ijk}(t)$ of $\mathcal{Z}$-space maps to the line $\hat{\ell}(t)$ of $\mathcal{Y}$-space that is tangent to C' at the point $\psi(t)$. As a result of the orientations defined on C' and τ_{ijk} along with the remarks of the Section 2.7.6, we can easily deduct the following lemma, equivalent to lemma 13 for hyperbolic trisectors.

Lemma 19 *There is a 1-1 correspondence between the order of appearance of the existing vertices among $v_{ijka}, v_{ikja}, v_{ijkb}$ and v_{ikjb} on the oriented elliptic trisector τ_{ijk} and the order of appearance of the existing points $\psi(v_{ijka})$, $\psi(v_{ikja})$, $\psi(v_{ijkb})$ and $\psi(v_{ikjb})$ on the oriented circle C'.*

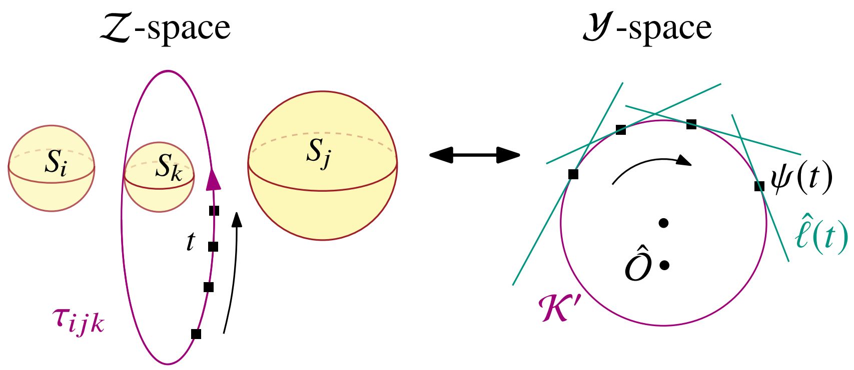

FIGURE 3.2: As t moves on the elliptic trisector τ_{ijk} following its positive orientation, its image $\psi(t)$ moves on the oriented circle C'. The image of the Apollonius sphere $\mathcal{A}_{ijk}(t)$ in $\mathcal{Y}$-space is the line $\hat{\ell}(t)$ that remains tangent to C'. Since all $\mathcal{A}_{ijk}(t)$ are finite, $\hat{\ell}(t)$ must leave $\hat{O}$ on its positive side, therefore $\hat{O}$ must lie inside C'. Apparently, there is a 1-1 and onto correspondence between the trisector τ_{ijk} in $\mathcal{Z}$-space and C' in $\mathcal{W}$-space.

This lemma enables us to follow similar strategy as the one presented in Section 2.4.6 to order the Apollonius vertices $v_{ijkl}, v_{ikjm}, v_{ijkq}$ and v_{ikjq} on the oriented τ_{ijk}. Then, using the remarks of Section 2.5, we deduce the intersection type of e_{ijklm} and $\mathcal{SR}(S_q)$ and therefore answer the EDGECONFLICT predicate.

Shadow Regions on elliptic trisectors

Let us assume that the trisector of the spheres S_i, S_j and S_k is elliptic and that $k < i, j$. After reducing the radii of all sites S_i, S_j, S_k and S_n, for $n \in \{l, m, q\}$, and then inverting through the point C_k, we consider the position of $\hat{S_n}$ against the semicone $\mathcal{K}^\star$ defined by $\hat{S_i}$ and $\hat{S_j}$. The cases where $\hat{S_n}$ point-touches or is full tangent to the cone from the inside are considered degenerate and are beyond the scope of this chapter. Therefore, we can assume that either 0 or 2 distinct planes commonly tangent to the spheres $\hat{S_i}, \hat{S_j}$ and $\hat{S_n}$ may exist. Due to the location of O, any such plane will correspond to an external Apollonius sphere of S_i, S_j, S_k and S_n in $\mathcal{Z}$-space. As an immediate result, either 0 or 2 such Apollonius spheres must exist.

Since the existence of v_{ijkl} (resp., v_{ikjm}) derives from the assumption that e_{ijklm} is a valid Voronoi edge, at least one Apollonius sphere of the sites S_i, S_j, S_k and S_l (resp., S_m) exist. Applying the previous remark, there must also exist another one; this sphere must be centered at the Apollonius vertex v_{ikjl} (resp., v_{ijkm}). We have therefore concluded that all vertices $\{v_{ijkl}, v_{ikjl}, v_{ijkm}, v_{ikjm}\}$ must exist and moreover, using the same argument, that either both or neither of $\{v_{ijkq}, v_{ikjq}\}$ exist.

According to the analysis of Section 2.4.5, the number of boundary points of $\mathcal{SR}(S_n)$ represent how many of the vertices $\{v_{ijkn}, v_{ikjn}\}$ exist, i.e., 0 or 2 based on our remarks. It is straightforward to deduce that, when seen as an interval, $\mathcal{SR}(S_n)$ must either be $\emptyset$ or $\mathbb{R}$, in case of 0 boundary points, or (χ, ϕ), in case of 2 boundary points. The notation we use for elliptic trisectors differs a little from the one used for hyperbolic ones:

- $\emptyset$ indicates that $\mathcal{SR}(S_n)$ does not contain any point of τ_{ijk},
- $\mathbb{R}$ symbolizes that $\mathcal{SR}(S_n)$ consists of all points of τ_{ijk} and
- (χ, ϕ) denotes that $\mathcal{SR}(S_n)$ is an arc on τ_{ijk}. This arc contains all points t of the trisector such that $v_{ikjn} \prec t \prec v_{ijkn}$; this fact derives from the proof of lemma 15 and the proper orientation of τ_{ijk}.

All these remarks sum up to the equivalent of lemma 15 for elliptic trisectors.

Lemma 20 *The type of the shadow region $\mathcal{SR}(S_n)$ of a sphere S_n on an elliptic trisector is one of the following: $\emptyset, \mathbb{R}$ or (χ, ϕ). In the latter case, χ and ϕ correspond to the points v_{ikjn} and v_{ijkn} of the oriented trisector τ_{ijk} respectively.*

3.2 The EDGECONFLICT predicate for Elliptic Trisectors

In this section, we provide a detailed algorithm that is used to answer the EDGECONFLICT $(S_i, S_j, S_k, S_l, S_m, S_q)$ predicate assuming that τ_{ijk} is an elliptic trisector. The main idea behind this algorithm is already presented in Section 2.5; our primary goal is to determine the intersection type of the edge e_{ijklm} (or simply e) and the shadow region $\mathcal{SR}(S_q)$.

If the topological form of $\mathcal{SR}(S_q)$, denoted by $SRT(q)$, is either $\emptyset$ or $\mathbb{R}$ then we can immediately return NoConflict or EntireEdge respectively. However if $SRT(q)$ is (χ, ϕ), then both v_{ijkq} and v_{ikjq} must exist and we may have to determine the order of appearance of the vertices $\{v_{ijkq}, v_{ikjq}, v_{ijkl}, v_{ikjm}\}$ on the oriented trisector. The ordering of these Apollonius vertices on τ_{ijk} is acquired from the respective ordering of their images on C', in $\mathcal{Y}$-space. Using this information, we can ultimately deduce the intersection type of $e_{ijklm} \cap \mathcal{SR}(S_q)$ and return it as the answer of the EDGECONFLICT predicate (see Section 2.5).

At this point, it is wise to recall that the possible answers to the EDGECONFLICT predicate, as presented in Section 2.3 and 2.5, are the following:

- NoConflict: no portion of e is destroyed by the insertion of S_q in the Apollonius diagram of the five sites.
- EntireEdge: the entire edge e is destroyed by the addition of S_q in the Apollonius diagram of the five sites.
- LeftVertex: a subsegment of e adjacent to its origin vertex (v_{ijkl}) disappears in the Apollonius diagram of the six sites.
- RightVertex: is the symmetric case of the LeftVertex case; a subsegment of e adjacent to the vertex v_{ikjm} disappears in the Apollonius diagram of the six sites.

- BothVertices: subsegments of e adjacent to its two vertices disappear in the Apollonius diagram of the five sites.
- Interior: a subsegment in the interior of e disappears in the Apollonius diagram of the five sites.

Lastly, it is quite convinient that the Existence (S_i, S_j, S_k, S_q) predicate can be evaluated as described in Section 2.7.3 without any modifications for elliptic trisectors. This subpredicate will return whether 0 or 2 of the vertices $\{v_{ikjq}, v_{ijkq}\}$ exist; assuming no degeneracies, these are the only possible answers.

Step 1 We evaluate $E = \textsc{Existence}(S_i, S_j, S_k, S_q)$, as described in Section 2.7.3. The predicate's outcome is either 0 or 2, assuming no degeneracies. If $E = 0$ go to Step 2 otherwise, if $E = 2$ go to step 3.

Step 2 We evaluate $I = \textsc{InSphere}(S_i, S_j, S_k, S_l, S_q)$ (or InSphere $(S_i, S_k, S_j, S_m, S_q)$); if $I = +$, the EdgeConflict predicate's outcome is NoConflict otherwise, if $I = -$, the outcome is EntireEdge.

Explanation: Since none of $\{v_{ijkq}, v_{ikjq}\}$ exist, the topological form $SRT(q)$ of $\mathcal{SR}(S_q)$ is either $\emptyset$ or $\mathbb{R}$. In the first case, the EdgeConflict predicate must return NoConflict and in the latter EntireEdge. Moreover, in the case where $SRT(q) = \emptyset$, none of the points of τ_{ijk}, including v_{ijkl} and v_{ikjm}, belong to $\mathcal{SR}(S_q)$, i.e., the Apollonius sphere centered at these points do not intersect S_q. Therefore, in this case both InSphere outcomes must be $+$. In the opposite case, both Apollonius vertices v_{ijkl} and v_{ikjm} would belong to $\mathcal{SR}(S_q)$ and equivalently, the corresponding Apollonius spheres would intersect S_q.

Step 3 We evaluate $I_1 = \textsc{InSphere}(S_i, S_j, S_k, S_l, S_q)$ and $I_2 = \textsc{InSphere}(S_i, S_k, S_j, S_m, S_q)$. If $(I_1, I_2) = (-, +)$ or $(+, -)$ return LeftVertex or RightVertex respectively. Otherwise, if $(I_1, I_2) = (+, +)$ or $(-, -)$ go to Step 4.

Explanation: Since both v_{ijkq} and v_{ikjq} exist, the topological form $SRT(q)$ of $\mathcal{SR}(S_q)$ is (χ, ϕ). If $(I_1, I_2) = (-, +)$ then the "left" vertex v_{ijkl} of the edge e_{ijklm} lies inside $\mathcal{SR}(S_q)$ and the "right" vertex v_{ikjm} does not. Due to the simple topological forms of $\mathcal{SR}(S_q)$ and the Voronoi edge, we deduce that $\mathcal{SR}(S_q)$ intersects only the left part of the edge hence the EdgeConflict predicate's outcome is LeftVertex. In the symmetric case where $(I_1, I_2) = (+, -)$, the shadow region $\mathcal{SR}(S_q)$ intersects only the right part of the edge and the predicate's answer is RightVertex.

Step 4 We evaluate $I_3 = \textsc{InSphere}(S_i, S_j, S_k, S_m, S_l)$. If $I_3 = -$ go to Step 5 otherwise, if $I_3 = +$ go to Step 6.

Explanation: We break down our analysis into two cases A and B, which we study in Steps 5 and 6 respectively. In case A (resp., B), v_{ijkm} belongs (resp., does not belong) to the shadow region $\mathcal{SR}(S_l)$ on the trisector since the Apollonius sphere

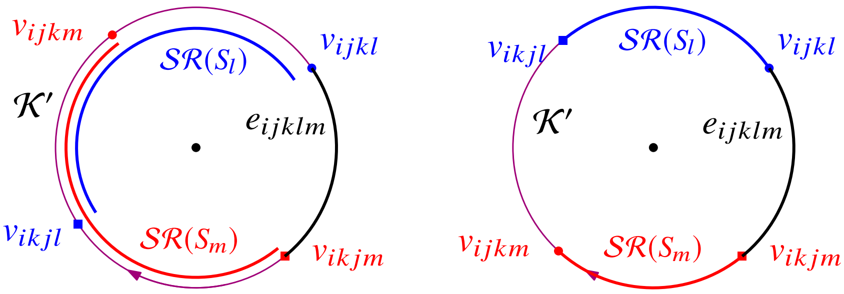

FIGURE 3.3: Since the Voronoi edge e_{ijklm} is bounded by v_{ijkl} and v_{ikjm} on the oriented circle trisector τ_{ijk}, the points of e_{ijklm} can not lie in the shadow regions of S_l and S_m. Since the latter two are bounded by v_{ikjl} and v_{ikjm} on the "left" and v_{ijkl} and v_{ijkm} on the "right" respectively, the only possible orderings of these vertices are the ones described in Case A(left figure) or Case B(right figure).

centered at v_{ijkm} intersects S_l as the outcome of the INSPHERE predicate suggests. Since $\mathcal{SR}(S_l) := \{t \in \tau_{ijk} : v_{ikjl} \prec t \prec v_{ijkl}\}$, a quick conclusion to be drawn is that,

- in case A, $v_{ijkl} \prec v_{ikjm} \prec v_{ikjl} \prec v_{ijkm}$ (see Figure 3.3(left)), whereas
- in case B, $v_{ijkl} \prec v_{ikjm} \prec v_{ijkm} \prec v_{ikjl}$ (see Figure 3.3(right)).

As stated before, $a \prec b \prec c$ denotes that you will meet a, b and c in this order while *positively* traversing the trisector, starting from point a.

Important notice: For the sake of clarity, the notation of the figures of this Section was simplified. Although all figures represent configurations in $\mathcal{Y}$-space, the Apollonius vertices are denoted by $v_{\alpha\beta\gamma\delta}$ instead of $\psi(v_{\alpha\beta\gamma\delta})$, where $\{\alpha, \beta, \gamma, \delta\} \in \{i, j, k, l, m, q\}$. Moreover, the image of the shadow region of S_α is denoted by $\mathcal{SR}(S_a)$ instead of $\psi(\mathcal{SR}(S_a))$ and the image of the Voronoi edge is denoted by e_{ijklm}. Lastly, although the images of the shadow regions of S_l, S_m and S_q lie on the circle C', sometimes they are drawn as if they lied on a concentric inner circle.

Step 5 We determine if all of the following predicates return $+$:

- I_4 = INSPHERE$(S_i, S_k, S_j, S_q, S_l)$,
- I_5 = INSPHERE$(S_i, S_k, S_j, S_q, S_m)$,
- I_6 = INSPHERE$(S_i, S_j, S_k, S_q, S_l)$ and,
- I_7 = INSPHERE$(S_i, S_j, S_k, S_q, S_m)$.

If all I_4 to I_7 are positive then

- return BothVertices, if $I_1 = I_2 = -$ or
- return Interior, if $I_1 = I_2 = +$.

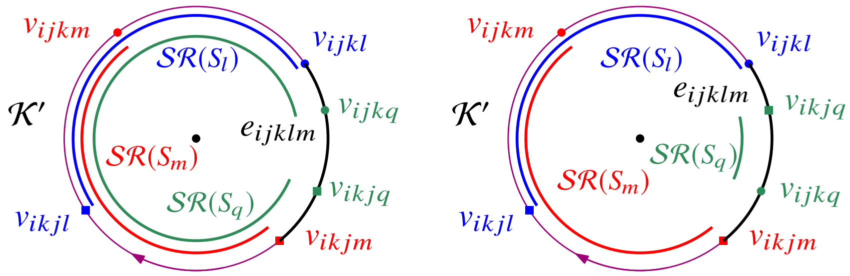

FIGURE 3.4: If all INSPHERE predicate I_4 to I_7 are positive then the vertices v_{ijkq} and v_{ikjq} do not lie on neither $\mathcal{SR}(S_l)$ nor $\mathcal{SR}(S_m)$ and therefore must lie on the edge e_{ijklm}. In this configuratoin, if $I_1 = I_2 = -$, both endpoints of the edge e_{ijklm} lie on $\mathcal{SR}(S_q)$ and the EDGECONFLICT predicate must return BothVertices (left figure). If $I_1 = I_2 = +$, both endpoints of the edge e_{ijklm} lie outside $\mathcal{SR}(S_q)$ and the EDGECONFLICT predicate must return Interior (right figure).

Otherwise, if at least one of I_4 to I_7 is negative, then

- return EntireEdge, if $I_1 = I_2 = -$ or
- return NoConflict, if $I_1 = I_2 = +$.

