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A study of codimension-2 gravitational defects with Gauss-Bonnet action

Master Thesis

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September 2010

Acknowledgements

I am greatly indebted to G. Kofinas for his patient instructions throughout this master thesis and guidance on this very interesting subject. I would also like to thank N. Tsamis for his help during the semester. I want to thank Y. Konstantinou and Y. Iatrakis for fruitful discussions.

Contents

Chapter 1

Introduction

In this thesis our study on extra dimensions refers to gravitational defects (p-branes) embedded in a higher dimensional bulk. The crucial point in our approach is that the system brane/bulk is considered to be dynamical (the opposite of probe), i.e. we take into account the back reaction of the brane to the bulk and the dynamical variables are the bulk metric $g_{\mu\nu}$ and the embedding fields x^{μ}. We will particularly focus on codimension-2 defects with the assumption of a conical singularity on the 3-brane.
This is an interesting theoretical setting on its own, but it has also been motivated by modern theories (strings, supergravity), where our 4-dimensional world is assumed to reside in 10 or 11 dimensions and we would like to derive p-brane solutions beyond the probe limit.
Especially the codimension-1 case (hypersurfaces) in the context of gravity is well-known and has been studied since Israel [1]. The matching conditions of this case are

$$K_{ab} - K h_{ab} = 8\pi G T_{ab}, \tag{1.1}$$

where K_{ab} is the extrinsic curvature and h_{ab} the induced metric on the brane; they are derived by focusing on the parallel components of the bulk Einstein equations at the position of the brane and isolating the distributional terms with the singular structure $\delta(y)$ (y the extra dimension). For a review on codimension-1 braneworlds see [2].
This is similar to what happens in mathematics in general when we have delta function sources in the equations and for example in electromagnetism we

obtain the junction conditions in terms of the surface sources

$$[\mathbf{E}_{\parallel}] = \mathbf{E}^+_{\parallel} - \mathbf{E}^-_{\parallel} = 0 \quad , \quad [\mathbf{E}_{\perp}] = \mathbf{E}^+_{\perp} - \mathbf{E}^-_{\perp} = 4\pi\sigma\mathbf{n} \tag{1.2}$$

and

$$[\mathbf{B}_{\perp}] = \mathbf{B}^+_{\perp} - \mathbf{B}^-_{\perp} = 0 \quad , \quad [\mathbf{B}_{\parallel}] = \mathbf{B}^+_{\parallel} - \mathbf{B}^-_{\parallel} = 4\pi\mathbf{j} \times \mathbf{n}\,. \tag{1.3}$$

On the contrary, the treatment of a codimension-2 defect is quite different and the situation becomes even more complicated for higher codimensions. Therefore, our starting point will be the understanding of the codimension-2 case with the hope that even higher codimensions could be examined accordingly in the future.

Furthermore, six dimensional gravity or supergravity has been recently [3], [4] proposed to offer a mechanism for understanding the smallness of the vacuum energy. In this scenario, a codimension-2 object induces a conical singularity, and the cancellation occurring between the brane tension and the bulk gravitational degrees of freedom gives rise to a vanishing effective cosmological constant. To see this, we write the higher dimensional Einstein equations

$$\mathcal{R}^{\mu}{}_{\nu} - \frac{1}{2}\mathcal{R}\delta^{\mu}_{\nu} = \kappa^2_D \mathcal{T}^{\mu}{}_{\nu} \tag{1.4}$$

with a localized energy-momentum tensor (brane tension)

$$\mathcal{T}^{\mu}{}_{\nu} = \begin{pmatrix} T^{a}{}_{b} = -\lambda\delta^{a}_{b} & 0 \\ 0 & T^{\alpha}{}_{\beta} = 0 \end{pmatrix}.$$

Then, the parallel ab and normal $\alpha\beta$ components of the equations take the form respectively

$$\mathcal{R}^{a}{}_{b} = -\kappa^2_D \lambda \frac{D-3-p}{D-2}\delta^{a}_{b} \qquad , \qquad \mathcal{R}^{\alpha}{}_{\beta} = \kappa^2_D \lambda \frac{p+1}{D-2}\delta^{\alpha}_{\beta} \tag{1.5}$$

and for $D = 6, p = 3$ we get $\mathcal{R}^{a}{}_{b} = 0$ independently of the value of brane tension λ. So, a Minkowski solution naturally arises for the universe, contrary to the case of codimension-1 [5], where a Minkowski solution arises with the cost of a fine-tuning between the brane tension and the bulk cosmological constant. Of course, the universe finally has a small effective cosmological constant which is not exactly zero (small de Sitter expansion) whose explanation was suggested [6] that would be facilitated by the quantum corrections within the context of 6-dimensional supergravity.

