University of Crete

Department of Physics

Bachelor's Thesis

Correlation functions in the AdS/CFT correspondence

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Abstract

The AdS_{n+1}/CFT_n correspondence is a fascinating aspect of superstring theory. This thesis aims to calculate the 2 and 3-point functions of a scalar field using both conformal field theory and classical gravity in AdS space, employing the supergravity approximation. We firstly demonstrate the essence of this duality and highlight the type of information that can be correlated between the 2 theories. We show that the results obtained from the classical methods are consistent with the predictions of the CFT.

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Notations

In Poincaré coordinates, capital indices $(M, N, \dots)$ sum over all coordinates, while lowercase indices $(\mu, \nu, \dots)$ sum over all coordinates except the radial depicted as z
Metric signature: $(-, +, \dots, +)$
Units: $\hbar = c = 1$. In a few cases we also use $R = 1$, where R is the radius of curvature of the AdS spacetime
We only use one conventional dimensionful unit: *mass = energy = 1/length = 1/time*

Contents

1 Introduction: What is the AdS/CFT correspondence?

Physicists had reasons to believe that there exists a relation between gravity theories and quantum field theories in lower dimensions. This idea did not come out of nowhere. For example, let us recall the holographic principle. The Hawking-Bekenstein entropy is

$$S = \frac{k_B \, c^3 \, A}{4 \, G_N \, \hbar}$$

One would expect that entropy, being an extensive quantity, would depend on the *volume* of a black hole and not from its event's horizon *surface* as shown above. But this relation suggests that the information contained in a volume of space can be represented as a theory on the boundary of that space. This leads to a broader conjecture that:

(n+1)-dimensional quantum gravity theory $\longleftrightarrow$ lower dimensional QFT in its boundary

However, this is a generalized hypothesis that suggests a more generic holographic duality. In a more specific context, it has been proposed that **any complete quantum gravity theory in an asymptotically AdS spacetime could define a CFT on its boundary**. Meaning, in the more special case where the quantum gravity theory is asymptotically AdS, the corresponding QFT is conformally invariant, hence it is a CFT. A quantum gravity theory should be well approximated by general relativity in low energies, and its lagrangian density is generally coupled to all kinds of matter, like scalar fields, fermions, or more exotic stuff like strings and membranes. While the AdS/CFT correspondence provides a powerful framework for exploring these ideas, it remains a conjecture and an area of active research rather than an established fact.

Interestingly however, there are a few known unquestionable holographic dualities that hold true. The AdS/CFT correspondence was first proposed by Juan Maldacena in [1]. But he mainly focused on a certain case. More specifically, he investigated a solution in 10-dimensional type IIB Supergravity (the D3-brane) that can be approximated by a metric of the form $AdS_5 \times S^5$. He observed that the CFT defined in the boundary of the AdS_5 is no other than the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory.

Under the scope of this thesis, we will not explore complex cases like the above. We will assume a classical gravity theory coupled only to scalar fields, meaning we will solve the classical equations of motion without taking into account any quantum corrections. We will firstly present a few basic concepts of the Anti de Sitter (AdS) space, as well as QFT and more importantly CFT. Since there exists a correspondence between AdS_{n+1} and CFT_n, it is natural to expect that physical quantities on the CFT can be calculated using classical gravity tools in the bulk, instead of standard CFT methods. In this thesis, our objective is to calculate the 2 and 3-point functions of scalar fields using classical gravity and show that the results are indeed as the CFT predicted.

2 Anti-de-Sitter Spacetime

In the framework of this thesis, the spacetime that we will mainly be involved with is the Anti-de-Sitter (AdS) spacetime. The AdS_n spacetime is an n_{th} dimensional spacetime with a constant negative curvature.

2.1 Derivation of (n,1) dimensional AdS

We can derive the AdS_{n+1} metric by 2 ways, one as a solution of the Einstein's Field Equations, and the other is to define it by embedding, meaning placing a constraint in an $n + 2$ Minkowski spacetime.

2.1.1 Derivation by the Einstein Field Equations

By taking the following action

$$S = \frac{1}{16\pi G_N} \int d^{n+1}x\sqrt{-g}(\mathcal{R} - 2\Lambda), \qquad \Lambda < 0 \tag{2.1}$$

and doing calculus of variations as shown in appendix A, we get the following equations of motion:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{2.2}$$

and by doing a contraction we conclude that the Ricci scalar is

$$\mathcal{R} = 2\Lambda \cdot \frac{d+1}{d-1} \tag{2.3}$$

One of the solutions that satisfy the equations in (2.2), assuming a completely empty space is the AdS spacetime. This is the most symmetric solution for these equations.

For example, in (1,3) dimensions, one can check that the vacuum solution of 2.2 is the metric

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \frac{\Lambda}{3}r^2} + r^2 d\Omega_2^2 \tag{2.4}$$

For $\Lambda < 0$, this is an AdS_{3+1} space written in spherical coordinates.

2.1.2 Derivation by embedding

To define Ads space by embedding, we start by taking (n+2) dimensional flat spacetime in cartesian coordinates.

$$ds^2 = -(dX^0)^2 - (dX^{n+1})^2 + \sum_{i=1}^{n}(dX^i)^2 \tag{2.5}$$

and place the following constraint:

$$(X^0)^2 + (X^{n+1})^2 - \sum_{i=1}^{n}(X^i)^2 = R^2 \tag{2.6}$$

which is basically a hyperboloid embedded in n+2 dimensions.

We can see that a good parametrization that satisfies the constraint (2.6) is the following:

$$X^0 = R\cosh\rho\,\sin\tau \quad X^{n+1} = R\cosh\rho\,\cos\tau \quad X^i = R\sinh\rho\,\hat{X}^i \quad where \quad \sum_{i=1}^{i=n}(\hat{X}^i)^2 = 1 \tag{2.7}$$

where for the components X^i we used hyperspherical coordinates. Applying this parametrization to the metric 2.5 we get the following induced metric:

$$ds^2 = R^2(-\cosh^2\rho\,d\tau^2 + d\rho^2 + \sinh^2\rho\,d\Omega_{n-1}^2), \quad \rho > 0, \quad \tau \in [0, 2\pi] \tag{2.8}$$

One could observe that τ is periodic. Thus, in order to avoid closed time-like curves, we can expand τ in $\mathbb{R}$, meaning $\tau \in \mathbb{R}$ without loss of generality, since the metric 2.8 for $\tau \in \mathbb{R}$ satisfies the equations of motion (2.2) regardless. This set of coordinates is called *global coordninates* and they are used to describe the whole of AdS space.

However, there is also another set of coordinates that is useful for describing the AdS space, called *Poincaré coordinates.* The parametrization needed to acquire the metric written in these coordinates is

$$X^{\mu=0,\dots n-1} = \frac{R}{z}x^{\mu}, \quad X^{n+1} = \frac{R}{2z}\left(1 + |x^{\mu}|^2 + z^2\right), \quad X^{n} = \frac{R}{2z}\left(1 - |x^{\mu}|^2 - z^2\right) \tag{2.9}$$

where $|x^{\mu}|^2 = x^{\mu}x_{\mu} = x^i x_i - (x^0)^2 = \eta_{\mu\nu}x^{\mu}x^{\nu}$. This parametrization also satisfies the constraint 2.6 and the metric in these coordinates is given by

$$ds^2 = R^2\frac{dz^2 + d\vec{x}^2}{z^2} \quad \text{where} \quad d\vec{x}^2 = dx^i dx_i - (dx^0)^2 = \eta_{\mu\nu}dx^{\mu}dx^{\nu} \tag{2.10}$$

This parametrization does not cover the whole space but only a patch of it, known as Poincaré patch, shown in figure 1b. This set of coordinates will be significantly useful later on.

2.2 Conformal Compactification of AdS

It is of great importance to us to study the topology of the AdS spacetime after a conformal compactification. For our purposes we will assume the spacetime is described in global coordinates, as shown in (2.8). In order to study the conformal compactification of AdS, it is convenient to make the transformation

$$\sinh\rho = \tan\theta \qquad \theta \in \left[0, \frac{\pi}{2}\right) \tag{2.11}$$

With this simple transformation one can easily show that the metric transforms as

$$ds^2 = \frac{R^2}{\cos^2\theta}\left(-d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega^2_{n-1}\right) \tag{2.12}$$

Now, by a Weyl transformation [1], we can eliminate the factor in the front, and thus the metric reads as

$$ds^2 = -d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega^2_{n-1} \tag{2.13}$$

From the above expression we can see that the topology of this metric is that of a n-dimensional ball times $\mathbb{R}$, i.e $\mathbb{B}_n \times \mathbb{R}$ where θ plays the role of the radial coordinate. That means that the boundary of the AdS after conformal compactification is $\mathbb{S}^{n-1} \times \mathbb{R}$. **We refer to a spacetime as asymptotically AdS if its boundary structure matches that of AdS once it undergoes conformal compactification.** Since now we know the boundary structure of AdS, we can illustrate it in the images below.

The hyperbolic disk depicted in figure 1a is defined as a disk with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2} \tag{2.14}$$

[1] A Weyl transformation is a local rescaling at the metric tensor, i.e,

$$g_{\mu\nu}(\vec{x}) \to \Omega(\vec{x})\, g_{\mu\nu}(\vec{x})$$

From (2.7) and (2.11) we see that the correct transformations to the cartesian coordinates are

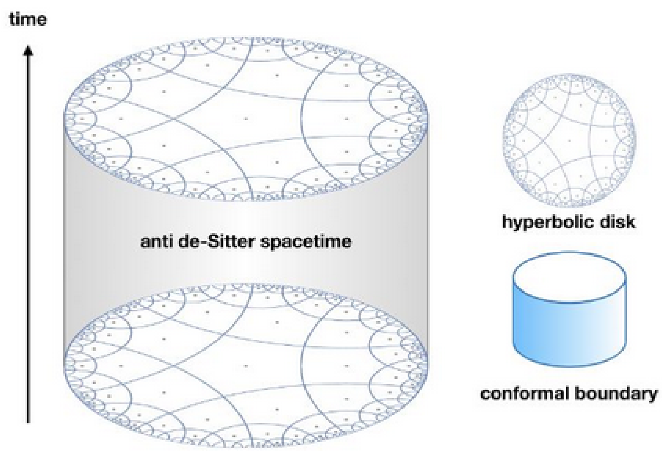
$$x = \tan\theta\,\cos\phi \tag{2.15}$$

$$y = \tan\theta\,\sin\phi \tag{2.16}$$

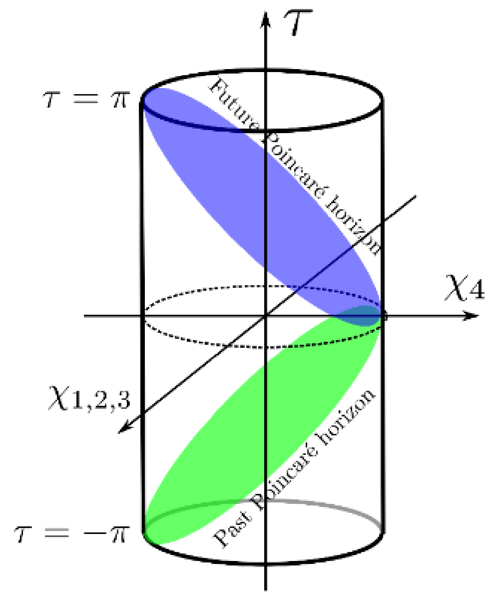
One can easily show that the above transformation in the hyperbolic metric (2.14) yields

$$ds^2 = \frac{d\theta^2 + \sin^2 d\phi^2}{(\cos^2\theta - \sin^2\theta)^2} \xrightarrow{Weyl} ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \tag{2.17}$$

We observe that the line element of the hyperbolic disk matches that of a conformally compactified (2,1)-dimensional AdS for $d\tau = 0$. Therefore, the hyperbolic disk describes the conformally compactified AdS metric at a given time.



(a) A (2,1)-dimensional AdS spacetime depicted as a stack of hyperbolic disks. Picture taken from this site



(b) The Poincaré patch in the AdS spacetime. Picture taken from [2]

2.3 Geodesics of 2-d AdS in Poincaré coordinates

We will now investigate the geodesics of a 2-dimensional Euclidean AdS. The line element as well as the langrangian density is

$$ds^2 = \frac{dz^2 + dx^2}{z^2} \qquad R = 1 \tag{2.18}$$

$$\mathcal{L} = \frac{\dot{z}^2 + \dot{x}^2}{z^2} = 1 \quad \text{where we adapt the convention} \quad \dot{w} = \frac{dw}{ds} \tag{2.19}$$

By solving the Euler-Lagrange equations

$$\frac{d}{ds}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) = \frac{\partial\mathcal{L}}{\partial x} \qquad \frac{\partial\mathcal{L}}{\partial x} = 0 \tag{2.20}$$

$$\frac{d}{ds}\left(\frac{\partial\mathcal{L}}{\partial\dot{z}}\right) = \frac{\partial\mathcal{L}}{\partial z} \tag{2.21}$$

as well as using equation (2.19), we derive

$$\dot{z} = \sqrt{z^2 - L^2 z^4} \tag{2.22}$$

$$\dot{x} = L z^2 \tag{2.23}$$

where L is an arbitrary constant. These equations are quite straightforward to solve. The outcome is

$$z = \frac{1}{L}\operatorname{sech}(s) \tag{2.24}$$

$$x - x_0 = \frac{1}{L}\tanh(s) \tag{2.25}$$

$$x - x_0 = \frac{1}{L}\sqrt{1 - L^2 z^2} \tag{2.26}$$

where x_0 is also an arbitrary constant. It is fascinating to realize that the geodesics are actually half circles with a radius 1 and a center located at the point $(x_0, 0)$

$$(x - x_0)^2 + z^2 = \frac{1}{L^2} \qquad 0 < z < \frac{1}{L} \tag{2.27}$$

We can now use the geodesic equations to determine the geodesic distance between any 2 points (x, z) and (x', z'). The actual result is (see p. 53 Eq. 6.7 of [3])

$$\mathrm{d} = \int_{(x',z')}^{(x,z)} ds = \int_{z'}^{z} \frac{dz}{z\sqrt{1 - L^2 z^2}} = u + 1 + \sqrt{(u+1)^2 - 1} \qquad u = \frac{(x - x')^2 + (z - z')^2}{2zz'} \tag{2.28}$$

where u is called the *chordal* distance. In the context of a hyperboloid (AdS space), the chordal distance refers to the straight-line distance between two arbitrary points in the ambient space that embeds the hyperboloid. This means that rather than measuring the distance along the surface of the hyperboloid (the geodesic distance), chordal distance measures the direct Euclidean distance between the points in the higher-dimensional space. This equation is of great importance to us as we will see later on.

