Doctoral Thesis

# Controlling the Dynamics of Optical Beams 

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#### Abstract

Since the theoretical prediction and experimental demonstration of optical Airy beams [Opt. Lett., 32(8):979-981,(2007), Phys. Rev. Lett., 99:213901,(2007)], accelerating waves have been established as a very important tool in the field of optics. The last dozen of years accelerating beams have attracted a lot of interest due to their intriguing properties, and were extensively studied both from a theoretical and an experimental perspective. Originally, Airy beams were proposed in quantum mechanics in the seminal work of Berry and Balazs [Am. J. Phys. 47(3):264-267, (1979)], showing that the potential-free Schrödinger equation admits propagation-invariant solutions in the form of accelerating Airy wavepackets. Beyond Airy beams, another family of diffraction-free waves was proposed and experimentally observed by Durnin, the well-known Bessel beams [J. Opt.Soc. Am. A, 4(4):651-654, (1987), Phys. Rev. Lett., 58:1499-1501, (1987)]. Due to their resilience to diffraction-spreading and the uniformity of their amplitude, such beams were also exploited in many applications. Furthermore, in the nonparaxial domain where rays and thus beams can bend at large angles, diffraction free beams accelerating along circular, elliptical, exponential and general power-law trajectories were demonstrated. In another concept, abruptly autofocusing waves mainly represented by Airy beams with radial symmetry, propagate along parabolic trajectories while focusing most of their energy right before a target.

In this dissertation, we focus on engineering the properties of optical waves. We focus in the case of propagation-invariant fields of the Airy and Bessel type and on different classes of accelerating waves. We engineer their fundamental properties such as their amplitude, their width and their trajectory. Furthermore, we examine the focusing characteristics of abruptly autofocusing waves. The possibility of optimizing their focusing features is of our particular interest.

To begin with, we study the generation of accelerating waves in the paraxial domain, whose propagation defining properties such as trajectory, maximum amplitude and beam-width will be predesigned. In the case of the power-law trajectories, the propagation of such beams is described by Airy-type solutions which are directly expressed in terms of the geometric properties of the preselected path.

Additionally, we investigate the propagation of accelerating beams in the nonparaxial domain. In this case we study accelerating beams along circular, elliptic and power-law curves. Our solutions indicate that independently of the trajectory assumed, the dynamics of the beam near the caustic are described by Airy-type functions. Our formulas are expressed in an elegant and practical way and highlight the dependence to the curvature of the predesigned trajectory, among other geometrical features. In particular, we show that the generation of accelerating beams along nonparaxial trajectories with pre-engineered amplitude and beam-width, is possible.

Moreover, we consider the propagation of abruptly autofocusing waves in the paraxial domain. Specifically, we emphasize on the propagation of such beams along convex but otherwise arbitrary predefined trajectories. Furthermore, in order to optimize their focusing characteristics, we properly modulate the important parameters such as the initial amplitude, the curvature of the trajectory, and the distance from the optical axis on the input plane, in order to achieve higher intensity contrast at the focus along with damped oscillatory behavior after the focal point.

Beyond accelerating beams, we also study the case of Bessel beams of zeroth order and higher-order optical vortices of the Bessel-type. We propose a method for generating such beams, exhibiting pre-engineered maximum amplitude and beam-width or hollowcore radius along the propagation distance. In both cases, numerical results agree well with the theoretical model developed.


## $\Pi \varepsilon \rho i ́ \lambda \eta \psi \eta$


 99:213901,(2007)], то $\varepsilon \nu \delta \iota \alpha \varphi \varepsilon ́ \rho о \nu ~ \tau \omega \nu ~ \varepsilon \rho \varepsilon \cup \nu \eta \tau \omega \nu \nu ~ \gamma \iota \alpha ~ \tau \eta ~ \mu \varepsilon \lambda \varepsilon ́ \tau \eta ~ \chi u \mu \alpha ́ \tau \omega \nu ~ \alpha v \tau \eta ́ s ~ \tau \eta \varsigma ~ \mu о \rho \varphi \eta ́ s ~$




 $\varepsilon \xi \omega \tau \varepsilon \rho \iota$ кои́ $\delta \cup \nu \alpha \mu \iota x$ ои́ $\delta \varepsilon ́ \chi \varepsilon \tau \alpha \iota ~ \lambda u ́ \sigma \varepsilon \iota \varsigma ~ \tau u ́ \pi о u ~ A i r y ~[A m . ~ J . ~ P h y s . ~ 47(3): 264-267, ~(1979)] . ~ П \varepsilon ́ \rho \alpha ~$

 Durnin [J. Opt.Soc. Am. A, 4(4):651-654, (1987), Phys. Rev. Lett., 58:1499-1501, (1987)], $\tau \alpha$










 тט́лои Bessel. П $\rho \alpha \dot{\lambda \lambda \lambda \eta \lambda \alpha ~ \vartheta \alpha ~ \varepsilon \sigma \tau \iota \alpha ́ \sigma о \cup \mu \varepsilon ~ \chi \alpha \iota ~ \sigma \varepsilon ~ \gamma \varepsilon \nu \iota х о ́ \tau \varepsilon \rho \varepsilon \varsigma ~ \chi \alpha \tau \eta \rho ү о \rho i ́ \varepsilon \varsigma ~ \varepsilon \pi \iota \tau \alpha \chi ט \nu o ́-~}$








 $\tau \eta \varsigma ~ \pi \rho о ж \alpha \theta$ о $\iota \iota \mu \varepsilon ́ \nu \eta \varsigma ~ \tau \rho о \chi \iota \alpha ́ \varsigma . ~$

Е $\tau \iota \pi \rho o ́ \sigma \theta \varepsilon \tau \alpha, \vartheta \alpha \mu \varepsilon \lambda \varepsilon \tau \eta ́ \sigma о \cup \mu \varepsilon ~ \tau \eta ~ \delta \iota \alpha ́ \delta o \sigma \eta ~ \varepsilon \pi \iota \tau \alpha \chi \nu \nu o ́ \mu \varepsilon \nu \omega \nu \chi \nu \mu \alpha ́ \tau \omega \nu$ $\sigma \tau o \mu \eta \pi \alpha \rho-$






 $\nu \alpha \iota \iota \delta \iota \alpha i \tau \varepsilon \rho \alpha$ бף $\mu \nu \tau \iota x o ́ \varsigma . ~ \Sigma \tau \alpha \pi \lambda \alpha i \sigma \iota \alpha$ $\tau \eta \varsigma ~ \varepsilon \rho \gamma \alpha \sigma i \alpha \varsigma \mu \alpha \varsigma \mu \varepsilon \lambda \varepsilon \tau \alpha ́ \mu \varepsilon ~ \tau \eta ~ \delta \iota \alpha ́ \delta o \sigma \eta ~ \tau \varepsilon ́-$


 тоv xú $\mu \alpha \tau о \varsigma ~ \sigma \tau о ~ \varepsilon \sigma \tau \iota \alpha x o ́ ~ \sigma \eta \mu \varepsilon i ́ o, ~ x \alpha \iota ~ \nu \alpha ~ \varepsilon \pi \iota \tau ט ́ \chi о ৩ \mu \varepsilon ~ \tau \eta \nu ~ \alpha ́ \alpha \mu \varepsilon \sigma \eta ~ \alpha \pi o ́ \sigma \beta \varepsilon \sigma \eta ~ \tau \eta \varsigma ~ \mu \varepsilon \tau \alpha ́ \alpha ~ \alpha \pi о ~$ $\alpha$ бо́.






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Taking this chance, I wish to thank sincerely all the people that contributed to this dissertation. First and foremost, I would like to express my gratitude to my advisor Professor Nikolaos Efremidis for giving me the opportunity to work along with him. His perspicacity, perception and deep knowledge of science in general were evident to me as early as my undergraduate studies, when i had the chance to attend his lectures. During my graduate studies his other virtues like his utter passion for research, his ability to explore every detail while staying true to the core of the problem and being precise, along with his profound knowledge of the field of optics were apparent. His distinctive personality, his modesty and his courtesy complement his ethics. The progress of my research has been benefited not only from his immense working energy but also from his constant interest as well. His patience along with his willingness to discuss with me all of my concerns was really important. The ability to clarify even my most complicated questions with an appropriate answer at hand, while at the same time providing me with fruitful comments is outstanding. But most importantly, I will always be indebted to Dr. Efremidis for the role model of a scientist, a professor and an individual he set for me.

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## Contents

Abstract ..... i
$\Pi \varepsilon \rho i ́ \lambda \eta \psi \eta$ ..... ii
Acknowledgements ..... iii
1 Introduction ..... 1
2 Fundamental concepts of Optics ..... 6
2.1 Introduction to Maxwell's equations ..... 6
2.2 Helmholtz equation ..... 9
2.3 The Green's function of the 2D Helmholtz equation ..... 10
2.4 The Rayleigh-Sommerfeld formulation of diffraction ..... 15
2.5 The Fresnel diffraction integral ..... 21
2.6 Paraxial approximation ..... 25
2.7 Method of stationary phase ..... 26
3 Propagation of accelerating beams in the paraxial domain ..... 29
3.1 Introduction ..... 29
3.2 Amplitude and trajectory/beam-width engineering ..... 30
3.3 Numerical Results ..... 34
4 Propagation of accelerating beams in the nonparaxial domain ..... 39
4.1 Introduction ..... 39
4.2 Amplitude and trajectory/beam-width engineering ..... 40
4.3 Numerical Results ..... 43
5 Propagation of abruptly autofocusing beams in the paraxial domain ..... 51
5.1 Introduction ..... 51
5.2 Amplitude and trajectory/beam-width engineering ..... 52
5.3 Numerical Results ..... 58
6 Tunable self-similar Bessel-like beams of arbitrary order ..... 64
6.1 Introduction ..... 64
6.2 Amplitude and width engineering of zeroth-order Bessel beams ..... 66
6.3 Numerical Results ..... 68
6.4 Amplitude and hollow-core radius engineering of higher-order Bessel beams ..... 68
6.5 Numerical Results ..... 71
7 Conclusions ..... 74
A Propagation dynamics of paraxial accelerating beams ..... 76
B Propagation dynamics of nonparaxial accelerating beams ..... 79
C Propagation dynamics of paraxial abruptly autofocusing beams ..... 85
Bibliography ..... 89

Dedicated to my family

## Chapter 1

## Introduction

Curved and accelerating beams have been established as a cornerstone in the field of Optics. Since 2007, when the prediction [1] and first experimental demonstration [2] of accelerating waves in the form of Airy beams accomplished, a lot of research has been focused on both theoretical and experimental studies (see for example the recent review article [3]). Of course, this outcome would not have been possible without the seminal work of Berry and Balazs - presented almost fourty years ago - showing that the Schrödinger equation in the absence of any external potential admits solutions in the form of the Airy function [4]. Interestingly, Airy wavepackets are the only non-spreading solutions of the potential-free Schrödinger equation in the case of one transverse dimension, and as such their amplitude profile remains invariant in the transverse plane during propagation. Perhaps, beyond the diffraction-free character, the most intriguing feature of such wavepackets is the ability to accelerate in the absence of any external field, i.e., they propagate following a parabolic trajectory. Despite the exciting properties they exhibit, Airy wavepackets remained unexploited for decades in the field of optics, mainly due to difficulties in their experimental realization, arising from their infinite transverse extent and thus the infinite energy they carry.

On the other hand, in higher dimensions diffraction-free waves in the form of Bessel beams have been proposed and experimentally demonstrated almost thirty years ago [5, 6]. Interestingly, such beams have already found numerous applications in many fields (see for example the review article [7]). Beyond Bessel beams of zeroth and higher order, other families of non-diffracting waves exist such as Mathieu beams along with waves based on parabolic cylinder functions [8, 9]. While as
we stated earlier, non-diffracting wavefunctions firstly emerged in the context of quantum mechanics, the equivalence of the potential-free Schrödinger equation with the paraxial wave equation, constitutes a solid ground for the mathematical description of non-diffracting beams in the optical domain. As in the case of the Airy beams with one transverse dimension, in higher dimensions non-diffracting beams have infinite energy from which their diffraction-free character arise. Obviously, infinite-energy carrying waves are not experimentally feasible, and thus a truncation factor is utilized to make the energy of such beams finite. The best example of this class of waves is the well-known quasi-Bessel beams named after the ideal (infinite energy) Bessel beams. While the latter propagate in a diffraction-free manner, their finite energy counterparts tend to diffract after they are truncated. However, quasiBessel beams can be properly designed in order to suppress diffraction along the intended propagation distance, meaning that the variations in the amplitude and in the radial extent that naturally occur through this procedure are negligible.

In the second chapter of this dissertation we introduce some of the basic mathematical concepts in the field of Optics. Starting from the Maxwell's equations which govern the propagation of any electromagnetic wave, we derive the wave equation and its time-independent counterpart; the Helmholtz's equation. The mathematical background necessary to describe the propagation of beams in the nonparaxial domain is shown next, namely the Rayleigh-Sommerfeld formulation of diffraction. The latter consists an integral representation of the solution of the Helmholtz's equation, based upon the Green's function of the problem. Here, we emphasize on the Green's function of the Helmholtz's equation in a 2D domain. We proceed with the Fresnel diffraction integral and the paraxial approximation, which are tools that are widely used in the context of this thesis. We select to conclude this chapter with the method of stationary phase, a powerful technique which is utilized when one encounters integral representations as those mentioned above but also, beyond Optics, in many other research fields.

In the third chapter of this thesis, we study the propagation of accelerating optical waves following power-law trajectories in the paraxial domain. The three fundamental features that characterize the propagation of any optical beam are the trajectory, the amplitude and its beam-width. In this respect, we would like to
generate paraxial accelerating beams whose amplitude and beam-width will be fullycontrollable while they propagate along a curved pre-designed trajectory. To obtain control over these parameters we engineer both the amplitude and the phase of the propagating beam on the input plane. Utilizing asymptotic methods in a region close to the beam's caustic, we were able to express the phase and the amplitude of the propagating beam in terms of the initial phase and the initial amplitude respectively. The only constraints that arise through our procedure are that the initial phase of the beam should be strictly monotone(the beam must follow a convex trajectory), and the amplitude should be a slowly varying function.

In the fourth chapter of this thesis, we generalize the results obtained from the previous chapter to the nonparaxial regime. In contrast with the paraxial domain, nonparaxial accelerating beams can bend at large angles and thus the notion as to what is diffraction-free must be modified accordingly. Specifically, we can define a beam to be diffraction-free if its amplitude profile remains invariant in the plane that is normal to the trajectory. Taking advantage of the large bending angles that the nonparaxial regime supports, we study the propagation of accelerating beams following circular and elliptic trajectories as well as beams forming power-law curves. As in the previous chapter, we are seeking to independently control the trajectory and the maximum amplitude along the trajectory of the propagating beam. Numerical results obtained are in excellent agreement with the theoretical model both in terms of the trajectory and in terms of the amplitude/beam-width. The functional form of the trajectory can be selected arbitrarily as long as it is convex, while the intended amplitude along the trajectory should be relatively slowly varying.

In the fifth section of this report, we discuss the propagation of abruptly autofocusing beams. Such beams have the ability to concentrate most of their energy at a specific point while maintaining a relatively low-intensity profile until their focus. While Airy beams with radial symmetry were the first family of waves utilized to demonstrate this unique property, several other classes of radially accelerating beams can be employed too. More specifically, by modulating the phase of the propagating beam on the input plane, we can generate abruptly autofocusing waves that follow general power-law trajectories, while exhibiting tighter focus along with more abrupt
and higher contrast. Closed form expressions describing the three stages of propagation of the autofocusing beam were obtained, namely before the focus, at the focus, and after the focus. Emphasis was also given to the third stage of propagation, i.e., after the focus, where the optimization of the focusing properties of the beam achieved by damping the oscillatory behavior, while at the same time the focal contrast increased by orders of magnitude.

In the sixth part of this thesis, we investigate the propagation of Bessel beams of zeroth- and higher-order of the first kind. Specifically, we demonstrate a simple and effective way to independently control both the beam-width/hollow-core radius and the maximum amplitude of such beams. Our methodology is based on engineering the amplitude and the phase of the propagating beam on the input plane. Interestingly, we express the phase and the amplitude of the optical wave with connection to the initial phase and the initial amplitude respectively. Furthermore, typical constraints that arise through our procedure such as the monotonicity of the phase and the slow variation of the amplitude, addressed. We report that our numerical results are in excellent agreement with the theoretical model both in terms of the intended beam-width/hollow-core radius, and in terms of the pre-designed amplitude.

The seventh chapter concludes this dissertation by summarizing our main results.

## List of publications

During my graduate studies we have published the following articles:

1. Michael Goutsoulas, Domenico Bongiovanni, Denghui Li, Zhigang Chen, and Nikolaos K. Efremidis. Tunable self-similar Bessel-like beams of arbitrary order. Opt. Lett., 45(7):1830-1833, Apr 2020.
2. Raluca-Sorina Penciu, Yujie Qiu, Michael Goutsoulas, Xiaopei Sun, Yi Hu, Jingjun Xu, Zhigang Chen, and Nikolaos K. Efremidis. Observation of microscale nonparaxial optical bottle beams. Opt. Lett., 43(16):3878-3881, Aug 2018.
3. Michael Goutsoulas, Raluca-Sorina Penciu, and Nikolaos K. Efremidis. Independent amplitude and trajectory/beam-width control of nonparaxial beams. Opt. Express, 26(15):18969-18974, Jul 2018.
4. Michael Goutsoulas and Nikolaos K. Efremidis. Precise amplitude, trajectory, and beam-width control of accelerating and abruptly autofocusing beams. Phys. Rev. A, 97:063831, Jun 2018.

## Chapter 2

## Fundamental concepts of Optics

### 2.1 Introduction to Maxwell's equations

Maxwell's equations are a set of 4 equations that govern the field of electrodynamics. These equations describe the propagation of electric and magnetic fields, and determine their interaction with matter. More specifically, Maxwell's set of equations includes, Gauss's laws for electric fields and magnetism, along with Faraday's law of induction and Ampere's Law. Despite that at his original work those equations were not so clearly stated, Maxwell was able to utilize them in order to approximate the speed of light in vacuum, which later proved to be within $5 \%$ of the correct answer.

In 1884, twenty years after the original form of Maxwell's equations posed, Oliver Heaviside and Willard Gibbs introduced a more compact and easily understandable form of those equations. The set of 4 equations they introduced, is considered today as the modern version of Maxwell's equations

$$
\begin{align*}
& \boldsymbol{\nabla} \cdot \mathbf{E}=\frac{\rho}{\epsilon} \\
& \boldsymbol{\nabla} \cdot \mathbf{B}=0 \\
& \boldsymbol{\nabla} \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}  \tag{2.1}\\
& \boldsymbol{\nabla} \times \mathbf{B}=\mu \mathbf{J}+\mu \epsilon \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\rho$ is the charge density, $\mathbf{J}$ is the current density, $\epsilon$ is the permittivity, and $\mu$ is the permeability of the medium.

In the absence of free charges or currents in a medium such as vacuum, we can assume that $\rho=0$ and $\mathbf{J}=0$. This assumptions are vastly used in Optics, leading to the simplified Maxwell's equations in the following form

$$
\begin{align*}
& \nabla \cdot \mathbf{E}=0 \\
& \nabla \cdot \mathbf{B}=0 \\
& \nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}  \tag{2.2}\\
& \boldsymbol{\nabla} \times \mathbf{B}=\mu \epsilon \frac{\partial \mathbf{E}}{\partial t} .
\end{align*}
$$

To obtain the electromagnetic wave equation for an optical wave, we begin from the above set of equations and firstly take the curl of the equation describing the Faraday's law

$$
\begin{align*}
\boldsymbol{\nabla} \times(\boldsymbol{\nabla} \times \mathbf{E})=\boldsymbol{\nabla} \times\left(-\frac{\partial \mathbf{B}}{\partial t}\right)=-\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{B})=-\frac{\partial}{\partial t}\left(\mu \epsilon \frac{\partial \mathbf{E}}{\partial t}\right) & =-\mu \epsilon \frac{\partial^{2} \mathbf{E}}{\partial t^{2}}  \tag{2.3}\\
& =-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{E}}{\partial t^{2}}
\end{align*}
$$

where for the last equality we have utilized the substitution $\mu \epsilon=1 / c^{2}$. Before we proceed, we would like to expand the leftmost term of the above equation. To begin with, let $(\boldsymbol{\nabla} \times \mathbf{M})=\mathbf{N}$, such that $\boldsymbol{\nabla} \times(\boldsymbol{\nabla} \times \mathbf{M})=\boldsymbol{\nabla} \times \mathbf{N}$, and

$$
\mathbf{N}=\left|\begin{array}{ccc}
\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
M_{x} & M_{y} & M_{z}
\end{array}\right|=\hat{\mathbf{i}}\left(\frac{\partial M_{z}}{\partial y}-\frac{\partial M_{y}}{\partial z}\right)-\hat{\mathbf{j}}\left(\frac{\partial M_{z}}{\partial x}-\frac{\partial M_{x}}{\partial z}\right)+\hat{\mathbf{k}}\left(\frac{\partial M_{y}}{\partial x}-\frac{\partial M_{x}}{\partial y}\right) .
$$

Then we can calculate

$$
\boldsymbol{\nabla} \times \mathbf{N}=\left|\begin{array}{ccc}
\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\left(\frac{\partial M_{z}}{\partial y}-\frac{\partial M_{y}}{\partial z}\right) & -\left(\frac{\partial M_{z}}{\partial x}-\frac{\partial M_{x}}{\partial z}\right) & \left(\frac{\partial M_{y}}{\partial x}-\frac{\partial M_{x}}{\partial y}\right)
\end{array}\right|
$$

which results to

$$
\begin{aligned}
& \boldsymbol{\nabla} \times \mathbf{N}=\left[\frac{\partial}{\partial y}\left(\frac{\partial M_{y}}{\partial x}-\frac{\partial M_{x}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial M_{z}}{\partial x}-\frac{\partial M_{x}}{\partial z}\right)\right] \hat{\mathbf{i}} \\
&-\left[\frac{\partial}{\partial x}\left(\frac{\partial M_{y}}{\partial x}-\frac{\partial M_{x}}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial M_{z}}{\partial y}-\frac{\partial M_{y}}{\partial z}\right)\right] \hat{\mathbf{j}} \\
&+\left[-\frac{\partial}{\partial x}\left(\frac{\partial M_{z}}{\partial x}-\frac{\partial M_{x}}{\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{\partial M_{z}}{\partial y}-\frac{\partial M_{y}}{\partial z}\right)\right] \hat{\mathbf{k}}
\end{aligned}
$$

or much simpler

$$
\begin{gathered}
\boldsymbol{\nabla} \times \mathbf{N}=\left[\frac{\partial^{2} M_{y}}{\partial x \partial y}-\frac{\partial^{2} M_{x}}{\partial y^{2}}+\frac{\partial^{2} M_{z}}{\partial x \partial z}-\frac{\partial^{2} M_{x}}{\partial z^{2}}\right] \hat{\mathbf{i}}+\left[\frac{\partial^{2} M_{x}}{\partial y \partial x}-\frac{\partial^{2} M_{y}}{\partial x^{2}}+\frac{\partial^{2} M_{z}}{\partial y \partial z}-\frac{\partial^{2} M_{y}}{\partial z^{2}}\right] \hat{\mathbf{j}} \\
+\left[\frac{\partial^{2} M_{x}}{\partial z \partial x}-\frac{\partial^{2} M_{z}}{\partial x^{2}}-\frac{\partial^{2} M_{z}}{\partial y^{2}}+\frac{\partial^{2} M_{y}}{\partial z \partial y}\right] \hat{\mathbf{k}} .
\end{gathered}
$$

To continue with, we know that $\boldsymbol{\nabla} \cdot \mathbf{M}=\left(\frac{\partial M_{x}}{\partial x}+\frac{\partial M_{y}}{\partial y}+\frac{\partial M_{z}}{\partial z}\right)$, so

$$
\begin{aligned}
& \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{M})=\frac{\partial}{\partial x}\left(\frac{\partial M_{x}}{\partial x}+\frac{\partial M_{y}}{\partial y}+\frac{\partial M_{z}}{\partial z}\right) \hat{\mathbf{i}} \\
& +\frac{\partial}{\partial y}\left(\frac{\partial M_{x}}{\partial x}+\frac{\partial M_{y}}{\partial y}+\frac{\partial M_{z}}{\partial z}\right) \hat{\mathbf{j}} \\
& +\frac{\partial}{\partial z}\left(\frac{\partial M_{x}}{\partial x}+\frac{\partial M_{y}}{\partial y}+\frac{\partial M_{z}}{\partial z}\right) \hat{\mathbf{k}},
\end{aligned}
$$

which leads to

$$
\begin{gathered}
\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{M})=\left(\frac{\partial^{2} M_{x}}{\partial x^{2}}+\frac{\partial^{2} M_{y}}{\partial x \partial y}+\frac{\partial^{2} M_{z}}{\partial x \partial z}\right) \hat{\mathbf{i}}+\left(\frac{\partial^{2} M_{x}}{\partial y \partial x}+\frac{\partial^{2} M_{y}}{\partial y^{2}}+\frac{\partial^{2} M_{z}}{\partial y \partial z}\right) \hat{\mathbf{j}} \\
+\left(\frac{\partial^{2} M_{x}}{\partial z \partial x}+\frac{\partial^{2} M_{y}}{\partial z \partial y}+\frac{\partial^{2} M_{z}}{\partial z^{2}}\right) \hat{\mathbf{k}}
\end{gathered}
$$

Taking all the above into account, we can express the curl of the curl of a vector field as

$$
\begin{aligned}
\boldsymbol{\nabla} \times(\boldsymbol{\nabla} \times \mathbf{M}) & =\left[\left(\frac{\partial^{2} M_{y}}{\partial x \partial y}+\frac{\partial^{2} M_{z}}{\partial x \partial z}+\frac{\partial^{2} M_{x}}{\partial x^{2}}\right)-\left(\frac{\partial^{2} M_{x}}{\partial x^{2}}+\frac{\partial^{2} M_{x}}{\partial y^{2}}+\frac{\partial^{2} M_{x}}{\partial z^{2}}\right)\right] \hat{\mathbf{i}} \\
& +\left[\left(\frac{\partial^{2} M_{x}}{\partial y \partial x}+\frac{\partial^{2} M_{z}}{\partial y \partial z}+\frac{\partial^{2} M_{y}}{\partial y^{2}}\right)-\left(\frac{\partial^{2} M_{y}}{\partial x^{2}}+\frac{\partial^{2} M_{y}}{\partial y^{2}}+\frac{\partial^{2} M_{y}}{\partial z^{2}}\right)\right] \hat{\mathbf{j}} \\
& +\left[\left(\frac{\partial^{2} M_{x}}{\partial z \partial x}+\frac{\partial^{2} M_{y}}{\partial z \partial y}+\frac{\partial^{2} M_{z}}{\partial z^{2}}\right)-\left(\frac{\partial^{2} M_{z}}{\partial x^{2}}+\frac{\partial^{2} M_{z}}{\partial y^{2}}+\frac{\partial^{2} M_{z}}{\partial z^{2}}\right)\right] \hat{\mathbf{k}}
\end{aligned}
$$

or equivalently

$$
\begin{equation*}
\boldsymbol{\nabla} \times(\boldsymbol{\nabla} \times \mathbf{M})=\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{M})-\nabla^{2} \mathbf{M} \tag{2.4}
\end{equation*}
$$

In our case $\mathbf{M}$ represents the electric field $\mathbf{E}$, and thus we can utilize Eqs. (2.3), (2.4) in order to obtain the wave equation in the following form

$$
\begin{align*}
& \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{E}=-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{E}}{\partial t^{2}} \\
& \Rightarrow \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E})-\nabla^{2} \mathbf{E}=-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{E}}{\partial t^{2}} \\
& \Rightarrow\left(\nabla^{2}-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}\right) \mathbf{E}=0, \tag{2.5}
\end{align*}
$$

where for the last step we have assumed zero charge density thus, $\rho=0$ and $(\boldsymbol{\nabla} \cdot \mathbf{E})=0$. To conclude with, we would like to note that we could also derive a wave equation for the magnetic field using the same approach. The corresponding equation reads $\left(\nabla^{2}-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}\right) \mathbf{B}=0$.