Explanation: If $I_1 = I_2 = -$, then both parts adjecent to the endpoints of the Voronoi edge will no longer exist in the updated Voronoi diagram. In this scenario, either a part in the middle will still remain (BothVertices) or all of the edge will cease to exist (EntireEdge). In the former case, both v_{ikjq} and v_{ijkq} ought to lie between v_{ijkl} and v_{ikjm} on the trisector (see Figure 3.4(left)). However, since we are in Step 5, it holds that $v_{ijkl} \prec v_{ikjm} \prec v_{ikjl} \prec v_{ijkm}$. As a result, an equivalent expression of $v_{ijkl} \prec v_{ikjq}, v_{ijkq} \prec v_{ikjm}$ would be that $v_{ikjq}, v_{ijkq} \notin \mathcal{SR}(S_l)$ and $v_{ikjq}, v_{ijkq} \notin \mathcal{SR}(S_m)$. Lastly, these four expressions amount to all INSPHERE outcomes I_4 to I_7 being positive.

In a similar way, if $I_1 = I_2 = +$ then both parts adjecent to the endpoints of the Voronoi edge will remain in the updated Voronoi diagram. In such scenario, either a part in the middle will cease to exist (Interior) or all of the edge will remain intact (NoConflict). The former case is equivalent to both v_{ikjq} and v_{ijkq} lying between v_{ijkl} and v_{ikjm} on the trisector (see Figure 3.4(right)). As stated above, such a geometric configuration amounts to all INSPHERE outcomes I_4 to I_7 being positive.

Step 6 We determine if all of the following predicates return +:

- $I_4 = \text{INSPHERE}(S_i, S_k, S_j, S_q, S_l)$,
- $I_5 = \text{INSPHERE}(S_i, S_k, S_j, S_q, S_m)$,
- $I_6 = \text{INSPHERE}(S_i, S_j, S_k, S_q, S_l)$ and,

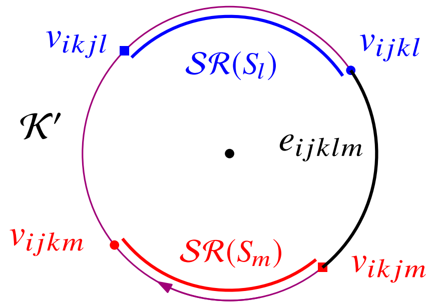

FIGURE 3.5: If all INSPHERE predicate I_4 to I_7 are positive then the vertices v_{ijkq} and v_{ikjq} do not lie on neither $\mathcal{SR}(S_l)$ nor $\mathcal{SR}(S_m)$. In Case B, this is equivalent to v_{ijkq} and v_{ikjq} lying either on the edge e_{ijklm} or in-between v_{ijkm} and v_{ikjl}. Further analysis is required to determine in which part of $\mathcal{K}^\star$ they actually lie.

- I_7 = INSPHERE$(S_i, S_j, S_k, S_q, S_m)$.

If at least one of them is negative, go to Step 6a otherwise go to Step 6b.

Explanation: For Steps 6,6a and 6b, it holds that $v_{ijkl} \prec v_{ikjm} \prec v_{ijkm} \prec v_{ikjl}$ hence all INSPHERE predicates I_4 to I_7 being positive implies that the vertices v_{ikjq} and v_{ijkq} lie either between v_{ijkl} and v_{ikjm} or between v_{ijkm} and v_{ikjl} (see Figure 3.5).

Step 6a If $I_1 = I_2 = -$ return EntireEdge otherwise, if $I_1 = I_2 = +$ return NoConflict.

Explanation: In this Step, v_{ijkq} and v_{ikjq} can not both lie in-between the endpoints v_{ijkl} and v_{ikjm} of the Voronoi edge e_{ijklm} and therefore the EDGECONFLICT predicate's answers BothVertices and Interior are not feasible. As a result, if $I_1 = I_2 = -$ the EDGECONFLICT predicate returns EntireEdge whereas, if $I_1 = I_2 = +$ the predicate returns NoConflict.

Step 6b Go to Step 6bA.

Analysis behind Step 6: In this Step, we have to determine if both or none of v_{ijkq} and v_{ikjq} lie in-between v_{ijkl} and v_{ikjm} on the oriented trisector. In order to distinguish among the two possible scenarios of Step 6b, we follow a similar analysis with the one presented in Section 2.7.6 when we wanted to decide between OrderCase 1 and 6.

Since we are in Step 6b, all INSPHERE outcomes I_4 to I_7 are positive and $I_1 = I_2$. A geometric consequence of these results is that while traversing the trisector the encountered Apollonius vertices will appear consecutively in pairs $\{v_{ijkl}, v_{ikjl}\}$, $\{v_{ijkm}, v_{ikjm}\}$ and $\{v_{ijkq}, v_{ikjq}\}$. Equivalently, both vertices v_{ijkq} and v_{ikjq} appear either in-between v_{ijkl} and v_{ikjm} (i.e., on the Voronoi edge) or in-between v_{ijkm} and v_{ikjl}. This remark proves that the order of appearance of these "pairs" is crucial for our analysis.

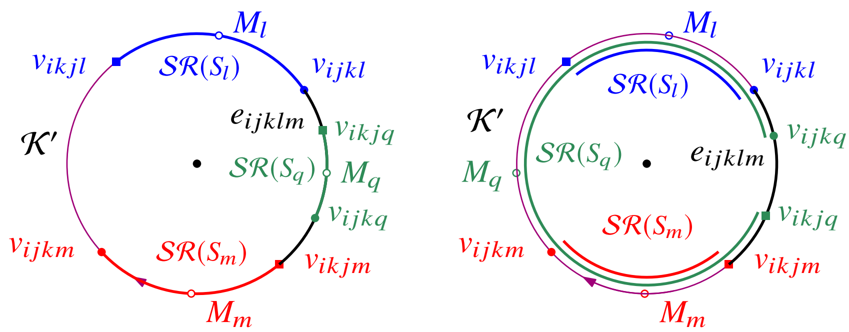

FIGURE 3.6: If both v_{ijkq}, v_{ikjq} lie on the edge e_{ijklm}, i.e., in-between the points v_{ijkl} and v_{ikjm}, then either a Interior (left figure) or a BothVertices (right figure) configuration arises. We can distinguish which is the actually case if can determine the order of appearance of M_l, M_m and M_q, which are the centers of the corresponding shadow regions $\mathcal{SR}(S_l)$, $\mathcal{SR}(S_m)$ and $\mathcal{SR}(S_q)$. In the Interior scenario, $M_l \prec M_q \prec M_m$ and $I_1 = I_2 = +$ whereas in the BothVertices case, $M_l \prec M_m \prec M_q$ and $I_1 = I_2 = -$.

To obtain this ordering, we will have to study the corresponding configuration in $\mathcal{Y}$-space. Let us denote by M_n, for $n \in \{l, m, q\}$, the middle point of the image of $\mathcal{SR}(S_n)$ in $\mathcal{Y}$-space. The order of appearance of the aforementioned pairs is equivalent to the order of appearance of M_l, M_m and M_q on C'.

Given the ordering of these points and the INSPHERE outcomes I_1 and I_2 we can ultimately determine if v_{ijkq} and v_{ikjq} lie or not in-between v_{ijkl} and v_{ikjm}. For example, assume we evaluate that $M_l \prec M_q \prec M_m$ and that $I_1 = I_2 = +$. Due to I_1 and I_2 being positive, it is known that the shadow region $\mathcal{SR}(S_q)$ either lies in the middle part of the Voronoi edge (and the outcome is Interior) or it does not intersect the edge at all (and the outcome is NoConflict). In the former case both v_{ijkq} and v_{ikjq} would lie in-between v_{ijkl} and v_{ikjm} and therefore $M_l \prec M_q \prec M_m$. In the latter case, both v_{ijkq} and v_{ikjq} would have to lie in-between v_{ijkm} and v_{ikjl} hence it would hold that $M_l \prec M_m \prec M_q$.

Using the same arguements, we can deduce that

- v_{ijkq} and v_{ikjq} lie in-between v_{ijkl} and v_{ikjm} if either $I_1 = I_2 = +$ and $M_l \prec M_q \prec M_m$ (see Figure 3.6(left)) or $I_1 = I_2 = -$ and $M_l \prec M_m \prec M_q$ (see Figure 3.6(right)) whereas,
- v_{ijkq} and v_{ikjq} do not lie in-between v_{ijkl} and v_{ikjm} if either $I_1 = I_2 = +$ and $M_l \prec M_m \prec M_q$ (see Figure 3.7(left)) or $I_1 = I_2 = -$ and $M_l \prec M_q \prec M_m$ (see Figure 3.7(right)) .

Let us now focus on how one may determine the order of M_l, M_m and M_q on C'. Firstly, we reflect on the fact that the the open ray from the center $\hat{\mathcal{A}}$ of C' towards

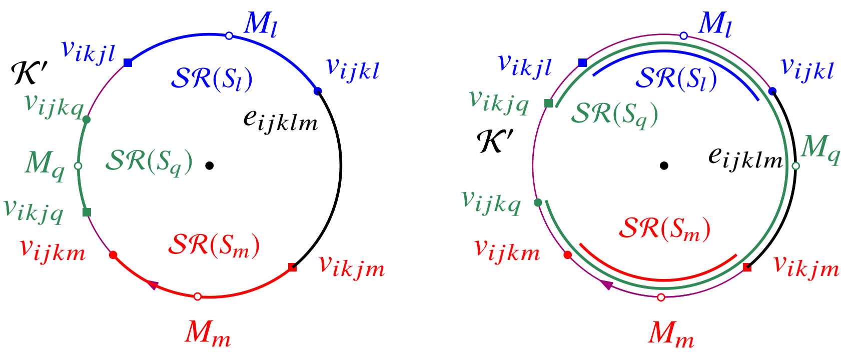

Figure 3.7: If both v_{ijkq}, v_{ikjq} lie in-between the points v_{ijkm} and v_{ikjl}, then either a NoConflict (left figure) or a EntireEdge (right figure) configuration arises. We can distinguish which is the actually case if can determine the order of appearance of M_l, M_m and M_q, which are the centers of the corresponding shadow regions $\mathcal{SR}(S_l)$, $\mathcal{SR}(S_m)$ and $\mathcal{SR}(S_q)$. In the NoConflict scenario, $M_l \prec M_m \prec M_q$ and $I_1 = I_2 = +$ whereas in the BothVertices case, $M_l \prec M_q \prec M_m$ and $I_1 = I_2 = -$.

M_n, for $n \in \{l, m, q\}$, goes through the center $\hat{C}_n$ of the circle $\hat{S}_n$. Consequently, the order of appearance of M_l, M_m and M_q on C' is the order of appearance of the corresponding open rays, if we apply the same orientation with τ_{ijk} (see Figure 3.8). The latter ordering in $\mathcal{Y}$-space can be deduced using orientation predicates of the points $C^\star_\mu$ of $\mathcal{W}$-space for $\mu \in \{i, j, l, m, q\}$. The required predicates to obtain the ordering are presented in Steps 6bA-C; the same analysis is followed in Section 2.7.6.

Step 6bA We evaluate $\Pi = O_1 \cdot O_2$, where $O_1 = \text{Orient3D}(C^\star_q, C^\star_i, C^\star_j, C^\star_l)$ and $O_2 = \text{Orient3D}(C^\star_m, C^\star_i, C^\star_j, C^\star_l)$. If $\Pi > 0$ go to Step 6bB, otherwise go to Step 6bC.

Explanation: If $\Pi > 0$ then both M_q and M_m lie on the same side of C' with respect to the line that goes through $\hat{\mathcal{A}}$ and M_l.

Step 6bB. We evaluate $O_3 = \text{Orient3D}(C^\star_q, C^\star_i, C^\star_j, C^\star_m)$. If $O_3 < 0$ then go to Step 6d if $I_1 = I_2 = +$ or go to Step 6c if $I_1 = I_2 = +$. Otherwise, if $O_3 > 0$ then go to Step 6c if $I_1 = I_2 = +$ or go to Step 6d if $I_1 = I_2 = +$.

Explanation: Both M_q and M_m lie on the same side of C' with respect to the line that goes through $\hat{\mathcal{A}}$ and M_l. If $O_3 < 0$, then in $\mathcal{Y}$-space it must hold that M_q lies on the right half-plane defined by the array that goes from $\hat{\mathcal{A}}$ to M_m. In this configuration, it must necessarily hold that $M_l \prec M_m \prec M_q$. On the other hand, if $O_3 > 0$, we deduce that $M_l \prec M_q \prec M_m$.

Step 6bC. If $\mathsf{sign}(O_1) > \mathsf{sign}(O_2)$ then go to Step 6d if $I_1 = I_2 = +$ or go to Step 6c if $I_1 = I_2 = +$. Otherwise, go to Step 6c if $I_1 = I_2 = +$ or go to Step 6d if $I_1 = I_2 = +$.

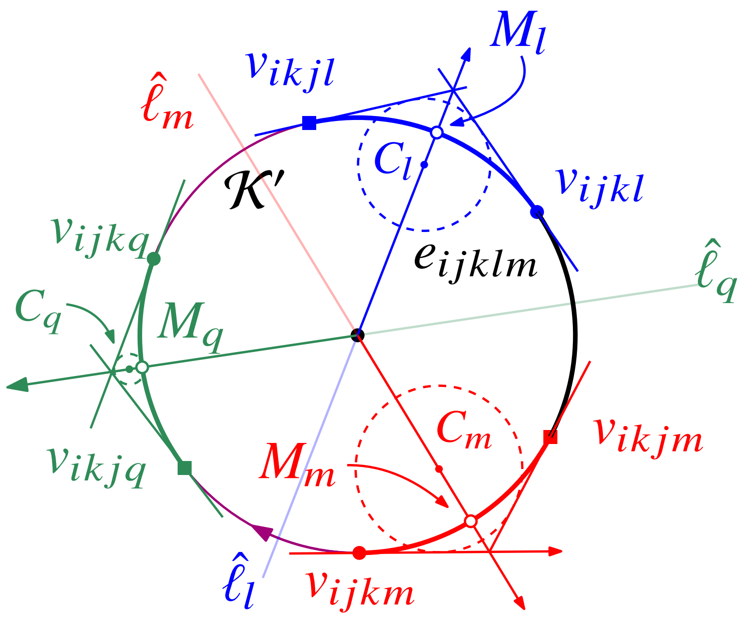

FIGURE 3.8: To order the points M_l, M_m and M_q on $\mathcal{K}^\star$, we consider some major facts. Firstly, as proven in Section 2.7.6, the image of C_n, for $n \in \{l, m, q\}$, in $\mathcal{Y}$-space must lie on the ray starting from the center of $\mathcal{K}^\star$ and going through M_n. As a consequence, the line $\hat{\ell}_n$ of $\mathcal{Y}$-space defined by this ray corresponds in $\mathcal{W}$-space to a plane containing the centers $C_i^\star, C_i^\star$ and $C_n^\star$. We shall denote the the halfspace of $\mathcal{Y}$-space defined by $\hat{\ell}_n$ and the point M_n if it moves infinitesimally following (resp., opposite) the positive orientation of τ_{ijk} as the *right* (resp., *left*) *side of* $\hat{\ell}_n$. Due to the orientation of $\mathcal{K}^\star$, for all points $N^\star$ of $\mathcal{W}$-space that satisfy ORIENT3D$(N^\star, C_i^\star, C_j^\star, C_n^\star) < 0$, the respective image $\hat{N}$ in $\mathcal{Y}$-space must lie on the right side of $\hat{\ell}_n$. The ORIENT3D predicate is known to be equivalent to ORIENT3D(N, C_i, C_j, C_n) from previous sections. Ultimately, we can safely conclude that $M_l \prec M_m \prec M_q$ if C_m and C_q lie on the right and left side of $\hat{\ell}_l$ respectively. Otherwise, C_m and C_q must lie on $\hat{\ell}_l$ or on the same side; in either case, checking if C_q lies on the left or right side of $\hat{\ell}_m$ determines the ordering of M_l, M_m and M_q.

Explanation: It is known that M_q and M_m do not lie on the same side of C' with respect to the line that goes through $\hat{\mathcal{A}}$ and M_l. If $\mathsf{sign}(O_1) > \mathsf{sign}(O_2)$ then M_q (resp., M_m) must lie on the right (resp., left) half-plane defined by the array that goes from $\hat{\mathcal{A}}$ to M_l. Therefore, we easily conclude that $M_l \prec M_m \prec M_q$. Using the same arguements, if $\mathsf{sign}(O_1) > \mathsf{sign}(O_2)$ does not hold, we deduce that $M_l \prec M_q \prec M_m$.