3 Basic elements of Quantum Field Theory

We covered the introduction to the AdS spacetime but in order to properly investigate correlation functions in the AdS/CFT correspondence we also need to investigate a few basic principles of QFT.

Starting from the familiar classical picture, the dynamics of a scalar field $\phi(x^\mu)$ can be accurately determined by its equations of motion (EOMs) that are derived by its action:

$$S = \int d^n x \mathcal{L}(\phi, \partial_\mu \phi) \tag{3.1}$$

The EOMs can be obtained by doing calculus of variations:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \tag{3.2}$$

However, the picture is different in quantum field theory, since we cannot accurately determine the value of a field in a given point in spacetime.

3.1 Path integral formulation, partition and correlation functions

In QFT, we use a formulation called path integral. Path integral generalizes the concept of the classical action, where instead of having a unique classical trajectory, we have a functional integral over all possible quantum trajectories, using it to compute quantum amplitudes and probability densities.

In QFT, we promote the fields as operators ($\varphi(x) \to \hat{\varphi}(x)$) and use them to calculate the vacuum expectation values (VEV) of said fields. These can be calculated by the path integral:

$$\langle 0|\hat{\varphi}(x_1)|0\rangle = \int \mathcal{D}\varphi(x)\, e^{iS[\varphi]}\varphi(x_1) \tag{3.3}$$

where $\mathcal{D}\varphi(x)$ is the differential of this functional integral and it resembles the integration of all the possible configurations of φ at every point in the given spacetime and, according to [4],is equal to

$$\mathcal{D}\varphi(x) = \prod_i \int d\varphi(x_i) \tag{3.4}$$

We can now define the correlation function, or n-point function, computed by:

$$G_n(x_1,\ldots,x_n) = \frac{\langle 0|T\{\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n)\}|0\rangle}{\langle 0|0\rangle} = \frac{\int \mathcal{D}\varphi(x)\, e^{iS[\varphi]}\varphi(x_1)\ldots\varphi(x_n)}{\int \mathcal{D}\varphi(x)e^{iS[\varphi]}} \tag{3.5}$$

where T{ ...} is the time-ordering operator, so that earlier time field operators are on the right side of later time field operators and thus acting first on the vacuum state. The vacuum state $\langle 0|0\rangle$ takes care of the normalization. **The correlation functions of a scalar field φ quantifies the correlation between an excitation of the scalar field at a point $\mathbf{x_1}$ with another excitation at a point $\mathbf{x_n}$.**

Another useful concept in quantum field theory is the partition function, defined by:

$$Z[J] = \int \mathcal{D}\varphi(x)\, e^{iS[\varphi]+i\int d^4x J(x)\varphi(x)} \tag{3.6}$$

where $J(x)$ is a background field coupled to the scalar field $\varphi(x)$, and acts as a source for its excitations.

It turns out it is easier for calculations to make the following transformation in the action

$$t \to -i\tau \quad \text{and} \quad S_E[x(\tau)] = -iS[x(t \to -i\tau)] \tag{3.7}$$

This is called a Wick rotation, and is used to transform the metric of our signature from Lorentzian to Euclidean. S_E is the Euclidean action. Later on we will assume Euclidean AdS for our calculations since it is more advantageous.

This is convenient definition and transformation, since now we can calculate the correlation functions using an alternative approach:

$$G_n(x_1,\ldots,x_n) = (-1)^n \frac{1}{Z[J=0]}\cdot\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)} Z[J]\Big|_{J=0} =$$

$$= \int \mathcal{D}\varphi(x)\, e^{-S_E[\varphi]}\varphi(x_1)\ldots\varphi(x_n) \tag{3.8}$$

This is the definition for an n-point function. However, a convenient calculation is the one of the *connected* correlation functions, that is, the correlation functions excluding the disconnected terms. The generating functional of the connected correlation functions is given by:

$$\mathcal{Z} = -\log Z \tag{3.9}$$

and the connected correlation functions of a scalar operator $\mathcal{O}$ will be given by

$$\langle \mathcal{O}(x_1)\dots\mathcal{O}(x_n)\rangle = (-1)^n \frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1)\dots\delta J(x_n)}\bigg|_{J=0} \tag{3.10}$$

Indeed for the 2-point function:

$$\langle \mathcal{O}(\vec{x}_1)\,\mathcal{O}(\vec{x}_2)\rangle = \frac{\partial^2 \mathcal{Z}}{\partial J(\vec{x}_1)\,\partial J(\vec{x}_2)}\bigg|_{J=0} = G_2(\vec{x}_1,\vec{x}_2) - \langle \mathcal{O}(\vec{x}_1)\rangle\,\langle \mathcal{O}(\vec{x}_2)\rangle \tag{3.11}$$

having establisehd a few basic consepts of QFT that will be useful for us, we move on to CFT.

3.2 From QFT to CFT

A CFT (Conformal Field Theory) is a QFT that is invariant under conformal transformations. A conformal transformation is a transformation that preserves the angles and by definition leaves the metric invariant up to a scale.[2]

$$g'_{\mu\nu}(\vec{x}\,') = \Omega^2(\vec{x})\,g_{\mu\nu}(x) \tag{3.12}$$

and consists of the following transformations:

$x'^{\mu} = x^{\mu} + b^{\mu}$	translation
$x'^{\mu} = \lambda\, x^{\mu}$	dilation
$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$	Lorentz transformation
$x'^{\mu} = \dfrac{x^{\mu} - b^{\mu}\,\vec{x}^2}{1 - 2\,\vec{b}\cdot\vec{x} + \vec{b}^2\,\vec{x}^2}$	Special Conformal Transformation

Table 1: List of transformations that are considered to be conformal.

Special Conformal Transformations (SCT) are nothing more than an inversion $x'^{\mu} = \dfrac{x^{\mu}}{\vec{x}^2}$ followed by a translation $x'^{\mu} = x^{\mu} + a^{\mu}$, and finally another inversion. SCTs can also be written in the form

$$\frac{x'^{\mu}}{\vec{x}\,'^2} = \frac{x^{\mu}}{\vec{x}^2} - a^{\mu}$$

Generally, a **scalar** operator of conformal dimension Δ in an n-dimensional CFT under a conformal transformation transforms as:

$$\mathcal{O}(\vec{x}) = \left|\frac{\partial \vec{x}\,'}{\partial \vec{x}}\right|^{\Delta/n} \mathcal{O}(\vec{x}\,') \tag{3.13}$$

[2]It is important not to confuse conformal transformations with Weyl trnasformations. A conformal transformation is a space-time transformation in the coordinates which leaves the metric invariant up to scale and thus preserves angles. A Weyl transformation *actively* rescales the metric.

More information about CFTs can be found in [5]. For our purposes, we are now able to calculate 2 and 3-point functions of a scalar field in a simple way using only the conformal invariance of the CFT.

3.2.1 The 2-point function of a scalar through CFT

Firtly we will demonstrate how we can calculate the 2-point function of 2 scalar fields. We consider 2 scalar fields $\varphi_1(x)$ and $\varphi_2(x)$ as well as 2 local operators generated by said fields, $\mathcal{O}_1(x)$ and $\mathcal{O}_2(x)$. Our objective is to calculate the correlation function at the points $\vec{x_1}$ and $\vec{x_2}$ respectively. We also assume that the local operator $\mathcal{O}_1(x)$ has conformal dimension Δ_1 while $\mathcal{O}_2(x)$ has dimension Δ_2.

Additionally, if we make a scaling transformation in our coordinates ($x^\mu \to \lambda\, x^\mu$) we expect (due to 3.13) that the 2-point function will be transformed in the following manner:

$$\langle \mathcal{O}_1(\vec{x}_1)\, \mathcal{O}_2(\vec{x}_2)\rangle = \lambda^{\Delta_1+\Delta_2}\, \langle \mathcal{O}_1(\lambda\vec{x}_1)\, \mathcal{O}_2(\lambda\vec{x}_2)\rangle \tag{3.14}$$

Furthermore, due to rotation and translation invariance, the correlation function must only depend on the spacetime distance between the 2 points, $|\vec{x_1} - \vec{x_2}|$ meaning

$$\langle \mathcal{O}_1(\vec{x}_1)\, \mathcal{O}_2(\vec{x}_2)\rangle = f(|\vec{x_1} - \vec{x_2}|) \tag{3.15}$$

Combining 3.14 and 3.15, we conclude that the 2-point function must be equal to

$$\langle \mathcal{O}_1(\vec{x}_1)\, \mathcal{O}_2(\vec{x}_2)\rangle = \frac{C_{12}}{|\vec{x_1} - \vec{x_2}|^{\Delta_1+\Delta_2}} \tag{3.16}$$

Indeed, this kind of result is invariant under boosts, dilations and translations.

The only invariance we have not used yet is the SCT which will further constrain our 2-point function.

One can show that an SCT for the distance separating 2 points is given by

$$|\vec{x}'_1 - \vec{x}'_2| = \frac{|\vec{x_1} - \vec{x}_2|}{\sqrt{\left(1 - 2\vec{b}\cdot\vec{x}_1 + \vec{b}^2\,\vec{x}_1^2\right)\left(1 - 2\vec{b}\cdot\vec{x}_2 + \vec{b}^2\,\vec{x}_2^2\right)}} \tag{3.17}$$

By making such a transformation at 3.16, and demanding the result to be the same (due to conformal invariance), we derive

$$\frac{C_{12}}{|\vec{x}_1 - \vec{x}_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{\gamma_1^{\Delta_1}\,\gamma_2^{\Delta_2}}\frac{(\gamma_1\,\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|\vec{x}_1 - \vec{x}_2|^{\Delta_1+\Delta_2}} \tag{3.18}$$

where

$$\gamma_i = 1 - 2\,\vec{b}\cdot\vec{x}_i + \vec{b}^2\,\vec{x}_i^2 \tag{3.19}$$

We can clearly see that for equation 3.18 to stand true, the dimensions must necessarily be equal, meaning $\Delta_1 = \Delta_2 = \Delta$. So the final result for the 2-point function is

$$\langle \mathcal{O}_1(\vec{x}_1)\, \mathcal{O}_2(\vec{x}_2)\rangle = \frac{C_{12}}{|\vec{x}_1 - \vec{x}_2|^{2\Delta}} \tag{3.20}$$

We have successfully derived the behavior of the 2-point function of scalar fields, with the exception of determining the constant factor. We will now move on on the 3-point function.

3.2.2 The 3-point function of a scalar through CFT

The case for the 3-point function is a bit more complicated, though the methodology used is precisely the same as before. A scale transformation would mean:

$$\langle \mathcal{O}_1(\vec{x}_1)\, \mathcal{O}_2(\vec{x}_2)\, \mathcal{O}_3(\vec{x}_3)\rangle = \lambda^{\Delta_1+\Delta_2+\Delta_3} \langle \mathcal{O}_1(\lambda\vec{x}_1)\, \mathcal{O}_2(\lambda\vec{x}_2)\, \mathcal{O}_3(\lambda\vec{x}_3)\rangle \tag{3.21}$$

Once again translation and rotation invariance imply that the 3-point function must solely depend on the distances between 2 points out of the 3 points. That means that the correlation function in general depends on all possible permutations of $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$. These are $|\vec{x}_1 - \vec{x}_2|$, $|\vec{x}_2 - \vec{x}_3|$ and $|\vec{x}_1 - \vec{x}_3|$.

Thus, our 3-point function must be of the form

$$\langle \mathcal{O}_1(\vec{x}_1)\, \mathcal{O}_2(\vec{x}_2)\, \mathcal{O}_3(\vec{x}_3)\rangle = \frac{C_{123}}{|\vec{x}_1 - \vec{x}_2|^a\, |\vec{x}_2 - \vec{x}_3|^b\, |\vec{x}_1 - \vec{x}_3|^c} \tag{3.22}$$

Due to 3.21, the powers must obey

$$a + b + c = \Delta_1 + \Delta_2 + \Delta_3 \tag{3.23}$$

Now as before, all we are left to do is apply the last transformation invariance, an SCT, to determine the powers a, b, c. After such a transformation, we find

$$\frac{C_{123}}{|\vec{x}_1 - \vec{x}_2|^a\, |\vec{x}_2 - \vec{x}_3|^b\, |\vec{x}_1 - \vec{x}_3|^c} = \frac{C_{123}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} \frac{(\gamma_1\gamma_2)^{a/2}(\gamma_2\gamma_3)^{b/2}(\gamma_1\gamma_3)^{c/2}}{|\vec{x}_1 - \vec{x}_2|^a\, |\vec{x}_2 - \vec{x}_3|^b\, |\vec{x}_1 - \vec{x}_3|^c} \tag{3.24}$$

in order for equation 3.24 to stand true, the following equations must also be true

$$a + c = 2\Delta_1 \quad a + b = 2\Delta_2 \quad b + c = 2\Delta_3 \tag{3.25}$$

From these, we can accurately derive the powers

$$a = \Delta_1 + \Delta_2 - \Delta_3, \qquad b = \Delta_2 + \Delta_3 - \Delta_1, \qquad c = \Delta_1 + \Delta_3 - \Delta_2 \tag{3.26}$$

Therefore, the 3-point function is given by

$$\langle \mathcal{O}_1(\vec{x}_1)\, \mathcal{O}_2(\vec{x}_2)\, \mathcal{O}_3(\vec{x}_3)\rangle = \frac{C_{123}}{|\vec{x}_1 - \vec{x}_2|^{\Delta_1+\Delta_2-\Delta_3}\, |\vec{x}_2 - \vec{x}_3|^{\Delta_2+\Delta_3-\Delta_1}\, |\vec{x}_1 - \vec{x}_3|^{\Delta_1+\Delta_3-\Delta_2}} \tag{3.27}$$

We also derived the 3-point function. Unfortunately, we cannot follow the same methodology for calculating higher-point functions because conformal invariance will not be sufficient. But we have gained the results we need for the objective of this thesis.

We have made an appropriate introduction in AdS and CFT, and we now have the needed tools for introducing the AdS/CFT correspondence and calculate a few correlations functions implementing a methodology that uses gravity in the bulk and not the boundary CFT.

