Nonlinear behavior of the extracellular matrix and long-range propagation of cell induced matrix displacements

Maria Proestaki

Supervisor: Phoebus Rosakis

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Department of Mathematics and Applied Mathematics

University of Crete

Heraklion

Greece

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Introduction

Living tissues are not made up solely of cells. A substantial part of their volume is extracellular space, which is largely filled by a network of macromolecules, constituting the extracellular matrix (ECM). The extracellular matrix is composed of two major classes of biomolecules: glycosaminoglycans, and fibrous proteins (fibers) which include collagen, elastin, fibronectin, and laminin. These components are secreted locally and assembled into the organized meshwork that is the extracellular matrix. Cells are connected with these fibers, hence when cells contract and change shape, they deform the fibers, and therefore the extracellular matrix. Thus, cell contraction induce matrix displacements. Cells can detect and respond to substrate strains, created by matrix displacements, which is known as cellular mechanosensing [23] [5]. This response is dependent on matrix stiffness[8] [20] [12] [24] [22]. Therefore, neighboring cells can detech each other through the displacements of extracellular matrix.

Since matrix displacements are important for cellular mechanosensing, finding the decay of these displacements would be essential. If we suppose that the extracellular matrix is composed of linear elastic material, we can use Linear Elasticity to find the decay. According to Linear Elasticity, cell-induced matrix displacements decay with order $O(r^{-1})$. Unfortunately, experiments [17] [15] [16] that have been held using confocal microscopy and digital volume correlation [6], revealed that cell-induced displacements scale as $u(r) \sim r^{-n}$, where n is in the range 0.2-0.5. This finding indicates that cell-induced matrix displacements decay slower that linear elasticity predicts(r^{-1}). We believe that this discord between experimentally observed

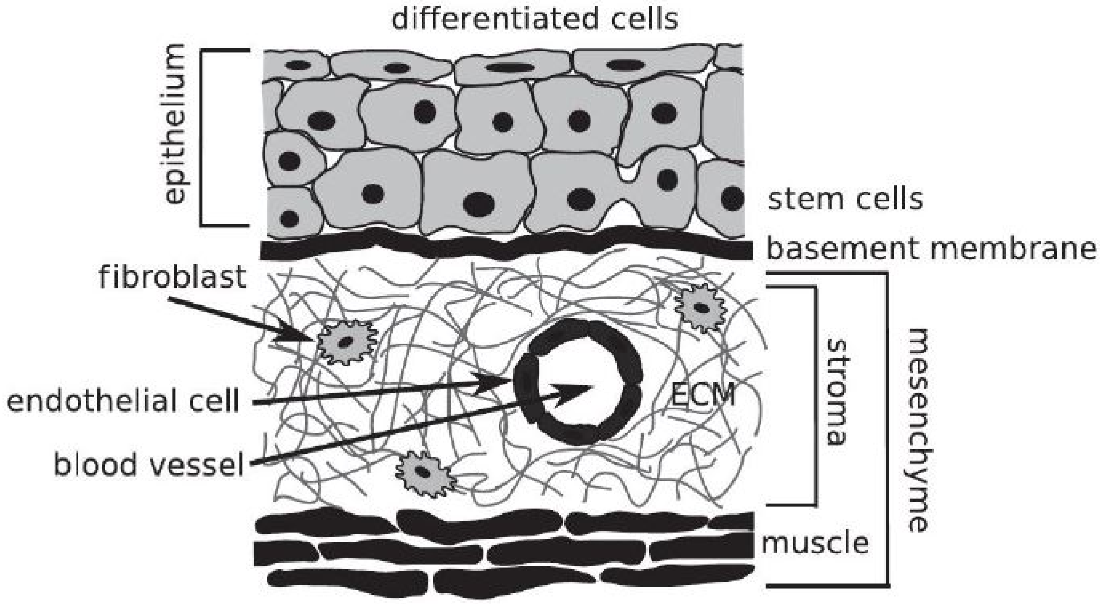


Figure 1: Drawing of the cellular microenvironment. Cells (fibroblasts) are connected with the fibers of ECM. [4]

displacement scaling and the prediction of linear elasticity, is a result of the buckling of fibers under compression [15]. In order to check if the aforementioned findings are closely related, we construct a constitutive model, which describes a material that buckles in compression, and provide much better agreement between theory and experiments [21].

It is well known from Euler that a macroscopic rod of length L and bending rigidity k, undergoes a buckling instability (buckles) if the compressional force F exceeds a certain threshold value which is $\frac{k}{L^2}$, because the rod can no longer support the compressive force and buckles. Buckling means that if one pushes the ends of the rod towards each other, the rod bends (buckles) easily without resisting compression load.

Such buckling instabilities also play a role in biological systems, whenever

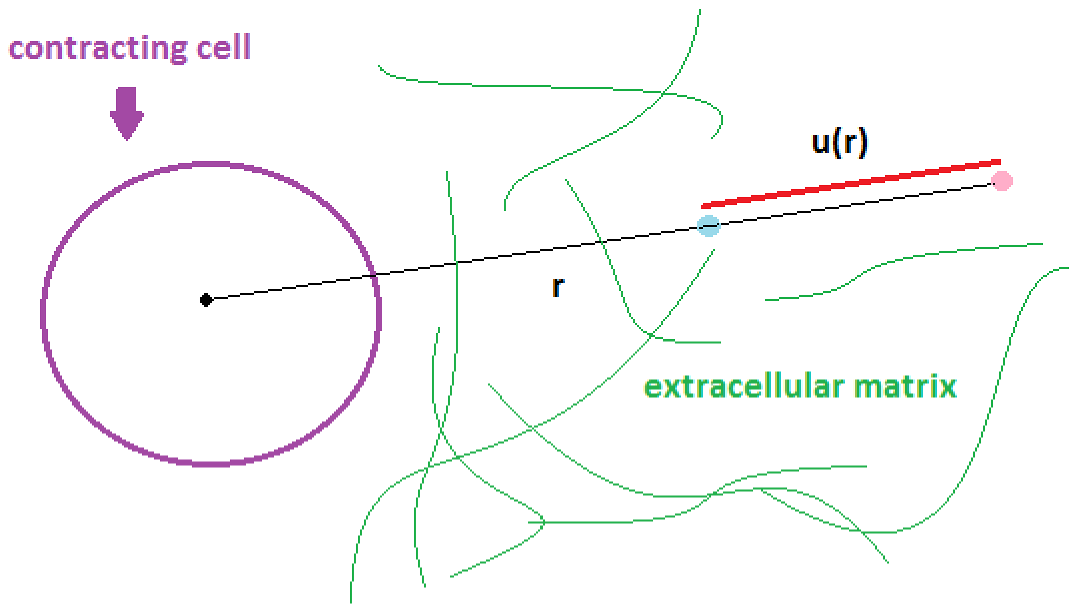


Figure 2: Displacement of a point in the ECM when cell contracts.

semiflexible polymers, such as fibers, are under a compressive load. [11] [2] [10] This compressive load is formed by cell contraction. Direct visual observations and analysis of the 3D structures of fibrin networks at different compressive strains showed buckling of filaments along the direction of compressive load. Each filament has very low resistance to buckling. Compressive loads can be generated by the polymerization of filaments or by molecular motors. Both processes can generate forces in the piconewton range. Upon buckling, filaments become more compliant, and as a result, the elasticity of the network gradually decreases with compression. [2]. Thus, buckling of filaments can change the mechanical response of the network, causing a loss of stiffness under compression.

In what way cells exploit this long-range propagation range of matrix

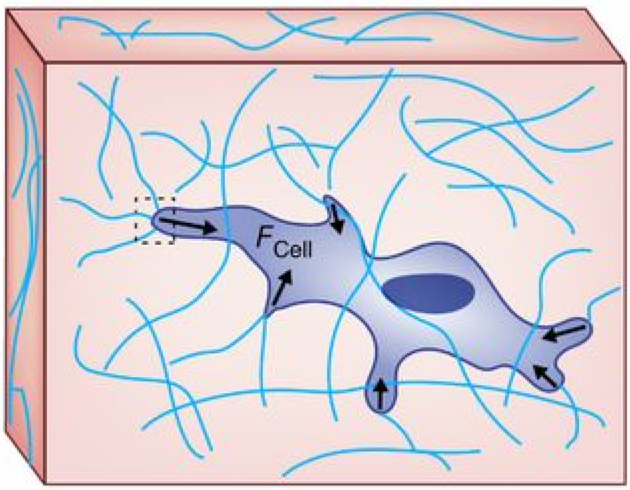


Figure 3: Traction forces applied by cells induce deformation to the 3D cell substrate and are balanced by reaction stresses within the substrate (not shown for clarity) [18]

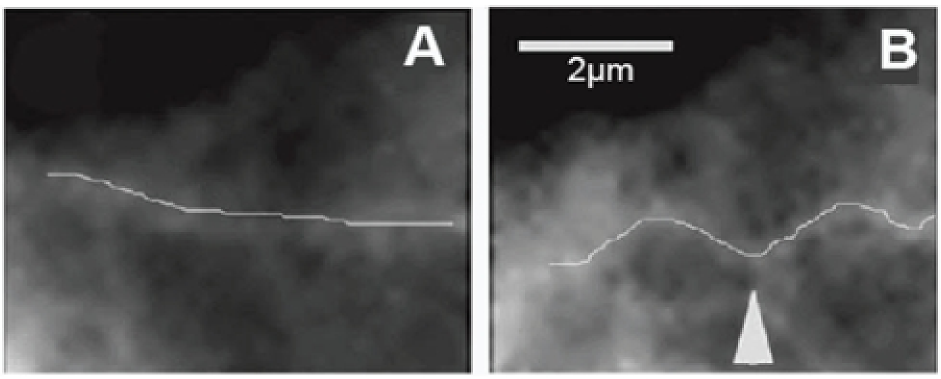


Figure 4: Buckling of a fiber due to cell contraction.(A). The fiber is fairly straight before the application of the load, and (B). it buckles under compressional load. The scale bar is 2mm. [14]

displacements to detect each other? Experiments [15] have shown that cells whose distance from each other is of the order of 10 cell diameters, form bonds (tethers) consisting of aligned and densely packed matrix fibres, which extend far beyond the cell's protrusion. It is believed that tethers are the mechanism cells use to detect or even approach their neighbors.

2 Constitutive Model

2.1 Constitutive Law

To begin with, we consider small deformations. Mathematically [7] [1], a body is deformed via a mapping **f** that carries each material point **x** into a point

$$\mathbf{y}=\mathbf{f}(\mathbf{x}).$$

Now, the vector

$$\mathbf{u}(\mathbf{x})=\mathbf{f}(\mathbf{x})\text{-}\mathbf{x} \tag{2.1.1}$$

represents the **displacement** of x. (Fig.2).

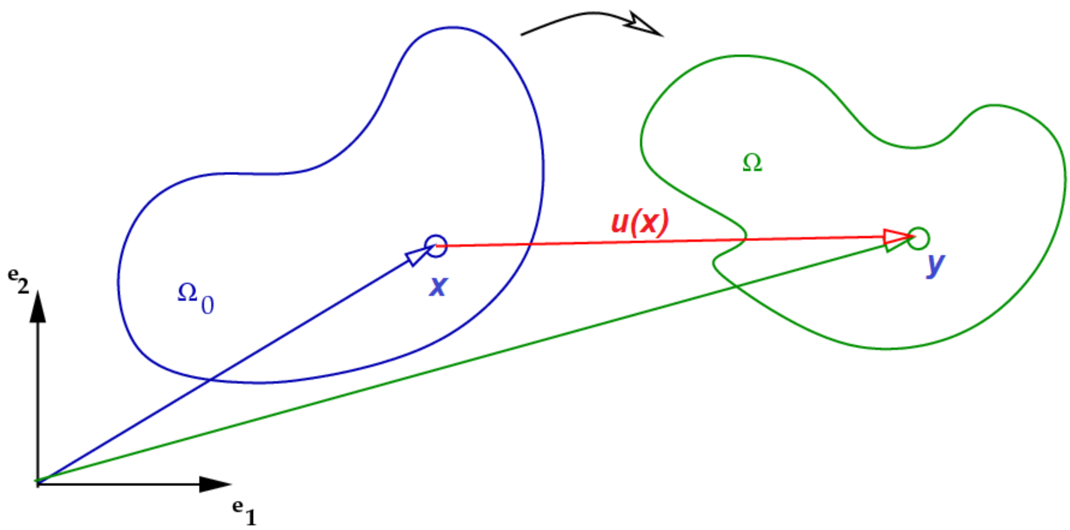


Figure 5: Displacement of x.

