Curvature in Sub-Riemannian Geometry

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Contents

Chapter 1

Introduction

A Riemannian manifold is a differentiable manifold endowed with an inner product at its tangent bundle. In a Sub-Riemannian manifold, there is a distribution with a fibre inner product. Recall that a distribution is a smooth family of k-planes, i.e., a linear subbundle of the tangent bundle of the manifold. The distribution shall be called the horizontal tangent space and objects tangent to it shall be called horizontal. In a Sub-Riemannian geometry, the distance travelled between two points is defined as in Riemannian geometry but here, we are allowed to travel only along horizontal curves which join the two points. Thus define a finite distance if the Lie bracket generating condition holds.

The awakening of Sub-Riemannian geometry is found in the work of Caratheodory on the Mathematical foundations of Thermodynamics and is related to Carnot's Thermodynamic laws. This is the reason why Gromov for instance, refers to Sub-Riemannian geometry as Carnot-Caratheodory geometry. Caratheodory's theorem is about codimension one distributions. Such a distribution is defined by a single Plaffian equation $\omega = 0$, where ω is a nowhere vanishing 1-form. It is called integrable if through each point there passes a hypersurface which is everywhere tangent to the distribution. Formally, an integrable distribution is involuntive according to Frobenius's theorem; for codimension one distributions, this means that locally, there exist functions λ and f such that $\omega = \lambda df$. In this case, any horizontal path passing through a point p_0 must lie in $S = f^{-1}(f(p_0))$. Consequently, pairs of points p_0 and p'_0 that lie in different hypersurfaces cannot be connected by a horizontal path. Caratheodory's theorem is the converse of this statement.

Theorem 1.1. (C. Caratheodory) Let M be a connected manifold endowed with a real analytic codimension one distribution. If there exist two points that cannot be connected by a horizontal path then the distribution is integrable.

Caratheodory's theorem may be read backwards: if a codimension one distribution is not integrable, then any two points can be connected with a horizontal path. In distributions of arbitrary codimension, this generalises to what is known as Chow's theorem. This theorem is considered the cornerstone of Sub-Riemannian geometry. We first recall Frobenius' integrability theorem: A distribution of codimension k is called integrable if through each point passes a k-dimensional horizontal submanifold. It is called involutive, if for every X and Y horizontal vector fields, the Lie bracket is horizontal. Frobenius theorem asserts that a distribution is integrable if and only if is involutive. We are interested in the opposite extreme of integrability; a bracket generating or completely non-integrable distribution is such that any tangent vector field may be written as the sum of iterated Lie Brackets $[X_1, [[X_2, [X_3, ...]]]$ of horizontal vector fields.

Theorem 1.2. (Chow) For a completely non-integrable distribution on a connected manifold, any two points can be connected by a horizontal path.

It follows that on a connected Sub-Riemannian manifold whose underlying distribution is completely non-integrable, the distance between any two points is finite since there exists at least one horizontal curve joining these two points. Summing up, Sub-Riemannian geometry is a Riemannian geometry together with a constraint on admissible directions of movements. In Riemannian geometry any smoothly embedded curve has locally finite length. In Sub-Riemannian geometry, a curve failing to satisfy the obligation of the constraint has necessarily infinite length.

One of the key tools in Riemannian Geometry is the Levi-Civita connection. It allows us to combine local differential equations in global ones and is useful in most modern descriptions of curvature and geodesics and underlies many computational methods in differential geometry. Taking into account that Sub-Riemannian Geometry is a natural generalization of Riemannian Geometry, we would ask whether there exists a connection in Sub-Riemannian Geometry which has similar properties as Levi-Civita connection. We operate under the assumption that the Sub-Riemannian metric on the horizontal bundle has been extended to a Riemannian metric on the whole space. This allows us to define a vertical bundle. Our first aim is to find a general Sub-Riemannian analogue for the Levi-Civita connection and establish its basic properties. There has been recently made effort to define such geometrically useful connections in Sub-Riemannian geometry. This necessity arose from the fact that the study of Sub-elliptic PDE and Sub-Riemannian manifolds was by necessity local in nature. In order to obtain more global results a new globally defined connection is used. In the special cases of Riemannian and strictly pseudoconvex pseudohermitian manifolds, this connection coincides with the Levi-Civita and the Tanaka-Webster connections respectively. Furthermore, any covariant derivative of any horizontal vector field will be independent of the choice of Riemann extension.

The above discussion brings us to curvature, which is the main subject of this thesis. Due to the close relationship between the curvature with the Levi-Civita connection, the topology of M and the analysis of the Laplacian on M in Riemannian Geometry, the next natural step is to study curvature for the associated connections described above. Some of the symmetries from the Riemannian Geometry are preserved others are not. These symmetries would in turn allow us to study Ricci curvature and the properties associated with it. Again someone would be able to see the importance of these generalizations due to their importance in Riemannian geometry. For the same reason as above, this applies for the gradient, the Hessian and the Laplacian. The importance can also be seen by the applications of Sub-Riemannian geometry in physics such as: The Heisenberg group and the Carnot group in general, the sphere S^3, some holonomic mechanical systems that arise in physics as well as examples of Sub-Riemannian manifolds of constant horizontal curvature and a horizontal Einstein gravitational tensor field.

This thesis is organised as follows:

In Chapter 2 we present the main definitions that will be used throughout the text. In Chapter 3 we define the connections that will be used throughout the text. In Chapter 4 we discuss the notions of Sub-Riemannian curvature and sectional curvature as well as Bianchi Identities. In Chapter 5 we define the Sub-Riemannian Ricci curvature and the notion of horizontal scalar curvature, prove the Contracted Bianchi Identity, define the gradient of a tensor, the Hessian, the horizontal Laplacian and vertical rigidity. Finally, in Chapter 6, we discuss the differences between the Sub-Riemannian connection and the Levi-Civita connection and state a Sub-Riemannian version of the Bonnet-Myers theorem.

This thesis is primarily based in [1], which is considered one of the most recent developments on the subject.

Chapter 2

Sub-Riemannian manifolds: Definitions and Examples

In this Chapter, we start with the definitions in Section 2.1. In Section 2.3 we present some significant examples. as well as the definition of a tensor B (when zero in some texts the manifolds with this property are called nearly Riemannian manifolds) used in order to contract to tensors $C^{(j)}$, which are then defined. Furthermore, we define j-traces, as well as j-normality of a metric extension, strict normality and normality of VM. We then present an example of a 4-dimensional Carnot group and study its properties. We define the interior multiplication and present a formula about $d\omega$, which we prove. We proceed to an example of a strictly pseudoconvex pseudohermitian manifold with a Reeb vector field, where we present the Levi-metric and use a Lemma in order to obtain Tanaka's definition of normal for strictly pseudoconvex pseudohermitian manifolds.

2.1 Definitions

Below we state the necessary definitions that will be used throughout the text:

Definition 2.1. A *Sub-Riemannian manifold* is a smooth manifold M, a smooth constant rank distribution $HM \subset TM$ and a smooth inner product $< \cdot, \cdot >$ on HM, that is the horizontal bundle.

Definition 2.2. A *Sub-Riemannian manifold with Complement*, henceforth *SRC-manifold*, is a Sub-Riemannian manifold together with a smooth bundle VM such that $HM \oplus VM = TM$. The bundle VM is known as the *vertical bundle*.

Remark 2.3. Usually the following Lie bracket generating condition holds for the subbundle $\Delta = HM$ along with the above definitions:
A distribution $\Delta \subseteq TM$ is called bracket generating if any local frame $\{X_1, \dots, X_n\}$ for Δ together with all of its iterated Lie brackets

$$[X_i, X_j], [X_i, [X_j, X_k]], \dots$$

spans TM.

Definition 2.4. Two SRC-manifolds M, N are *SRC-isometric* if there exists a diffeomorphism

$$\pi : M \mapsto N,$$

such that

$$\begin{aligned} \pi_* HM &= HN, \\ \pi_* VM &= VN, \\ < \pi_* X, \pi_* Y >_N &= < X, Y >_M, \end{aligned}$$

for all horizontal vectors X, Y.

Definition 2.5. A SRC-manifold $(M, HM, VM, < \cdot, \cdot >)$ is *r-graded* if there are smooth constant rank bundles $V^{(j)}$, $0 < j \leq r$, such that

$$VM = V^{(1)} \oplus \cdots \oplus V^{(r)}$$

and

$$HM \oplus V^{(j)} \oplus \left[HM, V^{(j)}\right] \subseteq HM \oplus V^{(j)} \oplus V^{(j+1)},$$

for all $0 \leq j \leq r$. Here we have adopted the convention that $V^{(0)} = HM$ and $V^{(k)} = 0$ for $k > r$.

Remark 2.6. 1. This condition above in the Definition 2.5 refers to the sheaves of local sections of the bundle.

2. The above Lie bracket in the Definition 2.5 has meaning only for local vector fields.

Definition 2.7. The grading is *j-regular* if

$$HM \oplus V^{(j)} \oplus \left[HM, V^{(j)}\right] = HM \oplus V^{(j)} \oplus V^{(j+1)}$$

and *equiregular* if it is j-regular for $0 \leq j \leq r$.

Definition 2.8. A *metric extension for an r-graded vertical complement* is a Riemannian metric g of $< \cdot, \cdot >$ that makes the split

$$TM = HM \bigoplus_{1 \leq j \leq r} V^{(j)},$$

orthogonal.

For convenience of notation, we shall denote a section of $V^{(k)}$ by $X^{(k)}$ and set

$$\widehat{V}^{(j)} = \bigoplus_{k \neq j} V^{(k)}.$$

If the metric extension has been chosen, then $\widehat{V}^{(j)} = (V^{(j)})^{\perp}$ is defined to be the orthogonal complement of $V^{(j)}$. For convenience, we shall often also extend the notation $< \cdot, \cdot >$ to the whole tangent space using it interchangeably with g.

Remark 2.9. Every SRC-manifold that admits an r-grading also admits k-gradings for all $1 \leq k < r$. To see this, set

$$\widetilde{V}^{(j)} = V^{(j)},\ 0 \leq j < k,$$
$$\widetilde{V}^{(k)} = \bigoplus_{j \geq k} V^{(j)}.$$

Definition 2.10. The unique 1-grading on each SRC-manifold, $V^{(1)} = VM$ is called the *basic grading*.

2.2 The tensor B

Definition 2.11. If a metric extension g has been chosen, we define

$$B(X,Y,Z) = (\mathscr{L}_Z g)(X,Y) = Zg(X,Y) + g([X,Z],Y) + g([Y,Z],X),$$

for vector fields X, Y, Z.

Remark 2.12. Recall that the definition for a tensor T for vector fields $X_1, \dots, X_i, \dots, X_r$, is that $T(X_1, \dots, X_r)$ is linear in each argument, meaning

$$T(X_1, \dots, fX+hY, \dots, X_r) = fT(X_1, \dots, X, \dots, X_r)+hT(X_1, \dots, Y, \dots, X_r),$$

where f, h are real valued functions on M.

Remark 2.13. We can prove that

$$(\mathscr{L}_Z g)(X,Y) = Zg(X,Y) + g([X,Z],Y) + g([Y,Z],X),$$

Proof. We will use the formula, the proof of which follows the lines in [12],

$$(\mathscr{L}_Z T)(X_1, ..., X_k) = Z(T(X_1, ..., X_k)) - \sum_{i=1}^{k} T(X_1, ..., \mathscr{L}_Z X_i, ..., X_k),$$

where T is a $(0,k)$ tensor and Z, $X_1, ..., X_k$ are vector fields.

For $k = 2$ and $T = g$ we apply the above formula and obtain

$$\begin{aligned} (\mathscr{L}_Z g)(X,Y) &= Z(g(X,Y)) - g(X, \mathscr{L}_Z Y) - g(\mathscr{L}_Z X, Y) \\ &= Z(g(X,Y)) - g(X,[Z,Y]) - g([Z,X],Y) \\ &= Z(g(X,Y)) + g([Y,Z],X) + g([X,Z],Y), \end{aligned}$$

due to the fact that $\mathscr{L}_Z X$ is the Lie Bracket. This is the desired result. $\square$

Remark 2.14. We observe that B is not tensorial in general and so cannot be viewed as a map on vectors rather than vector fields. This arises from the fact that

$$\begin{aligned} &B(fX,hY,Z) = (\mathscr{L}_Z g)(fX,hY) \\ &= Zg(fX,hY) + g([fX,Z],hY) + g([hY,Z],fX) \\ &= Z(fhg(fX,hY)) + hg([fX,Z],Y) + fg([hY,Z],X), \end{aligned}$$

is not equal to

$$fhB(X,Y,Z) = fh(\mathscr{L}_Z g)(X,Y) = fhZg(X,Y) + fhg([X,Z],Y) + fhg([Y,Z],X).$$

However, we can define a symmetric tensor $B^{(j)}$ by setting $B^{(j)}(X,Y,Z) = B(X,Y,Z)$ for $X, Y \in V^{(j)}, Z \in \widehat{V}^{(j)}$ and declaring $B^{(j)}$ to be zero on the orthogonal complement of $V^{(j)} \times V^{(j)} \times \widehat{V}^{(j)}$. We can then contract these to tensors

$$C^{(j)} : TM \times TM \mapsto V^{(j)}$$

defined by

$$g(C^{(j)}(X,Y), Z^{(j)}) = B^{(j)}(X_j, Z^{(j)}, Y_{\widehat{j}}),$$

where X_j is the projection of X to $V^{(j)}$ and $Y_{\widehat{j}}$ the projection to $\widehat{V}^{(j)}$.

Additionally, we can define *j-traces*, by

$$\mathrm{tr}_j B^{(j)}(Z) = \sum_j B^{(j)}(E_i^{(j)}, E_i^{(j)}, Z),$$

where $\{E_i^{(j)}\}$ are (local) orthonormal frames for $V^{(j)}$.

Definition 2.15. Suppose that M is an r-graded SRC-manifold with metric extension g.

- We call the metric extension *j-normal with respect to the grading* if $B^{(j)} \equiv 0$.
- We call the metric extension *strictly normal with respect to the grading* if it is j-normal for all $0 \le j \le r$.
- We shall say VM is normal, if every metric extension and grading is 0-normal.

Remark 2.16. The tensors $B^{(0)}$ and $C^{(0)}$ depend only on the underlying SRC-structure and are independent of the choice of grading and metric.

2.3 Examples

2.3.1 Carnot groups

A Carnot group (of step r) is Lie group, whose Lie algebra $\mathfrak{g}$ is stratified in the sense that

$$\begin{aligned}
&\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{r-1},\\
&[\mathfrak{g}_0, \mathfrak{g}_j] = \mathfrak{g}_{j+1}, \quad j = 1, \ldots, r,\\
&\mathfrak{g}_r = 0,
\end{aligned}$$

together with a left invariant metric $< \cdot, \cdot >$ on $HM = \mathfrak{g}_0$, the left translates of $\mathfrak{g}_0$.

The vertical bundle VM consists of the left-translates of

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{r-1}.$$

In addition to the basic grading, there is then a natural equiregular $(r-1)$-grading defined by setting $V^{(j)}$ to be the left translates of $\mathfrak{g}_j$.

2.3.2 An example from four dimensions

Let M be the 4-dimensional Carnot group with Lie algebra induced by the global left invariant vector fields X, Y, T, S with bracket structures

$$[X, Y] = T, \quad [X, T] = S,$$

and all others being zero. Then

$$\begin{aligned}
B(T, S, X) &= (\mathscr{L}_X g)(T, S)\\
&= Xg(T, S) + g([T, X], S) + g([S, X], T)\\
&= 0 + (-1) + 0\\
&= -1,
\end{aligned}$$

with all others vanishing.

Proof. We have that

$$B^{(j)}(X,Y,Z) = Zg(X,Y) + g([X,Z],Y) + g([Y,Z],X).$$

We observe that the first term in our case is always zero due to the fact that it is of the form $Z(0)$ or $Z(1)$. Therefore

$$B^{(j)}(X,Y,Z) = g([X,Z],Y) + g([Y,Z],X).$$

We also observe that in this case we have

$$B^{(j)}(X,Y,Z) = B^{(j)}(Y,X,Z).$$

We now proceed to the computations. We have

$$\begin{aligned}
B(X,Y,T) &= g([X,T],Y) + g([Y,T],X) = g(S,Y) + g(0,X) = 0+0=0\\
B(X,Y,S) &= g([X,S],Y) + g([Y,S],X) = g(0,Y) + g(0,X) = 0+0=0\\
B(S,T,X) &= g([S,X],T) + g([T,X],S) = g(0,T) + g(-S,S) = 0-1=-1\\
B(S,T,Y) &= g([S,Y],T) + g([T,Y],S) = g(0,T) + g(0,S) = 0+0=0.
\end{aligned}$$

The computation is complete. $\square$

Furthermore

$$[T,X] = -[X,T] = -S, \quad [S,X] = 0.$$

Now M admits an equiregular 2-grading defined by

$$V^{(1)} =< T >, \quad V^{(2)} =< S > .$$

Let g be the metric making the global frame orthonormal. Then g is strictly normal with respect to this 2-grading.

Proof. Using the same reasoning for $B^{(j)}$ as in the last proof we have:

$$\begin{aligned}
B^{(0)}(X,Y,T) &= g([X,T],Y) + g([Y,T],X) = g(S,Y) + g(0,X) = 0+0=0\\
B^{(0)}(X,Y,S) &= g([X,S],Y) + g([Y,S],X) = g(0,Y) + g(0,X) = 0+0=0\\
B^{(1)}(S,S,X) &= g([S,X],T) + g([S,X],S) = 2g(0,T) = 0\\
B^{(1)}(S,S,Y) &= g([S,Y],S) + g([S,Y],S) = 2g(0,S) = 0\\
B^{(2)}(T,T,X) &= g([T,X],T) + g([T,X],T) = 2g(-S,T) = 0\\
B^{(2)}(T,T,Y) &= g([T,Y],T) + g([T,Y],T) = 2g(0,T) = 0.
\end{aligned}$$

To prove equiregularity, by definition, we need to show for $0 \leq j \leq 2$ that

$$HM \oplus V^{(j)} \oplus [HM, V^{(j)}] = HM \oplus V^{(j)} \oplus V^{(j+1)}.$$

For $j = 0$, $V^{(1)} =< T >$ and $V^{(2)} =< S >$ we have $[HM, V^{(0)}] =< T >$ and therefore

$$HM \oplus V^{(0)} \oplus < T >= HM \oplus V^{(0)} \oplus V^{(1)},$$

or equivalently

$$HM \oplus < X, Y > \oplus < T >= HM \oplus < X, Y > \oplus < T > .$$

Therefore it is 0-regular.