4 The correspondence

We consider a scalar field $\phi(\vec{x}, z)$ living in the bulk, whose boundary values play the role of the sources for local operators $\mathcal{O}(\vec{x})$ that are descendants of the primary operators $\varphi(\vec{x})$. So the background fields $J(\vec{x})$ (the sources) in 3.6, are now portrayed by a scalar field $\phi(\vec{x}, z)$ in the bulk, and the scalar field $\varphi(\vec{x})$ is portrayed by the local operators $\mathcal{O}(\vec{x})$. Thus, we can correspond the scalar field ϕ in the CFT in the following manner:

$$\text{Field in the bulk} \quad \longleftrightarrow \quad \text{source in the boundary CFT}$$

More accurately, supposing the scalar field $\phi(\vec{x}, z)$ living in the bulk has a boundary condition

$$\phi(\vec{x}, z \to 0) = \phi_0(\vec{x})$$

we can write

$$\mathcal{Z}_{bulk}[\phi(\vec{x}, z \to 0) = \phi_0(\vec{x})] = \int \mathcal{D}\varphi(x)\, e^{-S[\varphi]+i\int d^n x \phi_0(x)\mathcal{O}(x)} \tag{4.1}$$

Meaning the generating functional in the QFT is equal to the boundary value of the gravity partition function. The partition function of gravity would generally be a complex object of the form[6]

$$Z_{bulk}[g, \phi] = \int \mathcal{D}\varphi\, \mathcal{D}g\, e^{-\frac{1}{\hbar}S[g,\varphi]-\frac{1}{\hbar}\int d^n x \phi(x)\mathcal{O}(x)} \tag{4.2}$$

Fortunately, it is possible to greatly simplify this expression. In the classical limit, we can approximate this expression in the following manner:

$$Z_{bulk} \simeq e^{-\frac{1}{\hbar}S[\phi]} \tag{4.3}$$

In the classical picture we can take $\hbar \to 0$, and thus, the largest contribution from the exponential in 4.3 comes from the smallest value of the action, meaning from the saddle points. So, the value of the partition function is mostly governed by the scalar fields that satisfy $\frac{\delta S}{\delta \varphi} = 0$, i.e, the EOMs.

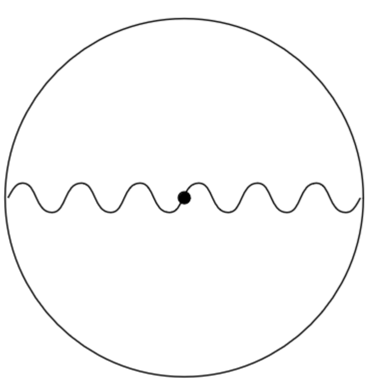
Equation 4.3 now makes the calculation for the connected correlation functions a lot easier since (we use Planck units where $\hbar = 1$)

$$\mathcal{Z} = -\log Z_{bulk} = S[\phi] \tag{4.4}$$

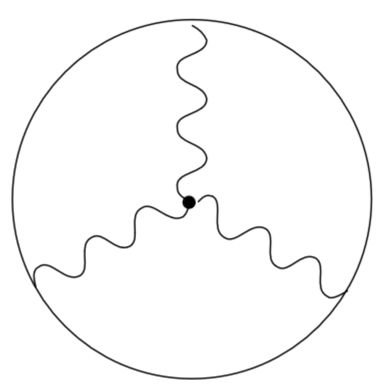
That means that the generating functional in the CFT is the classical gravity action in the bulk.

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle = (-1)^n \frac{\delta^n S}{\delta\phi(x_1) \dots \delta\phi(x_n)} \bigg|_{\phi(x)=0} \tag{4.5}$$

An easy way to illustrate the interaction of the operators is by **Witten diagrams**. A Witten diagram is a visual tool used to compute correlation functions in conformal field theory by utilizing bulk information. The Witten diagrams of 2 and 3-point functions are given below.



(a) Witten diagram of a 2-point function. Picture taken from [7]



(b) Witten diagram of a 3-point function. Picture taken from [7]

Since we are solving the classical EOMs, the interactions are tree-level interactions. No loop interactions where included since these demand quantum corections in the bulk which we do not investigate.

5 The 2-point function of a scalar through gravity

Let us consider a real scalar field $\phi(\vec{x})$ acting as a source for the local operators $\mathcal{O}$ in the CFT. To properly compute the 2-point function of a scalar field, one must find the EOMs for the field and solve them to derive the propagator. We will consider the AdS to be euclidean and use the methodology depicted in [8].

5.1 Calculation through momentum space

For the calculation of the 2-point function only quadratic terms are needed in the action. The action is:

$$S = \frac{1}{2}\int d^n x \int_\epsilon^\infty dz \sqrt{g}\left(\nabla_M \phi \nabla^M \phi + m^2 \phi^2\right) \tag{5.1}$$

Once again, by calculus of variations we can acquire the EOMs for ϕ which are

$$(\Box - m^2)\phi = 0 \tag{5.2}$$

Now, using the identity $\Gamma^M_{KM} = \frac{1}{\sqrt{g}}\partial_K \sqrt{g}$ one can show that (5.2) can be written as:

$$\frac{1}{\sqrt{g}}\partial_M(\sqrt{g}g^{MN}\partial_N \phi) - m^2\phi = 0 \tag{5.3}$$

In order to solve this equation, we need to go to the Fourier space for momentum on $\mathbb{R}^n$, where $\delta^{\mu\nu}\partial_\mu\partial_\nu = p^2$ and $\phi(\vec{x}, z) \to \widetilde{\phi}(\vec{p}, z)$. Thus for 5.3 we have:

$$\frac{1}{\sqrt{g}}\partial_M(\sqrt{g}g^{MN}\partial_N \phi) - m^2\phi = \frac{1}{\sqrt{g}}\partial_z\left(\sqrt{g}\, g^{zz}\partial_z \widetilde{\phi}\right) + \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\, g^{\mu\nu}\partial_\nu \widetilde{\phi}\right) - m^2\phi = 0 \tag{5.4}$$

Now using the fact that $\sqrt{g} = \left(\frac{R}{z}\right)^{n+1}$ and $g^{\mu\nu} = \frac{z^2}{R^2}\delta^{\mu\nu}$, we get the following equation:

$$z^{n+1}\partial_z\left(z^{1-n}\partial_z \widetilde{\phi}\right) - p^2 z^2 \widetilde{\phi} - m^2 R^2 \widetilde{\phi} = 0 \tag{5.5}$$

To solve (5.5) we do the following transformations:

$$x = pz, \quad \widetilde{\phi} = z^{n/2}\widetilde{\Phi} = x^{n/2}p^{-n/2}\widetilde{\Phi}$$

and get a familiar form:

$$x^2\,\partial_x^2\,\widetilde{\Phi} + x\,\partial_x \widetilde{\Phi} - \left(x^2 + m^2R^2 + \frac{n^2}{4}\right) = 0 \tag{5.6}$$

This is the modified Bessel equation, with known solutions. So the solution is of the form:

$$\widetilde{\phi}(\vec{p}, z) = c_1 z^{n/2} I_a(pz) + c_2 z^{n/2} K_a(pz), \quad a = \sqrt{\frac{n^2}{4} + m^2R^2} \tag{5.7}$$

Using a few known properties for the solutions of the modified Bessel equation (Appendic C), we derive

$$\widetilde{\phi}(\vec{p}, z \to 0) \sim z^{\alpha_\pm} \quad \text{where} \quad \alpha_\pm = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} + m^2 R^2} \tag{5.8}$$

We impose regularity on the interior of AdS, meaning the solution must not diverge for $z \to \infty$. Since the Bessel function $I_a(pz)$ is irregular at $z \to \infty$, we only keep the function $K_a(pz)$ as a solution. Meaning

$$\widetilde{\phi}(pz) = c\, z^{n/2} K_a(pz), \qquad \widetilde{\phi}(\vec{p}, z \to 0) \sim z^{\alpha_-} \tag{5.9}$$

So $\phi(\vec{x}, z)$, for each mode of the solution is:

$$\phi(\vec{x}, z) = e^{i\vec{p}\cdot\vec{x}}\, \widetilde{\phi}(pz) \tag{5.10}$$

Since from equation 4.2 the integral in the exponent must be dimensionless,

$$\int d^n x \phi(x) \mathcal{O}(x),$$

and since we know the dimension of the scalar field ϕ is α_- (from 5.9), the dimension of the local operators $\mathcal{O}(\vec{x})$ must be

$$\Delta = n - \alpha_- = \alpha_+ \tag{5.11}$$

From that definition, we conclude that

$$m^2\, R^2 = \Delta(\Delta - n) \tag{5.12}$$

This equation means that for a given value of the mass, there exists 2 different conformal dimensions. Thus in an AdS space there can be 2 corresponding CFTs. For the sake of simplicity, we will assume

$$\Delta > n/2$$

to avoid double quantization. In order to now properly solve the differential equation, we must impose a boundary condition:

$$\phi(\vec{x}, z = \epsilon) = \epsilon^{n-\Delta} \phi_0(\vec{x}) = \epsilon^{n-\Delta} e^{i\vec{p}\vec{x}}, \qquad z = \epsilon \to 0 \tag{5.13}$$

Now we can accurately calculate the coefficient in 5.9 and derive the following bulk-to-boundary propagator:

$$\phi(\vec{x}, z) \equiv K_{\vec{p}}(\vec{x}, z) \equiv \widetilde{K}_{\vec{p}}(\vec{x}, z)\, e^{i\,\vec{p}\cdot\vec{x}} \equiv \epsilon^{n-\Delta} \frac{z^{n/2} K_{\Delta-n/2}(pz)}{\epsilon^{n/2} K_{\Delta-n/2}(p\,\epsilon)}\, e^{i\,\vec{p}\cdot\vec{x}} \tag{5.14}$$

But that is just one mode of the solution. The general solution would be of the form:

$$\phi(\vec{x}, z) = \int d^n p\, \lambda_{\vec{p}}\, e^{i\,\vec{p}\cdot\vec{x}}\, \widetilde{K}_{\vec{p}}(\vec{x}, z) \tag{5.15}$$

Our objective is to calculate the 2-point function using equation, 4.5, so our first objective would be to simplify the action. Equation 5.1 can be rewritten as

$$S = \frac{1}{2} \int d^n x \int_\epsilon^\infty dz \left(-\phi\sqrt{g}\overbrace{(\Box - m^2)\phi}^{0} + \partial_M \left(\sqrt{g}\, \phi\, \partial^M \phi \right) \right) \tag{5.16}$$

The first term is the EOM so, naturally, it vanishes. Since we also take the fields to vanish at infinity in $\vec{x}$, so we are left with

$$S = \frac{1}{2}\left(\int_\epsilon^\infty dz \cancelto{0}{\int d^n x\, \partial_\mu \left(\sqrt{g}\,\phi\,\partial^\mu \phi\right)} \;+\; \int d^n x \int_\epsilon^\infty dz\, \partial_z \left(\sqrt{g}\, g^{zz}\,\phi\,\partial_z \phi\right)\right) \tag{5.17}$$

So the final and simplified form of the action is

$$S = \frac{1}{2}\int d^n x \left[\sqrt{g}\, g^{zz}\,\phi\,\partial_z \phi\right]_\epsilon^\infty \tag{5.18}$$

Plugging in the general solution for ϕ (5.15), we get

$$S = \frac{1}{2}\int d^n x\, e^{i(\vec{p}+\vec{q})} \int d^n p\, d^n q\, \lambda_{\vec{p}}\,\lambda_{\vec{q}}\,\mathcal{F} = \frac{1}{2}\,(2\pi)^n \int d^n p\, d^n q\, \lambda_{\vec{p}}\,\lambda_{\vec{q}}\,\delta^n(\vec{p}+\vec{q})\mathcal{F} \tag{5.19}$$

where

$$\mathcal{F} = -\left(\frac{R}{z}\right)^{n-1} \widetilde{K}_{\vec{p}}\,\partial_z \widetilde{K}_{\vec{q}}\Big|_{z=\epsilon} = -\left(\frac{R}{z}\right)^{n-1} \widetilde{K}_{\vec{p}}\,\partial_z \widetilde{K}_{\vec{p}}\Big|_{z=\epsilon} \tag{5.20}$$

It is important to state that from now on we choose convenient units where $R = 1$ as well. We have successfully calculated the action[3], however, we did so in momentum space. Unfortunately, equation 4.5 calculates the connected n-point function in position space, so we need to change it to momentum space. We find the procedure is almost identical. Specifically:

$$\int d^n x J(x)\mathcal{O}(x) = \int d^n x\, e^{i(\vec{p}+\vec{q})x} \int dp\, dq\, J(p)\mathcal{O}(q) = (2\pi)^n \int J(-p)\mathcal{O}(p)$$

So even in momentum space, we solely need to take the derivatives with respect to the sources $J(-p)$ to calculate the connected correlation functions.

It is obvious that the classical gravity action must depend on the sources. The only variables that contain this information in 5.19 are the Fourier coefficients $\lambda_{\vec{p}}$ and $\lambda_{\vec{q}}$, since the rest of the terms are simple functions that contain no dynamical value. Therefore, the connected 2-point function is:

$$\langle \mathcal{O}(\vec{p})\,\mathcal{O}(\vec{q})\rangle = \frac{\partial^2 S}{\partial\lambda_{\vec{p}}\,\partial\lambda_{\vec{q}}}\bigg|_{\lambda_{\vec{p}}=\lambda_{\vec{q}}=0} = \frac{1}{2}(2\pi)^n \int d^n p'\, d^n q' \left(\frac{\partial\lambda_{\vec{p'}}}{\partial\lambda_{\vec{p}}}\,\frac{\partial\lambda_{\vec{q'}}}{\partial\lambda_{\vec{q}}} + \frac{\partial\lambda_{\vec{p'}}}{\partial\lambda_{\vec{q}}}\,\frac{\partial\lambda_{\vec{q'}}}{\partial\lambda_{\vec{p}}}\right)\delta^n(\vec{p'}+\vec{q'})\mathcal{F} \tag{5.21}$$

The partial derivatives result to delta functions, meaning the integrals vanish. Eventually, the outcome is

$$\langle \mathcal{O}(\vec{p})\,\mathcal{O}(\vec{q})\rangle = (2\pi)^n \delta^n(\vec{p}+\vec{q})\mathcal{F} \tag{5.22}$$

This is the value of the 2-point function in the momentum space. We need to transform the result to position space in order for our calculation to be complete. To do that, we must calculate the following integral:

$$\langle \mathcal{O}(\vec{x})\,\mathcal{O}(\vec{y})\rangle = \int d^n p\, d^n q e^{i\,\vec{p}\cdot\vec{x}}\, e^{i\,\vec{q}\cdot\vec{y}}\,\langle \mathcal{O}(\vec{p})\,\mathcal{O}(\vec{q})\rangle \tag{5.23}$$

[3] Because of the presence of the delta function in 5.19 we can switch $\vec{q}$ with $-\vec{p}$ without loss of generality. Additionally, due to the properties of the modified Bessel function of the second kind, we can write $\widetilde{K}_{\vec{p}} = \widetilde{K}_{-\vec{p}}$. So, $\widetilde{K}_{\vec{p}} = \widetilde{K}_{\vec{q}}$

Now plugging equation 5.22 into 5.23 we arrive at the following outcome:

$$\langle \mathcal{O}(\vec{x})\,\mathcal{O}(\vec{y})\rangle = (2\pi)^n \int d^n p\, e^{i\,\vec{p}\cdot(\vec{x}-\vec{y})}\,\mathcal{F} \tag{5.24}$$

We can greatly simplify this expression by more closely examining $\mathcal{F}$ for $z = \epsilon \to 0$. First, we need to investigate the behavior of the K_a Bessel function. We must be cautious, since this function has different expansions depending on the nature of a. Generally, $a \in \mathbb{R}$, but in Appendix C we can see that the case where $a \notin \mathbb{Z}$ differs from the case where $a \in \mathbb{Z}$.