If we apply the gradient with respect to **x** to the above Equation (**??**), we get that

$$\nabla f(x) = \nabla x + \nabla u(x) \tag{2.1.2}$$

which is equivalent to

$$F = I + \nabla u \tag{2.1.3}$$

Small deformations are considered when the displacement gradient $\mid \nabla u \mid$ is small. If $\mid \nabla u \mid$ is small, then $F \simeq I$. The deformation gradient tensor F characterizes all geometric changes, i.e. deformations of lengths, rotations, angles, etc. Thus, all the deformations of lengths, rotations, angles, are small.

We model the cell plus the matrix as a system that occupies the whole 2D space $\mathbb{R}^2$. We firstly consider that the matrix is composed of linear elastic homogeneous isotropic material, in order to use Linear Elasticity. Therefore, the displacement field is $\boldsymbol{u}$: $\mathbb{R}^2 \to \mathbb{R}^2$.

The infinitensimal strain tensor is

$$\boldsymbol{E} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T), \tag{2.1.4}$$

and its components are

$$E_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.1.5}$$

The stress - strain relation for linear elastic isotropic material, which models the matrix, for infinitensimal deformations is

$$\boldsymbol{S} = \lambda(tr\boldsymbol{E})\mathbf{1} + 2\mu\boldsymbol{E} \tag{2.1.6}$$

where λ and μ are the *Lame* constants and $\mathbf{1}$ is the identity tensor.

The stress - strain relation for Linear Elasticity in components is written as follows

$$S_{\mathrm{ij}} = \lambda E_{\mathrm{kk}}\delta_{\mathrm{ij}} + 2\mu E_{\mathrm{ij}} \tag{2.1.7}$$

The Einstein summation convention is used.

Symmetric E and S have 2 eigenvalues ϵ_1, ϵ_2 and σ_1, σ_2 and two corresponding eigenvectors υ_1, υ_2, which are called *principal directions* of E and S, respectively. The particular basis consisting of the eigenvectors is called *principal basis for E or S*. Thus, the component matrix of tensor E in its principal basis is

$$E = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}$$

and the component matrix of tensor S in its principal basis is

$$S = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

So, we can write Eq.(2.1.3) as

$$S = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \lambda(\epsilon_1 + \epsilon_2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\mu \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \tag{2.1.8}$$

It is apparent that

$$\sigma_1 = (\lambda + 2\mu)\epsilon_1 + \lambda\epsilon_2, \tag{2.1.9}$$

$$\sigma_2 = (\lambda + 2\mu)\epsilon_2 + \lambda\epsilon_1 \tag{2.1.10}$$

Therefore, we can summarize that

$$\sigma_i = C_{ij}\epsilon_j, \tag{2.1.11}$$

where

$$C = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix},$$

$\alpha = \lambda + 2\mu$, $\beta = \lambda$

In order to check the positive definiteness of C, we calculate the eigenvalues λ and we find that

$$\lambda_1 = \alpha + \beta, \tag{2.1.12}$$

$$\lambda_2 = \alpha - \beta \tag{2.1.13}$$

We want all the eigenvalues to be positive, so for $\alpha > |\beta|$ the C is positive definite.

According to the *Spectral Theorem*, one can express the strain tensor E using tensor's eigenvalues and eigenvectors as follows:

$$E = \epsilon_1(\upsilon_1 \otimes \upsilon_1) + \epsilon_2(\upsilon_2 \otimes \upsilon_2) \tag{2.1.14}$$

Thus, 2.1.6 can be written as

$$\boldsymbol{S} = \lambda(\epsilon_1 + \epsilon_2)\mathbf{1} + 2\mu[\epsilon_1(\upsilon_1 \otimes \upsilon_1) + \epsilon_2(\upsilon_2 \otimes \upsilon_2)] \tag{2.1.15}$$

Remember that the eigenvectors of the stress tensor S are ν_i and the eigenvectors of the strain tensor E are υ_i, $i = 1, 2$. Let's apply inner product between 2.1.6 and the eigenvector υ_1 of the strain tensor E. We conclude that

$$\boldsymbol{S}\upsilon_1 = \lambda(tr\boldsymbol{E})\upsilon_1 + 2\mu\boldsymbol{E}\upsilon_1, \tag{2.1.16}$$

But we know that $\boldsymbol{E}\upsilon_1 = \epsilon_1\upsilon_1$, so we can write 2.1.16 as

$$\boldsymbol{S}\upsilon_1 = [\lambda(trE) + 2\mu\epsilon_1]\upsilon_1 \tag{2.1.17}$$

We know that

$$\sigma_1 = \lambda(trE) + 2\mu\epsilon_1 \tag{2.1.18}$$

Thus, υ_1 is common eigenvector of tensor E and tensor S.

$$\boldsymbol{S}\upsilon_1 = \sigma_1 \upsilon_1 \tag{2.1.19}$$

Doing the same procedure as before, using the eigenvector υ_2 of the strain tensor E this time, we find that

$$\boldsymbol{S}\upsilon_2 = \sigma_2 \upsilon_2 \tag{2.1.20}$$

These findings show that eigenvectors υ_i of the strain tensor E, are common for both E and stress tensor S.

Recalling that we would like to construct a constitutive law for a material (fibrin) that loses stiffness in compression [21] due to microbuckling, we construct a piecewise linear function which expresses the nonlinearity as shown in 2.1 Specifically,

$$Z_\rho(x) = \begin{cases} x, & x \geq 0 \\ \rho x, & x < 0, \end{cases} \tag{2.1.21}$$

where

$$0 \leq \rho \leq 1$$

is the constant *compression stiffness ratio*. The graph of Z_ρ is the curve 2.1.

The *stress-strain relation* for Linear Elasticity in 1D has the following form

$$\sigma = \alpha\epsilon$$

,where α is am elastic modulus. We choose for our model the Stress-strain relation to be:

$$\sigma = Z_\rho(\alpha\epsilon)$$

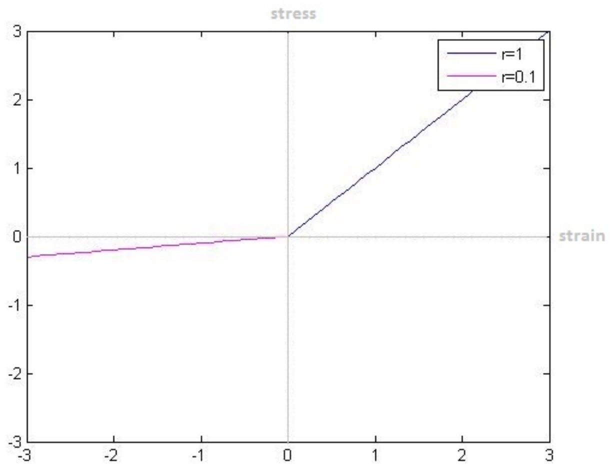


Figure 6: Stress-strain curve in one dimension for a material that loses stiffness in compression. The 1D stress-strain relation is $\sigma = Z_\rho(\alpha\epsilon)$, where σ is the principal stress, ϵ is the principal strain and α is an elastic constant. Here we choose $\rho = 0.1$. Horizontal axis: strain ϵ in percent. Vertical axis: $\frac{\sigma}{\alpha}$.

using our piecewise function Z_ρ.

Thus,

$$\sigma = \alpha Z_\rho(\epsilon) = \begin{cases} \alpha\epsilon, & \epsilon \geq 0 \\ \rho\alpha\epsilon, & \epsilon < 0 \end{cases} \tag{2.1.22}$$

Principal stress is a piecewise linear function of the principal strain in a way that expresses the non linear behavior of 2.1.

Now, we want to find a Strain Energy function $W(\epsilon)$, such that $\sigma = \partial W/\partial\epsilon$. Strain Energy function $W(\epsilon)$ expresses the energy stored (internal energy) in the structure which is equal to the work done by external load. The latter is expressed by Complementary Energy function $U(\sigma)$ and has the property that $\frac{\partial U(\sigma)}{\partial \sigma} = \epsilon$. In the case of linear elasticity, the strain energy and

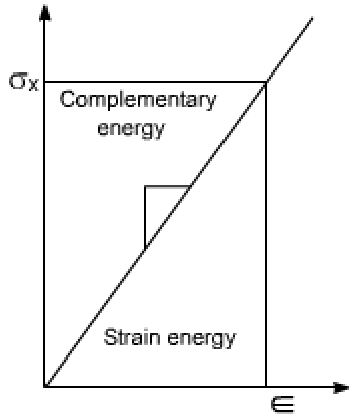


Figure 7: *The stress strain diagram. The area enclosed by the inclined line and the vertical axis is called the complementary strain energy. For a linearly elastic material the complementary strain energy and elastic strain energy are the same.*

its complementary counterpart are equal.
Strain energy density and its complementary are given by

$$W(\epsilon) = \int_0^{\epsilon} \sigma(\eta) d\eta, \tag{2.1.23}$$

$$U(\sigma) = \int_0^{\sigma} \hat{\epsilon}(\eta) d\eta. \tag{2.1.24}$$

We choose our 1D stress - strain relation to be $\sigma = \alpha Z_\rho(\epsilon)$, where α is a modulus. We continue calculations to find W:

$$W(\epsilon) = \int_0^{\epsilon} \alpha Z_\rho(\eta) d\eta \tag{2.1.25}$$

and using 2.1.22 we conclude that

$$W(\epsilon) = \begin{cases} \frac{1}{2}\alpha\epsilon^2, & \epsilon \geq 0 \\ \\ \frac{1}{2}\alpha(\sqrt{\rho}\epsilon)^2, & \epsilon < 0, \end{cases} \tag{2.1.26}$$

Also, one can see that

$$Z_{\sqrt{\rho}}(\epsilon) = \begin{cases} \epsilon, & \epsilon \geq 0 \\ \\ \sqrt{\rho}\epsilon, & \epsilon < 0, \end{cases} \tag{2.1.27}$$

while

$$Z^2_{\sqrt{\rho}}(\epsilon) = \begin{cases} \epsilon^2, & \epsilon \geq 0 \\ \\ (\sqrt{\rho}\epsilon)^2, & \epsilon < 0, \end{cases} \tag{2.1.28}$$

Therefore, Strain Energy function for a compression-weakening material in 1D takes the following form

$$W(\epsilon) = \frac{1}{2} Z^2_{\sqrt{\rho}}(\epsilon). \tag{2.1.29}$$

$U(\sigma)$ is the Legendre Transform of $W(\epsilon)$:

$$U(\sigma) = \sigma\epsilon - W(\epsilon), \tag{2.1.30}$$

where $\epsilon = \hat{\epsilon}(\sigma)$ is the strain-stress relation. We continue calculations to find U:

From the 1D stress - strain relation we have that

$$\sigma = \begin{cases} \alpha\epsilon, & \epsilon \geq 0 \\ \\ \alpha\rho\epsilon, & \epsilon < 0 \end{cases} \tag{2.1.31}$$

So,

$$\epsilon = \begin{cases} \alpha^{-1}\sigma, & \sigma \geq 0 \\ \\ \alpha^{-1}\rho^{-1}\sigma, & \sigma < 0 \end{cases} \tag{2.1.32}$$

Now this can be re-written as

$$\epsilon = \alpha^{-1} Z_{\rho^{-1}}(\sigma) \tag{2.1.33}$$

and is called the *Strain - Stress Relation.*

Now using 2.1.32 and that $U'(\sigma) = \epsilon$, we have that

$$U(\sigma) = \int_0^{\sigma} \alpha^{-1} Z_{\rho^{-1}}(\eta) d\eta, \tag{2.1.34}$$

which is equivalent to

$$= \begin{cases} \frac{1}{2}\alpha^{-1}\sigma^2, & \sigma \geq 0 \\ \\ \frac{1}{2}\alpha^{-1}(\frac{1}{\sqrt{\rho}}\sigma)^2, & \sigma < 0 \end{cases} \tag{2.1.35}$$