Step 6c If $I_1 = I_2 = +$ return Interior otherwise, if $I_1 = I_2 = -$ return BothVertices. **Explanation:** In this configuration, both v_{ikjq} and v_{ijkq} lie on the Voronoi edge e_{ijklm}. This implies that if $I_1 = I_2 = +$ then the shadow region of S_q is an interval in the middle of e_{ijklm} and therefore the predicate returns Interior. If $I_1 = I_2 = -$ then $\mathcal{SR}(S_q)$ would be all the trisector except the aforementioned interval in the middle of of e_{ijklm} and the predicate must return BothVertices in this case.

Step 6d If $I_1 = I_2 = +$ return NoConflict otherwise, if $I_1 = I_2 = -$ return EntireEdge. **Explanation:** In this configuration, v_{ikjq} and v_{ijkq} can not both lie on the Voronoi edge e_{ijklm}. As a result, if $I_1 = I_2 = +$ then the predicate returns NoConflict since Interior is infeasible. Additionally, if $I_1 = I_2 = -$ the predicate returns is EntireEdge as the alternative answer, BothVertices, is not possible.

3.3 Algebraic degrees and Conclusions

Using the algorithm presented in the previous section, we can answer the EdgeConflict predicate in case of elliptic trisectors, assuming all sub-predicates return a non degenerate answer. A layout of all possible subpredicates used, aside from primitives such as orientation tests, is is shown in Figure 3.9.

Note that the highest algebraic degree needed in the evaluation of the subpredicates used is 10. Moreover, during Steps 6bA-C we may have to evaluate orientation tests of the form Orient3D($\hat{S}_a, \hat{S}_i, \hat{S}_j, \hat{S}_b$), for $a, b \in \{l, m, q\}$. In Section 2.7.6 (paragraph *Algebraic Cost to resolve the Cases A, B or C*), we proved that

$$\text{Orient3D}(C_b^\star, C_i^\star, C_j^\star, C_a^\star) = \mathsf{sign}(D_{bija}^{uvw}) = \mathsf{sign}(p_i^\star p_j^\star p_a^\star p_b^\star)\mathsf{sign}(E_{bija}^{xyzp}) \tag{3.1}$$

$$= \mathsf{sign}(E_{bija}^{xyzp}), \tag{3.2}$$

where E_{bija}^{xyzp} is an expression of algebraic degree 5 on the input quantities. Since the evaluation of the InSphere predicate is the most degree demanding operation of the algorithm presented in Section 3.2, we have proven the following theorem.

Theorem 3 *The* EdgeConflict *predicate for hyperbolic trisectors can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

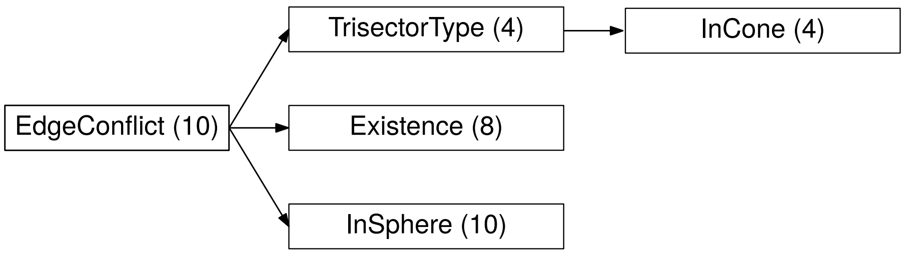

FIGURE 3.9: The layout of predicates and their subpredicates used to answer the EDGECONFLICT predicate for elliptic trisectors. The number next to each predicate corresponds to its algebraic cost. It is assumed that every subpredicate returns a non-degenerate answer.

Chapter 4

Degenerate Case Analysis

In this chapter, the qualitative symbolic perturbation scheme that was introduced in Section 1.2 is applied to resolve the degeneracies for the InCone, Distance, Shadow and Existence predicates in the Sections 4.1, 4.2, 4.3 and 4.4 respectively. The analysis of the respective predicates under the no-degeneracy assumption can be found in previous Sections 2.7.1, 2.7.4, 2.7.5 and 2.7.3 respectively.

4.1 The InCone$^{\epsilon}$ Predicate

As described in Section 2.4.2, there are four possible outcomes of the predicate InCone(S_i, S_j, S_k). In case the predicate returns one of the non-degenerate answers Outside or Inside, then the outcome of InCone(S_i, S_j, S_k) is trivialy the same. However, further analysis is required for the evaluation of the InCone$^{\epsilon}$ predicate if InCone returns one of the degenerate answers OnePointTouch or CircleTouch.

If InCone returned OnePointTouch then S_k lies inside the semi-cone defined by S_i and S_j and is also internally tangent to it at a single point. To resolve the degeneracy, we perturb the spheres in the order we defined in Section 1.2. If $k > i, j$ then the site S_k, after its infinitesimal inflation, will properly intersect the cone and therefore the perturbed predicate must return Outside.

If k has not the maximum index then either i or j has. Since a name exchange of S_i and S_j does not affect the InCone predicate, let us assume that $i > j, k$. Under this assumption, we first need to examine if t_k is identical with either t_i or t_j, where t_n denotes the tangency point of the cone and the sphere S_n, for $n \in \{i, j, k\}$. Indeed, if $t_k \equiv t_i$ or t_j, then it must hold that S_k lies within and is internally tangent to S_i or S_j respectively. In this scenario, if S_i is inflated, S_k would lie strictly inside the cone and the perturbed predicate would return Inside. To determine if this is the corresponding geometric configuration, we only need to check if $t_k \equiv t_n$, for $n \in \{i, j\}$. The last expression is equivalent to $d(C_k, C_n) = r_n - r_k$ and $(x_k - x_n)^2 + (y_k - y_n)^2 + (z_k - z_n)^2 = (r_k - r_n)^2$ and therefore is a 2-degree demanding operation.

Finally, if t_i, t_j and t_k are distinct points they must be collinear and it must also hold that, after the inflation of S_i, the sphere S_k will lie strictly inside the cone, thus the corresponding

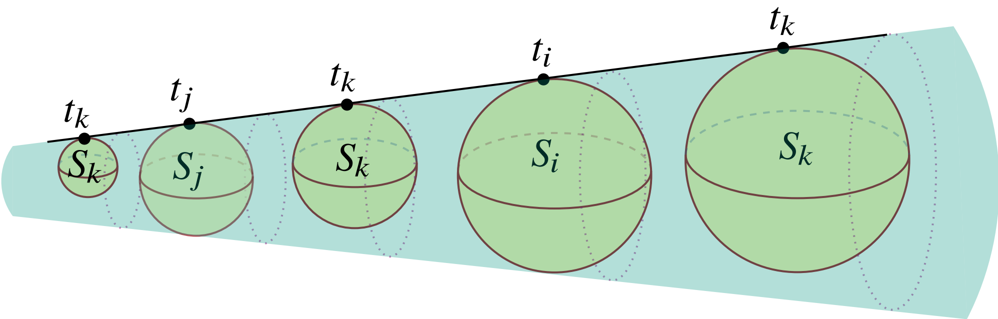

FIGURE 4.1: The spheres S_k in the center and on the right lie in such position that t_i and t_k are on the same side with respect to t_j. After S_i is perturbed, the cone defined by S_i and S_j will no longer intersect such spheres. The outcome of INCONE(S_k, S_j, S_i) is OnePointTouch only for the sphere S_k on the left; the respective outcome for the other two spheres is Outside.

INCONE$^\epsilon$ predicate would return Inside, if and only if t_i and t_k lie on the same side with respect to t_j (see Figure 4.1). In this geometric configuration and since S_k in known to be internally tangent to the cone of S_i and S_j, it is apparent that the outcome of INCONE(S_k, S_j, S_i) is Outside. On the other hand, if t_j is in-between t_i and t_k then the respective outcome would be OnePointTouch, since $i > j$. Summarizing, the outcome of the examined INCONE$^\epsilon$ predicate is Inside or Outside if INCONE(S_k, S_j, S_i) answers Outside or OnePointTouch respectively.

Lastly, we resolve the INCONE$^\epsilon$ predicate in the scenario that the corresponding INCONE predicate returned CircleTouch. Notice that the centers of the spheres S_i, S_j and S_k have to be collinear in this case. Apparently, if $k > i, j$, the sphere S_k would intersect the cone after being inflated and the outcome of the INCONE$^\epsilon$ predicate would be Outside. If this is not the case, we can assume that $i > j, k$ since a name exchange of S_i and S_j does not affect the predicates' answer.

Due to the collinearity of the centers C_i, C_j and C_k, the tangency points t_i, t_j and t_k must be distinct. Otherwise, if any two of these tangency points coincide, the respective spheres would also coincide, yielding a contradiction ; S_i, S_j and S_k are assumed to be distinct. Lastly, we observe that, after S_i is perturbed, the sphere S_k will intersect the cone, and therefore the corresponding INCONE$^\epsilon$ predicate would return Outside, if and only if C_j lies in-between the points C_i and C_k (see Figure 4.2). Otherwise, S_k would lie completely inside the cone and the INCONE$^\epsilon$ predicate would return Inside. Subsequently, INCONE$^\epsilon$ predicate should return Outside if the parallel vectors $\overrightarrow{C_jC_i}$ and $\overrightarrow{C_jC_k}$ have opposite directions or equivalently $(x_i - x_j, y_i - y_j, z_i - z_j) = \lambda(x_k - x_j, y_k - y_j, z_k - z_j)$ for some $\lambda < 0$, which is a 1-degree demanding operation. If the previous equality holds for some $\lambda > 0$ then the INCONE$^\epsilon$ predicate should return Inside.

The analysis of this section is summarized in the following algorithm that answers the INCONE$^\epsilon(S_i, S_j, S_k)$ predicate.

Step 1 If $I =$ INCONE(S_i, S_j, S_k) is Outside or Inside, return I. Otherwise go to Step 2.

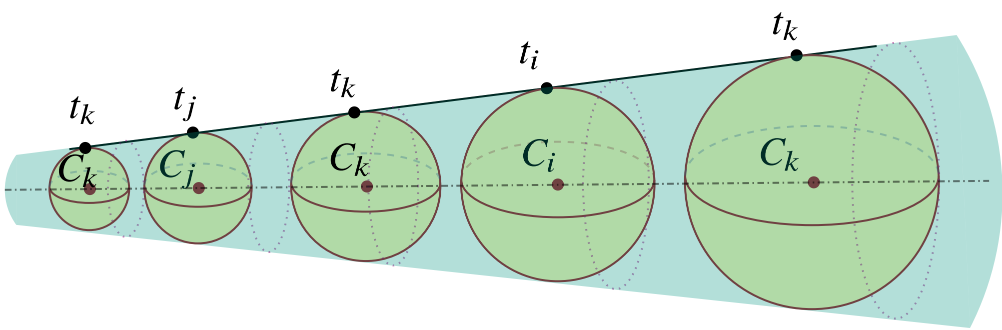

FIGURE 4.2: The spheres S_k in the center and on the right lie in such position that t_i and t_k are on the same side with respect to t_j. After S_i is perturbed, the cone defined by S_i and S_j will no longer intersect such spheres. t_i and t_k are on the same side with respect to t_j $C_j$if and only if C_i and C_k are on the same side with respect to C_j.

Step 2 If $k > i, j$ return Outside. Otherwise, assume $i > j, k$. If I is OnePointTouch or CircleTouch then go to Step 3 or 5 respectively.

Step 3 If $(x_k - x_n)^2 + (y_k - y_n)^2 + (z_k - z_n)^2 = (r_k - r_n)^2$ for some $n \in \{i, j\}$, return Inside. Otherwise go to Step 3.

Step 4 Evaluate I_2 = INCONE(S_k, S_j, S_i); if I_2 is Outside return Inside otherwise, return Outside.

Step 5 If all three equations $\mathsf{sign}(x_i - x_j) = \mathsf{sign}(x_k - x_j)$, $\mathsf{sign}(y_i - y_j) = \mathsf{sign}(y_k - y_j)$ and $\mathsf{sign}(z_i - z_j) = \mathsf{sign}(z_k - z_j)$ hold, return Inside. Otherwise, return Outside.

To summarize the analysis of this section, we have shown that in order to resolve the degeneracy of the INCONE(S_i, S_j, S_k) predicate and determine INCONE$^\epsilon(S_i, S_j, S_k)$ we may have to evaluate an additional INCONE predicate or perform various operations of lower algebraic cost. Consequently, we have proven the following lemma.

Lemma 21 *The* INCONE$^\epsilon$ *predicate can be evaluated by determining the sign of quantities of algebraic degree at most 4 (in the input quantities).*

4.2 The DISTANCE$^\epsilon$ Predicate

When the predicate DISTANCE(S_i, S_j, S_k, S_a) is called, it returns the tuple of signs (d_1, d_2), where d_1 and d_2 correspond to the distance of the sphere S_a from the planes Π^-_{ijk} and Π^+_{ijk}. As described in Sections 2.4.3 and 2.7.4, these planes are cotagent to all sites S_i, S_j and S_k, and both exist only in the case τ_{ijk} is a hyperbolic trisector.

The outcome of the DISTANCE predicate is consider degenerate if either one or both of $\{d_1, d_2\}$ equal 0. In these degenerate configurations, either one (Case A) or both (Case B) of the planes Π^-_{ijk} and Π^+_{ijk} are also tangent to S_a. We break down our analysis and study these two cases separately.

Case A. Let δ denote the d_ν, for $\nu \in \{1, 2\}$, that equals zero and δ' denotes the other one. In the case studied, there exists only one plane $\Pi : ax + by + cz + d = 0$ that is commonly tangent to S_n for all $n \in \{i, j, k, a\}$. The answer of the respective DISTANCE$^\epsilon$ predicate will be the same as DISTANCE if δ is replaced with $\Delta \in \{-, +\}$. To decide if $\Delta = -$ or $+$, we will have to determine whether S_a intersects or not respectively the plane Π after the perturbation scheme is applied.

Let t_n denote the tangency point of the plane Π with the sphere S_n, for $n \in \{i, j, k, a\}$. If $a > i, j, k$ then, after the perturbation, the inflated sphere S_a will intersect the plane Π and therefore we shall return $\Delta = -$. Otherwise, we cyclically permute the sites S_i, S_j and S_k such that $i > j, k, a$; notice that such a name exchange does not alter the outcome of the DISTANCE predicate.

Since $i > j, k, a$, S_i is the sphere that will be initially perturbed. Notice that if $t_a \equiv t_i$, the perturbation will result in S_a not intersecting Π, hence $\Delta = +$. In case $t_a \equiv t_j$ then either $r_a > r_j$ or $r_a < r_j$; the radii r_a and r_j can not be the same since the spheres S_a and S_j are considered to be distinct. If it holds that $r_a > r_j$ then S_j must lie inside and be tangent to S_a. In this configuration, Π will intersect S_a after the perturbation hence $\Delta = -$. Respectively, if $r_a < r_j$ then S_a must lie inside and be tangent to S_j; in this configuration we return $\Delta = +$ as S_a does not intersect Π after the perturbation. The same analysis can be applied in the case $t_a \equiv t_k$; it either holds that $r_a > r_k$ or $r_a < r_k$ resulting in $\Delta = -$ or $\Delta = +$ respectively.

Let us now consider the case where t_j, t_k and t_a are all distinct and additionally collinear points. Let us consider the relative position of S_a and the semi-cone defined by S_j and S_k; the outcome of INCONE(S_j, S_k, S_a) must be either OnePointTouch or Outside. In the former case, S_a will not intersect Π after the perturbation and moreover, we know that $\delta' = +$ as S_a does not intersect any other plane cotagent to S_j and S_k. In the latter case, S_a will intersect Π after the perturbation and we also know that $\delta' = -$ since, in this configuration, S_a intersected all other planes cotagent to S_j and S_k. Therefore, in both scenarios, we should return $\Delta = \delta'$.

Lastly, we consider the case where t_i, t_j, t_k and t_a are all distinct and t_a does not lie on the line $\ell(t_j, t_k)$ defined by t_j and t_k. Since two distinct planes commonly tangent to S_i, S_j and S_k exist, it also holds that t_i does not lie on $\ell(t_j, t_k)$. Therefore, on the plane Π, the points t_i and t_a lie either on the same or on different sides with respect to the line $\ell(t_j, t_k)$. In the former case, S_a will not intersect Π after the perturbation, thus $\Delta = +$. In the latter case respectively, we get $\Delta = -$ since S_a intersects Π after applying the perturbation scheme.