Firstly we will investigate the case where $a \notin \mathbb{Z}$. From C.6 we know that for $z = \epsilon \to 0$

$$K_a(pz) = c_1\,(pz)^{-a}\left(1 + \cdots + c_2\,(pz)^{2a} + \dots\right) \tag{5.25}$$

where c_1 and c_2 are the constants evaluated at C.6. Thus, $\widetilde{K}_{\vec{p}}$ defined in 5.14, can be expanded to

$$\widetilde{K}_{\vec{p}} = c_3\,z^{n/2}K_{\Delta-n/2}(pz) = c_1\,c_3\,p^{n/2-\Delta}z^{n-\Delta}\left(1 + \cdots + c_2\,(pz)^{2\Delta-n} + \dots\right) \tag{5.26}$$

where as c_3 we define the constant in 5.14, meaning

$$c_3 = \frac{\epsilon^{n-\Delta}}{\epsilon^{n/2}K_{\Delta-n/2}(p\epsilon)} \overset{\epsilon\to 0}{\simeq} \frac{1}{c_1}\,p^{\Delta-n/2} \tag{5.27}$$

Now using the fact that $\widetilde{K}_{\vec{p}}\,(z=\epsilon) = \epsilon^{n-\Delta}$, as well as equation 5.26, we can find that for $\mathcal{F}$:

$$\mathcal{F} = -z^{1-n}\widetilde{K}_{\vec{p}}\,\partial_z\widetilde{K}_{\vec{p}}\Big|_{z=\epsilon} = \textit{analytic} - c_1\,c_2\,c_3\,\Delta\,p^{\Delta-n/2} + \textit{subleading} \tag{5.28}$$

and plugging 5.27 into the the expression, we can rewrite 5.28 to:

$$\mathcal{F} = \textit{analytic} - c_2\,n\,(\,\vec{p}\,)^{2\Delta-n} + \textit{subleading} \tag{5.29}$$

All of the summation terms but one, are either *subleading* or *analytic*. *Subleading* are the terms that contain a factor of the form $\#\,\epsilon^k$ where $k > 0$, and thus vanish when $\epsilon \to 0$.

Analytic are the terms that diverge in ϵ, but they are also terms that contain an even power of p, meaning they are terms of the form $\#\,p^{2k}, \quad k \in \mathbb{N}$. Terms of this kind result to delta functions once transformed back to position space, meaning they are contact terms that serve no physical value and can be discarded.

To show that, we need to transform terms of the form $\#\,p^{2k}$ to position space. To do that, we must calculate the integral

$$I_{2k}(\vec{\xi}\,) = \int d^n p\, e^{i\,\vec{p}\cdot\vec{\xi}}\,(\vec{p}\,)^{2k} = \int d^n p\, e^{i\,\vec{p}\cdot\vec{\xi}}\,|\vec{p}\,|^{2k} = \int d^n p\, e^{i\,\vec{p}\cdot\vec{\xi}}\,(p_\lambda\,p^\lambda)^k, \quad k \in \mathbb{N} \tag{5.30}$$

One can easily show that 5.30 can be reformulated as follows [4]

$$I_{2k}(\vec{\xi}\,) = (-\Box)^k\,I_0(\vec{\xi}\,) \tag{5.31}$$

[4] This is simple to demonstrate considering the case where k=1. More specifically:

$$-\Box\,I_0(\vec{\xi}\,) = -i\int d^n p\,g^{\mu\nu}p_\nu\nabla_\mu\,e^{i\,p_\lambda\xi^\lambda} = \int d^n p\,e^{i\,p_\lambda\xi^\lambda}\,p_\mu\,p^\mu = I_2(\vec{\xi}\,)$$

But we know that $I_0(\vec{\xi}\,) = (2\pi)^n\, \delta^n(\vec{\xi}\,)$, meaning

$$I_{2k}(\vec{\xi}\,) = (2\pi)^n\, (-\Box)^k\, \delta^n(\vec{\xi}\,), \quad k \in \mathbb{N} \tag{5.32}$$

Conclusively, the *analytic* terms, give rise to δ functions. In our case, $\vec{\xi} = \vec{x} - \vec{y}$, so the terms in the 2-point function will be of the form $\#\,\delta(\vec{x} - \vec{y})$ (i.e it is a contact term).

Correlation functions only make physical sense when $\vec{x} \neq \vec{y}$. We do not examine the case where $\vec{x} = \vec{y}$. Under this scope, the *analytic* terms are discarded, while the *subleading* ones vanish, meaning there exists only one term in $\mathcal{F}$ that is neither *subleading* (because the orders of ϵ cancel out) nor *analytic* (because p is raised in a real and not an even power). That means that the 2-point function in 5.24 can be expressed as

$$\langle \mathcal{O}(\vec{x})\, \mathcal{O}(\vec{y}) \rangle = -(2\pi)^n\, c_2\, n \int d^n p\, e^{i\,\vec{p}\cdot(\vec{x}-\vec{y})} \left(p_\lambda\, p^\lambda\right)^{\Delta - n/2} \tag{5.33}$$

This integral has the same form as the one in 5.30, with the only difference being $k \in \mathbb{R}$. To solve this integral, we must first perform a substitution $\vec{\xi} = \vec{x} - \vec{y}$ as well as

$$u_\lambda = p_\lambda \sqrt{|\xi^\mu\, \xi_\mu|} = p_\lambda\, |\vec{\xi}| \tag{5.34}$$

and so the integral takes the form

$$I_{2k} = \frac{1}{|\vec{\xi}|^{n+2k}} \int d^n u\, e^{i\,\vec{u}\cdot\hat{\xi}}\, (u^\mu u_\mu)^k \qquad \hat{\xi} = \frac{\vec{\xi}}{|\vec{\xi}|} \tag{5.35}$$

We consider hyper-spherical coordinates in n dimensions. In order to simplify the integral, we will assume that the z axis will be in the direction of $\vec{\xi}$, and thus, in spherical coordinates, $\vec{u}\cdot\hat{\xi} = r\, \cos\theta$. The integral simplifies to

$$I_{2k} = \frac{1}{|\vec{\xi}|^{n+2k}} \int d\Omega_{n-2} \int_0^\infty \int_0^\pi dr\, d\theta\, r^{2k+n-1}\, \sin\theta\, e^{ir\cos\theta} \tag{5.36}$$

One can quite easily perform the integral on θ. By doing so, we end up with

$$I_{2k} = \frac{2}{|\vec{\xi}|^{n+2k}} \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty dr\, \sin r\, r^{2k+n-2} \tag{5.37}$$

In order to evaluate the integral

$$Y = \int_0^\infty dr\, \sin r\, r^{2k+n-2} \tag{5.38}$$

some knowledge in complex analysis is required.

Consider the below complex function

$$f(z) = z^{a-1}\, e^{-cz} \tag{5.39}$$

Integrated in the following contour (a quarter annulus)

$$\oint_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz + \int_{C_3} f(z)\, dz + \int_{C_4} f(z)\, dz \tag{5.40}$$

$$C_1(t) \to z = t, \quad dz = dt, \quad t \in (\epsilon, R)$$

$$C_2(\theta) \to z = R\, e^{i\theta}, \quad dz = iR\, e^{i\theta}\, d\theta \quad \theta \in (0, b)$$

$$C_3(t) \to z = t\, e^{ib}, \quad dz = e^{ib}\, dt \quad t \in (\epsilon, R)$$

$$C_4(\theta) \to z = \epsilon\, e^{i\theta}, \quad dz = i\,\epsilon\, e^{i\theta} d\theta \quad \theta \in (0, b)$$

It is important to note that the integral has no poles inside our contour, thus

$$\int_{\epsilon}^{R} t^{a-1}\, e^{-ct}\, dt + iR^a e^{i\theta a} \int_0^b e^{-cR\, e^{i\theta}}\, d\theta + e^{iba} \int_R^{\epsilon} t^{a-1}\, e^{-ct\, e^{ib}} dt + i\epsilon^a \int_b^0 e^{i\theta a}\, e^{-c\epsilon\, e^{i\theta}} d\theta = 0 \tag{5.41}$$

For $R \to \infty$ and $\epsilon \to 0$ we notice

$$iR^a \int_0^b e^{ia\theta}\, e^{-cR\, e^{i\theta}}\, d\theta \overset{R\to\infty}{\longrightarrow} 0 \tag{5.42}$$

$$i\, \epsilon^a \int_b^0 e^{i\theta a}\, e^{-c\epsilon\, e^{i\theta}} \overset{e\to 0}{\longrightarrow} 0 \tag{5.43}$$

thus from equation (5.40) we conclude that

$$e^{iba} \int_0^{\infty} t^{a-1}\, e^{-ct\, e^{ib}}\, dt = \int_0^{\infty} t^{a-1}\, e^{-ct}\, dt \tag{5.44}$$

The integral on the right can be easily evaluated in terms of the Gamma function. Equation (5.44) now takes the form

$$\int_0^{\infty} t^{a-1}\, e^{-ct\, e^{ib}} = e^{-iba} c^{-a} \Gamma(a) \tag{5.45}$$

Now assuming $b = \pi/2$ and $c = 1$, we have

$$\int_0^{\infty} t^{a-1}\, e^{-it} = e^{-iba}\, \Gamma(a) \tag{5.46}$$

Finally, by only taking the imaginary part and setting $2k = 2\Delta - 1$ and $a = 2k + n - 1 = 2\Delta - 1$, a familiar integral appears:

$$Y = \int_0^{\infty} t^{2k+n-2}\, \sin t\, dt = \sin\left(\pi\left(\Delta - 1/2\right)\right) \Gamma(2\Delta - 1) = 2^{2\Delta-2} \sqrt{\pi}\, \frac{\Gamma(\Delta)}{\Gamma\left(\frac{3}{2} - \Delta\right)} \tag{5.47}$$

Hence,

$$I_{2k} = \frac{2}{|\vec{\xi}|^{2\Delta}} \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} Y = \frac{2^{2\Delta}\, \pi^{n/2}\, \Gamma(\Delta)}{\Gamma\left(\frac{3}{2} - \Delta\right)\, \Gamma\left(\frac{n-1}{2}\right)} \frac{1}{|\vec{x} - \vec{y}|^{2\Delta}} \tag{5.48}$$

In our case, $2k = 2\Delta - n$, thus the complete result of the 2-point function in 5.33 is

$$\langle \mathcal{O}(\vec{x})\, \mathcal{O}(\vec{y}) \rangle = -\frac{2^{2n}\, \pi^{3n/2}\, n\, \Gamma(n/2 - \Delta)}{\Gamma(\Delta - n/2)} \frac{\Gamma(\Delta)}{\Gamma\left(\frac{3}{2} - \Delta\right)\, \Gamma\left(\frac{n-1}{2}\right)} \frac{1}{|\vec{x} - \vec{y}|^{2\Delta}} \tag{5.49}$$

It is important not to forget that this is for the case where $\Delta - n/2 \notin \mathbb{Z}$. We will now verify that the same result stands true for the case where $\Delta - n/2 \in \mathbb{Z}$. For $a \in \mathbb{Z}$, we know from Appendix C that K_a behaves as

$$K_a(x) = \tilde{c}_1\, x^{-a}(1 + \cdots + \tilde{c}_2\, x^{2a}\, \log x + \dots) \tag{5.50}$$

Following the exact same methodology as before, we find for $\mathcal{F}$

$$\mathcal{F} = analytic + n\, \tilde{c}_2\, p^{2\Delta-n}\, (\log p - \log z) + subleading \tag{5.51}$$

We can immediately see why this expression is problematic. For $z \to 0$, there exists a logarithmic divergence in our 2-point function. This is called a type-B anomaly which we will not delve into here. More information about it can be found in [9]. In our case, we will discard the anomaly, hence the integral we have to calculate is

$$I'_{2k} = \int d^n p \, e^{i\,\vec{p}\cdot\vec{x}} \, p^{2k} \, \log|p| \qquad k \in \mathbb{N} \tag{5.52}$$

We will firstly look into the case where $k = 0$. By following the same procedure shown from (5.34)to (5.37), the integral takes the form

$$I'_0 = \frac{1}{|\vec{\xi}|^n} \left(\frac{4\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty dr \, \sin r \, r^{n-2} \, \log r \, dr - \log|\xi| \int d^n u \, e^{i\vec{x}\cdot\hat{\xi}} \right) \tag{5.53}$$

The second term gives rise to a delta function $\delta^n(\hat{\xi})$ and is thus a contact term that can be discarded.