One can note that

$$Z^2_{\frac{1}{\sqrt{\rho}}}(\sigma) = \begin{cases} \sigma^2, & \sigma \geq 0 \\ \\ \frac{1}{\rho}\sigma^2, \end{cases} \tag{2.1.36}$$

Thus, Complementary Strain Energy function for a compression-weakening material in 1D has the following form

$$U(\sigma) = \frac{1}{2}\alpha^{-1} Z^2_{\frac{1}{\sqrt{\rho}}}(\sigma) \tag{2.1.37}$$

Let's examine 2D now. Strain Energy function $W(\epsilon_1, \epsilon_2)$ has the property that

$$\sigma_i = \frac{\partial W}{\partial \epsilon_i} \tag{2.1.38}$$

Also, Complementary Energy Density $U(\sigma_1, \sigma_2)$ has the property that

$$\frac{\partial U(\sigma_1, \sigma_2)}{\partial \sigma_i} = \epsilon_i \tag{2.1.39}$$

Suppose a curve C that begins from point $A(0,0)$ and ends at point $B(\epsilon_1, \epsilon_2)$.

$$\begin{aligned}
&\int_{C[A,B]} \sigma_1(\eta_1, \eta_2) d\eta_1 + \sigma_2(\eta_1, \eta_2) d\eta_2 \\
&= \int_{C[A,B]} \frac{\partial W(\eta_1, \eta_2)}{\partial \eta_1} d\eta_1 + \frac{\partial W(\eta_1, \eta_2)}{\partial \eta_2} d\eta_2 \\
&= \int_{C[A,B]} \nabla W \cdot (d\eta_1, d\eta_2) \\
&= \int_{C[A,B]} W(B) - W(A) \\
&= W(\epsilon_1, \epsilon_2)
\end{aligned}$$

where $\nabla W = \left(\frac{\partial W(\eta_1,\eta_2)}{\partial \eta_1}, \frac{\partial W(\eta_1,\eta_2)}{\partial \eta_2} \right)$. Thus, the Strain Energy function is written as

$$W(\epsilon_1, \epsilon_2) = \int_{C[A,B]} \sigma_1(\eta_1, \eta_2) d\eta_1 + \sigma_2(\eta_1, \eta_2) d\eta_2 \tag{2.1.40}$$

Also,

$$\begin{aligned}
&\int_{C[A,B]} \epsilon_1(\eta_1, \eta_2) d\eta_1 + \epsilon_2(\eta_1, \eta_2) d\eta_2 \\
&= \int_{C[A,B]} \frac{\partial U(\eta_1, \eta_2)}{\partial \eta_1} d\eta_1 + \frac{\partial U(\eta_1, \eta_2)}{\partial \eta_2} d\eta_2 \\
&= \int_{C[A,B]} \nabla U \cdot (d\eta_1, d\eta_2) \\
&= \int_{C[A,B]} U(B) - U(A) \\
&= U(\epsilon_1, \epsilon_2)
\end{aligned}$$

where $\nabla U = \left(\frac{\partial U(\eta_1,\eta_2)}{\partial \eta_1}, \frac{\partial U(\eta_1,\eta_2)}{\partial \eta_2} \right)$. Therefore, the Complementary Energy function is written as

$$U(\sigma_1, \sigma_2) = \int_{C[A,B]} \epsilon_1(\eta_1, \eta_2) d\eta_1 + \epsilon_2(\eta_1, \eta_2) d\eta_2 \tag{2.1.41}$$

The stress - strain relations are invertible to the form $\epsilon_i = \hat{\epsilon}(\sigma_1, \sigma_2)$ and again U is the Legendre transform of W:

$$U(\sigma_1, \sigma_2) = \sigma_i \epsilon_i - W(\epsilon_1, \epsilon_2) \tag{2.1.42}$$

The linear elastic Stress - Strain relation in 2D is

$$\sigma_i = C_{ij}\epsilon_j, i, j = 1, 2. \tag{2.1.43}$$

Multiplying the above equation by C^{-1} results in

$$C_{ki}^{-1}\sigma_i = C_{ki}^{-1}C_{ij}\epsilon_j \tag{2.1.44}$$

which is equivalent to

$$C_{ki}^{-1}\sigma_i = \delta_{kj}\epsilon_j \quad \Longrightarrow \quad C_{ki}^{-1}\sigma_i = \epsilon_k \tag{2.1.45}$$

where δ is the *Kronecker delta*.

And if we set k=i and i=j, we have that:

$$\epsilon_i = C_{ij}^{-1}\sigma_j \tag{2.1.46}$$

Therefore, if we substitute 2.1.46 into 2.1.42, we conclude that

$$U(\sigma_1, \sigma_2) = C_{ij}^{-1}\sigma_i\sigma_j - W(\epsilon_1, \epsilon_2) \tag{2.1.47}$$

In the case of *Linear Elasticity*, U has the following form:

$$U(\sigma_1, \sigma_2) = \frac{1}{2}C_{ij}^{-1}\sigma_i\sigma_j, \tag{2.1.48}$$

Let's now use our piecewise function 2.1.21 in our findings and replace σ_i by $Z_{\frac{1}{\sqrt{\rho}}}(\sigma_i)$. Then, the *Complementary Energy* will be

$$U(\sigma_1, \sigma_2) = \frac{1}{2}C_{ij}^{-1}Z_{\frac{1}{\sqrt{\rho}}}(\sigma_i)Z_{\frac{1}{\sqrt{\rho}}}(\sigma_j) \tag{2.1.49}$$

which can be written as

$$U(\sigma_1, \sigma_2) = \frac{1}{2} C_{11}^{-1} Z_{\frac{1}{\sqrt{\rho}}}^2(\sigma_1) + \frac{1}{2} C_{22}^{-1} Z_{\frac{1}{\sqrt{\rho}}}^2(\sigma_2) + C_{12}^{-1} Z_{\frac{1}{\sqrt{\rho}}}(\sigma_1) Z_{\frac{1}{\sqrt{\rho}}}(\sigma_2) \quad (2.1.50)$$

But there is a problem here. Despite the fact that $Z_{\frac{1}{\sqrt{\rho}}}^2(\sigma)$ is continuously differentiable in σ, $Z_{\frac{1}{\sqrt{\rho}}}(\sigma)$ is not. Thus, the third term of 2.1.50 turns out to be problematic, because it is not continuously differentiable and we should replace it. The term $Z_{\frac{1}{\sqrt{\rho}}}(\sigma_1) Z_{\frac{1}{\sqrt{\rho}}}(\sigma_2)$ can take 3 different values, which are

$$Z_{\frac{1}{\sqrt{\rho}}}(\sigma_1) Z_{\frac{1}{\sqrt{\rho}}}(\sigma_2) = \begin{cases} \sigma_1 \sigma_2, & \sigma_1 \geq 0, \sigma_2 \geq 0 \\ \rho \sigma_1 \sigma_2, & \sigma_1 \geq 0, \sigma_2 < 0 \quad or \quad \sigma_1 < 0, \sigma_2 \geq 0 \\ \rho^2 \sigma_1 \sigma_2, & \sigma_1 < 0, \sigma_2 < 0 \end{cases} \quad (2.1.51)$$

We choose the simplest coupling $\sigma_1 \sigma_2$, so the *Complementary Energy Density* for our constitutive model is:

$$U_\rho(\sigma_1, \sigma_2) = \frac{1}{2} M_{11} Z_{\frac{1}{\sqrt{\rho}}}^2(\sigma_1) + \frac{1}{2} M_{22} Z_{\frac{1}{\sqrt{\rho}}}^2(\sigma_2) + M_{12} \sigma_1 \sigma_2 \quad (2.1.52)$$

where M components depend on σ_i signs. U is piecewise quadratic, continuously differentiable and its partial derivative with respect to σ_i, ($\epsilon_i = \frac{\partial U}{\sigma_i}$) depends on σ_i in a piecewise linear fashion, with a change of slope when σ_i changes sign.

$$M = \frac{1}{\alpha^2 - \beta^2} \begin{pmatrix} \alpha & -\beta \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

When σ_i change sign, the matrix M has different components. There are 4 different combinations of different signs of σ_i, so M can take 4 different values.

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad \begin{pmatrix} \frac{M_{11}}{\rho} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad \begin{pmatrix} \frac{M_{11}}{\rho} & M_{12} \\ M_{21} & \frac{M_{22}}{\rho} \end{pmatrix}, \quad \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & \frac{M_{22}}{\rho} \end{pmatrix} \tag{2.1.53}$$

Using 2.1.52, 2.1.39 and noting that $M_{11} = M_{22}$, we get the Strain - stress relations:

$$\epsilon_i = M_{11} Z_{\frac{1}{\sqrt{\rho}}}(\sigma_i) + M_{12}\sigma_j \tag{2.1.54}$$

In a similar way, we find the *Strain Energy Density W* for a compression-weakening material:

$$W_\rho(\epsilon_1, \epsilon_2) = M_{ij}^{-1}\epsilon_i\epsilon_j - \frac{1}{2}M_{ij}^{-1}\epsilon_i\epsilon_j = \frac{1}{2}M_{ij}^{-1}\epsilon_i\epsilon_j. \tag{2.1.55}$$

To give an example, let's find the Stress - strain relation for a compression-weakening material when $\sigma_1 > 0$ and $\sigma_2 < 0$. For this case:

$$U(\sigma_1, \sigma_2) = \frac{1}{2}M_{11}\sigma_1^2 + \frac{1}{2}M_{22}\frac{\sigma_2^2}{\rho} + \frac{1}{2}M_{12}\sigma_1\sigma_2 \quad and \tag{2.1.56}$$

So, M matrix will have the following form:

$$\begin{pmatrix} \alpha & \beta \\ \beta & \frac{\alpha}{\rho} \end{pmatrix}$$

The corresponding Strain Energy function is

$$W(\epsilon_1, \epsilon_2) = \frac{1}{2}M_{ij}^{-1}\epsilon_i\epsilon_j \tag{2.1.57}$$

where M^{-1} has the following form

$$M^{-1} = \frac{(\alpha^2 - \beta^2)\rho}{\alpha^2 - \beta^2\rho}\begin{pmatrix} \frac{\alpha}{\rho} & \beta \\ \beta & \alpha \end{pmatrix}$$

Also, we know that

$$\frac{\partial W}{\partial \epsilon_i} = \sigma_i$$

So, if we differentiate the first equation twice, firstly with respect to ϵ_1 and after with respect to ϵ_2, we get the Stress-Strain relation for a compression-weakening material:

$$\sigma_1 = \frac{(\alpha^2 - \beta^2)\rho}{\alpha^2 - \beta^2\rho}\left(\frac{\alpha\epsilon_1}{\rho} + \beta\epsilon_2\right), \quad \sigma_2 = \frac{(\alpha^2 - \beta^2)\rho}{\alpha^2 - \beta^2\rho}\left(\beta\epsilon_1 + \alpha\epsilon_2\right) \tag{2.1.58}$$

Furthermore, using our Constitutive Law, we can form the Tensor Stress-Strain Relations. Specifically, we know that

$$\hat{\sigma_i}(\epsilon_1, \epsilon_2) = \frac{\partial W_\rho(\epsilon_1, \epsilon_2)}{\partial \epsilon_i} \tag{2.1.59}$$

