# POROUS MEDIUM/SLOW DIFFUSION EQUATION WITH NONLINEAR SOURCE AND 3RD TYPE BOUNDARY CONDITIONS 

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## 1. ABSTRACT

In the present paper, we obtain a new a priori estimate of the solution of the initial-boundary value Problem for the Porous medium equation with non-linear source. Also, we present the conditions guaranteeing the existence of a global classical solution of this Problem as well as the cases for which the solution may blow up (the last is discussed in [1,2]). We have to establish an a priori estimate of the already studied heat type problem with Dirichlet conditions instead (see [3]). The main tool which is going to be utilized for finding an a priori estimate and constructing an upper bound for the solution, is the maximum principle.

## 2. INTRODUCTION

Nowadays, PDEs constitute one of the most significant areas of both Theoretical and Applied Mathematics. This happens due to the fact that the utilization of Equations with Partial Derivatives has become more frequent in Physical, Technological, Economical and other Applied Sciences. Furthermore, the abundance of new Problems, Questions and Theories created in Theoretical Mathematics and other Sciences, demand additional research on the general study and solution of these Equations.

The porous medium equation is the most simple example of nonlinear parabolic type equations and strongly resembles the behaviour of heat equation. There are a number of physical applications where this model appears in a natural way, mainly to describe processes involving fluid flow, heat transfer or diffusion. Let's give 2 examples below.

The solution $u$ can represent either the scaled density (see [4]) or the absolute temperature (see [5]).

- As a first example, we are about to see briefly how the porous medium equation is constructed with $u$ to be the density of matter.

Physical Background: Consider an ideal gas flowing isentropically in a homogeneous porous medium. The flow is governed by the following 3 laws.

1/ Equation of state:

$$
p=p_{0} \rho^{\alpha},
$$

where $p=p(x, t)$ is the pressure, $\rho=\rho(x, t)$ is the density and $\alpha \in[1, \infty)$ and $p_{0} \in \mathbf{R}^{+}$are constants. Here $x \in \mathbf{R}^{d}$ for some $d \geq 1$.

2/ Conservation of mass:

$$
\kappa \frac{\partial \rho}{\partial t}+\operatorname{div}(\rho \mathbf{v})=0
$$

where $\mathbf{v}=\mathbf{v}(x, t)$ is the velocity vector and $\kappa \in \mathbf{R}^{+}$is the porosity of the medium (namely, the volume fraction available to the gas).

## 3/ Darcy's Law:

$$
\nu \mathbf{v}=-\mu \nabla p
$$

where $\nu \in \mathbf{R}^{+}$is the viscosity of the gas and $\mu \in \mathbf{R}^{+}$is the permeability of the medium.

Note that Darcy's law, is an empirically derived law, which replaces the usual conservation of momentum in the standard (Navier-Stokes) description of gas flow.

If we eliminate $p$ and $\mathbf{v}$ from the equations and scale away all of the resulting constants, we obtain the porous medium equation:

$$
\frac{\partial u}{\partial t}=\triangle\left(u^{q}\right), \quad q=1+\alpha \geq 2
$$

In more detail by Darcy's law, we have that:

$$
\mathbf{v}=-\frac{\mu}{\nu} \nabla p
$$

By the equation of state, we have:

$$
-\frac{\mu}{\nu} p=-\frac{\mu p_{0}}{\nu} \rho^{\alpha} \Rightarrow-\frac{\mu}{\nu} \nabla p=-\frac{\mu p_{0} \alpha}{\nu} \rho^{\alpha-1} \nabla \rho \Rightarrow \mathbf{v}=-\frac{\mu p_{0} \alpha}{\nu} \rho^{\alpha-1} \nabla \rho
$$

Finally, by the conservation of mass, we have:

$$
\begin{gathered}
\frac{\partial \rho}{\partial t}+\frac{1}{\kappa} \operatorname{div}\left(\rho\left(-\frac{\mu p_{0} \alpha}{\nu}\right) \rho^{\alpha-1} \nabla \rho\right)=0 \Rightarrow \frac{\partial \rho}{\partial t}=\frac{\mu p_{0} \alpha}{\kappa \nu} \operatorname{div}\left(\rho^{\alpha} \nabla \rho\right) \Rightarrow \\
\Rightarrow \frac{\partial \rho}{\partial t}=\frac{\mu p_{0} \alpha}{(\alpha+1) \kappa \nu} \triangle\left(\rho^{\alpha+1}\right) \Rightarrow \frac{\partial \rho}{\partial t}=c \triangle\left(\rho^{\alpha+1}\right)
\end{gathered}
$$

The last equation holds, because:

$$
\triangle\left(\frac{\rho^{\alpha+1}}{\alpha+1}\right)=\operatorname{div}\left(\rho^{\alpha} \nabla \rho\right)
$$

We can freely choose

$$
c=\frac{\mu p_{0} \alpha}{(\alpha+1) \kappa \nu}=1
$$

by scaling and we receive the desired result for $\rho=u, q=\alpha+1$.

The quantity $u$, represents a scaled density and so it is natural to assume that $u \geq 0$.

- As a second example, we are about to see briefly how the porous medium equation is constructed with solution $u$ to be the absolute temperature.

Let $u(x, t)$ be the temperature at the point $x=\left(x_{1}, \ldots, x_{n}\right) \in \Omega \subset \mathbf{R}^{n}$ at the time $t$. Then the integral:

$$
\int_{\Omega} \rho c u d x
$$

where $\rho>0$ is the density and $c>0$ is the heat capacity, gives us the total heat contained inside $\Omega$. According to Fourier's law, the temperature flows from the warmer to the colder regions of $\Omega$, based on the vector field:

$$
\mathbf{F}=-\kappa \nabla u
$$

where $\kappa>0$ is the coefficient of heat conductivity. According to the conservation of energy law, we have that the total heat change is determined by the heat flow through the boundary $\partial \Omega$ and by the heat sources $f$ located in $\Omega$, namely:

$$
\frac{d}{d t} \int_{\Omega} \rho c u d x=\int_{\partial \Omega} \kappa \nabla u \cdot \eta d s+\int_{\Omega} f d x
$$

where $\eta$ is the unitary outward normal vector on the boundary $\partial \Omega$. Applying the divergence theorem, we have:

$$
\begin{aligned}
& \frac{d}{d t} \int_{\Omega} \rho c u d x=\int_{\Omega} \operatorname{div}(\kappa \nabla u) d x+\int_{\Omega} f d x \Rightarrow \\
& \quad \Rightarrow \int_{\Omega}\left((\rho c u)_{t}-\operatorname{div}(\kappa \nabla u)-f\right) d x=0
\end{aligned}
$$

Since $\Omega$ is arbitrary, we conclude to the equation:

$$
(\rho c u)_{t}-\operatorname{div}(\kappa \nabla u)=f
$$

If we consider that the quantities $\rho, c, \kappa$ are constants, then for the temperature $u$ we have that:

$$
u_{t}-k \Delta u=\tilde{f}, k=\frac{\kappa}{\rho c}, \tilde{f}=\frac{f}{\rho c} .
$$

This is called heat equation. Now, if we let $k=q u^{q-1}$, then the previous equation receives the form:

$$
u_{t}-\operatorname{div}\left(q u^{q-1} \nabla u\right)=f \Rightarrow u_{t}-\triangle u^{q}=f
$$

We should also emphasize that in the case that $u$ is the absolute temperature we observe the following:

If the term $u^{p}$ exists on the RHS of the porous medium equation (possibly multiplied by a constant), it plays the role of a non-linear source which provides energy to the system.

On the other hand, if the term $-u^{p}$ exists on the RHS of the porous medium equation (possibly multiplied by a constant), a non-linear absorption process takes place.

The term $u^{p}$ plays the role of a source that can for example emit fire. The higher the temperature of the fire, the more energy is produced.

The porous medium equation, arises in many other applications, for example in the theory of ionized gases at high temperature for values $q>1$ and in various models in plasma physics for values $q<1$. For $q=1$, the equation is the classical equation of heat conduction.

Our goal is to find an a priori estimate of the solution $u$ and state the cases for which the solution exists globally.

We will also refer to the conditions under which the solution may blow up (proved in [1,2]).

We shall mention that the most simple Problem which trivially blows up is the following:

$$
\begin{cases}u^{\prime}(t)=u^{p}(t) & , p>0 \\ u(0)=1 & , t>0\end{cases}
$$

The solution of the Problem above is:

$$
u(t)=\frac{1}{\sqrt[p-1]{-(p-1) t+1}}
$$

Now, if we take the limit to $\frac{1}{p-1}$, the solution blows up only for $p>1$. Namely,

$$
\lim _{t \rightarrow \frac{1}{p-1}} u(t)=\infty
$$

The term $u^{p}$ makes the solution blow up. So, the term $\triangle u^{q}$ tends to counterbalance $u^{p}$, in order the latter to prevent the first from making
the solution blow up and therefore the two terms are trying to cancel each other out.