### 2.2 Helmholtz equation

The Helmholtz equation is a time-independent linear partial differential equation which usually arises when studying wave propagation phenomena or other physical problems for example in quantum mechanics, etc. In order to derive it, we begin from the wave equation Eq. (2.5)

$$
\begin{equation*}
\left(\nabla^{2}-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}\right) u(\vec{r}, t)=0 \tag{2.6}
\end{equation*}
$$

More specifically we assume that Eq. (2.6) admits solutions of the form $u(\vec{r}, t)=$ $U(\vec{r}) e^{-i \omega t}$, where $\omega^{2} / c^{2}=k_{0}^{2}$ and $k_{0}=2 \pi / \lambda$. Calculating the second derivative with respect to $t$ and substituting back to Eq. (2.6), we obtain the Helmholtz equation in the following form

$$
\begin{equation*}
\left(\nabla^{2}+k_{0}^{2}\right) U(\vec{r})=0 \tag{2.7}
\end{equation*}
$$

Interestingly enough, we can show that the wave equation admits plane wave solutions of the form $u(\mathbf{r}, t)=e^{i(\vec{k} \vec{r}-\omega t)}$, where the wavevector $\vec{k}$ is defined as $\vec{k}=k_{x} \hat{\mathbf{x}}+k_{y} \hat{\mathbf{y}}+k_{z} \hat{\mathbf{z}}$, and $\vec{k}^{2}=k_{x}^{2}+k_{y}^{2}+k_{z}^{2}=k_{0}^{2}$.

### 2.3 The Green's function of the 2D Helmholtz equation

As we will see next, the Green's function is a very useful tool that we can exploit in order to solve various problems in the field of optics, including emission, scattering and diffraction of waves. Here, we are mainly interested in the diffraction of scalar waves governed by the homogeneous Helmholtz equation

$$
\begin{equation*}
\nabla^{2} \psi+k_{0}^{2} \psi=0 \tag{2.8}
\end{equation*}
$$

where $k_{0}=\omega / c$. The Green's function is defined as usual by

$$
\begin{equation*}
\nabla^{2} G+k_{0}^{2} G=-\delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right) \tag{2.9}
\end{equation*}
$$

where $\delta(\mathbf{r})$ is the Dirac's delta function. Before we proceed with the calculation of the function $G$, we will begin our analysis with some mathematical preliminaries. Firstly, the Fourier- and inverse Fourier- transform couple in two spatial dimensions is defined as

$$
\begin{gathered}
\mathcal{F}\{f(\mathbf{r})\}=\hat{f}(\mathbf{k})=\frac{1}{2 \pi} \int_{\mathbb{R}^{2}} f(\mathbf{r}) e^{-i \mathbf{k r}} d^{2} r \\
\mathcal{F}^{-1}\{\hat{f}(\mathbf{k})\}=f(\mathbf{r})=\frac{1}{2 \pi} \int_{\mathbb{R}^{2}} \hat{f}(\mathbf{k}) e^{i \mathbf{k r}} d^{2} k .
\end{gathered}
$$

Some useful relation occur when considering the Fourier transform of the derivative of a function. More specifically borrowing some useful identities from the one dimensional case we obtain

$$
\begin{gathered}
\mathcal{F}\left\{f^{\prime}(x)\right\}=i k \hat{f}(k) \\
\mathcal{F}\left\{f^{\prime \prime}(x)\right\}=-k^{2} \hat{f}(k) \\
\mathcal{F}\left\{\delta\left(x-x^{\prime}\right)\right\}=\frac{1}{2 \pi} \int_{\mathbb{R}} \delta\left(x-x^{\prime}\right) e^{-i k x} d x=\frac{1}{2 \pi} e^{-i k x^{\prime}} .
\end{gathered}
$$

Taking all the above into consideration, we can express the Fourier transfor of Eq. (2.8) in the following form

$$
-\mathbf{k}^{2} \hat{G}(\mathbf{k})+k_{0}^{2} \hat{G}(\mathbf{k})=-\frac{1}{2 \pi} e^{-i \mathbf{k} \mathbf{r}^{\prime}} \Rightarrow \hat{G}(\mathbf{k})=\frac{1}{2 \pi} \frac{e^{-i \mathbf{k \mathbf { r } ^ { \prime }}}}{\mathbf{k}^{2}-k_{0}^{2}}
$$

and by taking the inverse Fourier transform of $\hat{G}(\mathbf{k})$ we obtain

$$
\begin{equation*}
G(\boldsymbol{\rho})=\frac{1}{(2 \pi)^{2}} \int_{\mathbb{R}^{2}} \frac{e^{i \mathbf{k} \cdot \boldsymbol{\rho}}}{\mathbf{k}^{2}-k_{0}^{2}} d^{2} \mathbf{k}, \tag{2.10}
\end{equation*}
$$

where $\rho=\mathbf{r}-\mathbf{r}^{\prime}$. The trick in order to calculate the above integral is to assume that the $k_{x}$ component of the wavevector $\mathbf{k}$ defined in Cartesian coordinates as $\mathbf{k}=k_{x} e_{x}+k_{y} e_{y}$, extends in the direction of $\rho$, and thus Eq. (2.10) becomes,

$$
\begin{equation*}
G(\boldsymbol{\rho})=\frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} e^{i k_{x} \rho} d k_{x} \int_{-\infty}^{\infty} \frac{d k_{y}}{k_{y}^{2}+k_{x}^{2}-k_{0}^{2}} \tag{2.11}
\end{equation*}
$$

Now we can calculate the integral over $k_{y}$ by applying the method of contour integration. In this respect let us select the upper half-plane as our closed path of integration, where the contour is closed by a semicircle of infinite radius ( $R \rightarrow \infty$ ), [see Fig. (2.1)].


FIGURE 2.1: Integration contour for $\left|k_{x}\right|>k_{0}$, (taken from [16]).
Furthermore, in order to properly calculate the above integral we need to determine the sign of the difference of the "constant" terms $k_{x}, k_{0}$ of the denominator $k_{y}^{2}+k_{x}^{2}-k_{0}^{2}$. In this respect we consider two separate cases: $\left|k_{x}\right|>k_{0}$ and $\left|k_{x}\right|<k_{0}$. For $\left|k_{x}\right|>k_{0}$, we can express the denominator of the integral as $k_{y}^{2}+$ $\left(k_{x}^{2}-k_{0}^{2}\right)=\left(k_{y}+i \sqrt{k_{x}^{2}-k_{0}^{2}}\right)\left(k_{y}-i \sqrt{k_{x}^{2}-k_{0}^{2}}\right)$, which leads to two poles namely $k_{y}= \pm i \sqrt{k_{x}^{2}-k_{0}^{2}}$. However, only the "plus" signed pole lies in the selected contour of integration, and therefore the main contribution to the integral will be given from the residue of the pole i.e.,
$\int_{-\infty}^{\infty} \frac{d k_{y}}{k_{y}^{2}+k_{x}^{2}-k_{0}^{2}}=2 \pi i \lim _{k_{y} \rightarrow i \sqrt{k_{x}^{2}-k_{0}^{2}}} \frac{\left(k_{y}-i \sqrt{k_{x}^{2}-k_{0}^{2}}\right)}{\left(k_{y}+i \sqrt{k_{x}^{2}-k_{0}^{2}}\right)\left(k_{y}-i \sqrt{k_{x}^{2}-k_{0}^{2}}\right)}=\frac{\pi}{\sqrt{k_{x}^{2}-k_{0}^{2}}}$

Before continuing with the next case, we believe it is instructive to analyze the above procedure more. Genuinely, using the contour selected in Fig. (2.1), the integral over $k_{y}$ would be

$$
\begin{equation*}
\oint \frac{d k_{y}}{k_{y}^{2}+k_{x}^{2}-k_{0}^{2}}=\lim _{R \rightarrow \infty} \int_{-R}^{R} \frac{d k_{y}}{k_{y}^{2}+k_{x}^{2}-k_{0}^{2}}+\lim _{R \rightarrow \infty} \int_{C} \frac{d z}{z^{2}+k_{x}^{2}-k_{0}^{2}}, \tag{2.12}
\end{equation*}
$$

where for the second integral we have made the substitution to polar coordinates $k_{y}=z=R e^{i \theta}$, and the integration is over the semicircle with infinitely large radius, namely $C$. Earlier we argued that the main contribution to the integral stems from the residue at the pole [i.e., the first integral of Eq. (2.12)], implying that the second integral vanishes. This can be shown explicitly as

$$
\lim _{R \rightarrow \infty} \int_{0}^{\pi} \frac{i R e^{i \theta}}{\left(R e^{i \theta}\right)^{2}+k_{x}^{2}-k_{0}^{2}} d \theta \approx \lim _{R \rightarrow \infty}\left(\frac{1}{R}\right) \int_{0}^{\pi} \frac{i e^{i \theta}}{\left(e^{i \theta}\right)^{2}+k_{x}^{2}-k_{0}^{2}} d \theta=0
$$

Considering now the next case, namely $\left|k_{x}\right|<k_{0}$, we can express the denominator in the form $k_{y}^{2}-\left(k_{0}^{2}-k_{x}^{2}\right)=\left(k_{y}+\sqrt{k_{0}^{2}-k_{x}^{2}}\right)\left(k_{y}-\sqrt{k_{0}^{2}-k_{x}^{2}}\right)$. Following the same procedure as before we deduce that there exist two real poles, $k_{y}= \pm \sqrt{k_{0}^{2}-k_{x}^{2}}$. In this case, there are 4 possible configurations for the selected contour of integration. Figure (2.2) illustrates the different available choices of the closed paths. In Fig. [2.2(a)] we pass the contour over the negative pole, while the inverse choice is made for the positive pole. Thus, the contribution of the negative pole is zero as it is excluded from the contour of integration, while for the positive pole we obtain utilizing the residue theorem that

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d k_{y}}{k_{y}^{2}+k_{x}^{2}-k_{0}^{2}}=2 \pi i \lim _{k_{y} \rightarrow \sqrt{k_{0}^{2}-k_{x}^{2}}} \frac{\left(k_{y}-\sqrt{k_{0}^{2}-k_{x}^{2}}\right)}{\left(k_{y}+\sqrt{k_{0}^{2}-k_{x}^{2}}\right)\left(k_{y}-\sqrt{k_{0}^{2}-k_{x}^{2}}\right)}=\frac{\pi i}{\sqrt{k_{0}^{2}-k_{x}^{2}}} \tag{2.13}
\end{equation*}
$$

For the second contour configuration shown in Fig. [2.2(b)], we have made the inverse selection, thus including the negative pole and excluding the positive pole. The contribution to the integral is

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d k_{y}}{k_{y}^{2}+k_{x}^{2}-k_{0}^{2}}=2 \pi i \lim _{k_{y} \rightarrow-\sqrt{k_{0}^{2}-k_{x}^{2}}} \frac{\left(k_{y}+\sqrt{k_{0}^{2}-k_{x}^{2}}\right)}{\left(k_{y}+\sqrt{k_{0}^{2}-k_{x}^{2}}\right)\left(k_{y}-\sqrt{k_{0}^{2}-k_{x}^{2}}\right)}=-\frac{\pi i}{\sqrt{k_{0}^{2}-k_{x}^{2}}} \tag{2.14}
\end{equation*}
$$



FIGURE 2.2: All the available integration contours in the case where $\left|k_{x}\right|<k_{0}$, (taken from [16]).

To calculate the contribution to our integral from the contour configuration shown in Fig. [2.2(c)], we observe that both poles were excluded from the closed path, and thus inevitably from the Cauchy's Principal Value Theorem the total contribution is zero. When both poles are included, we observe from Eqs. (2.13), (2.14) that the residues are equal but with opposite signs, thus the net contribution is zero again.

We now return to Eq. (2.11) to implement the integration with respect to $k_{x}$. Due to the two different cases that occurred, namely $\left|k_{x}\right|>k_{0}$, and $\left|k_{x}\right|<k_{0}$, we need to separate the integration intervals with respect to them, while checking the convergence at $k_{x}=k_{0}$. The latter is trivial because the integration over $k_{y}$ yields $\int_{\mathbb{R}} d k_{y} / k_{y}^{2}=\left.\left[-1 / k_{y}\right]\right|_{-\infty} ^{\infty}=0$, which makes no contribution. Thus Eq. (2.11) can be expressed as

$$
G(\boldsymbol{\rho})=\frac{1}{(2 \pi)^{2}}\left[\int_{\left|k_{x}\right|>k_{0}} \frac{\pi e^{i k_{x} \rho}}{\sqrt{k_{x}^{2}-k_{0}^{2}}} d k_{x} \pm \int_{\left|k_{x}\right|<k_{0}} \frac{i \pi e^{i k_{x} \rho}}{\sqrt{k_{0}^{2}-k_{x}^{2}}} d k_{x}\right],
$$

or in their expanded form

$$
\begin{align*}
G(\rho)=\frac{1}{(2 \pi)^{2}}\left[\int_{-\infty}^{-k_{0}} \frac{\pi e^{i k_{x} \rho}}{\sqrt{k_{x}^{2}-k_{0}^{2}}} d k_{x}+\int_{k_{0}}^{\infty} \frac{\pi e^{i k_{x} \rho}}{\sqrt{k_{x}^{2}-k_{0}^{2}}} d k_{x}\right. \\
\left.\quad \pm\left(\int_{-k_{0}}^{0} \frac{i \pi e^{i k_{x} \rho}}{\sqrt{k_{0}^{2}-k_{x}^{2}}} d k_{x}+\int_{0}^{k_{0}} \frac{i \pi e^{i k_{x} \rho}}{\sqrt{k_{0}^{2}-k_{x}^{2}}} d k_{x}\right)\right] \tag{2.15}
\end{align*}
$$

or more conveniently as

$$
G(\boldsymbol{\rho})=\frac{I_{1}+I_{2} \pm\left(I_{3}+I_{4}\right)}{(2 \pi)^{2}}
$$

For clarity we choose to study each integral separately. We begin from the the leftmost,

$$
\begin{aligned}
& I_{1}=\int_{-\infty}^{-k_{0}} \frac{\pi e^{i k_{x} \rho}}{\sqrt{k_{x}^{2}-k_{0}^{2}}} d k_{x}=\int_{\infty}^{k_{0}} \frac{\pi e^{-i k_{x} \rho}}{\sqrt{k_{x}^{2}-k_{0}^{2}}} d\left(-k_{x}\right) \\
&=\int_{k_{0}}^{\infty} \frac{\pi}{\sqrt{k_{x}^{2}-k_{0}^{2}}}\left[\cos \left(k_{x} \rho\right)-i \sin \left(k_{x} \rho\right)\right] d k_{x}
\end{aligned}
$$

Following a similar approach we can express $I_{2}$ in the following form

$$
I_{2}=\int_{k_{0}}^{\infty} \frac{\pi}{\sqrt{k_{x}^{2}-k_{0}^{2}}}\left[\cos \left(k_{x} \rho\right)+i \sin \left(k_{x} \rho\right)\right] d k_{x} .
$$

For the other couple of integrals we proceed as follows,

$$
I_{3}=\int_{0}^{k_{0}} \frac{i \pi}{\sqrt{k_{0}^{2}-k_{x}^{2}}}\left[\cos \left(k_{x} \rho\right)-i \sin \left(k_{x} \rho\right)\right] d k_{x}
$$

and

$$
I_{4}=\int_{0}^{k_{0}} \frac{i \pi}{\sqrt{k_{0}^{2}-k_{x}^{2}}}\left[\cos \left(k_{x} \rho\right)+i \sin \left(k_{x} \rho\right)\right] d k_{x} .
$$

Interestingly enough "sin" terms vanish upon summation, leading to a much simpler expression for $G$,

$$
\begin{equation*}
G(\boldsymbol{\rho})=\frac{I_{1}+I_{2} \pm\left(I_{3}+I_{4}\right)}{(2 \pi)^{2}}=\frac{1}{(2 \pi)^{2}}\left[2 \pi \int_{k_{0}}^{\infty} \frac{\cos \left(k_{x} \rho\right)}{\sqrt{k_{x}^{2}-k_{0}^{2}}} d k_{x} \pm 2 \pi i \int_{0}^{k_{0}} \frac{\cos \left(k_{x} \rho\right)}{\sqrt{k_{0}^{2}-k_{x}^{2}}} d k_{x}\right] . \tag{2.16}
\end{equation*}
$$

Assuming a change of variables $k_{x}=k_{0} u$, the first integral leads to a known integral representation of the Neumann function of zeroth order [17],

$$
\begin{equation*}
\int_{k_{0}}^{\infty} \frac{\cos \left(k_{x} \rho\right)}{\sqrt{k_{0}^{2}-k_{x}^{2}}} d k_{x}=\int_{1}^{\infty} \frac{\cos \left(k_{x} \rho\right)}{k_{0} \sqrt{u^{2}-1}} k_{0} d u=-\frac{\pi}{2} \Upsilon_{0}\left(k_{0} \rho\right) \tag{2.17}
\end{equation*}
$$

For the second integral we assume the change of variable, $k_{x}=k_{0} \cos \theta$, which also leads to a well-known integral representation of the Bessel function of zeroth order,

$$
\begin{aligned}
& \int_{0}^{k_{0}} \frac{\cos \left(k_{x} \rho\right)}{\sqrt{k_{0}^{2}-k_{x}^{2}}}=\int_{\pi / 2}^{0}-\frac{\cos \left(k_{0} \rho \cos \theta\right)}{k_{0} \sqrt{1-\cos ^{2} \theta}} k_{0} \sin \theta d \theta \\
&=\int_{0}^{\pi / 2} \cos \left(k_{0} \rho \cos \theta\right) d \theta=\frac{\pi}{2} J_{0}\left(k_{0} \rho\right) .
\end{aligned}
$$

Finally the Green's function of the Helmholtz equation can be expressed as

$$
G(\boldsymbol{\rho})=\frac{1}{2 \pi}\left[-\frac{\pi}{2} Y_{0}\left(k_{0} \boldsymbol{\rho}\right)+i \frac{\pi}{2} \mathrm{~J}_{0}\left(k_{0} \boldsymbol{\rho}\right)\right]=\frac{i}{4}\left[\mathrm{~J}_{0}\left(k_{0} \boldsymbol{\rho}\right)+i Y_{0}\left(k_{0} \boldsymbol{\rho}\right)\right],
$$

or equivalently

$$
\begin{equation*}
G(\boldsymbol{\rho})=\frac{i}{4} H_{0}^{(1)}\left(k_{0} \boldsymbol{\rho}\right) . \tag{2.18}
\end{equation*}
$$

### 2.4 The Rayleigh-Sommerfeld formulation of diffraction

In this section we will develop the foundations of scalar diffraction for 2D optical waves. While the 3D interpretation of the Kirchhoff and Rayleigh-Sommerfeld integrals has been extensively studied in the literature, here we emphasize on the two-dimensional problem which is of our particular interest.

Let us begin with the Green's Theorem, which is assumed to be the bedrock of the wave theory of diffraction. This theorem can be stated as follows:

Let $U(P)$ and $G(P)$ be any two complex-valued functions of position, and let $\Gamma$ be a closed surface surrounding a $2 D$ domain $\Omega$. If $U, G$, and their first and second partial derivatives are single-valued and continuous within and on $\Omega$

$$
\iint_{\Omega}\left(U \nabla^{2} G-G \nabla^{2} U\right) d s=\int_{\Gamma^{\prime}}\left(U \frac{\partial G}{\partial n}-G \frac{\partial U}{\partial n}\right) d l
$$

where $\frac{\partial}{\partial n}$ signifies a partial derivative in the outward normal direction at each point on $\Gamma^{\prime}$.


FIGURE 2.3: Illustration of the integration domain, (taken from [18]).

Heading now to the problem under consideration, we would like to calculate the field at any point $P_{0}$ in $\Omega$, with respect to the optical disturbance on the boundary of $\Omega$ (i.e., $\Gamma^{\prime}$ ). A schematic representation is shown in Fig. (2.3). Assuming that $U$ is a scalar function which satisfies the homogeneous Helmholtz equation Eq. (2.8), and G is the corresponding Green's function, we obtain for the first integral of the Green's Theorem that

$$
\iint_{\Omega}\left(U \nabla^{2} G-G \nabla^{2} U\right) d s=\iint_{\Omega}\left[U\left(-k_{0}^{2} G\right)-G\left(-k_{0}^{2} U\right)\right] d s=0
$$

Thus, the Theorem reduces to

$$
0=\int_{\Gamma^{\prime}}\left(U \frac{\partial G}{\partial n}-G \frac{\partial U}{\partial n}\right) d l,
$$

where $\Gamma^{\prime}$ is consisted of two curves, namely $\Gamma$ and $\Gamma_{\epsilon}$. The latter is a circle centered at $P_{0}$ with radius $\epsilon$, while the former is a smooth but otherwise arbitrary curve. As a result $\Gamma^{\prime}$ can be expressed as $\Gamma^{\prime}=\Gamma+\Gamma_{\epsilon}$, leading to

$$
\begin{equation*}
-\int_{\Gamma_{e}}\left(U \frac{\partial G}{\partial n}-G \frac{\partial U}{\partial n}\right) d l=\int_{\Gamma}\left(U \frac{\partial G}{\partial n}-G \frac{\partial U}{\partial n}\right) d l . \tag{2.19}
\end{equation*}
$$

At this point we should highlight that the normal outward vector $\overrightarrow{\mathbf{n}}$ points conventionally outwards in $\Gamma$, but "inwards" and towards to $P_{0}$ on $\Gamma_{\epsilon}$. In order to proceed with the calculation of the integrals, we first recall the Green's function $G$ of the 2D Helmholtz equation obtained in the previous section, at a point $P_{1}$ in $\Gamma^{\prime}$,

$$
G\left(P_{1}\right)=(-i / 4) H_{0}^{(1)}\left(k_{0} r_{01}\right)
$$

and

$$
\frac{\partial G\left(P_{1}\right)}{\partial n}=(-i / 4) k_{0} \frac{\partial H_{0}^{(1)}\left(k_{0} r_{01}\right)}{\partial r_{01}} \cos \left(\vec{n}, \overrightarrow{r_{01}}\right),
$$

where $H_{0}^{(1)}$ is a Hankel function of the first kind of zeroth order, $\cos \left(\vec{n}, \overrightarrow{0_{01}}\right)$ represents the cosine of the angle between the outward normal vector $\overrightarrow{\mathbf{n}}$ and the vector $\overrightarrow{r_{01}}$ joining $P_{0}$ to $P_{1}$. Furthermore when considering the integration over the inner curve $\Gamma_{\epsilon}, P_{1}$ lies on $\Gamma_{\epsilon}$ and thus $\cos \left(\vec{n}, \overrightarrow{r_{01}}\right)=-1$, and the above equations become

$$
G\left(P_{1}\right)=(-i / 4) H_{0}^{(1)}\left(k_{0} \epsilon\right) \quad \text { and } \quad \frac{\partial G\left(P_{1}\right)}{\partial n}=-\left.(-i / 4) k_{0} \frac{\partial H_{0}^{(1)}\left(k_{0} r_{01}\right)}{\partial r_{01}}\right|_{r_{01}=\epsilon}
$$

Taking the limit of arbitrary small $\epsilon$, the left-hand side of Eq. (2.19) reads

$$
\begin{aligned}
\lim _{\epsilon \rightarrow 0}[- & \left.\int_{\Gamma_{\epsilon}} U \frac{\partial G}{\partial n}-G \frac{\partial U}{\partial n} d l\right]=\lim _{\epsilon \rightarrow 0}\left[-\int\left(U \frac{\partial G}{\partial n}-G \frac{\partial U}{\partial n}\right) \epsilon d \theta\right] \\
& =\lim _{\epsilon \rightarrow 0}\left[-2 \pi \epsilon(-i / 4)\left(-\left.k_{0} U\left(P_{0}\right) \frac{\partial H_{0}^{(1)}\left(k_{0} r_{01}\right)}{\partial r_{01}}\right|_{r_{01}=\epsilon}-H_{0}^{(1)}\left(k_{0} \epsilon\right) \frac{\partial U}{\partial n}\right)\right],
\end{aligned}
$$

where we have used that $d \theta=d l / \epsilon, d \theta$ is the infinitesimal angle element and the integration is over the entire angular space. In order to simplify the above results, we make the following approximations [17]

$$
\begin{gathered}
H_{0}^{(1)}\left(k_{0} \epsilon\right)=1+i \frac{2}{\pi} \log \left(\frac{k_{0} \epsilon}{2}\right), \\
\left.\frac{\partial H_{0}^{(1)}\left(k_{0} r_{01}\right)}{\partial r_{01}}\right|_{r_{01}=\epsilon}=-\frac{k_{0} \epsilon}{4}+i \frac{2}{\pi} \frac{1}{k_{0} \epsilon}
\end{gathered}
$$

which lead to

$$
\lim _{\epsilon \rightarrow 0}\left[-\int_{\Gamma_{\epsilon}} U \frac{\partial G}{\partial n}-G \frac{\partial U}{\partial n} d l\right]=-2 \pi \epsilon(-i / 4)\left[-k_{0} U\left(P_{0}\right)\left(i \frac{2}{\pi} \frac{1}{k \epsilon}\right)\right]=U\left(P_{0}\right) .
$$

Finally, Eq. (2.19) can be expressed as

$$
\begin{equation*}
U\left(P_{0}\right)=\int_{\Gamma}\left(U \frac{\partial G}{\partial n}-G \frac{\partial U}{\partial n}\right) d l . \tag{2.20}
\end{equation*}
$$

Taking the above equation as our starting point, we now consider the problem of diffraction of light by an infinite opaque screen with an open aperture $\Sigma$ [see Fig. (2.4)]. In this case too the closed surface $\Gamma$ is consisted of two curves, namely $S_{1}$ and $S_{2}$. Here, $S_{1}$ is lying directly behind the diffraction element, while $S_{2}$ is a circle of infinite radius $R \rightarrow \infty$ around the observation point $P_{0}$, enclosing the domain. Utilizing Eq. (2.20) we obtain

$$
\begin{equation*}
U\left(P_{0}\right)=\int_{S_{1}}\left(U \frac{\partial G}{\partial n}-G \frac{\partial U}{\partial n}\right) d l+\int_{S_{2}}\left(U \frac{\partial G}{\partial n}-G \frac{\partial U}{\partial n}\right) d l \tag{2.21}
\end{equation*}
$$