For $j = 1$, $V^{(1)} =< T >$ and $V^{(2)} =< S >$ we have $[HM, V^{(1)}] =< S >$ and therefore

$$HM \oplus V^{(1)} \oplus < S >= HM \oplus V^{(1)} \oplus V^{(2)},$$

or equivalently

$$HM \oplus < T > \oplus < S >= HM \oplus < T > \oplus < S > .$$

Therefore it is 1-regular.

For $j = 2$, $V^{(1)} =< T >$ and $V^{(2)} =< S >$ we have $[HM, V^{(2)}] =< 0 >= \{0\}$ and therefore

$$HM \oplus V^{(2)} \oplus < 0 >= HM \oplus V^{(2)} \oplus V^{(3)},$$

or equivalently

$$HM \oplus < S > \oplus \{0\} = HM \oplus < S > \oplus 0.$$

Therefore it is 2-regular, proving equiregularity and strict normality. □

Remark 2.17. The above metric is not 1-regular with respect to the basic grading. If that was the case, we get

$$\tilde{B}^{(0)} \equiv 0,$$

but

$$\tilde{B}^{(1)}(T, S, X) = -1.$$

Thus the metric is 0-normal with respect to the basic grading.

Proof. By taking into consideration the above proofs, we have

$$\begin{aligned}
\widetilde{B}^{(0)}(X,Y,T) &= g([X,T],Y)+g([Y,T],X)=g(S,Y)+g(0,X)=0+0=0\\
\widetilde{B}^{(0)}(X,Y,S) &= g([X,S],Y)+g([Y,S],X)=g(0,Y)+g(0,X)=0+0=0\\
\widetilde{B}^{(0)}(Y,X,T) &= g([Y,T],X)+g([X,T],Y)=g(0,Y)+g(S,X)=0+0=0\\
\widetilde{B}^{(0)}(Y,X,S) &= g([Y,S],X)+g([X,S],Y)=g(0,Y)+g(0,X)=0+0=0,
\end{aligned}$$

and

$$\widetilde{B}^{(1)}(T,S,X)=Tg(S,X)+g([T,X],S)+g([S,X],T)=0+g(-S,S)+g(0,T)=-1.$$

Therefore the calculation is complete. □

Remark 2.18. Any step r Carnot group with a bi-invariant metric extension is strictly normal with respect to the equiregular $(r-1)$-grading, but is only 0-normal with respect to the basic grading.

2.3.3 Example of a strictly pseudoconvex pseudohermitian manifold

In order to proceed with the example will have to deal with some preliminaries

Definition 2.19. In this text ι is the *interior multiplication* or *contraction operator*, thus

$$\iota_T\omega(\cdot)=\omega(T,\cdot).$$

Remark 2.20. It is well known that:
If $\omega\in\Omega^1 M$ and $X,Y\in\mathscr{X}(M)$, then

$$d\omega(X,Y)=X(\omega(Y))-Y(\omega(X))-\omega([X,Y]).$$

Proof. A proof is presented in [17] in the Bibliography (page 24 Corollary 3.4). □

Example 2.21. Let (M,J,ω) be a strictly pseudoconvex pseudohermitian manifold, (see [6]) with characteristic vector field T (called Reeb vector field) such that $\omega(T)=1$, $\iota_T d\omega=0$ (in other words $T\in\ker d\omega$).

Due to Darboux's theorem T is unique up to change of coordinates.

The horizontal bundle HM is defined to be the kernel of the 1-form ω or in other words

$$\ker \omega := HM.$$

An immediate consequence of the above formula for $d\omega$ and the defining properties of T is that $[T, HM] \subset HM$. When J is extended to TM by defining $JT = 0$, the Levi metric

$$g(A, B) = d\omega(A, JB) + \omega(A)\omega(B),$$

can be viewed as an extension of the Sub-Riemannian metric

$$< X, Y >= d\omega(X, JY),$$

with

$$VM =< T > .$$

As VM is one dimensional, the basic grading is the only grading admitted and since $[T, HM] \subset HM$ we see $B^{(1)} = 0$ trivially. Thus the Levi metric is always 1-normal and so strict normality is equivalent to 0-normality.

Lemma 2.22. The following equalities hold true in this example:

$$\begin{aligned}
&< [T, X], Y >= - < [[T, X], JY], T >,\\
&- < [[T, X], JY], T >=< [[X, JY], T], T > + < [[JY, T], X], T >,\\
&< [[X, JY], T], T > + < [[JY, T], X], T >= T < X, Y > + < [JY, T], JX > .
\end{aligned}$$

Proof. As for our first equality we have

$$\begin{aligned}
< [T, X], Y > &= d\omega([T, X], JY) + \omega([T, X])\omega(Y)\\
&= d\omega([T, X], JY) + 0\\
&= [T, X]\omega(JY) - JY\omega([T, X]) - \omega([[T, X], JY]])\\
&= -\omega([[T, X], JY]).
\end{aligned}$$

and

$$\begin{aligned}
< [[T, X], JY], T > &= d\omega([T, X], JY], JT) + \omega([[T, X], JY])\omega(T)\\
&= \omega([[T, X], JY]).
\end{aligned}$$

The second equation is obtained by applying the Jacobi's Identity

$$[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0.$$

The third equality can be obtained if we take into account the two previous equalities together with the fact that

$$B(X,Y,T)=Tg(X,Y)+g([X,T],Y)+g([Y,T],X)=0,$$

or, in other words

$$T<X,Y>+<[X,T],Y>+<[Y,T],X>=0,$$

because

$$\begin{aligned}
&0=T<X,Y>+<[X,T],Y>+<[Y,T],X>\\
&=T(d\omega(X,JY)+\omega(X)\omega(Y))+(d\omega([X,T],JY)+\omega([X,T])\omega(Y))\\
&+(d\omega([Y,T],JX)+\omega([Y,T])\omega(JX))\\
&=T(d\omega(X,JY)+0)+(d\omega([X,T],JY)+0)+(d\omega([Y,T],JX)+0)\\
&=T(X\omega(JY)-JY\omega(X)-\omega([X,JY]))\\
&+([X.T]\omega(JY)-JY\omega([X,T])-\omega([[X,T],JY]))\\
&+([Y,T]\omega(JX)-JX\omega([Y,T])-\omega([[Y,T],JX]))\\
&=(0-0-0)+(0-0-0)+(0-0-0)\\
&=0.
\end{aligned}$$

If the above are combined with the fact that

$$<[JY,T],JX>=<[-X,T],Y>=<[T,X],Y>,$$

and with Corollary 2.20 we get

$$\begin{aligned}
<X,Y> &= d\omega(X,JY)+\omega(x)\omega(Y)=d\omega(X,JY)+0\\
&= d\omega(X,JY)=X\omega(JY)-JY\omega(X)-\omega([X,JY])\\
&= X\omega(-X)-JY\omega(X)-\omega([X,-X])\\
&= 0.
\end{aligned}$$

$\square$

Combining all the above equations we have

$$\begin{aligned} <[T,X],Y> &= -<[[T,X],JY],T> \\ &= <[[X,JY],T],T> + <[[JY,T],X],T]> \\ &= T<X,Y> + <[JY,T],JX> . \end{aligned}$$

This implies that 0-normality is equivalent to

$$<[T,X],Y> = -<[T,JY],JX> .$$

Since we also have

$$<[T,X],Y> = -<J[Y,T],JX>,$$

we obtain that this is equivalent to

$$[T,JY] = J[T,Y],$$

which is Tanaka's definition of normal for a strictly pseudoconvex pseudohermitian manifold. For more see in [8].

Chapter 3

Connections on SRC-manifolds

In Chapter 3, first we prove the existence of a unique connection for an r-graded SRC-manifold with some desired traits. After this, an example of global orthonormal frames of the horizontal and vertical bundle is presented along some formulas. We then continue our work from the example 4-dimensional Carnot group from Chapter 2 and utilize the above formulas in order to obtain results. We present an example of a flat, equiregular, strictly normal SRC-manifold. We return to the example of a strictly pseudoconvex pseudohermitian manifold from Chapter 2, present the Tanaka-Webster connection and prove the torsion symmetry of the connection.

The tensors $C^{(j)}$ provide the essential ingredient for the definition of our connections, which we are to follow. The idea boils down to using the Levi-Civita connection on each component $V^{(j)}$ and using projections of the Lie derivative mixed components. In general, this will not produce a connection compatible with the metric, but we can use the tensors $C^{(j)}$ to adjust it appropriately.

Lemma 3.1. If g is a metric extension of an r-graded SRC-manifold, then there exists a unique connection $\nabla^{(r)}$ on TM such that

- $\nabla^{(r)}$ is compatible with the metric.
- $\widehat{V}^{(j)}$ is parallel for all j.

- Let $\mathrm{Tor}^{(r)}$ be the torsion associated to $\nabla^{(r)}$:

  $$\mathrm{Tor}^{(r)}(A,B) = \nabla^{(r)}_A B - \nabla^{(r)}_B A - [A,B].$$

  Then $\mathrm{Tor}^{(r)}(V^{(j)}, V^{(j)}) \subseteq \widehat{V}^{(j)}$, for all j.

- $< \mathrm{Tor}^{(r)}(X^{(j)}, Y^{(k)}), Z^{(j)} > = < \mathrm{Tor}^{(r)}(Z^{(j)}, Y^{(k)}), X^{(j)} >$ for all j, k.

Furthermore, if X, Y are horizontal vector fields, then

$$\nabla^{(r)} X \text{ and } \mathrm{Tor}^{(r)}(X,Y),$$

are independent of the choice of grading and extension g. (They do however depend on choice of VM.)

Proof. For a vector field Z, we denote the orthogonal projections of Z to $V^{(j)}$ by Z_j. Define a connection $\nabla^{(r)}$ as follows: for X, Y, Z sections of $V^{(j)}$ and T a section of $\widehat{V}^{(j)}$ set:

a)
$$< \nabla^{(r)}_X Y, \widehat{V}^{(j)} > = 0,$$

for X, Y sections of $V^{(j)}$.

b) We use Koszul's Formula to define $\nabla^{(r)}$

$$\begin{aligned} 2 < \nabla^{(r)}_X Y, Z > &= X < Y, Z > + Y < Z, X > - Z < X, Y > \\ &\quad - < X, [Y,Z] > - < Y, [X,Z] > + < Z, [X,Y] >, \end{aligned}$$

for X, Y, Z sections of $V^{(j)}$.

c) Finally
$$\nabla^{(r)}_T Y = [T, Y]_j + \frac{1}{2} C^{(j)}(Y, T),$$

for X, Y, Z sections of $V^{(j)}$ and T section of $\widehat{V}^{(j)}$.

(Recall that $HM = V^{(0)}$.)

We will prove that this defines a connection with the desired properties.

For the first and second assertion we will be keeping in mind the work done in [13] (pages 49-56) and will make use of the following formula (found in page 51)

$$\frac{DV^{(j)}}{dt} = \sum_j \frac{d\upsilon_j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} \upsilon_j \nabla_{X_i} X_j.$$

For the first assertion we take into account that for $j \neq i$ we have

$$0 = \frac{d}{dt} < V^{(j)}, V^{(k)} > = < \frac{DV^{(j)}}{dt}, V^{(k)} > + < V^{(j)}, \frac{DV^{(k)}}{dt} > .$$

We also recall that

$$< \nabla^{(r)}_{X_i} X_j, \widehat{V}^{(j)} > = 0,$$

due to the first condition from part a) of the definition of the connection.

In addition, we take into account

$$< X_j, \widehat{V}^{(j)} > = 0,$$

due to the definition of the metric in Definition 2.1, Chapter 2.

Therefore, we have

$$< \frac{DV^{(j)}}{dt}, \widehat{V}^{(j)} > = 0,$$

for every j.

Furthermore, for $k = j$ we have from the Koszul formula (part b) of the definition) that

$$\begin{aligned}
2 < \nabla^{(r)}_X Y, Z > &= X < Y, Z > + Y < Z, X > - Z < X, Y > \\
&- < X, [Y, Z] > - < Y, [X, Z] > + < Z, [X, Y] > \\
2 < \nabla^{(r)}_X Z, Y > &= X < Z, Y > + Z < Y, X > - Y < X, Z > \\
&- < X, [Z, Y] > - < Z, [X, Y] > + < Y, [X, Z] >
\end{aligned}$$

and if we add these two equations together we get

$$X < Y, Z > = < \nabla^{(r)}_X Y, Z > + < \nabla^{(r)}_X Z, Y >$$

and therefore we have proved metric compatibility for every j and the second assertion.

For the third assertion for $X, Y, Z \in V^{(j)}$, using the Koszul formula we have that

$$\begin{aligned}
&2 < \mathrm{Tor}(X,Y), Z >= 2 < \nabla^{(r)}_X Y - \nabla^{(r)}_Y X - [X,Y], Z > \\
&= 2 < \nabla^{(r)}_X Y, Z > -2 < \nabla^{(r)}_Y X, Z > -2 < [X,Y], Z > \\
&= X < Y, Z > +Y < Z, X > -Z < X, Y > \\
&- < X, [Y,Z] > - < Y, [X,Z] > + < Z, [X,Y] > \\
&-(X < Z, Y > +Z < Y, X > -Y < X, Z > \\
&- < X, [Z,Y] > - < Z, [X,Y] > + < Y, [X,Z] >) \\
&-2 < [X,Y], Z > \\
&= 0,
\end{aligned}$$

due to the fact that all terms cancel each other out.

This happens for every $Z \in V^{(j)}$, therefore

$$\mathrm{Tor}^{(r)}(V^{(j)}, V^{(j)}) \subseteq \widehat{V}^{(j)}.$$

For the last assertion, we notice that for $k = j$ we have that both sides are equal to zero, due to the third assertion. For $k \neq j$ we have using the third equation in the definition and the definition of $C^{(j)}$ found in Section 2.2 of Chapter 2 that

$$\begin{aligned}
&< \mathrm{Tor}^{(r)}(X^{(j)}, Y^{(k)}), Z^{(j)} >=< \nabla^{(r)}_{X^{(j)}} Y^{(k)} - \nabla^{(r)}_{Y^{(k)}} X^{(j)} - [X^{(j)}, Y^{(k)}], Z^{(j)} > \\
&=< \nabla^{(r)}_{X^{(j)}} Y^{(k)}, Z^{(j)} > - < \nabla^{(r)}_{Y^{(k)}} X^{(j)}, Z^{(j)} > - < [X^{(j)}, Y^{(k)}], Z^{(j)} > \\
&=< [X^{(j)}, Y^{(k)}]_k, Z^{(j)} > +\frac{1}{2} < C^{(k)}(Y^{(k)}, X^{(j)}), Z^{(j)} > \\
&- < [Y^{(k)}, X^{(j)}]_j, Z^{(j)} > -\frac{1}{2} < C^{(j)}(X^{(j)}, Y^{(k)}), Z^{(j)} > - < [X^{(j)}, Y^{(k)}], Z^{(j)} > \\
&= 0 + \frac{1}{2}(B^{(k)}(Y^{(k)}_k, Z^{(j)}_k, X^{(j)}_{\widehat{k}})) - \frac{1}{2}(B^{(j)}(X^{(j)}_j, Z^{(j)}_j, Y^{(k)}_{\widehat{j}})) \\
&= \frac{1}{2}(B^{(k)}(Y^{(k)}_k, Z^{(j)}_k, X^{(j)}_{\widehat{k}})) - \frac{1}{2}(B^{(j)}(X^{(j)}_j, Z^{(j)}_j, Y^{(k)}_{\widehat{j}})) \\
&= 0 - \frac{1}{2}(B^{(j)}(X^{(j)}_j, Z^{(j)}_j, Y^{(k)}_{\widehat{j}})) \\
&= -\frac{1}{2}(Y^{(k)} < X^{(j)}, Z^{(j)} > + < [X^{(j)}, Y^{(k)}], Z^{(j)} > + < [Z^{(j)}, Y^{(k)}], X^{(j)} >,
\end{aligned}$$

as well as

$$
\begin{aligned}
&< \mathrm{Tor}^{(r)}(Z^{(j)}, Y^{(k)}), X^{(j)} > =< \nabla^{(r)}_{Z^{(j)}} Y^{(k)} - \nabla^{(r)}_{Y^{(k)}} Z^{(j)} - [Z^{(j)}, Y^{(k)}], X^{(j)} > \\
&=< \nabla^{(r)}_{Z^{(j)}} Y^{(k)}, X^{(j)} > - < \nabla^{(r)}_{Y^{(k)}} Z^{(j)}, X^{(j)} > - < [Z^{(j)}, Y^{(k)}], X^{(j)} > \\
&=< [Z^{(j)}, Y^{(k)}]_k, X^{(j)} > + \frac{1}{2} < C^{(k)}(Y^{(k)}, Z^{(j)}), X^{(j)} > \\
&- < [Y^{(k)}, Z^{(j)}]_j, X^{(j)} > - \frac{1}{2} < C^{(j)}(Z^{(j)}, Y^{(k)}), X^{(j)} > - < [Z^{(j)}, Y^{(k)}], X^{(j)} > \\
&= 0 + \frac{1}{2}(B^{(k)}(Y^{(k)}_k, X^{(j)}_k, Z^{(j)}_{\widehat{k}})) - \frac{1}{2}(B^{(j)}(Z^{(j)}_j, X^{(j)}_j, Y^{(k)}_{\widehat{j}})) \\
&= \frac{1}{2}(B^{(k)}(Y^{(k)}_k, X^{(j)}_k, Z^{(j)}_{\widehat{k}})) - \frac{1}{2}(B^{(j)}(Z^{(j)}_j, X^{(j)}_j, Y^{(k)}_{\widehat{j}})) \\
&= 0 - \frac{1}{2}(B^{(j)}(Z^{(j)}_j, X^{(j)}_j, Y^{(k)}_{\widehat{j}})) \\
&= -\frac{1}{2}(Y^{(k)} < Z^{(j)}, X^{(j)} > + < [Z^{(j)}, Y^{(k)}], X^{(j)} > + < [X^{(j)}, Y^{(k)}], Z^{(j)} >).
\end{aligned}
$$

Therefore all the assertions were proved.