Now, we are about to summarise what are the following sections of this paper going to include.

- In the 3rd section we are going to present some known results upon the global solvability and the situations in which the solution of the Problem above might blow up.
- In the 4th section we state the Problem we deal with. We establish the a priori estimate for the global classical solution $u$ of our Problem which is going to be proved in the next section.
- In the 5th section we are going to prove that a global classical solution of this Problem exists. The proof is detailed and concerns not only one, but also higher dimensions.
- In the 6th section we are going to see the Problem from the aspect of Physics. Although the boundary Robin type condition slightly changes, the main results remain the same.
- In the last section we are going to condense what was done in the previous sections.

In the later section we are going to study the following Problem:

$$
\begin{equation*}
u_{t}-\triangle u^{q}=k(t, x) u^{p}, \quad \text { in } \quad Q_{T}=(0, T) \times \Omega, \quad \Omega \subset \mathbf{R}^{n} \tag{1}
\end{equation*}
$$

where $q>1, p>0,0 \leq k(t, x) \leq \kappa, T>0, x=\left(x_{1}, \ldots, x_{n}\right)$ coupled with the initial and boundary conditions,

$$
\begin{gather*}
u(x, 0)=u_{0}(x)>0,  \tag{2}\\
\frac{\partial u}{\partial \eta}(x, t)+\phi(x, t, u)=0, \\
x \in \partial \Omega \times(0, T)
\end{gather*}
$$

respectively. Assume that $\Omega$ is always bounded.

## 3. SUMMARY OF SOME PREVIOUS RESULTS

## I)

RESULTS OF GALAKTIONOV
In [1], the Problem studied is the following:

$$
\begin{gather*}
u_{t}=\triangle u^{q}+u^{p}, t>0, x \in \Omega ; q-1>0, p>1  \tag{4}\\
u(0, x)=u_{0}(x) \geq 0, x \in \Omega ; u(t, x)=0, t \geq 0, x \in \partial \Omega \tag{5}
\end{gather*}
$$

With $u_{0} \in C(\bar{\Omega}), u_{0}^{q} \in H_{0}^{1}(\Omega)$.
It is known from [1,2], that solutions of this problem may blow-up in finite time. The global solvability (namely for arbitrary $T>0$, was proved in [1] for $k \equiv 1$ (see [3] for definition of $k$ ) and homogeneous boundary conditions in the following 3 cases:
a) Global solvability for $\mathrm{p}<\mathrm{q}$ :

Theorem 3.1. Let $p<q$. Then Problem (4),(5), has a global solution and it's true that:

$$
\begin{gather*}
u^{\frac{q+1}{2}} \in L^{\infty}\left(0, T ; L^{2}(\Omega)\right), \frac{\partial}{\partial t} u^{\frac{q+1}{2}} \in L^{2}\left(0, T ; L^{2}(\Omega)\right)  \tag{6}\\
u^{q} \in L^{\infty}\left(0, T ; H_{0}^{1}(\Omega)\right) \tag{7}
\end{gather*}
$$

b) Global solvability for $\mathrm{p}=\mathrm{q}$ :

Define as, $\lambda_{1}$ : the 1 st eigenvalue and $w_{1}$ : the 1 st eigenfunction of the Problem:

$$
\begin{equation*}
\triangle w_{j}+\lambda_{j} w_{j}=0, x \in \Omega ; w_{j} \in H_{0}^{1}(\Omega), j=1,2, \ldots \tag{8}
\end{equation*}
$$

The eigenvalues $\lambda_{j}$, can be ordered in an increasing order. Then the eigenvalue $\lambda_{1}$ is simple and the corresponding eigenfunction $w_{1}(x)>$ 0 in $\Omega$.

Theorem 3.2. Let $p=q$. Then if the domain $\Omega$ is such that $\lambda_{1}>$ 1, the Problem (4),(5), has a global solution, which satisfies (6),(7). Furthermore,

$$
\begin{equation*}
\|u\|_{L^{q+1}(\Omega)}=O\left(t^{-\frac{1}{q-1}}\right) \rightarrow 0, t \rightarrow \infty \tag{9}
\end{equation*}
$$

If, $\lambda_{1}<1$, then $\forall u_{0}(x) \not \equiv 0$, the Problem (4),(5) has no global solutions and $\exists T_{0} \in\left(0, T_{*}\right]$, where:

$$
\begin{equation*}
T_{*}=\left\{\left(1-\lambda_{1}\right)(q-1)\left\|w_{1}\right\|_{L^{1}(\Omega)}^{-(q-1)}\left(u_{0}, w_{1}\right)^{q-1}\right\}^{-1}<\infty \tag{10}
\end{equation*}
$$

such that $\left(u(t), w_{1}\right) \rightarrow \infty$, as $t \rightarrow T_{0}^{-}$.
c) Global solvability for $\mathrm{p}>\mathrm{q}$ :

In this case, the set $W$ (defined above Theorem 3.4) is non-empty. Below we suppose that: $p>q$, for $\mathrm{N}=1,2$ and $p \in\left(q, q \frac{N+2}{N-2}\right)$ for $N \geq 3$.

The next theorem concerns the local solvability of the Problem (4),(5).

Theorem 3.3. Let

$$
q \leq p<q+\frac{2(q+1)}{N}
$$

Then, $\exists T_{*}>0$, constant such that on the interval $\left[0, T_{*}\right]$ the Problem (4),(5) has a solution, which satisfies the inclusions (6),(7).

Define the set as follows:
$W=\left\{v:|v|^{q-1} v \in H_{0}^{1}(\Omega) ; 0 \leq J(\lambda v)<d, \lambda \in[0,1]\right\}$.
Theorem 3.4. Let $p>q$ for $N=1,2$ and

$$
q<p<q+\frac{2(q+1)}{N} \text { for } N \geq 3
$$

Assume that the initial function $u_{0}$ in (5) is such that $u_{0} \in W$. Then $\forall T>0$, there exists a generalized solution of the Problem (4),(5) which satisfies the inclusions (6),(7) and belongs to $\bar{W}, \forall t \geq 0 \overline{(W}$ is the closure of $W$ in the set: $\left\{v\left||v|^{q-1} v \in H_{0}^{1}(\Omega)\right\}\right.$ ).

## We define the following formulas:

i) $W=W_{*} \cup\{0\}$

Where

$$
W_{*}=\left\{\left.v| | v\right|^{q-1} v \in H_{0}^{1}(\Omega), a(v)-b(v)>0, J(v)<d\right\} .
$$

The functions $J(v), a(v), b(v)$ and the quantity $d<\infty$, are defined below:
ii ) $\quad J(v)=\frac{1}{2} a(v)-\frac{q}{p+q} b(v)$,
iii)

$$
a(v)=\sum_{i=1}^{N}\left\|\frac{\partial}{\partial x_{i}}\left(|v|^{q-1} v\right)\right\|_{L^{2}(\Omega)}^{2}, b(v)=\int_{\Omega} \Phi(v(x)) d x
$$

where $\Phi(u)=(\max \{0, u\})^{p+q}$
iv)

$$
d=\inf _{|v|^{q-1}} \sup _{0}^{1}(\Omega) \text { } \sup _{\lambda>0} J(\lambda v)>0, v \not \equiv 0
$$

v) $\quad J(\lambda v)=\frac{1}{2} \lambda^{2 q} a(v)-\frac{q}{p+q} \lambda^{p+q} b(v)$.

Theorem 3.5. Let $q<p<q+\frac{2(q+1)}{N}$ and

$$
J\left(u_{0}\right)<0,\left\|u_{0}\right\|_{L^{q+1}(\Omega)}>0 .
$$

Then, the Problem (4),(5), has no global solution and we can find $T_{0} \in$ ( $0, T_{*}$ ], where

$$
T_{*}=\frac{p+q}{(p-1)(p-q)}(m(\Omega))^{\frac{p-1}{q+1}}\left\|u_{0}\right\|_{L^{q+1}(\Omega)}^{1-p}<\infty
$$

such that:

$$
\lim _{t \rightarrow T_{0}^{-}}\|u(t)\|_{L^{q+1}(\Omega)}=\infty
$$

(blow up).

## II ) <br> RESULTS OF ANDERSON-DENG

In [2], the global solvability of Problem (11)-(13) (see immediately below) with homogeneous boundary conditions in the one dimensional case ( $n=1, x \in(0,1)$ ), was proved under 3 same assumptions which are described below.

## A. PRELIMINARY ARGUMENTS CONSIDERING GLOBAL EXISTENCE AND BLOW-UP FOR THE PROBLEM

In [2], the Problem studied is the following:

$$
\begin{array}{lr}
u(0, t)=u(1, t)=0, & t>0 \\
u(x, 0)=u_{0}(x) \geq 0, & 0 \leq x \leq 1 \tag{11}
\end{array}
$$

and as well as the auxiliary problem (14)-(16), which is a special case of (11)-(13).