FIGURE 2.4: Schematic representation of diffraction by a plane screen [19].
We begin our analysis by calculating the second integral (i.e., over $S_{2}$ ). In this respect we make use of the following approximations, valid for $R \gg 1$,

$$
\begin{gathered}
G(R)=(-i / 4) H_{0}^{(1)}\left(k_{0} R\right) \approx(-i / 4) \sqrt{\frac{2}{\pi k_{0} R}} \exp \left[i\left(k_{0} R-\pi / 4\right)\right], \\
\frac{\partial G}{\partial n}=\frac{\partial G}{\partial R} \approx(-i / 4) \sqrt{\frac{2}{\pi k_{0} R}} i k \exp \left[i\left(k_{0} R-\pi / 4\right)\right] .
\end{gathered}
$$

Based upon the above approximations we can express the integral over $S_{2}$ in the following form

$$
\begin{gathered}
\int_{S_{2}} U \frac{\partial G}{\partial n}-G \frac{\partial U}{\partial n} d l=\int_{S_{2}}(-i / 4) \sqrt{\frac{2}{\pi k_{0} R}} \exp \left[i\left(k_{0} R-\pi / 4\right)\right]\left(i k_{0} U-\frac{\partial U}{\partial n}\right) d l \\
=\int_{S_{2}}(-i / 4) \sqrt{\frac{2}{\pi k_{0}}} \exp \left[i\left(k_{0} R-\pi / 4\right)\right] \sqrt{R}\left(i k_{0} U-\frac{\partial U}{\partial n}\right) d \theta,
\end{gathered}
$$

where as before, $d \theta=d l / R, d \theta$ is the infinitesimal angle element and the integration is over the entire angular space. Taking all the above into consideration, we end up with the well-known Sommerfeld outgoing radiation condition in two dimensions, or

$$
\begin{equation*}
\lim _{R \rightarrow \infty}\left[\sqrt{R}\left(i k_{0} U-\frac{\partial U}{\partial n}\right)\right]=0 \tag{2.22}
\end{equation*}
$$

Thus the integral over $S_{2}$ vanishes for infinitely large $R$, and the only contribution to Eq. (2.21) stems from the integral over $S_{1}$,

$$
\begin{equation*}
U\left(P_{0}\right)=\int_{S_{1}}\left(U \frac{\partial G}{\partial n}-G \frac{\partial U}{\partial n}\right) d l \tag{2.23}
\end{equation*}
$$

At this point the contribution of Rayleigh and Sommerfeld to the theory of scalar diffraction becomes evident. Sommerfeld proposed that either $G$ or $\frac{\partial G}{\partial n}$ can be vanished over $S_{1}$. This means that either $U$ or $\frac{\partial U}{\partial n}$ will be eliminated respectively, thus obviating the need for imposing boundary conditions on both $U$ and $\frac{\partial U}{\partial n}$ at the same time. Nevertheless, the particular choice of the function $G$ must satisfy the homogeneous Helmholtz equation and fulfill Sommerfeld's radiation condition. A valid choice of the function $G$ as Sommerfeld suggested will be the following: Suppose $G$ is generated not only by a point source located at $P_{0}$, but also simultaneously by a second point source at a position $\tilde{P}_{0}$ which is the mirror image of $P_{0}$ on the opposite side of the diffracting element [see Fig. 2.5)]. This assumption is valid because the mirror source lies outside the contour of integration under consideration, and thus does not add to the value of the integral. The particular choice of the Green's function reads

$$
\begin{equation*}
G_{-}\left(P_{1}\right)=(-i / 4)\left[H_{0}^{(1)}\left(k_{0} r_{01}\right)-H_{0}^{(1)}\left(k_{0} r_{01}\right)\right], \tag{2.24}
\end{equation*}
$$



Figure 2.5: The Rayleigh-Sommerfeld representation of diffraction by a plane screen , (taken from[19]).
which vanishes in $S_{1}$ (assuming an infinitesimally thin opaque screen). As a result, it is sufficient to assume that only $U$ is vanishing in $S_{1} \backslash \Sigma$ leading to

$$
\begin{equation*}
U\left(P_{0}\right)=\int_{\Sigma} U \frac{\partial G_{-}}{\partial n} d l \tag{2.25}
\end{equation*}
$$

which is known as the first Rayleigh-Sommerfeld solution. Interestingly, we can further simplify the above formula for the particular choice of the function $G$. To accomplish this we note that,

$$
\begin{gathered}
\frac{\partial G_{-}}{\partial n}=\left(-\frac{i}{4}\right) \frac{\partial}{\partial r_{01}}\left[H_{0}^{(1)}\left(k_{0} r_{01}\right)\right] \cos \left(\overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{r}_{01}}\right)+\left(\frac{i}{4}\right) \frac{\partial}{\partial \tilde{r_{01}}}\left[H_{0}^{(1)}\left(k_{0} \tilde{0}\right)\right] \cos \left(\overrightarrow{\mathbf{n}}, \overrightarrow{r_{01}}\right) \\
=2\left(-\frac{i}{4}\right) k_{0} \frac{\partial}{\partial r_{01}}\left[H_{0}^{(1)}\left(k_{0} r_{01}\right)\right] \cos \left(\overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{r}_{01}}\right)=2 \frac{\partial G}{\partial n},
\end{gathered}
$$

where we have utilized that on $\Sigma \cos \left(\overrightarrow{\mathbf{n}}, r_{\tilde{01}}\right)=-\cos \left(\overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{r}_{\mathbf{0 1}}}\right)$ and $r \tilde{01}=r_{01}$. Finally the field at a point $P_{0}$ behind a planar screen in the two-dimensional configuration is given by

$$
\begin{equation*}
U\left(P_{0}\right)=2 \int_{\Sigma} U \frac{\partial G}{\partial n} d l \tag{2.26}
\end{equation*}
$$

When considering a 3D configuration, instead of using a Green's function in the form of a Hankel function, we can assume spherical waves of unit amplitude or more specifically $G(R)=\frac{\exp \left(i k_{0} R\right)}{R}$. Following the same procedure as before, the equation
above can be expressed as

$$
\begin{equation*}
U\left(P_{0}\right)=-\frac{1}{2 \pi} \iint_{\Sigma} U \frac{\partial G}{\partial n} d \sigma, \tag{2.27}
\end{equation*}
$$

where now $\Sigma$ is the surface of integration, and $\sigma$ is the infinitesimal surface element.

### 2.5 The Fresnel diffraction integral

Seeking for solutions of the paraxial Helmholtz equation, the Fresnel diffraction integral provides us exact and in many cases closed form expressions for the propagation of paraxial optical waves. Despite that the Rayleigh-Sommerfeld formulation of diffraction consists a more robust tool in studying wave-propagation phenomena, the Fresnel diffraction integral is considered as a very good approximation in the case of rays propagating close and at small angles with respect to the optical axis. The physical description of diffraction of light reads: Considering the illumination of an aperture with coherent light (e.g., spherical waves). We would like to obtain an expression for the amplitude of the optical field in the diffraction pattern placed at a normal distance $z$ away from the aperture. In more details, as we see in Fig. 2.6, we assume that the illuminating aperture is placed in the $(\xi, \eta, z=0)$ plane, whereas the diffraction pattern is formed at a distance $z$ away from the diffracting aperture, in the $(x, y)$ plane, parallel to $(\xi, \eta)$. The distance $r_{d}$, between two points $P_{1}$ and $P_{0}$ of the illuminating plane and the observation plane can be explicitly calculated by

$$
\begin{equation*}
r_{d}=\sqrt{(x-\xi)^{2}+(y-\eta)^{2}+z^{2}} \tag{2.28}
\end{equation*}
$$

and thus the contained angle between $r_{d}$ and a normal vector pointing towards positive $z$, is given using simple trigonometric formulas by $\cos \theta=z / r_{d}$. Utilizing Eq. (2.27), we can calculate the optical field at the observation plane $\psi(x, y)$ for a given optical wave at the input plane $\psi(\xi, \eta)$ from

$$
\begin{equation*}
\psi(x, y, z)=\frac{z}{i \lambda} \iint_{-\infty}^{\infty} \psi(\xi, \eta, 0) \frac{\exp \left(i k_{0} r_{d}\right)}{r_{d}^{2}} d \xi d \eta \tag{2.29}
\end{equation*}
$$



Figure 2.6: The diffraction geometry, showing the illuminating aperture positioned at $(\xi, \eta, z=0)$, and the diffraction pattern constructed at $(x, y, z)$, (taken from[19]).

At this point we should highlight that our calculations are valid as long as the width of the aperture is larger than the wavelength of the illuminating beam, and that the distance between the initial and the observation plane is much larger than the wavelength, $r_{d} \gg \lambda$.

While the evaluation of Eq. (2.29) is complicated, Fresnel's approximation contributed in order to simplify the procedure significantly. His method is based on approximating the distance between $P_{0}$ and $P_{1}$ through the binomial expansion of Eq. (2.28). In order to avoid long formulas in the series expansion we assume

$$
\rho_{d}^{2}=(x-\xi)^{2}+(y-\eta)^{2},
$$

which leads to

$$
\begin{equation*}
r_{d}=\sqrt{\rho_{d}^{2}+z^{2}} \Leftrightarrow r_{d}=z \sqrt{1+\frac{\rho_{d}^{2}}{z^{2}}} . \tag{2.30}
\end{equation*}
$$

Then, calculating the binomial expansion of the square root while retaining terms up to first order we obtain

$$
\begin{equation*}
\sqrt{1+\chi}=1+\frac{1}{2} \chi \tag{2.31}
\end{equation*}
$$

which indicates that $r_{d}$ can be simply approximated by $r_{d} \approx z\left[1+\rho_{d}^{2} /\left(2 z^{2}\right)\right]$ or equivalently in terms of the variables of the reference frame as

$$
\begin{equation*}
r_{d} \approx z\left[1+\frac{1}{2}\left(\frac{x-\xi}{z}\right)^{2}+\frac{1}{2}\left(\frac{y-\eta}{z}\right)^{2}\right] \tag{2.32}
\end{equation*}
$$

Before we proceed with the direct substitution of the above equation to Eq. (2.29), we believe it is more instructive to first address some aspects of this procedure. In this respect, we would like to discuss whether the full form of Eq. (2.32) is necessary, or we can further simplify it in order to obtain a more simple and easy to use expression. We begin with the phase term or more specifically $e^{i k_{0} r_{d}}$, where rapidly growing errors can be generated easily. This is because the phase terms are exhibiting very fast oscillations in general, where small variations of their argument leads to significantly modified behavior of both their real and imaginary parts. Adding to this the fact that $k$ can be selected to be very large depending on the wavelength of the optical wave, we conclude that both terms of the binomial expansion are necessary for this particular case. On the other hand, $r_{d}$ is also appearing on the denominator of Eq. (2.29). However in that case, $1 / r_{d}^{2}$ is a much smoother function than the sinusoidal function $e^{i k_{0} r_{d}}$, and thus for brevity we can neglect the terms appearing in the square brackets of Eq. (2.32) at small cost (i.e., without generating a rapidly growing error). Finally, taking into account all the above, we can express the optical field at the observation plane by utilizing Eqs. (2.29), (2.32) as

$$
\begin{equation*}
\psi(x, y, z)=\frac{e^{i k_{0} z}}{i \lambda z} \iint_{-\infty}^{\infty} \psi(\xi, \eta) \exp \left\{\frac{i k_{0}}{2 z}\left[(x-\xi)^{2}+(y-\eta)^{2}\right]\right\} d \xi d \eta . \tag{2.33}
\end{equation*}
$$

These type of integrals are also appearing often when we study wave propagation problems with radial symmetry, such as the propagation of abruptly autofocusing beams. In this respect, we believe it is useful to obtain a similar expression for the propagation of radially symmetric beams utilizing the more convenient radial rather than Cartesian coordinates. To begin with, we consider the polar coordinates $r=$ $\sqrt{x^{2}+y^{2}}$ and $\theta=\arctan (y / x), \rho=\sqrt{\xi^{2}+\eta^{2}}$ and $\phi=\arctan (\eta / \xi)$, where $x=$ $r \cos \theta, \xi=\rho \cos \phi$ and $y=r \sin \theta, \eta=\rho \sin \phi$. When using Cartesian coordinates an infinitesimal area element can be calculated as $d A=d x d y$, whereas in the case
of polar coordinates the Jacobian determinant of the conversion formula has to be considered

$$
\operatorname{det} \frac{\partial(\xi, \eta)}{\partial(\rho, \phi)}=\left|\begin{array}{ll}
\frac{\partial \xi}{\partial \rho} & \frac{\partial \xi}{\partial \phi} \\
\frac{\partial \eta}{\partial \rho} & \frac{\partial \eta}{\partial \phi}
\end{array}\right|=\left|\begin{array}{cc}
\cos \phi & -\rho \sin \phi \\
\sin \phi & \rho \cos \phi
\end{array}\right|=\rho .
$$

Thus, an area element in polar coordinates can be expressed as $d A=\rho d \rho d \phi$. Furthermore from the phase terms involved in the Fresnel integrand we obtain

$$
\begin{aligned}
(x-\xi)^{2}+(y-\eta)^{2} & =r^{2}+\rho^{2}-2 r \rho \cos \theta \cos \phi-2 r \rho \sin \theta \sin \phi \\
& =r^{2}+\rho^{2}-2 r \rho \cos (\phi-\theta)
\end{aligned}
$$

Substituting the latter to Eq. (2.33) we obtain

$$
\begin{equation*}
\psi(r, \theta, z)=\frac{e^{i k_{0} z}}{i \lambda z} \int_{0}^{\infty} \psi(\rho) \exp \left[\frac{i k_{0}}{2 z}\left(r^{2}+\rho^{2}\right)\right] \rho d \rho \int_{0}^{2 \pi} \exp \left[-i \frac{k_{0} r \rho}{z} \cos (\phi-\theta)\right] d \phi \tag{2.34}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
\psi(r, \theta, z)=\frac{k_{0} e^{i k_{0} z}}{i z} \int_{0}^{\infty} \rho \psi(\rho) \mathrm{J}_{0}\left(\frac{k_{0} r \rho}{z}\right) \exp \left(i k_{0} \frac{r^{2}+\rho^{2}}{2 z}\right) d \rho \tag{2.35}
\end{equation*}
$$

where $k_{0}=2 \pi / \lambda$, and we have also utilized the integral representation of a zerothorder Bessel function

$$
\begin{equation*}
\mathrm{J}_{0}(x)=\frac{1}{2 \pi} \int_{0}^{2 \pi} \exp (-i x \cos s) d s \tag{2.36}
\end{equation*}
$$

Interestingly, we encounter similar type of integrals when studying the propagation optical vortices. More specifically, we consider the more general case of an optical vortex with topological charge $n$, and assume as our starting point Eq. (2.34). By decomposing the input wave excitation into amplitude and phase as $\psi(\rho, v)=$ $A(\rho) e^{i \phi(\rho)+i n v}$, where $v=\theta-s$, we can rewrite Eq. (2.34) as

$$
\begin{equation*}
\psi=\frac{e^{i k_{0} z+i n \theta}}{i \lambda z} \int_{0}^{\infty} \int_{0}^{2 \pi} \rho A(\rho) e^{i \phi-i n s+i k_{0} \frac{r^{2}+\rho^{2}-2 r p \operatorname{coss} s}{2 z}} d s d \rho \tag{2.37}
\end{equation*}
$$

Implementing first the integration over $s$, and utilizing the integral representation of a Bessel function of order $n$ in the following form

$$
\begin{equation*}
\mathrm{J}_{n}(x)=\frac{i^{-n}}{2 \pi} \int_{0}^{2 \pi} \exp (-i n s-i x \cos s) d s \tag{2.38}
\end{equation*}
$$

we can express the Fresnel-type integral of diffraction as

$$
\begin{equation*}
\psi(r, \theta, z)=\frac{k_{0} e^{i k_{0} z+i n \theta}}{i^{n+1} z} \int_{0}^{\infty} \rho A(\rho) \mathrm{J}_{n}\left(\frac{k r \rho}{z}\right) e^{i \phi+i k_{0} \frac{r^{2}+\rho^{2}}{2 z}} d \rho \tag{2.39}
\end{equation*}
$$

Interestingly, we note that in the special case of zero vorticity, $n=0$ and $\psi(\rho)=$ $A(\rho) e^{i \phi(\rho)}$, thus Eq. (2.35) is recovered.

### 2.6 Paraxial approximation

We consider the $(2+1)$ D Helmholtz equation which governs the propagation dynamics of any optical wave

$$
\begin{equation*}
\left(\nabla^{2}+k_{0}^{2}\right) \psi(x, y, z)=0 \tag{2.40}
\end{equation*}
$$

where $x, y$ and $z$ are the transverse and the longitudinal coordinates respectively, $k_{0}$ is the wavnumber and $\psi$ is the wavefunction. We choose $\psi$ to modulate accordingly to $\psi=u(x, y, z) e^{i k_{0} z}$. Substituting the latter to Eq. (2.40) we end up with

$$
\begin{equation*}
\frac{\partial^{2}}{\partial z^{2}}\left(u(x, y, z) e^{i k_{0} z}\right)+\left[\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}\right]\left(u(x, y, z) e^{i k_{0} z}\right)+k_{0}^{2} u(x, y, z) e^{i k_{0} z}=0 . \tag{2.41}
\end{equation*}
$$

Before we continue, for clarity let us calculate the involved derivatives with respect to $z$ separately

$$
\frac{\partial}{\partial z}\left[u(x, y, z) e^{i k_{0} z}\right]=\frac{\partial u(x, y, z)}{\partial z} e^{i k_{0} z}+i k_{0} u(x, y, z) e^{i k_{0} z}
$$

and

$$
\begin{align*}
\frac{\partial^{2}}{\partial z^{2}}\left[u(x, y, z) e^{i k_{0} z}\right] & =\frac{\partial}{\partial z}\left[\frac{\partial u(x, y, z)}{\partial z} e^{i k_{0} z}+i k_{0} u(x, y, z) e^{i k_{0} z}\right] \\
& =\frac{\partial^{2} u(x, y, z)}{\partial z^{2}} e^{i k_{0} z}+2 i k_{0} \frac{\partial u(x, y, z)}{\partial z} e^{i k_{0} z}-k_{0}^{2} u(x, y, z) e^{i k_{0} z} \tag{2.42}
\end{align*}
$$

Considering an important interpretation of geometrical optics, that in the paraxial approximation $u(x, y, z)$ varies so slowly with $z$, that is its second derivative is relatively small compared to the other terms or equivalently,

$$
\left|\frac{\partial^{2} u}{\partial z^{2}}\right| \ll\left|k \frac{\partial u}{\partial z}\right|,
$$

and thus Eq. (2.42) can be approximated by

$$
\frac{\partial^{2}}{\partial z^{2}}\left[u(x, y, z) e^{i k_{0} z}\right]=2 i k_{0} \frac{\partial u(x, y, z)}{\partial z} e^{i k_{0} z}-k_{0}^{2} u(x, y, z) e^{i k_{0} z}
$$

Taking into consideration all the above, we can now express the Helmholtz equation under the paraxial approximation in the following form

$$
\begin{equation*}
\nabla_{\perp}^{2} u+2 i k_{0} \frac{\partial u}{\partial z}=0 . \tag{2.43}
\end{equation*}
$$

### 2.7 Method of stationary phase

When considering a differential equation, the derivation of an exact solution in closed form is very rare. In this "unfortunate" case when the solution of the differential equation of interest cannot be expressed in terms of elementary functions, "fortunately" it is possible sometimes to obtain the intended solution as an integral representation. Such integral representations can be utilized to describe the physical phenomenon under observation in many interdisciplinary fields, such as in fluid mechanics, electromagnetic and acoustic waves, and many more. Here we emphasize in the field of optics, where as we have already seen in previous sections in the case of the Fresnel approximation and in the Rayleigh-Sommerfeld formulation of diffraction, integral representation of the following form

$$
\begin{equation*}
I(k)=\int_{-\infty}^{\infty} f(t) e^{i k \phi(t)} d t \tag{2.44}
\end{equation*}
$$

appear rather frequently in the mathematical modeling of the propagation of any optical wave. Furthermore, the asymptotic expansion of integral representations is proved to be a very useful tool not only as it simplifies the calculation of such integrals, but also because all of the special functions such as Bessel, Airy, Gamma, parabolic
cylindrical, and hypergeometric which frequently encountered as particular solutions of the wave equation, have integral representations. Interestingly, the asymptotic properties of these special functions are defined from their integral representations. Depending on the exact form of Eq. (2.44), various techniques have been developed to evaluate such integrals, for example integration by parts, Laplace's method, and the method of steepest descent. Here, we mainly focus on the method of stationary phase [20]. Before proceeding with the details of this method, we would like to note that Eq. (2.44) is just the Fourier transform of $f$ when $\phi(t)=t$.

Let us begin with the necessary assumptions, that $f$ is continuous, $\phi$ is twice differentiable and has a single stationary point in the interval $[a, b]$ at $t=c$, such that $\phi^{\prime}(c)=0$ and $\phi^{\prime \prime}(c) \neq 0$. The leading behavior of Eq. (2.44) can be calculated from

$$
I(k) \sim \int_{c-R}^{c+R} f(c) \exp \left\{i k\left[\phi(c)+\frac{(t-c)^{2}}{2} \phi^{\prime \prime}(c)\right]\right\} d t
$$

where $R$ is small and $k$ is very large. The procedure of calculating the above integral can be simplified significantly by assuming the change of variables

$$
\mu \tau^{2}=(t-c)^{2} \frac{\phi^{\prime \prime}(c)}{2} k, \quad d t=d \tau \sqrt{\frac{2}{\left|\phi^{\prime \prime}(c)\right| k^{\prime}}}
$$

where $\mu=\operatorname{sgn} \phi^{\prime \prime}(c)$. Thus, we can rewrite the above integral as

$$
I(k) \sim f(c) e^{i k \phi(c)} \sqrt{\frac{2}{\left|\phi^{\prime \prime}(c)\right| k}} \int_{-R \sqrt{k\left|\phi^{\prime \prime}(c)\right| / 2}}^{R \sqrt{k\left|\phi^{\prime \prime}(c) / 2\right|}} \exp \left[i \mu \tau^{2}\right] d \tau
$$

For infinitely large $k$ the last integral reduces to $\int_{-\infty}^{\infty} \exp \left(i \mu \tau^{2}\right) d \tau$, which can be evaluated exactly as

$$
\begin{gathered}
\int_{-\infty}^{\infty} e^{i \mu \tau^{2}}=\int_{-\infty}^{0} e^{i \mu \tau^{2}} d \tau+\int_{0}^{\infty} e^{i \mu \tau^{2}} d \tau=\int_{\infty}^{0} e^{i \mu \tau^{2}} d(-\tau)+\int_{0}^{\infty} e^{i \mu \tau^{2}} d \tau=2 \int_{0}^{\infty} e^{i \mu \tau^{2}} d \tau \\
=2 e^{\frac{i \pi}{4}} \int_{0}^{\infty} e^{-\mu r^{2}} d r=2\left[\int_{0}^{\infty} \int_{0}^{\infty} e^{-\mu\left(x^{2}+y^{2}\right)} d x d y\right]^{1 / 2} \\
=2\left[\int_{0}^{\pi / 2} \int_{0}^{\infty} e^{-\mu \rho^{2}} \rho d \rho d \theta\right]^{1 / 2}=\sqrt{\pi} e^{\frac{i \pi \mu}{4}}
\end{gathered}
$$

where in the second line we have assumed the change of variables $\tau=r e^{i \pi / 4}$, and in the third line we have transformed to polar coordinates $\rho, \theta$. Finally the large $k$
behavior of Eq. (2.44) is given by

$$
\begin{equation*}
I(k) \sim e^{i k \phi(c)} f(c) \sqrt{\frac{2 \pi}{k\left|\phi^{\prime \prime}(c)\right|}} e^{\frac{i \pi \mu}{4}} . \tag{2.45}
\end{equation*}
$$

In Fig. (2.7) we depict the real part of the integrand quantity of Eq. (2.44) [i.e., $\left.f(t) e^{i k \phi(t)}\right]$, for various functions of the phase $\phi(t)$. Furthermore, for simplicity we have selected $f(t)=k=1$. In the first subfigure we choose $\phi(t)=t$ and thus the real part results to $\operatorname{Re}\left\{f(t) e^{i k t}\right\}=\cos t$. Interestingly, in this case $\phi^{\prime}(t)=1$ and thus we have no stationary points. In the second subfigure the phase is $\phi(t)=(t-5)^{2}$ which results to a stationary point at $t=5$. As we can see the integrand near the stationary point, clearly oscillates slower in comparison of the leftmost and rightmost regions. This indicates that the contribution to the integrals from these regions is negligible compared to the contribution from the neighborhood of the stationary point. Similar results can be extracted from the third and fourth subfigures, where the phase is $\phi(t)=(t+5)^{3}$ and $\phi(t)=t^{4}$ respectively.


FIGURE 2.7: Illustration of the oscillating functions $\cos t, \cos (t-5)^{2}, \cos (t+5)^{3}$ and $\cos t^{4}$. Except from the first subfigure, all other cases exhibit stationary points from which the main contribution to the leading behavior of Eq. (2.44) stems.

## Chapter 3

## Propagation of accelerating beams in the paraxial domain

### 3.1 Introduction

In 1979 Berry and Balazs predicted that the potential-free Schrödinger equation admits solutions of the Airy type [4]. Those solutions are diffraction-free and propagate along parabolic trajectories. Almost thirty years later, optical Airy waves were predicted and experimentally observed by Siviloglou and Christodoulides [1,2]. This was a significant breakthrough in the field of Optics and inspired many theoretical and experimental studies. Over the last dozen of years a lot of effort has been put on optical waves with pre-engineered trajectories, which has resulted in generating novel classes of waves and very promising applications. Importantly, different classes of paraxial accelerating beams following convex but otherwise predefined arbitrary trajectories can be generated by engineering the phase profile of an optical wave [21, 22, 23]. In a different configuration, the propagation of Bessel-like beams, which by definition propagate along a straight line, can be significantly altered in order to follow even convex trajectories [24,25]. Mainly owing to their significant applications a new class of accelerating beams introduced, which exhibit an almost constant intensity until a specific point where it suddenly autofocuses, resulting to a radical increased intensity contrast by orders of magnitude. This class of beams is known as abruptly autofocusing waves, with their main representative being the radially symmetric Airy beams [26].

While there are some works discussing particular cases of amplitude manipulation of accelerating beams [27, 28], in the bibliography since today there is no systematic method resulting to the independent control of both the trajectory and the amplitude along the trajectory of an accelerating beam. This is mainly due to the fact that most studies have been focused on engineering the trajectory, and do not take into account other important beam parameters such as the amplitude and the beam width. The prospect of being able to independently control such parameters will be important in applications such as particle manipulation, where the absence of deviations in the maximum amplitude of the beam is crucial, in order to transport particles through radiation pressure efficiently.

In this chapter, we show an effective method to generate fully-controllable accelerating beams in the paraxial domain, by modulating their propagation parameters such as trajectory, amplitude and beam-width. To gain control over those parameters, we need to engineer both the amplitude and the phase on the input plane. We obtained closed-form expressions describing the propagation of such beams, which explicitly show that the amplitude of the propagating beam can be expressed with respect to the initial amplitude and the geometrical characteristics of the trajectory, while the beam-width solely depends on the curvature of the trajectory. Through our procedure, we demonstrate that the generation of accelerating beams exhibiting predesigned maximum intensity and beam-width along predefined curved trajectories, is possible.