Also, using *Spectral Representation* the Strain Tensor E can be written as

$$E = \sum_{i=1}^{2} \epsilon_i v_i \otimes v_i. \tag{2.1.60}$$

If $\hat{\sigma_i}(\epsilon_1, \epsilon_2)$ are the eigenvalues of tensor $\hat{S}(E)$, then $\hat{S}(E)$ can be written as

$$\hat{S}(E) = \sum_{i=1}^{2} \hat{\sigma_i}(\epsilon_1, \epsilon_2) v_i \otimes v_i, \tag{2.1.61}$$

with the same eigenvectors as Stress tensor. Thus the Tensor Stress-Strain relation is

$$S = \hat{S}(E) \tag{2.1.62}$$

Also,

$$\hat{\epsilon_i}(\sigma_1, \sigma_2) = \frac{\partial U_\rho(\sigma_1, \sigma_2)}{\partial \sigma_i} \tag{2.1.63}$$

using *Spectral Representation* the Stress Tensor S can be written as

$$S = \sum_{i=1}^{2} \sigma_i v_i \otimes v_i, \tag{2.1.64}$$

So, from 2.1.63 and 2.1.64 we conclude that

$$\hat{E}(S) = \sum_{i=1}^{2} \hat{\epsilon}_i(\sigma_1, \sigma_2) v_i \otimes v_i, \tag{2.1.65}$$

so the Tensor Strain Stress relation is

$$E = \hat{E}(S). \tag{2.1.66}$$

2.2 Mechanical Behavior

Now we will examine the mechanical behavior of our model. Specifically, an isotropic linear elastic material is described in general by the following set of equations in 2D using linearized theory of elasticity (Hooke's Law):

$$\epsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) \tag{2.2.1}$$

$$\epsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) \tag{2.2.2}$$

$$\epsilon_{12} = \frac{\sigma_{12}}{2G} \tag{2.2.3}$$

where E is the Young's Modulus, ν is the Poisson's Ratio and $G = \frac{E}{2(1+\nu)}$ is the shear modulus. Specifically, *Young's modulus* is a measure of stiffness of an elastic material and is defined as the ratio of stress (force per unit area) along an axis to strain (ratio of deformation over initial length) along that axis. Also, *Poisson's ratio* is the negative ratio of transverse to axial strain (fraction of expansion divided by the fraction of compression). When a material is compressed in one direction, it usually tends to expand in the other direction perpendicular to the direction of compression (2D). This phenomenon is called the Poisson effect. Poisson's ratio ν is a measure of this effect. *Shear modulus* is is defined as the ratio of shear stress to the shear strain and describes the material's response to shear stress.

In case of Uniaxial Stress, where $\sigma_2 = 0$, σ_{12} the general Hooke's Law equations specialize to:

$$\sigma_1 = E\epsilon_1, \tag{2.2.4}$$

$$\epsilon_2 = -\frac{\nu}{E}\sigma_1 = -\nu\epsilon_1 \tag{2.2.5}$$

Using our piecewise function Z_ρ 2.1.21, one could write 2.2.4 as:

$$\sigma_1 = Z_\rho(E\epsilon_1) = \begin{cases} E\epsilon_1, & E\epsilon_1 \geq 0 \\ \rho E\epsilon_1, & E\epsilon_1 < 0, \end{cases} \tag{2.2.6}$$

where $0 \leq \rho < 1$. When $\epsilon_1 < 0$, i.e compression, the Young's Modulus has lower value than in tension. So, there is a loss of stiffness in compression. Similarly, 2.2.5 can be written as:

$$\epsilon_2 = Z_\rho(-\nu\epsilon_1) = \begin{cases} -\nu\epsilon_1, & -\nu\epsilon_1 \geq 0 \\ -\rho\nu\epsilon_1, & -\nu\epsilon_1 < 0, \end{cases} \tag{2.2.7}$$

Again, when in compression $\epsilon_1 < 0$, the Poisson's ratio has lower value than in tension. So, again the loss of stiffness in compression is confirmed.

Let's see what's happening in 2D simple shear. For a homogeneous deformation the displacements in simple shear are [13]

$$u_1(x) = \gamma x_2, \quad u_2(x) = 0 \tag{2.2.8}$$

where γ is the amount of shear. The Strain Tensor E is defined as

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.2.9}$$

so in this case of simple shear we have that

$$E_{11} = 0, \quad E_{12} = \frac{\gamma}{2}, \quad E_{21} = \frac{\gamma}{2}, \quad E_{22} = 0 \tag{2.2.10}$$

which form the following matrix

$$E = \begin{pmatrix} 0 & \frac{\gamma}{2} \\ \frac{\gamma}{2} & 0 \end{pmatrix}.$$

The eigenvalues of the matrix above are:

$$\epsilon_1 = \frac{\gamma}{2}, \quad \epsilon_2 = -\frac{\gamma}{2} \tag{2.2.11}$$

Thus, principal strains always have different signs. Specifically, when $\gamma > 0$,the principal strains lie in the fourth sector of the principal strain plane, while for $\gamma < 0$ the principal strains lie in the second sector. For these 2 cases, different values of the M matrix are used for calculating the energy function. For $\sigma_1 > 0$ and $\sigma_2 < 0$, the Stress-Strain relation is: 2.2.5

$$\sigma_1 = \frac{(\alpha^2 - \beta^2)\rho}{\alpha^2 - \beta^2\rho}\left(\frac{\alpha\epsilon_1}{\rho} + \beta\epsilon_2\right), \quad \sigma_2 = \frac{(\alpha^2 - \beta^2)\rho}{\alpha^2 - \beta^2\rho}\left(\beta\epsilon_1 + \alpha\epsilon_2\right) \tag{2.2.12}$$

One can calculate the components of Stress Tensor S, in the same basis as Strain Tensor E. The eigenvectors of E are

$$[1 \quad 1], [1 \quad -1]$$

Thus, Stress Tensor S will be formulated from the following product:

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \tag{2.2.13}$$

After calculations, Stress Tensor Matrix takes the following form

$$S = \begin{pmatrix} \frac{\alpha(\alpha^2-\beta^2)(1-\rho)\gamma}{4(\alpha^2-\beta^2\rho)} & \frac{(\alpha^2-\beta^2)[\alpha-\beta\rho+(\alpha-\beta)\rho]\gamma}{4\alpha^2-\beta^2\rho)} \\ \frac{(\alpha^2-\beta^2)[\alpha-\beta\rho+(\alpha-\beta)\rho]\gamma}{4\alpha^2-\beta^2\rho)} & \frac{\alpha(\alpha^2-\beta^2)(1-\rho)\gamma}{4(\alpha^2-\beta^2\rho)} \end{pmatrix} \tag{2.2.14}$$

where S_{11}=S_{22} are the normal stresses and S_{12}=S_{21} are the shear stresses. If G is the Shear modulus and γ the amount of shear:

$$G = \frac{S_{12}}{\gamma} \tag{2.2.15}$$

Thus,

$$\frac{(\alpha^2 - \beta^2)[\alpha - \beta\rho + (\alpha - \beta)\rho]\gamma}{4\alpha^2 - \beta^2\rho)} = G\gamma \tag{2.2.16}$$

which is equivalent to

$$G = \frac{(\alpha^2 - \beta^2)[\alpha - \beta\rho + (\alpha - \beta)\rho]}{4\alpha^2 - \beta^2\rho)} \tag{2.2.17}$$

Also, if N is the Normal stress modulus, then

$$N = \frac{S_{11}}{\gamma} \tag{2.2.18}$$

Thus,

$$N = \frac{\alpha(\alpha^2 - \beta^2)(1 - \rho)}{4(\alpha^2 - \beta^2\rho)} \tag{2.2.19}$$

In *Linear Elasticity Theory* the normal stresses S_{11}, S_{22} in simple shear are zero. In our case, when $\rho = 1$ the normal stresses vanish and they are in agreement with the Linear Elasticity. But in the case of $\rho < 1$ normal stresses are not negligible, so the Poynting Effect, which is defined next, occurs.

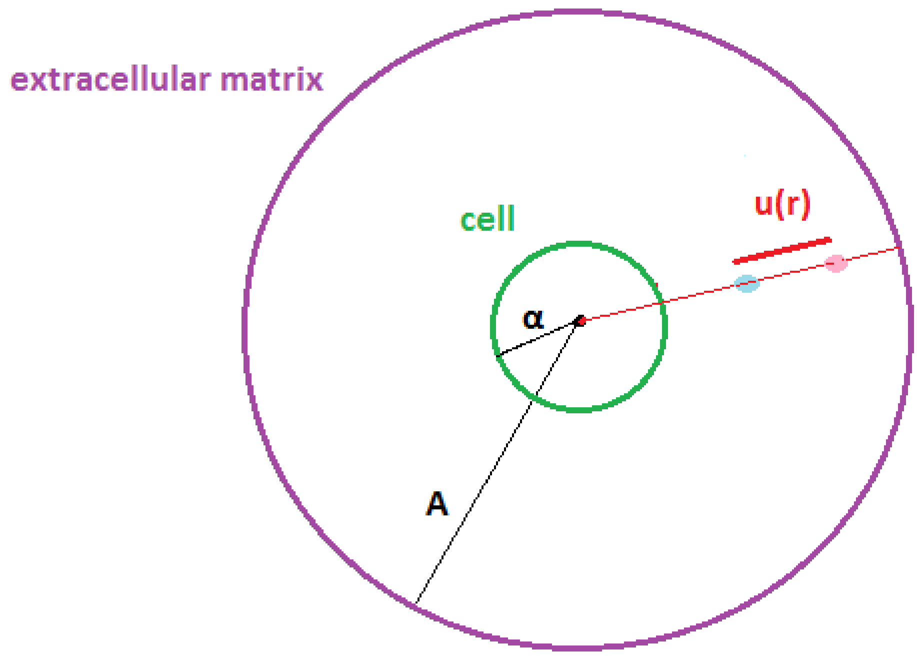
The positive Poynting effect occurs when the sheared faces tend to spread apart, and hence a compressive stress $S_{11} < 0$ is necessary to counteract this tendency and maintain the deformation. A good example is the wet sand. The tendency of wet sand (a compacted granular material) is to dilate (expand in volume) as it is sheared. This occurs because the grains in a compacted state are interlocking and therefore do not have the freedom to move around one another. When stressed, a lever motion occurs between adjacent grains, which produces a bulk expansion of the material. That's why when a person walks on the beach, the wet sand apperas to dry up around his foot. The deformation caused by the foot expands the sand under it and the water in the sand moves to fill the new space between the grains. On the other hand, the negative Poynting effect (or reverse Poynting effect) is obtained when the sheared faces tend to draw together.(move closer together) Thus, a tensile stress $S_{11} > 0$ is required to counteract this tendency and maintain the deformation.

Here for $\rho < 1$ the normal stresses are positive, so our case belongs to the second category, which is the Negative Poynting Effect. There have been conducted many experiments on semiflexible biopolymer gels, i.e. networks that make up the cytoskeleton of cells and the extracellular matrix, whereby the reverse of the usual (positive) Poynting effect was observed. [3]

3 The Contracting Cell Problem in 2D

3.1 Geometric Model of Cell

In our two-dimensional problem, a cell takes up the shape of a disk with radius α. We model an annulus that surrounds the cell with its bigger radius A, which represents the matrix. Also, $\mathbf{x}$ is the position vector and $r = |\mathbf{x}|$ is the radial distance from the center of the cell.