Namely we have the Problem:

$$
\begin{array}{cc}
u_{t}=\left[\phi(u)_{x}+\epsilon g(u)\right]_{x}+k f(u), & 0<x<1, t>0 \\
u(0, t)=u(1, t)=0, & t>0 \\
u(x, 0)=u_{0}(x) \geq 0, \quad \text { 10 } & 0 \leq x \leq 1
\end{array}
$$

The one dimensional Problem in [3] receives the following form:

$$
\begin{array}{cc}
u_{t}-\left(u^{q}\right)_{x x}=k(t, x) u^{p}, & Q_{T}=(0, T) \times \Omega \\
u_{0}=\left.u\right|_{\Gamma_{T}}=\left.\phi\right|_{\Gamma_{T}} \geq 0, & \Gamma_{T}=\Omega \cup([0, T] \times \partial \Omega)
\end{array}
$$

with $q>1, p>0,0 \leq k(t, x) \leq \kappa, T>0$
Our goal is to match the problem in [3], separately with each different case (11)-(13) and (14)-(16) written before, respectively. This match can happen only in one dimension.

We observe that:
i) Problem (17),(18) is of the form (11)-(13) if we utilize that $k(x, t) \equiv 1$. Additionally, in Problem (11)-(13), we shall choose

$$
\phi(x, t, u)=u^{q}, g(x, t, u)=0, f(x, t, u)=u^{p}
$$

in order to receive the form of Problem (17),(18) and to study it properly.
ii) Problem (17),(18) is of the form (14)-(16) if we utilize that $k(x, t) \equiv k$ and that everything is only a function of $u$. Additionally, in Problem (14)-(16), we shall choose $\phi(u)=u^{q}, g(u)=0, f(u)=u^{p}$.
iii) We should take into account that the results further are received only when $x \in(0,1)$.
So, $Q_{T}=(0, T) \times(0,1)$ and $\Gamma_{T}=(0,1) \cup([0, T] \times\{0,1\})$.
In order to present the results of [2], we should initially list some necessary hypotheses with which, Problem (17),(18) must be accompanied:
The first is that $\phi, g, f$, must be smooth. Then:
$\phi, f \in \mathbf{C}(\Omega \times[0, \infty) \times \mathbf{R})$
$\phi_{u}, f_{u} \in \mathbf{C}(\Omega \times[0, \infty) \times \mathbf{R} \backslash 0)$
$\phi(0)=f(0)=0$
$\phi_{u}(u)>0, \quad$ for $u>0$
$u_{0} \in L^{\infty}(\Omega)$

For Problem (17),(18), every condition (H2)-(H4) is satisfied trivially in a logical way by definition of the functions $\phi, g, f$.

In order to examine the global existence of the Problem, the following additional hypotheses to (H1), will be necessary:
(H2) $\exists q>0$ and $\kappa \in C([0, \infty))$ with $\kappa>0$ such that

$$
\phi_{u}(x, t, u) \geq \kappa(t) u^{q-1}, \forall x \in[0,1], t \geq 0, u>0
$$ $\exists s, M \geq 0$ and $b \in C([0, \infty))$ such that

$$
\begin{equation*}
\left|\phi_{u x}\right| \leq b(t) u^{s}, \forall x \in[0,1], t \geq 0, u \geq M \tag{H3}
\end{equation*}
$$

(H4) $\quad \exists p, M \geq 0$ and $k \in C([0, \infty))$ such that

$$
f,\left|\phi_{x x}\right| \leq k(t) u^{p}, \forall x \in[0,1], t \geq 0, u \geq M
$$

The main result on global existence for solutions of Problem (11)(13) may be defined by theorem 3.6, where the definitions of all intervening constants and functions is rather lengthy and can be found in Section 3:

Theorem 3.6. a) Assume that conditions (H1)-(H4) are satisfied with $q \geq 1$ or $p \leq 1$. Let $r \geq \frac{3}{2}$ be some given constant with $r>p$, $r \geq q+1, r>q-2 s-1$ and $r \geq-q+2 s+1$. Define:

$$
N=\left\{z>0:-\theta \kappa(t)+\frac{\omega(t)}{z^{r+q-1}}+\frac{\bar{\kappa} b^{2}(t)}{\kappa(t)} z^{\bar{s}}+\Delta(t) z^{\frac{(r+q-1)(p-q)}{q}}<0, \forall t \geq 0\right\} .
$$

(The set $N$ consisting of all positive elements $z$ such that the inequality inside the brackets is satisfied)
If $\operatorname{int}(N) \neq \emptyset$, then for

$$
\left[\int_{0}^{1} u_{0}^{r}(x) d x\right]^{\frac{1}{r}} \in\left(x_{1}, x_{2}\right) \subset \operatorname{int}(N)
$$

the solution of Problem (11)-(13), $u(x, t)$, with $u(x, 0)=u_{0}(x)$, exists, $\forall t \geq 0$. In fact:

$$
\lim _{t \rightarrow \infty} \sup \left[\int_{0}^{1} u_{0}^{r}(x) d x\right]^{\frac{1}{r}} \leq x_{1}
$$

Where $x_{1}$ : is the smallest non negative root of the the quantity:

$$
N=-\theta \kappa+\frac{\omega}{z^{r+q-1}}+\Delta z^{\frac{(r+q-1)(p-q)}{q}}
$$

b) If we have the Problem (14)-(16), then the set $N$, becomes:

$$
N=\left\{z>0:-\theta \kappa+\frac{\omega}{z^{r+q-1}}+\Delta z^{\frac{(r+q-1)(p-q)}{q}}<0\right\} .
$$

Hence we obtain, that if $p<q$ or $p=q$ and $k$ is sufficiently small, then all solutions of (14)-(16) are global. On the other hand, if $p>$ $q \geq 1$ and M is sufficiently small, then the solutions with small initial data, are global.

A similar result is obtained by applying Theorem 3.6 to the Problem:

$$
\begin{gather*}
u_{t}=\left[\phi(t, u)_{x}+g(x, t, u)\right]_{x}+f(x, t, u), \quad 0<x<1, t>0  \tag{19}\\
u(0, t)=u(1, t)=0,  \tag{20}\\
u(x, 0)=u_{0}(x) \geq 0, \tag{21}
\end{gather*}
$$

Where diffusion is spatially independent. For the Problem (19)(21), all solutions are global, if $p<q$ and $\frac{k(t)}{\kappa(t)}$ is bounded $\forall t \geq 0$ or if $p=q$ and $\frac{k(t)}{\kappa(t)}$ is sufficiently small $\forall t \geq 0$. On the other hand, if $p>q \geq 1, M$ is sufficiently small and $\frac{\kappa(t)}{k(t)}$ is bounded $\forall t \geq 0$, then the solutions with small initial data, are global.

In order to give our results on global existence for Problem (14)(16), it is more efficient to discuss them in the context of power laws, namely:
$\phi(u)=u^{q}, g(u)=u^{n}$ (which is 0 in our occasion) and $f(u)=u^{p}$.
By theorem 1.1, we see that all solutions of such a problem are global if $q>p$ or if $p=q$ and $k$ is sufficiently small. If $\epsilon=0$ and $q<p$, then there are solutions that blow up in finite time. Further, if $p=q>1$ and $k>0$ is large, then all nontrivial solutions of (14)-(16), blow up in finite time. If $p>q \geq 1$ (and $p>n$ ), then there are solutions of (14)-(16) which are global and others which blow up in finite time.

In [7], the global existence of all solutions for a general differential inclusion form of (11)-(13) in N dimensions, is established, where $\phi=\phi(u), f=f(u)$, combined with hypotheses such as $f(x, t, u) \leq$ $A+B u$ for $u>0, A, B>0$.

Also if $\phi=\phi(u), g=0, f=f(u)$, this conditions suffice and don't prevent all solution to be global. Therefore, he shows that for the

N -dimensional model (11)-(13) with the condition $\phi=\phi(u), g=0$, $f=f(u)$ has global solutions.

It is also shown that

$$
\int_{0}^{\infty}[f(s)]^{-1} d s=\infty
$$

is a sufficient condition for all solutions to exist globally.
On the other hand, if

$$
\int_{0}^{\infty}[f(s)]^{-1} d s<\infty \text { and } \lim _{s \rightarrow \infty} s^{-1}\left(f \circ \phi^{-1}\right)(s)
$$

is sufficiently large, then it is shown that large solutions blow up in finite time (see subsection E ).