However, codimension-2 gravitational defects in Einstein-Hilbert gravity were found to be inconsistent in general [7] and the reason is based on the new form of singularities $\frac{\delta(r)}{r}$ appeared (wilder than $\delta(y)$). Although the generic analysis is quite complicated, however this can be easily understood in the case of an axially symmetric defect where the bulk metric takes the form

$$ds_6^2 = dr^2 + L^2(x,r)d\theta^2 + h_{ab}(x,r)dx^a dx^b \,. \tag{1.6}$$

The function $L(x,r)$ is expanded as $L(x,r) = \beta_1(x)r + \mathcal{O}(r^2)$ where $\beta_1(x) < 1$ is the deficit angle of the conical singularity necessary to balance the distributional source. So, $L'(x,0^+) = \beta_1(x)$ and assuming $L'(x,0) = 1$ we have the discontinuity $L'(x,0^+) - L'(x,0) = \beta_1 - 1$, thus $L''(x) = \big(\beta_1(x) - 1\big)\delta(r)$. Now, the parallel components of the higher-dimensional Einstein tensor is of the form $\mathcal{G}_{ab} = \frac{L''}{L}g_{ab} + \ldots$ where all the dotted terms are regular. Therefore, the Einstein equation $\mathcal{G}_{ab} = \kappa_6^2 T_{ab}\frac{\delta(r)}{2\pi\beta_1 r}$ leads to the matching condition $T_{ab} = \frac{2\pi}{\kappa_6^2}(\beta_1 - 1)g_{ab}$, which means that in Einstein gravity only a brane tension is compatible. Indeed, this is the case for example of the special solution of the cosmic string in four dimensions (1-brane) [8], where the tension of the string exactly adjusts the deficit angle of the cone $\lambda = \frac{2\pi}{\kappa_6^2}(1 - \beta_1)$.

Since then, the problem has remained unsolved. Several efforts have been made to show the mathematical reasons for this inconsistency, alternative formulations for handling the distributions have also been proposed, or other approaches either modify the equations of motion for the defect, or consider thick defects (where a cut-off is introduced that makes the problem consistent but scheme-dependent), or study the original problem perturbatively (relatively successfully).

It was not until 2004 when it was suggested [9] that the problem of inconsistency might not be the pathology of codimension-2 itself but the simplicity of the Einstein-Hilbert Langrangian density. So, the higher-order six-dimensional Gauss-Bonnet term was added in the action. This term

$$\mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\kappa\lambda}\mathcal{R}^{\mu\nu\kappa\lambda}, \tag{1.7}$$

which makes sense only for $D > 4$ (for $D = 4$ it is a topological invariant [10]) contains particular quadratic combinations of the Riemann tensor in such a way that it contributes to the equations of motion only through second derivatives of the metric. Moreover, this term has been shown to have a well-defined perturbation operator around flat background and is free of ghosts [11] (also [12]). As far as the extensions of the Gauss-Bonnet term are concerned, the number of independent 2-rank tensors which are symmetric and divergence free depend crucially from the number of dimensions and these tensors are known as Lovelock terms [13]. The corresponding terms in the action which give these terms are the Euler densities. So, for $D = 4$ one includes in the action the zeroth density $\mathfrak{L}_o$ (cosmological constant) and the first density $\mathfrak{L}_1$ (Einstein-Hilbert term). For $D = 5, 6$ also $\mathfrak{L}_2$ (Gauss-Bonnet term) becomes non-trivial, for $D = 7, 8$ the third density $\mathfrak{L}_3$ (cubic in curvature) is also included, e.t.c. Of course, if arbitrary combinations of the higher curvature terms were added, the equations would contain higher than second derivatives which is complicated, unnatural and probably pathological.

Some effort was made towards this direction in [14], [15], [16], [17]. The problem turns out to be very complicated because beyond the matching conditions all the other equations have to be derived and their compatibility to be checked at the brane location. For the case of cosmology with an axially symmetric defect the consistency was indeed checked in [18]. A variety of cosmologies were derived on the brane with the freedom of one arbitrary function of time, reflecting to the choice of boundary/asymptotic conditions. This non-uniqueness does not imply any sort of deficiency of the theory, but it expresses the fact that from the viewpoint of the brane observer the system of the effective equations is consistent but not closed. This feature is qualitatively similar to what happens in codimension-1 cosmology [19], where one arbitrary integration constant appears.