For axial symmetry, displacement fields are

$$u(x) = u(r)\frac{x}{r}$$

There are no shear stesses and shear strains in the case of axial symmetry, so we seek only definitions for the radial and hoop strains. The radial strain is the derivative of radial displacement with respect to radial direction. There is established a definition for the hoop strain in an axisymmetric situation, as the ration between the change in circumference and the original circumference. With a radial displacement of u_r at a radius r, this is

$$E_{\theta\theta} = \frac{2\pi(r + u_r) - 2\pi r}{2\pi r} = \frac{u_r}{r} \tag{3.1.1}$$

Therefore, the Radial and Hoop Strain Components are:

$$E_{r\theta} = \frac{du_r}{dr}, \quad E_{\theta\theta} = \frac{u_r}{r}, \quad E_{r\theta} = 0. \tag{3.1.2}$$

3.2 Equilibrium Equations

In order to formulate the equilibrium equations, we will need the partial and second partial derivatives for polar coordinates with respect to x and y. To begin with we know that:

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2} \tag{3.2.1}$$

$$y = r\sin\theta \qquad \theta = \arctan\left(\frac{y}{x}\right) \tag{3.2.2}$$

Thus,

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \frac{2x}{2\sqrt{x^2+y^2}}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \frac{r\cos\theta}{r}\frac{\partial}{\partial r} + \frac{\partial}{\partial x}\arctan\left(\frac{y}{x}\right)\frac{\partial}{\partial \theta}$$
$$* = \cos\theta\frac{\partial}{\partial r} + \left(-\frac{y}{x^2+y^2}\right)\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{r\sin\theta}{r^2}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}. \tag{3.2.3}$$

* In order to calculate $\frac{\partial}{\partial x}\arctan\left(\frac{y}{x}\right)$, we set $\alpha = \arctan(\beta)$, where $\beta = \frac{y}{x}$. Then,

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \alpha}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{1}{1+\beta^2}\left(-\frac{y}{x^2}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}. \tag{3.2.4}$$

Also, with respect to y:

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \frac{2y}{2\sqrt{x^2+y^2}}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \frac{r\sin\theta}{r}\frac{\partial}{\partial r} + \frac{\partial}{\partial y}\arctan\big(\frac{y}{x}\big)\frac{\partial}{\partial \theta}$$
$$* = \sin\theta\frac{\partial}{\partial r} + \frac{x}{x^2+y^2}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{r\cos\theta}{r^2}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}. \tag{3.2.5}$$

* Working as before with respect to y, we have that

$$\frac{\partial \alpha}{\partial y} = \frac{\partial \alpha}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{1}{1+\beta^2}\big(\frac{1}{x}\big) = \frac{1}{x(1+\beta^2} = \frac{1}{x\big(1+\big(\frac{y}{x}\big)^2} = \frac{x}{x^2+y^2}. \tag{3.2.6}$$

Now, let's find the Second Partial Derivatives:

$$\frac{\partial^2}{\partial x^2} = \big(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\big)\big(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$
$$= \cos\theta\frac{\partial}{\partial r}\big(\cos\theta\frac{\partial}{\partial r}\big) - \cos\theta\frac{\partial}{\partial r}\big(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\big) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\big(\cos\theta\frac{\partial}{\partial r}\big) + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\big(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\big)$$
$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \sin^2\theta\big(\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \sin^2\theta\big(\frac{1}{r^2} + \frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\big) + \sin 2\theta\big(\frac{1}{r^2}\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial \theta}\big). \tag{3.2.7}$$

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta\frac{\partial^2}{\partial r^2} + \cos^2\theta\big(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\big) - \sin 2\theta\big(\frac{1}{r^2}\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial \theta}\big). \tag{3.2.8}$$

$$\frac{\partial^2}{\partial x\partial y} = -\sin\theta\cos\theta\big(-\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\big)-\cos 2\theta\big(\frac{1}{r^2}\frac{\partial}{\partial \theta}-\frac{1}{r}\frac{\partial^2}{\partial r\partial \theta}\big). \tag{3.2.9}$$

The General Equations of Equilibrium in cartesian 2D are:

$$\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} = 0 \tag{3.2.10}$$

$$\frac{S_{yx}}{\partial x} + \frac{\partial S_{yy}}{\partial y} = 0 \tag{3.2.11}$$

where

$$S_{xx} = S_{rr}\cos^2\theta + S_{\theta\theta}\sin^2\theta - S_{r\theta}\sin 2\theta \tag{3.2.12}$$

$$S_{yy} = S_{rr}\sin^2\theta + S_{\theta\theta}\cos^2\theta + S_{r\theta}\sin 2\theta \tag{3.2.13}$$

$$S_{xy} = \sin\theta\cos\theta(S_{rr} - S_{\theta\theta}) + S_{r\theta}\cos 2\theta \tag{3.2.14}$$

Therefore, applying these and 3.2.3, 3.2.5 to the 2D Equations of Equilibrium, we have that:

$$\cos\theta[\frac{\partial S_{rr}}{\partial r} + \frac{1}{r}\frac{\partial S_{r\theta}}{\partial\theta} + \frac{1}{r}(S_{rr} - S_{\theta\theta})] - \sin\theta[\frac{\partial S_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial\theta} + \frac{2S_{r\theta}}{r}] = 0 \tag{3.2.15}$$

$$\sin\theta[\frac{\partial S_{rr}}{\partial r} + \frac{1}{r}\frac{\partial S_{r\theta}}{\partial\theta} + \frac{1}{r}(S_{rr} - S_{\theta\theta})] + \cos\theta[\frac{\partial S_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial\theta} + \frac{2S_{r\theta}}{r}] = 0 \tag{3.2.16}$$

which is similar to the system:

$$\frac{\partial S_{rr}}{\partial r} + \frac{1}{r}\frac{\partial S_{r\theta}}{\partial\theta} + \frac{1}{r}(S_{rr} - S_{\theta\theta}) = 0 \tag{3.2.17}$$

$$\frac{\partial S_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial\theta} + 2\frac{S_{r\theta}}{r} = 0 \tag{3.2.18}$$

So, the above system represents the Equilibrium Equations in Polar Coordinates. However, in our case we have that $S_{r\theta} = 0$, so the above system can be written as follows:

$$\frac{\partial S_{rr}}{\partial r} + \frac{1}{r}(S_{rr} - S_{\theta\theta}) = 0 \tag{3.2.19}$$

$$(rS_{rr}(r))' = S_{\theta\theta}(r) \tag{3.2.20}$$

3.3 Formulation of the Equation

We suppose that the cell shrinks, so the first boundary condition is

$$u(\alpha) = -u_0 \tag{3.3.1}$$

where u_0 is a positive constant. Also, we suppose that the outside boundary of the matrix is traction free, so we have a second boundary condition

$$S_{rr}(A) = 0. \tag{3.3.2}$$

The solution of the corresponding linear elastic problem, for $\rho = 1$, has the property that

$$S_{rr} > 0,\ S_{\theta\theta} < 0 \tag{3.3.3}$$

. for a ¡ r ¡ A. We adopt these inequalities for our compression - weakening material and we will verify them later.

Substituting $E_{rr} = u'(r)$ and $E_{\theta\theta} = \frac{u(r)}{r}$ into the Stress - Strain Relations 2.2.5, and the result into the Equilibrium 3.2.20 we have that

$$\left(r\frac{h(\rho)\alpha u'(r)}{\rho} + \frac{rh(\rho)\beta u(r)}{r} \right)' = h(\rho)\beta \dot{u}(r) + h(\rho)\alpha\frac{u(r)}{r} \tag{3.3.4}$$

After calculations, the above Equation is similar to a *2nd order Linear ODE for u(r)*:

$$r^2 u''(r) + u'(r)r - u(r)\rho = 0 \tag{3.3.5}$$

where $\alpha < r < A$.

3.4 Solution of the ODE

After calculations we find that r^{ξ} and $r^{-\xi}$ are solutions of the ODE(78). Also, the Wronskian of these two solutions is

$$Wronskian(r^{\xi}, r^{-\xi}) = \begin{bmatrix} r^{\xi} & r^{-\xi} \\ \xi r^{\xi-1} & -\xi r^{-\xi-1} \end{bmatrix} = -2r^{\xi}\xi\frac{r^{-\xi}}{r} \neq 0$$

Thus, r^{ξ} and $r^{-\xi}$ are linearly independent solutions of the ODE 3.3.5 and the general solution has the following form

$$u(r) = C_1 r^{-\xi} + C_2 r^{\xi} \tag{3.4.1}$$

where $\xi = \sqrt{\rho}$.

Now, we turn to the Boundary Conditions. From the first one, we have that

$$C_1\alpha^{-\xi} + C_2\alpha^{\xi} = -u_0 \tag{3.4.2}$$

Also, from the second boundary condition we get that

$$C_1 = C_2\frac{-\alpha\xi A^{\xi} - \beta\rho A^{\xi}}{\beta\rho A^{-\xi} - \xi\alpha A^{-\xi}} \tag{3.4.3}$$

Substituting 3.4.3 into 3.4.3 we get that:

$$C_1 = -u_0\frac{A^{2\xi}\alpha^{\xi}(\alpha+\beta\xi)}{\alpha^{2\xi}(\alpha-\beta\xi)+A^{2\xi}(\alpha+\beta\xi)}, \tag{3.4.4}$$

$$C_2 = -u_0\frac{\alpha^{\xi}(\alpha-\beta\xi)}{\alpha^{2\xi}(\alpha-\beta\xi)+A^{2\xi}(\alpha+\beta\xi)}. \tag{3.4.5}$$

Therefore, the radial displacement $u(r)$ is written as follows:

$$u(r) = -u_0\frac{\left(\alpha+\beta\xi\right)\left(\frac{r}{A}\right)^{-\xi} + \left(\alpha-\beta\xi\right)\left(\frac{r}{A}\right)^{\xi}}{\left(\alpha+\beta\xi\right)\left(\frac{\alpha}{A}\right)^{-\xi} + \left(\alpha-\beta\xi\right)\left(\frac{\alpha}{A}\right)^{\xi}} \tag{3.4.6}$$

Also, the radial and hoop stress are, respectively:

$$S_{rr}(r) = (\frac{u_0}{\alpha})\frac{\xi(\alpha^2-\beta^2)[(\frac{r}{A})^{-\xi-1}-(\frac{r}{A})^{\xi-1}]}{(\alpha+\beta\xi)(\frac{\alpha}{A})^{-\xi-1}+(\alpha-\beta\xi)(\frac{\alpha}{A})^{\xi-1}} \tag{3.4.7}$$

$$S_{\theta\theta}(r) = (-\frac{u_0}{\alpha})\frac{\xi^2(\alpha^2-\beta^2)[(\frac{r}{A})^{-\xi-1}+(\frac{r}{A})^{\xi-1}}{(\alpha+\beta\xi)(\frac{\alpha}{A})^{-\xi-1}+(\alpha-\beta\xi)(\frac{\alpha}{A})^{\xi-1}} \tag{3.4.8}$$

where $a < r < A$.

In order to check if the radial stress $\sigma_r(r) > 0$ and the hoop stress $\sigma_\theta(r) < 0$ 3.3.3, we can simplify the form of radial stress by multiplying the term $A^{\xi-1}a^{\xi+1}$ with the above form of radial stress. After calculations we conclude that

$$u_r(r) = \frac{u_0\xi a^{\xi}(\alpha^2-\beta^2)r^{-\xi-1}(A^{2\xi}-r^{2\xi})}{a^{2\xi}(\alpha-\beta\xi)+A^{2\xi}(\alpha+\beta\xi)}. \tag{3.4.9}$$

Doing the same calculations for the hoop stress, we conclude that

$$u_\theta(r) = -\frac{u_0\xi^2 a^{\xi}(\alpha^2-\beta^2)r^{-\xi-1}(A^{2\xi}+r^{2\xi})}{a^{2\xi}(\alpha-\beta\xi)+A^{2\xi}(\alpha+\beta\xi)}. \tag{3.4.10}$$

We know that $u_0 > 0$ since the solution is the contractile displacement(contraction) and that $\alpha = \lambda + 2\mu$, $\beta = \lambda$, so we conclude that $\alpha \pm \beta\xi > 0$. Also, since $a \leq r \leq A$, we conclude that $A^{2\xi} \pm r^{2\xi} \leq 0$. Lastly, we remember that $\rho < 1$. Based on the above inequalities, we conclude that the radial stress $\sigma_r(r) > 0$ and that the hoop stress $\sigma_\theta(r) < 0$, so 3.3.3 is verified.

In the special case of $\rho = 1$ (Linear Elasticity), the displacement takes the form

$$u(r) = C_1 r^{-1} + C_2 r \tag{3.4.11}$$

which is equivalent to the final form

$$u(r) = -u_0\frac{(\alpha+\beta\xi)(\frac{r}{A})^{-1}+(\alpha-\beta\xi)(\frac{r}{A})}{(\alpha+\beta\xi)(\frac{a}{A})^{-1}+(\alpha-\beta\xi)(\frac{a}{A})}, \tag{3.4.12}$$

where again $a \leq r \leq A$.