For the case of reaction-diffusion ( $g \equiv 0$ ) we have that an N dimensional version of (11)-(13) is investigated with $\phi=\phi(u)$ and $g \equiv$ 0 . Assumptions placed on the diffusion term, require appropriate lower and upper bounds on $\phi^{\prime}(u)$. One of the main results established is,

$$
f(x, t, u) \operatorname{sgn}(u) \leq C\left(1+(|\phi(u)|)^{\kappa}\right),
$$

for some constants $C$ and $0<\alpha<1$, then all solutions are global. Also, if $\kappa=1$ and the spacial domain is sufficiently small, then all solutions are again global.

## B. LOCAL EXISTENCE AND UNIQUENESS

The following results concern only the Problem (11)-(13):
Theorem 3.7. (Local Existence and continuation). Assume that conditions (H1) are satisfied. For some $T \equiv T\left(u_{0}\right)>0$, the Problem (11)-(13) has a nonnegative solution, $u(x, t)=u\left(x, t ; u_{0}\right)$ on $\Omega \times(0, T)$. Furthermore, if $T$ is redefined to be the supremum of all values $t$ such that $u$ is a solution of (11)-(13) on $\Omega \times(0, \bar{s}), \forall \bar{s} \in[0, t)$, then:

$$
\lim _{t \rightarrow T} \sup \left\{t+\|u(\cdot, t)\|_{\infty}\right\}=\infty
$$

Theorem 3.8. (Uniqueness and comparison). In addition to the assumptions of Theorem 3.7, suppose that $f_{u}$ is continuous $\forall u \in \boldsymbol{R}$ and either $\phi_{u}$ or $\phi_{u}^{-1}$ is bounded on $[0, \delta]$ for some $\delta>0$. Let $u\left(x, t ; u_{0}\right)$ and $v\left(x, t ; v_{0}\right)$ be non negative solutions of (11)-(13) on $\Omega \times(0, T)$, where $t<\min \left\{T\left(u_{0}\right), T\left(v_{0}\right)\right\}$. If $u_{0} \leq v_{0}$, then $u \leq v$ on $\Omega \times(0, T)$.
(In our occasion $(0,1) \times(0, T)$ is $\left.\Omega \times(0, T)=Q_{T}\right)$.

## C. GLOBAL EXISTENCE FOR PROBLEM (11)-(13)

In this section we will omit the proof of theorem 3.6. But we will focus on the constants and functions, utilized in Theorem 3.6 and we are going to define them. Also we should rearrange the functions and the constants used in Theorem 3.6 in [2], to be appropriately correct for Problem (17),(18). Statements (H1)-(H4) are all true and satisfied:

$$
\begin{gathered}
M(t)=0, \text { because } \frac{\partial^{2} \phi(x, t, u)}{\partial x^{2}}=\frac{\partial^{2} \phi(u)}{\partial x^{2}}=0 \\
N(t)=M^{r-1} \max _{x \geq 0, \frac{u}{M} \leq 1}\left\{k(x, t) u^{p}, 0\right\} \\
L(t)=0, \text { because } \frac{\partial^{2} \phi(x, t, v)}{\partial x \partial v}=\frac{\partial^{2} \phi(v)}{\partial x \partial v}=0 \\
\omega(t)=r N(t) \Rightarrow \omega(t)=r M^{r-1} \max _{x \geq 0, \frac{u}{M} \leq 1}\left\{k(x, t) u^{p}, 0\right\} \\
A=r q\left(\frac{r-1}{r+p-1}+1\right)^{\frac{(r+q-1)}{q}}(r+q-1)^{\frac{(r-q-1)}{q}} \\
\Delta(t)=A k^{\frac{(q+r-1)}{q}}(t) \kappa^{\frac{-(r-1)}{q}}(t) \\
\theta=\frac{r(r-1)}{2(r+q-1)^{2}} \\
\bar{\kappa}=\frac{r(r-1)}{(r-q+2 s+1)^{2}} \\
\bar{s}=2(s-q+1)^{2}
\end{gathered}
$$

## D. GLOBAL EXISTENCE FOR PROBLEM (14)-(16)

The main result of this section is the global existence of solutions to Problem (14)-(16) that subject to the following assumptions:

$$
\begin{array}{r}
\phi \in C^{1}((0, T)), \phi(0)=0, \phi^{\prime \prime}(u)=q(q-1) u^{q-2} \geq 0, \text { for } u>0  \tag{D.1}\\
\lim _{u \rightarrow \infty} \sup \frac{u \phi^{\prime \prime}(u)}{\phi^{\prime}(u)}=\frac{u q(q-1) u^{q-2}}{q u^{q-1}}=q-1<\infty \quad(D .2) \\
\lim _{u \rightarrow \infty} \sup \frac{f(u)}{(\phi(u))^{\kappa}}=\frac{u^{p}}{u^{\kappa q}}=u^{p-\kappa q}<\infty, \quad 0<\kappa \leq 1 \quad
\end{array}
$$

In the case of power laws for reaction, diffusion and convection, the next result amounts for the global existence of all solutions of (14)-(16) if either $q \geq 1, q>p$.

This shows that, in fact sufficiently strong convective terms may also result in a model problem for which all solutions are global.

Theorem 3.9. Under assumptions (D.1), (D.2), (D.3) with $\kappa<1$ and any $k>0$ or $\kappa=1$ and small $k$, the solutions of (14)-(16) are uniformly bounded on $\Omega \times[0, \infty)$.

## E. BLOW-UP IN FINITE TIME FOR PROBLEM (14)-(16)

In this section we show that solutions of (14)-(16) may become unbounded in finite time for a sufficiently strong source. In contrast to corresponding results for (14)-(16) with no convection term, the source term must now overcome both diffusion and convection in order for blow up to be possible in finite time. However, in the latter part of the section we show that (14)-(16) may have both global solutions and solutions which blow up in finite time, even if there are no nontrivial equilibria.

To begin, we introduce the following hypotheses on the relationship between reaction, diffusion and convection:

$$
\begin{gather*}
f \in C^{1}([0, T)), \quad f^{\prime \prime}(u) \geq 0, \text { for } u>0, \quad \int_{0}^{\infty} \frac{1}{f(u)} d u=0<\infty  \tag{19}\\
\phi(u) \leq C_{3}(f(u))^{\kappa}, \quad 0 \leq C_{4}(f(u))^{\beta}, \text { for } u>0,0<\kappa \leq 1,0<\beta<1  \tag{20}\\
\phi(u)=k_{2} f(u), \text { for } u \geq 0
\end{gather*}
$$

Under such assumptions, the global non existence (blow up) result for (14)-(16) now may be stated:

Theorem 3.10. i) Assume hypotheses (19) and (20) with $\kappa<1$, or (19) and (20) with $\kappa=1$ and large $k$. Let

$$
\lambda=\frac{1}{1-\beta} \text { and } \psi(x)=\frac{\pi}{2} \sin (\pi x)
$$

There exists $c_{0}=c_{0}\left(k, C_{3}, C 4\right)$ such that if

$$
\int_{0}^{1} u_{0}(x) \psi^{\lambda}(x) d x>c_{0}
$$

then the solution of (14)-(16), blows up in finite time.
ii) Suppose that (19) and (21) hold. If $k$ is sufficiently large, then the solution of (14)-(16) blows up in finite time for any nonnegative initial data $u_{0}(x)$ with $u_{0} \not \equiv 0$.

We close the section by summarizing what is currently known and what is not, concerning global existence and nonexistence for the reaction-diffusion-convection model governed by pure power laws.

In case $p \leq 1$, an easy supersolution argument allows the conclusion that all solutions are global. They may, however blow up in infinite time. On the other hand, when $q>p$ or $q=p$ and $k$ is sufficiently small, Theorem 3.6 implies global existence and uniform boundedness of any solution of (14)-(16). Whether or not there are solutions which become unbounded in infinite time, is actually not known.

When $p>1$ and $q>p$, then by Theorem 3.6 all solutions of (14)(16) are global, uniformly bounded and have mass which decays to 0 . This statement is also true if $q=p$ and $k$ is sufficiently small. If however, $q=p$ and $k>0$ is large, then all nontrivial solutions, blow up in finite time by Theorem 3.10. The behaviour of solutions when $k$ is between sufficiently small and large, is unknown. The case $q=p$ and $\epsilon=0$ the Problem (14)-(16) possesses a continuum of equilibrium states.

If $p>1, q \geq 1$ and $q<p$, then again by Theorem 3.6 all solutions of (14)-(16) are global, provided the initial state $u_{0}$ is small in an integral sense. The fast diffusion case $q<1$, cannot be dealt with here, only because the method converting an $L^{r}$ bound into an $L^{\infty}$ bound requires $q \geq 1$. However, the $L^{r}$ bounds derived herein are valid $\forall q>0$. Therefore, if it were the case that a solution becomes unbounded in finite time, the set of all $x \in \Omega$ such that $u(x, t)$ blows up, must be a set of measure 0 . If $\epsilon=0$ and $u_{0}$ is large, then the solution blows up in finite time.

In the situation of $q<p$, where $p>1$, Theorem 3.10 yields that if $u_{0}$ is large, then the solution blows up in finite time.