### 3.2 Amplitude and trajectory/beam-width engineering

The propagation dynamics of an optical beam in $(1+1) \mathrm{D}$ configuration is governed by the Fresnel diffraction integral,

$$
\begin{equation*}
\psi(x, z)=\frac{1}{(i \lambda z)^{1 / 2}} \int_{-\infty}^{\infty} \psi_{0}(\xi) \exp \left[i k \frac{(x-\xi)^{2}}{2 z}\right] d \xi \tag{3.1}
\end{equation*}
$$

where $x$ is the transverse and $z$ is the longitudinal propagation direction, $k=2 \pi / \lambda, \lambda$ is the wavelength, and $\psi_{0}(x)$ is the optical wave excitation on the input plane $(z=0)$. By decomposing $\psi_{0}$ into amplitude and phase as $\psi_{0}(x)=A(x) \exp [i \phi(x)]$, we obtain
the total phase $\Psi$ of the Fresnel integral in the following form

$$
\begin{equation*}
\Psi(\xi ; x, z)=\phi(\xi)+k \frac{(x-\xi)^{2}}{2 z} \tag{3.2}
\end{equation*}
$$

Expressing the total phase as above, is advantageous in applying the Stationary phase approximation to our problem. The first order stationarity of the phase, that is $\partial_{\xi} \Psi=0$, yields the ray equation which is $x=\xi+z \phi^{\prime}(\xi) / k$, while from the condition of the second order phase stationarity we can parametrically express the trajectory of the propagating beam in terms of $\xi_{c}$,

$$
\begin{equation*}
\left[x_{c}\left(\xi_{c}\right), z_{c}\left(\xi_{c}\right)\right]=\left(\xi_{c}-\frac{\phi^{\prime}\left(\xi_{c}\right)}{\phi^{\prime \prime}\left(\xi_{c}\right)},-\frac{k}{\phi^{\prime \prime}\left(\xi_{c}\right)}\right) \tag{3.3}
\end{equation*}
$$

where subscript $c$ stands for caustic. From our perspective, what seems to be more interesting is the inverse problem. More specifically, we would like to appropriately modulate the phase on the input plane in terms of a convex but otherwise arbitrary predefined trajectory of the form,

$$
\begin{equation*}
x_{c}=f\left(z_{c}\right) . \tag{3.4}
\end{equation*}
$$

In this respect, we consider that the trajectory is formed by an envelope of straight lines each one of them is tangent to a specific point $\left(x_{c}, z_{c}\right)$ of the beam's caustic and described by a ray equation, resulting to,

$$
\begin{equation*}
\frac{d \phi}{d \tilde{\zeta}}=k \frac{d f\left[z_{c}(\xi)\right]}{d z_{c}} \tag{3.5}
\end{equation*}
$$

where $z_{c}$ is calculated from

$$
\begin{equation*}
\xi=f\left(z_{c}\right)-z_{c} f^{\prime}\left(z_{c}\right) \tag{3.6}
\end{equation*}
$$

At this point we should mention that since in the general case of a forward propagating beam, a power-law trajectory is defined only for positive values of $z_{c}$, the second derivative of the initial phase should be negative, $\left(\phi^{\prime \prime}(\xi)<0\right)$, so as the parametric expression of $z_{c}$ from Eq. (3.3) to be well defined.

Furthermore, we are interested in obtaining an expression for the amplitude close to the caustic. Towards this direction, firstly we would like to obtain an analytic
expression describing the evolution of the field in a region near the caustic [29]. To accomplish this, we expand the phase in a Taylor series around the perturbed variables $x=x_{c}+\delta x, \xi=\xi_{c}+\delta \xi$, while we keep a constant value for $z=z_{c}$ and retain terms up to cubic order [i.e., $(\delta x)^{j}(\delta \xi)^{k}, j+k \leq 3$ ],

$$
\begin{align*}
\Psi(x, \xi) \approx & \Psi_{c}+\partial_{x} \Psi_{c}(\delta x)+\partial_{\xi} \Psi_{c}(\delta \xi) \\
& +\frac{1}{2}\left[\partial_{x}^{2} \Psi_{c}(\delta x)^{2}+2 \partial_{x} \partial_{\xi} \Psi_{c}(\delta x)(\delta \xi)+\partial_{\xi}^{2} \Psi_{c}(\delta \xi)^{2}\right] \\
& +\frac{1}{6}\left[\partial_{x}^{3} \Psi_{c}(\delta x)^{3}+3 \partial_{x}^{2} \partial_{\xi} \Psi_{c}(\delta x)^{2}(\delta \xi)+3 \partial_{x} \partial_{\xi}^{2} \Psi_{c}(\delta x)(\delta \xi)^{2}+\partial_{\xi}^{3} \Psi_{c}(\delta \xi)^{3}\right] . \tag{3.7}
\end{align*}
$$

Finally, assuming a slow variation of the amplitude A with respect to $\xi$ while substituting Eq. (3.7) back to Eq. (3.1), we obtain after integrating over $\delta \xi$,

$$
\begin{equation*}
\psi=2 A(\xi)\left(\frac{\pi^{4} z_{c}^{3} \kappa^{2}}{\lambda}\right)^{1 / 6} e^{i \Xi} \operatorname{Ai}\left(s\left(2 k^{2} \kappa\right)^{1 / 3} \delta x\right) \tag{3.8}
\end{equation*}
$$

where $\Xi=\phi+k \frac{\left(\frac{x_{c}-\xi}{}\right)^{2}}{2 z_{c}}-\frac{\pi}{4}$ and

$$
\begin{equation*}
\kappa\left(z_{c}\right)=\left|\frac{d^{2} f\left(z_{c}\right)}{d z_{c}^{2}}\right|, \tag{3.9}
\end{equation*}
$$

is the curvature of the trajectory in the paraxial approximation, $s=\operatorname{sgn}\left[\frac{d^{2} f\left(z_{c}\right)}{d z_{c}^{2}}\right]$ is the sign of the curvature, Ai is the Airy function, and for simplicity we have replaced $\xi_{c}$ with $\xi$.

Having obtained a formula which describes the behavior of the field near the trajectory of the propagating beam, we can now extract some very useful results. To begin with, surprisingly the solution obtained from Eq. (3.8), indicates that close to the caustic the optical wave is described by an Airy function that varies linearly with $\delta x$, independently of the trajectory initially assumed. Furthermore the beam width which can be expressed as

$$
w(z)=\frac{1}{\left[2 k^{2} \kappa(z)\right]^{1 / 3}},
$$

modulates inversely with the cubic root of the paraxial curvature of the trajectory. From this we deduce that a beam has constant width if and only if the curvature of its trajectory is constant. This statement suggests that the only family of accelerating
waves that have constant width, are those of the Airy type following a general parabolic trajectory of the form $x_{c}=s_{0}+s_{1} z+s_{2} z^{2}$, with $s_{j}$ being arbitrary constants. As we mentioned at the beginning of the analysis, the amplitude of the beam along the curved trajectory is of our particular interest. Equation (3.8) implies that it depends not only on the initial amplitude $A(\xi)$ but also to the geometrical properties of the trajectory: It is proportional to the beam curvature $\kappa$ and the distance from the incident plane $z_{c}$.

This procedure led us to generate beams with predefined trajectory or beam width. Furthermore, by appropriately selecting the amplitude on the input plane we can also design beams with pre-engineered maximum amplitude along the trajectory. To this end, utilizing Eq. (3.8) we can choose the initial amplitude as

$$
\begin{equation*}
A(\xi)=\frac{U\left[z_{c}(\xi)\right]}{2.3}\left(\frac{\lambda}{\kappa^{2} z_{c}^{3}}\right)^{1 / 6} \tag{3.10}
\end{equation*}
$$

where $U(z)=\{\max (|\psi(x, z)|), x \in \mathbb{R}\}$, is simply the intended amplitude along the trajectory. We note that the above equation occurred after some minor approximations, however it is selected as it nicely illustrates the dependence of the amplitude of the beam on the initial amplitude and the geometrical characteristics of the trajectory, as mentioned before. Furthermore, numerical results presented next highlight the accuracy of this formula.

Although different classes of convex trajectories can result to closed-form expressions, here we only consider the case of power-law trajectories,

$$
\begin{equation*}
x_{c}=f\left(z_{c}\right)=\beta z_{c}^{\alpha}, \tag{3.11}
\end{equation*}
$$

with $\alpha>1$. Following the calculations presented earlier and more specifically by utilizing Eqs. (3.5), (3.6) we firstly obtain the phase on the input plane,

$$
\begin{equation*}
\phi(\xi)=\frac{-k \beta^{1 / \alpha} \alpha^{2}}{(\alpha-1)^{1-1 / \alpha}} \frac{(-\xi)^{2-1 / \alpha}}{2 \alpha-1} \tag{3.12}
\end{equation*}
$$

for $\xi<0$. Additionally, for the amplitude on the input plane Eq. (3.10) can equivalently expressed as,

$$
\begin{equation*}
A\left[z_{c}(\xi)\right]=\frac{U\left(z_{c}\right)}{2.3}\left(\frac{\lambda}{[\alpha(\alpha-1) \beta]^{2} z_{c}^{(2 \alpha-1)}}\right)^{1 / 6} \tag{3.13}
\end{equation*}
$$

where we assumed that the intended amplitude along the curved trajectory varies relatively slowly with $\xi$, and $z_{c}(\xi)=\left[\frac{-\xi}{\beta(\alpha-1)}\right]^{1 / \alpha}$. A uniform constant amplitude along the trajectory i.e., $U\left(z_{c}\right)=C$, will be very useful in applications such as particle manipulation. In this respect, we can select the initial amplitude to modulate accordingly to $A(\xi) \propto \frac{1}{(-\xi)^{(2 \alpha-1) /(6 \alpha)}}$. Interestingly, for the case of a parabolic trajectory $(\alpha=2)$, we recover the characteristic amplitude profile $A(\xi) \propto \frac{1}{(-\xi)^{1 / 4}}$ of the Airy function. The properties of the width of the propagating beam can be clarified by expressing $w$ as

$$
w(z)=\frac{1}{\left[2 k^{2} \beta \alpha(\alpha-1) z^{\alpha-2}\right]^{1 / 3}},
$$

which indicates that it decreases with $z$ for $\alpha>2$, and increases with $z$ for $\alpha<2$. More importantly, we conclude that the only trajectory that the beam-width remains invariant with the propagation distance is the parabolic, as it was expected, recalling that Airy beams following parabolic trajectories are the only non-diffracting solutions of the paraxial wave equation with one transverse dimension.

Finally, before proceeding with our simulations we note that in all figures the coordinates used were normalized. To obtain dimensionless parameters we scale the transverse and the longitudinal coordinates as $x \rightarrow x_{0} x$ and $z \rightarrow k x_{0}^{2} z$ respectively, while we assumed $k=1 \mathrm{in}$ all cases.

### 3.3 Numerical Results

To begin with our first propagating scenario, we assumed an accelerating beam following a parabolic trajectory of the form mentioned in Eq. (3.11) with $\alpha=2$ and $\beta=10^{-2}$. In Fig. 3.1 we show the evolution dynamics (first column), the maximum amplitude (second column), and various intensity cross-sections selected at different propagation distances (third and fourth column). In the three rows, different functional forms of the maximum amplitude along the caustic $U\left(z_{c}\right)$ are
selected. More specifically in the first row the amplitude is constant, while in the second and third row we have selected a Gaussian and a sinusoidal modulation of the amplitude profile. In all cases the predesigned amplitude along the trajectory (black circles) agrees well with the numerical result obtained. Although, in the first and second row a small propagation distance is necessary before the maximum amplitude reaches the predicted value. More specifically, during this transition distance the numerical value of the maximum amplitude is underestimated, although it gradually increases in order to match the theoretical value. This issue is not apparent in the second row where the selected functional form of $U(z)$ i.e., a Gaussian amplitude profile, exhibits smooth changes in comparison with the two other cases. In the third and fourth column, a comparison between the transverse intensity cross sections with the theoretical model Eq. (3.8) represented by black circles is shown. Interestingly, the theoretical prediction is in excellent agreement with the numerical data both in terms of the amplitude and in terms of the width. Additionally our formula describes well not only the main lobe (the neighborhood of the area where the caustic is formed), but also several other consecutive lobes. This result is surprisingly accurate taken into account that in our calculation $d x$ is taken to be small. The predesigned trajectory is represented by a white and black dashed curve in the figures of the first column, while horizontal and vertical dashed lines indicate the propagation distance at which each cross-section is selected.

To continue with the next case, we assumed another power-law trajectory with $\alpha=3 / 2$ and $\beta=1 / 5$. The configuration of Fig. 3.2 is similar to the previous case. However, here we have utilized three different functional forms for the intended amplitude along the trajectory. More specifically in the first row we have selected a constant amplitude profile, while in the second and the third row an elevated Gaussian and a sinusoidal respectively. Although smaller, the transition region until the beam reaches its intended maximum amplitude is evident in this case too, mostly in the first two rows. Again the fact that the selected functional form of $U(z)$ in the third row varies slowly with $z$, minimizes this effect. Moreover, the intensity cross sections shown at the third and fourth column demonstrate the comparison of the numerical data with the theoretical prediction [Eq. (3.8)]. The theoretical predicted curve (black circles) agrees well with the numerical results (solid line), both as it


Figure 3.1: Accelerating beams following a power-law trajectory [Eq. (3.11) with $\alpha=2$ and $\beta=10^{-2}$ ]. In the first column we can see the amplitude dynamics and the theoretical prediction for the trajectory (dashed curve). In the second column the maximum amplitude as a function of the propagation distance (solid curve) along with the theoretical prediction (shown in circles) is depicted. In the third and fourth columns, cross sections of the beam intensity at different propagation distances are presented with the theoretical predictions shown in circles. In the three rows the theoretical maximum values of the field amplitudes are $U(z)=1, U(z)=\exp \left[-(z-160)^{2} / 100^{2}\right]$, and $U(z)=1+0.5 \sin ^{2}[(z-160) / 80]$, respectively. The horizontal dashed lines in the first column, correspond to the cross sections shown in the third and fourth columns.
concerns the amplitude and as it concerns the frequency of the first and subsequent lobes. In each row both cross sections shown, were selected at different propagation distances, but pictured at the same intensity scale in order to visualize the modulation of the selected amplitude profile. White-black dashed curve shown in the subfigures of the first column, represents the predesigned trajectory of the propagating beam.

To highlight the potential of our method, as a final case we study the propagation of an accelerating beam following a cubic trajectory. In Fig. 3.3 three different functional forms for the amplitude along the trajectory $U\left(z_{c}\right)$ were presented. More specifically, in the first row we have selected a constant amplitude, while in the second and third row a sinusoidal and a sigmoid modulation of the maximum amplitude are shown respectively. As in the previous cases, in the last two columns we show the comparison between numerical data and the theoretical model for various intensity cross-sections selected at different propagation distances. Numerical results obtained agree well with the theoretical predicted values from Eq. (3.8). Horizontal and vertical


FIGURE 3.2: Accelerating beams following a power-law trajectory [Eq. (3.11) with $\alpha=3 / 2$ and $\beta=1 / 5$ ]. In the first column we can see the amplitude dynamics and the theoretical prediction for the trajectory (dashed curve). In the second column the maximum intensity as a function of the propagation distance (solid curve) along with the theoretical prediction (shown in circles) is depicted. In the third and fourth columns, cross sections of the beam intensity at different propagation distances are presented with the theoretical predictions shown in circles. In the three rows the theoretical maximum values of the field amplitudes are $U(z)=1$, $U(z)=0.5+0.5 \exp \left[-(z-160)^{2} / 110^{2}\right]$ and $U(z)=0.1 z+6 \sin ^{2}(z / 80)$, respectively. The horizontal dashed lines in the first column, correspond to the cross sections shown in the third and fourth columns.
dashed-lines shown in the first and second column respectively, indicate the exact propagation distance at which each cross section is pictured. Despite the overall good resemblance between the theoretical model and numerical results obtained, some deviations are evident mainly in the subfigures showing the maximum amplitude along the propagation distance in the first and second row. Having demonstrated all three cases with respect to the power-law trajectory we have enough results to address this phenomenon.

In order to have some results to reference at, we shall compare numerical simulations in Fig. (3.3) with the relative simulations in Fig. (3.2). More specifically, we would like to discuss the case of the constant amplitude shown in Figs. [3.3(a), 3.2(a)]. We can see that for the cubic trajectory the numerical value of the maximum amplitude converges to the theoretical value at approximately $z=100$, while in the second case the convergence of the numerical results with the predicted values has been achieved at $z=60$ already. Thus the transition region for the latter case is


Figure 3.3: Accelerating beams following a cubic trajectory [Eq. (3.11) with $\alpha=3$ and $\beta=1 / 32000$ ]. In the first column we can see the amplitude dynamics and the theoretical prediction for the trajectory (dashed curve). In the second column the maximum intensity as a function of the propagation distance (solid curve) along with the theoretical prediction (shown in circles) is depicted. In the third and fourth columns, cross sections of beam intensity at different propagation distances are presented with the theoretical predictions shown in circles. In the three rows the predicted maximum values of the field amplitudes are $U(z)=1$, $U(z)=z+15 \sin (z / 20)$, and $U(z)=1+\tanh [0.015(z-160) / 2]$, respectively. The horizontal dashed lines in the first column, correspond to the cross sections shown in the third and fourth columns.
significantly smaller compared to the former. This can be explained by considering that at early stages of propagation the paraxial curvature $\kappa$ is smaller in the trajectory shown in Fig. (3.3) compared to Fig. (3.2).

## Chapter 4

## Propagation of accelerating beams in the nonparaxial domain

### 4.1 Introduction

Over the last dozen of years, a lot of different classes of accelerating waves with engineered trajectories have been generated, both in the paraxial and in the nonparaxial regime [21,22,23]. In this chapter we focus on the propagation of nonparaxial accelerating beams. Such beams are advantageous in the sense that their trajectory can bend at very large angles [23, 27, 30]. In this respect, accelerating beams following circular trajectories and exhibiting non-diffracting features were proposed in the form of Bessel-like beams [31,32]. Additionally, new classes of accelerating beams exhibiting shape-preserving propagation on elliptic and parabolic trajectories in the form of Mathieu and Weber wavefunctions, were observed [33, 34, 35]. Abruptly autofocusing waves propagating under nonparaxial settings, exhibiting improved focusing characteristics were demonstrated [36]. Furthermore, Bessel-like beams propagating along curved nonparaxial trajectories were theoretically proposed [37]. Quite recently, nonparaxial accelerating beams following various convex trajectories have been studied and their phase profiles were calculated explicitly [27].

In the case of beams propagating along straight lines such as Bessel beams, it was shown that their longitudinal intensity can be engineered to almost any desired shape [38]. For beams propagating along curves, while the control of their trajectory has been accomplished, there are still propagation-defining features such as the amplitude and beam-width that up to now remain unexplored. In the bibliography there are
some works discussing the amplitude engineering of accelerating waves, such as the propagation of accelerating beams with a periodic amplitude profile [28], and the generation of Airy beams considering an amplitude correction factor to compensate for propagation losses [39]. However, to our knowledge no systematic method exists in order to independently control the trajectory, the amplitude along the trajectory, and the width of an accelerating beam.

Here we propose an effective method for generating nonparaxial accelerating beams that follow convex but otherwise arbitrary predesigned trajectories, while their amplitude profile is pre-engineered. To accomplish this, we need to appropriately modulate the amplitude and the phase on the input plane. The analytic expressions obtained in order to describe the propagation of such beams, indicate that the resulting beam-width solely depends on the curvature of the trajectory. As a result, curved accelerating beams in the nonparaxial regime with fully-controllable propagation features such as trajectory/beam-width and maximum amplitude, are possible. Beams exhibiting those highly desirable features can find numerous applications in research fields including, particle manipulation [40, 41, 42, 43, 44], micromachining [45, 46], filamentation [47, 48] and electric discharge generation [49], where the precise control over the properties of the beam is important. Finally we note that, despite we only study trajectories yielding to closed-form expressions for the phase profile of the propagating beam, our methodology can expand to the case of any arbitrary convex trajectory by applying numerical calculations.

### 4.2 Amplitude and trajectory/beam-width engineering

We consider the propagation of a monochromatic optical wave in a homogeneous isotropic medium with refractive index $n$. The associated electric field is selected to be polarized along the $y$-direction while, it depends only on the transverse coordinate $x$ and the longitudinal coordinate $z$ as, $E=\hat{y} \psi(x, x) e^{-i \omega t}$. The propagation dynamics of an arbitrary initial condition $\psi_{0}(\xi)$ is governed by the Rayleigh-Sommerfeld integral

$$
\begin{equation*}
\psi(x, z)=2 \int_{-\infty}^{\infty} \psi_{0}(\xi) \frac{\partial G(x, z ; \xi)}{\partial z} d \tilde{\xi} \tag{4.1}
\end{equation*}
$$

where $G(x, z ; \xi)=(-i / 4) H_{0}^{(1)}(k R), H_{0}^{(1)}$ is a Hankel function, $k=\frac{n \omega}{c}=\frac{2 \pi}{\lambda}$, $R=\sqrt{(x-\xi)^{2}+z^{2}}$, and $\psi_{0}$ is decomposed into amplitude and phase as $\psi_{0}(x)=$ $A(x) e^{i \phi(x)}$. Thus, the total phase involved in the Rayleigh-Sommerfeld integral is given by

$$
\begin{equation*}
\Psi(x, \xi ; z)=\phi(\xi)+k \sqrt{(x-\xi)^{2}+z^{2}} \tag{4.2}
\end{equation*}
$$

Expressing the total phase as above, is advantageous in applying the Stationary phase approximation to our problem. The first order stationarity of the phase, that is $\partial_{\xi} \Psi=0$, yields the ray equation which is $x=\xi+\frac{R \phi(\xi)}{k}$, while from the condition of the second order phase stationarity we obtain the prediction for the trajectory. However, we are more interested in solving the inverse problem. More specifically, we would like to determine the phase that is required in order to generate a beam with a convex but otherwise arbitrary trajectory $x_{c}=f\left(z_{c}\right)$. In this respect, we consider that the trajectory is formed by an envelope of straight lines, each one of them is tangent to a specific point $\left(x_{c}, z_{c}\right)$ on the beam's caustic, and is described by a ray equation resulting to,

$$
\begin{equation*}
\frac{d \phi}{d \tilde{\xi}}=k \frac{f^{\prime}\left(z_{c}\right)}{\left[1+\left(f^{\prime}\left(z_{c}\right)\right)^{2}\right]^{1 / 2}}, \tag{4.3}
\end{equation*}
$$

where prime denotes the derivative, and $z_{c}(\xi)$ is calculated from $\xi=f\left(z_{c}\right)-z_{c} f^{\prime}\left(z_{c}\right)$. As it was shown in [27], Eq. (4.3) can be analytically integrated in the case of convex accelerating trajectories such as circular, elliptic, power-law and exponential. In a different case, numerical calculations should be utilized.

In order to describe the propagation of a nonparaxial accelerating beam with fully controllable propagation parameters such as trajectory amplitude and beam-width, an expression for the evolution of the field near the caustic is necessary. To accomplish this, we expand the total phase in a Taylor series in the neighborhood close to the caustic described by the variables, $\xi=\xi_{c}+\delta \xi, x=x_{c}+\delta x$, while we keep $z$ constant at $z=z_{c}$ and retain terms up to cubic order [i.e., $(\delta \xi)^{j}(\delta x)^{k}, j+k \leq 3$ ],

$$
\begin{align*}
\Psi(x, \xi) \approx & \Psi_{c}+\partial_{x} \Psi_{c}(\delta x)+\partial_{\tilde{\xi}} \Psi_{c}(\delta \xi) \\
& +\frac{1}{2}\left[\partial_{x}^{2} \Psi_{c}(\delta x)^{2}+2 \partial_{x} \partial_{\tilde{\xi}} \Psi_{c}(\delta x)(\delta \xi)+\partial_{\tilde{\xi}}^{2} \Psi_{c}(\delta \xi)^{2}\right] \\
& +\frac{1}{6}\left[\partial_{x}^{3} \Psi_{c}(\delta x)^{3}+3 \partial_{x}^{2} \partial_{\tilde{\xi}} \Psi_{c}(\delta x)^{2}\left(\delta_{\xi}\right)+3 \partial_{\tilde{\xi}}^{2} \partial x \Psi_{c}(\delta \xi)^{2}(\delta x)+\partial_{\tilde{\xi}}^{3} \Psi_{c}(\delta \xi)^{3}\right] . \tag{4.4}
\end{align*}
$$

Finally, assuming a slow variation of the amplitude $A$ on the input plane so as to $A\left(\xi_{c}+\delta \xi\right)=A\left(\xi_{c}\right)$, while substituting Eq. (4.4) back to Eq. (4.1) we obtain after integrating over $\delta \xi$,

$$
\begin{equation*}
\psi=2 A(\xi)\left(\frac{\pi^{4} R_{c}^{3} \kappa^{2}}{\lambda}\right)^{1 / 6} e^{i \Xi} \mathrm{Ai}\left[-s \frac{z_{c}}{R_{c}}\left(2 k^{2} \kappa\left(z_{c}\right)\right)^{1 / 3} \delta x\right] \tag{4.5}
\end{equation*}
$$

where $s=\operatorname{sgn}\left(z_{c}^{\prime}\right), R_{c}=R\left(x_{c}, z_{c}, \xi_{c}\right), \Xi=k R_{c}+\phi-\frac{\pi}{4}$ and for simplicity we have replaced $\xi_{c}$ with $\xi$. Furthermore the curvature of the trajectory is defined by

$$
\begin{equation*}
\kappa\left(z_{c}\right)=\frac{\mid f^{\prime \prime}\left(z_{c} \mid\right.}{\left[1+\left(f^{\prime}\left(z_{c}\right)\right)^{2}\right]^{3 / 2}}=\frac{z_{c}^{2}}{R_{c}^{2}\left|z_{c}^{\prime}(\xi)\right|} . \tag{4.6}
\end{equation*}
$$

At this point we would like to make some comments on the results we obtained. To begin with, as in the paraxial regime the field near the caustic is described by an Airy function which varies in a linear manner with $\delta x$, independently of the trajectory assumed. Secondly, by direct inspection of Eq. (4.5), we deduce that the amplitude along the trajectory $\left[U\left(z_{c}(\xi)=\max |\psi|: x \in \mathbb{R}\right]\right.$, depends both on the initial amplitude and on the geometrical features of the trajectory. Taking the above into account, we can control the maximum amplitude of the propagating beam by appropriately selecting the amplitude on the input plane as

$$
\begin{equation*}
A(\xi)=F(\xi) \frac{U\left(z_{c}(\xi)\right)}{2.3}\left(\frac{\lambda}{R_{c}^{3} \kappa^{2}\left(z_{c}\right)}\right)^{1 / 6} \tag{4.7}
\end{equation*}
$$

where $F(\xi)$ is a sigmoid function properly selected to smoothly annihilate the amplitude in the region where the phase and the curvature were not defined. While some minor approximations have been applied in order to derive Eq. (4.7), the expression for the initial amplitude in this form is preferred in order to highlight its dependence
to both the intended maximum amplitude and the geometrical properties of the trajectory. Furthermore, numerical results presented next verify the accuracy of this formula.