For the uniqueness, suppose that connections ∇ and ∇' satisfy the required properties and set

$$A(Z, W) = \nabla_W Z - \nabla'_W Z.$$

Then for sections X, Y, Z of $V^{(j)}$, since the torsion terms are in $\widehat{V}^{(j)}$ we

$$
\begin{aligned}
< A(X, Y), Z > &=< \nabla_X Y - \nabla'_X Y, Z > \\
&= \frac{1}{2}(X < Y, Z > + Y < Z, X > - Z < X, Y > \\
&- < X, [Y, Z] > - < Y, [X, Z] > + < Z, [X, Y] >) \\
&- \frac{1}{2}(X < Y, Z > + Y < Z, X > - Z < X, Y > \\
&- < X, [Y, Z] > - < Y, [X, Z] > + < Z, [X, Y] >) = 0
\end{aligned}
$$

and using

$$\nabla_T Y = [T, Y]_j + \frac{1}{2} C^{(j)}(Y, T),$$

we have

$$\begin{aligned}
&< A(T,X), Y >=< \nabla_T X - \nabla'_T X > \\
&=< [T,X]_j + \frac{1}{2}C^{(j)}(Y,T) - ([T,X]_j + \frac{1}{2}C^{(j)}(Y,T)), Y > \\
&= 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
< A(X,Y), Z > &= - < Y, A(X,Z) > \\
&=< Y, A(Z,X) > \\
&=< A(Z,Y), X > \\
&=< A(Y,Z), X > \\
&= - < Z, A(X,Y) > \\
&(= 0).
\end{aligned}$$

Similarly if T is a section of $\widehat{V}^{(j)}$,

$$\begin{aligned}
< A(T,X), Y > &= - < X, A(T,Y) > \\
&= - < X, \mathrm{Tor}(T,Y) - \mathrm{Tor}'(T,Y) > \\
&=< \mathrm{Tor}(T,X) - \mathrm{Tor}'(T,X), Y > \\
&=< A(Y,Z), X > \\
&= - < A(T,X), Y > \\
&(= 0).
\end{aligned}$$

Thus $A = 0$, and the connection ∇ is the unique connection with the desired properties. The required independence from g follows from Koszul Formula and the independence of B and $C^{(0)}$ from the metric extension. □

Remark 3.2. Furthermore if $\overline{\nabla}$ is the Levi-Civita connection for g, then for sections X, Y of $V^{(j)}$, we have

$$\nabla^{(r)}_X Y = (\overline{\nabla}_X Y)_{(j)}.$$

Remark 3.3. An r-grading induces a family of connections $\nabla^{(1)}, ..., \nabla^{(r)}$ associated to each possible sub-grading. Each of these connections agrees with the others in the sense that

$$\nabla^{(j)} X^{(k)} = \nabla^{(r)} X^{(k)}$$

whenever $1 \leq k < j$. In particular, for horizontal vector fields this means

$$\nabla^{(1)} X = \nabla^{(2)} X = ... = \nabla^{(r)} X.$$

Thus the differences between the connections can be viewed as a choice in how to differentiate vertical vectors. The connection $\nabla^{(1)}$ associated to the basic grading is referred to as the basic connection. We shall denote the basic connection by ∇.

Corollary 3.4. If M admits an r-grading, then

1. $\mathrm{Tor}^{(r)}(V^{(j)}, V^{(j)}) = 0$ if and only if $V^{(j)}$ is integrable,
2. $\mathrm{Tor}^{(r)}(HM, V^{(j)}) \subset HM \oplus V^{(j)} \oplus V^{(j+1)}$ for all j.

If the r-grading is j-normal then

$$\mathrm{Tor}^{(r)}(TM, V^{(j)}) \subseteq \widehat{V}^{(j+1)}.$$

If the r-grading is 0-normal and j-normal then

$$\mathrm{Tor}^{(r)}(HM, V^{(j)}) \subseteq V^{(j+1)},$$

with the equality holding if and only if the grading is j-regular.

Proof. For the first part we recall from [13] in the Bibliography (pages 50-51) the following formulas

$$\begin{aligned}
\nabla_X Y &= \sum_k (\sum_{i,j} x_i y_j \Gamma^k_{ij} + X(y_k)) X_k, \\
\nabla_Y X &= \sum_q (\sum_{i,j} x_i y_j \Gamma^q_{ji} + X(y_i)) X_q.
\end{aligned}$$

From this and the fact that the torsion is zero, we have

$$[X, Y] = [\sum_i x_i X_i, \sum_j y_j Y_j] = \sum_{i,j} x_i y_j [X_i, X_j],$$
$$[X, Y] = \nabla_X Y - \nabla_Y X = \sum_k c_k X_k,$$

where
$$c_k = (\sum_{i,j} x_i y_j \Gamma^k_{ij} + X(y_k)) - (\sum_{i,j} x_i y_j \Gamma^q_{ji} + X(y_i)).$$
Therefore we have an involution and by Frobenius Theorem found in [6] in the Bibliography (Specifically, found in page 159) we have that $V^{(j)}$ is integrable.

For the second part we have from the part c) of the definition of the connection and the grading, for Z, Q arbitrary that

$$\begin{aligned}
< \nabla_X Y, Z > &= < [X,Y]_j + \frac{1}{2}C^{(j)}(Y,X), Z > \\
&= < [X,Y]_j, Z > + < \frac{1}{2}C^{(j)}(Y,X), Z > \\
&:= < [X,Y]_j, Z > + \frac{1}{2}B^{(j)}(Y_j, Z_j, X_{\widehat{j}}) \\
&= 0 + \frac{1}{2}B^{(j)}(Y_j, Z_j, X_{\widehat{j}}) \\
&= \frac{1}{2}B^{(j)}(Y_j, Z_j, X_{\widehat{j}}), \\
< \nabla_Y X, Q > &= < [Y,X]_h + \frac{1}{2}C^{(h)}(X,Y), Q > \\
&= < [Y,X]_h, Q > + < \frac{1}{2}C^{(h)}(X,Y), Q > \\
&:= < [Y,X]_h, Q > + \frac{1}{2}B^{(h)}(X_h, Q_h, Y_{\widehat{h}}) \\
&= 0 + \frac{1}{2}B^{(h)}(X_h, Q_h, Y_{\widehat{h}}) \\
&= \frac{1}{2}B^{(h)}(X_h, Q_h, Y_{\widehat{h}}),
\end{aligned}$$
$$[X,Y] \subset [HM, V^{(j)}] \subset V^{(j+1)},$$
therefore we have that (corresponding to each equation above)
$$\begin{aligned}
&Z \in \widehat{V}^{(j)}, \\
&Q \in \widehat{V}^{(0)} := \widehat{HM}, \\
&[X,Y] \subset [HM, V^{(j)}] \subset V^{(j+1)},
\end{aligned}$$
for every j and therefore we have
$$\mathrm{Tor}^{(r)}(HM, V^{(j)}) = HM \oplus V^{(j)} \oplus [HM, V^{(j)}] \subset HM \oplus V^{(j)} \oplus V^{(j+1)},$$

for every j. Therefore the second part was proved.

If the r-grading is j-normal then

$$0 = B^{(j)}(X,Y,Z) = B(X_j, Y_j, Z_{\widehat{j}}) = Z_{\widehat{j}} g(X_j, Y_j) + g([X_j, Z_{\widehat{j}}], Y_j) + g([Y_j, Z_{\widehat{j}}], X_j).$$

Therefore for $Z \in V^{(j)}$ and

$$\begin{aligned} &< \mathrm{Tor}^{(r)}(X,Y), Z > = < \nabla_X Y - \nabla_Y X - [X,Y], Z > \\ &= < \nabla_X Y, Z > - < \nabla_Y X, Z > - < [X,Y], Z > . \end{aligned}$$

We have two cases.

The first case is $X \in \widehat{V}^{(j)}$ and $Y \in V^{(j)}$.

If we use the last formula from the definition of the connection we have

$$\begin{aligned} < \nabla_X Y, Z > &= < [X,Y]_j + \frac{1}{2} C^{(j)}(Y,X), Z > \\ &= < [X,Y]_j, Z > + < \frac{1}{2} C^{(j)}(Y,X), Z > \\ &= < [X,Y]_j, Z > + \frac{1}{2} B^{(j)}(Y,Z,X) \\ &= < [X,Y]_j, Z > + 0 \\ &= < [X,Y]_j, Z >, \\ < \nabla_Y X, Z > &= < [Y,X]_{\widehat{j}} + \frac{1}{2} C^{(\widehat{j})}(X,Y), Z > \\ &= < [Y,X]_{\widehat{j}}, Z > + < \frac{1}{2} C^{(\widehat{j})}(X,Y), Z > \\ &= < [Y,X]_{\widehat{j}}, Z > + \frac{1}{2} B^{(\widehat{j})}(X,Z,Y) \\ &= < [Y,X]_{\widehat{j}}, Z > + 0 \\ &= < [Y,X]_{\widehat{j}}, Z >, \\ < [X,Y], Z > &= < [X,Y]_j, Z > + < [X,Y]_{\widehat{j}}, Z > . \end{aligned}$$

The second case is $X, Y \in V^{(j)}$.

We have from the first Lemma of this section that

$$\mathrm{Tor}^{(r)}(V^{(j)}, V^{(j)}) \subseteq \widehat{V}^{(j)},$$

which is our case and therefore proved above.

If the r-grading is 0-normal and j-normal we have

$$0 = B^{(0)}(X,Y,Z) = B(X_h, Y_h, Z_{\widehat{h}}) = Z_{\widehat{h}} g(X_h, Y_h) + g([X_h, Z_{\widehat{h}}], Y_h) + g([Y_h, Z_{\widehat{h}}], X_h),$$
$$0 = B^{(j)}(X,Y,Z) = B(X_j, Y_j, Z_{\widehat{j}}) = Z_{\widehat{j}} g(X_j, Y_j) + g([X_j, Z_{\widehat{j}}], Y_j) + g([Y_h, Z_{\widehat{j}}], X_j).$$

and

$$< \mathrm{Tor}^{(r)}(X,Y), Z > = < \nabla_X Y, Z > - < \nabla_Y X, Z > - < [X,Y], Z >,$$

as above.

For $Z \in \widehat{V}^{(j+1)}$, $X \in V^{(0)} := HM$ and $Y \in V^{(j)}$ we have from the last equation of the definition of the connection and the first equation of the same definition and the grading that

$$\begin{aligned}
< \nabla_X Y, Z > &= < [X,Y]_j + \frac{1}{2} C^{(j)}(Y,X), Z > \\
&= < [X,Y]_j, Z > + < \frac{1}{2} C^{(j)}(Y,X), Z > \\
&= < [X,Y]_j, Z > + \frac{1}{2} B^{(j)}(Y,Z,X) \\
&= < [X,Y]_j, Z > + 0 \\
&= < [X,Y]_j, Z > \\
&= 0, \\
< \nabla_Y X, Z > &= < [Y,X]_h + \frac{1}{2} C^{(h)}(X,Y), Z > \\
&= < [Y,X]_h, Z > + < \frac{1}{2} C^{(h)}(X,Y), Z > \\
&= < [Y,X]_h, Z > + \frac{1}{2} B^{(h)}(X,Y,Z) \\
&= < [Y,X]_h, Z > + 0 \\
&= < [Y,X]_h, Z > \\
&= 0, \\
< [X,Y], Z > &:= < [X,Y]_{j+1}, Z > + < [X,Y]_{\widehat{j+1}}, Z > = 0.
\end{aligned}$$

Therefore we have shown

$$\mathrm{Tor}^{(r)}(HM, V^{(j)}) \subseteq V^{(j+1)}.$$

We recall that j-regularity means that for all $0 \leq j \leq r$ we have

$$HM \oplus V^{(j)} \oplus [HM, V^{(j)}] = HM \oplus V^{(j)} \oplus V^{(j+1)}.$$

Therefore if $[HM, V^{(j)}] = V^{(j+1)}$ and repeat the above process we obtain

$$\mathrm{Tor}^{(r)}(HM, V^{(j)}) = V^{(j+1)},$$

which was the desired result. $\square$

Example 3.5. Suppose that HM has a global orthonormal frame $\{X_i\}$ and VM has global orthonormal frame $\{T_\beta\}$ with the following bracket identities:

$$\begin{aligned} [X_i, X_j] &= c_{ij}^k X_k + c_{ij}^\alpha T_\alpha, \\ [X_i, T_\beta] &= c_{i\beta}^k X_k + c_{i\beta}^\alpha T_\alpha, \\ [T_\gamma, T_\beta] &= c_{\gamma\beta}^k X_k + c_{\gamma\beta}^\alpha T_\alpha. \end{aligned}$$

Remark 3.6. This is always the case locally.

Then using the basic grading and connection we have

a) VM is normal if and only if $c_{i\beta}^k = -c_{k\beta}^i$.

b) g is strictly normal if and only if $c_{i\beta}^\alpha = -c_{i\alpha}^\beta$ and $c_{i\beta}^k = -c_{k\beta}^i$.

c) g is vertically rigid if and only if $\sum c_{i\beta}^\beta = 0$.

And if a), b), c) hold then we have

d) $\nabla_{X_i} X_j = \frac{1}{2}(c_{ij}^k + c_{ki}^j + c_{kj}^i)X_k, \quad \mathrm{Tor}(X_i, X_j) = -c_{ij}^\alpha T_\alpha,$

e) $\nabla_{T_\beta} X_j = \frac{1}{2}(c_{k\beta}^j - c_{j\beta}^k)X_k,$

f) $\nabla_{X_j} T_\beta = \frac{1}{2}(c_{j\beta}^\alpha - c_{j\alpha}^\beta)T_\alpha,$

g) $\nabla_{T_\gamma} T_\beta = \frac{1}{2}(c_{\gamma\beta}^\alpha + c_{\alpha\gamma}^\beta + c_{\alpha\beta}^\gamma)T_\alpha, \qquad \mathrm{Tor}(T_\gamma, T_\beta) = -c_{\gamma\beta}^k X_k,$

h) $\mathrm{Tor}(X_j, T_\beta) = -\frac{1}{2}(c_{k\beta}^j + c_{j\beta}^k)X_k - \frac{1}{2}(c_{j\beta}^\alpha + c_{j\alpha}^\beta)T_\alpha.$

Proof. We begin by observing that

$$[X_i, X_j] = c_{ij}^k X_k + c_{ij}^\alpha T_\alpha,$$
$$[X_j, X_i] = c_{ji}^k X_k + c_{ji}^\alpha T_\alpha,$$

If we add these equations we get, due to the cancelation of Lie brackets:

$$c_{ij}^k = c_{ji}^k,$$
$$c_{ij}^\alpha = c_{ji}^\alpha.$$

Furthermore, we observe

$$[T_\gamma, T_\beta] = c_{\gamma\beta}^k X_k + c_{\gamma\beta}^\alpha T_\alpha,$$
$$[T_\beta, T_\gamma] = c_{\beta\gamma}^k X_k + c_{\beta\gamma}^\alpha T_\alpha.$$

and therefore

$$c_{\gamma\beta}^k = c_{\gamma\beta}^\alpha,$$
$$c_{\beta\gamma}^\alpha = c_{\beta\gamma}^\alpha.$$

We will use the aforementioned relations below.

For part a) of the example we have that the normality of VM is equivalent to $B^{(0)}(X_i, X_j, T_\beta) = 0$. We have

$$\begin{aligned} 0 &= B^{(0)}(X_i, X_j, T_\beta) \\ &= T_\beta g(X_i, X_j) + g([X_i, T_\beta], X_j) + g([X_j, T_\beta], X_i) \\ &= 0 + g(c_{i\beta}^k X_k + c_{i\beta}^\alpha T_\alpha, X_j) + g(c_{j\beta}^r X_r + c_{j\beta}^m T_m, X_i) \\ &= c_{i\beta}^k g(X_k, X_j) + c_{j\beta}^r g(X_r, X_i) \\ &= c_{i\beta}^k \delta_j^k + c_{j\beta}^r \delta_i^r, \end{aligned}$$

where δ is the Kronecker delta.
If we sum over k and r the last equation is equivalent to

$$c_{i\beta}^k + c_{j\beta}^i = 0,$$

where $k = j$ and $r = i$. Therefore it is equivalent to

$$c_{i\beta}^k + c_{k\beta}^i = 0,$$

which is what was required.
For part b) of the example we have that g is strictly normal if and only if

$$B^{(0)}(X_i, X_j, T_\beta) = 0$$

and

$$B^{(1)}(T_\beta, T_\gamma, X_i) = 0.$$

The second condition of part b) can be obtained from part a). For the other condition we have

$$\begin{aligned}
0 &= B^{(1)}(T_\beta, T_\gamma, X_i) \\
&= X_i g(T_\beta, T_\gamma) + g([T_\beta, X_i], T_\gamma) + g([T_\gamma, X_i], T_\beta) \\
&= 0 - g(c^k_{i\beta} X_k + c^\alpha_{i\beta} T_\alpha, T_\gamma) - g(c^r_{i\gamma} X_r + c^m_{i\gamma} T_m, T_\beta) \\
&= -c^\alpha_{i\beta} g(T_\alpha, T_\gamma) - c^m_{i\gamma} g(T_m, T_\beta) \\
&= -c^\alpha_{i\beta} \delta^\alpha_\gamma - c^m_{i\gamma} \delta^m_\beta,
\end{aligned}$$

where δ is again the Kronecker delta.
If we sum over α and m the last equation is equivalent to

$$c^\alpha_{i\beta} + c^m_{i\gamma} = 0,$$

where $\gamma = \alpha$ and $m = \beta$.
Therefore it is equivalent to

$$c^\alpha_{i\beta} + c^\beta_{i\alpha} = 0,$$

which is what was required.

For part c) of the example we have that g is vertically rigid if and only if

$$\sum_{i,\beta} (B^{(1)}(T_\beta, T_\beta, X_i)) = 0.$$

We have

$$\begin{aligned}
B^{(1)}(T_\beta, T_\beta, X_i) &= 0 + g([T_\beta, X_i], T_\beta) + g([T_\beta, X_i], T_\beta) \\
&= 2g([T_\beta, X_i], T_\beta) \\
&= -2g(c^k_{i\beta} X_k + c^\alpha_{i\beta} T_\alpha, T_\beta) \\
&= -2c^\alpha_{i\beta} g(T_\alpha, T_\beta) \\
&= -2c^\alpha_{i\beta} \delta^\alpha_\beta.
\end{aligned}$$

The required formula is now obtained after we take sums.

To prove $d)$ we write:

$$\nabla_X Y = \sum_k < \nabla_X Y, X_k > X_k.$$

Then Koszul's Formula gives:

$$\begin{aligned} 2 < \nabla_{X_i} X_j, X_k > &= X_k < X_i, X_j > + X_j < X_k, X_i > - X_k < X_i, X_j > \\ &\quad - < X_i, [X_j, X_k] > - < X_j, [X_i, X_k] > + < X_k, [X_i, X_j] > \\ &= - < X_i, c^{\lambda}_{jk} X_\lambda + c^{\alpha}_{jk} T^\alpha > - < X_j, c^{s}_{ik} X_s + c^{\beta}_{ik} T^\beta > \\ &\quad + < X_k, c^{m}_{ij} X_m + c^{\gamma}_{ij} T^\gamma > \\ &= -c^{\lambda}_{jk} \delta^{\lambda}_{\iota} - c^{s}_{ik} \delta^{s}_{j} + c^{m}_{ij} \delta^{m}_{k}. \end{aligned}$$

Summing up, we have

$$-c^{\lambda}_{jk} - c^{s}_{ik} + c^{m}_{ij} = c^{j}_{ik} + c^{i}_{kj} + c^{k}_{ij} = c^{k}_{ij} + c^{j}_{ki} + c^{i}_{kj}$$

and if we divide by 2 we get the required result.