Finally, if $q<p$, then by Theorem 3.9 all solutions of (14)-(16) exist $\forall t$. Here, we see the reversal of solutions which blow up in finite time if $\epsilon=0$ to global solutions when $\epsilon>0$.

## III )

 RESULTS OF TERSENOVHere, we have to accentuate that the procedure for the proof of the estimate we are about to follow, is based on the results of [3], which are as stated briefly below:

In [3], the porous medium equation with nonlinear source is being studied. Consider the parabolic type equation :

$$
\begin{equation*}
u_{t}-\triangle u^{q}=k(t, x) u^{p}, \text { in } Q_{T}=(0, T) \times \Omega, \Omega \subset \mathbf{R}^{n} \tag{22}
\end{equation*}
$$

where $q>1, p>0,0 \leq k(t, x) \leq \kappa, T>0$, coupled with the initial and boundary conditions:

$$
\begin{equation*}
\left.u\right|_{\Gamma_{T}}=\left.\phi\right|_{\Gamma_{T}} \geq 0, \quad \Gamma_{T}=\Omega \cup([0, T] \times \partial \Omega) \tag{23}
\end{equation*}
$$

which imply that $u \geq 0$ in $Q_{T}$.
As it was mentioned before, from [1,2], the solutions of this problem may blow-up in finite time.

The global solvability was proved (see [1]) for $k \equiv 1$ and for homogeneous boundary conditions in 3 different cases mentioned previously as well as the blow-up conditions.

The goal here is to obtain a new a priori estimate of the solution and to propose the conditions that guarantee the global solvability. For simplicity, in order to work with classical solution, it is supposed that : $\left.u\right|_{\Gamma_{T}}=\left.\phi\right|_{\Gamma_{T}}>0$, which implies that $u>0$ in $Q_{T}$.

Assume that $\phi$ is continuous and $k$ is a continuously differentiable function. The domain $\Omega$ satisfies the exterior sphere condition, with

$$
\Omega \subset\left\{x \in \mathbf{R}^{n}:\left|x_{i}\right| \leq l_{i}, i=1, \ldots, n\right\}
$$

and without loss of generality suppose that $l_{1}=\min _{i}\left\{l_{i}\right\}$. We define the constant $K$ as follows:
$K=\max \left\{\left(\frac{\kappa}{q} l_{1}^{2(p-q+1)} 2^{2-p-q} \frac{(4 q-3)^{p}}{(q-1)^{q-1}}\right)^{\frac{1}{q-p}}, \frac{m}{2 l_{1}^{2}(q-1)}\right\}$, for $q \neq p$
and

$$
K=\frac{m}{2 l_{1}^{2}(q-1)}, \text { for } q=p \quad, \quad \text { where } m=\left.\max \phi\right|_{\Gamma_{T}}
$$

Theorem 3.11. There exists a global classical solution of Problem (1),(2) which satisfies the estimate

$$
0<u(t, x) \leq \frac{l_{1}^{2}}{2}(4 q-3) K, \forall x \in \Omega \text { and } t \geq 0
$$

in the following 3 cases:
a/ if $q>p$,
b/ if $q=p$, and provided that the following inequality holds:

$$
l_{1}^{2} \leq \frac{q}{\kappa} \frac{(4 q-4)^{q-1}}{(4 q-3)^{q}}
$$

c/ if $q<p$, and provided that the following inequality holds:

$$
m \leq 2 l_{1}^{2}(q-1)\left(\frac{\kappa}{q} l_{1}^{2(p-q+1)} 2^{2-p-q} \frac{(4 q-3)^{p}}{(q-1)^{q-1}}\right)^{\frac{1}{q-p}} .
$$

Remark. a/ In the case $q=p$, the smallness type restriction on the size of the domain is only in one direction and the a priori estimate of the Theorem receives the form:

$$
0<u(t, x) \leq \frac{4 q-3}{4 q-4} m
$$

b/ In the case $q<p$ we do not need any additional restrictions on p for $n \geq 3$.

Eventually, in comparison with the results concerning the global solvability in [1], the results in [3] respectively for $k(x, t) \equiv 1$, become as follows:
a) If $\mathrm{p}<\mathrm{q}$, then there exists a global classical solution of the Problem (22),(23).
b) If $\mathrm{p}=\mathrm{q}$, then there exists a global classical solution of the Problem (22),(23) if the following inequality is satisfied:

$$
l_{1}^{2} \leq \frac{q}{\kappa} \frac{(4 q-4)^{q-1}}{(4 q-3)^{q}}
$$

c) If $p>q$, then there exists a global classical solution of the Problem (22),(23) if the following inequality is satisfied:

$$
m \leq 2 l_{1}^{2}(q-1)\left(\frac{\kappa}{q} l_{1}^{2(p-q+1)} 2^{2-p-q} \frac{(4 q-3)^{p}}{(q-1)^{q-1}}\right)^{\frac{1}{q-p}}
$$

## 4. FORMULATION OF OUR RESULT

Now we shall present the main subject of study of this paper. It is the same equation as before with the boundary condition to be the only difference. We are going to find an a priori estimate of the new Problem and guarantee the existence of classical and global solution of the Problem. We base the result on the maximum principle.

Definition. We define as classical solution of Problem (24)-(26) the function $u$ which satisfies the following 2 properties:
$i / u>0$ in $Q_{T}$,
ii/ $u \in C^{2,1}\left(\overline{Q_{T}}\right)$
We study the porous medium, slow diffusion $(q \geq 1)$ equation with nonlinear source and Robin initial-boundary conditions. We consider the parabolic type equation :

$$
\begin{equation*}
u_{t}-\triangle u^{q}=k(t, x) u^{p}, \quad \text { in } Q_{T}=(0, T) \times \Omega, \quad \Omega \subset \mathbf{R}^{n} \tag{24}
\end{equation*}
$$

where $q>1, p>0,0<k(t, x) \leq \kappa, T>0$ coupled with the initial and boundary conditions respectively:

$$
\begin{align*}
& 0<\epsilon \leq u(x, 0)=u_{0}(x) \leq m, \quad x \in \Omega  \tag{25}\\
& \frac{\partial u}{\partial \eta}(x, t)+\phi(x, t, u)=0, \quad x \in \partial \Omega \times(0, T) \tag{26}
\end{align*}
$$

Assume that $\phi$ is continuous and $k$ is a continuously differentiable function. The domain $\Omega$ satisfies the exterior sphere condition, with

$$
\Omega \subset\left\{x \in \mathbf{R}^{n}:\left|x_{i}\right| \leq l_{i}, i=1, \ldots, n\right\}
$$

and without loss of generality suppose that $l_{1}=\min _{i}\left\{l_{i}\right\}$.
We define the constant $K$ as follows:
$K=\max \left\{\left(\frac{\kappa}{q} l_{1}^{2(p-q+1)} 2^{2-p-q} \frac{(4 q-3)^{p}}{(q-1)^{q-1}}\right)^{\frac{1}{q-p}}, \frac{m}{2 l_{1}^{2}(q-1)}\right\}, \quad$ for $q \neq p$
and

$$
K=\frac{m}{2 l_{1}^{2}(q-1)}, \quad \text { for } q=p \quad, \quad \text { where } \quad m=\left.\max u_{0}(x)\right|_{\Omega}
$$

Theorem 4.1. If $\phi(x, t, u)>\frac{m}{2 l_{1}(q-1)}>0$, for $u>m>0$ and $\phi(x, t, u)<0$, for $u<\epsilon$, then there exists a global classical solution of Problem (24)-(26) which satisfies the estimate,

$$
0<u(t, x) \leq \frac{l_{1}^{2}}{2}(4 q-3) K, \forall x \in \Omega \text { and } t \geq 0
$$

in the following 3 cases:
$a /$ if $q>p$,
b/ if $q=p$, and the following inequality is satisfied:

$$
l_{1}^{2} \leq \frac{q}{\kappa} \frac{(4 q-4)^{q-1}}{(4 q-3)^{q}}
$$

c/ if $q<p$, and the following inequality is satisfied:

$$
m \leq 2 l_{1}^{2}(q-1)\left(\frac{\kappa}{q} l_{1}^{2(p-q+1)} 2^{2-p-q} \frac{(4 q-3)^{p}}{(q-1)^{q-1}}\right)^{\frac{1}{q-p}}
$$

Remark. In the case $q=p$, the smallness type restriction on the size of the domain is only in one direction and estimate above takes the form:

$$
0<u(x, t) \leq \frac{4 q-3}{4 q-4} m
$$

Because,

$$
u(x, t) \leq \frac{l_{1}^{2}}{2}(4 q-3) K=\frac{l_{1}^{2}}{2}(4 q-3) \frac{m}{2 l_{1}^{2}(q-1)}=\frac{4 q-3}{4 q-4} m
$$

The inequalities in the previous Theorem constitute the smallness type restrictions for $l_{1}$ and $m$ respectively to the cases $b$ and $c$.