Moreover from Eq. (4.5) we obtain an analytic expression for the beam width,

$$
\begin{equation*}
w(z)=\frac{R_{c}}{z_{c}\left[2 k^{2} \kappa(z)\right]^{1 / 3}}, \tag{4.8}
\end{equation*}
$$

which clearly indicates the dependence of the width to both the curvature and the geometrical features of the trajectory. More importantly we can simplify the above expression by assuming a coordinate system that follows the beam trajectory $\left(x_{c}, z_{c}\right)$. In this particular case Eq. (4.8) can be simplified significantly and expressed in the following form

$$
\begin{equation*}
w(z)=\frac{1}{\left[2 k^{2} \kappa(z)\right]^{1 / 3}} . \tag{4.9}
\end{equation*}
$$

In view of Eq. (4.9), we deduce that as the curvature increases the beam width becomes smaller. We should also mention that since the circle is the only trajectory with constant curvature, only beams propagating along circular trajectories exhibit constant beam widths. Interestingly Eqs. (4.8), (4.9), highlight the possibility to generate nonparaxial accelerating waves with predesigned beam width, by engineering the curvature of the intended trajectory.

### 4.3 Numerical Results

In the more general case of a convex trajectory $x_{c}=f\left(z_{c}\right)$ along with an arbitrary predesigned maximum amplitude $U(z)$, our method is still applicable although numerical calculations may be necessary. For the purposes of this chapter we restrict ourselves to different classes of convex accelerating trajectories, which lead to closedform expressions for the initial phase. To begin with, let us consider the propagation of nonparaxial accelerating beams following circular trajectories, formed in the first quadrant of the circle and described by

$$
\begin{equation*}
x_{c}=f\left(z_{c}\right)=\sqrt{R_{0}^{2}-z_{c}^{2}} \tag{4.10}
\end{equation*}
$$

where $R_{0}$ is the caustic radius and $z_{c}$ is given by

$$
\begin{equation*}
z_{c}(\xi)=\left(R_{0} / \xi\right) \sqrt{\xi^{2}-R_{0}^{2}} \tag{4.11}
\end{equation*}
$$

Utilizing Eq. (4.3), we obtain the phase on the input plane at the following form

$$
\begin{equation*}
\phi(\xi)=k R_{0}\left[\operatorname{arcsec}\left(\xi / R_{0}\right)-\sqrt{\left(\xi / R_{0}\right)^{2}-1}\right] \tag{4.12}
\end{equation*}
$$

while the initial amplitude is given in terms of the intended amplitude along the trajectory, from Eq. (4.7).

In Fig. 4.1 we show our results for the propagation of nonparaxial accelerating beams following a circular trajectory with $R_{0}=5000 \lambda$. In the three rows, different functional forms for the amplitude along the trajectory $U\left(z_{c}\right)$ are selected. More specifically, in the first row the maximum amplitude is constant, while in the second and third row a Gaussian-type and a sigmoid modulation of the maximum amplitude is depicted respectively. In all cases presented, numerical results agree well with the theoretical model. In the first two rows, at early stages of propagation the numerical values of the maximum amplitude are smaller in comparison with the analytic prediction $U\left(z_{c}\right)$, however they gradually increase in order to converge to the predicted values. We note that this transition region is negligible compared to the whole propagation distance. Interestingly, this is not the case in the third row where the functional form of $U\left(z_{c}\right)$ exhibits slow variations. Furthermore, some discrepancies occur at $z=R_{0}$, where the rays that form the caustic are emitting from $\xi$ that goes to infinity. To counteract this issue an infinite aperture will be necessary which unfortunately is not realizable. Finally, various intensity cross-sections are presented in the third column along with the theoretical prediction in circles. Surprisingly, Eq. (4.5) describes well not only the main-lobe( beam-width), but also several other consecutive lobes despite the small $\delta x$ assumed. The red(white) dashed line shown in the figures of the second(first) column represents the propagation distance where each intensity cross-section is selected. Additionally, the white-red dashed curve shown in the figures of the first column depict the predesigned trajectory of the propagating beam.


Figure 4.1: Propagation of nonparaxial beams accelerating along a circular trajectory $x_{c}=\sqrt{R_{0}^{2}-z_{c}^{2}}$ with $R_{0}=5000 \lambda$. In the first column the amplitude dynamics along with the theoretical prediction for the trajectory (white-red dashed curve) are depicted. In the second column we show the comparison of the numerical maximum amplitude along the propagation distance (black line) with the theoretical prediction (black circles). In the third column, various intensity profiles (black line) compared with the theoretical model (black circles) at the propagation distance indicated by a white-dashed line in each figure of the first column. For the amplitude along the trajectory, in the three rows, $U=1$, $U=\left[2-\exp \left(-\left(z-R_{0} / 2\right)^{2} /(1500 \lambda)^{2}\right)\right], U=\left[0.5+0.5 \tanh \left(\left(z-R_{0} / 2\right)\left(10^{3} \lambda\right)\right)\right]$, respectively. In all figures $F(\xi)=\operatorname{sig}\left[\left(\xi-R_{0}\right) /(5 \lambda)\right]$.

For our second case, we consider the propagation of nonparaxial accelerating beams following elliptic trajectories of the form

$$
\begin{equation*}
x_{c}=f\left(z_{c}\right)=\left[R_{0}^{2}-\left(z_{c} / \alpha\right)^{2}\right]^{1 / 2} \tag{4.13}
\end{equation*}
$$

Following the procedure described earlier, we obtain a closed-form expression for the phase on the input plane

$$
\begin{equation*}
\phi(\xi)=k R_{0}\left[\mathrm{E}\left(\sin ^{-1} u\left(R_{0}\right) \mid \tau\right)-\mathrm{E}\left(\sin ^{-1} u(\xi) \mid \tau\right)\right] \tag{4.14}
\end{equation*}
$$

where $u(x)=x /\left(R_{0} \sqrt{\tau}\right), \tau=1-\alpha^{2}$, and $\mathrm{E}(\phi \mid m)$ is the elliptic integral of the second kind for $\alpha<1$ and

$$
\begin{equation*}
\phi=k R_{0}\left[\mathrm{E}\left(i \sinh ^{-1} u(\xi) \mid \tau\right)-\mathrm{E}\left(i \sinh ^{-1} u\left(R_{0}\right) \mid \tau\right)\right] \tag{4.15}
\end{equation*}
$$

where $u(x)=x /(R \sqrt{-\tau})$, for $\alpha>1$. Finally, the initial amplitude of the propagating beam can be obtained in terms of the intended amplitude along the trajectory from Eq. (4.7).

Our results for the propagation of nonparaxial beams following elliptic trajectories are illustrated in Fig. 4.2 for $R_{0}=3000 \lambda$ and $\alpha=3 / 2$. In the three rows, different functional forms for the amplitude along the trajectory $U\left(z_{c}\right)$ are selected. In particular, in the first row we set the intended amplitude to unity, while in the second and third row a Gaussian-type and a sigmoid modulation of the maximum amplitude were assumed respectively. As in the previous case, in the first and third rows a small transition region is necessary for the maximum amplitude to converge to its predicted value. Furthermore, in all rows some discrepancies appear when $z_{0}$ is approaching $R_{0}$. This can be explained by considering that the caustic in the neighborhood of $z=R_{0}$, is constructed by rays which emitted from infinity. However, we should note that our numerical results are in excellent agreement with the theoretical model. More specifically, in the second column we compare the numerical values of the maximum amplitude along the propagation distance (solid line) with the theoretically predicted maximum amplitude (black circles). Furthermore, in the third column various intensity cross sections are presented comparing our numerically computed intensity profiles (solid line) with the theoretical model [Eq. (4.5)]. Remarkably, our theoretical predictions represented by black circles agree well not only with the amplitude of the oscillations, but also with their frequency as well. Each cross section is selected at a propagation distance indicated by the white(red) dashed line in the figures of the first(second) column.


FIGURE 4.2: Propagation of nonparaxial beams accelerating along an elliptic trajectory $x_{c}=$ $\sqrt{R_{0}^{2}-\left(z_{c} / \alpha\right)^{2}}$ with $R_{0}=3000 \lambda$ and $\alpha=3 / 2$. In the first column the amplitude dynamics along with the theoretical prediction for the trajectory (white-red dashed curve) are depicted. In the second column we show the comparison of the numerical maximum amplitude along the propagation distance (black line) with the theoretical prediction (black circles). In the third column, various intensity profiles (black line) compared with the theoretical model (black circles) at the propagation distance indicated by a white-dashed line in each figure of the first column. For the amplitude along the trajectory, in the three rows, $U=1, U=$ $\left[2-\exp \left(-\left(z-3 R_{0} / 4\right)^{2} /(1500 \lambda)^{2}\right)\right], U=\left[3-2 \tanh \left(\left(z-3 R_{0} / 4\right)(500 \lambda)\right)\right]$, respectively. In all figures $F(\xi)=\operatorname{sig}\left[\left(\xi-R_{0}\right) /(5 \lambda)\right]$.

To continue with, we consider the case of general power-law trajectories of the form

$$
\begin{equation*}
x_{c}=f\left(z_{c}\right)=\alpha z_{c}^{\beta}, \tag{4.16}
\end{equation*}
$$

for $\alpha>0$ and $\beta>1$.

As in the previous cases, we obtain a closed form expression for the phase of the beam at the input plane

$$
\begin{equation*}
\phi(\xi)=\frac{\alpha \beta^{2} k \zeta(\alpha \zeta)^{\frac{1}{\beta}-1}}{2 \beta-1}{ }_{2} \mathrm{~F}_{1}\left(\alpha, b ; c ;-\alpha^{2 / \beta} \beta^{2} \zeta^{\frac{2}{\beta}-2}\right), \tag{4.17}
\end{equation*}
$$

where ${ }_{2} \mathrm{~F}_{1}$ is a hypergeometric function, $\zeta=\frac{\beta-1}{-\zeta}, \alpha=\frac{1}{2}, b=1+\frac{1}{2(\beta-1)}$, and $c=$ $2+\frac{1}{2(\beta-1)}$. In order to pre-engineer the maximum amplitude of the propagating beam, we select the initial amplitude from Eq. (4.7).

In Fig. 4.3 we show our results for the propagation of nonparaxial beams accelerating along a parabolic trajectory with $\alpha=10^{-4} / \lambda$ and $\beta=2$. The formation of this figure is similar to the previous cases presented. More specifically, in the first row a constant amplitude along the trajectory is presented, while in the second and third row a Gaussian type and sinusoidal modulation of the maximum amplitude is selected respectively. Furthermore, numerical results obtained are in excellent agreement with the theoretical model. In particular, the numerical values of the maximum amplitude converge fast enough to the predicted values (second column), while the intensity profiles plotted in the third column are described accurately by the theoretical model developed. Interestingly, while our solution is obtained in the neighborhood near the caustic (the main-lobe of the propagating beam), Eq. (4.5) predicts the oscillatory behavior of the main and the other consecutive lobes accurately. Additionally, in contrast with the figures presented before, the rays forming the caustic in the case of power-law trajectories do not bend at a $90^{\circ}$ angle, thus our numerical results agree well with the theoretical model up to the simulated propagation distance. The predesigned trajectory is represented by the white-red dashed curve in the figures of the first column. The red(white) dashed line in the subfigures of the second(first) column indicates the cross-section at which each intensity profile is selected. In all figures theoretical results shown are represented by black circles. As we mentioned earlier, the only constraint through our procedure is that the intended maximum amplitude should be a relatively slowly varying function.

Finally, we assume a propagating scenario where nonparaxial beams were designed to accelerate along cubic trajectories. As in the previous case, Eqs. (4.17), (4.7) can be utilized to obtain the initial phase and the initial amplitude respectively. In Fig.


FIGURE 4.3: Propagation of nonparaxial beams accelerating along a parabolic trajectory $x_{c}=\alpha z_{c}^{\beta}$ with $\alpha=10^{-4} / \lambda$ and $\beta=2$. In the first column the amplitude dynamics along with the theoretical prediction for the trajectory (white-red dashed curve) are depicted. In the second column we show the comparison of the numerical maximum amplitude along the propagation distance (black line) with the theoretical prediction (black circles). In the third column, various intensity profiles (black line) compared with the theoretical model (black circles) at the propagation distance indicated by a white-dashed line in each figure of the first column. For the amplitude along the trajectory, in the three rows, $U=1$, $U=\left[4-3 \exp \left(-(z-5000 \lambda)^{2} /(3000 \lambda)^{2}\right)\right], U=\left[0.75 * z / \lambda+750 \sin ^{2}(0.001 z / \lambda)\right] / 2000$, respectively. In all figures $F(\xi)=\operatorname{sig}[(-\xi) /(5 \lambda)]$.
4.4 we present our results for the propagation of nonparaxial beams along a cubic trajectory with $\alpha=10^{-8}$ and $\beta=3$. A different functional form of the amplitude along the trajectory is illustrated in each row. More specifically, in the first row we set the maximum amplitude to unity, while in the second and third row a Gaussian-type and a sinusoidal-type modulation of the maximum amplitude is selected respectively. In the first column we depict the amplitude dynamics along with the predesigned trajectory represented by the white-red dashed curve. In the second column, our
numerical results for the maximum amplitude with respect to the propagation distance are compared with the theoretical prediction, while in the third column various intensity profiles are compared with the theoretical model [Eq. (4.5)]. We note that all numerical results agree well with the theoretical predictions. Although, compared to the results presented in Fig. 4.3 the propagation distance needed for the beam in order to converge to its predesigned maximum amplitude is notably greater. This can be explained because at early stages of propagation, the curvature of the selected parabolic trajectory is greater than the curvature of this particular cubic trajectory.


FIGURE 4.4: Propagation of nonparaxial beams accelerating along a parabolic trajectory $x_{c}=$ $\alpha z_{c}^{\beta}$ with $\alpha=10^{-8} / \lambda$ and $\beta=3$. In the first column the amplitude dynamics along with the theoretical prediction for the trajectory (white-red dashed curve) are depicted. In the second column we show the comparison of the numerical maximum amplitude along the propagation distance (black line) with the theoretical prediction (black circles). In the third column, various intensity profiles (black line) compared with the theoretical model (black circles) at the propagation distance indicated by a white-dashed line in each figure of the first column. For the amplitude in the three rows, $U=1, U=\left[0.5+1.5 \exp \left(-(z-5000 \lambda)^{2} /(1500 \lambda)^{2}\right)\right]$,

$$
U=\left[1+\sin ^{2}(0.0007(z-5000 \lambda))\right] . \text { In all figures } F(\xi)=\operatorname{sig}[(-\xi) /(5 \lambda)]
$$

## Chapter 5

## Propagation of abruptly

## autofocusing beams in the paraxial

## domain

### 5.1 Introduction

The ability to control the propagation defining parameters of an optical wave, such as its amplitude, trajectory and beam width, is crucial in order to design beams with specific properties. Obviously, being able to optimize the focusing characteristics of a beam is rather important in the field of optics. Beyond optics, beams that can abruptly focus their energy at a specific point, while at the same time they exhibit a low intensity profile until their focus, have already been exploited in applications realted to biomedicine and laser nanosurgery. The first family of beams utilized to demonstrate those interesting features, were the radially symmetric Airy waves [26]. Such beams exhibit a low intensity profile until their designed focal point, where suddenly their maximum intensity is increased by orders of magnitude. Along with the abrupt autofocusing features, Airy beams with radial symmetry accelerate along a parabolic trajectory in the same fashion with their Cartesian counterparts, and were experimentally observed in [41, 50]. Different classes of abruptly autofocusing waves that accelerate to their focal point along arbitrary trajectories have been generated by engineering their phase profile on the input plane [22]. Pre-engineered abruptly autofocusing beams have been linked to advantageous focusing properties over the conventional autofocusing Airy waves, such as more abrupt focusing along with
enhanced contrast at the focus. The propagation stage after the focus is also of particular interest. Abruptly autofocusing waves exhibit an oscillatory behavior after their focal point, which in some cases might be not desirable. In applications such as in medical laser treatments, it would be favorable this oscillatory behavior to be damped, in order to prevent harming any unwanted tissue for example. In this respect, beams exhibiting an almost constant intensity peak until their focus are highly desirable. Lately, abruptly autofocusing beams have been studied in the nonparaxial regime too, where beams bending at large angles are supported. Benefited from this aspect, abruptly autofocusing waves have been generated, while exhibiting tighter focus along with smaller spot size and enhanced intensity contrast as compared to the paraxial case [36]. The highly desirable characteristics of abrupt focus along with the high contrast of the intensity at the focus, were exploited in applications such as particle manipulation [41], filamentation [51], and in creating ablation spots in materials [50].

In this chapter, we study the propagation of abruptly autofocusing waves in the paraxial domain. In order to better analyze their focusing characteristics, we decompose their propagation dynamics into three stages. The first stage corresponds to the region before the focus, while in the second and third stage we study the beam's dynamics at the focus and in the region after the focus respectively. We find analytic formulas describing the propagation of such beams at each stage. In our numerical results, we simulate the propagation of abruptly autofocusing beams along different power-law trajectories. Furthermore, to optimize their focusing properties we present different choices for the amplitude modulation on the input plane, leading to larger intensity contrast at the focal point, along with radically damped oscillatory behavior after the focus.

### 5.2 Amplitude and trajectory/beam-width engineering

The propagation dynamics of abruptly autofocusing waves with radial symmetry, can be described by the Fresnel-type diffraction integral in the following form

$$
\begin{equation*}
\psi(r, z)=\frac{k}{i z} \int_{0}^{\infty} \rho \psi_{0}(\rho) \mathrm{J}_{0}\left(\frac{k \rho r}{z}\right) \exp \left(i k \frac{\rho^{2}+r^{2}}{2 z}\right) d \rho \tag{5.1}
\end{equation*}
$$

where $r, \rho$ are the radial coordinates, $z$ is the longitudinal propagation direction, $\psi_{0}$ is the optical wave excitation on the input plane, and $k$ is the wavenumber. By decomposing $\psi_{0}$ into amplitude and phase as $\psi_{0}(\rho)=A(\rho) \exp [i \phi(\rho)]$, while applying large argument asymptotics for the Bessel function $\mathrm{J}_{0}(x) \approx \sqrt{\frac{1}{2 i \pi x}}\left(e^{i x}+i e^{-i x}\right)$, we obtain the total phase $\Psi$ of the integrand in the following form

$$
\begin{equation*}
\Psi(r ; \rho, z)=\phi(\rho)+k \frac{(r-\rho)^{2}}{2 z} \tag{5.2}
\end{equation*}
$$

To derive the equation above, we assumed that the caustic is formed by rays propagating "backwards" and towards to the optical axis, thus neglecting $e^{i x}$ term of the approximation of the Bessel function mentioned above. Those rays are described by a ray equation, which can be obtained by applying the method of Stationary phase to Eq. (5.2). More specifically, the first order of the phase stationarity yields $r=\rho+z \phi^{\prime}(\rho) / k$, while from the condition of the second order phase stationarity, we can parametrically express the trajectory of the propagating beam as a function of $\rho_{c}$

$$
\begin{equation*}
\left[r_{c}\left(\rho_{c}\right), z_{c}\left(\rho_{c}\right)\right]=\left(\rho_{c}-\frac{\phi^{\prime}\left(\rho_{c}\right)}{\phi^{\prime \prime}\left(\rho_{c}\right)^{\prime}},-\frac{k}{\phi^{\prime \prime}\left(\rho_{c}\right)}\right) \tag{5.3}
\end{equation*}
$$

where subscript $c$ stands for caustic. Importantly, we can also solve the inverse problem of determining the required phase as a function of the convex but otherwise arbitrary predefined trajectory of the form

$$
\begin{equation*}
r_{c}=f\left(z_{c}\right) \tag{5.4}
\end{equation*}
$$

In this direction, firstly we consider that the trajectory is formed by an envelope of straight lines, each one of them is tangent to a specific point $\left(r_{c}, z_{c}\right)$ of the beam's caustic, and described by a ray equation, resulting to

$$
\begin{equation*}
\frac{d \phi}{d \rho}=k \frac{d f\left[z_{c}(\rho)\right]}{d z_{c}} \tag{5.5}
\end{equation*}
$$

where $z_{c}(\rho)$ is calculated from

$$
\begin{equation*}
\rho_{c}=f\left(z_{c}\right)-z_{c} f^{\prime}\left(z_{c}\right) \tag{5.6}
\end{equation*}
$$

Furthermore, since in the general case of an arbitrary power-law trajectory the caustic is formed only for positive values of $z_{c}$, the second derivative of the phase on the input plane should be negative $\left(\phi^{\prime \prime}\left(\rho_{c}\right)<0\right)$, so as the parametric expression of $z_{c}$ from Eq. (5.3) to be well defined.

At this point, we would like to briefly discuss the main results we have obtained so far. Interestingly, the resulted equations namely Eqs. (5.3)-(5.6) are identical to Eqs. (3.3)-(3.6) under the substitution $x \rightarrow r$ and $\xi \rightarrow \rho$. This similarity is not a coincidence at all. More specifically, the rays that form the caustic, propagate in a linear fashion and are not affected by the configuration of the space i.e., Cartesian or radial coordinates.

In order to describe the propagation of abruptly autofocusing beams following arbitrary power-law trajectories, we need an explicit formula for the evolution of the beam in the neighborhood of the caustic. In this respect, we expand the total phase in a Taylor series around the perturbed variables $r=r_{c}+\delta r, \rho=\rho_{c}+\delta \rho$, while we keep a constant value for $z=z_{c}$ and retain terms up to cubic order [i.e., $\left.(\delta r)^{j}(\delta \rho)^{k}, j+k \leq 3\right]$,

$$
\begin{align*}
\Psi(r, z) \approx & \Psi_{c}+\partial_{r} \Psi_{c}(\delta r)+\partial_{\rho} \Psi_{c}(\delta \rho) \\
& +\frac{1}{2}\left[\partial_{r}^{2} \Psi_{c}(\delta r)^{2}+2 \partial_{r} \partial_{\rho} \Psi_{c}(\delta r)(\delta \rho)+\partial_{\rho}^{2} \Psi_{c}(\delta \rho)^{2}\right] \\
& +\frac{1}{6}\left[\partial_{r}^{3} \Psi_{c}(\delta r)^{3}+3 \partial_{r}^{2} \partial_{\rho} \Psi_{c}(\delta r)^{2}(\delta \rho)+3 \partial_{r} \partial_{\rho}^{2} \Psi_{c}\left(\delta_{r}\right)(\delta \rho)^{2}+\partial_{\rho}^{3} \Psi_{c}(\delta \rho)^{3}\right] \tag{5.7}
\end{align*}
$$

Finally, assuming a slowly varying envelope for the amplitude $A$ so as to $A\left(\rho_{c}+\delta \rho\right)=$ $A\left(\rho_{c}\right)$, while substituting Eq. (5.7) back to Eq. (5.1), we obtain after integrating over $\delta_{\rho}$,

$$
\begin{equation*}
\psi=2 A(\rho) \sqrt{\frac{\rho}{i r}}\left(\frac{\pi^{4} z_{c}^{3} \kappa^{2}}{\lambda}\right)^{1 / 6} e^{i \Xi} \mathrm{Ai}\left[-\left(2 k^{2} \kappa\right)^{1 / 3} \delta r\right] \tag{5.8}
\end{equation*}
$$

where $\Xi=\phi(\rho)+k \frac{\left(r_{c}-\rho_{c}\right)^{2}}{2 z_{c}}$, and $g=\frac{d f\left(z_{c}\right)}{d z_{c}}, \kappa=\frac{d^{2} f\left(z_{c}\right)}{d z_{c}^{2}}$ being the slope and the curvature of the trajectory respectively. In the expressions showed above we have substituted for simplicity $\rho_{c} \rightarrow \rho$, a replacement that we will retain for the rest of this chapter. In accordance with the solution obtained in the second chapter of this dissertation in the case of accelerating waves, Eq. (5.8) depends both on the initial amplitude of
the propagating beam and on the geometrical features of the predesigned trajectory. Although, this solution is obtained under the large argument assumption for the Bessel function $(k r \rho / z \gg 1)$, which collapses near the optical axis i.e., as $r \rightarrow 0$, thus failing to describe the propagation of the beam near the focus. This issue was expected, as from Eq. (5.8) we observe that the amplitude depends on $1 / \sqrt{r}$. However, the aforementioned equation successfully describes the evolution of the propagating beam before the focal point.

The collapse of the caustic on the optical axis, indicates that an expression obtained in the neighborhood of the focal point of the beam might be more accurate. To obtain this, we assume the perturbed variables $\rho=\rho_{c}+\delta \rho, z=z_{c}+\delta z$, while we keep $r$ constant at $r=r_{c}$. We note that this approximation is designed so as to describe the behavior of the optical wave both at the early stages of its propagation and close to the focus. In this respect, we begin our analysis based on the same large argument asymptotics we developed earlier $(k r \rho / z \gg 1)$, and retain terms up to third order in our Taylor series expansion of the total phase

$$
\begin{align*}
\Psi(\rho, z) \approx & \Psi_{c}+\partial_{\rho} \Psi_{c}(\delta \rho)+\partial_{z} \Psi_{c}(\delta z) \\
& +\frac{1}{2}\left[\partial_{\rho}^{2} \Psi_{c}(\delta \rho)^{2}+2 \partial_{\rho} \partial_{z} \Psi_{c}(\delta \rho)(\delta z)+\partial_{z}^{2} \Psi_{c}(\delta z)^{2}\right] \\
& +\frac{1}{6}\left[\partial_{\rho}^{3} \Psi_{c}(\delta \rho)^{3}+3 \partial_{\rho}^{2} \partial_{z} \Psi_{c}(\delta \rho)^{2}(\delta z)+3 \partial_{\rho} \partial_{z}^{2} \Psi_{c}(\delta \rho)(\delta z)^{2}+\partial_{z}^{3} \Psi_{c}(\delta z)^{3}\right] \tag{5.9}
\end{align*}
$$

Substituting Eq. (5.9) back to Eq. (5.1) we obtain after integrating over $\delta \rho$

$$
\begin{equation*}
\psi\left(r_{c}, z\right)=A(\rho)\left(\frac{2 \kappa}{k}\right)^{1 / 3}\left(\frac{2 \pi k \rho z_{c}^{2}}{i r_{c} z}\right)^{1 / 2} e^{i \Xi} \mathrm{Ai}\left[\left(2 k^{2} \kappa\right)^{1 / 3} g\left(z_{c}\right) \delta z\right] \tag{5.10}
\end{equation*}
$$

where now $\Xi=\phi(\rho)+k \frac{\left(\rho-r_{c}\right)^{2}}{2 z_{c}}$. Although, since the amplitude in this case too is proportional to $1 / \sqrt{r}$, we conclude that the expression obtained above diverges as $r \rightarrow 0$, in the same fashion as Eq. (5.8). However, Eq. (5.10) describes accurately the behavior of the propagating beam in the region $0<z<z_{f}$ (i.e., just before the focus).

In order to bypass this issue - that our solution collapses at the optical axis - we utilize the same expansion as before $\rho=\rho_{c}+\delta \rho, z=z_{c}+\delta z$, but now we set $r=0$. This will lead to us to an asymptotic expression which will be valid close to the focus.