This formula is used in order to find the torsion part of this point. We have

$$\begin{aligned} &\mathrm{Tor}(X_i, X_j) = \nabla_{X_i} X_j - \nabla_{X_j} X_i - [X_i, X_j] \\ &= \frac{1}{2}(c^{k}_{ij} + c^{j}_{ki} + c^{i}_{kj}) X_k - \frac{1}{2}(c^{k}_{ji} + c^{i}_{kj} + c^{i}_{kj}) X_k - (c^{k}_{ij} X_k + c^{\alpha}_{ij} T_\alpha) \\ &= -(c^{i}_{kj} + c^{k}_{ij}) X_k - c^{\alpha}_{ij} T_\alpha = c^{\alpha}_{ij} T_\alpha. \end{aligned}$$

For part $e)$ we make use of the following formula:

$$\nabla_T Y = [T, Y]_j + \frac{1}{2} C^{(j)}(Y, T),$$

used in the proof of Lemma 3.1.

We have

$$\nabla_{T_\beta} X_j = [T_\beta, X_j]_h + \frac{1}{2} C^{(h)}(X_j, T_\beta).$$

We also have

$$
\begin{aligned}
&< [T_\beta, X_j]_h, X_k >= - < [X_j, T_\beta], X_k > \\
&= - < c^r_{j\beta} X_r + c^m_{j\beta} T_m, X_k > \\
&= -c^r_{j\beta}\delta^r_k,
\end{aligned}
$$

as well as

$$
\begin{aligned}
&< C^{(h)}(X_j, T_\beta), X_k >= B^{(h)}(X_j, X_k, T_\beta) \\
&= T_\beta g(X_j, X_k) + g([X_j, T_\beta], X_k) + g([X_k, T_\beta], X_j) \\
&= 0 + g(c^m_{j\beta} X_m + c^q_{j\beta} T_q, X_k) + g(c^s_{k\beta} X_s + c^r_{k\beta} T_r, X_j) \\
&= c^m_{j\beta}\delta^m_k + c^s_{k\beta}\delta^s_j.
\end{aligned}
$$

If we combine these two formulas we have

$$< [T_\beta, X_j]_h, X_k > + \frac{1}{2} < C^{(h)}(X_j, T_\beta), X_k > .$$

If we sum over β and j we obtain

$$< \nabla_{T_\beta} X_j, X_k >=< [T_\beta, X_j]_h + \frac{1}{2} C^{(h)}(X_j, T_\beta), X_k >= -c^k_{j\beta} + \frac{1}{2}(c^j_{k\beta} + c^k_{j\beta}) = \frac{1}{2}(c^j_{k\beta} - c^k_{j\beta}),$$

which provides the desired result.

To prove the other points we use the same methods as used in the proof of a), b), c), d) and e) and combining these with the new results that arise as we progress. $\square$

To illustrate some important behavior, we shall highlight a particular case of the previous example

Example 3.7. Let M be the 4 dimensional Carnot group of Example 2.3.2 of Chapter 2. Using the basic grading, we can easily compute that

$$
\begin{aligned}
&\nabla_X T = S - \frac{1}{2}S = \frac{1}{2}S, \\
&\nabla_X S = 0 - \frac{1}{2}T = -\frac{1}{2}T, \\
&\mathrm{Tor}(X, Y) = -T, \\
&\mathrm{Tor}(X, T) = -\frac{1}{2}S, \\
&\mathrm{Tor}(X, S) = -\frac{1}{2}T.
\end{aligned}
$$

All the other covariant derivatives of frame elements vanish.

Proof. We observe that

$$[X, Y] = 0 \cdot X + 0 \cdot Y + 1 \cdot T + 0 \cdot S,$$
$$[X, T] = 0 \cdot X + 0 \cdot Y + 0 \cdot T + 1 \cdot S,$$

therefore using the previously mentioned formulas, we have

$$\begin{aligned}
\nabla_X T &= \frac{1}{2}(c^{\alpha}_{11} - c^{1}_{1\alpha})T_{\alpha} \\
&= \frac{1}{2}(c^{1}_{11} - c^{1}_{11})T + \frac{1}{2}(c^{2}_{11} - c^{1}_{12})S \\
&= 0 + \frac{1}{2}(1 - 0)S = \frac{1}{2}S, \\
\nabla_X S &= \frac{1}{2}(c^{\alpha}_{12} - c^{\alpha}_{12})T_{\alpha} \\
&= \frac{1}{2}(c^{1}_{12} - c^{2}_{11})T + \frac{1}{2}(c^{2}_{12} - c^{2}_{12})S \\
&= \frac{1}{2}(0 - 1)T + 0 = -\frac{1}{2}T, \\
\mathrm{Tor}(X, Y) &= Tor(X_1, X_2) \\
&= = c^{\alpha}_{11}T_{\alpha} \\
&= -c^{1}_{11}T_1 - c^{2}_{11}T_2 \\
&= -1 \cdot T + 0 = -T,
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tor}(X, T) &= \mathrm{Tor}(X_1, T_1) \\
&= = \frac{1}{2}(c^{1}_{k1} + c^{k}_{11})X_k - \frac{1}{2}(c^{\alpha}_{11} + c^{1}_{1\alpha})T_{\alpha} \\
&= -\frac{1}{2}(c^{1}_{11} + c^{1}_{11})X - \frac{1}{2}(c^{1}_{21} + c^{2}_{11})Y \\
&= -\frac{1}{2}(c^{1}_{11} + c^{1}_{11})T - \frac{1}{2}(c^{2}_{11} + c^{1}_{12})S \\
&= -\frac{1}{2}(0 + 0)X - \frac{1}{2}(1 - 1)Y \\
&= -\frac{1}{2}(0 + 0)T - \frac{1}{2}(1 + 0)S \\
&= -\frac{1}{2}S,
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Tor}(X,S) &= \mathrm{Tor}(X_1,T_2)\\
&= = \frac{1}{2}(c_{k2}^1 + c_{12}^k)X_k - \frac{1}{2}(c_{12}^\alpha + c_{1\alpha}^2)T_\alpha\\
&= -\frac{1}{2}(c_{12}^1 + c_{12}^1)X - \frac{1}{2}(c_{21}^1 + c_{12}^2)Y\\
&= -\frac{1}{2}(c_{12}^1 + c_{11}^2)T - \frac{1}{2}(c_{12}^1 + c_{12}^2)S\\
&= -\frac{1}{2}(0+0)X - \frac{1}{2}(0+0)Y\\
&= -\frac{1}{2}(0-1)T - \frac{1}{2}(0+0)S\\
&= -\frac{1}{2}T.
\end{aligned}
$$

□

Remark 3.8. That the basic covariant derivatives of the natural vertical frame do not vanish is typical of non-step 2 Carnot groups.

Remark 3.9. However, if we use the more refined 2-grading, then all covariant derivatives of the frame elements vanish and the only non-trivial behavior occurs in the torsion.

Proof. We calculate with the fomrulas mentioned in the first lemma that all the covariant derivatives vanish.Then we have

$$
\begin{aligned}
&\mathrm{Tor}^{(2)}(X,Y) = -[X,Y] = -T,\\
&\mathrm{Tor}^{(2)}(X,T) = -[X,T] = -S,\\
&\mathrm{Tor}^{(2)}(X,S) = -[X,S] = 0.
\end{aligned}
$$

□

Example 3.10. Let $M = \mathbb{R}^4$ with the following global orthonormal frames for HM and VM

$$
\begin{aligned}
&X = \frac{\partial}{\partial x},\\
&Y = \frac{\partial}{\partial y} + \sin x\frac{\partial}{\partial t} - \cos x\frac{\partial}{\partial s},\\
&T = \cos x\frac{\partial}{\partial t} + \sin x\frac{\partial}{\partial s},\\
&S = -\sin x\frac{\partial}{\partial t} + \cos x\frac{\partial}{\partial s}.
\end{aligned}
$$

Then we calculate

$$[X, Y] = T = -[X, S],$$
$$[X, T] = S,$$

with all other Lie brackets vanishing.

It is then possible to check that this is a strictly normal extension for the basic grading and that the only non-trivial covariant derivatives are then $\nabla_X T = S$ and $\nabla_X S = T$. This is an example of a flat, equiregular, strictly normal SRC-manifold with step greater than 2.

Example 3.11. Let (M, J, ω) be a strictly pseudoconvex pseudohermitian manifold.

Recall Example 2.19, from which we now continue our work.

We introduce the following definition:

Definition 3.12. The Tanaka-Webster connection is a unique connection such that:

a) $\omega, d\omega$ and J are parallel

and the torsion satisfies

b) $\mathrm{Tor}(X, Y) = d\omega(X, Y)T$,

c) $\mathrm{Tor}(T, JX) = -J\mathrm{Tor}(T, X)$,

where X, Y are horizontal.

For the existence and the uniqueness of the Tanaka-Webster connection see [8] (Mainly Proposition 3.1 page 29).

The only defining property of the basic connection not clearly satisfied by the Tanaka-Webster connection is torsion symmetry, which we will now prove.

Proof. But if we choose X, Y as any horizontal vector fields then the Jacobi identity and $[T, HM] \subset HM$ implies

$$0 = \omega([T,[X,JY]] + [JY,[T,X]] + [X,[JY,T]])$$

and we also have

$$\begin{aligned}
&-T < X, Y > + < [T,X], Y > - < [JY,T], JX > \\
&= -T < X, Y > + < [T,X], Y > - < [-X,T], -Y > \\
&= -T < X, Y > +2 < [T,X], Y > \\
&= -T(d\omega(X,JY) + \omega(X)\omega(Y)) \\
&+2(d\omega([T,X],Y) + \omega([T,X])\omega(Y)) \\
&= 0 + 2(d\omega([T,X],Y) + 0) \\
&= 2(([T,X])\omega(Y) - Y\omega([T,X]) - \omega([[T,X],Y]) \\
&= 2(0 - 0 - 0) \\
&= 0.
\end{aligned}$$

For this reason

$$\begin{aligned}
&0 = \omega([T,[X,JY]] + [JY,[T,X]] + [X,[JY,T]]) \\
&= -T < X, Y > + < [T,X], Y > - < [JY,T], JX > \\
&= - < \nabla_T X, Y > - < \nabla_T Y, X > + < [T,X], Y > - < [JY,T], JX > \\
&= - < \mathrm{Tor}(T,X), Y > - < J\nabla_T Y, JX > - < [JY,T], JX >,
\end{aligned}$$

keeping in mind that due to the third part of the definition of the connection $< \nabla_X T, Y >= 0$ and because $(\nabla_X J)Y = 0, \forall J$ is equivalent to

$$< (\nabla_X JY), Z >=< J(\nabla_X Y), Z >,$$

we obtain

$$\begin{aligned}
&- < \mathrm{Tor}(T,X), Y > - < J\nabla_T Y, JX > - < [JY,T], JX > \\
&= - < \mathrm{Tor}(T,X), Y > + < X, \nabla_T Y > + < \nabla_T JY, JX > \\
&- < \mathrm{Tor}(T,JY), JX > \\
&= - < \mathrm{Tor}(T,X), Y > - < \mathrm{Tor}(T,JY), JX > .
\end{aligned}$$

From the definition of Tanaka-Webster connection we have

$$- < \mathrm{Tor}(T,X), Y > + < \mathrm{Tor}(T,Y), X >= 0,$$

which implies

$$< \mathrm{Tor}(T, X), Y >=< \mathrm{Tor}(T, Y), X > .$$

Thus the Tanaka-Webster connection satisfies the requirements of the basic connection.

For more information about these kind of structures see [8], [9], [10] and [21]. $\square$

Chapter 4

Curvature and the Bianchi Identities

In Chapter 4 we start with the definitions of Sub-Riemannian curvature and also set K^S, a type of sectional curvature. We then introduce the definitions of a horizontally and vertically flat and flat extension and examples for each case. We define a cyclic permutation of F as $\mathfrak{S}$, as well as torsion of second-order. We continue with Symmetries of the Curvature Tensor, which are proved and Bianchi Identities such as the Algebraic Bianchi Identity, along with properties, as well as the Horizontal Bianchi Identity with the corresponding proofs. We then present a Corollary, which is another symmetry of the K^S, which is proved and used in the rest of the text. Furthermore, we define the Covariant derivative and we present and prove the Differential Horizontal Bianchi Identity.

4.1 Curvature Tensor and Sectional Curvature

The Sub-Riemannian curvature tensor for a SRC-manifold with extension g is defined by

$$R(A, B)C = \nabla_A \nabla_B C - \nabla_B \nabla_A C - \nabla_{[A,B]} C,$$

where ∇ is the basic connection.
We also set

$$K^S(A, B, C, D) =< R(A, B)C, D > .$$

We note that for any vectors $A, B \in TM$, the restriction of the $(1,1)$-tensor $R(A, B)$ to HM is independent of the choice of extension g.
This definition immediately yields notions of flatness in Sub-Riemannian Geometry.

Definition 4.1.

1. We say that an M is *horizontally flat* if $K^S(\cdot, \cdot, HM, \cdot) = 0$, for any extension g.
2. We say that a particular extension is *vertically flat* if $K^S(\cdot, \cdot, VM, \cdot) = 0$.
3. We say that a particular extension is *flat* if $K^S = 0$.

Remark 4.2. A SRC-manifold is horizontally flat if and only if in a neighborhood of every point $p \in M$ there is a local orthonormal frame $\{E_i\}$ for HM such that $\nabla E_i = 0$. If HM is integrable, this local frame can be chosen to be a coordinate frame.

A similar result holds for a vertically flat extension g and VM.

Proof. The proof of this Remark follows the lines of the proof of [7] (Specifically, Theorem 7.3 page 89). □

Example 4.3. Every step r Carnot group is horizontally flat for the basic grading and flat for the $(r-1)$-grading. The SRC-manifolds considered in Example 2.3.2 and Example 3.10 are both flat.

Proof. The result can be obtained through the computation that has already been made in the examples. □

For convenience of notation, it is useful to define the following

Definition 4.4. If S is any set and $F : S^k \mapsto L$ is any map into a vector space L, we define $\mathfrak{S}F$ to be the *sum of all cyclic permutations of* F. For example if $k = 3$, then

$$\mathfrak{S}F(X, Y, Z) = F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y).$$

Remark 4.5. An example of the cyclic construction in action is a compressed form of the Jacobi identity for vector fields, namely $\mathfrak{S}([X, [Y, Z]]) = 0$. We shall use it primarily to efficiently describe symmetries of the curvature tensor.

Definition 4.6. The *second-order torsion of* ∇ is the (3,1)-tensor

$$\mathrm{Tor}_2(A, B, C) = \mathrm{Tor}(A, \mathrm{Tor}(B, C)).$$

We are now in a position to discuss the fundamental questions of curvature symmetries. Many of the properties of the Riemannian curvature tensor go through unchanged, with exactly the same proof.

4.2 Symmetries of the Curvature Tensor

Lemma 4.7. The Sub-Riemannian curvature tensor has the following symmetries

1. $K^S(A, B, C, D) = -K^S(A, B, D, C)$.
2. $K^S(A, B, C, D) = -K^S(B, A, C, D)$.
3. $K^S(TM, TM, HM, VM) = 0$.

Proof. The two first symmetries are a reproduction of the Riemannian curvature proofs contained in any text on Riemannian geometry (For example see [13] page 91 Proposition 2.5).

The third equality arises from the fact that we have an inner product (on the horizontal bundle) of the form $< \cdot, Q >, Q \in VM$, therefore it is zero due to the fact that:

$$\begin{aligned}
&K^S(X,Y,Z,W) =< R(X,Y)Z, W > \\
&=< \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W > \\
&= 0.
\end{aligned}$$

□

However, many symmetry properties of the Riemannian curvature tensor require additional assumptions in the Sub-Riemannian case. Most of these symmetries are naturally related to the Bianchi Identities.

4.3 Bianchi Identities

4.3.1 Algebraic Bianchi Identities

We will need some preliminaries first:

Remark 4.8. The following hold:

$$\begin{aligned}
&\mathrm{Tor}(A, B + C) = \mathrm{Tor}(A, B) + \mathrm{Tor}(A, C), \\
&\mathrm{Tor}(A, -B) = \mathrm{Tor}(-A, B) = -\mathrm{Tor}(A, B) = \mathrm{Tor}(B, A).
\end{aligned}$$

Proof. The proof follows from the linearity property of connections. □

Lemma 4.9. For any vector fields X, Y, Z,

$$\mathfrak{S} R(X,Y)Z = -\mathfrak{S}\mathrm{Tor}_2(X,Y,Z) + \mathfrak{S}(\nabla \mathrm{Tor}(X,Y,Z)).$$

Where $\nabla \mathrm{Tor}(X,Y,Z) = \nabla_Z \mathrm{Tor}(X,Y)$.

Furthermore:

1. If $X, Y, Z \in V^{(j)}$ then
$$\mathfrak{S}(\nabla\mathrm{Tor}(X, Y, Z)) \in \widehat{V}^{(j)}.$$
2. If $X, Y, Z \in V^{(j)}$ and the grading is j-normal, then
$$-\mathfrak{S}\mathrm{Tor}_2(X, Y, Z) \in \widehat{V}^{(j)}.$$
3. If $X, Y, Z \in V^{(j)}$ and the grading is j-normal and $\widehat{V}^{(j)}$ is integrable then
$$-\mathfrak{S}\mathrm{Tor}_2(X, Y, Z) \in \widehat{V}^{(j)}.$$

Proof. The equality is proven as follows:
$$\mathfrak{S}R(X, Y)Z = \mathfrak{S}(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z).$$
From the definition of $\mathfrak{S}$ and its linearity this is equal to:
$$\begin{aligned}&\mathfrak{S}(\nabla_Z\nabla_X Y - \nabla_Z\nabla_Y X - \nabla_{[X,Y]}Z)\\ &= \mathfrak{S}(\nabla_Z(\nabla_X Y - \nabla_Y X) - \nabla_{[X,Y]}Z).\end{aligned}$$
Also, from the definition of torsion this is equal to:
$$\begin{aligned}&\mathfrak{S}(\nabla_Z([X, Y] + \mathrm{Tor}(X, Y)) - \nabla_{[X,Y]}Z)\\ &= \mathfrak{S}((\nabla_Z[X, Y] - \nabla_{[X,Y]}Z) + \nabla_Z\mathrm{Tor}(X, Y)).\end{aligned}$$
Again from the definition of torsion this is equal to:
$$\mathfrak{S}([Z[X, Y]] + \mathrm{Tor}(Z, [X, Y]) + \nabla_Z\mathrm{Tor}(X, Y)) := A.$$
From the Jacobi identity we have:
$$\mathfrak{S}([Z[X, Y]]) = [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$
We also have from the definition of the covariant derivative that:
$$(\nabla\mathrm{Tor})(X, Y, Z) = (\nabla_Z\mathrm{Tor})(X, Y) = \nabla_Z\mathrm{Tor}(X, Y) - \mathrm{Tor}(\nabla_Z X, Y) - \mathrm{Tor}(X, \nabla_Z Y)$$
and therefore if we apply $\mathfrak{S}$ and use it's linearity we have
$$\mathfrak{S}(\mathrm{Tor}(\nabla_Z X, Y) + \mathrm{Tor}(X, \nabla_Z Y) + (\nabla_Z\mathrm{Tor})(X, Y)) = \mathfrak{S}(\nabla_Z\mathrm{Tor}(X, Y)),$$

where we keep in mind that $(\nabla_Z \mathrm{Tor})(X,Y) = (\nabla \mathrm{Tor})(X,Y,Z)$.