For the sake of clarity, we summarize the analysis of Case A as follows.

Step 1 If $a > i, j, k$ then return $\Delta = -$. Otherwise, assume that $i > j, k, a$ and go to Step 2.

Step 2 If $t_a \equiv t_i$, return $\Delta = +$. Otherwise go to Step 2.

Step 3 If $t_a \equiv t_j$, return $\Delta = -$ if $r_a > r_j$, or $\Delta = -$ if $r_a < r_j$. Otherwise go to Step 4.

Step 4 If $t_a \equiv t_k$, then return $\Delta = -$ if $r_a > r_k$ or $\Delta = +$ if $r_a < r_k$. Otherwise go to Step 5.

Step 5 If t_j, t_k and t_a are collinear, return $\Delta = \Delta'$. Otherwise go to Step 6.

Step 6 If t_i and t_a are on the same side of $\ell(t_j, t_k)$, return $\Delta = +$. Otherwise, if t_i and t_a are on different sides of $\ell(t_j, t_k)$, return $\Delta = -$.

For Steps 2 to 4, we have to determine whether $t_a \equiv t_n$, for $n \in \{i, j, k\}$. This is equivalent to the spheres S_a and S_n being internally tangent and therefore $(x_a - x_n)^2 + (y_a - y_n)^2 + (z_a - z_n)^2 = (r_a - r_n)^2$, which is a 2-degree demanding operation.

For Steps 5 and 6, the relative position of the points t_a and t_i and the line $\ell(t_j, t_k)$ on Π is required to resolve the degeneracy. Initially, we consider the algebraic expression $ax + by + cz + d = 0$ of the plane Π and we assume, without loss of generality, that $a^2 + b^2 + c^2 = 1$. Since all sites S_n, for $n \in \{i, j, k, a\}$ are tangent to Π, it must hold that $ax_n + by_n + cz_n + d = r_n$. Using Crammer's rule, we evaluate the vector $V = (a, b, c) = V'/D^{xyz1}_{ijka}$, where $V' = (D^{ryz}_{ijka}, D^{xrz}_{ijka}, D^{xyr}_{ijka})$; it is well known that V' is perpendicular with the plane Π.

We now consider the plane Π' that contains the line $\ell(t_j, t_k)$ and is perpendicular the plane Π. This plane is necessarily parallel to the vector V' and must contain both points C_j and C_k. As an immediate result, we deduce that t_a is collinear with t_j and t_k if and only if C_a lies on the plane Π', i.e., if $C_jC_a \cdot (V' \times C_jC_k) = 0$. The last expression can be rewritten as $D^{ryz}_{ijka} D^{yz}_{akj} - D^{xrz}_{ijka} D^{xz}_{ajk} + D^{xyr}_{ijka} D^{xy}_{ajk} = 0$ which is a 6-degree demanding operation.

Lastly, for Step 6, we can determine whether t_i and t_a are on the same side of $\ell(t_j, t_k)$, as this is equivalent to C_i and C_a lying on the same side of the plane Π'. In this case, the expressions $E_i = C_jC_i \cdot (V' \times C_jC_k)$ and $E_a = C_jC_a \cdot (V' \times C_jC_k)$ must have the same sign. Since $E_\nu = D^{ryz}_{ijk\nu} D^{yz}_{\nu kj} - D^{xrz}_{ijk\nu} D^{xz}_{\nu jk} + D^{xyr}_{ijk\nu} D^{xy}_{\nu jk}$, for $\nu \in \{i, a\}$, we can determine if $\mathsf{sign}(E_i) = \mathsf{sign}(E_a)$ using 6-fold operations.

Case B. In this configuration, S_a it tangent to both planes Π^-_{ijk} and Π^+_{ijk} and therefore it must hold that the centers of the spheres S_n, for $n \in \{i, j, k, a\}$, lie on the same plane, denoted by Π'. The symmetry in this geometric configuration indicates that, after the perturbation, the site S_a will intersect either both or none of the planes Π^-_{ijk} and Π^+_{ijk} and as a subsequence, the result R of DISTANCE$^\epsilon$ will be $(-, -)$ or $(+, +)$ respectively.

We first consider the case where $a > i, j, k$. In this case, S_a will be perturbed first and as a result it will intersect both Π^-_{ijk} and Π^+_{ijk}. It is apparent that the result of the perturbed predicate will be $(-, -)$ in this scenario.

If it does not hold that $a > i, j, k$, we cyclically permute S_i, S_j and S_k such that $i > j, k, a$. As stated before, this name exchange does not alter the outcome of the DISTANCE predicate. We now break down our analysis depending on whether the centers C_j, C_k and C_a are collinear or not. Note that these points are collinear if and only if $C_jC_k \times C_jC_a = \vec{0}$ or equivalently $D^{xy}_{akj} = D^{xz}_{akj} = D^{yz}_{akj} = 0$, which is a 2-degree demanding operation.

If C_j, C_k and C_a are not collinear, then the sphere S_a will intersect both Π^-_{ijk} and Π^+_{ijk} after perturbing S_i if and only if both C_a and C_i lie, on the plane Π', on the same side with respect to the line $\ell(C_j, C_k)$. A simple way to examine if the last property holds is via the use of an auxiliary point P. This point P is selected among $\{(0,0,0),(0,0,1),(0,1,0),(1,0,0)\}$ such that $O_i = \textsc{Orient3D}(P, C_j, C_k, C_i)$ does not equal zero , i.e., P does not lie on Π'. This is plausible since the possible four candidates for P are not coplanar and therefore at least one does not lie on Π'. Observe now that C_a and C_i lie on the same side with respect to the line $\ell(C_j, C_k)$ if and only if the signs of O_i and $O_a = \textsc{Orient3D}(P, C_j, C_k, C_a)$ are the same, which is a 3-degree demanding operation.

Lastly, we consider the case where C_j, C_k and C_a are collinear. In this configuration, even after perturbing S_i, S_a still remains commonly tangent to both planes Π^-_{ijk} and Π^+_{ijk}. Since the degeneracy is not yet resolved, the perturbation of a second site among S_j, S_k and S_a is required. If $a > j, k$, we will trivially have to return $(-,-)$ as S_a will intersect both cotagent planes. If $j > a, k$, S_a will intersect both Π^-_{ijk} and Π^+_{ijk} if and only if C_k lies in-between C_j and C_a or equivalently $\vec{C_jC_k} = \lambda \cdot \vec{C_jC_a}$ for some $\lambda > 0$. Respectively, if $k > a, j$, S_a will intersect both cotagent planes if and only if C_j lies in-between C_k and C_q or equivalently $\vec{C_jC_k} = \lambda \cdot \vec{C_jC_a}$ for some $\lambda < 0$. Notice that since C_j, C_k and C_a are known to be collinear, the sign of λ in the last two cases is positive if and only if $\mathsf{sign}(x_a - x_j) = \mathsf{sign}(x_k - x_j)$, $\mathsf{sign}(y_a - y_j) = \mathsf{sign}(y_k - y_j)$ and $\mathsf{sign}(z_a - z_j) = \mathsf{sign}(z_k - z_j)$, which is a 1-degree demanding operation.

The analysis of Case B is summarized for clarity in the following steps.

Step 1 If $a > i, j, k$ then return $(-,-)$. Otherwise, assume that $i > j, k, a$ and go to Step 2.

Step 2 Evaluate $D^{xy}_{akj}, D^{xz}_{akj}$ and D^{yz}_{akj}; if not all equal 0, go to Step 3 otherwise go to Step 4.

Step 3 Pick a point $P \in \{(0,0,0),(0,0,1),(0,1,0),(1,0,0)\}$ such that $P \notin \Pi(C_i, C_j, C_k)$. Evaluate $O_i = \textsc{Orient3D}(P, C_j, C_k, C_i)$ and $O_a = \textsc{Orient3D}(P, C_j, C_k, C_a)$. Return $(+,+)$ if $\mathsf{sign}(O_i) = \mathsf{sign}(O_a)$ otherwise, return $(-,-)$.

Step 4 If $a > j, k$ return $(-,-)$. Otherwise, if $j > a, k$ go to Step 5 or else, if $k > a, j$ go to Step 6.

Step 5 If all three equalities $\mathsf{sign}(x_a - x_j) = \mathsf{sign}(x_k - x_j)$, $\mathsf{sign}(y_a - y_j) = \mathsf{sign}(y_k - y_j)$ and $\mathsf{sign}(z_a - z_j) = \mathsf{sign}(z_k - z_j)$ hold, then return $(-,-)$, otherwise return $(+,+)$.

Step 6 If all three equalities $\mathsf{sign}(x_a - x_j) = \mathsf{sign}(x_k - x_j)$, $\mathsf{sign}(y_a - y_j) = \mathsf{sign}(y_k - y_j)$ and $\mathsf{sign}(z_a - z_j) = \mathsf{sign}(z_k - z_j)$ hold, then return $(+,+)$, otherwise return $(-,-)$.

To summarize the analysis of this section, we have shown that in order to evaluate the $\textsc{Distance}^\epsilon(S_i, S_j, S_k, S_a)$ predicate we have call the respective $\textsc{Distance}$ predicate and perform additional operations of algebraic cost at most 6. Consequently, we have proven the following lemma.

Lemma 22 *The* $\textsc{Distance}^\epsilon$ *predicate can be evaluated by determining the sign of quantities of algebraic degree at most 6 (in the input quantities).*

4.3 The SHADOWREGION$^\epsilon$ Predicate for Hyperbolic Trisectors

Based on the analysis of Sections 2.4.5 and 2.7.5, the outcome of the SHADOW(S_i, S_j, S_k, S_a) predicate is the topological form $SRT(S_a)$ of $\mathcal{SR}(S_a)$ on the hyperbolic trisector τ_{ijk} if seen as an interval or union of intervals.

We have proven in previous sections that the boundary points of $\mathcal{SR}(S_a)$ correspond to the Apollonius vertices v_{ijka} and/or v_{ikja} and that in non-degenerate configurations only the following outcomes of the SHADOW predicate are plausible: $\emptyset, \mathbb{R}, (-\infty, \phi), (\chi, +\infty)$, (χ, ϕ) and $(-\infty, \phi) \cup (\chi, +\infty)$. A useful remark is stated in Lemma 15: *the endpoints χ and ϕ correspond to v_{ikja} and v_{ijka} respectively*. As in previous sections, the notation "$\mathcal{SR}(S_a) = \emptyset$" will be equivalent to "$\mathcal{SR}(S_a)$ is of type $\emptyset$" and "$SRT(S_a) = \emptyset$".

Regarding degenerate shadow regions, they can be classified into two categories. The first category involves shadow regions where χ and/or ϕ are allowed to coincide with $\pm\infty$ but not with each other; we shall call these cases *degeneracies of type A*. The shadow regions where χ and ϕ, and therefore v_{ikja} and v_{ijka}, coincide are called *degeneracies of type B*.

To resolve degenerate shadow regions, we must first be able to detect them and their type. As shown in Section 2.7.5, there is a close relation between the outcome of SHADOW(S_i, S_j, S_k, S_a) and the outcomes of the EXISTENCE and DISTANCE predicates, given the same input. Assuming no degeneracies, type of $\mathcal{SR}(S_a)$ is deduced by combining the latter two outcomes, as shown in Table 4.1. If the EXISTENCE and DISTANCE predicates return a combination that does not appear in Table 4.1, then we have a degenerate shadow region $\mathcal{SR}(S_a)$ and our second step is to determine its type.

EXISTENCE (S_i, S_j, S_k, S_a)	DISTANCE (S_i, S_j, S_k, S_a)	$\mathcal{SR}(S_a)$
0	$(+,+)$	$\emptyset$
	$(-,-)$	$\mathbb{R}$
1	$(+,-)$	$(\chi, +\infty)$
	$(-,+)$	$(-\infty, \phi)$
2	$(+,+)$	(χ, ϕ)
	$(-,-)$	$(-\infty, \phi) \cup (\chi, +\infty)$

TABLE 4.1: If the outcome of EXISTENCE(S_i, S_j, S_k, S_a) and DISTANCE(S_i, S_j, S_k, S_a) is provided, we can safely deduce the form of the non-degenerate shadow region $\mathcal{SR}(S_a)$ on the hyperbolic trisector τ_{ijk} as shown in this table. In all other possible combinations of these two predicates, we have a degenerate $\mathcal{SR}(S_a)$.

Crucially, type B degeneracies occur only if $v_{ikja} \equiv v_{ijka}$. In such cases, during the evaluation of the EXISTENCE(S_i, S_j, S_k, S_a) predicate as presented in Section 2.7.3, a double root of either $M(d)$ or $L(c)$ must appear. This double root indicates that there is a "double plane" $\Pi^{\star}_{ijk}$: $au + bv + cw + d = 0$ cotagent to $S^{\star}_i, S^{\star}_j$ and $S^{\star}_a$; this is the image of the coinciding spheres $\mathcal{A}_{ijk}(v_{ikja})$ and $\mathcal{A}_{ijk}(v_{ijka})$ in $\mathcal{Y}$-space.

Based on the analysis of Section 2.7.3, such double roots may exist only if $\Delta_M = 0$ in all possible cases. Therefore, degenerate shadow regions are of type A if and only if Δ_M does not equal zero otherwise, they are of type B. Depending on its type, we follow the analysis of the corresponding section to resolve the degeneracy.

Degeneracies of type A. Firstly, we examine the degenerate shadow regions of type A. In these cases, we reflect on all possible non-degenerate forms of $\mathcal{SR}(S_a)$ that contain ϕ and/or χ; these are $(-\infty, \phi), (\chi, +\infty), (\chi, \phi)$ and $(-\infty, \phi) \cup (\chi, +\infty)$. For each of these possible forms, we consider the outcome of the corresponding Distance and Existence predicates if ϕ and/or χ coincided with $\pm\infty$ and what will happen after applying the perturbation scheme. Beforehand, we make three crucial remarks.

Remark 1. For all cases that we have to study, one can observe that if ϕ and/or χ tend to $-\infty$ (resp., $+\infty$) then Π^{-}_{ijk} (resp., Π^{+}_{ijk}) becomes the Apollonius sphere of S_i, S_j, S_k and S_a. In this configuration, S_a must necessarily be tangent to Π^{-}_{ijk} (resp., Π^{+}_{ijk}) yielding $d_1 = 0$ (resp., $d_2 = 0$), where (d_1, d_2) denotes the answer of the Distance(S_i, S_j, S_k, S_a) predicate.

Remark 2. After applying the perturbation scheme, the endpoint(s) among $\{\phi, \chi\}$ that coincided with $\pm\infty$ will move infinitesimally on the trisector τ_{ijk}. Thus, if an endpoint initially coincided with $\pm\infty$ then, after the perturbation it will either move infinitesimally towards o_{ijk} and become "finite" or it will move further away from o_{ijk} and truly become "infinite". In both cases, the degeneracy is resolved. To distinguish between the two possible scenarios, we have to consider the possible outcomes of the perturbed Distance predicate and make a connection with the possible shadow regions types that arise after the substition of the endpoint(s) with its finite or infinite form(s).

Remark 3. The Existence(S_i, S_j, S_k, S_a) predicate returns how mane of the Apollonius vertices $\{v_{ijka}, v_{ikja}\}$ exist and correspond to *finite* Apollonius spheres. Therefore, although ϕ and/or χ correspond to the existence of v_{ijka} and v_{ikja} respectively, the Existence predicate does not count them if they coincide with $\pm\infty$.

Using these three remarks, we may analyse all possible degenerate shadow regions of type A and draw our conclusions. The following two examples are indicative of the rest of the analysis that is summarized in Table 4.2.

Example 1. Let us consider the case where $\mathcal{SR}(S_a) = (-\infty, \phi)$ and ϕ coincides with $+\infty$. As a first observation, $\mathcal{SR}(S_a)$ has no "finite" boundary and therefore Existence(S_i, S_j, S_k, S_a) must return 0, based on Remark 3. As of Remark 1, the Distance(S_i, S_j, S_k, S_a) predicate should return (d_1, d_2), where $d_2 = 0$. Finally, using Remark 2 we know that the perturbed shadow region $\mathcal{SR}^{\epsilon}(S_a)$ will be either $(-\infty, \phi)$ or $(-\infty, +\infty) = \mathbb{R}$. In the former case, Distance$^{\epsilon}(S_i, S_j, S_k, S_a)$, will be $(-, +)$ whereas in the latter, it will be $(-, -)$. We can now safely deduce that $d_1 = -$ in both cases and that $\mathcal{SR}^{\epsilon}(S_a)$ can be identified by the outcome of Distance$^{\epsilon}$.