We calculate the integral

$$Y' = \int_0^\infty \sin z \, \log z \, z^{n-2} \, dz = \mathrm{Im}\left(\int_0^\infty e^{iz} \, \log z \, z^{n-2} \, dz \right) \tag{5.54}$$

We have to be careful calculating this integral since the logarithm has a branch cut at $(0, \infty)$. The contour we will use is a semi circle with the following parametrizations

$$\begin{aligned} C_1(t) &\to z = t + i\epsilon, \quad dz = dt, \quad t \in (\epsilon, R) \\ C_2(\theta) &\to z = R\,e^{i\theta} + i\epsilon, \quad dz = iR\,e^{i\theta}\,d\theta \quad \theta \in \left(0, \frac{\pi}{2}\right) \\ C_3(t) &\to z = t\,e^{i\frac{\pi}{2}}, \quad dz = e^{i\frac{\pi}{2}}\,dt \quad t \in (\epsilon, R) \end{aligned} \tag{5.55}$$

The integral in the closed contour has no poles and thus vanishes. Furthermore, for $R \to \infty$ the contribution from $C_2(t)$ also vanishes similarly as before. If we also take $\epsilon \to 0$ we arrive at the following expression

$$\int_0^\infty e^{iz} \, \log z \, z^{n-2} \, dz = e^{i\frac{\pi}{2}(n-1)} \int_0^\infty e^{-t} \, t^{n-2} \, \log t \, dt + e^{i\frac{n\pi}{2}} \int_0^\infty e^{-t} \, t^{n-2} \, dt \tag{5.56}$$

By observing that

$$\frac{\partial}{\partial n} \left(\int_0^\infty e^{-t} \, t^{n-2} \, dt \right) = \int_0^\infty e^{-t} \, t^{n-2} \, \log t \, dt \tag{5.57}$$

we conclude that

$$\int_0^\infty e^{iz} \, \log z \, z^{n-2} \, dz = e^{i\frac{\pi}{2}(n-1)} \, \Psi(n-1) \, \Gamma(n-1) + \frac{\pi}{2} \, e^{i\frac{n\pi}{2}} \, \Gamma(n-1) \tag{5.58}$$

where

$$\Psi(z) = \frac{1}{\Gamma(z)} \frac{d\Gamma}{dz}$$

is the digamma function. By only taking the imaginary part we conclude that

$$\begin{aligned} Y' &= \sin\left(\frac{\pi}{2}(n-1)\right) \, \Psi(n-1) \, \Gamma(n-1) + \frac{\pi}{2} \, \sin\left(\frac{n\pi}{2}\right) \, \Gamma(n-1) = \\ &= \Gamma(n-1) \left(\frac{\pi \, \Psi(n-1)}{\Gamma\left(\frac{n-1}{2}\right) \, \Gamma\left(\frac{3-n}{2}\right)} + \frac{\pi^2}{2} \frac{1}{\Gamma\left(\frac{n}{2}\right) \, \Gamma\left(1 - \frac{n}{2}\right)} \right) \end{aligned} \tag{5.59}$$

Thus for I'_{2k} we find

$$I'_0 = \frac{1}{|\xi|^n} \frac{4\,\pi^{\frac{n-1}{2}}\,\Gamma(n-1)}{\Gamma\left(\frac{n-1}{2}\right)} \left(\frac{\pi\,\Psi(n-1)}{\Gamma\left(\frac{n-1}{2}\right)\,\Gamma\left(\frac{3-n}{2}\right)} + \frac{\pi^2}{2} \frac{1}{\Gamma\left(\frac{n}{2}\right)\,\Gamma\left(1-\frac{n}{2}\right)} \right) \tag{5.60}$$

As for I'_{2k}, one can show, as before, that

$$I'_{2k} = (-\Box)^n\, I'_0 = \frac{1}{|\xi|^{2k}}\, I'_0 \tag{5.61}$$

The final result for the 2-point function where $\Delta - n/2 \in \mathbb{Z}$ is

$$\langle \mathcal{O}(\vec{x})\, \mathcal{O}(\vec{y}) \rangle = \frac{A}{|\vec{x}-\vec{y}|^{2\Delta}} \tag{5.62}$$

$$A = \frac{(-1)^{\Delta - n/2}\, n\, 2^{2n-2\Delta+3}\, \pi^{\frac{3n-1}{2}}}{(\Delta - n/2)!} \frac{\Gamma(n-1)}{\Gamma\left(\frac{n-1}{2}\right)} \left(\frac{\pi\,\Psi(n-1)}{\Gamma\left(\frac{n-1}{2}\right)\,\Gamma\left(\frac{3-n}{2}\right)} + \frac{\pi^2}{2} \frac{1}{\Gamma\left(\frac{n}{2}\right)\,\Gamma\left(1-\frac{n}{2}\right)} \right) \tag{5.63}$$

We can see that the behavior of the function is the same as in (5.49). We would expect for the coefficient to be the same, so we conclude that there must be some error in our calculations. Nonetheless, the behavior of the 2-point function agrees to the result (3.16) up to a constant as governed by the CFT.

5.1.1 Calculation by deriving the Bulk-to-boundary propagator

In the previous chapters we managed to derive the 2-point function of a scalar through gravity. One can also compute it with an alternative way using a similar, but different, technique. We once again take the same quadratic action as before, simplified at 5.18 and attempt to solve the Klein-Gordon equation for ϕ in a different way, using the Kernel function, or, as it is called under this context, the "bulk-to-boundary" propagator.

Once again, the boundary condition used is

$$\phi(\vec{x}, z \to 0) = z^{n-\Delta} \phi_0(\vec{x}) \tag{5.64}$$

Using that, complete solution for ϕ works out to be:

$$\phi(\vec{x}, z) = \int d^n y\, K_\Delta(\vec{x}-\vec{y}, z)\, \phi_0(\vec{y}) \tag{5.65}$$

with the Kernel function satisfying:

$$(-\Box + m^2) K_\Delta = 0, \qquad K_\Delta(\vec{x}-\vec{y}, z \to 0) = z^{n-\Delta}\, \delta^n(\vec{x}-\vec{y}) \tag{5.66}$$

In order to solve this differential equation we need to examine the box operator more closely. To be explicit, for **scalars**, it transforms to:

$$\Box = g^{MN} \nabla_M \nabla_N = g^{MM}\, \partial_M\, \partial_M - g^{MM} \Gamma^{\Lambda}_{MM}\, \partial_\Lambda G \tag{5.67}$$

Now using 2 very well known properties of the Christoffel symbols (Eqs. 3.214 and 3.216, p. 147, in [10])

$$\Gamma^{\Lambda}_{MM} = -\frac{1}{2}\, g^{\Lambda\Lambda}\, \partial_{\Lambda}\, g_{MM} \qquad \Lambda \neq M \tag{5.68}$$

$$\Gamma^{M}_{MM} = \partial_M \left(\log \sqrt{|g_{MM}|}\right) \tag{5.69}$$

The d'Alambert operator works out to be

$$\Box = z^2\, \partial_z^2 - z(d-1)\, \partial_z + z^2\, \delta^{\mu\nu}\partial_\mu\partial_\nu \tag{5.70}$$

A perceptive reader can observe that

$$K_\Delta = z^\Delta$$

satisfies the Klein-Gordon equation (but not the boundary condition). In order for it to satisfy the boundary condition as well, one can detect that we need to transfer the point at infinity ($z \to \infty$) to $z \to 0$. To do that, we use an inversion.[5]

$$z \to \frac{z}{z^2 + \vec{x}^2}, \qquad \vec{x} \to \frac{\vec{x}}{z^2 + \vec{x}^2} \tag{5.71}$$

By doing so, the Kernel function becomes:

$$K_\Delta(\vec{x}, z) = c\left(\frac{z}{z^2 + \vec{x}^2}\right)^{\Delta} \tag{5.72}$$

The constant c can now be fixed so that the solution satisfies the boundary condition. We implement 5.72 into 5.65 and obtain:

$$\phi(\vec{x}, z) = c\int d^n y \left(\frac{z}{z^2 + (\vec{x} - \vec{y})^2}\right)^{\Delta} \phi_0(\vec{y}) = c\, z^{n-\Delta} \int d^n u\, \frac{\phi_0(\vec{x} - z\vec{u})}{(1 + u^2)^\Delta} \tag{5.73}$$

For $z \to 0$, equation 5.73 translates to

$$\phi(\vec{x}, z \to 0) = c\, z^{n-\Delta} \int d^n u\, \frac{\phi_0(\vec{x})}{(1 + u^2)^\Delta} = c\, z^{n-\Delta}\, \frac{\pi^{n/2}\,\Gamma(\Delta - n/2)}{\Gamma(\Delta)}\, \phi_0(\vec{x}) \tag{5.74}$$

But the boundary condition demands that

$$\phi(\vec{x}, z \to 0) = z^{n-\Delta}\phi_0(\vec{x})$$

therefore

$$c = \frac{\Gamma(\Delta)}{\pi^{n/2}\,\Gamma(\Delta - n/2)} \tag{5.75}$$

Now that we have the exact solution for K_Δ, we can investigate how our solution behaves at the boundary. Generally, the Klein-Gordon equation that we have is a second order O.D.E, so we expect 2 linearly independent solutions with different starting powers, a and b. In our case, we already

[5] This is allowed without loss of generality since an inversion keeps the AdS metric invariant, since it is an isometry as discussed in section 8.1 of [3]

know one starting power since this information is contained in the boundary condition. Thus, we expect our final solution to behave near the boundary as:

$$\phi(\vec{x},z) = \phi_0(\vec{x})(z^{n-\Delta} + \dots) + \#\,(z^b + \dots) \tag{5.76}$$

where ... are the series expansion in our solutions and b is the starting power of the second solution.

Now, to also determine the second starting power, we have to follow a procedure similar to 5.73.

$$\phi(\vec{x},z) = c\,z^{\Delta} \int d^n y \, \frac{\phi_0(\vec{y})}{\left(1 + \left(\frac{z}{\vec{x}-\vec{y}}\right)^2\right)^{\Delta}} \frac{1}{(\vec{x}-\vec{y})^{2\Delta}} \overset{z\to 0}{\simeq} z^{\Delta}\cdot c \int d^n y \, \frac{\phi_0(\vec{y})}{(\vec{x}-\vec{y})^{2\Delta}} \tag{5.77}$$

Conclusively, ϕ behaves at the boundary as

$$\phi(\vec{x},z) = \phi_0(\vec{x})(z^{n-\Delta} + \dots) + \phi_1(\vec{x})(z^{\Delta} + \dots) \tag{5.78}$$

where

$$\phi_1(\vec{x}) = c \int d^n y \, \frac{\phi_0(\vec{y})}{(\vec{x}-\vec{y})^{2\Delta}} \tag{5.79}$$

This is an interesting outcome considering the fact that $\phi_1(\vec{x})$ is a non-local functional of $\phi_0(\vec{x})$, meaning we need to know the value of ϕ_0 in all of the boundary in order to calculate ϕ_1 at one point.

Now that we have calculated ϕ, we insert it in the action

$$S = \frac{1}{2} \int d^n x \, \frac{\phi\,\partial_z \phi}{z^{n-1}} \Bigg|_{z\to 0}^{\infty} \tag{5.80}$$

This calculation will lead up of terms of the form $\#\,\phi_0^2$, $\#\,\phi_1^2$ and $\#\,\phi_0\,\phi_1$. We will attempt to show that for both 1 and 2-point functions, only terms of the form $\#\,\phi_0\,\phi_1$ contribute to the result, the rest vanish.

For 1-point functions:

$$\begin{aligned} \frac{\delta}{\delta\phi_0(\vec{x})}\left(\int d^n x' \phi_0^2(\vec{x'})\right)\Bigg|_{\phi_0=0} &= 2\int d^n x' \, \phi_0(\vec{x'}) \, \frac{\delta\phi_0(\vec{x'})}{\delta\phi_0(\vec{x})}\Bigg|_{\phi_0=0} = \\ &= 2\int d^n x' \phi_0(\vec{x'}) \delta^n(x-x')\Bigg|_{\phi_0=0} = \phi_0(\vec{x})\Bigg|_{\phi_0=0} = 0 \end{aligned} \tag{5.81}$$

We gain a similar result for ϕ_1^2. Specifically:

$$\begin{aligned} \frac{\delta}{\delta\phi_0(\vec{x})}\left(\int d^n x' \phi_1^2(\vec{x'})\right)\Bigg|_{\phi_0=0} &= \int d^n x' \, \phi_1(\vec{x}') \int d^n y \frac{\frac{\delta\phi_0(\vec{y})}{\delta\phi_0(\vec{x})}}{(\vec{x}'-\vec{y})^{2\Delta}}\Bigg|_{\phi_0=0} = \\ &= \int d^n x' \frac{\phi_1(\vec{x}')}{(\vec{x}'-\vec{x})^{2\Delta}}\Bigg|_{\phi_0=0} = 0 \end{aligned} \tag{5.82}$$

For 2-point functions:

$$\begin{aligned} \frac{\delta^2}{\delta\phi_0(\vec{x})\,\delta\phi_0(\vec{y})}\left(\int d^n x' \, \phi_0^2(\vec{x}')\right)\Bigg|_{\phi_0=0} &= 2\frac{\delta}{\delta\phi_0(\vec{x})} \int d^n x' \phi_0(\vec{x}') \frac{\delta\phi_0(\vec{x}')}{\delta\phi_0(\vec{y})}\Bigg|_{\phi_0=0} = \\ &= 2\frac{\delta}{\delta\phi_0(\vec{x})} \int d^n x' \, \phi_0(\vec{x}') \, \delta^n(\vec{x}'-\vec{y})\Bigg|_{\phi_0=0} = \delta^n(\vec{x}-\vec{y}) \end{aligned} \tag{5.83}$$

This is a contact term and thus is discarded.

The case for terms of the form ϕ_1^2 is even more subtle. If we carefully take into consideration the powers of z, we deduce:

$$\# \, z^{2\Delta - n} \, \phi_1^2$$

but since $\Delta > n/2 \Rightarrow 2\Delta - n > 0$ we conclude

$$\# \, z^{2\Delta - n} \, \phi_1^2 \xrightarrow{z \to 0} 0 \tag{5.84}$$

Eventually, the action (with just one term remaining) can be expressed as:

$$S = \frac{n}{2} \int d^n x \, \phi_0(\vec{x}) \, \phi_1(\vec{x}) = \frac{n \cdot c}{2} \int d^n x \, d^n y \frac{\phi_0(\vec{x}) \, \phi_0(\vec{y})}{|\vec{x} - \vec{y}|^{2\Delta}} \tag{5.85}$$

The 1-point function is:

$$\langle \mathcal{O}(\vec{x}) \rangle = \frac{\delta S}{\delta \phi_0(\vec{x})} \bigg|_{\phi_0 = 0} = n \cdot \phi_1(\vec{x}) \big|_{\phi_0 = 0} = 0 \tag{5.86}$$

As expected.