Applying the same methodology as before while retaining the dominant terms of our Taylor series expansion, we end up with

$$
\begin{equation*}
\psi(0, z)=A(\rho)\left(\frac{2 \kappa}{k}\right)^{1 / 3} \frac{2 \pi k \rho z_{c}}{i z} e^{i \Xi} \operatorname{Ai}\left[\left(2 k^{2} \kappa\right)^{1 / 3} g\left(z_{c}\right) \delta z\right] \tag{5.11}
\end{equation*}
$$

where $\Xi$ is the same as in Eq. (5.10), as it is well-defined for $r_{c}=0$.
By directly comparing the expressions obtained for the propagation of the optical wave until the focus and exactly at the focus, namely Eq. (5.10) and Eq. (5.11) respectively, we can extract some very useful results. Firstly, the phase factor in both solutions is the same. Secondly and even more importantly, the argument of the Airy function is the same in both expressions. Thus, both equations describe the propagation of a beam with the same qualitative features, such as trajectory beam-width etc. What differs is the amplitude. While there exist numerous ways to combine those expressions, we select the following formula in order to obtain a global asymptotic describing the overall behavior of the optical wave, from the early stages of its propagation until its focal point

$$
\begin{equation*}
\psi(r, z)=A(\rho)\left(\frac{2 \kappa}{k}\right)^{1 / 3} \frac{2 \pi k z_{c} \rho}{\left[2 \pi i k \rho z r_{c}-z^{2}\right]^{1 / 2}} \operatorname{Ai}\left[\left(2 k^{2} \kappa\right)^{1 / 3} g\left(z_{c}\right) \delta z\right] \tag{5.12}
\end{equation*}
$$

This particular form of Eq. (5.12) is preferred due to its simplicity and accuracy. More specifically, by substituting $r_{c}=0$ we obtain our initial expression for the amplitude at the focus [i.e., Eq. (5.11)], while by assuming that the first order of the expression in the square root dominates over the second term, we can recover Eq. (5.10). Furthermore, the validity of this global asymptotic is confirmed by our numerical simulations presented later.

In order to obtain the exact propagation distance where the maximum intensity of the beam is observed, we argue that the terms consisting the amplitude in Eq. (5.12) vary relatively slow with $\rho$, when compared with the fast oscillating Airy function. Thus, the terms that determine the position of the focal point are solely those that appear in the argument of the Airy function. In this respect, we can estimate the location of $z_{f}$ by setting this particular argument equal to -1 i.e., the point where the Airy function gets its maximum value. Following this procedure, we can express $z_{f}$
in the following form

$$
\begin{equation*}
z_{f}=z_{c}-\frac{1}{\left[2 k^{2} \kappa\left(z_{c}\right)\right]^{1 / 3} g\left(z_{c}\right)} \tag{5.13}
\end{equation*}
$$

The above equation clearly indicates that the exact location of the focus is not the point where the trajectory intersects the optical axis. From Eq. (5.13) we conclude that the focal distance is shifted from $z_{c}$ towards larger values of $z$, by an amount that is inversely proportional to the slope and the curvature of the trajectory, given that $g\left(z_{c}\right)<0$. Furthermore, to obtain an expression for the amplitude at this particular point, we set $r_{c}=0$ and by utilizing Eq. (5.12) we obtain that

$$
\begin{equation*}
\left|\psi_{\max }\left(z_{f}\right)\right| \approx 2 \pi \rho A(\rho)\left(2 k^{2} \kappa\right)^{1 / 3} \operatorname{Ai}(-1) \tag{5.14}
\end{equation*}
$$

From Eq. (5.14) we conclude that there are only three parameters defining the maximum intensity of the optical wave at the focus. That is, the distance from the axis on the input plane $\rho$ of the ray that converges to the focus, the initial amplitude $A(\rho)$, and the curvature of the trajectory at the focus $\kappa\left(z_{c}\right)$.

Until now, we have obtained various formulas describing the dynamics of the optical wave from the early stages of its propagation up to the focus $z_{f}$, while the region after the focal point remains unexplored. In this direction, we observe that after the focus the maximum amplitude is no longer formed along the trajectory of the caustic. Interestingly, at the focal point a beam transformation occurs, directly affecting the maximum amplitude of the optical wave, which will be lying in a region close to the optical axis. Applying the condition of the phase stationarity of the first order to Eq. (5.2) we obtain

$$
\begin{equation*}
\frac{k \rho}{z}+\phi^{\prime}(\rho)=0 \tag{5.15}
\end{equation*}
$$

where we have assumed that $k r \rho / z$ is relatively small for $z>z_{f}$, and thus terms depending on $r$ can be neglected from our calculations. As we will show later, for a given trajectory we can calculate explicitly the corresponding phase on the input plane, which can be utilized to obtain the solutions of the equation above. Equation (5.15) supports two real valued solutions $\rho_{1}, \rho_{2}$, provided that they do not exceed the aperture $r_{a}$ (i.e., $\rho_{1}<\rho_{2}<\rho_{a}$ ). Defining by $z_{c, j}=-k / \phi^{\prime \prime}\left(\rho_{j}\right)$, the position in the longitudinal direction where the rays emitted from $\rho_{j}$ contribute to the caustic,
we obtain that $z_{c, 1}<z<z_{c, 2}$. This means that one of the rays contributes to the caustic before and the other after the selected value of $z$. Employing a stationary phase method we obtain

$$
\begin{align*}
\psi(r, z)= & \sum_{j=1,2} \rho_{j} A\left(\rho_{j}\right)\left|\frac{2 \pi k z_{c, j}}{z\left(z_{c, j}-z\right)}\right|^{1 / 2} \mathrm{~J}_{0}\left(\frac{k r \rho_{j}}{z}\right) \\
& \times \exp \left\{i\left[\frac{k\left(r^{2}+\rho_{j}^{2}\right)}{2 z}+\phi\left(\rho_{j}\right)+\left(\mu_{j}-2\right) \frac{\pi}{4}\right]\right\}, \tag{5.16}
\end{align*}
$$

where $\mu_{j}=(-1)^{j}$, is defined by $\mu_{j}=\operatorname{sgn}\left(z_{c, j}-z\right)$. Because Eq. (5.16) is consisted of a superposition of partial solutions which give rise to destructive interference, the maximum intensity is not always located at the optical axis. However, it can always be found in an area that is close to the input plane. For large enough $z$ [i.e., $z \gg z_{f}$ ], $\rho_{2}$ corresponds to rays emitted outside the aperture and thus do not contribute to the amplitude. This suggests that the second term $(j=2)$ may be excluded from the calculation of Eq. (5.16), which means that the destructive interference is canceled, as only $(j=1)$ terms are retained, and the maximum amplitude will be located exactly at the origin,

$$
\begin{equation*}
\left|\psi_{\max }(z)\right|=\rho_{1} A\left(\rho_{1}\right)\left|\frac{2 \pi k z_{c, 1}}{z\left(z_{c, 1}-z\right)}\right|^{1 / 2} . \tag{5.17}
\end{equation*}
$$

### 5.3 Numerical Results

To proceed with our numerical results, in all our simulation we study the propagation of abruptly autofocusing beams accelerating along a power-law trajectory of the form

$$
\begin{equation*}
r_{0}-r_{c}=\beta z_{c}^{\alpha}, \tag{5.18}
\end{equation*}
$$

where $r_{0}$ is the radius of the Airy ring on the input plane. Utilizing Eqs. (5.5), (5.6), (5.18) we can express the initial phase of the beam as

$$
\begin{equation*}
\phi(\rho)=\frac{-k \beta^{1 / \alpha} \alpha^{2}}{(\alpha-1)^{1-1 / \alpha}} \frac{\left(\rho-r_{0}\right)^{2-1 / \alpha}}{2 \alpha-1} \tag{5.19}
\end{equation*}
$$

while the amplitude on the input plane is defined by a monotonically decreasing exponential function in the following form

$$
\begin{equation*}
A(r)=A_{0} \operatorname{sig}\left(\frac{r-r_{0}}{w_{1}}\right) \operatorname{sig}\left(\frac{r_{a}-r}{w_{2}}\right) e^{c\left(r_{0}-r\right)} \tag{5.20}
\end{equation*}
$$

and the sigmoid function sig is arbitrary selected as

$$
\operatorname{sig}(x)=\left\{\begin{array}{cl}
\tanh (x), & x \geq 0  \tag{5.21}\\
0, & x \leq 0
\end{array}\right.
$$

where $w_{j}{ }^{\prime}$ s are the slopes of the sigmoid function and $r_{a}$ is the selected aperture.
In Fig. 5.1 we show our results for the propagation of an abruptly autofocusing beam following a parabolic trajectory with $\alpha=2$ and $\beta=1$. In the left column we depict the spectrum along with the initial amplitude as an inset Fig. [5.1 (a)], while the amplitude dynamics of the propagating beam with the prediction of the trajectory (white-black dashed curve) are presented in (b). In (c) we compare the numerically obtained intensity contrast along the propagation distance, with the theoretical model developed. More specifically, we utilize Eq. (5.12) for $z \leq z_{f}$ and Eq. (5.16) for $z>z_{f}$. Our numerical results agree well with the designed theoretical model, although some deviations exist in the region near the focus where the theoretically predicted curve is steeper than the numerical. Furthermore, a slightly increased contrast is predicted from our theoretical estimations. We argue that both of those deviations arise due to the diffraction effects that have not taken into account in our theoretical model. In the right panel various intensity cross-sections are presented. Each intensity profile shown, is selected at a propagation distance indicated by the white-dashed lines drawn in Fig. [5.1(b)]. In the upper row we depict a couple of intensity profiles selected at propagation distances before the focus. As our approximation methods were applied in a region near the caustic, our theoretical model [Eq. (5.8)] describes accurately the main lobe, while minor deviations occur for the subsequent lobes. In the lower row the intensity profiles shown are selected exactly at the focus and after the focus respectively. For the theoretical predicted curves we have utilized Eq. (5.12). We see that, at the focus the theoretical values exceed slightly the numerical values, as we mentioned earlier, while after the focus the agreement between the numerical
data and the theoretical model is excellent. We note that all theoretical predictions are represented by circles.


FIGURE 5.1: An abruptly autofocusing beam following a parabolic trajectory with $\alpha=2$, $\beta=1, r_{0}=20, c=0.06$ and $w_{1}=1$. In (a) we see the spectrum and the amplitude profile on the input plane. In (b) the three-dimensional wave dynamics are depicted along with the theoretical prediction for the trajectory (white-black dashed curve). In (c) the intensity contrast is presented as a function of the propagation distance along with the theoretical prediction (shown in circles). In the last two columns we depict the intensity profile of the horizontal cross sections shown in (b) with the theoretical predictions shown in circles. Specifically in the upper row the cross sections are taken before the focus, and the analytic expression is obtained from Eq. (5.8). In the lower row, (f) is computed at the focus, and (g) is computed after the focus, whereas the theoretical estimates are given by Eq. (5.12).

To continue with, let us consider the propagation of an abruptly autofocusing beam following a different power-law trajectory. In Fig. (5.2) we show typical results for the case of a cubic trajectory with $\alpha=3$ and $\beta=1$. The formation of this figure is the same as before. More specifically, in the left panel we depict the spectrum and the initial amplitude as an inset [Fig. 5.2(a)], while in (b) the amplitude dynamics with the predesigned trajectory (white-black dashed curve) are presented. In (c) we show the comparison of the numerically obtained intensity contrast as a function of the propagation distance, with the theoretical model developed. For our theoretical predictions we have utilized Eq. (5.12) for $z \leq z_{f}$ and Eq. (5.16) for $z>z_{f}$. Numerical results obtained agree well with the theoretical estimations, although some deviations appear near the focus, where the slope of the theoretically predicted intensity contrast is steeper than the numerical. Furthermore, from our theoretical model, higher maximum intensity contrast values were predicted. In the right panel we present various intensity cross sections selected at different propagation distances, indicated
by the white dashed lines in Fig. [5.2(b)]. In the upper row the intensity profiles depicted are selected before the focus. Our designed formulas describe accurately not only the main lobe of the optical wave, but also the oscillatory frequency of the subsequent lobes. Although, some deviations appear in the amplitude of the latter. In the lower row, the transverse intensity profiles shown are selected at the focus and after the focus respectively. Our numerical results are in excellent agreement with the theoretical model, except from the slightly higher intensity values predicted at the focus. We note that in Fig. [5.2(e)], the maximum intensity contrast is enhanced compared to parabolic case, and this is explained by the higher curvature of the cubic trajectory at the focus, along with the larger value of $\rho_{c}$.


FIGURE 5.2: An abruptly autofocusing beam following a cubic trajectory with $\alpha=3, \beta=1$, $r_{0}=10, c=0.05$ and $w_{1}=1$. In (a) we see the spectrum and the amplitude profile on the input plane. In (b) the three-dimensional wave dynamics are depcited along with the theoretical prediction for the trajectory (white-black dashed curve). In (c) the intensity constrast is presented as a function of the propagation distance along with the theoretical prediction (shown in circles). In the last two columns we depict the intensity profile of the horizontal cross sections shown in (b) with the theoretical predictions shown in circles. Specifically in the upper row the cross sections are taken before the focus, and the analytic expression is obtained from Eq. (5.12). In the lower row, (f) is computed at the focus and (g) is computed after the focus, whereas the theoretical estimates are given by Eq. (5.16).

As we mentioned in the introduction of this chapter, beyond the control of the trajectory and the intensity contrast of abruptly autofocusing waves, we are also interested in optimizing their focusing properties. In this respect, we would like to damp the oscillatory behavior of the propagating beam after the focus, while an enhanced intensity contrast at the focus will be highly desirable. To accomplish the first of our goals we can decrease the aperture properly, so as the the amplitude


FIGURE 5.3: Intensity contrast as a function of the propagation distance. In the insets the initial amplitude profiles are shown. The parameters for the trajectory are the same as in Fig. 5.2. In (a) the amplitude on the input plane is the same as in Fig. 5.2, but the aperture is reduced to $r_{a}=45$ and $w_{2}=1$. In (b) the amplitude is constant $c=0$, whereas $r_{a}=45$ and

$$
w_{2}=5 .
$$

of the oscillations after the focus is reduced, while the maximum intensity contrast remains unaffected. In Fig. 5.3 we use the same parameters presented in the case of the cubic trajectory. Thus, using Fig. [5.2 (c)] as a reference, we can conclude that the dampening of the oscillatory behavior of the optical wave is achieved, while the maximum intensity contrast is the same. This is due to the reduced aperture assumed, [compare the inset of Fig. [5.3(a)] with the inset of Fig. [5.2(a)]. In order to obtain larger maximum intensity contrast at the focus, we firstly utilize Eq. (5.14) in order to obtain

$$
\begin{equation*}
\frac{I_{\max }\left(z=z_{f}\right)}{I_{\max }(z=0)}=17.98\left(\frac{\left(k^{2} \kappa\right)^{1 / 3} \rho_{c} A\left(\rho_{c}\right)}{\max [A(\rho)]}\right)^{2} . \tag{5.22}
\end{equation*}
$$

From the equation above we deduce that the contrast at the focus depends on $\rho_{c}$, on the curvature of the trajectory at the focus $\kappa\left(z_{c}\right)$, and on the fraction of the amplitude at the focus over the maximum value of the initial amplitude, $A\left(\rho_{c}\right) / \max [A(\rho)]$. At this point, we should highlight that a trajectory with different geometrical features such as increased curvature at the focus and larger values of $\rho_{c}$, would lead to an
enhanced intensity contrast at the focus. However, in our simulation we utilized the same parameters as in the case of the cubic trajectory and achieved larger values of the intensity contrast, by increasing the value of $A\left(\rho_{c}\right) / \max (A)$ up to unity. More specifically, in Fig. [5.3(b)] we select a constant amplitude while we radically reduce the aperture compared to Fig. 5.2. This results to an enhanced intensity contrast at the focus, while we also reduced the amplitude of the oscillations after the focus. In both figures Eq. (5.22) is represented by circles.

## Chapter 6

## Tunable self-similar Bessel-like

## beams of arbitrary order

### 6.1 Introduction

Propagation-invariant (PI) beams exhibiting an invariant intensity profile as they propagate, have attracted a lot of interest. However, such feature does not come at no cost. In their ideal form, PI beams carry infinite energy and thus are not experimentally feasible. Finite power truncations of such beams are experimentally realizable, and can preserve their robust propagation character for several diffraction lengths. There are two main classes of PI optical fields, namely the Bessel beams proposed by Durnin [5, 6], and the Airy beams. When considering those two families of beams, their most important characteristic is the high intensity lobe, which propagates along a straight line and a parabola respectively. Evidently, the main lobe of PI fields is of significant importance and its properties have been extensively studied for both theoretical and experimental purposes. In this respect, several applications benefit from the use of PI beams, ranging from particle manipulation [52], microscopy and imaging [53, 54], and free-space optical communications [55].

Beyond zeroth order Bessel beams which have their maximum intensity ring positioned on the propagation axis, all the other higher order beams exhibit a zero on-axis intensity surrounded by concentric rings. The dark spot observed at their center is expressed through the azimuthial phase term $\exp (i n \theta)$, and is associated with the order $n$ and the intrinsic vorticity of the Bessel beam. Higher order Bessel beams carrying Orbital Angular Momentum (OAM) first realized experimentally in
[56], but later works achieved higher efficiency by using an axicon [57, 58, 59]. This led to exciting applications in the areas of plasma generation and filamentation [60], particle manipulation [44, 61], and free space optical communications [62].

Engineering the properties of Bessel beams has been an issue of fundamental importance in terms of applications. In this respect, several works have considered how to engineer the trajectory of Bessel beams. Specifically, the idea of snaking a beam was proposed in [63]. Spiraling Bessel beams were proposed and observed in [64, 65]. Similar principles were used in [66] for a snaking beam capable of propagating around obstacles. Helicon beams result from the superposition of standard Bessel beams [67, 68, 69]. A generic approach that addresses the problem of generating a Bessel beam that follows, not particular classes of paths, but generic arbitrary trajectories was proposed in [24]. It was followed by an experimental observation [25] and generalizations in the non-paraxial domain [37] and in the case of vortex Bessel-like beams [44]. In [70] a technique was proposed to engineer the axial profile of Bessel beams (axial intensity and lateral cross section) in the Fourier space. Pin beams, a class of Bessel-like beams with engineered width that decreases with the propagation distance, exhibit robust propagation through atmospheric turbulence over kilometric distances [12].

Here we propose a method for the generation of vortex Bessel-like beams with the tunable parameters being the hollow core radius and the maximum amplitude. This is achieved by engineering the amplitude and the phase of the optical wave on the input plane in real space. Our method is also applied in the case of zeroth order Bessel-like beams, in which case the width as well as the axial (maximum) intensity are fully controllable. The optical waves considered here take an invariant form under suitable stretching transformations and, thus, they can be considered as self-similar. Our theoretical results are in excellent agreement with direct numerical simulations.

### 6.2 Amplitude and width engineering of zeroth-order Bessel beams

Let us start by considering the Fresnel diffraction integral for the dynamics of an optical beam in a dielectric medium in cylindrical coordinates

$$
\begin{equation*}
\psi(r, \theta, z)=\frac{e^{i \frac{k r^{2}}{2 z}} e^{i n \theta}}{i \lambda z} \int_{0} d \rho \int_{0}^{2 \pi} d s A(\rho) \rho e^{i \phi-i n s+i k \frac{\rho^{2}-2 r \rho \cos s}{2 z}} \tag{6.1}
\end{equation*}
$$

where $k=2 \pi / \lambda$ is the wavenumber, $\lambda$ is the optical wavelength, $(r, \theta, z)$ are cylindrical coordinates with $(r, \theta)$ being the transverse polar parameters and $z$ being the longitudinal distance, $(\rho, \xi)$ are the polar coordinates on the input plane, and $s=\theta-\xi$. In Eq. (6.1) the initial condition is decomposed into amplitude and phase as $\psi_{0}=A(\rho) e^{i \phi(\rho)+i n \xi}$, where $n$ is the topological charge. Let us first focus in the case of beams without vorticity $n=0$. By making the assumptions that (i) the amplitude $A(\rho)$ is a slowly varying function, (ii) a single ray emerges from each radial location $\rho$, and (iii) $r \ll z /(k \rho)$, we derive the ray equation

$$
\begin{equation*}
\phi^{\prime}(\rho)=-k \rho / z \tag{6.2}
\end{equation*}
$$

and the following relation for the dynamics of the optical wave [15]

$$
\begin{equation*}
\psi(r, z)=\frac{\rho A(\rho) e^{i \Psi_{0}}}{i z} \frac{(2 \pi k)^{1 / 2}}{\left|\frac{1}{z}-\frac{1}{z_{c}}\right|^{1 / 2}} \mathrm{~J}_{0}\left(\frac{k r \rho}{z}\right) \tag{6.3}
\end{equation*}
$$

In Eq. (6.3) $\Psi_{0}=\phi(\rho)+k\left(r^{2}+\rho^{2}\right) /(2 z)+\mu \pi / 4, \mu=\operatorname{sgn}\left(1 / z-1 / z_{c}\right)$, and we have defined

$$
\begin{equation*}
z_{c}=-k / \phi^{\prime \prime}(\rho) \tag{6.4}
\end{equation*}
$$

In addition, we define the width of the Bessel beam as

$$
\begin{equation*}
W=2 z /(k \rho) \tag{6.5}
\end{equation*}
$$

After some calculations, we can express Eq. (6.3) in the simpler form

$$
\begin{equation*}
\psi(r, z)=\left(2 \pi k \rho \rho^{\prime}(z)\right)^{1 / 2} \frac{A(\rho) e^{i \psi_{0}}}{i} \mathrm{~J}_{0}\left(\frac{2 r}{W}\right) \tag{6.6}
\end{equation*}
$$

where we have selected

$$
\begin{equation*}
\rho^{\prime}(z)>0 \tag{6.7}
\end{equation*}
$$

and thus $\mu=\operatorname{sgn}\left(\rho^{\prime}(z)\right)=1$. We relate the full width at half intensity maximum (FWHM) with $W$ via $W_{0}=w_{0} W$, where $w_{0}$ is the width of the main lobe of $J_{0}(2 r)$ at half intensity maximum.

The ray picture of the zeroth-order Bessel-like beams consists of rays emitted from expanding concentric circles with different inclinations that intersect along a focal line that passes perpendicularly through the center of the circles. The condition given by Eq. (6.7) results to a $1-1$ correspondence between the location of the ray on the input plane and the on-axis focal distance $z$. Furthermore, since $\rho^{\prime}(z)>0$ then as $\rho$ increases $z$ also increases. The scenario where $\rho^{\prime}(z)<0$ also provides such a $1-1$ correspondence but, in this case, rays from smaller $\rho$ focus at larger distances $z$. This leads to the problem of ray interference that reduces the quality of the resulting beam. A more convenient form of Eq. (6.7) is

$$
\begin{equation*}
W(z)-W^{\prime}(z) z>0 . \tag{6.8}
\end{equation*}
$$

The on-axis maximum amplitude of the Bessel beam is related to the amplitude on the input plane as

$$
\begin{equation*}
U(z)=\left(2 k \pi \rho \rho^{\prime}(z)\right)^{1 / 2} A(\rho) . \tag{6.9}
\end{equation*}
$$

The above calculations can be used to generate Bessel beams with preassigned width $W(z)$ and maximum amplitude $U(z)$ as a function of the propagation distance. Depending on the complexity of $W(z)$ the calculations can be carried out analytically or numerically.

Let us discuss some specific examples. For a power-law beam width

$$
W(z)=a+b z^{c}
$$

Eq. (6.8) is satisfied as long as $a+b(1-c) z^{c}>0$. In particular, for a linearly varying (increasing or decreasing) beam width $(c=1)$ the phase $\phi(\rho)=\rho(k b \rho-4) /(2 a)$ is single valued provided that $a>0$. On the other hand, if $a=0$ Eq. (6.8) is satisfied when $c<1$. The required phase on the input plane is then given by $\phi=-k \rho^{2}[(1-c) /(1-2 c)](2 /(k b \rho))^{1 /(1-c)}$ for $c \neq 1 / 2$, whereas for $c=1 / 2$ we have $\phi=-4 \log (\rho) /\left(b^{2} k\right)$. In the particular case where $c=-1$ the width is inversely proportional to the propagation distance. These solutions are called pin-like beams and were recently examined in detail in [11]. Note that for $c=-1$, the phase exponent $3 / 2$ is characteristic of the Airy beam.

### 6.3 Numerical Results

In all our simulations we normalize the transverse radial coordinate according to $r_{0}$ and the longitudinal coordinate according to $z_{0}=k r_{0}^{2}$. In addition, since in many cases $A(\rho) \rightarrow \infty$ as $\rho \rightarrow 0$, to eliminate the artificial singularity, the amplitude on the input plane is multiplied with a hyperbolic tangent function with argument proportional to $\rho$. In Fig. (6.1) we select the beam width to increase linearly with the propagation distance $(W=a+b z)$. We see that there is an excellent agreement between the theoretical and the numerical results. Note that the width of the Bessel beam on the output is 6 times larger as compared to the initial plane.

### 6.4 Amplitude and hollow-core radius engineering of higherorder Bessel beams

In the case of vortex Bessel-like beams, rather than engineering the width of the main (first) lobe of the beam, it is more useful to engineer the inner radius of the cylindrical high intensity surface that surrounds the void region as a function of the propagation distance $R_{f}(z)$. In particular, the calculated value of $\left|\psi\left(R_{f}\right)\right|^{2}$ is selected to be half of the maximum intensity of the first Bessel ring. We start by applying a stationary phase approximation to the radial coordinate, $\rho$, of Eq. (6.1). As a result we obtain the ray equation

$$
\begin{equation*}
\phi^{\prime}(\rho)+k(\rho-r \cos s) / z=0 \tag{6.10}
\end{equation*}
$$



FIGURE 6.1: First row: (a) Propagation dynamics of a self-similar zeroth-order Bessel beam whose width is set to increase linearly as $W+a+b z$ for $a=0.05, b=0.006, w_{0}=1.13$ and $U(z)=1$. In (b) and (c) the maximum intensity and FWHM along $z$ are shown respectively. Second row: Various intensity cross sections are selected at propagation distances indicated by the white dashed-dotted lines in (a). Numerical (theoretical) results are represented by green solid line (black circles) in the corresponding subfigures.
which is utilized to integrate the Fresnel integral over $\rho$. Subsequently, we apply a stationary phase approximation to the angular variable, $s$, leading to the additional relation for the ray dynamics

$$
\begin{equation*}
n=k r \rho \sin s / z \tag{6.11}
\end{equation*}
$$

By directly integrating the Fresnel integral over $s$, we derive the following equation for the amplitude dynamics

$$
\begin{equation*}
\psi(r, \theta, z)=\frac{\rho A(\rho) e^{i \Psi_{n}}}{i^{1+n_{z}}} \frac{(2 \pi k)^{1 / 2}}{\left|\frac{1}{z}-\frac{1}{z_{c}}\right|^{1 / 2}} \mathrm{~J}_{n}\left(\frac{k r \rho}{z}\right) \tag{6.12}
\end{equation*}
$$

where $\Psi_{n}=\Psi_{0}+n \theta$. It is worth pointing out that Eq. (6.12) can be considered as a generalization of Eq. (6.3). The trajectory of a ray on the $r-z$ plane is independent from the launch angle on the input plane. In particular, by eliminating the angle $s$ from the ray Eqs. (6.10)-(6.11), we find that the rays follow the hyperbolic trajectory

$$
\begin{equation*}
r^{2}=\left(\rho+\frac{z}{k} \phi^{\prime}(\rho)\right)^{2}+\left(\frac{n z}{k \rho}\right)^{2} \tag{6.13}
\end{equation*}
$$

We can determine the location of the focal ring from the relation $d r^{2} / d z=0$ leading to

$$
\begin{equation*}
\left(z, r_{f}\right)=\left(-\frac{k \phi^{\prime}(\rho) \rho^{3}}{n^{2}+\left(\phi^{\prime}(\rho)\right)^{2} \rho^{2}}, \frac{|n| \rho}{\left(n^{2}+\left(\phi^{\prime}(\rho)\right)^{2} \rho^{2}\right)^{1 / 2}}\right) \tag{6.14}
\end{equation*}
$$

Furthermore, by using Eq. (6.13) we can now express $\rho$ as a function of the propagation distance

$$
\begin{equation*}
\rho(z)=\left[r_{f}^{2}(z)+\left(\frac{n z}{k r_{f}(z)}\right)^{2}\right]^{1 / 2} \tag{6.15}
\end{equation*}
$$

The ray picture of the higher-order Bessel-like beams proposed here, consists of rays emitted at skewed angles from expanding concentric circles with different inclinations that generate hyperbolic surfaces. The minimum radius of the rays from the axis $r_{f}$ is achieved at $z_{f}$.