Also, the stresses in case of Linear Elasticity take the following forms

$$S_{rr}(r) = u_0 \frac{a(\alpha^2 - \beta^2)r^{-2}(A^2 - r^2)}{a^2(\alpha - \beta) + A^2(\alpha + \beta)}, \tag{3.4.13}$$

$$S_{\theta\theta}(r) = -u_0 \frac{a(\alpha^2 - \beta^2)r^{-2}(A^2 + r^2)}{a^2(\alpha - \beta) + A^2(\alpha + \beta)}. \tag{3.4.14}$$

3.5 Decay of displacements

Let's remember the general solution 3.4.1

$$u(r) = C_1 r^{-\xi} + C_2 r^{\xi}$$

where $\xi = \sqrt{\rho}$. We want to find with what order the displacements decay, but the general solution is somewhat confusing. As r increases, the term $(C_1 r^{-\xi})$ decreases, while the term $(C_2 r^{\xi})$ increases. So, what will happen for larger r? Also, the constants C_1 and C_2 depend on A, where $\alpha \leq r \leq A$. Specifically, C_2 increases when A increases and in the case of an infinite matrix, i.e. $A \to \infty$, the constant $C_2 \to 0$. From the other hand, when $A \to \infty$, we cannot make predictions about the constant C_1 in this limit, because it contains A both in its numerator and denominator.

Let's now find a lower bound for the displacements. Specifically, we know that $u_0 > 0$ and that $\alpha + \beta\xi > 0$, $\alpha - \beta\xi > 0$. Also, since $\alpha = \lambda + 2\mu$, $\beta = \lambda$ and $\xi = \sqrt{\rho} < 1$, we conclude that $\alpha + \beta\xi > \alpha - \beta\xi$.

Thus, if we replace term $(\alpha - \beta\xi)$ with the term $(\alpha + \beta\xi)$, we get

$$| C_1 | = -C_1 \geq u_0 \frac{a^{\xi} A^{2\xi}(\alpha + \beta\xi)}{a^{2\xi}(\alpha + \beta\xi) + A^{2\xi}(\alpha + \beta\xi)}$$

$$= u_0 \frac{a^{\xi} A^{2\xi}}{a^{2\xi} + A^{2\xi}}. \tag{3.5.1}$$

Also, we know that $a < A$, so we get that

$$-C_1 \geq u_0 \frac{a^{\xi}}{2} \tag{3.5.2}$$

So,we conclude that

$$| u(r) |= -u(r) \geq \frac{1}{2}u_0(\frac{r}{a})^{-\xi} \tag{3.5.3}$$

Therefore, the above is the lower bound of the displacements.

Now, one could easily find an upper bound for the displacements. If we multiply 3.4.1 by the term $a^\xi A^\xi$ we have that

$$u(r) = -u_o\frac{A^{2\xi}(\frac{r}{a})^{-\xi} + a^{2\xi}(\alpha - \beta\xi)(\frac{r}{a})^{\xi}}{a^{2\xi}(\alpha - \beta\xi) + A^{2\xi}(\alpha + \beta\xi)}, \tag{3.5.4}$$

where $a \leq r \leq A$. If we subtract $a^{2\xi}(\alpha - \beta\xi)$ from the denominator of 3.5.4 we get that

$$| u(r) |\leq u_0\frac{A^{2\xi}(\frac{r}{a})^{-\xi} + a^{2\xi}(\alpha - \beta\xi)(\frac{r}{a})^{\xi}}{A^{2\xi}(\alpha + \beta\xi)}$$

which is equivalent to the following form

$$| u(r) |\leq u_0\Big(\frac{r}{a}\Big)^{-\xi} + u_0\frac{a^{2\xi}(\alpha - \beta\xi)(\frac{r}{a})^{\xi}}{A^{2\xi}(\alpha + \beta\xi)}. \tag{3.5.5}$$

We know that $0 < r \leq A$, so we have that

$$\Big(\frac{a}{A}^{2\xi}\Big) \leq \Big(\frac{a}{r}\Big)^{2\xi}. \tag{3.5.6}$$

One could replace the term $(\frac{a}{A})^{2\xi}$ of 3.5.5 with term $(\frac{a}{r})^{2\xi}$ resulting in

$$| u(r) |\leq u_0\Big(\frac{r}{a}\Big)^{-\xi} + u_0\Big(\frac{a}{r}\Big)^{2\xi}\Big(\frac{r}{a}\Big)^{\xi}\frac{(\alpha - \beta\xi)}{(\alpha + \beta\xi)}$$

$$= u_0\frac{\big(\frac{r}{a}\big)^{-\xi}(\alpha + \beta\xi + \alpha - \beta\xi)}{\alpha + \beta\xi}$$

which is equivalent to

$$| u(r) | \le u_0 \left(\frac{r}{a}\right)^{-\xi} \frac{2\alpha}{\alpha + \beta\xi}. \tag{3.5.7}$$

So, the above is the upper bound for the displacements. Therefore, we conclude that

$$\frac{1}{2} u_0 \left(\frac{r}{a}\right)^{-\xi} \le | u(r) | \le u_0 \frac{2\alpha}{(\alpha + \beta\xi)} \left(\frac{r}{a}\right)^{-\xi}. \tag{3.5.8}$$

We observe that the bounds are independent of the outside radius A and that the displacements decay with order $O(r^{-\xi}) = O(r^{-\sqrt{\rho}})$, where $\rho < 1$, despite the fact that the general solution contains a growing term. Thus, the decay of the displacements induced by a contracting cell in a matrix composed of compression weakening material is slower than in a linear elastic matrix where the displacements decay as $O(r^{-1})$.

In fact, in order to compare the decay of displacements in the compression weakening matrix and in the linear elastic one, we only use the lower bound of the displacements. That is to say, when we want to prove that a term x decays slower than another term y, i.e. has bigger value, we want to check its lower bound. If the value of the lower bound of x has bigger value than the term y we compare with, then it is true that the term x decays slower than the term y. We needed the upper bound in order to find the decay of the displacements. Let's now find bounds for the stresses. We have radial and hoop stress, so we will calculate the stress norm. Specifically,

$$S(r) = \sqrt{S_{rr}^2(r) + S_{\theta\theta}^2(r)}$$

Substituting 3.4.13, 3.4.14 into the above equation we conclude that

$$| S(r) | = \frac{u_0 \xi a^{\xi} (\alpha^2 - \beta^2) r^{-\xi-1} \sqrt{(A^{2\xi} - r^{2\xi})^2 - \xi^2 (A^{2\xi} + r^{2\xi})^2}}{a^{2\xi}(\alpha - \beta\xi) + A^{2\xi}(\alpha + \beta\xi)} \tag{3.5.9}$$

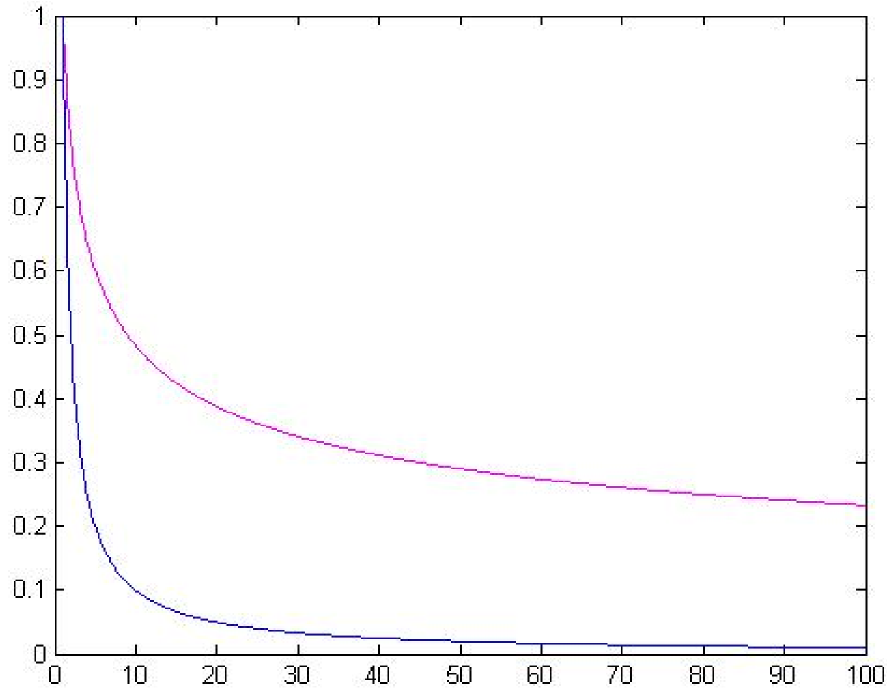


Figure 8: Graph of displacement $u(r)$ vs radial distance r. Blue line is without buckling and purple line is with buckling. The compression stiffness ratio is $\rho = 0.1$ here.

Since $a \le r \le A$, we know that $A^{2\xi} + r^{2\xi} \le 2A^{2\xi}$. After calculations we find that

$$\left(\frac{u_0}{a}\right)\frac{\xi^2(\alpha^2-\beta^2)}{2(\alpha+\beta\xi)}\left(\frac{r}{a}\right)^{-\xi-1} \le |\, S_r \,| \le \left(\frac{u_0}{a}\right)\frac{2\xi(\alpha^2-\beta^2)}{\alpha+\beta\xi}\left(\frac{r}{a}\right)^{-\xi-1} \qquad (3.5.10)$$

We note that the lower bound of the stresses decays with order $O(r^{-\xi-1})$, while the stresses in a linear elastic matrix decay with order $O(r^{-2})$. Thus, we conclude that stresses induced by a contracting cell in a matrix composed of compression weakening material, decay slower than in a linear elastic matrix. Again, there is no need to check the upper bound since the lower one decays slower than its linear elastic counterpart.

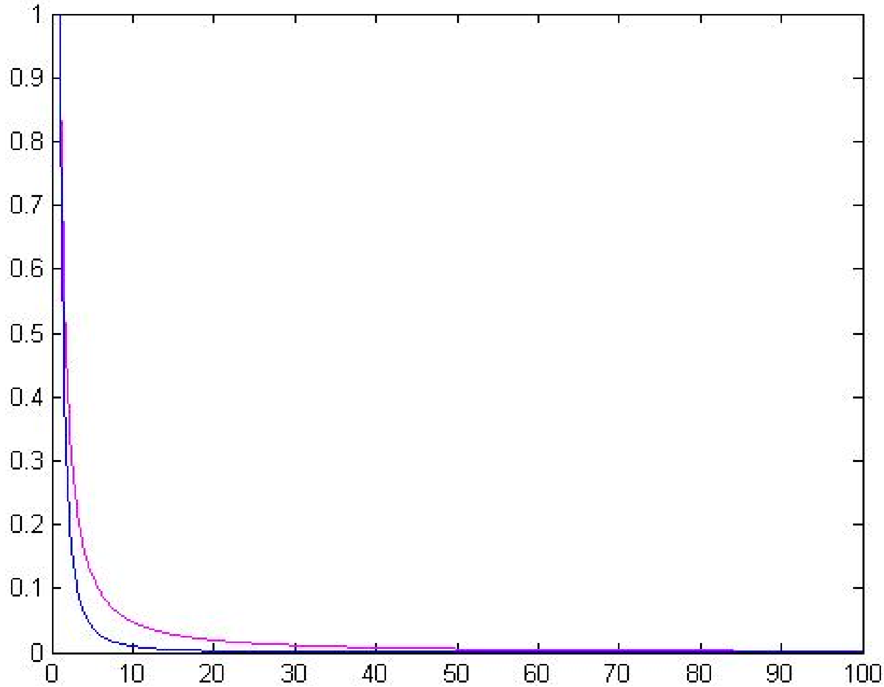


Figure 9: Graph of stresses vs radial distance r. Blue line is without buckling and purple line is with buckling. The compression stiffness ratio is $\rho = 0.1$ here.