The 2-point function would be (substituting the constant c):

$$\langle \mathcal{O}(\vec{x}) \, \mathcal{O}(\vec{y}) \rangle = \frac{\delta^2 S}{\delta \phi_0(\vec{x}) \, \delta \phi_0(\vec{y})} \bigg|_{\phi_0 = 0} = \frac{n \, \Gamma(\Delta)}{\pi^{n/2} \, \Gamma(\Delta - n/2)} \frac{1}{|\vec{x} - \vec{y}|^{2\Delta}} \tag{5.87}$$

This result once again verifies our 2 previous outcomes up to a consant, equations 3.20 and 5.49. Unfortunately, the constant we derived is only partially correct, as one can see the correct result in Eq. 56 of [11].

6 The 3-point function of a scalar through gravity

For the scalar 3-point function, the methodology that we are going to follow is the same as before, but we have to notice that the quadratic action is no longer suitable for us. For the 3-point function, we need an interaction term in the action. The Witten diagram would be tree-level since we are only solving the classical EOMs. Non-tree-level behavior requires quantum corrections that we do not investigate. For the 3-point function, the action that we are going to use is

$$S = \int d^n x \, dz \sqrt{g} \left(\frac{1}{2} \sum_{i=1}^{3} \left(\partial_M \phi_i \, \partial^M \phi_i + m_i^2 \phi_i^2 \right) + \lambda \, \phi_1 \, \phi_2 \, \phi_3 \right) \tag{6.1}$$

Where we use 3 different fields, ϕ_1, ϕ_2, ϕ_3 so that each one can act as a source to their corresponding operator $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$,. The last term is the interaction term, with the constant λ defining the strength of the interaction. By varying with respect to ϕ_i we find the following EOMs

$$(-\Box + m^2)\phi_1 = \lambda \, \phi_2 \, \phi_3 \tag{6.2}$$

$$(-\Box + m^2)\phi_2 = \lambda \, \phi_1 \, \phi_3 \tag{6.3}$$

$$(-\Box + m^2)\phi_3 = \lambda \, \phi_1 \, \phi_2 \tag{6.4}$$

These equations are non-linear and can only be solved pertubatively. We assume that the interaction is a first and not a zeroth order interaction, meaning we assume the expansion:

$$\phi_i = \phi_i^{(0)} + \phi_i^{(1)}, \qquad \lambda \equiv \lambda^{(1)} \tag{6.5}$$

Pertubation theory demands the following equations to be true:

$$(-\Box + m^2)\phi_i^{(0)} = 0, \qquad \phi_i^{(0)}(\vec{x}, z \to 0) = z^{n-\Delta}\, \phi_{i,0}^{(0)}(\vec{x}) \tag{6.6}$$

$$(-\Box + m^2)\phi_1^{(1)} = \lambda \phi_2^{(0)}\, \phi_3^{(0)}, \qquad \phi_1^{(1)}(\vec{x}, z \to 0) = 0 \tag{6.7}$$

$$(-\Box + m^2)\phi_2^{(1)} = \lambda \phi_1^{(0)}\, \phi_3^{(0)}, \qquad \phi_2^{(1)}(\vec{x}, z \to 0) = 0 \tag{6.8}$$

$$(-\Box + m^2)\phi_3^{(1)} = \lambda \phi_1^{(0)}\, \phi_2^{(0)}, \qquad \phi_3^{(1)}(\vec{x}, z \to 0) = 0 \tag{6.9}$$

To make our calculations easier, we use the same trick that we used before to simplify the action (equations 5.16 to 5.18). Finally, the actions transforms to:

$$S = \frac{1}{2} \int d^n x\, \frac{\phi_1\, \partial_z \phi_1 + \phi_2\, \partial_z \phi_2 + \phi_3\, \partial_z \phi_3}{z^{n-1}} \Bigg|_{z \to 0}^{\infty} + \frac{11\,\lambda}{2} \int d^n x\, dz\, \sqrt{g}\, \phi_1\, \phi_2\, \phi_3 \tag{6.10}$$

But this can be even more simplified considering we want to keep terms only up to the first order. So by imposing the expansion at 6.5, the action is rewritten as

$$S = \frac{1}{2} \int d^n x\, \frac{1}{z^{n-1}} \sum_{i=1}^{3} \left[\phi_i^{(0)}\, \partial_z \phi_i^{(0)} + \partial_z \left(\phi_i^{(0)}\, \phi_i^{(1)} \right) \right] \Bigg|_{z \to 0}^{\infty} + \frac{11\,\lambda}{2} \int d^n x\, dz\, \frac{\phi_1^{(0)}\, \phi_2^{(0)}\, \phi_3^{(0)}}{z^{n+1}} \tag{6.11}$$

The 3-point function will therefore be given by

$$\langle \mathcal{O}(\vec{x_1})\, \mathcal{O}(\vec{x_2})\, \mathcal{O}(\vec{x_3}) \rangle = - \frac{\delta^3 S}{\delta \phi_{1,0}^{(0)}(\vec{x}_1)\, \delta \phi_{2,0}^{(0)}(\vec{x}_2)\, \delta \phi_{3,0}^{(0)}(\vec{x}_3)} \Bigg|_{\phi_{1,0}^{(0)} = \phi_{2,0}^{(0)} = \phi_{3,0}^{(0)} = 0} \tag{6.12}$$

Our objective now is to calculate $\phi^{(0)}$ and $\phi^{(1)}$, governed by the equations 6.6 to 6.9. We have already solved equation 6.6, using the Kernel function, the bulk-to-boundary propagator:

$$\phi_i^{(0)}(\vec{x}, z) = \int d^n x'\, K_{\Delta_i}(\vec{x} - \vec{x}\,', z)\, \phi_{i,0}^{(0)}(\vec{x}\,') \tag{6.13}$$

$$K_{\Delta_i}(\vec{x} - \vec{x}\,', z) = C_i \left(\frac{z}{z^2 + (\vec{x} - \vec{x}\,')^2} \right)^{\Delta_i} \qquad C_i = \frac{\Gamma(\Delta_i)}{\pi^{n/2}\, \Gamma(\Delta_i - n/2)} \tag{6.14}$$

6.1 The bulk-to-bulk propagator

In order to account for the interaction in the bulk as well, which is a first order interaction, we need to solve equations (6.7) to (6.9). The solution is not trivial but can be done by using the Green's function, the bulk-to-bulk propagator. The solution will be of the form

$$\phi_1^{(1)}(\vec{x}, z) = \lambda \int d^n x'\, dz'\, \sqrt{g}\, G(\vec{x} - \vec{x}\,', z, z')\, \phi_2^{(0)}(\vec{x}\,', z')\, \phi_3^{(0)}(\vec{x}\,', z') \tag{6.15}$$

$$\phi_2^{(1)}(\vec{x}, z) = \lambda \int d^n x'\, dz'\, \sqrt{g}\, G(\vec{x} - \vec{x}\,', z, z')\, \phi_1^{(0)}(\vec{x}\,', z')\, \phi_3^{(0)}(\vec{x}\,', z') \tag{6.16}$$

$$\phi_3^{(1)}(\vec{x}, z) = \lambda \int d^n x'\, dz'\, \sqrt{g}\, G(\vec{x} - \vec{x}\,', z, z')\, \phi_1^{(0)}(\vec{x}\,', z')\, \phi_2^{(0)}(\vec{x}\,', z') \tag{6.17}$$

where the Green's function satisfies

$$(-\Box + m^2)G(\vec{x}, \vec{x}\,', z, z') = \frac{1}{\sqrt{g}}\,\delta^n(\vec{x} - \vec{x}\,')\,\delta(z - z') \tag{6.18}$$

This equation is a bit tricky but it can be solved, using equation 5.70 and a number of transformations. We will firstly solve it treating it as a **homogeneous** equation and temporarily discarding the delta function source (we will see more about it shortly). Meaning we will first attempt to solve

$$(-\Box + m^2)G(\vec{x}, \vec{x}\,', z, z') = 0 \tag{6.19}$$

Naturally, we expect for the Green's function to depend only on the geodesic distance between the points $(\vec{x}, z)$ and $(\vec{x}\,', z')$, so a transformation on the coordinates that implements the geodesic distance would greatly simplify the PDE. The geodesic distance for a 2-dimensional AdS is given by (2.28) but due to its algebraic complexity, it is more convenient to work with the choral distance u. So we apply the following transformation and calculate the transformed derivatives:

$$u = \frac{\delta_{\mu\nu}(x - x')^\mu\,(x - x')^\nu + (z - z')^2}{2zz'} = \frac{(\vec{x} - \vec{x}\,')^2 + (z - z')^2}{2zz'} \tag{6.20}$$

$$\partial_z = \left(-\frac{u}{z} + \frac{z - z'}{zz'}\right) \tag{6.21}$$

$$\partial_z^2 = \frac{1}{z^2}\left[2u + 1 - \frac{z - z'}{z'}\frac{d}{du} + \left(u^2 + \frac{(z - z')^2}{z'^2} - \frac{2u(z - z')}{z'}\right)\frac{d^2}{du^2}\right] \tag{6.22}$$

which result in the following ODE.

$$(u^2 + 2u)G'' + (d + 1)(u + 1)G' - m^2 G = 0 \tag{6.23}$$

This is remarkable since with a single transformation we managed to convert a PDE into an ODE. But our work is not finished yet, we still do not have a satisfactory form for our ODE. We use 2 more transformations on our independent variable u:

$$\xi = \frac{1}{u + 1} \tag{6.24}$$

$$w = \xi^2 \tag{6.25}$$

from which the ODE reshapes to

$$4w^2(1 - w)G'' + 2w(2 - d - 3w)G' - \Delta(\Delta - n)G = 0 \tag{6.26}$$

where we recalled that $m^2 = \Delta(\Delta - n)$. The final transformation needed is one in the dependent variable G. In particular,

$$G = w^{\Delta/2}\,Y \tag{6.27}$$

This final transformation allows us to write our ODE in a rather familiar form

$$w(1 - w)Y'' + \left[\Delta + 1 - \frac{n}{2} - \left(\frac{\Delta}{2} + \frac{\Delta}{2} + \frac{1}{2} + 1\right)w\right]Y' - \frac{\Delta(\Delta + 1)}{4}Y = 0 \tag{6.28}$$

This is the hypergeometric equation. That means that the solution for (6.28) (taking into account all the transformations we performed) is

$$G(\xi) = C\,\xi^{\Delta}\,{}_2F_1\left(\frac{\Delta}{2},\frac{\Delta+1}{2};\Delta+1-\frac{n}{2};\xi^2\right) \qquad \xi = \frac{2zz'}{z^2+(z')^2+(\vec{x}-\vec{x}\,')^2} \tag{6.29}$$

where $0<\xi\leq 1$, the equality ($\xi=1$) holds true for $(\vec{x},z)=(\vec{x}\,',z')$.

For $0\leq\xi<1$ equation (6.18) reduces to (6.19) since for $\xi\neq 1$ we can omit the delta function source. That means that the hypergeometric function is actually a solution of (6.18) (for $0\leq\xi<1$). All we need to do now is to more closely examine the behavior of the hypergeometric function at $\xi=1$. It is known that for $\xi\to 1$ we have

$$F(a,b;c;z)\sim(z-1)^{c-a-b} \qquad z\to 1 \tag{6.30}$$

In our case,

$${}_2F_1\left(\frac{\Delta}{2},\frac{\Delta+1}{2};\Delta+1-\frac{n}{2};\xi^2\right)\sim(\xi^2-1)^{(1-n)/2} \qquad \xi^2\to 1 \tag{6.31}$$

For $n>1$, then $\frac{1-n}{2}<0$, thus for $\xi\to 1$ the hypergeometric function diverges, just like the delta function. One could also verify that indeed the power $\frac{1-n}{2}$ is the appropriate one.[6] Therefore, since the Green's function diverges with the exact behavior predicted by the delta function, it is a solution for equation (6.18).

6.2 Calculation of the 3-point function

We will use the methodology depicted in [12]. The action written in the form of (6.11) essentially has 3 terms. The first one,

$$\frac{\phi_i^{(0)}\partial_z\phi_i^{(0)}}{z^{n-1}} \tag{6.32}$$

is identical to the action examined in the 2-point function case. Varying with respect to $\phi_{i,0}^{(0)}$ will result to

$$\frac{\delta}{\delta\phi_{i,0}^{(0)}(\vec{x}_i)}\left(\int d^n x\left[\frac{\phi_i^{(0)}\partial_z\phi_i^{(0)}}{z^{n-1}}\right]_{z\to 0}^{\infty}\right)\Bigg|_{\phi_{i,0}^{(0)}=0} = \langle\mathcal{O}_i(\vec{x}_i)\rangle = 0 \tag{6.33}$$

[6] Let us examine a 3-dimensional equivalent. It is known that for the Poisson equation with a delta-like source

$$\nabla^2 V = \frac{\delta(\vec{r}-\vec{r}\,')}{\varepsilon_0}$$

that the Green's function will be given by

$$G(\vec{r},\vec{r}\,') = -\frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}\,'|}$$

So if we assume a 3-dimensional AdS spacetime, we should expect equation (6.31) to result to a behavior analogous to

$$G\sim\frac{1}{|\vec{x}-\vec{x}\,'+z-z'|}$$

and indeed, for n=2 and $(\vec{x},z)\to(\vec{x}\,',z')$ we have

$$(\xi^2-1)^{-1/2}\sim u^{-1/2}\sim\frac{1}{|\vec{x}-\vec{x}\,'+z-z'|}$$

So our results are verified.

As we can see, from the first term emerges the sum of three 1-point functions that naturally vanish as shown in (5.86).

Moving on to the next term, that is a bit more delicate.

$$\left.\frac{\partial_z\left(\phi_i^{(0)}\,\phi_i^{(1)}\right)}{z^{n-1}}\right|_{z=\epsilon}^{\infty}=\left.\frac{\phi_i^{(1)}\partial_z\phi_i^{(0)}+\phi_i^{(0)}\partial_z\phi_i^{(1)}}{z^{n-1}}\right|_{z=\epsilon}^{\infty} \tag{6.34}$$

We have used a cut-off value for z. It is apparent that we need to examine the behavior of $\phi^{(0)}$ and $\phi^{(1)}$ for $z=\epsilon\to 0$ and $z\to\infty$.