Example 2. We consider the case where $\mathcal{SR}(S_a) = (\chi, \phi)$ and χ, ϕ coincide with $-\infty, +\infty$ respectively. As before, $\mathcal{SR}(S_a)$ has no "finite" boundary and therefore Existence (S_i, S_j, S_k, S_a) must return 0 based on Remark 3. The Distance(S_i, S_j, S_k, S_a) predicate should return $(0, 0)$ as Remark 1 suggests. Using Remark 2, we know that ϕ (resp., χ) will become either finite or infinite after the perturbation and therefore the possible forms of $\mathcal{SR}^\epsilon(S_a)$ are: $(-\infty, +\infty) = \mathbb{R}, (-\infty, \phi), (\chi, \infty)$ and (χ, ϕ). However, since the respective Distance$^\epsilon$ predicate is known to return either $(+, +)$ or $(-, -)$ in this scenario (see Section 4.2), only the cases $\mathbb{R}$ and (χ, ϕ) are plausible.

Using the Remarks 1-3 and similar analysis with the Examples 1 and 2, we construct the Table 4.2 for all possible degenerate shadow regions of type A. This table indicates that we can resolve a degeneracy of type A and determine the form of $\mathcal{SR}^\epsilon(S_a)$ using only the Existence, Distance and Distance$^\epsilon$ predicates.

E	D	D^ϵ	$\mathcal{SR}^\epsilon(S_a)$	Original $\mathcal{SR}(S_a)$
0	$(0,+)$	$(-,+)$	$(-\infty,\phi)$	$(-\infty,\phi)$ when $\phi \equiv -\infty$
		$(+,+)$	$\emptyset$	
	$(0,-)$	$(-,-)$	$\mathbb{R}$	$(\chi,+\infty)$ when $\chi \equiv -\infty$
		$(+,-)$	$(\chi,+\infty)$	
	$(+,0)$	$(+,-)$	$(\chi,+\infty)$	$(\chi,+\infty)$ when $\chi \equiv +\infty$
		$(+,+)$	$\emptyset$	
	$(-,0)$	$(-,-)$	$\mathbb{R}$	$(-\infty,\phi)$ when $\phi \equiv +\infty$
		$(-,+)$	$(-\infty,\phi)$	
	$(0,0)$	$(-,-)$	$\emptyset$	$(-\infty,\phi)\cup(\chi,+\infty)$ when $\phi \equiv -\infty$ and $\chi \equiv +\infty$
		$(+,+)$	$(-\infty,\phi)\cup(\chi,+\infty)$	
1	$(0,+)$	$(-,+)$	$(-\infty,\phi)$	(χ,ϕ) when $\chi \equiv -\infty$
		$(+,+)$	(χ,ϕ)	
	$(0,-)$	$(-,-)$	$(-\infty,\phi)\cup(\chi,+\infty)$	$(-\infty,\phi)\cup(\chi,+\infty)$ when $\phi \equiv -\infty$
		$(+,-)$	$(\chi,+\infty)$	
	$(+,0)$	$(+,-)$	$(\chi,+\infty)$	(χ,ϕ) when $\phi \equiv +\infty$
		$(+,+)$	(χ,ϕ)	
	$(-,0)$	$(-,-)$	$(-\infty,\phi)\cup(\chi,+\infty)$	$(-\infty,\phi)\cup(\chi,+\infty)$ when $\chi \equiv +\infty$
		$(-,+)$	$(-\infty,\phi)$	

Table 4.2: E, D and D^ϵ denote the outcomes of the Existence, Distance and Distance$^\epsilon$ predicates for input (S_i, S_j, S_k, S_a). The last column describes the original degenerate shadow region $\mathcal{SR}(S_a)$ that was studied.

Degeneracies of Type B. Lastly, we consider degeneracies of type B. In these degeneracies, the Apollonius vertices v_{ikja} and v_{ijka} coincide; this "double" vertex is denoted by V. Evidently, the degenerate shadow region $\mathcal{SR}(S_a)$ of S_a must be either (χ, ϕ) or $(-\infty, \phi) \cup (\chi, +\infty)$, where ϕ and χ essentially coincide. After the perturbation, these endpoints infinitesimally move "away" from each other and their initial position. Depending on the form of $\mathcal{SR}(S_a)$ and whether ϕ moves following the positive orientation of the trisector or not, we can deduce the perturbed shadow region $\mathcal{SR}^\epsilon(S_a)$ of S_a.

The analysis that allows us to determine $\mathcal{SR}^\epsilon(S_a)$ is subdivided into two main cases. In Case 1, V coincides with either $-\infty$ or $+\infty$ on the trisector whereas in Case 2 it does not. The outcome (d_1, d_2) of Distance(S_i, S_j, S_k, S_a) indicates if we are in Case 1 or 2. When an Apollonius vertex coincides with $-\infty$ (resp., $+\infty$) on the trisector then d_1 (resp., d_2) must equal 0, as S_a is tangent to Π^-_{ijk} (resp., Π^+_{ijk}). Therefore, if d_1 or d_2 equal 0, we follow the analysis of Case 1 otherwise, that of Case 2.

Case 1. If d_1 or d_2 equals 0, we immediately deduce that V, and therefore both ϕ and χ, coincide with $-\infty$ or $+\infty$ respectively. The same strategy with degeneracies of type A is followed here. Initially, we consider all possible scenarios regarding the endpoints ($\phi \equiv \chi \equiv -\infty$ or $\phi \equiv \chi \equiv +\infty$) and the shadow regions types ((χ, ϕ) and $(-\infty, \phi) \cup (\chi, +\infty)$). For each of the four possible subcases, we consider the perturbed shadow region $\mathcal{SR}^\epsilon(S_a)$ that derives after ϕ moves infinitesimally, following the positive or the negative orientation of the trisector. Due to the symmetry of this configuration in $\mathcal{W}$-space and becoming apparent from the analysis of Case 2 below, at the same time that ϕ moves infinitesimally on the trisector, the endpoint χ also moves infinitesimally on the opposite direction.

For each subcase studied, we consider the resulting perturbed shadow region $\mathcal{SR}^\epsilon(S_a)$ along with the outcomes of the the respective Distance and Distance$^\epsilon$ predicates. Let us consider two of these subcases; the analysis of the other two is are essentially the same with the one provided with minor modifications.

Subcase 1. Assume that the original shadow region type was (χ, ϕ) and $\phi \equiv \chi \equiv -\infty$. If ϕ moves following the positive orientation of τ_{ijk}, χ will move on the opposite direction. In this scenario, χ will cease to exist and ϕ will become "finite"; the resulting $\mathcal{SR}^\epsilon(S_a)$ will be $(-\infty, \chi)$. In the scenario where ϕ moves on the opposite direction of the one described above, $\mathcal{SR}^\epsilon(S_a)$ will essentially become $\emptyset$ as the right endpoint infinitesimally "becomes" $-\infty$.

Notice that the Distance predicate in this Subcase is $(0, d_2)$ and therefore the Distance$^\epsilon$ predicate must be either $(-, d_2)$ or $(+, d_2)$. If ϕ moves towards $+\infty$ or $-\infty$, then $\mathcal{SR}^\epsilon(S_a)$ becomes $(-\infty, \chi)$ or $\emptyset$ respectively. In these scenarios, it must hold that Distance$^\epsilon$ equals $(-, +)$ or $(+, +)$ respectively. Since in both cases, $+\infty$ is not part of the shadow region we can conclude that $d_2 = +$ in this Subcase (see Section 2.7.5).

Subcase 2. Assume that the original shadow region type was $(-\infty, \phi) \cup (\chi, +\infty)$ and $\phi \equiv \chi \equiv +\infty$. If ϕ moves following the positive orientation of τ_{ijk}, the perturbed shadow region becomes $\mathbb{R}$. Otherwise, if ϕ moves on the opposite direction, it will become "finite" whereas χ will move further towards ∞; the resulting $\mathcal{SR}^\epsilon(S_a)$ will be $(-\infty, \phi)$.

Notice that the Distance predicate in this Subcase is $(d_1, 0)$ and therefore the Distance$^\epsilon$ predicate must be either $(d_1, -)$ or $(d_1, +)$. If ϕ moves towards $+\infty$ or $-\infty$, then $\mathcal{SR}^\epsilon(S_a)$ becomes $\mathbb{R}$ or $(-\infty, \phi)$ respectively. In these scenarios, it must hold that Distance$^\epsilon$ equals

$(-,-)$ or $(-,+)$ respectively. Since in both cases, $-\infty$ is part of the shadow region we can conclude that $d_1 = -$ in this Subcase (see Section 2.7.5).

The results from the analysis of all four subcases is found in Table 4.3. Notice that since no finite Apollonius vertices exist, the outcome of the respective EXISTENCE predicate must equal 0. The results in this table imply that, for the case studied, the type of $\mathcal{SR}^{\epsilon}(S_a)$ can be deduced from the combination of the outcomes of the DISTANCE and DISTANCE$^{\epsilon}$ predicates alone.

E	D	D^{ϵ}	$\mathcal{SR}^{\epsilon}(S_a)$	Original $\mathcal{SR}(S_a)$
0	$(0,+)$	$(-,+)$	$(-\infty,\phi)$	(χ,ϕ) when $\phi \equiv \chi \equiv -\infty$
		$(+,+)$	$\emptyset$	
	$(0,-)$	$(-,-)$	$\mathbb{R}$	$(-\infty,\phi) \cup (\chi,+\infty)$ when $\phi \equiv \chi \equiv -\infty$
		$(+,-)$	$(\chi,+\infty)$	
	$(+,0)$	$(+,-)$	$(\chi,+\infty)$	(χ,ϕ) when $\phi \equiv \chi \equiv +\infty$
		$(+,+)$	$\emptyset$	
	$(-,0)$	$(-,-)$	$\mathbb{R}$	$(-\infty,\phi) \cup (\chi,+\infty)$ when $\phi \equiv \chi \equiv +\infty$
		$(-,+)$	$(-\infty,\phi)$	

TABLE 4.3: Degeneracies of type B, Case 1. E, D and D^{ϵ} denote the outcomes of the EXISTENCE, DISTANCE and DISTANCE$^{\epsilon}$ predicates for input (S_i, S_j, S_k, S_a). The last column describes the original degenerate shadow region $\mathcal{SR}(S_a)$ that was studied.

Case 2. In this case, the "double" Apollonius vertex V is necessarily finite, i.e., $V \in \tau_{ijk} \backslash \{\pm\infty\}$. The possible shadow region forms of $\mathcal{SR}(S_a)$ are the same as before, either (χ,ϕ) (Subcase 1) or $(-\infty,\phi) \cup (\chi,+\infty)$ (Subcase 2), with ϕ and χ actually coinciding. The same strategy that was used in Case 1 is also used for Case 2; we consider the perturbed shadow region $\mathcal{SR}^{\epsilon}(S_a)$ when ϕ and χ move infinitesimally on the trisector, on opposite directions. Let us now consider in detail these two subcases.

Subcase 1. If $\mathcal{SR}(S_a) = (\chi,\phi)$ and ϕ moves on the positive (resp., negative) direction of τ_{ijk}, the non-degenerate shadow region $\mathcal{SR}^{\epsilon}(S_a)$ of S_a will become (χ,ϕ) (resp., $\emptyset$). Since $\phi \equiv \chi \neq \pm\infty$, the EXISTENCE predicate returns 1 and both DISTANCE and DISTANCE$^{\epsilon}$ predicates return $(+,+)$.

Subcase 2. If $\mathcal{SR}(S_a) = (-\infty,\phi) \cup (\chi,+\infty)$ and ϕ moves on the positive (resp., negative) direction of τ_{ijk}, the non-degenerate shadow region $\mathcal{SR}^{\epsilon}(S_a)$ of S_a will become $\mathbb{R}$ (resp., $(-\infty,\phi) \cup (\chi,+\infty)$). Since $\phi \equiv \chi \neq \pm\infty$, the EXISTENCE predicate returns 1 and both DISTANCE and DISTANCE$^{\epsilon}$ predicates return $(-,-)$.

The main difference with Case 1 that can easily be spotted is that we can no longer draw conclusions regarding $\mathcal{SR}^{\epsilon}(S_a)$ from the outcomes of the DISTANCE and DISTANCE$^{\epsilon}$ predicates. Nevertheless, we are still able to distinguish the form of $\mathcal{SR}(S_a)$; we are in Subcase 1 if DISTANCE returned $(+,+)$ otherwise, we are in Subcase 2.

In either subcase, we have to apply the perturbation scheme to resolve the degeneracy. Among the sites S_i, S_j, S_k and S_a that may have to be perturbed, S_k has the minimum radius among the first three after a proper name exchange. Even if a tie break arise, we select S_k to be the sphere whose center is lexicographically smaller than the rest; in all cases, it will hold that $k < i, j$. Bearing that in mind and since we will prove that perturbing only one site suffices to resolve the degeneracy, S_k can be considered as fixed. Since the topology and and geometric objects of $\mathcal{W}$-space are greatly affected by the sphere S_k, as it is the sphere through which the inversion is made, we can safely consider that the perturbation of a sphere S_n of $\mathcal{Z}$-space, with $n \neq k$, is equivalent to the symbolic perturbation of the respective site $S_n^\star$ and $\hat{S_n}$ in $\mathcal{W}$-space and $\mathcal{Y}$-space respectively.

For convinience, we carry the rest of the analysis of Case 2 in the inverted $\mathcal{W}$-space. The *double* Apollonius vertex in $\mathcal{Y}$-space will correspond to a *double* plane $\Pi^\star$ commonly tangent to all $S_i^\star, S_j^\star$ and $S_a^\star$ in $\mathcal{W}$-space. Evidently, the sphere $S_a^\star$ must be tangent to the semi-cone $\mathcal{K}^\star$ defined by the former two spheres. As a consequence of Section 2.7.6, we are in Subcase 2 (resp., Subcase 1) if and only if all other planes tangent to this cone intersect (resp., do not intersect) $S_a^\star$. In $\mathcal{W}$-space, let ω_n, for $n \in \{i, j, a\}$, denote the tangency point of $S_n^\star$ and the plane $\Pi^\star$. Since S_i and S_j are considered to be distinct, ω_i and ω_j must also be distinct.

If $a > i, j, k$ then S_a and therefore $S_a^\star$ will be initially perturbed. In Subcase 1, some planes tangent to the cone will now properly intersect the perturbed sphere $S_a^{\star,\epsilon}$ and therefore $\mathcal{SR}^\epsilon(S_a)$ will become (χ, ϕ). In Subcase 2, all planes tangent to the cone will now properly intersect $S_a^{\star,\epsilon}$ hence $\mathcal{SR}^\epsilon(S_a)$ will become $\mathbb{R}$.

Otherwise, let us consider the case where $i > j, k, a$; the case where $j > i, k, a$ is treated in the same way if we name exchange S_i and S_j. If $\omega_a = \omega_i$ or equivalently $t_a = t_i$, the inflation of $S_i^\star$ will affect the cone $\mathcal{K}^\star$. In Subcase 1, all planes tangent to $\mathcal{K}^\star$ will not intersect $S_a^\star$, yielding $\mathcal{SR}^\epsilon(S_a) = \emptyset$ ((see Figure 4.3)). If we are in a Subcase 2 configuration, some of these planes will no longer intersect $S_a^\star$, hence $\mathcal{SR}^\epsilon(S_a) = (-\infty, \phi) \cup (\chi, +\infty)$ (see Figure 4.4). In a respective way, if $\omega_a = \omega_j$ or equivalently $t_a = t_j$, the perturbed shadow region $\mathcal{SR}^\epsilon(S_a)$ is $\emptyset$ or $\mathbb{R}$ in Subcase 1 or 2 respectively (see Figures 4.5 and 4.6).

Lastly, we consider the scenario where ω_i, ω_j and ω_a are all distinct points; these tangency points must lie on the same line due to the degenerate configuration we are studying. Using the same arguements as in the case where $\omega_a = \omega_i$, we conclude that, if ω_i and ω_a lie on the same side with respect to ω_j then $\mathcal{SR}^\epsilon(S_a)$ becomes $\emptyset$ or $(-\infty, \phi) \cup (\chi, +\infty)$ in the Subcase 1 or 2 respectively. Otherwise, if ω_i and ω_a lie on different sides with respect to ω_j then $\mathcal{SR}^\epsilon(S_a)$ becomes $\emptyset$ or $\mathbb{R}$ in the Subcase 1 or 2 respectively (see Figure 4.7 and 4.8).