Now we continue the work from A taking into consideration the above arguments:

$$A = \mathfrak{S}(\mathrm{Tor}(Z,[X,Y]) + \mathrm{Tor}(\nabla_Z X, Y) + \mathrm{Tor}(X, \nabla_Z Y)) + \mathfrak{S}(\nabla \mathrm{Tor})(X,Y,Z) := B$$

Therefore we have, due to the aforementioned Remark 4.8:

$$\begin{aligned}
&\mathfrak{S}(\mathrm{Tor}(Z,[X,Y] - \nabla_X Y + \nabla_Y X)) \\
&= \mathfrak{S}(\mathrm{Tor}(Z,[X,Y]) + \mathrm{Tor}(Z, -\nabla_X Y) + \mathrm{Tor}(Z, \nabla_Y X)) \\
&= \mathfrak{S}(\mathrm{Tor}(Z,[X,Y]) + \mathrm{Tor}(\nabla_Z X, Y) + \mathrm{Tor}(X, \nabla_Z Y)).
\end{aligned}$$

We now combine the previously mentioned results and continue from B:

$$B = \mathfrak{S}(\mathrm{Tor}(Z,[X,Y] + \nabla_X Y - \nabla_Y X)) + \mathfrak{S}(\nabla \mathrm{Tor})(X,Y,Z),$$

from the definition of the torsion and the torsion of second order we have

$$= -\mathfrak{S}\mathrm{Tor}_2(X,Y,Z) + \mathfrak{S}(\nabla \mathrm{Tor})(X,Y,Z),$$

which was the desired equality.

The other assertions in the lemma occur from analysing the terms $\mathfrak{S}\mathrm{Tor}_2$ and $\mathfrak{S}(\nabla \mathrm{Tor})$. Since both of them are tensorial, we can compute using normal and semi-normal frames.

1. First let X, Y, Z be elements of a seminormal frame for $V^{(j)}$ at p then

$$\begin{aligned}
\mathfrak{S}(\nabla \mathrm{Tor})(X,Y,Z) &= \mathfrak{S}((\nabla_Z \mathrm{Tor})(X,Y)) = \mathfrak{S}((\nabla_X \mathrm{Tor})(Y,Z)) \\
&= \mathfrak{S}(\nabla_X \mathrm{Tor}(Y,Z) - \mathrm{Tor}(\nabla_X Y, Z) - \mathrm{Tor}(Y, \nabla_X Z)).
\end{aligned}$$

   But we know that each torsion part belongs in $\widehat{V}^{(j)}$. Due to the fact that this is bundle parallel, we have obtained part 1 of the lemma.

2. We now proceed to part 2 of lemma. We operate under the assumption that the frame is j-normal and be able to use a normal frame at p. If X is an element of this frame then $\mathrm{Tor}(X, TM) \subset \widehat{V}^{(j)}$ then $\mathrm{Tor}(Z, \mathrm{Tor}(X,Y)) \subset \widehat{V}^{(j)}$ is of the form $\mathrm{Tor}(X, TM)$ where $X \in V^{(j)}$. Therefore part 2 is proven.

3. For the last part of the lemma we have assumed j-normal grading and integrability of $\widehat{V}^{(j)}$ and for $X, Y \in V^{(j)}$ and Z arbitrary we have

$$\begin{aligned}&\mathfrak{S}\mathrm{Tor}_2(X,Y,Z) = \mathfrak{S}\mathrm{Tor}(X, \mathrm{Tor}(Y,Z))\\ &= \mathrm{Tor}(X, \mathrm{Tor}(Y,Z)) + \mathrm{Tor}(Y, \mathrm{Tor}(Z,X)) + \mathrm{Tor}(Z, \mathrm{Tor}(X,Y))\\ &= \mathrm{Tor}(Z, \mathrm{Tor}(X,Y)).\end{aligned}$$

However we have that

$$\begin{aligned}&(\mathrm{Tor}(Z, \mathrm{Tor}(X,Y)))_j\\ = \;&(\nabla_Z(\mathrm{Tor}(X,Y)) - \nabla_{\mathrm{Tor}(X,Y)}Z - [Z, \mathrm{Tor}(X,Y)])_j\\ = \;&(-[Z, \mathrm{Tor}(X,Y)])_j = -[Z_{\widehat{j}}, \mathrm{Tor}(X,Y)]_j = 0,\end{aligned}$$

because $\widehat{V}^{(j)}$ is integrable. Therefore we have proved the lemma.

□

4.3.2 Horizontal Algebraic Biachi Identity

Corollary 4.10. If X, Y, Z, W are horizontal vector fields and VM is normal, then

$$< \mathfrak{S}R(X,Y)Z, W >= 0.$$

If VM is also integrable, then this can be relaxed to any three of X, Y, Z, W horizontal.

Proof. If we combine the previous Lemma 4.9 with the given condition we have

$$\begin{aligned}\mathfrak{S}R(X,Y)Z = -\mathfrak{S}\mathrm{Tor}_2(X,Y,Z) + \mathfrak{S}(\nabla\mathrm{Tor}(X,Y,Z)),\\ \mathfrak{S}(\nabla\mathrm{Tor}(X,Y,Z)) \in \widehat{V}^{(j)},\\ -\mathfrak{S}\mathrm{Tor}_2(X,Y,Z) \in \widehat{V}^{(j)},\end{aligned}$$

in any of the cases mentioned. But W is horizontal in the first case so

$$\begin{aligned}&< \mathfrak{S}R(X,Y)Z, W >=< -\mathfrak{S}\mathrm{Tor}_2(X,Y,Z) + \mathfrak{S}(\nabla\mathrm{Tor}(X,Y,Z)), W >\\ &=< -\mathfrak{S}\mathrm{Tor}_2(X,Y,Z), W > + < \mathfrak{S}(\nabla\mathrm{Tor}(X,Y,Z)), W >\\ &= 0.\end{aligned}$$

The most difficult case that may present is if X, Y, Z are given horizontal and W arbitrary. Then we have

$$\mathfrak{S}R(X,Y)Z = -\mathfrak{S}\mathrm{Tor}_2(X,Y,Z) + \mathfrak{S}(\nabla \mathrm{Tor}(X,Y,Z)) = 0 + 0 = 0.$$

Therefore the inner product is again zero. □

Corollary 4.11. If VM is normal and X, Y, Z, W are horizontal vector fields then

$$K^S(X,Y,Z,W) = K^S(Z,W,X,Y).$$

If VM is also integrable, then this can be relaxed to any three of X, Y, Z, W horizontal.

Proof. Using the Lemma 4.7 about the symmetries of the curvature tensor and the Horizontal Algebraic Bianchi Identity that was shown above, we have

$$\begin{aligned}
K^S(X,Y,Z,W) + K^S(Y,Z,X,W) + K^S(Z,X,Y,W) &=< \mathfrak{S}R(X,Y)Z, W >,\\
K^S(Y,Z,W,X) + K^S(Z,W,Y,X) + K^S(W,Y,Z,X) &=< \mathfrak{S}R(Y,Z)W, X >,\\
K^S(Z,W,X,Y) + K^S(W,X,Z,Y) + K^S(X,Z,W,Y) &=< \mathfrak{S}R(Z,W)X, Y >,\\
K^S(W,X,Y,Z) + K^S(X,Y,W,Z) + K^S(Y,W,X,Z) &=< \mathfrak{S}R(W,X)Y, Z >,
\end{aligned}$$

combined with these two relations from Lemma 4.7

$$\begin{aligned}
K^S(X,Y,Z,W) &= -K^S(X,Y,W,Z),\\
K^S(X,Y,Z,W) &= -K^S(Y,X,Z,W).
\end{aligned}$$

We see that due to Corollary 4.10 the first four equations are equal to zero. Also the first 4 equations if added have the terms of the first two columns canceled out giving us combined again with the two relations from Lemma 4.9 that

$$\begin{aligned}
&K^S(Z,X,Y,W) + K^S(W,Y,Z,X) + K^S(X,Z,W,Y) + K^S(Y,W,X,Z)\\
&= \mathfrak{S} < \mathfrak{S}R(X,Y)Z, W >,
\end{aligned}$$

is equivalent to

$$\begin{aligned}
&K^S(Z,X,Y,W) - K^S(Y,W,Z,X) + K^S(X,Z,W,Y) - K^S(Y,W,Z,X)\\
&= \mathfrak{S} < \mathfrak{S}R(X,Y)Z, W >,
\end{aligned}$$

which give us

$$2K^S(Z,X,Y,W) - 2K^S(Y,W,Z,X) = \mathfrak{S} < \mathfrak{S}R(X,Y)Z, W > .$$

If we use the Horizontal Algebraic Bianchi Identity we see that the right hand side of the last relation is zero, therefore we have that the following is equivalent

$$2K^S(Z,X,Y,W) - 2K^S(Y,W,Z,X) = \mathfrak{S} < \mathfrak{S}R(X,Y)Z, W >,$$

with

$$2K^S(Z,X,Y,W) - 2K^S(Y,W,Z,X) = 0,$$

which in turn is equivalent to

$$K^S(Z,X,Y,W) = K^S(Y,W,Z,X).$$

Which is the desired equality. □

4.3.3 Differential Horizontal Bianchi Identities

Definition 4.12. The *covariant derivative* for a horizontal (p,r) tensor A field on M is defined as follows:

$$\begin{aligned}(\nabla_X A)(\omega_1,\ldots,\omega_p,hX_1,\ldots,hX_r) &= X(A(\omega_1,\ldots,\omega_p,hX_1,\ldots,hX_r)) \\ &\quad -\sum_{i=1}^{p} A(\omega_1,\ldots,\nabla_X\omega_i,\ldots,\omega_p,hX_1,\ldots,hX_r) \\ &\quad -\sum_{j=1}^{r} A(\omega_1,\ldots,\omega_p,hX_1,\ldots,\nabla_X hX_j,\ldots,hX_r).\end{aligned}$$

Lemma 4.13. 1. For any vector fields X, Y, Z, W

$$\mathfrak{S}((\nabla_W R)(X,Y))Z = \mathfrak{S}(R(\mathrm{Tor}(X,W),Y))Z.$$

2. If VM is normal and integrable and $X,Y,Z,W,V \in HM$ then

$$(\nabla K^S)(X,Y,Z,W,V)+(\nabla K^S)(X,Y,W,V,Z)+(\nabla K^S)(X,Y,V,Z,W) = 0.$$

Proof. To prove our first equality, we observe first that

$$\begin{aligned}(\nabla_W R)(X,Y)Z &= \nabla_W(R(X,Y))Z - R(\nabla_W X,Y)Z \\ &\quad -R(X,\nabla_W Y)Z - R(X,Y)\nabla_W Z \\ &= [\nabla_W, R(X,Y)]Z - R(\nabla_W X,Y)Z - R(X,\nabla_W Y)Z.\end{aligned}$$

Due to the linearity of $\mathfrak{S}$ and by using the Jacobi Identity

$$\mathfrak{S}([\nabla_X,[\nabla_Y,\nabla_Z]]) = 0,$$

we have

$$\begin{aligned}\mathfrak{S}((\nabla_W R)(X,Y))Z &= \mathfrak{S}([\nabla_W, R(X,Y)])Z - \mathfrak{S}(R(\nabla_W X,Y))Z \\ &\quad -\mathfrak{S}(R(X,\nabla_W Y))Z.\end{aligned}$$

Due to the fact that

$$R(X,Y) = -R(Y,X)$$

and

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]},$$

we have

$$\begin{aligned}\mathfrak{S}((\nabla_W R)(X,Y))Z &= \mathfrak{S}([\nabla_W,[\nabla_X,\nabla_Y] - \nabla_{[X,Y]}])Z \\ &\quad -\mathfrak{S}(R(\nabla_W X,Y))Z \\ &\quad +\mathfrak{S}(R(\nabla_W Y,X))Z.\end{aligned}$$

Using the definition of $\mathfrak{S}$ and the Jacobi Identity we have

$$\begin{aligned}\mathfrak{S}((\nabla_W R)(X,Y))Z &= \mathfrak{S}([\nabla_W[\nabla_X,\nabla_Y]]) - \mathfrak{S}([\nabla_W,\nabla_{[X,Y]}])Z \\ &\quad -\mathfrak{S}(R(\nabla_W X,Y))Z + \mathfrak{S}(R(\nabla_X W,Y))Z.\end{aligned}$$

We now use the linearity properties of R and of $\mathfrak{S}$ as well as the definition of torsion we will obtain Using the definition of $\mathfrak{S}$ and the Jacobi Identity we have

$$\begin{aligned}&\mathfrak{S}((\nabla_W R)(X,Y))Z = -\mathfrak{S}([\nabla_W,\nabla_{[X,Y]}])Z + \mathfrak{S}(R(\nabla_X W - \nabla_W X,Y))Z \\ &= -\mathfrak{S}([\nabla_W,\nabla_{[X,Y]}])Z + \mathfrak{S}(R([X,W] + \mathrm{Tor}(X,W),Y))Z \\ &= -\mathfrak{S}([\nabla_W,\nabla_{[X,Y]}])Z + \mathfrak{S}(R([X,W],Y) + \mathfrak{S}(R(\mathrm{Tor}(X,W),Y))Z.\end{aligned}$$

However we have

$$R([X,W],Y) = \nabla_{[X,W]}\nabla_Y - \nabla_Y\nabla_{[X,W]} - \nabla_{[[X,W],Y]}$$

and now if we apply $\mathfrak{S}$ we have the Jacobi identity, due to linearity of the connections we have $\mathfrak{S}(\nabla_{[[X,W],Y]}) = 0$ combined with the fact that we have

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

and therefore

$$\begin{aligned}
&-\mathfrak{S}([\nabla_W, \nabla_{[X,Y]}])Z + \mathfrak{S}(R([X,W],Y) + \mathfrak{S}(R(\mathrm{Tor}(X,W),Y))Z \\
&= -\mathfrak{S}([\nabla_W, \nabla_{[X,Y]}])Z + \mathfrak{S}([\nabla_{[X,W]}, \nabla_Y] - \nabla_{[[X,W],Y]}) \\
&+\mathfrak{S}(R(\mathrm{Tor}(X,W),Y))Z \\
&= -\mathfrak{S}([\nabla_W, \nabla_{[X,Y]}])Z + \mathfrak{S}([\nabla_{[X,W]}, \nabla_Y]) + \mathfrak{S}(R(\mathrm{Tor}(X,W),Y))Z
\end{aligned}$$

and by using the definition of $\mathfrak{S}$ we have

$$\begin{aligned}
&= -\mathfrak{S}([\nabla_W, \nabla_{[X,Y]}])Z + \mathfrak{S}([\nabla_{[Y,X]}, \nabla_W]) + \mathfrak{S}(R(\mathrm{Tor}(X,W),Y))Z \\
&= -\mathfrak{S}([\nabla_W, \nabla_{[X,Y]}])Z + \mathfrak{S}([\nabla_W, \nabla_{[X,Y]}])Z + \mathfrak{S}(R(\mathrm{Tor}(X,W),Y))Z \\
&= \mathfrak{S}(R(\mathrm{Tor}(X,W),Y))Z.
\end{aligned}$$

We want to prove

$$0 = (\nabla K^S)(X,Y,Z,W,V) + (\nabla K^S)(X,Y,W,V,Z) + (\nabla K^S)(X,Y,V,Z,W),$$

which is equivalent to

$$0 = (\nabla_V K^S)(X,Y,Z,W) + (\nabla_Z K^S)(X,Y,W,V) + (\nabla_W K^S)(X,Y,V,Z).$$