We will firstly investigate the case for $z=\epsilon\to 0$.
It is known from the boundary condition itself that,

$$\phi_i^{(0)}(\vec{x},z=\epsilon)\overset{\epsilon\to 0}{=}\epsilon^{n-\Delta}\,\phi_{i,0}^{(0)}(\vec{x}) \tag{6.35}$$

For $\phi^{(1)}$ more calculations are needed. We recall that ξ, as governed by (6.29), behaves as

$$\xi\overset{\epsilon\to 0}{\sim}\epsilon \tag{6.36}$$

Therefore, since ${}_2F_1(a,b;c;z\to 0)\to 1$, the bulk-to-bulk propagator must have the following behavior

$$G(\xi\to 0)\sim\xi^{\Delta}\sim\epsilon^{\Delta} \tag{6.37}$$

Working out how this propagates to $\phi_i^{(1)}$ we find

$$\phi_i^{(1)}\overset{\epsilon\to 0}{\sim}\epsilon^{n+\Delta_i-\Delta_j-\Delta_k-1}\,A\left(\phi_{j,0}^{(0)},\phi_{k,0}^{(0)}\right)\qquad i\neq j\neq k \tag{6.38}$$

where $A\left(\phi_{j,0}^{(0)},\phi_{k,0}^{(0)}\right)$ is an integral containing all the boundary conditions of our fields $\phi_{j,0}^{(0)}$, $\phi_{k,0}^{(0)}$. Plugging equations (6.35) and (6.38) into (6.34) we observe that the second term transforms to

$$\#\,A\left(\phi_{j,0}^{(0)},\phi_{k,0}^{(0)}\right)\,\phi_{i,0}^{(0)}\,\epsilon^{n-\Delta_j-\Delta_k-1} \tag{6.39}$$

By varying with respect to $\phi_{i,0}^{(0)}$, $\phi_{j,0}^{(0)}$ and $\phi_{k,0}^{(0)}$, and setting $\phi_{i,0}^{(0)}=\phi_{j,0}^{(0)}=\phi_{k,0}^{(0)}=0$, the term **vanishes** regardless of the value of ϵ.

We examined the behavior of the second term when $z\to 0$ but in order for our calculation to be complete we need to also account for $z\to\infty$.
We understand from (6.13) that $\phi_i^{(0)}$ has the following expansion

$$\phi_i^{(0)}\overset{z\to\infty}{\sim}B\left(\phi_{i,0}^{(0)}\right)\,z^{-\Delta} \tag{6.40}$$

where $B\left(\phi_{i,0}^{(0)}\right)$ is another integral containing $\phi_{i,0}^{(0)}$.

The behavior of ξ is also straightforward, and equal to

$$\xi\overset{z\to\infty}{\sim}1/z\to 0 \tag{6.41}$$

Therefore the hypergeometric function has the same expansion as before. Similarly, the bulk-to-bulk propagator behaves as

$$G(\xi\to 0)\sim\xi^{\Delta}\sim z^{-\Delta} \tag{6.42}$$

and thus, for $\phi^{(1)}$

$$\phi^{(1)} \overset{z\to\infty}{\sim} C\left(\phi^{(0)}_{j,0}, \phi^{(0)}_{k,0}\right)\, z^{-\Delta_i-\Delta_j-\Delta_k-n-1} \tag{6.43}$$

where $C\left(\phi^{(0)}_{j,0}, \phi^{(0)}_{k,0}\right)$ is an integral similar to $A\left(\phi^{(0)}_{j,0}, \phi^{(0)}_{k,0}\right)$.

Finally, for the behavior of the second term (6.34) for $z \to \infty$ we discover

$$\#\, B\left(\phi^{(0)}_{i,0}\right)\, C\left(\phi^{(0)}_{j,0}, \phi^{(0)}_{k,0}\right)\, z^{-2\Delta_1-\Delta_j-\Delta_k-2n-1} \tag{6.44}$$

This term **vanishes** either by varying with respect to $\phi^{(0)}_{i,0}$, $\phi^{(0)}_{j,0}$ and $\phi^{(0)}_{k,0}$, and setting $\phi^{(0)}_{i,0} = \phi^{(0)}_{j,0} = \phi^{(0)}_{k,0} = 0$, or when $z \to \infty$ since it is raised in a negative power.

We have now fully examined the behavior of the second term (6.34) and have found out that it **vanishes** completely.

Consequently, the only term that will contribute to the overall result for the 3-point function is the last one. Given the recent developments, the 3-point function is rewritten as:

$$\langle \mathcal{O}(\vec{x_1})\, \mathcal{O}(\vec{x_2})\, \mathcal{O}(\vec{x_3})\rangle = -\frac{\delta^3}{\delta\phi^{(0)}_{1,0}(\vec{x}_1)\, \delta\phi^{(0)}_{2,0}(\vec{x}_2)\, \delta\phi^{(0)}_{3,0}(\vec{x}_3)} \left(\frac{11\,\lambda}{2}\int d^n x\, dz\, \sqrt{g}\, \phi^{(0)}_1\, \phi^{(0)}_2\, \phi^{(0)}_3\right)\Bigg|_{\phi^{(0)}_{1,0}=\phi^{(0)}_{2,0}=\phi^{(0)}_{3,0}=0} \tag{6.45}$$

But knowing (using (6.13)) that

$$\frac{\delta\phi^{(0)}_i}{\delta\phi^{(0)}_{i,0}} = K_{\Delta_i}(\vec{x} - \vec{x}_i, z) \tag{6.46}$$

we find

$$\langle \mathcal{O}(\vec{x_1})\, \mathcal{O}(\vec{x_2})\, \mathcal{O}(\vec{x_3})\rangle = -\frac{11\lambda}{2}\int d^n y\, dz\, \sqrt{g}\, K_{\Delta_1}(\vec{y} - \vec{x_1}, z)\, K_{\Delta_2}(\vec{y} - \vec{x_2}, z)\, K_{\Delta_3}(\vec{y} - \vec{x_3}, z) \tag{6.47}$$

where we renamed the variables $\vec{x}$ as $\vec{y}$ to avoid confusion. We can use translation symmetry to move the position of $\vec{x}_3$ to 0. Meaning we perform the translation

$$\vec{x_1} \to \vec{x_1} - \vec{x_3} \qquad \vec{x_2} \to \vec{x_2} - \vec{x_3} \qquad \vec{x_3} \to 0 \tag{6.48}$$

followed by an inversion,

$$y^{\mu} = \frac{y'^{\mu}}{\vec{y}\,'^2 + z'^2}, \qquad z = \frac{z'}{\vec{y}\,'^2 + z'^2} \qquad \vec{x}_i = \frac{\vec{x}\,'_i}{(\vec{x}\,'_i)^2} \tag{6.49}$$

where the inversion of the points $\vec{x}_i \quad i = 1, 2, 3$ is independent of z because these points exist in the boundary of AdS.

Utilizing the relations at (6.49), the bulk-to-boundary propagator transforms as follows

$$K_{\Delta_i}(\vec{y} - \vec{x}_i, z) = |\vec{x}\,'_i|^{2\Delta_i}\, K(\vec{y}\,' - \vec{x}\,'_i, z') = \frac{1}{|\vec{x}_i|^{2\Delta_i}}\, K_{\Delta_i}(\vec{y}\,' - \vec{x}\,'_i, z') \tag{6.50}$$

We should be careful since the above transformation holds true solely for $i = 1, 2$ since $\vec{x}_3 = 0$ after the translation. So for K_{Δ_3}, the transformation is

$$K_{\Delta_3} = C_3\left(\frac{z}{z^2 + \vec{y}^2}\right)^{\Delta_3} = C_3\left(\frac{\dfrac{z'}{z'^2 + \vec{y}\,'^2}}{\dfrac{z'^2}{z'^2 + \vec{y}\,'^2} + \dfrac{\vec{y}\,'^2}{z'^2 + \vec{y}\,'^2}}\right)^{\Delta_3} = C_3\,(z')^{\Delta_3} \tag{6.51}$$

Moreover, we notice that

$$\left.\begin{array}{l}\sqrt{g} = \dfrac{1}{z^{n+1}} = \dfrac{(z'^2 + \vec{x}\,'^2)^{n+1}}{(z')^{n+1}} = \left(z'^2 + \vec{x}\,'^2\right)^{n+1} \sqrt{g'} \\ \\ d^n x\, dz = \dfrac{d^n x'}{(z'^2 + \vec{x}\,'^2)^n} \dfrac{dz'}{z'^2 + \vec{x}\,'^2} = \dfrac{d^n x'\, dz'}{(z'^2 + \vec{x}\,'^2)^{n+1}}\end{array}\right\} \sqrt{g}\, d^n x\, dz = \sqrt{g'}\, d^n x'\, dz' \tag{6.52}$$

This result is to be expected since inversion is an isometry in AdS. Finally, the 3-point function takes the form

$$-\frac{11\lambda}{2} \frac{1}{|\vec{x}_1|^{2\Delta_1}} \frac{1}{|\vec{x}_2|^{2\Delta_2}} C_1\, C_2\, C_3 \int d^n y'\, dz' \, \frac{(z')^{\Delta_1+\Delta_2+\Delta_3-n-1}}{[(z')^2 + (\vec{y}\,' - \vec{x}\,'_1)^2]^{\Delta_1}\; [(z')^2 + (\vec{y}\,' - \vec{x}\,'_2)^2]^{\Delta_2}} \tag{6.53}$$

The integral we have to calculate is challenging, but can be estimated using a number of transformations

$$I(a, b, c; \vec{x}_1, \vec{x}_2) = \int d^n x\, dz \frac{z^a}{[z^2 + (\vec{x} - \vec{x}_1)^2]^b\, [z^2 + (\vec{x} - \vec{x}_2)^2]^c} \tag{6.54}$$

First, it is trivial to show from the definition of the Gamma function, that this identity holds true

$$\frac{1}{Q^q} = \frac{1}{\Gamma(q)} \int_0^\infty ds\, s^{q-1}\, e^{-sQ} \tag{6.55}$$

Inserting this identity twice we get the following

$$I = \frac{1}{\Gamma(b)\,\Gamma(c)} \int_0^\infty dz \int_0^\infty ds \int_0^\infty dt \int_{\mathbb{R}^\ltimes} d^n x\, z^a\, s^{b-1}\, t^{c-1}\, e^{-V_1} \tag{6.56}$$

$$\begin{aligned} V_1 &= s(z^2 + (\vec{x} - \vec{x}_1)^2) + t(z^2 + (\vec{x} - \vec{x}_2)^2) = \\ &= s(z^2 + \vec{x}_1^2) + t(z^2 + \vec{x}_2^2) + (s+t)x^2 - 2\vec{x}(s\vec{x}_1 + t\vec{x}_2) \end{aligned} \tag{6.57}$$

We notice that one of the integrals simplifies since it is an n-dimensional gaussian integral:

$$\begin{aligned} \int d^n x\, e^{-V_1} &= e^{-s(z^2+\vec{x}_1^2)-t(z^2+\vec{x}_2^2)+\frac{(s\vec{x}_1+t\vec{x}_2)^2}{s+t}} \int_{\mathbb{R}^n} d^n x\, e^{-(s+t)\left(\vec{x}-\frac{s\vec{x}_1+t\vec{x}_2}{s+t}\right)^2} = \\ &= e^{V_2} \left(\frac{\pi}{s+t}\right)^{n/2} \end{aligned} \tag{6.58}$$

$$V_2 = \frac{(s\vec{x}_1 + t\vec{x}_2)^2}{s+t} - s(z^2 + \vec{x}_1^2) - t(z^2 + \vec{x}_2^2) = -\frac{z^2(s+t)^2 + st(\vec{x}_1 - \vec{x}_2)^2)}{s+t} \tag{6.59}$$

Thus it transforms to

$$I = \frac{\pi^{n/2}}{\Gamma(b)\,\Gamma(c)} \int_0^\infty ds \int_0^\infty dt\, s^{b-1}\, t^{c-1}\, (s+t)^{-n/2}\, e^{-\frac{st(\vec{x}_1-\vec{x}_2)^2)}{s+t}} \int_0^\infty dz\, z^a\, e^{-(s+t)z^2} \tag{6.60}$$

By making a $(s+t)z^2 \to z$ substitution, one can easily solve the latter integral, and rewrite I

$$I = \frac{\pi^{n/2}\,\Gamma(\frac{a+1}{2})}{2\Gamma(b)\,\Gamma(c)} \int_0^\infty ds \int_0^\infty dt\, s^{b-1}\, t^{c-1}\, (s+t)^{-\frac{a+1+n}{2}}\, e^{-\frac{st(\vec{x}_1-\vec{x}_2)^2}{s+t}} \tag{6.61}$$

Our job is not done yet, we have to perform 2 more substitutions:

$$s' = \frac{s}{(\vec{x}_1 - \vec{x}_2)^2} \qquad t' = \frac{t}{(\vec{x}_1 - \vec{x}_2)^2} \tag{6.62}$$

and

$$s' = uv \qquad t' = (1-v)u$$

$$\begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} = -u \tag{6.63}$$

The updated form of the integral is

$$I = \frac{\pi^{n/2}\,\Gamma\left(\frac{a+1}{2}\right)}{2\,\Gamma(b)\,\Gamma(c)} \frac{1}{|\vec{x}_1 - \vec{x}_2|^{2b+2c-a-1-n}} \int_0^\infty du\, u^{b+c-1-\frac{a+n+1}{2}}\, e^{-uv(1-v)} \int_0^1 dv\, v^{b-1}\,(1-v)^{c-1} \tag{6.64}$$

One final substitution

$$w = vu(1-v) \tag{6.65}$$

results to

$$I = \frac{\pi^{n/2}\,\Gamma\left(\frac{a+1}{2}\right)}{2\,\Gamma(b)\,\Gamma(c)} \frac{1}{|\vec{x}_1 - \vec{x}_2|^{2b+2c-a-1-n}} \int_0^\infty dw\, w^{b+c-1-\frac{a+n+1}{2}}\, e^{-w} \int_0^1 dv\, v^{\frac{a+n+1}{2}-c-1}\,(1-v)^{\frac{a+n+1}{2}-1-b} \tag{6.66}$$

From the definitions of the Gamma and Beta functions, we can easily compute these 2 integrals. Eventually, the final result is

$$I(a,b,c;\vec{x}_1,\vec{x}_2) = \frac{\pi^{n/2}\,\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(b+c-\frac{a+n+1}{2}\right)\Gamma(\frac{a+n+1}{2}-b)\,\Gamma(\frac{a+n+1}{2}-c)}{2\,\Gamma(b)\,\Gamma(c)\,\Gamma(a+n+1-b-c)} \cdot \frac{1}{|\vec{x}_1 - \vec{x}_2|^{2b+2c-a-1-n}} \tag{6.67}$$