3.6 Infinite matrix

Let's now examine the case of an infinite matrix, i.e. when $A \to \infty$. In this limit, the displacements take the following form:

$$u(r) = -u_0(\frac{r}{a})^{-\xi}, \tag{3.6.1}$$

where again $\xi = \sqrt{\rho}$. Also, when $A \to \infty$, the stresses are convertible to the following form :

$$S_{rr}(r) = \xi(\frac{u_0}{a})^{\xi}(\frac{r}{a})^{-\xi-1}\frac{\alpha^2 - \beta^2}{\alpha + \beta\xi}, \tag{3.6.2}$$

$$S_{\theta\theta}(r) = -\xi^2(\frac{u_0}{a})(\frac{r}{a})^{-\xi-1}\frac{\alpha^2 - \beta^2}{\alpha + \beta\xi} \tag{3.6.3}$$

We note that $S_{\theta\theta}(r) = -\xi S_{rr}(r)$.

Therefore, in the limit of a infinite matrix, where $A \to \infty$, the displacements decay again with order $O(r^{-\xi})$, while in a linear elastic matrix the displacements decay with order $O(r^{-1})$ in this limit. So, displacements induced by a contracting cell in an infinite matrix composed of compression weakening material, decay slower than the displacements in a infinite linear elastic matrix. Also, we note that stresses decay with order $O(r^{-\xi-1})$ when $A \to \infty$, while in a linear elastic matrix the stresses decay with order $O(r^{-2})$. So, we conclude that stresses induced by a contracting cell in an infinite matrix composed of compression weakening material, decay slower than stresses in a infinite linear elastic matrix.

3.7 Zero compression stiffness ratio

Let's see what's happening when compression stiffness ratio $\rho = 0$. We have set $\xi = \sqrt{\rho}$, so $\xi = 0$. Using the Constitutive Law we created in section 2 2.2.5, and setting $\rho = 0$, we get that:

$$S_{rr}(r) = \frac{\alpha^2 - \beta^2}{\alpha} E_{rr}(r), \tag{3.7.1}$$

$$S_{\theta\theta}(r) = 0. \tag{3.7.2}$$

The equilibrium equation takes the following form:

$$(rS_{rr}(r))' = 0 \tag{3.7.3}$$

Substituting 3.7.1 into 3.7.3 we have that:

$$(r\frac{\alpha^2 - \beta^2}{\alpha} E_{rr}(r))' = 0 \tag{3.7.4}$$

since the term $\frac{\alpha^2-\beta^2}{\alpha}$ is independent of r we can omit it, because the above derivative is with respect to r. Thus,

$$(rE_{rr}(r))' = 0$$

Also, since $E_{rr}(r) = u'(r)$ we conclude that

$$(ru'(r))' = 0. \tag{3.7.5}$$

The general solution of 3.7.5 is

$$u(r) = C_1 \log r + C_2. \tag{3.7.6}$$

Now, $u'(r) = \frac{C_1}{r}$, so the radial stress takes the form

$$S_{rr}(r) = \frac{\alpha^2 - \beta^2}{\alpha} \frac{C_1}{r} \tag{3.7.7}$$

In order to satisfy the boundary conditions $u(\alpha) = -u_0$ since the cell shrinks, and $S_{rr}(A) = 0$ because we consider the outside boundary of the matrix to be traction free,the values of constants C_1 and C_2 must be:

$$C_1 = 0, \quad C_2 = -u_0 \tag{3.7.8}$$

Therefore, the radial displacement takes the form:

$$u(r) = -u_0. \tag{3.7.9}$$

If we take the limit as $A \to \infty$ (infinite matrix), we have only one boundary condition $u(\alpha) = -u_0$, ($C_2 = -u_0$) since the matrix has no outside boundary. Thus, the radial displacement takes the form:

$$u(r) = C_1 \log r - u_0. \tag{3.7.10}$$

4 The Expanding Cell Problem in 2D

Let's examine an expanding cell instead of a contracting one. The first thing that changes is the first boundary condition, since the cell expands. Thus, we have that

$$u(a) = -u_0, \tag{4.0.1}$$

where now $u_0 < 0$. Again we assume that:

$$S_{rr}(r) < 0, \quad S_{\theta\theta}(r) > 0. \tag{4.0.2}$$

and we will verify them later. Therefore, from the above equations, we are moving to the second quadrant of the principal stress plane. This quadrant is the opposite of the one that we used in the precious case of a contracting cell, the fourth quadrant. Now, a different quadratic branch of the energy function is in force. The Strain Energy function is

$$W(E_1, E_2) = \frac{1}{2} M_{ij}^{-1} E_i E_j,$$

where now M^{-1} matrix is the inverse of the second M matrix 2.1.53 and has the following form:

$$M^{-1} = \frac{(\alpha^2 - \beta^2)\rho}{\alpha^2 - \beta^2\rho} \begin{pmatrix} \alpha & \beta \\ \beta & \frac{\alpha}{\rho} \end{pmatrix} \tag{4.0.3}$$

Again, $\frac{\partial W}{\partial E_i} = S_i$, so if we differentiate the first equation twice, firstly with respect to E_1 and after with respect to E_2, we get the following Stress-Strain relation:

$$S_1 = \frac{(\alpha^2 - \beta^2)\rho}{\alpha^2 - \beta^2\rho} (\alpha E_1 + \beta E_2), \tag{4.0.4}$$

$$S_2 = \frac{(\alpha^2 - \beta^2)\rho}{\alpha^2 - \beta^2\rho} (\beta E_1 + \frac{\alpha E_2}{\rho}). \tag{4.0.5}$$

The principal strains are:

$$E_{rr}(r) = u'(r), \quad E_{\theta\theta}(r) = \frac{u(r)}{r} \tag{4.0.6}$$

Also, the equilibrium equations take the following form:

$$(rS_{rr}(r))' = S_{\theta\theta}(r) \tag{4.0.7}$$

Thus, substituting 4.0.4, 4.0.5, 4.0.6 into 4.0.7 we get that:

$$(r\alpha u'(r) + \beta u(r))' = \beta u'(r) + \frac{\alpha u(r)}{\rho r} \tag{4.0.8}$$

which results *in a 2nd order linear ODE for u(r)*:

$$r^2 u''(r) + ru(r) - \rho^{-1}u(r) = 0, \tag{4.0.9}$$

where $a < r < A$. After calculations, we find that $r^{-\frac{1}{\xi}}$ and $r^{\frac{1}{\xi}}$ are solutions of the ODE 4.0.9. Also, the Wronskian of these two solutions is

$$Wronskian(r^{\frac{1}{\xi}}, r^{-\frac{1}{\xi}}) = \begin{bmatrix} r^{\frac{1}{\xi}} & r^{-\frac{1}{\xi}} \\ \frac{1}{\xi}r^{\frac{1}{\xi}-1} & -\frac{1}{\xi}r^{-\frac{1}{\xi}-1} \end{bmatrix}$$

$$= -\frac{1}{\xi}r^{-1} - \frac{1}{\xi}r^{-1} = -\frac{2}{\xi}r^{-1} \neq 0$$

Thus, $r^{-\frac{1}{\xi}}$ and $r^{\frac{1}{\xi}}$ are *linearly independent solutions* of the ODE 4.0.9 and the general solution has the following form:

$$u(r) = C_1 r^{-\frac{1}{\xi}} + C_2 r^{\frac{1}{\xi}}. \tag{4.0.10}$$

We have the first boundary condition because the cell expands

$$u(a) = -u_0,$$

where $u_0 < 0$. Also, we suppose that the outside boundary of the matrix is traction free, so the 2nd boundary condition is:

$$S_{rr}(A) = 0, \tag{4.0.11}$$

where $a < r < A$. We know that $S_{rr}(r) = \frac{(\alpha^2-\beta^2)\rho}{\alpha^2-\beta^2\rho}$, and that strains are $E_{rr}(r) = u'(r)$, $E_{\theta\theta}(r) = \frac{u(r)}{r}$. Thus, the second boundary condition can be written as:

$$\frac{\alpha}{\rho}u'(A) + \beta\frac{u(A)}{A} = 0 \tag{4.0.12}$$

Thus, the general solution $u(r)$ should obey the boundary conditions. From the first one, we have that

$$C_1 a^{-\frac{1}{\xi}} + C_2 a^{\frac{1}{\xi}} = -u_0 \tag{4.0.13}$$

Also, from the second we get that

$$C_1 = C_2\frac{(\frac{1}{xi}\alpha + \beta\rho)A^{\frac{1}{\xi}}}{\frac{1}{\xi}\alpha A^{-\frac{1}{\xi}} - \beta\rho A^{-\frac{1}{\xi}}} \tag{4.0.14}$$

Now, substituting 4.0.14 into 4.0.13 we get that:

$$C_1 = -u_0\frac{(\frac{1}{\xi}\alpha + \beta\rho)A^{\frac{1}{\xi}}}{(\frac{1}{\xi}\alpha + \beta\rho)(\frac{a}{A})^{-\frac{1}{\xi}} + (\frac{1}{\xi}\alpha - \beta\rho)(\frac{a}{A})^{\frac{1}{\xi}}} \tag{4.0.15}$$

and

$$C_2 = -u_0\frac{(\frac{1}{\xi}\alpha - \beta\rho)A^{-\frac{1}{\xi}}}{(\frac{1}{\xi}\alpha + \beta\rho)(\frac{a}{A})^{-\frac{1}{\xi}} + (\frac{1}{\xi}\alpha - \beta\rho)(\frac{a}{A})^{\frac{1}{\xi}}} \tag{4.0.16}$$

Therefore, if we substitute 4.0.15, 4.0.16 into the general solution 4.0.10, we conclude that the radial displacement takes the form:

$$u(r) = -u_0\frac{(\frac{1}{\xi}\alpha + \beta\rho)(\frac{r}{A})^{-\frac{1}{\xi}} + (\frac{1}{\xi}\alpha - \beta\rho)(\frac{r}{A})^{\frac{1}{\xi}}}{(\frac{1}{\xi}\alpha + \beta\rho)(\frac{a}{A})^{-\frac{1}{\xi}} + (\frac{1}{\xi}\alpha - \beta\rho)(\frac{a}{A})^{\frac{1}{\xi}}} \tag{4.0.17}$$

For an infinite matrix, the displacement takes the form

$$u(r) = -u_0(\frac{r}{a})^{-\frac{1}{\xi}}$$

We note that since $0 < \rho < 1$, we have that $\frac{1}{\xi} > 1$. Also, when $\rho \to 0$, we have that $\frac{1}{\xi} \to \infty$. After calculations, we find that the displacements decay with order $O(r^{-\frac{1}{\xi}})$ and that the stresses decay with order $O(r^{-\frac{1}{\xi}-1})$. Thus, both decay faster than their linear elastic counterparts, which are $O(r^{-1})$ and $O(r^{-2})$. respectively. We conclude that displacements and stresses induced by an expanding cell in a matrix composed of compression weakening material, decay faster than in a linear elastic matrix.

5 Discussion

5.1 A Constitutive Model for Fibrin

The first step in our attempt to test the hypothesis that displacements induced by a contracting cell in a matrix composed of compression weakening material, decay slower than in a linear elastic matrix, is done in subsection 2.1. Here, we construct a constitutive model for fibrin networks. Fibers buckle under compression, and they become more compliant. As a result, the elasticity of the fibrin network gradually decreases, i.e. there is a loss of stiffness in compression. Thus, the main thing that we want our model to capture, is buckling of fibers. We set a compression-stiffness ratio ρ which is $0 < \rho < 1$ for a compression-weakening material that buckles, and $\rho = 1$ for a linear elastic material without buckling. We create a special nonlinearity, in which each principal stress is a piecewise linear function of the principal strains in a way that clearly separates the behavior of a compression-weakening material in tension and in compression. 2.1 shows exactly this buckling behavior.

In subsection 2.2 we examine our model under homogeneous deformations. Firstly, we consider Uniaxial Stress. We find that the Young's modulus has lower value in compression than in tension, so the uniaxial stress-strain relation is the same as 2.1. Furthermore, we find that Poisson's ratio has lower absolute value in compression than in tension. Specifically, we find that $\frac{\nu_c}{\nu_t} = \rho$, where ν_c is the Poisson's ratio in compression and ν_t is the Poisson's ratio in tension. Again $0 < \rho < 1$. In other words, there is a weakening of the Poisson effect in uniaxial compression. Simulations of a discrete fiber network model agree with this weakening. Specifically, this network consists of fibers, each of them behaves as in 2.1. Simulatios for different values of *compression stiffness ratio* $\rho = 0.1, 0.3, 0.5, 0.7$, showed that the ratio of the Poisson's ration in compression to the Poisson's ratio in tension is $\frac{\nu_c}{\nu_t} = 0.74\rho$. So, there is a weakening of the Poisson effect in compression and is in qualitative agreement with the ratio of our model.