By using the Definition 4.12 of the Covariant derivative we have

$$
\begin{aligned}
&0 = (\nabla_V K^S)(X, Y, Z, W) + (\nabla_Z K^S)(X, Y, W, V) + (\nabla_W K^S)(X, Y, V, Z)\\
&= (\nabla_V (K^S(X, Y, Z, W)) - K^S(\nabla_V X, Y, Z, W) - K^S(X, \nabla_V Y, Z, W)\\
&-K^S(X, Y, \nabla_V Z, W) - K^S(X, Y, Z, \nabla_V W))\\
&+(\nabla_Z (K^S(X, Y, W, V)) - K^S(\nabla_Z X, Y, W, V) - K^S(X, \nabla_Z Y, W, V)\\
&-K^S(X, Y, \nabla_Z W, V) - K^S(X, Y, W, \nabla_Z V))\\
&+(\nabla_W (K^S(X, Y, V, Z)) - K^S(\nabla_W X, Y, V, Z) - K^S(X, \nabla_W Y, V, Z)\\
&-K^S(X, Y, \nabla_W V, Z) - K^S(X, Y, V, \nabla_W Z))\\
&= \nabla_V (K^S(X, Y, Z, W)) + \nabla_Z (K^S(X, Y, W, V)) + \nabla_W (K^S(X, Y, V, Z))\\
&-(K^S(X, \nabla_V Y, Z, W) + K^S(X, \nabla_Z Y, W, V) + K^S(X, \nabla_W Y, V, Z))\\
&-(K^S(X, Y, \nabla_V Z, W) + K^S(X, Y, W, \nabla_Z V) + K^S(X, Y, \nabla_Z W, V)\\
&+K^S(X, Y, V, \nabla_W Z) + K^S(X, Y, Z, \nabla_V W) + K^S(X, Y, \nabla_W V, Z))\\
&-(K^S(\nabla_V X, Y, Z, W) + K^S(\nabla_Z X, Y, W, V) + K^S(\nabla_W X, Y, V, Z))
\end{aligned}
$$

and by the compatibility of the metric, after applying Corollary 4.7, Corollary 4.11, the definition of K^S and use it's linearity we have

$$
\begin{aligned}
&\nabla_V (K^S(X, Y, Z, W)) + \nabla_Z (K^S(X, Y, W, V)) + \nabla_W (K^S(X, Y, V, Z))\\
&-(K^S(X, \nabla_V Y, Z, W) + K^S(X, \nabla_Z Y, W, V) + K^S(X, \nabla_W Y, V, Z))\\
&-(K^S(X, Y, \nabla_V Z, W) + K^S(X, Y, W, \nabla_Z V) + K^S(X, Y, \nabla_Z W, V)\\
&+K^S(X, Y, V, \nabla_W Z) + K^S(X, Y, Z, \nabla_V W) + K^S(X, Y, \nabla_W V, Z))\\
&-(K^S(\nabla_V X, Y, Z, W) + K^S(\nabla_Z X, Y, W, V) + K^S(\nabla_W X, Y, V, Z))\\
&= \nabla_V (K^S(Z, W, X, Y)) + \nabla_Z (K^S(W, V, X, Y)) + \nabla_W (K^S(V, Z, X, Y))\\
&-(K^S(Z, W, X, \nabla_V Y) + K^S(W, V, X, \nabla_Z Y) + K^S(V, Z, X, \nabla_W Y))\\
&-(K^S(\nabla_V Z, W, X, Y) + K^S(W, \nabla_Z V, X, Y) + K^S(\nabla_Z W, V, X, Y)\\
&+K^S(V, \nabla_W Z, X, Y) + K^S(\nabla_W V, Z, X, Y) + K^S(Z, \nabla_V W, X, Y))\\
&-(K^S(Z, W, \nabla_V X, Y) + K^S(W, V, \nabla_Z X, Y) + K^S(V, Z, \nabla_W X, Y))
\end{aligned}
$$

$$
\begin{aligned}
&\nabla_V(K^S(Z,W,X,Y)) + \nabla_Z(K^S(W,V,X,Y)) + \nabla_W(K^S(V,Z,X,Y))\\
&-(K^S(Z,W,X,\nabla_V Y) + K^S(W,V,X,\nabla_Z Y) + K^S(V,Z,X,\nabla_W Y))\\
&-(K^S(\nabla_V Z,W,X,Y) + K^S(W,\nabla_Z V,X,Y) + K^S(\nabla_Z W,V,X,Y)\\
&+K^S(V,\nabla_W Z,X,Y) + K^S(\nabla_W V,Z,X,Y) + K^S(Z,\nabla_V W,X,Y))\\
&-(K^S(Z,W,\nabla_V X,Y) + K^S(W,V,\nabla_Z X,Y) + K^S(V,Z,\nabla_W X,Y))\\
&= (\nabla_V(< R(Z,W)X,Y >) + \nabla_Z(< R(W,V)X,Y) >)\\
&+\nabla_W(< R(V,Z)X,Y >))\\
&-(K^S(Z,W,X,\nabla_V Y) + K^S(W,V,X,\nabla_Z Y) + K^S(V,Z,X,\nabla_W Y))\\
&-(K^S(\mathrm{Tor}(V,Z) + [V,Z],W,X,Y) + K^S(\mathrm{Tor}(Z,W) + [Z,W],V,X,Y)\\
&+K^S(\mathrm{Tor}(W,V) + [W,V],Z,X,Y))\\
&-(K^S(Z,W,\nabla_V X,Y) + K^S(W,V,\nabla_Z X,Y) + K^S(V,Z,\nabla_W X,Y))\\
&= (< \nabla_V(R(Z,W)X),Y > + < \nabla_Z(R(W,V)X),Y > + < \nabla_W(R(V,Z)X),Y >\\
&+(K^S(Z,W,X,\nabla_V Y) + K^S(W,V,X,\nabla_Z Y) + K^S(V,Z,X,\nabla_W Y))\\
&-(K^S(Z,W,X,\nabla_V Y) + K^S(W,V,X,\nabla_Z Y) + K^S(V,Z,X,\nabla_W Y))\\
&-(K^S(\mathrm{Tor}(V,Z),W,X,Y) + K^S([V,Z],W,X,Y)\\
&+K^S(\mathrm{Tor}(Z,W),V,X,Y) + K^S([Z,W],V,X,Y)\\
&+K^S(\mathrm{Tor}(W,V),Z,X,Y) + K^S([W,V],Z,X,Y))\\
&-(K^S(Z,W,\nabla_V X,Y) + K^S(W,V,\nabla_Z X,Y) + K^S(V,Z,\nabla_W X,Y))\\
&=< \nabla_V(R(Z,W)X),Y > + < \nabla_Z(R(W,V)X),Y > + < \nabla_W(R(V,Z)X),Y >\\
&-(K^S(\mathrm{Tor}(V,Z),W,X,Y) + K^S(\mathrm{Tor}(Z,W),V,X,Y) + K^S(\mathrm{Tor}(W,V),Z,X,Y)\\
&+K^S([V,Z],W,X,Y) + K^S([Z,W],V,X,Y) + K^S([W,V],Z,X,Y))\\
&-(K^S(Z,W,\nabla_V X,Y) + K^S(W,V,\nabla_Z X,Y) + K^S(V,Z,\nabla_W X,Y))\\
&=< \nabla_V(R(Z,W)X),Y > + < \nabla_Z(R(W,V)X),Y > + < \nabla_W(R(V,Z)X),Y >\\
&-(< R(\mathrm{Tor}(V,Z),W)X,Y > + < R(\mathrm{Tor}(Z,W),V)X,Y) + < R(\mathrm{Tor}(W,V),Z)X,Y >\\
&+ < R([V,Z],W)X,Y > + < R([Z,W],V)X,Y > + < R([W,V],Z)X,Y) >)\\
&-(< R(Z,W)\nabla_V X,Y > + < R(W,V)\nabla_Z X,Y > + < R(V,Z)\nabla_W X,Y >),
\end{aligned}
$$

or using the symbol $\mathfrak{S}$ and the definition and linearity of K^S we have

$$
\begin{aligned}
&< \nabla_V(R(Z,W)X), Y > + < \nabla_Z(R(W,V)X), Y > + < \nabla_W(R(V,Z)X), Y > \\
&-(< R(\mathrm{Tor}(V,Z),W)X, Y > + < R(\mathrm{Tor}(Z,W),V)X, Y) + < R(\mathrm{Tor}(W,V),Z)X, Y > \\
&+ < R([V,Z],W)X, Y > + < R([Z,W],V)X, Y > + < R([W,V],Z)X, Y) >) \\
&-(< R(Z,W)\nabla_V X, Y > + < R(W,V)\nabla_Z X, Y > + < R(V,Z)\nabla_W X, Y >) \\
&=< (\mathfrak{S}(\nabla_V(R(Z,W))))X, Y > -(< (\mathfrak{S}(R(\mathrm{Tor}(V,Z),W)))X, Y >) \\
&-(< (\mathfrak{S}(R([V,Z],W)))X, Y >) - (< (\mathfrak{S}(R(Z,W)\nabla_V))X, Y >),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&< (\mathfrak{S}(\nabla_V(R(Z,W)) - R(\mathrm{Tor}(V,Z),W) \\
&-R([V,Z],W) - R(Z,W)\nabla_V))X, Y > .
\end{aligned}
$$

We now focus on

$$
\mathfrak{S}(\nabla_V(R(Z,W)) - R(\mathrm{Tor}(V,Z),W) - R([V,Z],W) - R(Z,W)\nabla_V).
$$

We have

$$
\begin{aligned}
\mathfrak{S}(\nabla_V(R(Z,W))) &= \mathfrak{S}(\nabla_V\nabla_Z\nabla_W - \nabla_V\nabla_W\nabla_Z - \nabla_V\nabla_{[Z,W]}), \\
\mathfrak{S}(R([V,Z],W)) &= \mathfrak{S}(\nabla_{[V,Z]}\nabla_W - \nabla_W\nabla_{[V,Z]} - \nabla_{[[V,Z],W]}), \\
\mathfrak{S}(R(Z,W)\nabla_V) &= \mathfrak{S}(\nabla_Z\nabla_W\nabla_V - \nabla_W\nabla_Z\nabla_V - \nabla_{[Z,W]}\nabla_V).
\end{aligned}
$$

If we subtract the two last equations from the first and keep in mind the linearity of $\mathfrak{S}$ we have

$$
\begin{aligned}
&\mathfrak{S}(\nabla_V(R(Z,W)) - R([V,Z],W) - R(Z,W)\nabla_V) \\
&= \mathfrak{S}(\nabla_V(R(Z,W))) - \mathfrak{S}(R([V,Z],W)) - \mathfrak{S}(R(Z,W)\nabla_V) \\
&= \mathfrak{S}(\nabla_V\nabla_Z\nabla_W - \nabla_V\nabla_W\nabla_Z - \nabla_V\nabla_{[Z,W]}) \\
&-\mathfrak{S}(\nabla_{[V,Z]}\nabla_W - \nabla_W\nabla_{[V,Z]} - \nabla_{[[V,Z],W]}) \\
&-\mathfrak{S}(\nabla_Z\nabla_W\nabla_V - \nabla_W\nabla_Z\nabla_V - \nabla_{[Z,W]}\nabla_V) \\
&= (\mathfrak{S}(\nabla_V\nabla_Z\nabla_W) - \mathfrak{S}(\nabla_Z\nabla_W\nabla_V)) - (\mathfrak{S}(\nabla_V\nabla_W\nabla_Z) - \mathfrak{S}(\nabla_W\nabla_Z\nabla_V)) \\
&-(\mathfrak{S}(\nabla_V\nabla_{[Z,W]}) - \mathfrak{S}(\nabla_W\nabla_{[V,Z]})) - (\mathfrak{S}(\nabla_{[V,Z]}\nabla_W) - \mathfrak{S}(\nabla_{[Z,W]}\nabla_V)) \\
&+(\mathfrak{S}(\nabla_{[[Z,V],W]})) \\
&= 0 - 0 - 0 - 0 + 0 \\
&= 0.
\end{aligned}
$$

Therefore we obtained

$$\mathfrak{S}(\nabla_V(R(Z,W)) - R([V,Z],W) - R(Z,W)\nabla_V) = 0.$$

For this reason we have

$$\begin{aligned}
&< (\mathfrak{S}(\nabla_V(R(Z,W)) - R(\mathrm{Tor}(V,Z),W) \\
&-R([V,Z],W) - R(Z,W)\nabla_V))X,Y > \\
&=< (\mathfrak{S}(-R(\mathrm{Tor}(V,Z),W))X,Y > .
\end{aligned}$$

If we recall Remark 4.8, where some properties of Torsion are described, we have

$$\mathrm{Tor}(V,Z) = -\mathrm{Tor}(Z,V)$$

and therefore the equation above is equal to

$$< (\mathfrak{S}(R(\mathrm{Tor}(Z,V),W))X,Y > .$$

We now make the following observation by using Corollary 4.11

$$\begin{aligned}
< R(\mathrm{Tor}(Z,V),W)X,Y > &= K^S(\mathrm{Tor}(Z,V),X,Y) \\
&= K^S(X,Y,\mathrm{Tor}(Z,V),W).
\end{aligned}$$

We also know from Lemma 3.1 that

$$\mathrm{Tor}(V^{(j)},V^{(j)}) \subset \widehat{V}^{(j)}$$

and by part 1 and 3 of Corollary 4.7 that

$$K^S(TM,TM,HM,VM) = 0 = K^S(TM,TM,VM,HM).$$

In our case we have

$$< (\mathfrak{S}(R(\mathrm{Tor}(Z,V),W))X,Y >= 0,$$

which proves the Lemma. $\square$

Chapter 5

Ricci Curvature and Bochner Formula

In Chapter 5 we begin by defining the Sub-Riemannian Ricci curvature and the notion of the horizontal scalar curvature. We continue with some properties about Ricci curvature derived from previous results. We proceed to the Contracted Bianchi Identity. We then define the horizontal gradient of a tensor T, as well as the horizontal Hessian of T, the symmetric horizontal Hessian T and the horizontal Laplacian of T. We also define vertical rigidity. We then continue to a Lemma where equivalent forms of vertical rigidity are presented and proved and use it to advance to a Lemma 5.10, where a formula about the Laplacian is proved. The chapter is completed with the proof of a Sub-Riemannian generalization of Bochner Formula.

5.1 Ricci Curvature and properties

We will investigate the Sub-Riemannian notion of the Ricci curvature and create a geometrically defined horizontal Laplace operator. The relationship between the horizontal Laplacian and Ricci curvature will be studied and in the end the Sub-Riemannian version of Bonner-Myers theorem will be presented.

Definition 5.1. 1. The *Sub-Riemannian Ricci curvature of* ∇ is defined by

$$\mathrm{Ric}^S(A,B) = \sum_k K^S(A, X_k, X_k, B),$$

where by X_k we denote any horizontal orthonormal frame.

2. We now introduce the notion of the *horizontal scalar curvature* that is defined by

$$S_h = \mathrm{tr}_h \mathrm{Ric}^S = \sum_k \mathrm{Ric}^S(X_k, X_k).$$

Remark 5.2. 1. It should also be mentioned that the scalar curvature is independent of the choice of extension g as is the Ricci curvature when restriction to horizontal vector fields is made.

2. It should be mentioned that in general the Ricci curvature for the canonical connection is not symmetric.

Despite the above remark, if we use Corollary 4.11 and other already mentioned properties of the connection, we have

Lemma 5.3. 1. If VM is normal and $X, Y \in HM$ then

$$\mathrm{Ric}^S(X,Y) = \mathrm{Ric}^S(Y,X).$$

2. If VM is normal and integrable then

$$\mathrm{Ric}^S(VM, HM) = 0.$$

Proof. For the first part we have from the definition that

$$\mathrm{Ric}^S(X,Y) := \sum_k K^S(X, X_k, X_k, Y).$$

If we apply the Corollary 4.11 and then Lemma 4.7 we have

$$\sum_k K^S(X, X_k, X_k, Y) = \sum_k K^S(X_k, Y, X, X_k) = -\sum_k K^S(Y, X_k, X, X_k)$$

and if we apply the Lemma 4.7 again and use the definition we have

$$-\sum_k K^S(Y, X_k, X, X_k) = \sum_k K^S(Y, X_k, X_k, X) := \mathrm{Ric}^S(Y, X)$$

and therefore

$$\mathrm{Ric}^S(X, Y) = \mathrm{Ric}^S(Y, X).$$

For the second part we have from the definition that

$$\mathrm{Ric}^S(U, X) := \sum_k K^S(U, E_k, E_k, X)$$

and if we apply Lemma 4.7 twice we have

$$\sum_k K^S(U, E_k, E_k, X) = -\sum_k K^S(E_k, U, E_k, X) = \sum_k K^S(E_k, U, X, E_k)$$

and if we use Corollary 4.11 and the Lemma 4.7 after, we obtain

$$\sum_k K^S(E_k, U, X, E_k) = \sum_k K^S(X, E_k, E_k, U) = 0,$$

due to the fact that K^S is of the form $K^S(TM, TM, HM, VM) = 0$ and therefore

$$\mathrm{Ric}^S(U, X) = 0,$$

which was the desired result. □

Lemma 5.4. Suppose VM is normal and integrable, then for all horizontal vectors X

$$\nabla_X S_h = 2\sum(\nabla \mathrm{Ric}^S)(E_j, X, E_j),$$

where E_i is an orthonormal frame for HM. Equivalently,

$$\nabla S_h = 2\mathrm{tr}_h(\nabla \mathrm{Ric}^S).$$

Proof. Let X be any element of a normal frame at p. We recall the Differential Bianchi Identity

$$(\nabla_V K^S)(X, Y, Z, W) + (\nabla_Z K^S)(X, Y, W, V) + (\nabla_W K^S)(X, Y, V, Z) = 0.$$

We apply the Differential Bianchi Identity to E_i, E_j, E_j, E_i, X and sum over i and j. We have

$$(\nabla_X K^S)(E_i, E_j, E_j, E_i)+(\nabla_{E_j} K^S)(E_i, E_j, E_i, X)+(\nabla_{E_i} K^S)(E_i, E_j, X, E_j) = 0.$$

If we expand we obtain

$$\begin{aligned}
&(\nabla_X K^S)(E_i, E_j, E_j, E_i) + (\nabla_{E_j} K^S)(E_i, E_j, E_i, X) + (\nabla_{E_i} K^S)(E_i, E_j, X, E_j)\\
&= \nabla_X (K^S(E_i, E_j, E_j, E_i)) - K^S(\nabla_X E_i, E_j, E_j, E_i) - K^S(E_i, \nabla_X E_j, E_j, E_i)\\
&-K^S(E_i, E_j, \nabla_X E_j, E_i) - K^S(E_i, E_j, E_j, \nabla_X E_i)\\
&\nabla_{E_j}(K^S(E_i, E_j, E_i, X)) - K^S(\nabla_{E_j} E_i, E_j, E_i, X) - K^S(E_i, \nabla_{E_j} E_j, E_i, X)\\
&-K^S(E_i, E_j, \nabla_{E_j} E_i, X) - K^S(E_i, E_j, E_i, \nabla_{E_j} X)\\
&+\nabla_{E_i}(K^S(E_i, E_j, X, E_j)) - K^S(\nabla_{E_i} E_i, E_j, X, E_j) - K^S(E_i, \nabla_{E_i} E_j, X, E_j)\\
&-K^S(E_i, E_j, \nabla_{E_i} X, E_j) - K^S(E_i, E_j, X, \nabla_{E_i} E_j).
\end{aligned}$$

Furthermore, by applying the Lemma 4.7 we have

$$\begin{aligned}
&\nabla_{E_j}(K^S(E_i, E_j, E_i, X)) - K^S(\nabla_{E_j} E_i, E_j, E_i, X) - K^S(E_i, \nabla_{E_j} E_j, E_i, X)\\
&-K^S(E_i, E_j, \nabla_{E_j} E_i, X) - K^S(E_i, E_j, E_i, \nabla_{E_j} X)\\
&= -\nabla_{E_j}(K^S(E_j, E_i, E_i, X)) + K^S(E_j, \nabla_{E_j} E_i, E_i, X) + K^S(\nabla_{E_j} E_j, E_i, E_i, X)\\
&+K^S(E_j, E_i, \nabla_{E_j} E_i, X) + K^S(E_j, E_i, E_i, \nabla_{E_j} X)\\
&= -(\nabla_{E_j} K^S)(E_j, E_i, E_i, X)
\end{aligned}$$

and by applying the Lemma 4.7 again we have

$$\begin{aligned}
&\nabla_{E_i}(K^S(E_i, E_j, X, E_j)) - K^S(\nabla_{E_i} E_i, E_j, X, E_j) - K^S(E_i, \nabla_{E_i} E_j, X, E_j)\\
&-K^S(E_i, E_j, \nabla_{E_i} X, E_j) - K^S(E_i, E_j, X, \nabla_{E_i} E_j)\\
&= -\nabla_{E_i}(K^S(E_i, E_j, E_j, X)) + K^S(\nabla_{E_i} E_i, E_j, E_j, X) + K^S(E_i, \nabla_{E_i} E_j, E_j, X)\\
&+K^S(E_i, E_j, E_j, \nabla_{E_i} X) + K^S(E_i, E_j, \nabla_{E_i} E_j, X)\\
&= -(\nabla_{E_i} K^S)(E_i, E_j, E_j, X),
\end{aligned}$$

which means the sums over i, j of two last terms of the Identity are equal.