In our case, we have $I(\Delta_1+\Delta_2+\Delta_3-n-1, \Delta_1, \Delta_2; \vec{x}\,'_1, \vec{x}\,'_2)$. We also need to transform back from $\vec{x}\,'_i$ to $\vec{x}_i$ in order to arrive at a proper result. We observe that

$$|\vec{x}\,'_1 - \vec{x}\,'_2|^2 = (x_1'^{\mu} - x'_{2\mu})(x_1'^{\mu} - x'_{2\mu}) = |\vec{x}_1'|^2 + |\vec{x}_2'|^2 - 2\,\vec{x}_1\cdot\vec{x}_2 = \tag{6.68}$$

$$\frac{1}{|\vec{x}+1|^2} + \frac{1}{|\vec{x}_2|^2} - \frac{2\,\vec{x}_1\cdot\vec{x}_2}{|\vec{x}_1|^2|\vec{x}_2^2|} = \frac{|\vec{x}_1 - \vec{x}_2|^2}{|\vec{x}_1|^2|\vec{x}_2|^2} \tag{6.69}$$

If we also take into account the translation we formerly did in (6.48) and transform this back as well, the final result for the 3-point function would be

$$\langle \mathcal{O}(\vec{x_1})\,\mathcal{O}(\vec{x_2})\,\mathcal{O}(\vec{x_3})\rangle = \frac{\Lambda}{|\vec{x}_1 - \vec{x}_3|^{\Delta_1+\Delta_3-\Delta_2}\,|\vec{x}_2 - \vec{x}_3|^{\Delta_2+\Delta_3-\Delta_1}\,|\vec{x}_1 - \vec{x}_2|^{\Delta_1+\Delta_2-\Delta_3}} \tag{6.70}$$

$$\Lambda = -\frac{11\lambda}{4} \frac{\Gamma\left(\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)\right)\,\Gamma\left(\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)\right)\,\Gamma\left(\frac{1}{2}(\Delta_1+\Delta_3-\Delta_2)\right)\,\Gamma\left(\frac{1}{2}(\Delta_1+\Delta_2+\Delta_3-n)\right)}{\Gamma(\Delta_1-n/2)\,\Gamma(\Delta_2-n/2)\,\Gamma(\Delta_3-n/2)} \tag{6.71}$$

We can see that this outcome agrees with the behavior of the 3-point function indicated in (3.27) as well as with the exact result found in Eq. 25 of [12].

Appendices

A Derivation of the Einstein field equations

We will start by varying with respect to $g^{\mu\nu}$ on the Einstein-Hilbert action with a cosmological constant

$$S = \int d^{n+1}x \sqrt{g}\,(\mathcal{R} - 2\Lambda) \tag{A.1}$$

This will result to

$$\delta S_{EH} = \delta S_1 + \delta S_2 + \delta S_3 + \delta S_4 \tag{A.2}$$

$$\delta S_1 = \int d^{n+1}x \sqrt{g}\, g^{\mu\nu}\, \delta \mathcal{R}_{\mu\nu} \tag{A.3}$$

$$\delta S_2 = \int d^{n+1}x \sqrt{g}\, \mathcal{R}_{\mu\nu}\, \delta g^{\mu\nu} \tag{A.4}$$

$$\delta S_3 = \int d^{n+1}x\, \mathcal{R}\, \delta \sqrt{g} \tag{A.5}$$

$$\delta S_4 = -2 \int d^{n+1}x\, \Lambda\, \delta \sqrt{g} \tag{A.6}$$

The most challenging computation is the one of (A.3), the Ricci scalar. so we will begin with that. Firstly, using the definition of the Riemann tensor

$$\mathcal{R}^{\rho}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\lambda} + \Gamma^{\rho}_{\mu\sigma}\, \Gamma^{\sigma}_{\nu\lambda} - \partial_\nu \Gamma^{\rho}_{\lambda\mu} - \Gamma^{\rho}_{\nu\sigma}\, \Gamma^{\sigma}_{\lambda\mu} \tag{A.7}$$

one can show that a variation in the Riemann tensor will yield

$$\delta \mathcal{R}^{\rho}{}_{\lambda\mu\nu} = \nabla_\mu \delta\, \Gamma^{\rho}_{\nu\lambda} - \nabla_\nu \delta\, \Gamma^{\rho}_{\lambda\mu} \tag{A.8}$$

We have now established the necessary groundwork to demonstrate the variations on the Ricci tensor.

$$\delta \mathcal{R}_{\mu\nu} = \delta \mathcal{R}^{\rho}{}_{\mu\rho\nu} = \nabla_\rho \delta\, \Gamma^{\rho}_{\nu\mu} - \nabla_\nu \delta\, \Gamma^{\rho}_{\mu\rho} \tag{A.9}$$

This is known as the Palatini identity. We can now rewrite δS_1 as

$$\delta S_1 = \int d^{n+1}x \sqrt{g}\, \nabla_\rho \left(g^{\mu\nu}\, \delta\, \Gamma^{\rho}_{\nu\mu} - g^{\mu\rho}\, \delta\, \Gamma^{\sigma}_{\mu\sigma} \right) \tag{A.10}$$

In order to complete our analysis on the Ricci scalar we also need to further investigate the variations on the Christoffel symbols'. With the help of the Christoffel symbols definition

$$\Gamma^{\rho}_{\nu\mu} = \frac{1}{2}\, g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \tag{A.11}$$

and this straightforward identity

$$\delta^{\mu}_{\nu} = g_{\mu\sigma}\, g^{\sigma\nu} \Rightarrow \delta g^{\mu\nu} = -g^{\mu\sigma} g^{\lambda\nu}\, \delta g_{\sigma\lambda} \tag{A.12}$$

we can show that

$$\delta\, \Gamma^{\rho}_{\nu\mu} = \frac{1}{2}\, g^{\rho\sigma} \left(\nabla_\mu \delta g_{\nu\sigma} + \nabla_\nu \delta g_{\sigma\mu} - \nabla_\sigma \delta g_{\mu\nu} \right) \tag{A.13}$$

and with a few more algebraic calculations, we prove that

$$g^{\mu\nu}\,\delta\,\Gamma^{\rho}_{\nu\mu} - g^{\mu\rho}\,\delta\,\Gamma^{\sigma}_{\mu\sigma} = g_{\mu\nu}\,\nabla^{\rho}\,\delta g^{\mu\nu} - \nabla_{\sigma}\,\delta g^{\sigma\rho} \tag{A.14}$$

Taking the above into account, δS_1 takes the form

$$\delta S_1 = \int d^{n+1}x\,\sqrt{g}\,\nabla_{\rho}\left(g_{\mu\nu}\,\nabla^{\rho}\,\delta g^{\mu\nu} - \nabla_{\sigma}\,\delta g^{\sigma\rho}\right) = \tag{A.15}$$

$$= \int d^{n+1}x\,\partial_{\rho}\left(\sqrt{g}\left(g_{\mu\nu}\,\nabla^{\rho}\,\delta g^{\mu\nu} - \nabla_{\sigma}\,\delta g^{\sigma\rho}\right)\right) = 0 \tag{A.1}$$

We notice that this integral has a total derivative, and according to Stoke's theorem this is equal to the boundary contribution at infinity. But since we take the variation to vanish at infinity, δS_1 vanishes as well.

As for the rest of the terms, equation (A.4) is already in the appropriate form, but as for (A.5) and (A.6), we need to calculate the variation $\delta\sqrt{g}$.

Starting from the well known identity

$$\log g = Tr(\log g_{\mu\nu}) \Rightarrow \frac{\delta g}{g} = g^{\mu\nu}\,\delta g_{\mu\nu} \Rightarrow \delta g = -g\,g_{\mu\nu}\,\delta g^{\mu\nu} \tag{A.16}$$

We conclude that for $\sqrt{g}$ we have

$$\delta\sqrt{g} = \frac{\delta g}{2\sqrt{g}} = -\frac{1}{2}\sqrt{g}\,g_{\mu\nu}\,\delta g^{\mu\nu} \tag{A.17}$$

With that information we can also rewrite δS_3 and δS_4 in the proper form. Eventually, the variation in the action is written as

$$\delta S = \int d^{n+1}x\,\sqrt{g}\,\left(\mathcal{R}_{\mu\nu} - \frac{1}{2}\,\mathcal{R}\,g_{\mu\nu} + \Lambda\,g_{\mu\nu}\right)\delta g^{\mu\nu} \tag{A.18}$$

Since the EOMs are found from the local minima of the action, we demand

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0$$

hence the Einstein field equations are

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{A.19}$$

B Derivation of the equations of motion of a scalar field

In this appendix we will derive the equations of motion for a scalar field assuming a quadratic action of the form

$$S = \frac{1}{2}\int d^{n+1}x\,\sqrt{g}\,\left(\nabla_{\mu}\phi\,\nabla^{\mu}\phi + m^2\phi^2\right) \tag{B.1}$$

By varying with respect to ϕ, we get the following

$$\delta S = \int d^{n+1}x\,\sqrt{g}\,\left(\frac{1}{2}\,g^{\mu\nu}\,\partial_{\mu}(\delta\phi)\,\partial_{\nu}\phi + \frac{1}{2}\,g^{\mu\nu}\,\partial_{\mu}\phi\,\partial_{\nu}(\delta\phi) + m^2\,\phi\,\delta\phi\right) = \tag{B.2}$$

$$= \int d^{n+1}x\,\sqrt{g}\,\left(g^{\mu\nu}\,\partial_{\mu}(\delta\phi)\,\partial_{\nu}\phi + m^2\,\phi\,\delta\phi\right) \tag{B.3}$$

By integrating by parts and assuming that the variation vanishes at the boundary, the variation at the action transforms to

$$\delta S = \int d^{n+1}x \left(-\partial_\mu \left(\sqrt{g}\, g^{\mu\nu}\, \partial_\nu \phi\right) + m^2\phi\right)\, \delta\phi =$$

$$= \int d^{n+1}x\, \sqrt{g}\left(-\nabla_\mu \left(g^{\mu\nu}\, \nabla_\nu \phi\right) + m^2\phi\right) \delta\phi =$$

$$= \int d^{n+1}x\sqrt{g}\left(-\Box + m^2\right) \phi\, \delta\phi \tag{B.4}$$

By once again demanding

$$\frac{\delta S}{\delta \phi} = 0$$

we arrive at the famous Klein-Gordon equation

$$(-\Box + m^2)\phi = 0 \tag{B.5}$$

We can also add an interaction term in the action

$$S^{(3)} = \int d^{n+1}x\, \sqrt{g}\left(\frac{1}{2}\nabla_\mu\phi\, \nabla^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{3}\,\phi^3\right) \tag{B.6}$$

It is straightforward to show that

$$\frac{\delta}{\delta\phi}\left(\frac{\lambda}{3}\,\phi^3\right) = \lambda\,\phi^2$$

Therefore the EOMs take the form

$$(-\Box + m^2)\phi = \lambda\phi^2 \tag{B.7}$$

C Properties of modified Bessel functions

The function $I_a(x)$ is known as modified Bessel function of the first kind and is defined by:

$$I_a(x) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(a+k+1)} \left(\frac{x}{2}\right)^{a+2k} \tag{C.1}$$

The function $K_a(x)$ is known as modified Bessel function of the second kind and is defined by:

$$K_a(x) = \begin{cases} \dfrac{\pi}{2}\dfrac{I_{-a}(x) - I_a(x)}{\sin a\pi} & a \notin \mathbb{Z} \\[2ex] \dfrac{\pi}{2}\lim\limits_{p\to a}\dfrac{I_{-p}(x) - I_p(x)}{\sin p\pi} & a \in \mathbb{Z} \end{cases} \tag{C.2}$$

In greater detail, $K_a(x)$ for $a \in \mathbb{Z}$ can be written as

$$K_a(x) = (-1)^{a+1}\left(\log\frac{x}{2} + \gamma_E\right) I_a(x) + \frac{1}{2}\sum_{k=0}^{a-1}(-1)^k\,(a-k-1)!\left(\frac{x}{2}\right)^{2k-a} \tag{C.3}$$

$$+\frac{(-1)^a}{2}\sum_{k=0}^{\infty}\frac{\Phi(k) + \Phi(a+k)}{k!\,(a+k)!}\,\left(\frac{x}{2}\right)^{a+2k}, \quad a \in \mathbb{Z}$$

where $\gamma_E = \psi(z=1)$ with $\psi(z)$ being the Digamma function. $\Phi(n)$ is called the harmonic sum and is equal to

$$\Phi(n) = \sum_{k=1}^{n} \frac{1}{k} \qquad \Phi(0) = 0 \tag{C.1}$$

The asymptotic behaviors of the Bessel functions at $x \to 0$ are given by

$$I_a(x \to 0) \sim x^a, \quad K_a(x \to 0) \sim x^{-a} \quad \alpha > 0 \tag{C.4}$$

while for $x \to \infty$ we have

$$I_a(x \to \infty) \sim \frac{e^x}{\sqrt{x}}, \quad K_a(x \to \infty) \sim \frac{e^{-x}}{\sqrt{x}} \tag{C.5}$$

It is also useful for our calculations to examine the starting powers of $K_a(x)$ power series. It is clear from C.1 that the starting power of $I_a(x)$ is x^a. Additionally, from C.2 we can also derive the starting powers of $K_a(x)$ that work out to be x^a as well as x^{-a}. Therefore, for $x \to 0$, we are allowed to expand $K_a(x)$ in the following manner:

$$\begin{aligned} K_a(x) &= \frac{\pi}{2\sin(a\pi)} \left(\frac{1}{\Gamma(-a+1)} \left(\frac{x}{2}\right)^{-a} + \dots + \frac{1}{\Gamma(a+1)} \left(\frac{x}{2}\right)^{a} + \dots \right) \\ &= 2^{a-1}\,\Gamma(a)\, x^{-a} \left(1 + \dots + 2^{-2a} \frac{\Gamma(-a)}{\Gamma(a)} x^{2a} + \dots \right) \qquad a \notin \mathbb{N} \end{aligned} \tag{C.6}$$

$$K_a(x) = 2^{a-1}\,(a-1)!\, x^{-a} \left(1 + \dots + \frac{(-1)^{a+1}\, 2^{-a}}{2^{a-1}\, a!} x^{2a} \log x \right) \qquad a \in \mathbb{N} \tag{C.7}$$

where $\dots$ represent the rest of the summation terms.

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