For clarity we summarize the analysis for Case 2 into the following algorithm that returns the perturbed shadow region $\mathcal{SR}^\epsilon(S_a)$.

Step 1. If $a > i, j, k$ return $\mathcal{SR}^\epsilon(S_a)$ return (χ, ϕ) if we are in Subcase 1 or $\mathbb{R}$ if we are in Subcase 2. Otherwise, go to Step 2.

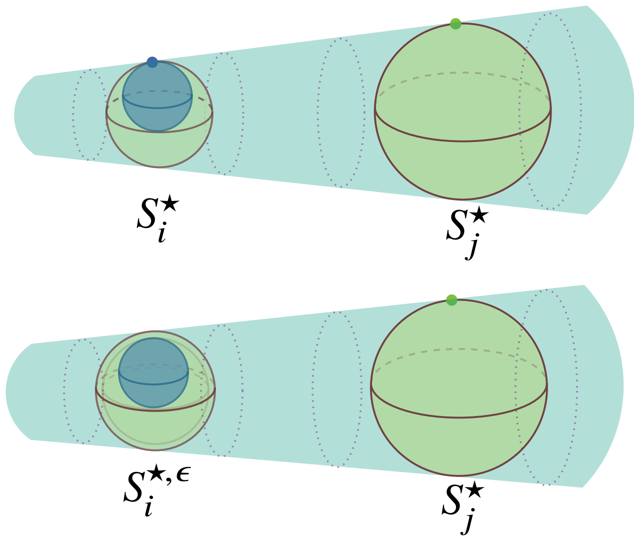

Figure 4.3: In Subcase 1, if $\omega_a = \omega_i$ then $S_a^\star$ lies inside and is tangent to $S_i^\star$. After $S_i^\star$ is perturbed, the cone defined by the perturbed sphere $S_i^{\star,\epsilon}$ and $S_j^\star$ will contain $S_a^\star$ and therefore $\mathcal{SR}^\epsilon(S_a)$ will be $\emptyset$.

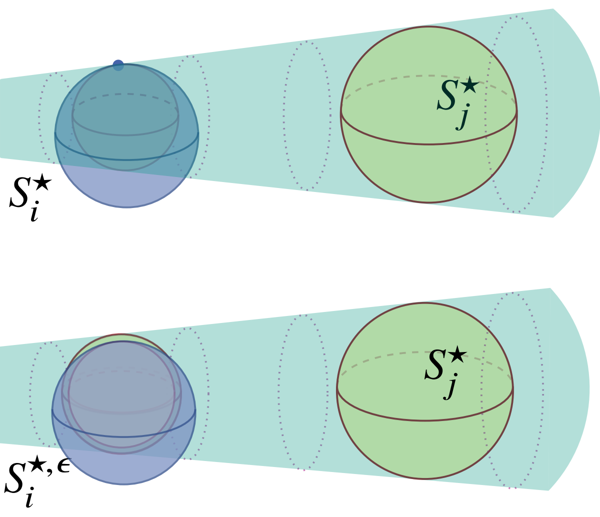

Figure 4.4: In Subcase 2, if $\omega_a = \omega_i$ then $S_i^\star$ lies inside and is tangent to $S_a^\star$. After $S_i^\star$ is perturbed, some planes tangent to the perturbed sphere $S_i^{\star,\epsilon}$ and $S_j^\star$ will not intersect $S_a^\star$ and therefore $\mathcal{SR}^\epsilon(S_a)$ will be $(-\infty, \phi) \cup (\chi, +\infty)$.

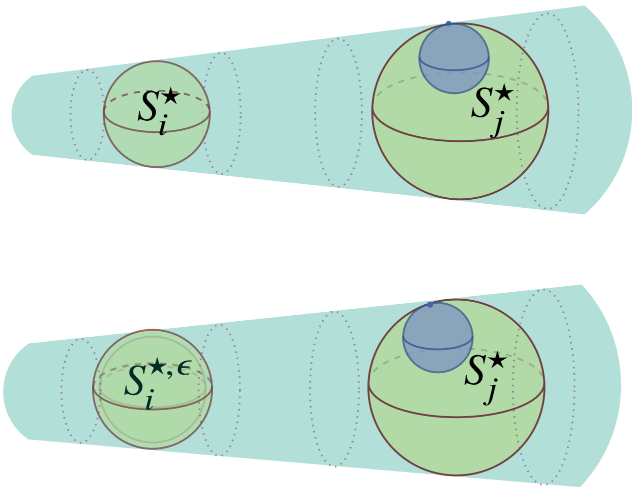

FIGURE 4.5: In Subcase 1, if $\omega_a = \omega_j$ then $S_a^\star$ lies inside and is tangent to $S_j^\star$. After $S_i^\star$ is perturbed, the cone defined by the perturbed sphere $S_i^{\star,\epsilon}$ and $S_j^\star$ will contain $S_a^\star$ and therefore $\mathcal{SR}^\epsilon(S_a)$ will be $\emptyset$.

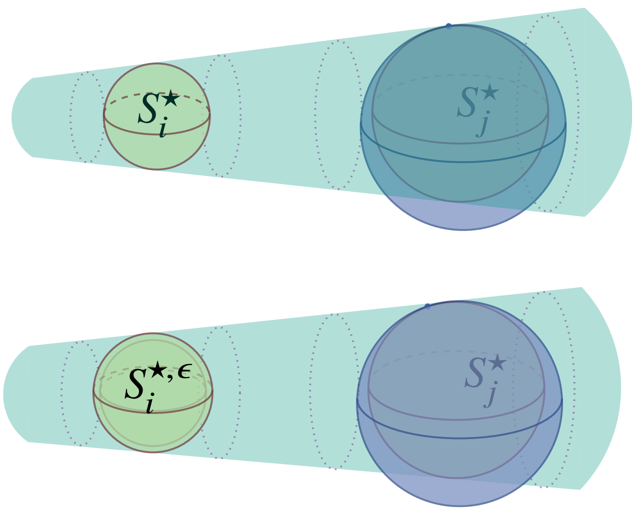

FIGURE 4.6: In Subcase 2, if $\omega_a = \omega_j$ then $S_j^\star$ lies inside and is tangent to $S_a^\star$. After $S_i^\star$ is perturbed, all planes tangent to the perturbed sphere $S_i^{\star,\epsilon}$ and $S_j^\star$ will intersect $S_a^\star$ and therefore $\mathcal{SR}^\epsilon(S_a)$ will be $\mathbb{R}$.

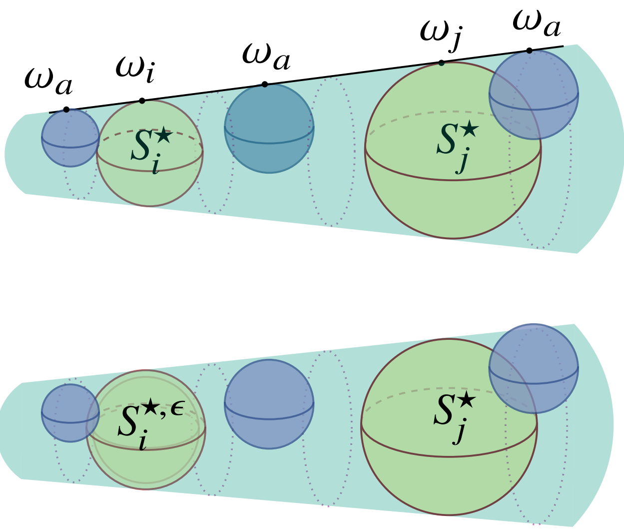

FIGURE 4.7: In Subcase 1, ω_a and ω_i are on the same side with respect to ω_j if $S_a^\star$ lie as the blue spheres on the left or the center. In this case, after $S_i^\star$ is perturbed to $S_i^{\star,\epsilon}$, all planes commonly tangent to $S_i^{\star,\epsilon}$ and $S_j^\star$ will not intersect $S_a^\star$ and therefore $\mathcal{SR}^\epsilon(S_a)$ is $\emptyset$. If $S_a^\star$ lies as the blue sphere on the right, the respective $\mathcal{SR}^\epsilon(S_a)$ is (χ, ϕ), as there exist some planes tangent to the perturbed cone that intersect $S_a^\star$.

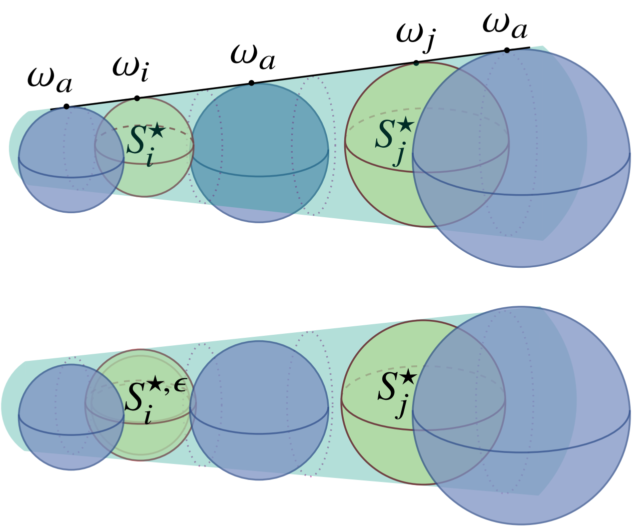

FIGURE 4.8: In Subcase 2, ω_a and ω_i are on the same side with respect to ω_j if $S_a^\star$ lie as the blue spheres on the left or the center. In this case, after $S_i^\star$ is perturbed to $S_i^{\star,\epsilon}$, some commonly tangent to $S_i^{\star,\epsilon}$ and $S_j^\star$ will not intersect $S_a^\star$ and therefore $\mathcal{SR}^\epsilon(S_a)$ is $(-\infty, \phi) \cup (\chi, +\infty)$. $S_a^\star$ lies as the blue sphere on the right, the respective $\mathcal{SR}^\epsilon(S_a)$ is $\mathbb{R}$ since all planes tangent to the perturbed cone will intersect $S_a^\star$.

Step 2. If $j > i, k, a$, name exchange S_i and S_j. Otherwise, since it already holds that $i > j, k, a$, go to Step 3.

Step 3. If $t_a = t_i$ then return $\emptyset$ if we are in Subcase 1, or $(-\infty, \phi) \cup (\chi, +\infty)$ if we are in Subcase 2. If $t_a = t_j$ then return $\emptyset$ if we are in Subcase 1, or $\mathbb{R}$ if we are in Subcase 2. Otherwise go to Step 4.

Step 4. Let ω_n denote the tangency points of $\mathcal{K}^\star$ and the inverted spheres S_n for $n \in \{i, j, a\}$. In the studied geometric configuration, ω_i, ω_j and ω_a are distinct and collinear. If ω_i and ω_a lie on the same side with respect to ω_j go to Step 4i, otherwise go to Step 4ii.

i. Return $\emptyset$ if we are in Subcase 1 or $(-\infty, \phi) \cup (\chi, +\infty)$ if we are in Subcase 2.

ii. Return (χ, ϕ) if we are in Subcase 1 or $\mathbb{R}$ if we are in Subcase 2.

Recall that $t_a = t_n$ holds, for $n \in \{i, j\}$, if and only if $(x_a - x_n)^2 + (y_a - y_n)^2 + (z_a - z_n)^2 = (r_a - r_n)^2$, which is a 2-degree demanding operation. The most difficult and degree-demanding operation is to evaluate whether ω_i and ω_a lie on the same side with respect to ω_j, in Step 4.

Since the tangency points ω_i, ω_j and ω_a lie on the same line $\ell^\star$ in $\mathcal{W}$-space, their preimages t_i, t_j and t_a must be co-circular with t_k in $\mathcal{Z}$-space; this is a direct result of the properties of the inversion transformation (see Section 1.4). Moreover, if C_{ijk} denotes the circle that contains the points t_n, for $n \in \{i, j, k, a\}$, then C_{ijk} must lie on the "double" Apollonius sphere $\mathcal{A}$ of S_i, S_j, S_k and S_a, centered at V.

Since $\ell^\star$ is the image of C_{ijk} in $\mathcal{Y}$-space, it holds that ω_i and ω_a lie on the same side with respect to ω_j on $\ell^\star$ if and only if t_a lies on the arc A of C_{ijk} that is bounded by t_j and t_k and contains t_i (see Figure 4.9).

In order to decide if t_a lies on A or not, we consider the relative position of the Apollonius vertex V and the circle C_{ijk}. We distinguish two different cases depending on whether V is coplanar with C_{ijk} or not, which is equivalent to all points C_i, C_j, C_k, C_a and V being coplanar or not respectively (see Figure 4.10). Therefore, we evaluate $\textsc{Orient3D}(C_i, C_j, C_k, C_a) = D^{xyz}_{ijka}$ and if it equals 0, we are in the former case, otherwise we are in the latter.

In the no-coplanar case, we can deduce that t_a lies on the arc A if and only if the orientation predicates $O_i = \textsc{Orient3D}(t_i, t_k, t_j, V)$ and $O_a = \textsc{Orient3D}(t_a, t_k, t_j, V)$ have the same sign. Note that, since it holds that $O_n = \textsc{Orient3D}(C_n, C_k, C_j, V)$, for $n \in \{i, a\}$, we do not need to evaluate the tangency points t_a, t_j and t_k explicitly (see Figure 4.11).

In the coplanar case, the Apollonius vertex V along with all points C_n and t_n, for $n \in \{i, j, k, a\}$, lie on the same plane. In this scenario, we select a point γ from the set $\{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$ such that $\gamma \notin \Pi(C_i, C_j, C_k)$ or equivalently $O_\gamma = \textsc{Orient3D}(C_i, C_j, C_k, \gamma) \neq 0$ and we denote $O_{\mu\nu} = \textsc{Orient3D}(C_\mu, \gamma, V, C_\nu)$, where $\mu, \nu \in \{a, k, j\}$ (see Figure 4.12).

Firstly, we evaluate O_{kj}; if it equals 0 then C_k, C_j and V are collinear. In this case, t_a does not lie on the arc A if and only if $\textsc{Orient3D}(C_i, C_j, C_k, \gamma) \cdot \textsc{Orient3D}(C_a, C_j, C_k, \gamma) < 0$.

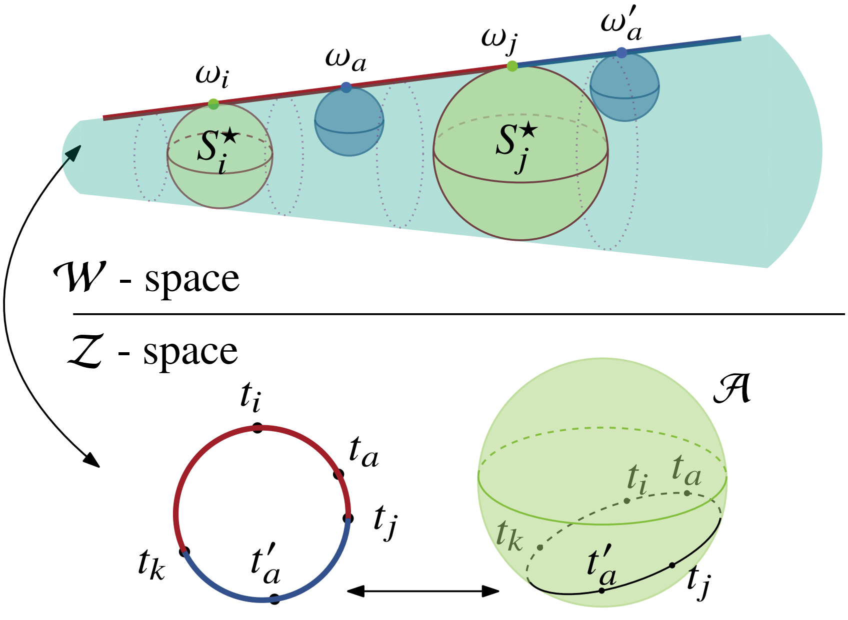

FIGURE 4.9: The line that contains ω_i and ω_j in $\mathcal{W}$-space, corresponds in $\mathcal{Y}$-space to a circle that contains the tangency points t_i, t_j and t_k, due to the properties of the inversion transformation. Since the image of t_k in $\mathcal{W}$-space is the point at infinity, there is an equvalency between the arc A (red arc) that is bounded by t_j and t_k and contains t_i and the (red) ray that starts from ω_j, *ends at the point at infinity which is the image of* t_k and contains ω_i. The arc A lies on the circle defined by the three points t_i, t_j and t_k, on the surface of the Apollonius sphere $\mathcal{A}$.