Therefore we have

$$\begin{aligned}
&\sum_{i,j}(\nabla_{E_j}K^S)(E_i,E_j,E_i,X)+\sum_{i,j}(\nabla_{E_i}K^S)(E_i,E_j,X,E_j)\\
&\sum_{i,j}(\nabla_{E_j}K^S)(E_i,E_j,E_i,X)+\sum_{j,i}(\nabla_{E_j}K^S)(E_j,E_i,X,E_i)\\
&=-2\sum_{i,j}(\nabla_{E_j}K^S)(E_j,E_i,E_i,X)=-2(\nabla_{E_j}\mathrm{Ric}^S)(E_j,X)\\
&=-2(\nabla\mathrm{Ric}^S)(E_j,X,E_j).
\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{i,j}(\nabla_X K^S)(E_i,E_j,E_j,E_i)+\sum_{i,j}(\nabla_{E_j}K^S)(E_i,E_j,E_i,X)+\\
&+\sum_{i,j}(\nabla_{E_i}K^S)(E_i,E_j,X,E_i)=0,
\end{aligned}$$

which is equivalent to

$$\sum_{i,j}(\nabla_X K^S)(E_j,E_i,E_i,E_j)-2(\nabla\mathrm{Ric}^S)(E_j,X,E_j)=0,$$

or in other words

$$\nabla_X S_h=2(\nabla\mathrm{Ric}^S)(E_j,X,E_j),$$

which was the desired formula. □

For the next Corollary recall Remark 2.3.

Corollary 5.5. Suppose that M is a connected SRC-manifold such that HM bracket generates, $\dim(HM)=d>2$ and that VM is normal and integrable.

If $\mathrm{Ric}^S(X,Y)=\lambda<X,Y>$ for all horizontal all vectors X,Y then λ must be constant.

Proof. Let E_i be a normal frame at $p\in M$. Then at p,

$$S_h=\sum_i \mathrm{Ric}^S(E_i,E_i)=\lambda d.$$

Keeping in mind Corollary 4.11, we have

$$2\mathrm{tr}_h(\nabla \mathrm{Ric}^S)(E_j) = 2\sum_{i,j}(\nabla_{E_i}\mathrm{Ric}^S)(E_j, E_i) = 2\sum_{i,j}(\nabla_{E_i}\mathrm{Ric}^S)(E_i, E_j) = 2E_j\lambda.$$

Since $E_j S_h = 2\mathrm{tr}_h(\nabla \mathrm{Ric}^S)(E_j)$, we must have $d = 2$ or $E_j\lambda = 0$. Therefore all horizontal vector fields annihilate λ. As HM bracket generates, this imples that λ is constant. □

5.2 Operators

One of our purposes is to use Bochner type results to study the relationships between curvature, geometry and topology on Sub-Riemannian manifolds. To use this theory, we shall need a geometrically defined subelliptic Laplacian.

Definition 5.6.

1. For a tensor T, the *horizontal gradient of T* is defined by

$$\nabla_h \mathrm{T} = \sum_i \nabla_{E_i}\mathrm{T} \otimes E_i,$$

where $\{E_i\}$ is an orthonormal frame for HM. The gradients ∇_{E_j} are defined similarly for $V^{(j)}$.

2. The *horizontal Hessian of T* is defined by

$$\nabla_h^2 \mathrm{T}(X, Y) = (\nabla_X\nabla_Y - \nabla_{\nabla_X Y})\mathrm{T},$$

for $X, Y \in HM$ and zero otherwise.

3. The *symmetric horizontal Hessian T* is defined to be

$$\Delta_h^{2,sym}\mathrm{T}(X, Y) = \frac{1}{2}(\nabla_h^2\mathrm{T}(X, Y) + \nabla_h^2\mathrm{T}(Y, X)).$$

4. The *horizontal Laplacian of* T is defined by

$$\Delta_h \mathrm{T} = \mathrm{tr}(\nabla_h^2 \mathrm{T}) = (\nabla_{E_i}\nabla_{E_i} - \nabla_{\nabla_{E_i}}E_i)\mathrm{T}.$$

The horizontal Laplacian does not behave as nicely as its Riemannian counterpart. To remedy this we will make below an additional assumption of the metric extension.

Definition 5.7. For a metric extension of an r-grading we define a 1-form $\mathfrak{R}_g$ by

$$\mathfrak{R}_g(\upsilon) = \sum_{j>0}\sum_i B^{(j)}(E_i^{(j)}, E_i^{(j)}, \upsilon_0),$$

where $E_i^{(j)}$ is an orthonormal frame for $V^{(j)}$.

We say that a complement VM is *vertically rigid* if there exists a metric extension g such that $\mathfrak{R}_g \equiv 0$.

Lemma 5.8. For an orientable SRC-maifold, the following are equivalent:

1. VM is vertically rigid.
2. There exists a volume form dV on M such that for any horizontal vector field
$$\mathrm{div}X = \mathrm{tr}_h(\nabla X) = \sum_i < \nabla_{E_i} X, E_i >,$$
where E_i is an orthonormal frame for HM.
3. Every metric extension g is vertically conformal to a metric $\tilde{g}$ with $\mathfrak{R}_{\tilde{g}}$.

Furthermore, if HM bracket generates, then the volume form in (b) is unique up to constant multiplication.

Proof. ((1) $\implies$ (2))

We first note that for the particular metric extension g with $\mathfrak{R}_g \equiv 0$, we have

$$\sum_{j>0}\sum_i < \mathrm{Tor}(E_i^{(j)}, X), E_i^{(j)} >= 0.$$

This is due to the fact that

$$0 = \mathfrak{R}_g = \sum_{j>0}\sum_i B^{(j)}(E_i^{(j)}, E_i^{(j)}, X).$$

We have that

$$\begin{aligned} B^{(j)}(E_i^{(j)}, E_i^{(j)}, X) &= X < E_i^{(j)}, E_i^{(j)} > + < [E_i^{(j)}, X], E_i^{(j)} > + < [E_i^{(j)}, X], E_i^{(j)} > \\ &= 2 < [E_i^{(j)}, X], E_i^{(j)} > \\ &= 0. \end{aligned}$$

We also have

$$\sum_{j>0}\sum_i < \mathrm{Tor}(E_i^{(j)}, X), E_i^{(j)} >= \sum_{j>0}\sum_i < \nabla_{E_i^{(j)}} X - \nabla_X E_i^{(j)} - [E_i^{(j)}, X], E_i^{(j)} > .$$

For $j > 0$ from the third formula of the definition of the connection we have

$$\begin{aligned} &< \nabla_{E_i^{(j)}} X - \nabla_X E_i^{(j)} - [E_i^{(j)}, X], E_i^{(j)} > \\ &=< \nabla_{E_i^{(j)}} X, E_i^{(j)} > - < \nabla_X E_i^{(j)}, E_i^{(j)} > - < [E_i^{(j)}, X], E_i^{(j)} > \\ &=< [E_i^{(j)}, X]_h + \frac{1}{2} C^{(h)}(X, E_i^{(j)}), E_i^{(j)} > \\ &- < [X, E_i^{(j)}]_j + \frac{1}{2} C^{(j)}(E_i^{(j)}, X), E_i^{(j)} > - < [E_i^{(j)}, X], E_i^{(j)} > \\ &= \frac{1}{2} B^{(h)}(X_h, (E_i^{(j)})_h, (E_i^{(j)})_{\widehat{h}}) - \frac{1}{2} B^{(j)}((E_i^{(j)})_j, (E_i^{(j)})_j, X_{\widehat{j}}) \\ &= 0 - < [E_i^{(j)}, X], E_i^{(j)} > \\ &= 0. \end{aligned}$$

Therefore we proved the above relation.

Now we recall the standard result from (see [4], appendix 6) that since ∇ is metric compatible and HM is parallel, the divergence operator for the metric volume from g satisfies

$$\begin{aligned} &\mathrm{div}_g X = \mathrm{tr}_h(\nabla + \mathrm{Tor})(X) \\ &= \mathrm{tr}_h(\nabla X) + \sum_{j>0}\sum_i < \mathrm{Tor}(E_i^{(j)}, X), E_i^{(j)} > \\ &= \mathrm{tr}_h(\nabla X) - \mathfrak{R}_g(X) \\ &= \mathrm{tr}_h(\nabla X). \end{aligned}$$

Thus we can set $dV = dV_g$.

((2) $\Longrightarrow$ (3))

We consider metrics vertically conformal to an arbitrary extension g,

$$g_\lambda = \begin{cases} g & \text{on } HM, \\ e^\lambda g & \text{on } VM. \end{cases}$$

Now if $dV_g = e^\mu dV$, then set $\lambda = -\frac{\mu}{dim(VM)}$ so dV_{g_λ}. Then for horizontal X

$$\mathrm{tr}_h(\nabla X) - \mathfrak{R}_{g_\lambda}(X) = \mathrm{div}_{g_\lambda} X = \mathrm{div} X = \mathrm{tr}_h(\nabla X),$$

so $\mathfrak{R}_{g_\lambda} \equiv 0$.

Since the proof of ((3) $\Longrightarrow$ (1)) is trivial, the proof of the lemma is complete.

For the uniqueness part, we note that if $\Omega = e^\lambda dV$ then for any horizontal X, we have $\mathrm{div}_\Omega X = \mathrm{div} X - X(\lambda)$. If the two divergences agree on the horizontal vector fields and HM bracket generates, this immediately implies that λ is a constant. □

Remark 5.9. For an orientable, vertically rigid SRC-manifold, there is then a 1-dimensional family of volume form for which $\mathrm{div} X = \mathrm{tr}(\nabla X)$. We shall often refer to such a volume form. Vertical rigidity therefore gives us a canonical notion of integration on a SRC-manifold that does not depend on the choice of metric extension.

As an immediate consequence, we have

Lemma 5.10. Suppose that M is orientable and VM is vertically rigid. Then on functions

$$\Delta = \sum_i E_i^2 + E_i \mathrm{div}(E_i),$$

where the divergence is taken with respect to a rigid volume form.

Proof. For functions we have that

$$\nabla(f) = \sum_i \nabla_{E_i}(f) \otimes E_i = \sum_i E_i(E_i(f)).$$

Therefore combined with the fact that

$$\mathrm{div}X = \sum_i < \nabla_{E_i} X, E_i >,$$

we have

$$\begin{aligned}&\Delta(f) = \mathrm{div}(grad(f))\\&= \sum_k < \nabla_{E_k}(\mathrm{grad}(f)), E_k >= \sum_k \sum_i < \nabla_{E_k}(E_i(f)E_i), E_k > .\end{aligned}$$

Due to the definition of connectors ($\nabla_X(fY) = X(f)Y + f\nabla_X Y$):

$$\nabla_{E_k}(E_i(f)E_i) = (E_k(E_i(f))E_k) + E_i(f)\nabla_{E_k} E_i,$$

we have

$$\begin{aligned}&= \sum_k \sum_i < E_k(E_i(f))E_i + E_i(f)\nabla_{E_k} E_i, E_k >\\&= \sum_k \sum_i (< E_k(E_i(f))E_i, E_k > + < E_i(f)\nabla_{E_k} E_i, E_k >)\\&= \sum_k \sum_i E_k(E_i(f)(< E_i, E_k >) + \sum_k \sum_i E_i(f)(< \nabla_{E_k} E_i, E_k >)\\&= \sum_i E_i(E_i(f)) + \sum_i E_i(f) \sum_k (< \nabla_{E_k} E_i, E_k >)\\&= \sum_i E_i(E_i(f)) + \sum_i E_i(f)\mathrm{div}(E_i) = \sum_i (E_i(E_i(f)) + E_i(f)\mathrm{div}(E_i)).\end{aligned}$$

Therefore we have obtained the required formula. □

5.3 Sub-Riemannian generalization of Bochner Formula

Before we prove the generalization, we introduce some terminology.

Definition 5.11. We define $J : TM \times TM \mapsto TM$ by

$$< J(A, Z), B >=< \mathrm{Tor}(A, B), Z > .$$

Remark 5.12. We recall that a vector field F is closed if and only if $A \mapsto < F, A >$ is a closed 1-form or in other words $\omega(A) =< F, A >$ is closed.

Due to the fact that the form is closed and if we recall from Corollary 2.20 that

$$d\omega(A, B) = A\omega(B) - B\omega(A) - \omega([A, B]),$$

from the compatability of the metric we obtain

$$\begin{aligned} 0 &= A < F, B > -B < F, A > - < F, [A, B] > \\ &= < \nabla_A F, B > + < F, \nabla_A B > \\ &\quad -(< \nabla_B F, A > + < F, \nabla_B A >) - < F, [A, B] > \\ &= < \nabla_A F, B > - < \nabla_B F, A > - < \mathrm{Tor}(B, A), F > . \end{aligned}$$

Combined with the fact that

$$< J(A, F), B >=< \mathrm{Tor}(A, B), F >= - < \mathrm{Tor}(B, A), F >= - < J(B, F), A >,$$

because as we have shown $\mathrm{Tor}(A, B) = -\mathrm{Tor}(B, A)$.

The above imply that

$$< \nabla_B F, A >=< \nabla_A F, B > - < J(B, F), A >=< \nabla_A F, B > + < J(A, F), B > .$$

Applying Lemma 5.8 on a vertically rigid SRC-manifold of the horizontal Laplacian allows the following generalisation of the classical Bochner formulas:

Theorem 5.13. If F is a closed vector field and F_k is the projection of F to $V^{(k)}$ then

$$\begin{aligned} &\frac{1}{2}\Delta_h |F_j|^2 = \mathrm{Ric}^S(F_j, F_h) + |\nabla_h F_j|^2 \\ &+ \sum_i (< E_i, \nabla_h^2 F_j(F_j, E_i) > -2 < \nabla_{E_i} F, \mathrm{Tor}(E_i, F_j) > \\ &+ < F, (\nabla \mathrm{Tor})(F_j, E_i, E_i) > - < F, \mathrm{Tor}_2(E_i, E_i, F_j) >), \end{aligned}$$

where E_i is any orthonormal horizontal frame.

Proof. We set $u = \frac{1}{2}|F_j|^2$; then for a horizontal vector field Y and an arbitrary vector field in $V^{(j)}$ we then have

$$\begin{aligned}
< \nabla_h u, Y > &:= < \mathrm{grad}_h u, Y > \\
&:= Y(u) \\
&= Y(\frac{1}{2} < F_j, F_j >) \\
&= < \nabla_Y F_j, F_j > \\
&= < \nabla_Y < F, Z >, F_j > \\
&= << \nabla_Y F, Z >, F_j > + << \nabla_Y Z, F >, F_j > \\
&= < \nabla_Y F, F_j > + < \nabla_Y Z, F_j > \\
&= < \nabla_Y F, F_j > +0 \\
&= < \nabla_{F_j} F, Y > + < J(F_j, F), Y > .
\end{aligned}$$

Therefore we obtain

$$\nabla_h u = \nabla_{F_j} F_h + J(F_j, F)_h,$$

where by h we mean horizontal.

In order to prove the main result we need two preliminaries. The first preliminary is for horizontal X, Y

$$\nabla_h^2 u(X, Y) = X < Y, \nabla_h u > - < \nabla_X Y, \nabla_h u >=< Y, \nabla_X \nabla_h u > .$$

The second preliminary is

$$\begin{aligned}
< \nabla_X J(F_j, F), X > &= X < J(F_j, F), X > - < J(F_j, F), \nabla_X X > \\
&= X < F, \mathrm{Tor}(F_j, X) > - < J(F_j, F), \nabla_X X > \\
&= < \nabla_X F, \mathrm{Tor}(F_j, X) > + < F, \nabla_X \mathrm{Tor}(F_j, X) > \\
&- < F, \mathrm{Tor}(F_j, \nabla_X X) > .
\end{aligned}$$

We now proceed to the proof of the Bochner Formula.