## 5. PROOF OF THEOREM 4.1

In order to prove theorem 4.1, we must first prove the following auxiliary proposition.

Proposition: If $\phi(x, t, u)<0$, for $u<\epsilon$, then the solution of Problem (24)-(26) is strictly positive ( $u>0$ ).

Proof : Consider the equation (24) and its auxiliary equation respectively:

$$
u_{t}=q u^{q-1} \triangle u+q(q-1) u^{q-2}|\nabla u|^{2}+k(t, x) u^{p}
$$

$$
u_{t}=\left|f_{1}(u)\right| \triangle u+q(q-1)|u|^{q-2}|\nabla u|^{2}+\left|f_{2}(u)\right| .
$$

Where, $f_{1}(u)=\left\{\begin{array}{ll}q u^{q-1} & , u \geq \epsilon \\ q \epsilon^{q-1} & , u<\epsilon\end{array} \quad, \quad f_{2}(u)= \begin{cases}k(x, t) u^{p} & , u \geq \epsilon \\ k(x, t) \epsilon^{p} & , u<\epsilon\end{cases}\right.$
Let,

$$
v(x, t)=u(x, t)-\epsilon, \epsilon>0 .
$$

Then, the auxiliary equation for the function $v$ becomes:

$$
v_{t}=\left|f_{1}(u)\right| \Delta v+q(q-1)|u|^{q-2}|\nabla u|^{2}+\left|f_{2}(u)\right| .
$$

- If $v$ attains its negative minimum at the internal point $\left(x_{0}, t_{0}\right)$ or at the top base, then:

$$
\nabla v=0, v<0, \Delta v \geq 0, v_{t} \leq 0
$$

So, by the auxiliary equation for $v$, we receive a contradiction, because on the LHS of the equation we have something non-positive since $v_{t} \leq 0$ and on the RHS we have something strictly positive since $\nabla v=0 \Rightarrow \nabla u=0, \Delta v \geq 0$ and $f_{2}(u)$ is a strictly positive quantity.

- For $\mathrm{t}=0: v(x, 0)=u(x, 0)-\epsilon \Rightarrow v(x, 0)=u_{0}(x)-\epsilon \geq 0$.
- Finally, if $v$ attains its negative minimum at the point $\left(x_{0}, t_{0}\right) \in$ $\partial \Omega \times(0, T)$, then at this point: $\frac{\partial v}{\partial \eta} \leq 0(\eta$ is the outward normal unit vector). Therefore,

$$
\frac{\partial u}{\partial \eta} \leq 0 \Rightarrow-\phi(x, t, u) \leq 0 \Rightarrow \phi(x, t, u) \geq 0
$$

Due to the assumption for $\phi$, we have that:

$$
\phi(x, t, u)<0, u<\epsilon
$$

Since the assumption for $\phi$ contradicts the above non-positive condition for $\phi$ at the lateral surface, then $v$ can't attain its negative minimum at the lateral surface.

Thus, since $v$ can't attain its negative minimum neither at the internal nor at the top base nor at the lateral surface, then it attains it either at the bottom base or $v \geq 0$. But at the bottom base $v \geq 0$. Then, necessarily $v \geq 0 \Rightarrow u-\epsilon \geq 0 \Rightarrow u \geq \epsilon \Rightarrow u>0$ at $Q_{T}$.

For example we can choose:

$$
\phi(x, t, u)=a(u-\epsilon), a>\frac{m}{m-\epsilon} \frac{1}{2 l_{1}(q-1)} .
$$

Now, we are ready to prove Theorem 4.1.
We use maximum principle and with proof by contradiction, we will have the desired result.

The methodology we utilize is as follows:

- Assume that the function $v(x, t)=u(x, t)-h\left(x_{1}\right)$ attains its positive maximum at the internal point $N$ or at the top base.

We will show that $v$ can't attain its positive maximum at the internal point $N$, hence 2 things can happen:
i/ Either $v$ attains its maximum at $\Gamma_{T}$ (lateral surface + bottom base ( $\mathrm{t}=0$ )).
ii/ Or $v \leq 0$, since $v$ can not be positive from the previous cases.
Rewrite equation (24) in the following form:

$$
\begin{equation*}
u_{t}=q u^{q-1} \triangle u+q(q-1) u^{q-2}|\nabla u|^{2}+k(t, x) u^{p} \tag{27}
\end{equation*}
$$

which results from the following computations:

$$
\nabla u^{q}=q u^{q-1} \nabla u
$$

and

$$
\triangle u^{q}=q u^{q-1} \triangle u+q(q-1) u^{q-2}|\nabla u|^{2} .
$$

Consider the auxiliary equation:

$$
\begin{equation*}
u_{t}=q u^{q-1} \triangle u+q(q-1) u^{q-2}|\nabla u|^{2}+k(t, x) g(u), \tag{28}
\end{equation*}
$$

where, $g(u)= \begin{cases}u^{p} & , u \leq \frac{l_{1}^{2}}{2}(4 q-3) K \\ \left(\frac{l_{1}^{2}}{2}(4 q-3) K\right)^{p} & , u>\frac{l_{1}^{2}}{2}(4 q-3) K\end{cases}$
We performed this procedure by considering the above auxiliary equation in order to obtain the classical solution of the Problem and furthermore because if we didn't, the solution might blow up eventually. The existence of a classical solution follows from the standard theory.

Our goal is to obtain the a priori estimate

$$
u \leq \frac{l_{1}^{2}}{2}(4 q-3) K
$$

for the solution of Problem (28),(25),(26) and by this to show that equations (27),(28) coincide.

Consider the functions:

$$
v(t, x)=u(t, x)-h\left(x_{1}\right), \quad h\left(x_{1}\right)=\frac{K}{2}\left(l_{1}^{2}-x_{1}^{2}\right)+2 l_{1}^{2}(q-1) K .
$$

Then the equation for the function $v(t, x)$, becomes:

$$
\begin{equation*}
v_{t}-q u^{q-1} \triangle v=q(q-1) u^{q-2}|\nabla u|^{2}+k(t, x) g(u)-q u^{q-1} K . \tag{29}
\end{equation*}
$$

The last is obvious, due to the following computations:

$$
\begin{aligned}
& v_{t}=u_{t}-h_{t}\left(x_{1}\right) \Rightarrow v_{t}=u_{t}, \\
& \triangle v=\triangle u-h^{\prime \prime}\left(x_{1}\right) \Rightarrow \Delta v=\triangle u+K, \\
& -q u^{q-1} \triangle v=-q u^{q-1} \triangle u-q u^{q-1} K .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& v_{t}-q u^{q-1} \triangle v=u_{t}-h_{t}(x)-q u^{q-1} \triangle u+q u^{q-1} h^{\prime \prime}\left(x_{1}\right) \Rightarrow \\
\Rightarrow & v_{t}-q u^{q-1} \triangle v=q(q-1) u^{q-2}|\nabla u|^{2}+k(t, x) g(u)-q u^{q-1} K .
\end{aligned}
$$

- Assume that the function $v$ attains its positive maximum at the point $N \in \overline{Q_{T}} \backslash \Gamma_{T}$. At this point, we have that $v>0$ and $\nabla v=0$, namely:

$$
v>0 \Rightarrow u>h=\frac{K}{2}\left(l_{1}^{2}-x_{1}\right)^{2}+2 l_{1}^{2}(q-1) K \geq 2 l_{1}^{2}(q-1) K .
$$

However, $\nabla v=0 \Rightarrow v_{x_{i}}=0, i=1, \ldots, n$. So,

$$
v_{x_{1}}=u_{x_{1}}-h^{\prime}\left(x_{1}\right) \Rightarrow u_{x_{1}}=\underset{\substack{x_{1}}}{h_{24}}=-K x_{1}, u_{x_{i}}=0, i=2, \ldots, n
$$

Thus, we have :

$$
\begin{aligned}
& v_{t}-\left.q u^{q-1} \triangle v\right|_{N}=q(q-1) u^{q-2}\left(-K x_{1}\right)^{2}+k(t, x) g(u)-\left.q u^{q-1} K\right|_{N} \\
& <q(q-1) u^{q-2} K^{2} l_{1}^{2}+\kappa\left(\frac{l_{1}^{2}}{2}(4 q-3) K\right)^{p}-\left.q u^{q-1} K\right|_{N} \\
& \left.=\left(q(q-1) u^{q-2} K^{2} l_{1}^{2}-\frac{q}{2} u^{q-1} K\right)+\left(\kappa\left[\frac{l_{1}^{2}}{2}(4 q-3) K\right]^{p}-\frac{q}{2} u^{q-1} K\right)\right)\left.\right|_{N} \\
& <\frac{q}{2} u^{q-2} K\left(2 l_{1}^{2}(q-1) K-u\right)+\left.\left(\kappa\left[\frac{l_{1}^{2}}{2}(4 q-3) K\right]^{p}-\frac{q}{2}\left[2 l_{1}^{2}(q-1) K\right]^{q-1} K\right)\right|_{N} \\
& <K^{p}\left[\kappa\left(\frac{l_{1}^{2}}{2}(4 q-3)\right)^{p}-\frac{q}{2}\left(2 l_{1}^{2}(q-1)\right)^{q-1} K^{q-p}\right] \leq 0 .
\end{aligned}
$$

Hence we obtain at $N$ that:

$$
v_{t}-q u^{q-1} \triangle v<0
$$

which is impossible.
On the other hand at $N$, we have that:

$$
\Delta v \leq 0, v_{t} \geq 0, u>0
$$

Thus,

$$
v_{t}-\left.q u^{q-1} \triangle v\right|_{N} \geq 0
$$

So, $v$ can't attain its positive maximum at $\bar{Q}_{T} \backslash \Gamma_{T}$.