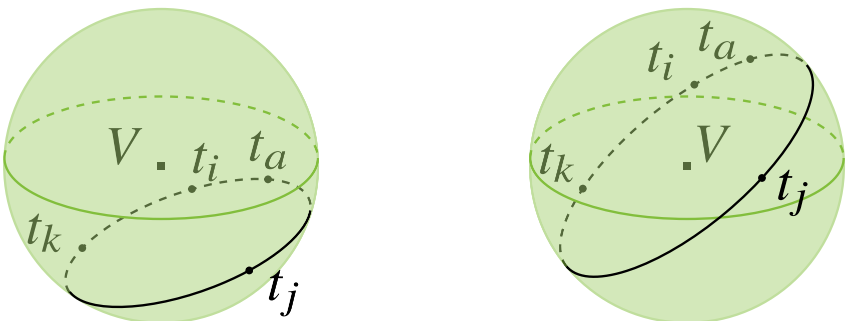

FIGURE 4.10: The Apollonius vertex V can either be coplanar with the tangency points t_i, t_j and t_k (right) or not (left). The way the degeneracy is resolved is severely affected depending on which of these cases applies.

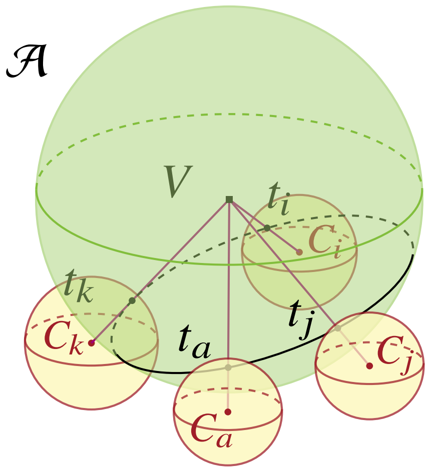

FIGURE 4.11: When the Apollonius vertex V is not coplanar with the tangency points t_i, t_j and t_k, we can determine if t_a and t_i lie on the same arcs bounded by t_k and t_j by checking various ORIENT3D predicates that involve V and three of the tangency points. The outcome of these predicates remain the same if the tangency point t_n is replaced with the respective center C_n, as shown in the figure.

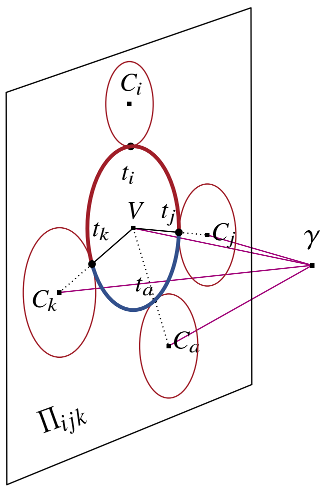

FIGURE 4.12: When the Apollonius vertex V is coplanar with the tangency points t_i, t_j and t_k, an auxiliary point γ outside the plane is used to determine if t_a lies on the (red) arc A or not. This is determined by evaluating various ORIENT3D predicates that involve V, γ and two of the tangency points. The outcome of these predicates remain the same if the tangency point t_n is replaced with the respective center C_n, as shown in the figure.

If $O_{kj} \neq 0$ then t_a does not lie on the arc A if and only if C_a and C_i lie on different sides of the angle (C_k, V, C_j). This is equivalent to either

- $O_{jk} \cdot O_{ak}, O_{kj} \cdot O_{aj} > 0$ and at least one of $O_{jk} \cdot O_{ik}, O_{kj} \cdot O_{ij}$ is non positive, Or
- $O_{jk} \cdot O_{ik}, O_{kj} \cdot O_{ij} > 0$ and at least one of $O_{jk} \cdot O_{ak}, O_{kj} \cdot O_{aj}$ is non positive.

Recollect that, if t_a lies or not on the arc A, then we can resolve resolve the degeneracy by following Step 4i or 4ii respectively. The most degree-demanding operation is the evaluation of the signs of $O_n = \text{ORIENT3D}(C_n, C_k, C_j, V)$ and $O_{\mu\nu} = \text{ORIENT3D}(C_\mu, \gamma, V, C_\nu)$, for $n \in \{i, a\}$ and $\mu, \nu \in \{a, k, j\}$ respectively. These ORIENT3D predicates can be expressed as

$$O_n = \frac{1}{w_V}(-x_V D^{yz}_{nkj} + y_V D^{xz}_{nkj} - z_V D^{xy}_{nkj} + w_V D^{xyz}_{nkj}) \tag{4.1}$$

$$O_{\mu\nu} = \frac{1}{w_V}(x_V D^{yz}_{\mu\gamma\nu} - y_V D^{xz}_{\mu\gamma\nu} + z_V D^{xy}_{\mu\gamma\nu} + w_V D^{xyz}_{nkj}) \tag{4.2}$$

where $V = \frac{1}{w_V}(x_V, y_V, z_V)$.

The coordinates of the double Apollonius vertex V can be evaluated from the analysis of the EXISTENCE(S_i, S_j, S_k, S_a) predicate (see Section 2.7.3), if we take into consideration three major facts.

1. The sphere that was used to apply the inversion was S_k and not S_a. Therefore, for all expressions of Section 2.7.3) that we will use below, we exchange k with a in all subscripts appearing.
2. Since we have a finite "double" Apollonius vertex, it holds that $\Delta_M = 0$ and $d \neq 0$. Moreover, the centers $C_i^\star, C_j^\star$ and $C_a^\star$ cannot be collinear since that would result in infinite number of planes being tangent the cone $\mathcal{K}^\star$, yielding a contradiction. Consequently, the quantities $E^{xyp}_{ija}, E^{xzp}_{ija}$ and E^{yzp}_{ija} are not all zero and therefore $M_2 > 0$.
3. As shown in [28], if $au + bv + cw + d = 0$ is the image of the Apollonius sphere of S_i, S_j, S_k, S_l in $\mathcal{W}$-space, then the coordinates of the Apollonius vertex in $\mathcal{Z}$-space is $(x_l - \frac{a}{2d}, y_l - \frac{b}{2d}, z_l - \frac{c}{2d})$.

Taking all the above facts into consideration, we can explicitly evaluate a, b, c and d in case $D^{uvw}_{ija} \neq 0$ and $\Delta_M = 0$. Indeed, $(a, b, c, d) = \frac{1}{\Phi}(A_a, A_b, A_c, A_d)$ where

$$A_a = -E^{xzp}_{ija} E^{zrp}_{ija} - E^{xyp}_{ija} E^{yrp}_{ija}, \tag{4.3}$$

$$A_b = E^{xyp}_{ija} E^{xrp}_{ija} - E^{yzp}_{ija} E^{zrp}_{ija}, \tag{4.4}$$

$$A_c = E^{yzp}_{ija} E^{yrp}_{ija} + E^{xzp}_{ija} E^{xrp}_{ija}, \tag{4.5}$$

$$A_d = E^{yzr}_{ija} E^{yzp}_{ija} + E^{xzr}_{ija} E^{xzp}_{ija} + E^{xyr}_{ija} E^{xyp}_{ija} \neq 0, \tag{4.6}$$

$$\Phi = (E^{xyp}_{ija})^2 + (E^{xzp}_{ija})^2 + (E^{yzp}_{ija})^2. \tag{4.7}$$

The following equations can also be obtained if one exchanges k with a and substitutes $\Gamma = 0$ in the expressions of Iordanov in [28]. Lastly, we can determine

$$V = (x_k - \frac{a}{2d}, y_k - \frac{b}{2d}, z_k - \frac{c}{2d}) = \frac{1}{2d}(2dx_k - a, 2dy_k - b, 2dz_k - c) \tag{4.8}$$

$$= \frac{1}{2A_d}(2A_d x_k - A_a, 2A_d y_k - A_b, 2A_d z_k - A_c) \tag{4.9}$$

and therefore $V = \frac{1}{w_V}(x_V, y_V, z_V)$, where $x_V = 2A_d x_k - A_a, y_V = 2A_d y_k - A_b, z_V = 2A_d z_k - A_c$ and $w_V = 2A_d$. Since x_V, y_V, z_V and w_V are quantities of algebraic degree 8,8,8 and 7 respectively, it follows that the signs O_n and $O_{\mu\nu}$ both require algebraic operations of degree 10 on the input quantities to be evaluated. Since this is the most degree demanding operation in the evaluation of ShadowRegion$^\epsilon$, we have proven the following lemma.

Lemma 23 *The* ShadowRegion$^\epsilon$ *predicate for hyperbolic trisectors can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

4.4 The Existence$^\epsilon$ Predicate for hyperbolic trisectors

The Existence$^\epsilon$ denotes the number of the "finite" endpoints of $\mathcal{SR}^\epsilon(S_a)$, after applying the perturbation scheme. Therefore, to determine the outcome of the Existence$^\epsilon(S_i, S_j, S_k, S_a)$ predicate, we need to call the ShadowRegion$^\epsilon$ predicate with the same input. It is assumed that the trisector of S_i, S_j and S_k is hyperbolic and therefore the outcome of the ShadowRegion$^\epsilon$ predicate is determined as described in Section 4.3.

- If ShadowRegion$^\epsilon$ returns $\emptyset$ or $\mathbb{R}$ then Existence$^\epsilon$ returns 0.
- If ShadowRegion$^\epsilon$ returns $(-\infty, \phi)$ or $(\chi, +\infty)$ then Existence$^\epsilon$ returns 1. Finally,
- If ShadowRegion$^\epsilon$ returns $(-\infty, \phi) \cup (\chi, +\infty)$ or (χ, ϕ) then Existence$^\epsilon$ returns 2.

Since the evaluation of the ShadowRegion$^\epsilon$ predicate requires 10-fold operations with respect to the input quantities, we have proven the following lemma.

Lemma 24 *The* Existence$^\epsilon$ *predicate for hyperbolic trisectors can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

4.5 The Existence$^\epsilon$ Predicate for elliptic trisectors

In the case where the trisector of the sites S_i, S_j and S_k is elliptic, it is clear based on the previous analysis that a non-degenerate shadow region type of a site S_a on τ_{ijk} is either $\emptyset, \mathbb{R}$ or (χ, ϕ), where $\mathbb{R}$ denotes the entire trisector. The first two shadow region types correspond to Existence$(S_i, S_j, S_k, S_a) = 0$ as they do not have "finite"boundary points. On the other hand, (χ, ϕ) is the only option for $\mathcal{SR}(S_a)$ if EXISTENCE$(S_i, S_j, S_k, S_a) = 2$ as the last relation denotes a two-finite-boundary shadow region and the option $(-\infty, \phi) \cup (\chi, +\infty)$ in not feasible in elliptic trisectors (in contrast with hyperbolic ones).

Therefore a clear indication of a degenerate EXISTENCE, and equivalently SHADOW, outcome is the case where EXISTENCE$(S_i, S_j, S_k, S_a) = 1$. In this scenario, the shadow region of S_a on the trisector τ_{ijk} is of the form (χ, ϕ) (Subcase 1) or $\mathbb{R}\backslash(\chi, \phi)$ ((Subcase 2) with $\chi \equiv \phi$. If the last shadow region form is viewed as $(-\infty, \phi) \cup (\chi, +\infty)$ (in the sense that the shadow region contains all points of τ_{ijk} except one or simply put, we may consider that $-\infty \equiv +\infty$) then we can determine the $\mathcal{SR}^{\epsilon}(S_a)$ based on the analysis of Section 4.3, Degeneracies of Type B, Case 2.

In order to apply the analysis of this section, we have determine if we are in Subcase 1 or 2. This is easily accomplished as the shadow region $\mathcal{SR}(S_a)$ in Subcase 1 denotes that all points of the trisector τ_{ijk} except one do not belong in $\mathcal{SR}(S_a)$ and therefore all but one Apollonius sphere of the sites S_i, S_j and S_k do not intersect S_a. Since v_{ijkl} and v_{ikjm} are considered to exist on τ_{ijk} due to the fact that e_{ijklm} is a valid Voronoi edge on the trisector we only need to evaluate $I_1 =$ INSPHERE$(S_i, S_j, S_k, S_l, S_a)$ and $I_2 =$ INSPHERE$(S_i, S_j, S_j, S_m, S_a)$. If at least one of $\{I_1, I_2\}$ is positive (hence the other is necessarily either positive or zero) the corresponding vertex $\{v_{ijkl}, v_{ikjm}\}$ does not belong to $\mathcal{SR}(S_a)$, indicating we are in Subcase 1. On the other hand, if at least one of $\{I_1, I_2\}$ is negative (hence the other is necessarily either negative or zero) the corresponding vertex $\{v_{ijkl}, v_{ikjm}\}$ belongs to $\mathcal{SR}(S_a)$, indicating we are in Subcase 2.

Since we can distinguish if we are in Subcase 1 or 2, the analysis of Section 4.3 can be applied, yielding one of the possible $\mathcal{SR}^{\epsilon}(S_a)$: $\emptyset, \mathbb{R}, (\chi, \phi)$ or $(-\infty, \phi) \cup (\chi, +\infty)$. The later result can be viewed as (χ, ϕ) as it denotes that all points of the elliptic trisector τ_{ijk} except for a segment belong to $\mathcal{SR}^{\epsilon}(S_a)$ (or simply put, since we considered $-\infty \equiv +\infty$).

Similar with Section 4.4 for hyperbolic trisectors, if $\mathcal{SR}^{\epsilon}(S_a)$ returned $\emptyset$ or $\mathbb{R}$, the corresponding EXISTENCE$^{\epsilon}$ predicate returns 0. Otherwise, if $\mathcal{SR}^{\epsilon}(S_a)$ returned (χ, ϕ), the corresponding EXISTENCE$^{\epsilon}$ predicate returns 2. Since the evaluation of the SHADOWREGION$^{\epsilon}$ predicate and the signs of I_1 and I_2 demand, in the worst cast, operations of degree 10 in the input quantities, we have proven the following lemma.

Lemma 25 *The* EXISTENCE$^{\epsilon}$ *predicate for elliptic trisectors can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

4.6 Perturbation for the remaining predicates

Using the analysis of Section 2.7.6, we can evaluate the ORDER predicate for hyperbolic trisectors and non-degenerate input. Since the predicate requires the call of INSPHERE, EXISTENCE, SHADOW and DISTANCE subpredicates, we can also answer it for degenerate input if we could determine the outcome of the perturbed INSPHERE$^{\epsilon}$, EXISTENCE$^{\epsilon}$, SHADOWREGION$^{\epsilon}$ and DISTANCE$^{\epsilon}$ subpredicates. However the current known algebraic degree of INSPHERE$^{\epsilon}$, also known as VertexConflict$^\epsilon$ predicate, is known to be 28 (cf. [15], result provided by the author). It is therefore apparent that the algebraic degree of the Order$^\epsilon$ predicate is subject to the following lemma.

Lemma 26 *The* Order$^\epsilon$ *predicate for hyperbolic trisectors can be evaluated by determining the sign of quantities of algebraic degree at* $\max\{8, d\}$ *(in the input quantities), where d is the algebraic degree of the* VertexConflict$^\epsilon$ *predicate.*

Using the algorithms presented in Section 2.5, we can answer the EdgeConflict, InfiniteLeftEdgeConflict and InfiniteRightEdgeConflict predicates for hyperbolic trisectors and non-degenerate inputs using the InSphere, Order, Existence, Shadow and Distance subpredicates. For degenerate inputs, we can use the outcomes of the respective perturbed versions of these subpredicates and the following lemma is proven.

Lemma 27 *The predicates* InfiniteLeftEdgeConflict$^\epsilon$, InfiniteRightEdgeConflict$^\epsilon$ *and* EdgeConflict$^\epsilon$ *for hyperbolic trisectors can be evaluated by determining the sign of quantities of algebraic degree at* $\max\{8, d\}$ *(in the input quantities), where d is the algebraic degree of the* VertexConflict$^\epsilon$ *predicate.*

Finally, the algorithm presented in Section 3 can answer the EdgeConflict predicate for elliptic trisectors and non-degenerate inputs using the InSphere, Order and Existence subpredicates. In case of degenerate inputs, we can answer the EdgeConflict$^\epsilon$ predicates using the respective outcomes of the VertexConflict$^\epsilon$, Order$^\epsilon$ and Existence$^\epsilon$ subpredicates and the following lemma is proven.