For a horizontal vector field X, we have by making use of the above preliminaries

$$\begin{aligned}
\nabla_h^2 u(X,X) &= <\nabla_X \nabla_h u, X> \\
&= <\nabla_X \nabla_{F_j} F_h, X> + <\nabla_X J(F_j,F)_h, X> \\
&= K^S(X,F_j,F_h,X) + <\nabla_{F_j}\nabla_X F_h, X> + <\nabla_{[X,F_j]} F_h, X> \\
&+ <\nabla_X J(F_j,F)_j, X> \\
&= K^S(X,F_j,F_h,X) + <\nabla_{F_j}\nabla_X F_h, X> \\
&+ <\nabla_{\nabla_X F_j - \nabla_{F_j} X - \mathrm{Tor}(X,F_j)} F_h, X> + <\nabla_X J(F_j,F)_h, X> \\
&= K^S(X,F_j,F_h,X) + <\nabla_{F_j}\nabla_X F_h - \nabla_{\nabla_{F_j} X} F_h, X> \\
&+ <\nabla_{\nabla_X F_j - \mathrm{Tor}(X,F_j)} F_h, X> + <\nabla_X J(F_j,F)_h, X> .
\end{aligned}$$

We now focus on

$$<\nabla_{\nabla_X F_j - \mathrm{Tor}(X,F_j)} F_h, X> .$$

We have

$$<\nabla_{\nabla_X F_j F_h - \mathrm{Tor}(X,F_j)} F_h, X> = <\nabla_{\nabla_X F_j} F_h, X> - <\nabla_{\mathrm{Tor}(X,F_j)} F_h, X>$$

and if we use the preliminaries we obtain

$$\begin{aligned}
&<\nabla_{\nabla_X F_j} F_h, X> - <\nabla_{\mathrm{Tor}(X,F_j)} F_h, X> \\
&= <\nabla_X F_j, \nabla_X F_j> + <F, \mathrm{Tor}(X, \nabla_X F_j)> \\
&-(<\nabla_X F, \mathrm{Tor}(X,F_j)> + <F, \mathrm{Tor}(X, \mathrm{Tor}(X,F_j))>),
\end{aligned}$$

due to the fact that

$$\begin{aligned}
<\nabla_X F_j, \nabla_X F_j> &= <\nabla_{\nabla_X F_j} F_j, X> + <J(\nabla_X F_j.F_j), X> \\
&= <\nabla_{\nabla_X F_j} F_j, X> + <\mathrm{Tor}(\nabla_X F_j, X), F_j> \\
&= <\nabla_{\nabla_X F_j} F_j, X> - <\mathrm{Tor}(X, \nabla_X F_j), F_j>
\end{aligned}$$

and

$$\begin{aligned}
<\nabla_{\mathrm{Tor}(X,F_j)} F, X> &= <\nabla_X F, \mathrm{Tor}(X,F_j)> + <J(X,F), \mathrm{Tor}(X,F_j)> \\
&= <\nabla_X F, \mathrm{Tor}(X,F_j)> + <\mathrm{Tor}(X, \mathrm{Tor}(X,F_j)), F> .
\end{aligned}$$

Therefore

$$
\begin{aligned}
&K^S(X,F_j,F_h,X)+<\nabla_{F_j}\nabla_X F_h-\nabla_{\nabla_{F_j}X}F_h,X>\\
&+<\nabla_{\nabla_X F_j-\mathrm{Tor}(X,F_j)}F_h,X>+<\nabla_X J(F_j,F)_h,X>\\
&=K^S(X,F_j,F_h,X)+<\nabla_{F_j}\nabla_X F_h-\nabla_{\nabla_{F_j}X}F_h,X>\\
&+(<\nabla_X F_j,\nabla_X F_j>+<F,\mathrm{Tor}(X,\nabla_X F_j)>\\
&-(<\nabla_X F,\mathrm{Tor}(X,F_j)>+<F,\mathrm{Tor}(X,\mathrm{Tor}(X,F_j))>))\\
&+<\nabla_X J(F_j,F)_h,X>\\
&=K^S(X,F_j,F_h,X)+<\nabla^2_h F_h(F_j,X),X>\\
&+(|\nabla_X F_j|^2+<F,\mathrm{Tor}(X,\nabla_X F_j)>\\
&-(<\nabla_X F,\mathrm{Tor}(X,F_j)>+<F,\mathrm{Tor}_2(X,X,F_j)>)\\
&+<\nabla_X J(F_j,F)_h,X>.
\end{aligned}
$$

We now show that

$$
\begin{aligned}
&<F,\mathrm{Tor}(X,\nabla_X F_j)>-<\nabla_X F,\mathrm{Tor}(X,F_j)>+<\nabla_X J(F_j,F)_h,X>\\
&=-2<\nabla_X F,\mathrm{Tor}(X,F_j)>+<F,(\nabla\mathrm{Tor})(F_j,X,X)>,
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&<F,\mathrm{Tor}(X,\nabla_X F_j)>+<\nabla_X F,\mathrm{Tor}(X,F_j)>+<\nabla_X J(F_j,F)_h,X>\\
&=<F,\nabla_X\mathrm{Tor}(F_j,X)>-<F,\mathrm{Tor}(F_j,\nabla_X X)>-<F,\mathrm{Tor}(\nabla_X F_j,X)>,
\end{aligned}
$$

or in other words

$$
\begin{aligned}
&<\nabla_X F,\mathrm{Tor}(X,F_j)>+<\nabla_X J(F_j,F)_h,X>\\
&=<F,\nabla_X\mathrm{Tor}(F_j,X)>-<F,\mathrm{Tor}(F_j,\nabla_X X)>.
\end{aligned}
$$

By metric compatibility we have

$$
\begin{aligned}
<\nabla_X J(F_j,F),X> &= X<J(F_j,F),X>-<J(F_j,F),\nabla_X X>\\
&= X<\mathrm{Tor}(F_j,X),F>-<\mathrm{Tor}(F_j,\nabla_X X),F>\\
&= <\nabla_X\mathrm{Tor}(F_j,X),F>+<\mathrm{Tor}(F_j,X),\nabla_X F>\\
&- <\mathrm{Tor}(F_j,\nabla_X X),F>.
\end{aligned}
$$

Therefore the above equation holds.

This implies that

$$\begin{aligned}
&K^S(X, F_j, F_h, X) + < \nabla_h^2 F_h(F_j, X), X > + |\nabla_X F_j|^2 \\
&(< \nabla_X F_j, \nabla_X F_j > + < F, \mathrm{Tor}(X, \nabla_X F_j) > \\
&-(< \nabla_X F, \mathrm{Tor}(X, F_j) > + < F, \mathrm{Tor}_2(X, X, F_j) >)) \\
&+ < \nabla_X J(F_j, F)_h, X >,
\end{aligned}$$

is equal to

$$\begin{aligned}
&K^S(X, F_j, F_h, X) + < \nabla_h^2 F_h(F_j, X), X > + |\nabla_X F_j|^2 \\
&-2 < \nabla_X F, \mathrm{Tor}(X, F_j) > + < F, (\nabla \mathrm{Tor})(F_j, X, X) > \\
&- < F, \mathrm{Tor}_2(X, X, F_j) > .
\end{aligned}$$

If we set $X = E_i$ and sum over i, we get the desired result. $\square$

Chapter 6

Comparison with the Riemannian Geometry

In the final Chapter, we compare the Sub-Riemannian connection to the Levi-Civita connection for a metric extension. Then the Bonnet-Myers is presented as well as a Sub-Riemannian version of the Theorem, which does not apply for Riemannian manifolds.

6.1 Comparison with the Riemannian Curvatures

We now make a refinement to the operator J (Specifically Definition 5.11 used in Chapter 5), presented in the proof of the Sub-Riemannian version of the Bochner formula

$$\begin{aligned} < J^1(A,B), C > &= < \mathrm{Tor}(A,C), B_1 >, \\ < J^0(A,B), C > &= < \mathrm{Tor}(A,C), B_0 >, \end{aligned}$$

where $B = B_0 + B_1$.

Lemma 6.1. For any SRC-manifold (with no restriction on $\dim VM$) the Levi-Civita connection associated to g can be computed from the basic connection for g as follows

$$
\begin{aligned}
\overline{\nabla}_X Y &= \nabla_X Y - \frac{1}{2}\mathrm{Tor}(X,Y) + J^1(X,Y),\\
\overline{\nabla}_T T &= \nabla_T T - \frac{1}{2}J^0(T,T),\\
\overline{\nabla}_T X &= \nabla_T X + \frac{1}{2}J^0(X,T) - \mathrm{Tor}(T,X)_1,\\
\overline{\nabla}_X T &= \nabla_X T + \frac{1}{2}J^0(X,T) - \mathrm{Tor}(X,T)_0.
\end{aligned}
$$

We now utilize the above formulas in order to obtain the following Corollary

Corollary 6.2. If X, Y are horizontal vector fields and T is a vertical vector field then

$$
\begin{aligned}
\overline{K}^S(X,Y,Y,X) &= K^S(X,Y,Y,X) - \frac{3}{4}|\mathrm{Tor}(X,Y)|^2\\
&\quad - < J^1(Y,Y), J^1(X,X) > + |J^1(X,X)|^2,\\
\overline{K}^S(T,X,X,T) &= K^S(T,X,X,T) + \frac{1}{4}|J^0(X,T)|^2,\\
&\quad + < \nabla\mathrm{Tor}(T,X,X) - \mathrm{Tor}(X,\mathrm{Tor}(X,T)), T >\\
&\quad + < \nabla\mathrm{Tor}(X,T,T), X > - |\mathrm{Tor}(X,T)_0|^2,\\
\overline{K}^S(X,Y,T,X) &= K^S(X,Y,T,X) + \frac{1}{2} < \nabla\mathrm{Tor}(Y,X,X), T >\\
&\quad + < \nabla\mathrm{Tor}(X,T,Y) - \nabla\mathrm{Tor}(Y,T,X), X > .
\end{aligned}
$$

This may not be the complete list of curvature terms, however if the properties of Riemannian and Sub-Riemannian curvatures and polarization identities it is sufficient to compute all sectional and Ricci curvatures for the case $\dim VM = 1$.

Remark 6.3. If M is strictly normal, then second equation from the Corollary 6.2 is equivalent to

$$\overline{K}^S(T,X,X,T) = \frac{1}{4}|J^0(X,T)|^2$$

and therefore, if HM is bracket generated of step 2, there will be at least one plane with positive sectional curvature. The Riemannian approach in order to generalise results concerning negative sectional curvatures is likely to be of high difficulty.

6.2 A Sub-Riemannian version of Bonnet–Myers theorem

In the early steps of Sub-Riemannian Geometry, Riemannian approximations were commonly made. By that we mean, for a chosen Riemannian extension $g = g_0 \oplus g_1$ we would re-scale as $g^\lambda = g_0 \oplus \lambda^2 g_1$ and study them for $\lambda \to \infty$. The idea is that by blowing up vertical directions, the movement in these directions would be extremely inefficient and therefore Riemannian geodesics would converge to Sub-Riemannian geodesics. However, this study is problematic regarding the effects of curvature and this kind of re-scaling enlarges vertical curvatures much more compared to the horizontal ones. This approach is only useful if $\lambda \to 0$.

In this section, Ricci and sectional curvatures of this re-scaled Riemannian metric will be presented in terms of the basic connection. To simplify things, we shall restrict our study to $\dim VM = 1$ and therefore only the basic grading will be needed.

In order to proceed, a Riemannian metric extension $g = g_0 \oplus g_1$ is chosen. The basic connection will be expressed in terms of this metric. From now on $E_1, \ldots, E_d$ will be an orthonormal frame for HM with respect to g and U will be a unit length vector in VM, with respect to g.

Provided that we only use constants for our re-scaling, it can be verified that the covariant derivatives for the basic connection associated to the re-scaled metric will not differ from the base metric. Therefore, by being careful to how the re-scaling is made on each term, we can obtain the Riemannian Ricci curvatures for the metrics $g^\lambda = g_0 \oplus \lambda^2 g_1$.

For $Y \in HM$ and $T \in VM$, with the inner products and norms computed in the unscaled metric we have

$$\begin{aligned}
\overline{\mathrm{Ric}}^{\lambda}(Y,Y) &= \lambda^0(\mathrm{Ric}(Y,Y) + <\nabla\mathrm{Tor}(U,Y,Y),U> - <\mathrm{Tor}_2(Y,Y,U),U>) \\
&\quad +\lambda^2(-\frac{1}{2}\sum_i |\mathrm{Tor}(E_i,Y)|^2) \\
&\quad +\lambda^{-2}(<\nabla\mathrm{Tor}(Y,U,U),Y> - |\mathrm{Tor}(Y,U)_0|^2 \\
&\quad +\sum_i(|J^1(E_i,Y)|^2 - <J^1(E_i,E_i),J^1(Y,Y)>)), \\
\overline{\mathrm{Ric}}^{\lambda}(Y,T) &= \lambda^0(\sum_i <\nabla\mathrm{Tor}(E_i,T,Y) - \nabla\mathrm{Tor}(Y,T,E_i),E_i>) \\
&\quad +\lambda^2(\frac{1}{2} < tr_h(\nabla\mathrm{Tor}(Y)),T>), \\
\overline{\mathrm{Ric}}^{\lambda}(T,T) &= \lambda^0(\sum_i <\nabla\mathrm{Tor}(E_i,T,T),E_i> - |\mathrm{Tor}(E_i,T)_h|^2) \\
&\quad +\lambda^2(\sum_i <\nabla\mathrm{Tor}(E_i,E_i,T),T>) \\
&\quad +\lambda^4(\sum_i \frac{1}{4}|J^0(E_i,T|^2).
\end{aligned}$$

In the case of a strictly normal SRC-manifold, these formulas come down to

$$\begin{aligned}
\overline{\mathrm{Ric}}^{\lambda}(Y,Y) &= \lambda^0\mathrm{Ric}(Y,Y) - \frac{\lambda^2}{2}\sum_i |\mathrm{Tor}(E_i,Y)|^2, \\
\overline{\mathrm{Ric}}^{\lambda}(Y,T) &= \frac{\lambda^2}{2} < tr_h(\nabla\mathrm{Tor}(Y)),T>, \\
\overline{\mathrm{Ric}}^{\lambda}(T,T) &= \frac{\lambda^4}{4}\sum_i |J^0(E_i,T)|^2 = \frac{\lambda^4}{4}\sum_{i,j}|\mathrm{Tor}(E_i,E_j)|^2.
\end{aligned}$$

Definition 6.4. The torsion bounds of M are defined by

$$k_{ij}^m = \sup\left\{|\mathrm{Tor}(X^{(i)},X^{(j)})_m|^2 : |X^{(i)},X^{(j)}| \le 1\right\}.$$

Remark 6.5. Note that $0 \le k_{ij}^m \le +\infty$.

We also recall the Bonnet-Myers theorem from Riemannian Geometry:

Theorem 6.6. Let M a complete, connected m-dimensional Riemannian manifold, $m \geq 2$. If there is an $R > 0$ such that

$$\mathrm{Ric}_p(X, X) \geq \frac{(m-1)}{R^2} \quad, \forall X \in T_pM, \quad p \in M,$$

then we have the following consequences:

1. $\mathrm{diam}(M) \leq \pi R$.
2. M is compact.
3. $\pi_1(M)$ is a finite group.

Theorem 6.7. Suppose g is strictly normal for the basic grading and VM is integrable. If $k^1_{00} < \infty$ and there are constants $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\mathscr{R}(A, A) \geq \rho_1 |A_0|^2 + \rho_2 |A_1|^2,$$

where

$$\mathscr{R}(A, A) = \mathrm{Ric}^S(A_0, A_0) + < A, tr_h(\nabla \mathrm{Tor}(A_0)) > + \frac{1}{4} \sum_{i,j} | < \mathrm{Tor}(E_i, E_j), A > |^2$$

and if we add the assumption that $\mathrm{dim} VM = 1$, then there are constants $\lambda, c \geq 0$ such that

$$\overline{Ric}^\lambda(A, A) \geq c g^\lambda(A, A),$$

for all vectors A, where g^λ is a re-scaled metric.

Remark 6.8. Note that from standard polarization arguments one can define

$$\mathscr{R}(A, B) = \frac{1}{4}(\mathscr{R}(A + B, A + B) - \mathscr{R}(A - B, A - B)).$$

If Theorem 6.7 is combined with the classical Bonnet-Myers (Theorem 6.6) we have

Corollary 6.9. Under the same assumptions as Theorem 6.7, we have that M is compact and has a finite fundamental group.

Definition 6.10. We define the following symmetric 2-tensors

$$\mathscr{B}(X,Y) = < \nabla \mathrm{Tor}(X,U,U), Y > - < \mathrm{Tor}(X,U)_h \mathrm{Tor}(Y,U)_h >$$

and

$$\mathscr{K}(X,Y) = \sum_i (< J^1(E_i,X), J^1(E_i,Y) > - < J^1(E_i,E_i), J^1(X,Y) >),$$

where U is a unit length vertical vector.

The tensor $\mathscr{B}$ is a genuinely SRC-invariant when $\dim VM = 1$, but has no good invariant generalization when $\dim VM > 1$.

Furthermore, $\mathscr{K}$ is only a vertically conformal SRC-invariant.

With these in mind we obtain the following theorem, which is a purely Sub-Riemannian result as the conditions are trivially false when restricted to Riemannian manifolds:

Theorem 6.11. Let M be an SRC-manifold with $\dim VM = 1$ and bounded curvature and torsion. If there are constants $a, b > 0$ such that for all horizontal vectors Y, we have

$$\mathrm{tr}_h \mathscr{B} \geq a|T|^2$$

and

$$\mathscr{B}(Y,Y) + \mathscr{K}(Y,Y) \geq b|Y|^2,$$

then M is compact and has finite fundamental group.

Proof. The condition of bounded curvature implies that for small λ there will be some, possibly large, constant M such that

$$2\overline{Ric}^\lambda(T,Y) \leq 2M|T||Y| \leq \frac{a}{4}|T|^2 + \frac{4M^2}{a}|Y|^2.$$

Since $\mathrm{tr}_h \mathscr{B} \geq a|T|^2$ globally, for sufficiently small λ, we have

$$\overline{Ric}^\lambda(T,T) \geq \frac{a}{2}|T|^2.$$

We also have with usage of metric compatibility, the formulas in the definition of the Connection, Corollary 3.4 and orthogonality we have

$$
\begin{aligned}
&(< \nabla \mathrm{Tor}(U,Y,Y),U > - < \mathrm{Tor}_2(Y,Y,U),U >)\\
&=< \nabla \mathrm{Tor}(U,Y,Y),U > - < \mathrm{Tor}_2(Y,Y,U),U >\\
&=< \nabla_Y \mathrm{Tor}(U,Y),U > - < \mathrm{Tor}(Y,\mathrm{Tor}(Y,U)),U >\\
&=< \nabla_Y \mathrm{Tor}(U,Y),U > -0\\
&=< \nabla_Y \nabla_U Y - \nabla_Y \nabla_Y U - \nabla_Y [U,Y],U >\\
&= Y < \nabla_U Y, U > - < \nabla_U Y, \nabla_Y U > -Y < \nabla_Y U, U > + < \nabla_Y U, \nabla_Y U >\\
&-Y < [U,Y],U > + < [U.Y],\nabla_Y U >\\
&= Y < \mathrm{Tor}(U,Y),U > + < \nabla_Y U, \nabla_Y U > + < [U.Y],\nabla_Y U >\\
&= 0 + < \nabla_Y U, \nabla_Y U > + < [U.Y],\nabla_Y U >\\
&=< \nabla_Y U - \nabla_U Y - [Y,U],\nabla_Y U >\\
&=< \mathrm{Tor}(Y,U),\nabla_Y U >\\
&= 0
\end{aligned}
$$

and since $\mathscr{B}(Y,Y) + \mathscr{K}(Y,Y) \geq b|Y|^2$, again for small λ, we have

$$\overline{Ric}^{\lambda}(Y,Y) \geq \frac{b}{2\lambda^2}|Y|^2.$$

Then for small enough λ we have

$$\overline{Ric}^{\lambda}(T+Y,T+Y) \geq \left(\frac{b}{2\lambda^2} - \frac{4M^2}{\alpha}\right)|Y|^2 + \frac{\alpha}{4}|T|^2.$$

Therefore we have for small enough λ

$$\overline{Ric}^{\lambda}(T+Y,T+Y) \geq c|T+Y|^2,$$

for some positive constant c. The result then follows from the classical Bonnet–Myers theorem. □

For more information about this Chapter see Chapters 4 and 5 in [1].

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