Next, we consider Simple Shear. We observe that for a compression weakening material ($\rho < 1$), there are normal stresses present, according to our model. These normal stresses vanish in absence of buckling, i.e. for a linear elastic material ($\rho = 1$). This existence of normal stresses in simple shear agrees with experiments for fibrin, whereby the Negative Poynting Effect (opposite of the Poynting Effect [19]) was observed [9]. Specifically, the negative Poyting effect is obtained, when the sheared faces tend to move closer together, i.e. under compression. Then, a hydrostatic tension must be applied to counteract the tendency of drawing together and maintain the deformation. The basic underlying mechanism of this model is the buckling of filaments [3], which behave as in 2.1, i.e. like compression weakening material.

5.2 Displacements induced by a contracting cell

In Section 3 we model the contracting cell as a disk with radius a, plus the matrix to be the annulus $a < r < A$. The matrix is consisted of fibers that buckle in compression, so it is composed of compression weakening material. Therefore, we consider that it is governed by our constitutive law that we constructed in Section 2. We find that the radial displacement takes the form $u(r) = C_1 r^{-\xi} + C_2 r^{\xi}$, where $\xi = \sqrt{\rho}$. In Subsection 3.2 we find that the displacements decay with order $O(R^{-\xi}$, despite the presence of the second growing term. In [15] Notbohm et al. developed a FE-based microstructural model consisting of a network of linear elements representing fibres. Buckling of fibers will refer to elements (fibers) obeying a stressstrain relation where the stiffness (slope) in compression is smaller than the stiffness under tension (Fig.2). The compression stiffness ratio is $\rho = 0.1$ in the simulations. Also, the fibrin networks have low connectivity, i.e. small value of coordinator number C, which is defined as the average number of fibers meeting at a node. Fibrin has often a value of $C = 3$, a fact that renders it a floppy network, since the critical value of rigidity for 2D networks is 4. [25] Simulations were performed for connectivities in the range $2.5 \leq C \leq 8$.

The displacement $u(r)$ was find to decay with order $O(r^{-n})$. Displacements are plotted for different connectivities (Fig.8). All curves show long-range propagation of displacements with slopes $\approx= -0.5$, except one case. Specifically, for the critical value of connectivity, C=4, displacements exhibit spatial inhomogeneities resulting in fluctuations. Furthermore, The findings reveal that generally the decay power n for networks composed of fibers that buckle, has lower value (by at least 0.4) than for networks consisted of fibers that resist buckling (Fig.9). Again there is the exception of the critical value of connectivity, C=4. For this value, findings showed that $n = 0.6$ for both

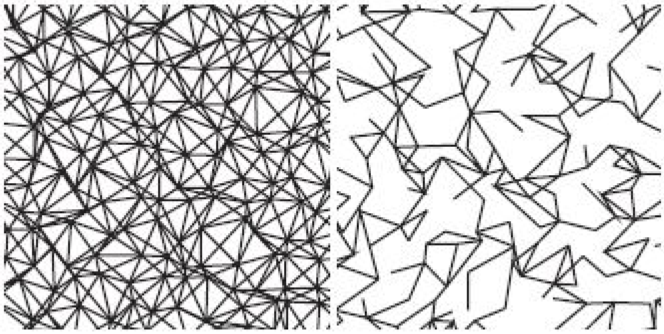


Figure 10: Left: Randomized fibrin network with C=8. Right: Fibrin network with C=3. [15]

networks with buckling or not.

Let's now compare the findings in [15](discrete model) with our continuum model. We remember that our term $\xi = \sqrt{\rho}$.

Comparison of the decay power ξ of our model with the decay power n of simulations.		
ρ	$\xi = \sqrt{\rho}$	n
0.03	0.173	0.226
0.3	0.548	0.532
0.1	0.316	0.337

Therefore, we conclude that the predictions of our model about the decay power are generally in agreement with the results of simulations. The displacements induced by a constracting cell in a matrix which consists of fibers that buckle (compression weakening material), decay slower than in a linear elastic matrix.

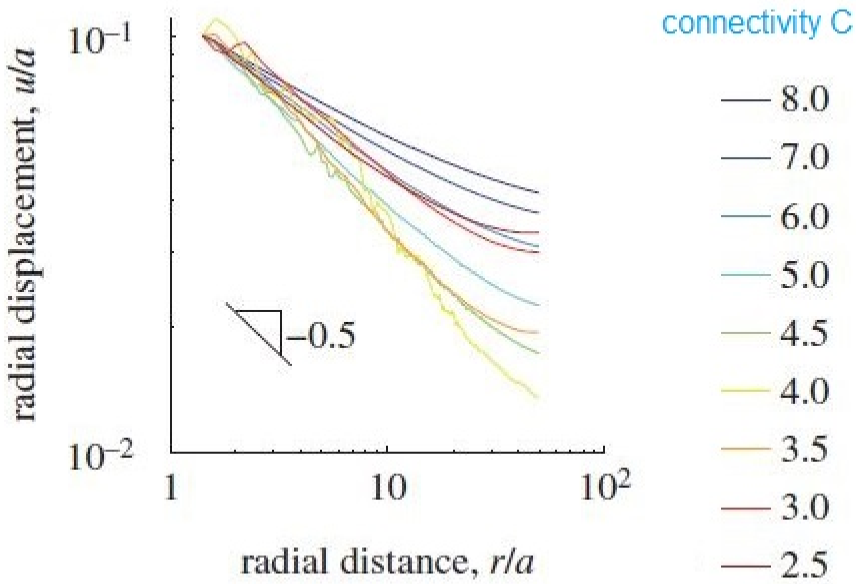


Figure 11: Displacements are plotted for simulations that used different connectivities. The radial displacement and the radial distance are normalized with radius a. [15]

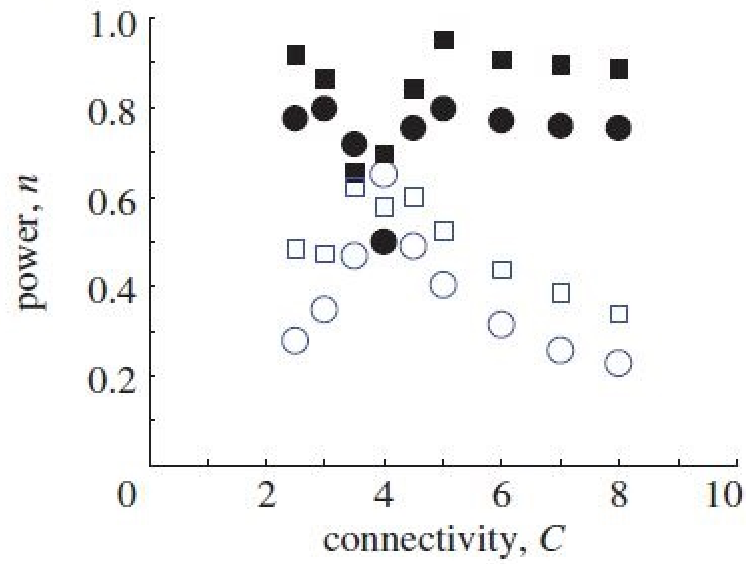


Figure 12: Decay power n versus connectivity C. Solid black symbols represent fibres that resist buckling($\rho = 1$), while open symbols represent fibres that buckle ($\rho = 0.1$). Circles show fits to $u = Ar^{-n}$ while squares show fits to $u = Ar^{-n} + Br^{n}$. [15]

5.3 Cells exploit slower decay of displacements

Experiments [15] have shown that cells whose distance from each other is of the order of 10 cell diameters, form bonds (tethers) consisting of aligned and densely packed matrix fibres, which extend far beyond the cell's protrusion 5.3.

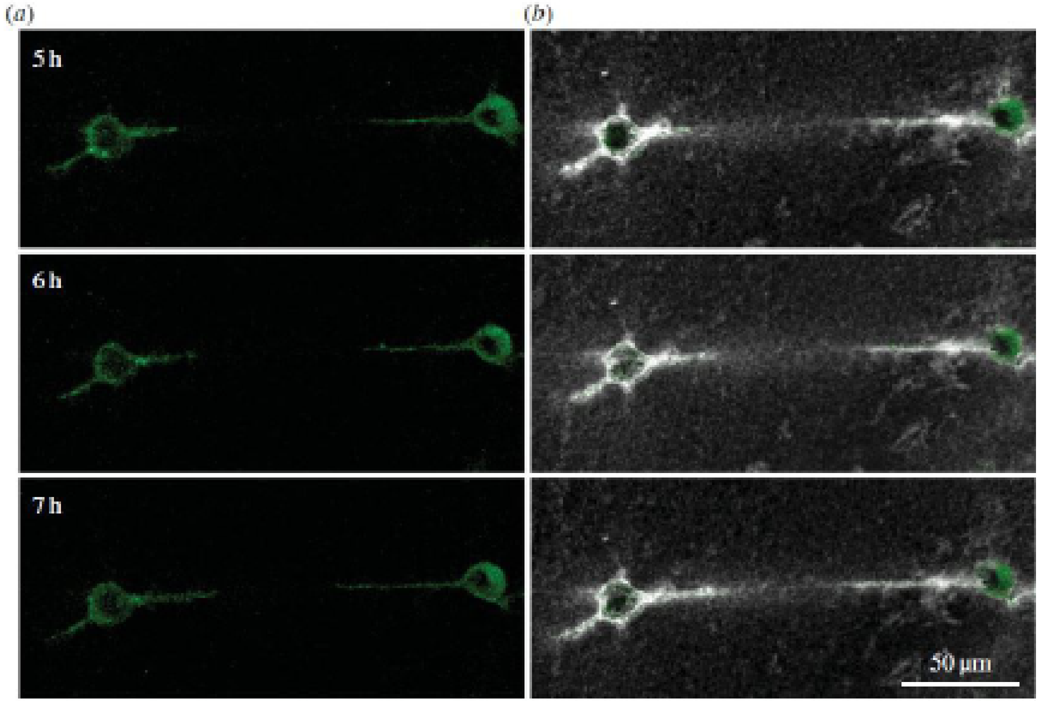


Figure 13: Pairs of cells spread toward one another along tethers.

5.4 Displacements induced by an expanding cell

In Section 4 we model an expanding cell as a disk with radius a, plus the matrix to be the annulus $a < r < A$. The matrix is cosisted of fibers that buckle in compression, so it is composed of compression weakening material. Therefore, we consider that it is governed by our constitutive law that we constructed in Section 2. We find that the radial displacements decay with order $O(r^{-\frac{1}{\xi}})$, where $\xi = \sqrt{\rho}$. Thus, displacements induced by an expanding cell in a matrix composed of compression weakening material, decay faster than in a linear elastic matrix. This finding is in agreement with a numerical calculation in [17]. Specifically, it was observed that when a cell expands, the displacements decay faster in a matrix composed of compression weakening material than in a linear elastic matrix. This provide additional explanation why cells prefer to contrant and not expand.

Let's compare the findings of simulations [15] with our model. We notice that our term now is $\frac{1}{\xi}$.

Comparison of the decay power $\frac{1}{\xi}$ of our model with the decay power n of simulations.		
ρ	$\frac{1}{\xi}$	n
0.03	5.77	5.17
0.3	1.83	1.87
0.1	3.16	3.16

Therefore, we conclude that the predictions of our model about the decay power are generally in agreement with the results of simulations. The displacements induced by an expanding cell in a matrix which consists of fibers that buckle (compression weakening material), decay faster than in a linear elastic matrix. Thus, expanding cells would not facilitate mechanosensing, since there would not be a chance for long ranged cell induced deformations of the matrix, that would form bands (tethers) between cells. This povide additional explanation why cells prefer to contract and not to expand.

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