- Now, we will show that $v$ can't attain its positive maximum at the lateral surface $\partial \Omega \times(0, T)$.

Assume that $v$ attains its positive maximum at the point $N_{1} \in$ $\partial \Omega \times(0, T)$. Then at this point, $\frac{\partial v}{\partial \eta} \geq 0$ and we have that:

$$
\begin{aligned}
& \frac{\partial u}{\partial \eta}-\frac{\partial h}{\partial \eta}\left(x_{1}\right) \geq 0 \Rightarrow \\
& \Rightarrow \frac{\partial u}{\partial \eta} \geq \frac{\partial h}{\partial \eta}\left(x_{1}\right) \Rightarrow \\
& \Rightarrow-\phi\left(x, t, u\left(N_{1}\right)\right) \geq \frac{\partial h}{\partial \eta}\left(x_{1}\right) \Rightarrow \\
& \Rightarrow-\phi\left(x, t, u\left(N_{1}\right)\right)-\frac{\partial h}{\partial x_{1}}\left(x_{1}\right) \cdot \mathbf{n} \geq 0 \Rightarrow \\
& \Rightarrow-\phi\left(x, t, u\left(N_{1}\right)\right)-\left(-K x_{1}, 0, \ldots, 0\right) \cdot\left(n_{1}, \ldots, n_{n}\right) \geq 0 \Rightarrow
\end{aligned}
$$

$$
\Rightarrow-\phi\left(x, t, u\left(N_{1}\right)\right)+K x_{1} n_{1} \geq 0
$$

Due to the assumption for $\phi$, we have that:

$$
-\phi\left(x, t, u\left(N_{1}\right)\right)+K x_{1} n_{1}<0
$$

Namely,

$$
\begin{aligned}
& \phi\left(x, t, u\left(N_{1}\right)\right)>K x_{1} n_{1} \Rightarrow \phi\left(x, t, u\left(N_{1}\right)\right)>\max \left(K x_{1} n_{1}\right)=K l_{1} \Rightarrow \\
& \Rightarrow \phi\left(x, t, u\left(N_{1}\right)\right)>\frac{m}{2 l_{1}^{2}(q-1)} l_{1} \Rightarrow \phi\left(x, t, u\left(N_{1}\right)\right)>\frac{m}{2 l_{1}(q-1)}>0
\end{aligned}
$$

for $u\left(N_{1}\right)>m>0$.
Since the assumption for $\phi$ contradicts the previous non-negative condition for $\phi$ at the lateral surface, then $v$ can not attain its positive maximum at the lateral surface.

- For $\mathrm{t}=0$, we apply the initial condition (25) for $v$ and we have:

$$
\begin{aligned}
& v(0, x)=u(0, x)-h\left(x_{1}\right) \Rightarrow \\
& \Rightarrow v(0, x)=u_{0}(x)-\frac{K}{2}\left(l_{1}^{2}-x_{1}^{2}\right)-2 l_{1}^{2}(q-1) K \Rightarrow \\
& \Rightarrow v(0, x)=u_{0}(x)-\frac{K}{2}\left(l_{1}^{2}-x_{1}^{2}\right)-m \Rightarrow \\
& \Rightarrow v(0, x)=u_{0}(x)-\frac{K}{2}\left(l_{1}^{2}-x_{1}^{2}\right)-\left.\max u_{0}(x)\right|_{\Omega} \Rightarrow \\
& \Rightarrow v(0, x) \leq-\frac{K}{2}\left(l_{1}^{2}-x_{1}^{2}\right) \leq 0 \Rightarrow \\
& \Rightarrow v(0, x) \leq 0 .
\end{aligned}
$$

Because, $u_{0}(x)-\left.\max u_{0}(x)\right|_{\Omega} \leq 0$.
However, since $v$ can not attain its positive maximum neither at the internal nor at the top base nor at the lateral surface, then either attains it at the bottom base or $v \leq 0$. But, at the bottom base $v \leq 0$. Therefore, eventually $v \leq 0$.

Taking into account the fact that $v \leq 0$, we conclude that:

$$
u(t, x) \leq h\left(x_{1}\right) \leq h(0)=\frac{K}{2} l_{1}^{2}+2 l_{1}^{2}(q-1) K=\frac{l_{1}^{2}}{2}(4 q-3) K
$$

Since $h$ attains maximum at 0 , through the 2 nd derivative rule $\left(h^{\prime \prime}\left(x_{1}\right)=-K<0\right)$.

Eventually, the inequality

$$
\kappa\left(\frac{l_{1}^{2}}{2}(4 q-3)\right)^{p}-\frac{q}{2}\left(2 l_{1}^{2}(q-1)\right)^{q-1} K^{q-p} \leq 0
$$

holds for the following 3 cases:
a) for $q>p$ : follows directly from the definition of $K$. Thus,

$$
\begin{aligned}
& \kappa\left(\frac{l_{1}{ }^{2}}{2}(4 q-3)\right)^{p}-\frac{q}{2}\left(2 l_{1}^{2}(q-1)\right)^{q-1}\left(\frac{\kappa}{q} l_{1}^{2(p-q+1)} 2^{2-p-q} \frac{(4 q-3)^{p}}{(q-1)^{q-1}}\right)^{\frac{q-p}{q-p}}= \\
& =\ldots=K 2^{-p} l_{1}^{2 p}(4 q-3)^{p}-2^{2} \frac{K}{q} l_{1}^{2 p}(4 q-3)^{p} \leq 0
\end{aligned}
$$

b) for $q=p$ : follows from the restriction on $l_{1}$. Thus,

$$
\frac{p^{p}}{2^{p} \kappa^{p-1}}\left(\frac{4 p-4}{4 p-3}\right)^{p(p-1)}-p^{p} 2^{p-2}(p-1)^{p-1} \kappa^{1-p} \frac{(4 p-4)^{(p-1)^{2}}}{(4 p-3)^{p(p-1)}} \leq 0,
$$

c) for $q<p$ : the inequality above takes the form:

$$
\begin{aligned}
& \kappa\left(\frac{l_{1}^{2}}{2}(4 q-3)\right)^{p} \leq \frac{q}{2}\left(2 l_{1}^{2}(q-1)\right)^{q-1} K^{q-p} \Rightarrow K^{q-p} \geq \frac{\kappa\left(\frac{l_{1}^{2}}{2}(4 q-3)\right)^{p}}{\frac{q}{2}\left(2 l_{1}^{2}(q-1)\right)^{q-1}} \Rightarrow \\
& \Rightarrow K^{p-q} \leq \frac{\frac{q}{2}\left(2 l_{1}^{2}(q-1)\right)^{q-1}}{\kappa\left(\frac{l_{1}{ }^{2}}{2}(4 q-3)\right)^{p}}
\end{aligned}
$$

which is satisfied by the definition of $K$ if:

$$
\frac{m}{2 l_{1}^{2}(q-1)} \leq\left(\frac{\kappa}{q} l_{1}^{2(p-q+1)} 2^{2-p-q} \frac{(4 q-3)^{p}}{(q-1)^{q-1}}\right)^{\frac{1}{q-p}}
$$

So, for example we can choose:

$$
\phi(x, t, u)=a u, \forall t \in(0, T), \forall u>h \Rightarrow \forall u>m
$$

Thus, $\frac{m}{2 l_{1}(q-1)}<a u \Rightarrow \frac{m}{2 l_{1}(q-1) u}<a \Rightarrow \frac{u}{2 l_{1}(q-1) u}<a \Rightarrow a>\frac{1}{2 l_{1}(q-1)}>0$.
Including the positivity of $u$, the selection can be:

$$
\phi(x, t, u)=a(u-\epsilon), a>\frac{m}{m-\epsilon} \frac{1}{2 l_{1}(q-1)} .
$$

The last inequality is applied, because:

$$
\begin{aligned}
& \Rightarrow h\left(l_{1}\right)=2 l_{1}^{2}(q-1) K \Rightarrow \\
& \Rightarrow h\left(l_{1}\right)=2 l_{1}^{2}(q-1) \frac{m}{2 l_{1}^{2}(q-1)} \Rightarrow \\
& \Rightarrow h\left(l_{1}\right)=m .
\end{aligned}
$$

## 6. PHYSICAL INTERPRETATION OF THE PROBLEM

Rewrite equation (1) in its divergence form:

$$
u_{t}=\operatorname{div}\left(q|u|^{q-1} \nabla u\right)+k(x, t) u^{p} .
$$

Where $q>1, p>0,0 \leq k(t, x) \leq \kappa, T>0$ with the initial and boundary conditions:

$$
\begin{equation*}
q u^{q-1} \frac{\partial u}{\partial \eta}(x, t)+\phi(x, t, u)=0, \quad x \in \partial \Omega \times(0, T) \tag{30}
\end{equation*}
$$

respectively.
Zero Neumann condition corresponds to the case when we have absolute insulation and thus there can not be heat flow through the boundary of $Q_{T}$. Therefore, blow up is inevitable and it is reasonable to assume that necessarily $\phi(x, t, u) \not \equiv 0$.