As in the case of zeroth order Bessel-like beams, we define the width of a vortex Bessel-like beam as

$$
\begin{equation*}
W=\frac{2 z}{k \rho}=-\frac{2 \phi^{\prime}(\rho) \rho^{2}}{n^{2}+\left(\phi^{\prime}(\rho)\right)^{2} \rho^{2}} . \tag{6.16}
\end{equation*}
$$

The derivative of the phase is related to the vortex trajectory through

$$
\begin{equation*}
\phi^{\prime}(\rho)=-n^{2} z /\left(k \rho r_{f}^{2}\right) \tag{6.17}
\end{equation*}
$$

We also satisfy the constraint for an increasing $\rho(z)$ [Eq. (6.8)].
Simplified formulas are derived by utilizing the inequality $\left|\phi^{\prime}(\rho)\right| \rho \gg|n|$, which is equivalent to the assumption that the radius of a ray on the input plane is much larger than the focal radius, $\rho \gg r_{f}$. We would like to point out that this latter approximation is valid in most of the relevant cases. Thus we obtain the following relation

$$
\left(z, r_{f}\right)=\left(-k \rho / \phi^{\prime}(\rho),-|n| / \phi^{\prime}(\rho)\right)
$$

for the focal coordinates. Note that $\left|\psi\left(r_{f}\right)\right|$ is in very good agreement with $\left|\psi\left(R_{f}\right)\right|$ and thus $r_{f} \approx R_{f}$. The largest deviation occurs for $n=1\left(R_{f} / r_{f} \approx 0.91\right)$ while for $n=2$ we have $R_{f} / r_{f} \approx 0.98$. In addition the width of the Bessel-like beam takes the simple form

$$
\begin{equation*}
W=-2 / \phi^{\prime}(\rho)=2 r_{f} /|n| . \tag{6.18}
\end{equation*}
$$

From Eq. (6.18), we see that $W$ and $r_{f}$ are proportional, as expected due to the selfsimilar nature of the solutions. Also from Eqs. (6.17), (6.18), we obtain the interesting relationship $r_{f}=(n / k)(z / \rho)$. It is worth mentioning that, in this approximation, the relation between the width $W$ and the phase $\phi$ is identical to the case of zero vorticity. Following the relevant calculations, it can be shown that the beam dynamics is given by

$$
\begin{equation*}
\psi(r, \theta, z)=\left(2 \pi k \rho \rho^{\prime}(z)\right)^{1 / 2} \frac{A(\rho) e^{i \Psi_{n}}}{i^{1+n}} \mathrm{~J}_{n}\left(\frac{2 r}{W}\right) . \tag{6.19}
\end{equation*}
$$

Equation (6.19), which holds when $\left|\phi^{\prime}(\rho) \rho\right| \gg|n|$, can be considered as a generalization of Eq. (6.6) for nonzero values of $n$. In order to achieve the intended maximum amplitude $U(z)$ along the propagation distance, we choose the amplitude on the input plane as

$$
\begin{equation*}
A(\rho)=U(z) /\left[\left(2 \pi k \rho \rho^{\prime}(z)\right)^{1 / 2} c_{1}\right] \tag{6.20}
\end{equation*}
$$

where $c_{1}=\max \left|/ J_{n}(r)\right|$.
We would like to point out that it is possible to extend the propagation distance of these solutions beyond the critical value $z=z_{\text {cr }}$ after which the solution does not satisfy Eq. (6.7). In particular, we can select a value $z_{m}<z_{\text {cr }}$ such that for $z>z_{m}$ the Bessel-like beam becomes a regular Bessel beam (having constant width).

### 6.5 Numerical Results

We have performed numerical simulations for different classes of self-similar vortex Bessel-like beams. In the first example shown in Fig. 6.2, we have selected the vortex radius to increase in a linear manner as $r_{f}=a+b z$ with positive $a$ and $b$, the maximum amplitude to be constant $U=1$, and the topological charge $n=3$. Interestingly, in accordance to our theoretical model, a 6 times increment over the initial radius and beam-width is achieved. We see that our theoretical predictions are in excellent agreement with our numerical results.

In the second example shown in Fig. 6.3, we have selected a vortex with topological charge $n=2$, and the core radius to decrease linearly with the propagation distance ( $r_{f}=a+b z$ with $a$ positive and $b$ negative). In addition, the maximum


FIGURE 6.2: First row: (a) Propagation dynamics of a self-similar Bessel beam carrying OAM whose width is set to increase linearly. Here $a=0.05, b=0.006, n=3$, and $U(z)=1$. The hollow core radius is represented by black dashed lines. In (b) and (c) the maximum intensity and $R_{f}$ along $z$ are shown respectively. Second row: Various intensity cross sections are selected at propagation distancs indicated by the white dashed-dotted lines in (a). Numerical (theoretical) results are represented by green solid line ( black circles) in the corresponding subfigures.
amplitude $U$ is the sum of a constant and a sinusoidal function. During propagation the initial radius decreases by $75 \%$.


FIGURE 6.3: First row: (a) Propagation dynamics of a self-similar Bessel beam carrying OAM whose width is set to decrease linearly as $W=a-b z$. Here, $a=0.5$ and $b=0.0075, n=-2$, and $U(z)=1+0.5 \sin (3 \pi z / 50)^{2}$. The hollow core radius is represented by black dashed lines. In (b) and (c) the maximum intensity and $R_{f}$ along $z$ are shown respectively. Second row: Various intensity cross sections are selected at propagation distances indicated by the white dashed-dotted lines in (a). Numerical (theoretical) results are represented by green solid line (black circles) in the corresponding subfigures.

In the final example, we have selected a hyperbolic secant modulation of the
hollow core radius. In Fig. 6.4, we depict results for the propagation of such a first order vortex Bessel beam, where the maximum intensity is selected to be constant. To ensure that $\rho(z)$ is an increasing function we have tested that Eq. (6.8) is fulfilled.


Figure 6.4: First row: (a) Propagation dynamics of a self-similar Bessel beam carrying OAM whose width is set to modulate as $W=a+b \operatorname{sech}\left(\gamma\left(z-z_{0}\right)\right)$. Here, $a=0.025$ and $b=\gamma=0.1, z_{0}=25, n=1$, and $U(z)=1$. The hollow core radius is represented by black dashed lines. In (b) and (c) the maximum intensity and $R_{f}$ along $z$ are shown respectively. Second row: Various intensity cross sections are selected at propagation distances indicated by the white dashed-dotted lines in (a). Numerical (theoretical) results are represented by green solid line ( black circles) in the corresponding subfigures.

## Chapter 7

## Conclusions

In the context of this dissertation, we study from a theoretical perspective the parameters defining the propagation of accelerating waves and how to engineer them properly. Now at the end of this work, let us summarize our main results.

In Chapter 3 we showed that the phase of an optical wave can be properly engineered, in order to design beams that accelerate along convex but otherwise arbitrary predefined paraxial trajectories. In addition, we obtained explicit formulas describing the behavior of the optical wave near the caustic. Our solutions proved to be of significant importance in developing expressions for the amplitude and the width of the propagating beam. More specifically, we showed that the maximum amplitude along the propagation distance depends on the initial amplitude and the geometrical characteristics of the trajectory, while the width of the optical wave is solely related to the curvature of the trajectory. To sum up with, we demonstrated a simple and efficient method to generate beams with predefined trajectory and beam width, while their maximum amplitude is fully-controllable.

In Chapter 4 we generalized our results presentd in the previous Chapter, to the case of nonparaxial beams following curved but otherwise predefined arbitrary trajectories. In the particular case of circular, elliptic and general power-law trajectories we obtained closed form expressions for the phase of the beam on the input plane. Beyond the trajectory engineering, we showed that it is also possible to predesign both the amplitude and the width of such beams. In order to accomplish this, we engineer both the phase and the amplitude on the input plane. Our analytic solutions demonstrate that independently of the trajectory assumed, the dynamics of the beam in a region close to the caustic are described by an Airy function. Importantly, we
showed that the width of the optical wave solely depends on the curvature of the trajectory, while the amplitude along the propagation distance is related to the initial amplitude and the geometrical features of the trajectory.

In Chapter 5 we designed abruptly autofocusing waves which accelerate along arbitrary power-law trajectories. We obtained analytic formulas describing accurately the propagation of such beams and identified the important parameters that affect their focal characteristics. Importantly, we can engineer those parameters in order to design abruptly autofocusing waves with higher intensity contrast at the focus, along with damped oscillatory behavior after their focal point. We conclude that in this case too, the propagation features such as amplitude and beam width are directly related to the geometrical characteristics of the trajectory.

In Chapter 6 we demonstrated a simple and efficient method to generate Bessellike beams of integer order, exhibiting tunable propagation defining characteristics. More specifically, we showed that we can engineer the beam-width of a zeroth-order Bessel beam, while for their higher order counterparts their hollow-core radius can be predesigned. Additionally, in both cases, we can also preselect the maximum amplitude of the beam along the propagation distance. To accomplish this, we must properly engineer the amplitude and the phase of the beam on the input plane. By applying asymptotic methods, we were able to extract closed form expressions describing the propagation of such beams. In this direction, we also address the physical constraints that arise through our procedure. Such restrictions dictate that the width/hollow-core radius of the beams should always be positive, while the rays forming the beams should be emanating from expanding concentric circles. Finally, our theoretical models developed were verified via numerical simulations for a variety of different cases.

## Appendix A

## Propagation dynamics of paraxial accelerating beams

In this Appendix we will demonstrate some basic aspects of the mathematical tools utilized in order to describe the propagation dynamics of accelerating beams in the paraxial domain. Beyond these mathematical formulas, we will also present how our solutions are directly linked to the geometrical features of the beam's trajectory.

We begin our analysis by recalling that the total phase involved in the Fresnel integrand [Eq.(3.1)] is given by,

$$
\Psi(\xi ; x, z)=\phi(\xi)+k \frac{(x-\xi)^{2}}{2 z}
$$

As we showed in the main text, in order to describe the dynamics of the beam in the neighborhood of the caustic, we expand the total phase in a Taylor series around an arbitrary point $\left(\xi_{c}, x_{c}\right)$ and retain terms up to third order

$$
\begin{equation*}
\Psi\left(\xi_{c}+\delta \xi, x_{c}+\delta x\right) \approx\left[\Psi_{c}+\partial_{x} \Psi_{c}(\delta x)+\frac{\partial_{x}^{2} \Psi_{c}}{2}(\delta x)^{2}\right]+\partial_{\xi x} \Psi_{c}(\delta \xi)(\delta x)+\frac{\partial_{\xi}^{3} \Psi_{c}}{6}(\delta \xi)^{3} \tag{A.1}
\end{equation*}
$$

where we have taken into account that $\partial_{\xi} \Psi_{c}=\partial_{\tilde{\xi}}^{2} \Psi_{c}=0$ due to the first and secondorder stationarity of the phase, and $\partial_{\xi \xi \xi x} \Psi_{c}=\partial_{\xi x x} \Psi_{c}=\partial_{x}^{3} \Psi_{c}=0$. Furthermore, by directly substituting to the total phase we also obtain $\partial_{\xi \overline{ } x} \Psi_{c}=-k / z_{c}=\phi^{\prime \prime}(\xi)$ and $\partial_{\tilde{\xi}}^{3} \Psi_{c}=\phi^{\prime \prime \prime}(\xi)$, where for simplicity we have replaced $\xi_{c}$ with $\xi$.

Utilizing the perturbed total phase obtained above, the Fresnel integral under the slowly varying envelope approximation can be expressed by

$$
\begin{equation*}
\psi\left(x_{c}+\delta x, z_{c}\right)=\frac{A(\xi)}{\left(i \lambda z_{c}\right)^{1 / 2}} e^{i \Phi} \int_{-\infty}^{\infty} \exp \left[i\left(\phi^{\prime \prime}(\xi)(\delta \xi)(\delta x)+\frac{1}{6} \phi^{\prime \prime \prime}(\xi)(\delta \xi)^{3}\right)\right] d(\delta \xi) \tag{A.2}
\end{equation*}
$$

where $\Phi$ is the terms in brackets in Eq. (A.1).
In order to obtain an analytic solution of the above integral, we will now present our methodology resulting to an Airy-type solution. This should be expected as the phase of the integrand is consisted by a third order polynomial. While the integral representation of an Airy function can be directly applied in this case as

$$
\begin{equation*}
\operatorname{Ai}(a x)=\frac{1}{2 \pi \alpha} \int_{-\infty}^{\infty} \exp \left[i\left(\frac{v^{3}}{3 \alpha^{3}}+x v\right)\right] d v, \tag{A.3}
\end{equation*}
$$

we will present the more general case of an arbitrary cubic polynomial in the form, $a t+b t^{2}+\frac{c}{3} t^{3}$. Obviously in order to exploit Eq. (A.3) we need to eliminate the quadratic term. In this respect we assume the substitution $t=u-\frac{b}{c}$, and thus

$$
\begin{aligned}
& a\left(u-\frac{b}{c}\right)+b\left(u-\frac{b}{c}\right)^{2}+\frac{c}{3}\left(u-\frac{b}{c}\right)^{3}= \\
& a u-\frac{a b}{c}+b u^{2}-\frac{2 b^{2}}{c} u+\frac{b^{3}}{c^{2}}+\frac{c}{3} u^{3}-b u^{2}+\frac{b^{2}}{c} u-\frac{b^{3}}{3 c^{2}} . \\
& =\frac{c}{3} u^{3}+\left(\frac{a c-b^{2}}{c}\right) u+\frac{2 b^{3}-3 a b c}{3 c^{2}}
\end{aligned}
$$

Finally we can express the integral of $\exp \left[i\left(a t+b t^{2}+c t^{3} / 3\right)\right]$ as

$$
e^{i\left(2 b^{3}-3 a b c\right) /\left(3 c^{2}\right)} \int_{-\infty}^{\infty} \exp \left[i\left(\frac{c}{3} u^{3}+\frac{a c-b^{2}}{c} u\right)\right] d u
$$

which when compared to Eq. (A.3) leads to $\alpha=c^{-1 / 3}$ and $x=\left(a c-b^{2}\right) / c$. To conclude, we obtain that

$$
\begin{equation*}
\int_{-\infty}^{\infty} e^{i\left(a s+b s^{2}+c s^{3} / 3\right)} d s=\frac{2 \pi e^{i\left(2 b^{3}-3 a b c\right) /\left(3 c^{2}\right)}}{|c|^{1 / 3}} \operatorname{Ai}\left(\frac{a c-b^{2}}{|c|^{4 / 3}}\right) \tag{A.4}
\end{equation*}
$$

As a direct consequence of the result obtained in Eq. (A.4), we state that Eq. (A.2) admits solutions of the Airy type which are described by

$$
\begin{equation*}
\psi\left(x_{c}+\delta x, z_{c}\right)=\frac{2 \pi A(\xi)}{\left.\left(i \lambda z_{c}\right)^{1 / 2} \mid \phi^{\prime \prime \prime}(\xi) / 2\right)\left.\right|^{1 / 3}} \operatorname{Ai}\left(\operatorname{sgn}\left(\left(\phi^{\prime \prime \prime}(\xi)\right) \frac{\phi^{\prime \prime}(\xi)}{\left|\phi^{\prime \prime \prime}(\xi) / 2\right|^{1 / 3}}(\delta x)\right) .\right. \tag{A.5}
\end{equation*}
$$

Interestingly we can express our results in terms of the geometrical features of the beam's trajectory. To begin with, we recall that the curvature of the trajectory is defined as $\kappa=\left|\frac{d^{2} f\left(z_{c}\right)}{d z_{c}^{c}}\right|$ and $s=\operatorname{sgn}\left[\kappa\left(z_{c}\right)\right]$ is the sign of the curvature. We begin our analysis by recalling that $\xi=f\left(z_{c}\right)-z_{c} f^{\prime}\left(z_{c}\right)$, where $f\left(z_{c}\right)$ is the beam's trajectory. Taking the derivative with respect to $z_{c}$ we obtain that $f^{\prime \prime}\left(z_{c}\right)=-\frac{1}{z_{c}} d z_{c}$. Thus the curvature can be rewritten as

$$
\begin{align*}
\kappa & =\left|f^{\prime \prime}\left(z_{c}\right)\right|=\left|-\frac{1}{z_{c}} \frac{1}{\frac{d z_{c}}{d \xi}}\right|=\left|\frac{\phi^{\prime \prime}(\xi)}{k} \frac{1}{\frac{d\left(-\frac{k}{\phi^{\prime}(\xi)}\right)}{d \xi}}\right|=\left|\frac{\phi^{\prime \prime}(\xi)}{k} \frac{\phi^{\prime \prime}(\xi)^{2}}{k \phi^{\prime \prime \prime}(\xi)}\right|  \tag{A.6}\\
& =\left|\frac{\phi^{\prime \prime}(\xi)^{3}}{2 k^{2}\left(\phi^{\prime \prime \prime}(\xi) / 2\right)}\right| \Rightarrow\left(2 k^{2} \kappa\right)^{1 / 3}=\frac{\left|\phi^{\prime \prime}(\xi)\right|}{\left|\phi^{\prime \prime \prime}(\xi) / 2\right|^{1 / 3}}
\end{align*}
$$

or equivalently

$$
\begin{equation*}
\left.\mid \phi^{\prime \prime \prime}(\xi) / 2\right)\left.\right|^{-1 / 3}=\frac{\left(2 k^{2} \kappa\right)^{1 / 3}}{\left|\phi^{\prime \prime}(\xi)\right|}=z_{c}\left(\frac{2 \kappa}{k}\right)^{1 / 3} \tag{A.7}
\end{equation*}
$$

Finally taking into account the last couple of equations, we can express Eq. (A.5) in terms of the geometrical features of the trajectory and the initial amplitude as

$$
\begin{equation*}
\psi\left(x_{c}+\delta x, z_{c}\right)=2 A(\xi)\left(\frac{\pi^{4} z_{c}^{3} \kappa^{2}}{\lambda}\right)^{1 / 6} e^{i \Xi} \operatorname{Ai}\left(s\left(2 k^{2} \kappa\right)^{1 / 3} \delta x\right) \tag{A.8}
\end{equation*}
$$

where $\Xi=\phi(\xi)+\frac{k\left(x_{c}-\zeta\right)^{2}}{2 z_{c}}-\frac{\pi}{4}$.

## Appendix B

## Propagation dynamics of

## nonparaxial accelerating beams

In this Appendix we will present an in-depth analysis regarding the propagation dynamics of nonparaxial accelerating beams. Here we show that independently of the assumed trajectory, the evolution of the optical wave can be described by Airy-type solutions which are directly linked to the geometrical features of the trajectory.

Let us begin with the Rayleigh-Sommerfeld integral

$$
\begin{equation*}
\psi(x, z)=2 \int_{-\infty}^{\infty} \psi_{0}(\xi) \frac{\partial G(x, z ; \xi)}{\partial z} d \xi \tag{B.1}
\end{equation*}
$$

where $\psi_{0}(x)=A(x) e^{i \phi(x)}$ is the optical wave on the input plane, $x$ is the transverse and $z$ is the longitudinal coordinate, $G(x, z ; \xi)=-(i / 4) H_{0}^{(1)}(k R)$, and $H_{0}^{(1)}$ is a Hankel function which can be approximated by $H_{0}^{(1)}(k R) \approx \sqrt{2 /(\pi k R)} e^{i(k R-\pi / 4)}$. Let us begin our analysis by calculating the derivative of $G$ with respect to $z$

$$
\begin{array}{r}
\frac{\partial G(x, z ; \xi)}{\partial z}=-(i / 4) \frac{\partial}{\partial z} H_{0}^{(1)}(k R)=-(i / 4) \frac{\partial}{\partial z}\left[\sqrt{\frac{2}{\pi k R}} e^{i(k R-\pi / 4)}\right] \\
=(-i / 4) \sqrt{\frac{2}{\pi k}} e^{-i \pi / 4}\left[\frac{i k z e^{i k R}}{R^{3 / 2}}-\frac{z e^{i k R}}{2 R^{5 / 2}}\right] e^{i k R}  \tag{B.2}\\
=(-i / 4) \sqrt{\frac{2}{\pi k}} e^{-i \pi / 4} \frac{i z(2 k R+i)}{2 R^{5 / 2}} e^{i k R} \\
=\frac{1}{4} \sqrt{\frac{2}{\pi k}} e^{-i \pi / 4} \frac{k z}{R^{3 / 2}} e^{i k R}
\end{array}
$$

where for the last step we have utilized that $2 k R \gg i$. Thus, Eq. (B.1) can be rewritten in the following form

$$
\begin{equation*}
\psi(x, z)=\sqrt{\frac{k}{2 \pi}} e^{-i \pi / 4} \int_{-\infty}^{\infty} z \frac{A(\xi)}{R^{3 / 2}} e^{i(k R+\phi)} d \xi \tag{B.3}
\end{equation*}
$$

The above equation is very useful in applying the method of stationary phase in order to approximate the dynamics of the propagating beam. In this respect, the total phase of the integrand is

$$
\begin{equation*}
\Psi(x, z ; \xi)=\phi(\xi)+k \sqrt{(x-\xi)^{2}+z^{2}} \tag{B.4}
\end{equation*}
$$

from where we can obtain the ray equation

$$
\begin{equation*}
\phi^{\prime}(\xi)=\frac{k(x-\xi)}{R} \Leftrightarrow x=\xi+R \phi^{\prime}(\xi) / k . \tag{B.5}
\end{equation*}
$$

Now by assuming that a line tangent to the trajectory $x_{c}=f\left(z_{c}\right)$ at an arbitrary point on the caustic $\left(x_{c}, z_{c}\right)$ describes the ray equation we obtain that

$$
\begin{equation*}
\xi=f\left(z_{c}\right)-z_{c} f^{\prime}\left(z_{c}\right) \tag{B.6}
\end{equation*}
$$

where we have also assumed that at $z=0 \Rightarrow x=\xi$. Interestingly we can express the derivative of the phase on the input plane with respect to the trajectory as

$$
\begin{equation*}
\phi^{\prime}(\xi)=\frac{k z_{c} f^{\prime}\left(z_{c}\right)}{\sqrt{z_{c}^{2} f^{\prime}\left(z_{c}\right)^{2}+z_{c}^{2}}}=\frac{k f^{\prime}\left(z_{c}\right)}{\sqrt{1+f^{\prime}\left(z_{c}\right)^{2}}} \tag{B.7}
\end{equation*}
$$

Furthermore, we define the curvature *of the trajectory as

$$
\begin{equation*}
\kappa\left(z_{c}\right)=\frac{\left|f^{\prime \prime}\left(z_{c}\right)\right|}{\left|1+f^{\prime}\left(z_{c}\right)^{2}\right|^{3 / 2}} \tag{B.8}
\end{equation*}
$$

where $f^{\prime \prime}\left(z_{c}\right)$ can be expressed through Eq. (B.6) as $f^{\prime \prime}\left(z_{c}\right)=-1 /\left[\left(z_{c}(\xi) z_{c}^{\prime}(\xi)\right]\right.$ and similarly $f^{\prime}\left(z_{c}\right)=\left(f\left(z_{c}\right)-\xi\right) / z_{c}$. By directly substituting the latter to Eq. (B.8) we can express the curvature of the selected trajectory as

$$
\begin{equation*}
\kappa\left(z_{c}\right)=\frac{\frac{1}{z_{c}}\left|\frac{1}{z_{c}^{\prime}(\xi)}\right|}{\left|1+\frac{\left(x_{c}-\xi\right)^{2}}{z_{c}^{2}}\right|^{3 / 2}}=\frac{z_{c}^{2}}{R_{c}^{3}\left|z_{c}^{\prime}(\xi)\right|} . \tag{B.9}
\end{equation*}
$$

To proceed with, we would like to obtain an expression describing the evolution dynamics of the propagating beam in the neighborhood of the caustic. To achieve this, we select an arbitrary point on the trajectory $\left(x_{c}, z_{c}\right)$ and expand the total phase in a Taylor series around $\left(\xi_{c}, x_{c}\right)$ and retain terms up to third order while we keep $z$ constant at $z=z_{c}$

$$
\begin{align*}
\Psi\left(\xi_{c}+\delta \xi, x_{c}+\delta x\right) & =\left[\Psi_{c}+\partial_{x} \Psi_{c}(\delta x)+\frac{1}{2} \partial_{x}^{2} \Psi_{c}(\delta x)^{2}+\frac{1}{6} \partial_{x}^{3} \Psi_{c}(\delta x)^{2}\right] \\
& +\partial_{\xi x} \Psi_{c}(\delta \xi)(\delta x)+\frac{1}{2} \partial_{\xi x x} \Psi_{c}(\delta \xi)(\delta x)^{2}+\frac{1}{2} \partial_{\xi \tilde{\xi} x} \Psi_{c}(\delta \xi)^{2}(\delta x) \\
& +\frac{1}{6} \partial_{\xi}^{3} \Psi_{c}(\delta \xi)^{3} . \tag{B.10}
\end{align*}
$$

In the above expansion we have considered that $\partial_{\tilde{\xi}} \Psi_{c}=0$ and $\partial_{\tilde{\xi}}^{2} \Psi_{c}=0 \Rightarrow \phi^{\prime \prime}(\tilde{\xi})=$ $-k z_{c}^{2} / R_{c}^{3}$. Now let us calculate each term of the Taylor series separately for clarity. Beginning from terms proportional only to $(\delta x)^{j}$ for $j=1,2,3$ we obtain

$$
\partial_{x} \Psi_{c}=\frac{k\left(x_{c}-\xi\right)}{R_{c}}=\phi^{\prime}(\xi), \quad \partial_{x}^{2} \Psi_{c}=\frac{k z_{c}^{2}}{R_{c}^{3}}=-\phi^{\prime \prime}(\xi), \quad \partial_{x}^{3} \Psi_{c}=-\frac{3 k z_{c}^{2}\left(x_{c}-\xi\right)}{R_{c}^{5}} .
$$

while for the terms proportional to $(\delta \zeta)^{j}(\delta x)^{k}$ for $k=0,1,2$

$$
\begin{aligned}
\partial_{\xi x} \Psi_{c} & =-\frac{k z_{c}^{2}}{R_{c}^{3}}=\phi^{\prime \prime}(\xi), & \partial_{\xi \xi \zeta} \Psi_{c}=\frac{3 k z_{c}^{2}\left(x_{c}-\xi\right)}{R_{c}^{5}}, \\
\partial_{\xi x x} \Psi_{c} & =-\frac{3 k z_{c}^{2}\left(x_{c}-\xi\right)}{R_{c}^{5}}, & \partial_{\xi}^{3} \Psi_{c}=\phi^{\prime \prime \prime}(\xi)+\frac{3 k z_{c}^{2}\left(x_{c}-\xi\right)}{R_{c}^{5}} .
\end{aligned}
$$