Lemma 28 *The* EdgeConflict$^\epsilon$ *predicate for elliptic trisectors can be evaluated by determining the sign of quantities of algebraic degree at* $\max\{8, d\}$ *(in the input quantities), where d is the algebraic degree of the* VertexConflict$^\epsilon$ *predicate.*

Chapter 5

Analysis of parabolic trisectors

In previous chapters, we analysed all arising predicates and subpredicates under the assumption that the trisector of the first three input spheres where either of hyperbolic or elliptic type. We will now consider in detail the case of a parabolic trisector, i.e., when the respective trisector is a parabola.

This type of trisector is considered to be degenerate; it only arises when one of the spheres that define it point lies inside and it's boundary point-touches the convex hull of the other two. Equivalently, if the the trisector τ_{ijk} is parabolic and we properly exchange the names of the sites S_i, S_j and S_k such that the latter has the smaller radius, then the outcome of the $\textsc{InCone}(S_i, S_j; S_k)$ is necessarily OnePointTouch.

This type of trisector can be viewed in two ways. We can either consider it to be an elliptic trisector that contains the point at infinity or we can view it as a hyperbolic trisector where the points at $\pm\infty$ coincide. Although both of these considerations seem to contradict with our common sense and the way we perceive a trisector in $\mathcal{Z}$-space, these remarks become clear when viewed in $\mathcal{W}$-space and $\mathcal{Y}$-space. When the trisector τ_{ijk} is a parabola, the point $O^\star$ of $\mathcal{W}$-space lies on the semi-cone defined by $S_i^\star$ and $S_j^\star$ and equivalently, the point $\hat{O}$ of $\mathcal{Y}$-space lies on the circle C'. Therefore, the image of τ_{ijk} in $\mathcal{Y}$-space is the circle C' that contains the image of the point of infinity of $\mathcal{Z}$-space.

If a perturbation scheme is applied, the outcome of $\textsc{InCone}^\epsilon(S_i, S_j; S_k)$ will be either Outside or Inside; in this case the pertubed spheres will define a hyperbolic or an elliptic trisector, respectively. In $\mathcal{Y}$-space, the perturbation of the spheres will cause the point $\hat{O}$ to move infinitesimally Outside or Inside the circle C' respectively. This result is indicative of the fact that a parabolic trisector can be viewed as an in-between state of elliptic and hyperbolic trisectors, as mentioned above.

Due to the close relation of the parabolic trisectors with the other two types, we can determine the outcome of predicates that involve a parabolic trisector using variations of the algorithms that evaluate them for the respective hyperbolic or elliptic type. In the following section, we present a list of these predicates and the corresponding modifications that have to be applied in each case.

5.1 Predicates for parabolic trisectors

The type of the trisector τ_{ijk} can be easily determined via the call of the TRISECTORTYPE(S_i, S_j, S_k). If the predicate returns *parabolic* then the following modifications have to be considered during the evaluation of the respective predicate. Notice that in the following list of predicates, only non-degenerate intermediate results are considered and that the orientation of a parabolic trisector is defined in the same way as an elliptic trisector (see Section 3.1).

DISTANCE(S_i, S_j, S_k, S_a) **:** This predicate can be evaluated as described in Section 2.7.4, after taking into consideration that the centers C_i, C_j and C_k are not collinear and that there exist only one plane commonly tagent to the first three spheres. As a consequence, if $D^{xyz}_{ijka} \neq 0$, the quadratic polynomial $\Lambda(\epsilon)$, in terms of ϵ, has a double root and therefore its only root is $\epsilon = -\frac{\Lambda_1}{2\Lambda_2}$, where $\Lambda_2 \neq 0$. It follows that the DISTANCE predicate has to return a single (double) sign, $\mathsf{sign}(\epsilon) = -\mathsf{sign}(\Lambda_1) = \mathsf{sign}(D^{yzr}_{ijka}D^{yz}_{ijk} + D^{xzr}_{ijka}D^{xz}_{ijk} - D^{xyr}_{ijka}D^{xy}_{ijk})$, which corresponds to the signed distance of S_a from the plane cotagent to the sites S_i, S_j and S_k. Alternatively, the predicate could return this *double* sign in a tuple, i.e., $(-\mathsf{sign}(\Lambda_1), -\mathsf{sign}(\Lambda_1))$, for consistency with the hyperbolic case. Lastly, if $D^{xyz}_{ijka} = 0$, we return $(\mathsf{sign}(\epsilon), \mathsf{sign}(\epsilon))$, where $\mathsf{sign}(\epsilon) = -\mathsf{sign}(D^{xyzr}_{ijka}) \cdot \mathsf{sign}(D^{xyz}_{ijk})$.

EXISTENCE(S_i, S_j, S_k, S_a) **:** This predicate can be evaluated as described in Section 4.1, without modifications. Remember that the EXISTENCE predicate only returns the number of *finite* Apollonius vertices that exist among $\{v_{ikja}, v_{ijka}\}$ on the trisector τ_{ijk}.

SHADOW(S_i, S_j, S_k, S_a) **:** The outcomes of the SHADOW predicate are the same with the hyperbolic case, with the only difference that we can consider that $\pm\infty$ coincide. Therefore, the only possible topological forms of the shadow region are $\emptyset, \mathbb{R} = (-\infty, +\infty), (-\infty, \phi) \cup (\chi, +\infty)$ and (χ, ϕ). Notice that the latter two cases indicate that the image of the shadow region of S_a on τ_{ijk} in $\mathcal{Y}$-space is an arc of C' that either contains $\hat{O}$ or not, respectively. We can answer the SHADOW predicate using the analysis presented in Section 2.7.5, although the cases $(-\infty, \phi)$ and $(\chi, +\infty)$ will never arise.

EDGECONFLICT($S_i, S_j, S_k, S_l, S_m, S_q$) **:** This predicate can be determined using the same algorithm presented for the respective elliptic case in Section 3.2.

Via the use of these modifications, we have proven the following lemma.

Lemma 29 *The* DISTANCE, EXISTENCE, SHADOW *and* EDGECONFLICT *predicate for parabolic trisectors can be evaluated by determining the sign of quantities of algebraic degree at most 6, 8, 8 and 10 (in the input quantities), respectively.*

In the case where an intermediate degenerate outcome arises during the evaluation of one of these predicates, we can resolve the degeneracy by applying a perturbation scheme. Below we present a list of perturbed predicates and how we can determine their outcome, in the case where τ_{ijk} is a parabola.

Distance$^{\epsilon}(S_i, S_j, S_k, S_a)$**:** The sphere S_a is tangent to the plane commonly tagent to the first three input spheres, i.e., the predicate Distance(S_i, S_j, S_k, S_a) returned 0. Let Δ denote the outcome of the respective perturbed predicate, we determine the sign of Δ, using similar arguements and notation with the ones presented in Section 4.2.

Step 1. If $a > i, j, k$, return $\Delta = -$. Otherwise, name exchange the sites S_i, S_j and S_k such that $i > j, k, a$ and go to Step 2.

Step 2. If $t_a \equiv t_i$ return $\Delta = +$, otherwise go to Step 3.

Step 3. If $t_a \equiv t_j$ return $\Delta = -$ if $r_a > r_j$ or $\Delta = -$ if $r_a < r_j$. Otherwise go to Step 4.

Step 4. If $t_a \equiv t_k$ return $\Delta = -$ if $r_a > r_k$ or $\Delta = -$ if $r_a < r_k$. Otherwise go to Step 5.

Step 5. If t_j, t_k and t_a are not collinear (equivalently $D^{xyz}_{ijka} \neq 0$) go to step 6. Otherwise we should return as Δ the sign of $-D^{xyzr}_{ijka}/D^{xyz}_{ijk}$ (see case $D^{xyz}_{ijka} = 0$ in Section 2.7.4) when the radius r_a of the sphere S_a infinitesimally becomes $r_a + \epsilon_a$ for some $\epsilon_a > 0$. In this case, $\mathsf{sign}(\Delta) = \mathsf{sign}(D^{xyz}_{ijk}) \cdot \mathsf{sign}(\epsilon_a D^{xyz}_{jka}) = \mathsf{sign}(D^{xyz}_{ijk}) \cdot \mathsf{sign}(D^{xyz}_{jka})$, since it originally holded that $D^{xyzr}_{ijka} = 0$ (the quantity $-D^{xyzr}_{ijka}/D^{xyz}_{ijk}$ was zero as we are studying a degenerate case).

Step 6. Since $D^{xyz}_{ijka} \neq 0$, we should return as Δ the $\mathsf{sign}(-\Lambda_1) = \mathsf{sign}(D^{yzr}_{ijka}D^{yz}_{ijk} + D^{xzr}_{ijka}D^{xz}_{ijk} - D^{xyr}_{ijka}D^{xy}_{ijk})$ when the radius r_a of the sphere S_a infinitesimally becomes $r_a + \epsilon_a$ for some $\epsilon_a > 0$. The last expression, initially equaled zero as we were in a degenerate case, but after substituting r_a with $r_a + \epsilon_a$, it becomes $\mathsf{sign}(\epsilon_a)\mathsf{sign}(D^{yz}_{jka}D^{yz}_{ijk} + D^{xz}_{jka}D^{xz}_{ijk} - D^{xy}_{jka}D^{xy}_{ijk}) = \mathsf{sign}(D^{yz}_{jka}D^{yz}_{ijk} + D^{xz}_{jka}D^{xz}_{ijk} - D^{xy}_{jka}D^{xy}_{ijk})$.

ShadowRegion$^{\epsilon}(S_i, S_j, S_k, S_a)$**:** The non-degenerate outcomes of the Shadow predicate for parabolic trisectors are $\emptyset, \mathbb{R} = (-\infty, +\infty), (-\infty, \phi) \cup (\chi, +\infty)$ and (χ, ϕ) (the last two cases, can be considered, for parabolic trisectors, as the set (χ, ϕ) that either contains the point at infinity or not, respectively). Degenerate shadow regions occur when ϕ and/or χ coincide with each other and/or the point at infinity.

The non-degenerate outcomes of the $S =$ Shadow(S_i, S_j, S_k, S_a) predicate are identified from the combination of the outcomes $E =$ Existence(S_i, S_j, S_k, S_a) and $D =$ Distance(S_i, S_j, S_k, S_a).

1. If $E = 2$ and $D = (+, +)$ then $S = (\chi, \phi)$.
2. If $E = 2$ and $D = (-, -)$ then $S = (-\infty, \phi) \cup (\chi, +\infty)$.

3. If $E = 0$ and $D = (+,+)$ then $S = \emptyset$.
4. If $E = 0$ and $D = (-,-)$ then $S = \mathbb{R} = (-\infty, +\infty)$.

All other combinations of S and E outcomes are either infeasible or yield a degenerate S outcome. In order to determine, if such a combination results in a degenerate shadow region and what the corresponding pertrubed shadow region would be, we follow an analysis similar with the one presented in Section 4.3. The feasible combinations are the following

Case 1. If $E = 0$ and $D = (0,0)$ then either $S = (\chi, \phi)$ where $\chi \equiv \phi \pm \infty$.

Case 2. If $E = 1$ and $D = (0,0)$ then $S = (\chi, \phi)$ where either $\chi \equiv -\infty$ or $\phi \equiv +\infty$.

Case 3. If $E = 1$ and $D = (+,+)$ or $(-,-)$ then $S = (-\infty, \phi) \cup (\chi, +\infty)$ or (χ, ϕ), where $\chi \equiv \phi \not\equiv \pm\infty$.

To determine the perturbed shadow region S^ϵ, we evaluate D^ϵ if $D = (0,0)$ and a non-zero I that corresponds to either InSphere$(S_i, S_j, S_k, S_n, S_a)$ or InSphere$(S_i, S_k, S_j, S_n, S_a)$ for some $S_n \not\equiv S_a$. Note that the Shadow predicate is assumed to be called during the evaluation of the EdgeConflict predicate and therefore the Apollonius vertices v_{ijkl} and v_{ikjm} and either one or both of $\{v_{ijkq}, v_{ikjq}\}$ exist on the trisector τ_{ijk}. The existence of these vertices guarantee that the corresponding InSphere predicates are well defined for the respective inputs.

To resolve degeneracies of Case 1, using the same arguements presented in Section 4.3, we consider D^ϵ. If $D^\epsilon = (+,+)$ then $S^\epsilon = \emptyset$ if $I = +$ or $S^\epsilon = (-\infty, \phi) \cup (\chi, +\infty)$ if $I = -$. Otherwise, if $D^\epsilon = (-,-)$ then $S^\epsilon = (\chi, \phi)$ if $I = +$ or $S^\epsilon = \mathbb{R} = (-\infty, +\infty)$ if $I = -$.

For degeneracies of Case 2, if $D^\epsilon = (+,+)$ or $(-,-)$ then $S^\epsilon = (\chi, \phi)$ or $(-\infty, \phi) \cup (\chi, +\infty)$, respectively.

Finally, degeneracies of Case 3 are handled exactly as the degeneracies of Type B, Case 2, Subcase 2, presented in Section 4.3. This is also the most difficult degenerate case to handle in both hyperbolic and elliptic trisectors.

Existence$^\epsilon(S_i, S_j, S_k, S_a)$ **:** The outcome of the respective Existence predicate is non-degenerate if and only if it does not equal 1 and the corresponding Distance predicate did not return 0 (or $(0,0)$). The outcome of the Existence$^\epsilon$ predicate can be obtained by the evaluation of $\mathcal{SR}^\epsilon$ = ShadowRegion$^\epsilon(S_i, S_j, S_k, S_a)$, as it expresses the number of finite boundary of the perturbed shadow region of S_a on the trisector τ_{ijk}:

1. If $\mathcal{SR}^\epsilon = \emptyset$ or $\mathbb{R} = (-\infty, +\infty)$, then the Existence$^\epsilon$ predicate returns 0. Otherwise,
2. If $\mathcal{SR}^\epsilon = (\chi, \phi)$ or $\mathbb{R} = (-\infty, \phi) \cup (\chi + \infty)$, then the Existence$^\epsilon$ predicate returns 2.

EDGECONFLICT$^\epsilon(S_i, S_j, S_k, S_l, S_m, S_q)$ **:** The predicate makes use of the outcomes of the perturbed predicates DISTANCE$^\epsilon$, EXISTENCE$^\epsilon$, SHADOWREGION$^\epsilon$, which can be evaluated as described above.

The following remarks prove the following theorem.

Theorem 4 *Let d denote the algebraic degree of the* VERTEXCONFLICT$^\epsilon$ *predicate. The* DISTANCE$^\epsilon$*,* EXISTENCE$^\epsilon$*,* SHADOWREGION$^\epsilon$ *and* EDGECONFLICT$^\epsilon$ *predicate for parabolic trisectors can be evaluated by determining the sign of quantities of algebraic degree at 5, 10, 10 and* $\max\{10, d\}$ *(in the input quantities), respectively.*

Chapter 6

Conclusion and Future Work

In this thesis, we presented a clever way of combining various subpredicates in order to answer the EdgeConflict predicate. The design of all predicates and primitives was made in such a way that the maximum algebraic cost of answering them would be as low as possible, on the input quantities. Based on current bibliography, the resulting degree 10 of the main predicate is quite small when compared with the respective 2D version of the EdgeConflict predicate (16 as shown in [22] and 6 as shown in [37]). Moreover, the fact that the inversion and perturbation techniques can be applied in the 2D Apollonius diagram suggests that a similar analysis with the respective 3D case will yield similar algebraic degrees with [37], for both degenerate and non-degenerate inputs. It is also remarkable that both the InSphere (equivalent to VertexConflict) and the EdgeConflict predicates share the same algebraic degree.

Through our attempt to answer the master predicate, various useful primitives were also developed. These tools can also be used in the context of an incremental algorithm that evaluates the Apollonius diagram of a set of spheres.

Ultimately, we would like to use the remarks that connect $\mathcal{Z}$-space with $\mathcal{Y}$-space to resolve the degenerate VertexConflict$^{\epsilon}$ predicate described in [15] with the lowest algebraic degree, using a qualitative perturbation scheme. The resulting degree will inevitably be an upper bound for the cost of the Order$^{\epsilon}$ and EdgeConflict$^{\epsilon}$ predicates for degenerate inputs.

Lastly, it is our intent to implement the algorithms presented in this thesis. This would consist a great step towards the exact construction of the 3D Apollonius diagram via an incremental algorithm and would provide information of the viability of such algorithms opposite existing implementations based on floating-point arithmetic.

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