According to the law of heat conduction (Fourier's law), the heat flux density is equal to the product of the heat conductivity $k$ and the negative gradient $-\nabla u$ of the absolute temperature $u$. Thus, from the aspect of Physics, boundary condition (31) is a more appropriate condition than (26).

Heat flux cannot be performed from the warmer to the colder regions. The laws of Thermodynamics state that absolute zero cannot be achieved, thus $u$ must be strictly positive. But, substantially this is a thing we have proved and agrees with the assumptions we defined.

Accordingly, even if the temperature on the boundary is arbitrary small, the heat flow from the boundary does not allow the formation of a zone with 0 temperature.

Thus equation

$$
\frac{\partial u}{\partial t}=\triangle\left(u^{q}\right)
$$

is uniformly parabolic in any region where $u$ is bounded away from 0 . But is degenerate at the point $u=0$. In standard Fickian diffusion theory, the diffusivity $q u^{q-1}$, vanishes together with $u$. The most striking manifestation of this non-linear degeneracy is that in porous medium flow, there is a finite speed of propagation of disturbances from rest. This is in stark contrast to the linear heat equation $(q=1)$, where there is an infinite speed of propagation.

Returning to the proof in section 5, if we divide the boundary condition by $q u^{q-1}$, we get:

$$
\frac{\partial u}{\partial \eta}(x, t)+\frac{\phi(x, t, u)}{q u^{q-1}}=0 \Rightarrow \frac{\partial u}{\partial \eta}(x, t)+\left.\phi_{1}(x, t, u)\right|_{\partial \Omega \times(0, T)}=0
$$

By setting:

$$
\phi_{1}(x, t, u)=\frac{\phi(x, t, u)}{q u^{q-1}}
$$

we observe that the Problem has the same form and nothing changes significantly in the Proof. Finally for this reason, the selection of $\phi$ is the only thing that changes.

- We apply the same procedure as in the section 5 but this time for the boundary condition (31). So again the only difference is whether $v$ attains its positive maximum at the point $N_{2} \in \partial \Omega \times(0, T)$ (which does not).

At this point where the positive maximum is attained: $\frac{\partial v}{\partial \eta} \geq 0$ and we have that:

$$
\begin{aligned}
& \frac{\partial u}{\partial \eta}-\frac{\partial h}{\partial \eta}\left(x_{1}\right) \geq 0 \Rightarrow \\
& \Rightarrow \frac{\partial u}{\partial \eta} \geq \frac{\partial h}{\partial \eta}\left(x_{1}\right) \Rightarrow \\
& \Rightarrow-\frac{1}{q u^{q-1}} \phi\left(x, t, u\left(N_{2}\right)\right) \geq \frac{\partial h}{\partial \eta}\left(x_{1}\right) \Rightarrow \\
& \Rightarrow-\frac{1}{q u^{q-1}} \phi\left(x, t, u\left(N_{2}\right)\right)-\frac{\partial h}{\partial x_{1}}\left(x_{1}\right) \cdot \mathbf{n} \geq 0 \\
& \Rightarrow-\frac{1}{q u^{q-1}} \phi\left(x, t, u\left(N_{2}\right)\right)-\left(-K x_{1}, 0, \ldots, 0\right) \cdot\left(n_{1}, \ldots, n_{n}\right) \geq 0 \Rightarrow \\
& \Rightarrow-\frac{1}{q u^{q-1}} \phi\left(x, t, u\left(N_{2}\right)\right)+K x_{1} n_{1} \geq 0
\end{aligned}
$$

The assumption for $\phi$, should be:

$$
-\frac{1}{q u^{q-1}} \phi\left(x, t, u\left(N_{2}\right)\right)+K x_{1} n_{1}<0 .
$$

Namely, $\frac{1}{q u^{q-1}} \phi\left(x, t, u\left(N_{2}\right)\right)>K x_{1} n_{1} \Rightarrow$

$$
\Rightarrow \frac{1}{q u^{q-1}} \phi\left(x, t, u\left(N_{2}\right)\right)>\max \left(K x_{1} n_{1}\right)=K l_{1} \Rightarrow
$$

$$
\begin{aligned}
& \Rightarrow \phi\left(x, t, u\left(N_{2}\right)\right)>q u^{q-1} K l_{1} \Rightarrow \\
& \Rightarrow \phi\left(x, t, u\left(N_{2}\right)\right)>q u^{q-1} \frac{m}{2 l_{1}^{2}(q-1)} l_{1} \Rightarrow \\
& \Rightarrow \phi\left(x, t, u\left(N_{2}\right)\right)>q u^{q-1} \frac{m}{2 l_{1}(q-1)}>0, \text { for } u\left(N_{2}\right)>m>0 .
\end{aligned}
$$

For the same reason as in Section 5, the assumption for $\phi$ contradicts the previous non-negative condition for $\phi$ at the lateral surface. Then, similarly $v \leq 0$.

So, for example we can choose:

$$
\phi(x, t, u)=a u^{q}, \forall t \in(0, T), \forall u>h \Rightarrow \forall u>m,|\mathbf{n}| \leq 1
$$

Thus, $q u^{q-1} \frac{m}{2 l_{1}(q-1)}<a u^{q} \Rightarrow q u^{q-1} \frac{m}{2 l_{1}(q-1) u^{q}}<a \Rightarrow$

$$
\Rightarrow q u^{q-1} \frac{u}{2 l_{1}(q-1) u^{q}}<a \Rightarrow a>\frac{q}{2 l_{1}(q-1)} .
$$

Including the positivity of $u$, the selection can be:

$$
\phi(x, t, u)=a(u-\epsilon)^{q}, a>\frac{m^{q}}{(m-\epsilon)^{q}} \frac{q}{2 l_{1}(q-1)} .
$$

- For the positivity of $u$ we have the same results and the only difference is in $\partial \Omega \times(0, T)$.

Namely, if $v$ attains its negative minimum at the point $\left(x_{0}, t_{0}\right) \in$ $\partial \Omega \times(0, T)$, then at this point: $\frac{\partial v}{\partial \eta} \leq 0$. Therefore,

$$
\frac{\partial u}{\partial \eta} \leq 0 \Rightarrow \frac{-\phi(x, t, u)}{q u^{q-1}} \leq 0 \Rightarrow \phi(x, t, u) \geq 0 .
$$

Due to the assumption for $\phi$, we have that:

$$
\phi(x, t, u)<0, u<\epsilon .
$$

Since the assumption for $\phi$ contradicts the above non-positive condition for $\phi$ at the lateral surface, then $v$ can't attain its negative minimum at the lateral surface.

Thus, since $v$ can't attain its negative minimum neither at the internal nor at the top base nor at the lateral surface, then it attains it either at the bottom base or $v \geq 0$. But at the bottom base $v \geq 0$. Then, necessarily $v \geq 0 \Rightarrow u-\epsilon \geq 0 \Rightarrow u \geq \epsilon \Rightarrow u>0$ at $Q_{T}$.

## 7. CONCLUSION

In this paper, we dealt with a non-linear parabolic type Problem. We examined the global solvability. We ensured the existence of classical solution of the Problem under some restrictions. Based on previous research studies, we applied proof by contradiction to find an upper bound for the solution. By using the maximum principle, we found an appropriate bound for the solution. It is known from other papers when the Problem with Dirichlet boundary conditions has global solution and when blows-up, every time with the appropriate value selection of the exponents $p, q$.

On the other hand, some things are not yet known, such as if we can find the explicit solution of this problem. Regardless of the big utility the method we develop presents, lacks generalization. In the end, we are hopeful that we will find a method which solves all types of Problems or at least a wide range of them.

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