We note that in all the above expressions we have replaced $\xi_{c}$ with $\xi$ for brevity. Furthermore we observe that the term $3 k z_{c}^{2}\left(x_{c}-\xi\right) / R_{c}^{5}$ appears in most of the involved terms and in this respect we define $U=3 k z_{c}^{2}\left(x_{c}-\xi\right) / R_{c}^{5}$, in order to simplify the procedure following. Taking all the above into account, heading back to Eq. (B.3) and substituting for the perturbed total phase we obtain

$$
\begin{array}{r}
\psi=\frac{A(\xi) z_{c}}{R_{c}^{3 / 2}} \sqrt{\frac{k}{2 \pi}} e^{i \Xi} \int \exp \left[i\left(\phi^{\prime \prime}(\xi)+\frac{1}{2} U(\delta x)\right)(\delta x)(\delta \xi)-\frac{i}{2} U(\delta x)(\delta \xi)^{2}\right.  \tag{B.11}\\
+
\end{array} \begin{array}{r}
\left.\frac{i}{6}\left(\phi^{\prime \prime \prime}(\xi)+U\right)(\delta \xi)^{3}\right] d(\delta \xi),
\end{array}
$$

where we have utilized the slowly varying envelope approximation and $\Xi=k R_{c}+$
$\phi-\frac{\pi}{4}$. An analytic solution of the above integral can be obtained utilizing Eq. (A.4) with the corresponding substitutions being $a=\phi^{\prime \prime}(\xi)(\delta x)+(U / 2)(\delta x)^{2}, b=$ $-(U / 2)(\delta x)$ and $c=\left(\phi^{\prime \prime \prime}(\xi)+U\right) / 2$. However, before we proceed with the involved calculations we find more instructive to obtain an expression for $\phi^{\prime \prime \prime}(\xi)$. To do so, we begin from the condition describing the first-order stationarity of the phase $\phi^{\prime}(\xi)=k\left(x_{c}-\xi\right) / R_{c}$, and taking the derivative with respect to $\xi$ in order to obtain $\phi^{\prime \prime}(\xi)$

$$
\begin{align*}
{\left[\phi^{\prime}(\xi)\right]^{\prime} } & =\frac{k\left(x_{c}^{\prime}-1\right) R_{c}-k\left(x_{c}-\xi\right) R_{c}^{\prime}}{R_{c}^{2}} \\
& =\frac{k\left(x_{c}^{\prime}-1\right) R_{c}^{2}-k\left(x_{c}^{\prime}-1\right)\left(x_{c}-\xi\right)^{2}-k z_{c} z_{c}^{\prime}\left(x_{c}-\xi\right)}{R_{c}^{3}}  \tag{B.12}\\
& =\frac{k\left(x_{c}^{\prime}-1\right) z_{c}^{2}-k z_{c} z_{c}^{\prime}\left(x_{c}-\xi\right)}{R_{c}^{3}}
\end{align*}
$$

Taking into account that $\phi^{\prime \prime}(\xi)=-k z_{c}^{2} / R_{c}^{3}$ we obtain

$$
\begin{array}{r}
\frac{k\left(x_{c}^{\prime}-1\right) z_{c}^{2}-k z_{c} z_{c}^{\prime}\left(x_{c}-\xi\right)}{R_{c}^{3}}=-\frac{k z_{c}^{2}}{R_{c}^{3}} \Rightarrow x_{c}^{\prime} z_{c}^{2}-z_{c} z_{c}^{\prime}\left(x_{c}-\xi\right)=0  \tag{B.13}\\
\Rightarrow x_{c}^{\prime}=\frac{z_{c}^{\prime}\left(x_{c}-\xi\right)}{z_{c}}
\end{array}
$$

while for the third derivative of the initial phase

$$
\begin{align*}
{\left[\phi^{\prime \prime}(\xi)\right]^{\prime} } & =\left(-\frac{k z_{c}^{2}}{R_{c}^{3}}\right)^{\prime}=\frac{-2 k z_{c} z_{c}^{\prime} r_{c}^{3}+3 k z_{c}^{2} R_{c}^{\prime} R_{c}^{2}}{R_{c}^{6}} \\
& =\frac{-2 k z_{c} z_{c}^{\prime} R_{c}^{3}+3 k z_{c}^{2} R_{c}\left(x_{c}-\xi\right)\left(x_{c}^{\prime}-1\right)+3 k z_{c}^{3} z_{c}^{\prime} R_{c}}{R_{c}^{6}} \\
& =\frac{-2 k z_{c} z_{c}^{\prime} R_{c}^{3}+3 k z_{c}^{2} R_{c}\left(x_{c}-\xi\right)\left[\frac{\left(x_{c}-\xi\right) z_{c}^{\prime}}{z_{c}}-1\right]+3 k R_{c}\left(R_{c}^{2}-\left(x_{c}-\xi\right)^{2}\right) z_{c} z_{c}^{\prime}}{R_{c}^{6}} \\
& =\frac{k z_{c} z_{c}^{\prime} R_{c}^{3}-3 k z_{c}^{2} R_{c}\left(x_{c}-\xi\right)}{R_{c}^{6}}=\frac{k z_{c}\left[R_{c}^{2} z_{c}^{\prime}+3\left(\xi-x_{c}\right) z_{c}\right]}{R_{c}^{5}} \tag{B.14}
\end{align*}
$$

[^0]After those brief calculations we can now return to the evaluation of Eq. (B.11). We begin with the terms involved in the argument of the Airy function

$$
\begin{aligned}
a c-b^{2} & =\frac{1}{2} \phi^{\prime \prime}\left(\phi^{\prime \prime \prime}+U\right)(\delta x)+\frac{1}{4} U \phi^{\prime \prime \prime}(\delta x)^{2}, \\
c^{4 / 3} & =\left[\frac{1}{2}\left(\phi^{\prime \prime \prime}+U\right)\right]^{4 / 3},
\end{aligned}
$$

where $\left(\phi^{\prime \prime \prime}+U\right)=k z_{c} z_{c}^{\prime} / R_{c}^{3}$. We will evaluate first the dominant terms (i.e., those proportional to $(\delta x)$ while terms proportional to $(\delta x)^{2}$ will be evaluated afterwards in order to complete our final result. In this respect,

$$
\begin{equation*}
\frac{\phi^{\prime \prime}\left(\phi^{\prime \prime \prime}+U\right) / 2}{\left[\left(\phi^{\prime \prime \prime}+U\right) / 2\right]^{4 / 3}}=\frac{\phi^{\prime \prime}}{\left[\left(\phi^{\prime \prime \prime}+U\right) / 2\right]^{1 / 3}}=-s \frac{k z_{c}^{2} / R_{c}^{3}}{\left[k z_{c}\left|z_{c}^{\prime}\right| /\left(2 R_{c}^{3}\right)\right]^{1 / 3}}=-s\left(\frac{2 k^{2} z_{c}^{5}}{R_{c}^{6}\left|z_{c}^{\prime}\right|}\right)^{1 / 3}, \tag{B.15}
\end{equation*}
$$

where we have defined $s$ as $s=\operatorname{sgn}\left(z_{c}^{\prime}\right)$. Proceeding with terms related to $(\delta x)^{2}$ we obtain

$$
\begin{align*}
W=\frac{1}{4} \frac{\phi^{\prime \prime \prime} U}{\left[\left(\phi^{\prime \prime \prime}+U\right) / 2\right]^{4 / 3}} & =\frac{s}{4} \frac{\left[\left(k z_{c} z_{c}^{\prime} R_{c}^{2}+3 k z_{c}^{2}\left(\xi-x_{c}\right)\right) / R_{c}^{5}\right]\left[3 k z_{c}^{2}\left(x_{c}-\xi\right) / R_{c}^{5}\right]}{\left[k z_{c}\left|z_{c}^{\prime}\right| /\left(2 R_{c}^{3}\right)\right]^{4 / 3}} \\
& =\frac{s}{4} \frac{3 k^{2} z_{c}^{3}\left(x_{c}-\xi\right)\left[R_{c}^{2} z_{c}^{\prime}+3\left(\xi-x_{c}\right) z_{c}\right] / R_{c}^{10}}{\left[k z_{c}\left|z_{c}^{\prime}\right| /\left(2 R_{c}^{3}\right)\right]^{4 / 3}} \\
& =3 s\left(x_{c}-\xi\right) \frac{\left[R_{c}^{2} z_{c}^{\prime}+3\left(\xi-x_{c}\right) z_{c}\right]}{R_{c}^{6}}\left(\frac{k^{2} z_{c}^{5}}{4\left|z_{c}^{\prime}\right|^{4}}\right)^{1 / 3} \tag{B.16}
\end{align*}
$$

Finally, taking all the above into account we can write the Airy function as

$$
\begin{equation*}
\mathrm{Ai}\left[-s\left(\frac{2 k^{2} z_{c}^{5}}{R_{c}^{6}\left|z_{c}^{\prime}\right|}\right)^{1 / 3}(\delta x)+W(\delta x)^{2}\right] \tag{B.17}
\end{equation*}
$$

or by keeping only the dominant terms while utilizing Eq. (B.9) as

$$
\begin{equation*}
\operatorname{Ai}\left(\frac{a c-b^{2}}{c^{4 / 3}}\right)=\operatorname{Ai}\left[-s \frac{z_{c}}{R_{c}}\left(2 k^{2} \kappa\left(z_{c}\right)\right)^{1 / 3}(\delta x)\right] \tag{B.18}
\end{equation*}
$$

To continue with the rest of the terms involved in Eq. (A.4) we will begin from the less computational demanding $c^{1 / 3}=s\left[k z_{c}\left|z_{c}^{\prime}\right| /\left(2 R_{c}^{3}\right)\right]^{1 / 3}$, and before proceed to those that form the pure phase terms we will introduce another substitution, namely
$S=\phi^{\prime \prime \prime}+U$, in order to further simplify our resulted formulas,

$$
\begin{align*}
\frac{2 b^{3}-3 a b c}{3 c^{2}} & =\frac{2(-U(\delta x) / 2)^{3}-3\left(\phi^{\prime \prime}(\delta x)+U(\delta x)^{2} / 2\right)(-U(\delta x / 2))\left[\left(\phi^{\prime \prime \prime}+U\right) / 2\right]}{3\left[\left(\phi^{\prime \prime \prime}+U\right) / 2\right]^{2}} \\
& =\frac{-U^{3}(\delta x)^{3} / 4+3 \phi^{\prime \prime} U(\delta x)^{2} S / 4+3 U^{2}(\delta x)^{3} S / 8}{3 S^{2} / 4} \\
& =\frac{U(\delta x)^{2}\left[6 \phi^{\prime \prime} S+U(\delta x)(3 S-2 U)\right]}{6 S^{2}} \\
& =\frac{U(\delta x)^{2}\left[6 \phi^{\prime \prime} S+U(\delta x)\left(S+2 \phi^{\prime \prime \prime}\right)\right]}{6 S^{2}}=\Delta \tag{B.19}
\end{align*}
$$

Finally considering Eqs. (B.17)- (B.19) we can express Eq. (B.11) as

$$
\begin{align*}
\psi\left(x_{c}+\delta x, z_{c}\right) & =\frac{2 \pi A(\xi) z_{c} \sqrt{\frac{k}{2 \pi}}}{c^{1 / 3}} e^{i(\Xi+\Delta)} \mathrm{Ai}\left[-s \frac{z_{c}}{R_{c}}\left(2 k^{2} \kappa\right)^{1 / 3}(\delta x)+W(\delta x)^{2}\right] \\
& =2 A(\xi)\left(\frac{\left(\pi z_{c}\right)^{4}}{\lambda R_{c}^{3}\left|z_{c}^{\prime}\right|^{2}}\right)^{1 / 6} e^{i(\Xi+\Delta)} \mathrm{Ai}\left[-s \frac{z_{c}}{R_{c}}\left(2 k^{2} \kappa\right)^{1 / 3}(\delta x)+W(\delta x)^{2}\right] \tag{B.20}
\end{align*}
$$

or more conveniently, by recalling the formula for the curvature while retaining only the dominant terms

$$
\begin{equation*}
\psi\left(x_{c}+\delta x, z_{c}\right)=2 A(\xi)\left(\frac{\pi^{4} R_{c}^{3} \kappa^{2}}{\lambda}\right)^{1 / 6} e^{i \Xi} \mathrm{Ai}\left[-s \frac{z_{c}}{R_{c}}\left(2 k^{2} \kappa\right)^{1 / 3}(\delta x)\right] \tag{B.21}
\end{equation*}
$$

## Appendix C

## Propagation dynamics of paraxial abruptly autofocusing beams

In this Appendix, we will give an insight in the mathematical techniques utilized in order to obtain the necessary formulas to describe the propagation of abruptly autofocusing beams in the paraxial domain. Here, by replacing the radial coordinates with their corresponding Cartesian coordinates [i.e., $r \rightarrow x, \rho \rightarrow \xi$ ] and following the relevant calculations presented in Appendix A, we obtain that the qualitatively features of the beam's propagation are described by the same equations as in Appendix A. This is expected as the rays cannot distinguish between Cartesian and radial coordinates. Although, due to the unique phenomena that arise during the propagation of abruptly autofocusing beams such as the caustic collapse on the optical axis, further investigation was necessary in order to in-depth describe the propagation dynamics of such beams.

Let us begin with Fresnel-type diffraction integral

$$
\begin{equation*}
\psi(r, z)=\frac{k}{i z} \int_{0}^{\infty} A(\rho) \sqrt{\frac{i z \rho}{2 \pi k r}} \exp \left[i \phi(\rho)+i k \frac{(r-\rho)^{2}}{2 z}\right] d \rho \tag{C.1}
\end{equation*}
$$

where we have assumed the optical wave excitation on the input plane $\psi_{0}(r)=$ $A(r) e^{i \phi(r)}$, and we have applied large argument asymptotics for the Bessel function while eliminating any forward propagating rays thus, $J_{0}(x)=\sqrt{\frac{1}{2 i \pi x}} i e^{-i x}$. We observe that the total phase of the integrand is given by

$$
\begin{equation*}
\Psi(\rho ; r, z)=\phi(\rho)+k \frac{(r-\rho)^{2}}{2 z} \tag{C.2}
\end{equation*}
$$

which is the same with the corresponding total phase obtained in Appendix A under the substitutions mentioned earlier.

In order to obtain an expression for the amplitude of the beam near the caustic, we expand the phase in a Taylor series around an arbitrary point $\left(\rho_{c}, r_{c}\right)$ as

$$
\begin{equation*}
\Psi\left(\rho_{c}+\delta \rho, r_{c}+\delta r\right) \approx\left[\Psi_{c}+\partial_{r} \Psi_{c}(\delta r)+\frac{\partial_{r}^{2} \Psi_{c}}{2}(\delta r)^{2}\right]+\partial_{\rho r} \Psi_{c}(\delta \rho)(\delta r)+\frac{\partial_{\rho}^{3} \Psi_{c}}{6}(\delta \rho)^{3} \tag{C.3}
\end{equation*}
$$

where we have retained terms up to third order and we have taken into account that $\partial_{\rho} \Psi_{c}=\partial_{\rho}^{2} \Psi_{c}=0$ due to the phase stationarity and also $\partial_{\rho \rho r} \Psi_{c}=\partial_{\rho r r} \Psi_{c}=\partial_{r}^{3} \Psi_{c}=$ 0 . Additionally $\partial_{\rho r} \Psi_{c}=\phi^{\prime \prime}(\rho)$ and $\partial_{\rho}^{3} \Psi_{c}=\phi^{\prime \prime \prime}(\rho)$, where for simplicity we have substituted $\rho_{c} \rightarrow \rho$. Utilizing the perturbed total phase obtained above, the Fresneltype diffraction integral in radial coordinates under the slowly varying envelope approximation can be expressed by

$$
\begin{equation*}
\psi\left(r_{c}+\delta r, z_{c}\right)=A(\rho) \sqrt{\frac{k \rho}{2 \pi i r z_{c}}} e^{I \Xi} \int_{0}^{\infty} \exp \left[i\left(\phi^{\prime \prime}(\rho)(\delta \rho)(\delta r)+\phi^{\prime \prime \prime}(\rho)(\delta \rho)^{3}\right)\right] d(\delta \rho) \tag{С.4}
\end{equation*}
$$

where $\Xi=\phi(\rho)+\frac{k\left(r_{c}-\rho\right)^{2}}{2 z_{c}}$. Following the same procedure presented in Appendix A, we can show that Eq. (C.4) admits solutions of the Airy-type given by

$$
\begin{equation*}
\psi\left(r_{c}+\delta r, z_{c}\right)=2 A(\rho) \sqrt{\frac{\rho}{i r}}\left(\frac{\pi^{4} z_{c}^{3} \kappa^{2}}{\lambda}\right)^{1 / 6} e^{i \Xi} \operatorname{Ai}\left(-\left(2 k^{2} \kappa\right)^{1 / 3} \delta r\right), \tag{C.5}
\end{equation*}
$$

Here we have defined the slope of the trajectory [i.e., $r_{c}=f\left(z_{c}\right)=r_{0}-\beta z_{c}^{\alpha}$ ] as $g\left(z_{c}\right)=d f\left(z_{c}\right) / d z_{c}=\left(r_{c}-\rho\right) / z_{c}$. The last part of the equality can be derived by considering the equation of a line tangent to the trajectory at an arbitrary point $\left(r_{c}, z_{c}\right)$, which reads $\left(r-r_{c}\right)=f^{\prime}\left(z_{c}\right)\left(z-z_{c}\right)$. Taking $z=0 \Rightarrow r=\rho$ and solving with respect to $f^{\prime}\left(z_{c}\right)$ completes the argument. Furthermore, the minus sign in the argument of the Airy function is a direct consequence of the negative curvature of the trajectory (i.e., $s=-1$ ). At this point we should highlight that, while Eq. (C.5) is very useful in describing the propagation dynamics of the beam before the focus, it diverges near the focal point as $r \rightarrow 0$.

In order to obtain an analytic formula for the amplitude of the propagating beam valid both at early stages of propagation and near the focus, we will expand the total
phase around an arbitrary point on the caustic, $\rho=\rho_{c}+\delta \rho, z=z_{c}+\delta z$ while this time we keep r constant at $r=r_{c}$. For the perturbed phase we retain terms up to third order

$$
\begin{array}{r}
\Psi\left(\rho_{c}+\delta \rho, z_{c}+\delta z\right)=\left[\Psi_{c}+\partial_{z} \Psi_{c}(\delta z)\right]+\partial_{\rho z} \Psi_{c}(\delta \rho)(\delta z)+\frac{1}{6} \partial_{\rho}^{3} \Psi_{c}(\delta \rho)^{3} \\
+\frac{1}{2} \partial_{\rho \rho z} \Psi_{c}(\delta \rho)^{2}(\delta z) \tag{C.6}
\end{array}
$$

where we have taken into account that $\partial_{\rho} \Psi_{c}=\partial_{\rho}^{2} \Psi_{c}=0$ and we have also eliminate terms proportional to $(\delta z)^{j}$ for $j>1$. Furthermore from the explicit expression of $\Psi$ we obtain that $\partial_{\rho z} \Psi_{c}=k\left(r_{c}-\rho\right) / z_{c}^{2}, \partial_{\rho}^{3} \Psi_{c}=\phi^{\prime \prime \prime}(\rho)$ and $\partial_{\rho \rho z} \Psi_{c}=-k / z_{c}^{2}$ where we have also made the substitution $\rho_{c} \rightarrow \rho$. Finally, utilizing the perturbed total phase while applying the slowly varying envelope approximation, the Fresnel integral becomes

$$
\begin{array}{r}
\psi\left(r_{c}, z_{c}+\delta z\right)=A(\rho) \sqrt{\frac{k \rho}{2 \pi i r_{c} z}} e^{i \Xi} \int_{0}^{\infty} \exp \left[i \left(k \frac{r_{c}-\rho}{z_{c}^{2}}(\delta \rho)(\delta z)-\frac{k}{2 z_{c}^{2}}(\delta \rho)^{2}(\delta z)\right.\right. \\
\left.\left.+\frac{1}{6} \phi^{\prime \prime \prime}(\rho)(\delta \rho)^{3}\right)\right] d(\delta \rho) . \tag{C.7}
\end{array}
$$

Utilzing Eq. (A.4) for $a=k \frac{r_{c}-\rho}{z_{c}^{2}}(\delta z), b=-\frac{k}{2 z_{c}^{2}}(\delta z)$ and $c=\frac{\phi^{\prime \prime \prime}(\rho)}{2}$ along with Eqs. (A.6), (A.7) we obtain our solution in the following form

$$
\begin{equation*}
\psi\left(r_{c}, z_{c}+\delta z\right)=A(\rho)\left(\frac{2 \kappa}{k}\right)^{1 / 3}\left(\frac{2 \pi k \rho z_{c}^{2}}{i r_{c} z}\right)^{1 / 2} e^{i \Xi} \operatorname{Ai}\left(\left(2 k^{2} \kappa\right)^{1 / 3} g\left(z_{c}\right) \delta z\right) \tag{C.8}
\end{equation*}
$$

where $\Xi=\phi(\rho)+\frac{k\left(r_{c}-\rho\right)^{2}}{2 z_{c}}$. Obviously this formula too, fails to describe the optical wave close to the focus as the denominator is proportional to $\sqrt{r}$, and thus the amplitude diverges as $r \rightarrow 0$. In order to obtain an expression for the amplitude of the propagating beam exactly at the focus we will perturb the phase of the optical wave around an arbitrary point on the caustic $\left(\rho_{c}, z_{c}\right)$, while we set $r=0$. In this case the simplified Fresnel-type integral in radial coordinates reads

$$
\begin{equation*}
\psi(0, z)=\frac{k}{i z} \int_{0}^{\infty} \rho A(\rho) \exp \left[i\left(\phi(\rho)+k \frac{\rho^{2}}{2 z}\right)\right] d \rho, \tag{С.9}
\end{equation*}
$$

where now the total phase is given by

$$
\begin{equation*}
\Psi(\rho, z)=\phi(\rho)+\frac{k \rho^{2}}{2 z} \tag{C.10}
\end{equation*}
$$

Expanding the above equation in a Taylor series, while keeping only the dominant terms we end up with

$$
\begin{align*}
& \Psi\left(\rho_{c}+\delta \rho, z_{c}+\delta z\right)=\Psi_{c}+\partial_{z} \Psi_{c}(\delta z)+\partial_{\rho z} \Psi_{c}(\delta \rho)(\delta z)+\frac{1}{6} \partial_{\rho}^{3} \Psi_{c}(\delta \rho)^{3} \\
& +\frac{1}{2} \partial_{\rho \rho z} \Psi_{c}(\delta \rho)^{2}(\delta z) \tag{C.11}
\end{align*}
$$

where we have considered that $\partial_{\rho} \Psi_{c}=\partial_{\rho}^{2} \Psi_{c}=0$ due to the conditions of the phase stationarity, and from the analytic formula of the total phase we obtain $\partial_{z} \Psi_{c}=$ $-k g^{2}\left(z_{c}\right) / 2, \partial_{\rho z} \Psi_{c}=-k \rho / z_{c}^{2}, \partial_{\rho}^{3} \Psi_{c}=\phi^{\prime \prime \prime}(\rho)$ and $\partial_{\rho \rho z} \Psi_{c}=-k / z_{c}^{2}$. As in the previous cases for simplicity we have replaced $\rho_{c}$ with $\rho$. Finally, utilizing the slowly varying envelope approximation we can write the Fresnel integral as

$$
\begin{array}{r}
\psi\left(0, z_{c}+\delta z\right)=\frac{k}{i z} A(\rho) \rho e^{i \Xi} \int_{0}^{\infty} \exp \left[i \left(-\frac{k \rho}{z_{c}^{2}}(\delta \rho)(\delta z)-\frac{k}{2 z_{c}^{2}}(\delta \rho)^{2}(\delta z)\right.\right. \\
+
\end{array} \begin{array}{r}
6  \tag{C.12}\\
\left.\left.\phi^{\prime \prime \prime}(\rho)(\delta \rho)^{3}\right)\right] d(\delta \rho)
\end{array}
$$

The above integral can be calculated explicitly utilizing Eq. (A.4) for $a=-\frac{k \rho}{z_{c}^{2}}(\delta z)$, $b=-\frac{k}{2 z_{c}^{2}}(\delta z)$ and $c=\frac{\phi^{\prime \prime \prime}(\rho)}{2}$. To conclude with, the propagation dynamics of the optical wave near the focal point can be obtained from

$$
\begin{equation*}
\psi\left(0, z_{c}+\delta z\right)=A(\rho)\left(\frac{2 \kappa}{k}\right)^{1 / 3} \frac{2 \pi k \rho z_{c}}{i z} e^{i \Xi} \mathrm{Ai}\left(\left(2 k^{2} \kappa\right)^{1 / 3} g\left(z_{c}\right) \delta z\right) \tag{C.13}
\end{equation*}
$$

where $\Xi$ is the same as in Eq. (C.8) for $r_{c}=0$.

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[^0]:    ${ }^{*}$ Assuming the trajectory $x_{c}=f\left(z_{c}\right)$, the angle $\theta$ between the tangent line and the $z$-axis, and the arclength $s=\int_{z_{0}}^{z_{1}} \sqrt{1+\left(d f / d z_{c}\right)^{2}} d z_{c}$, we can obtain the curvature of the trajectory as follows: $\kappa=\frac{d \theta}{d s}=$ $\frac{\left(d \theta / d z_{c}\right)}{\left(d s / d z_{c}\right)}$. We already know that $\tan \theta=d f\left(z_{c}\right) / d z_{c}=f^{\prime}\left(z_{c}\right)$ and thus $d(\tan \theta) / d z_{c}=f^{\prime \prime}\left(z_{c}\right)$. Applying the chain rule derivative we also obtain that $d(\tan \theta) / d z_{c}=\sec ^{2} \theta d \theta / d z_{c}$ or equivalently $d \theta / d z_{c}=$ $f^{\prime \prime}\left(z_{c}\right) / \sec ^{2} \theta$. Utilizing known trigonometric formulas we obtain that $d \theta / d z_{c}=f^{\prime \prime}\left(z_{c}\right) /\left(1+\tan ^{2} \theta\right)=$ $f^{\prime \prime}\left(z_{c}\right) /\left(1+\left(d f / d z_{c}\right)^{2}\right)$. Substituting back to $\kappa=d \theta / d s$ we obtain the curvature of the trajectory as $\kappa\left(z_{c}\right)=f^{\prime \prime}\left(z_{c}\right) /\left(1+\left(d f / d z_{c}\right)^{2}\right)^{3 / 2}$.