In this thesis we adopt a different philosophy to obtain the matching conditions. Contrary to the above conventional way of obtaining the matching conditions by varying the action with respect to the metric (equivalent to isolating the distributional terms in the equations), we here consider the variation with respect to the embedding fields (brane position coordinates) x^μ, but in such a way that the brane is dynamical and back-reacts to the bulk. The same style of variation was performed for Einstein gravity in [20], where new matching conditions were found along with a consistent but rather trivial cosmology. Of course, in the probe limit the matching conditions reduced to the Nambu-Goto equation of motion. Reminiscent of this process for Einstein gravity has appeared for codimension-1 in [21], where relaxed Israel matching conditions were obtained.

In the following, we consider Einstein-Gauss-Bonnet theory in six dimensions where the embeddibility of the brane will be determined by this sort of non-conventional equations of motion for the defect. We will derive the coupled system of equations at the brane position and investigate if the cosmological ansatz is mathematically consistent. It will be interesting to investigate under which condition, if any, the arising cosmology is unique.

Chapter 2

Submanifolds in spacetime

Consider a D-dimensional manifold M which is considered to be the spacetime bulk and a $(p+1)$-dimensional manifold Σ with $p+1 < D$, equipped with a map $\phi : \Sigma \longrightarrow M$, which is C^∞ and one-to-one, and its inverse $\phi^{-1} : \phi[\Sigma] \longrightarrow \Sigma$ which is also C^∞. Then, the image $\phi[\Sigma]$ is said to be an embedded submanifold (surface) of M of codimension $D-p-1 \equiv \mathcal{N}$. In this thesis, our main interest is for $D=6$, $p=3$ and therefore the submanifold $\phi[\Sigma]$ has codimension 2. Geometrical aspects of higher-codimensional surfaces can be found in [22].
In order to choose a unique connection for the manifold M, we assume metric compatibility

$$\nabla_\lambda g_{\mu\nu} = 0 \tag{2.1}$$

and torsion-free condition

$$T^\lambda_{\mu\nu} \equiv 2\Gamma^\lambda_{[\nu\mu]} - C^\lambda_{\mu\nu} = 0 \Leftrightarrow \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} \tag{2.2}$$

in a coordinate basis where $C^\lambda_{\mu\nu} = 0$ (see appendix). The above properties reflect to the submanifold $\phi[\Sigma]$. The general choice for the signature of the metric $g_{\mu\nu}$ is $(\underbrace{\varepsilon,\ldots,\varepsilon}_{\mathcal{N}},\underbrace{+1,\ldots,+1}_{p+1})$ with $\varepsilon = \pm 1$, representing the inner product in the space of vector fields

$$g(u,v) = g(u^\mu\partial_\mu, v^\nu\partial_\nu) = u^\mu v^\nu g(\partial_\mu,\partial_\mu) = u^\mu v^\nu g_{\mu\nu} = u^\mu v_\mu \tag{2.3}$$

where $\{\partial_\mu = \frac{\partial}{\partial x^\mu}\}$ is a coordinate basis and u^μ, v^μ are the vector fields' components $(\mu,\nu = 0,1,\ldots,D-1)$.

The inverse metric is $g^{\mu\nu}$ and indices $\mu, \nu, \ldots$ are raised (lowered) with $g^{\mu\nu}$ ($g_{\mu\nu}$) respectively.
The dual space of 1-form (covectors) is defined by the basis of differentials $\{dx^\mu\}$ with $dx^\mu(\partial_\mu) = \delta^\mu_\nu$.

For σ^i coordinates on $\Phi[\Sigma]$ with $i = 0, 1, \ldots, p$, we can define $p+1$ tangent vectors as

$$t_i = \frac{\partial}{\partial \sigma^i} = x^\mu{}_{,i}\partial_\mu \tag{2.4}$$

and therefore, the tangent vectors' components are

$$t_i{}^\mu = x^\mu{}_{,i} \ . \tag{2.5}$$

We can also choose an arbitrary (non-coordinate) basis $\{E_A\} = \{e_a, n_\alpha\}$, where $\{e_a\}$ are parallel vectors to the surface with $a = 0, 1, \ldots, p$ and $\{n_\alpha\}$ are normal vectors with $\alpha = 1, \ldots, \mathcal{N}$. These are expressed in terms of the coordinate basis $\{\partial_\mu\}$ as

$$e_a = e_a{}^\mu \partial_\mu \qquad\qquad n_\alpha = n_\alpha{}^\mu \partial_\mu \ , \tag{2.6}$$

where $e_a{}^\mu$ and $n_\alpha{}^\mu$ are the vectors' components respectively.
The metric g can be expressed in terms of the arbitrary parallel vectors as

$$g_{ab} = g(e_a, e_b) = g_{\mu\nu} e_a{}^\mu e_b{}^\nu = e_a{}^\mu e_{b\mu} \ , \tag{2.7}$$

in terms of the tangent $\{t_i\}$ vectors as

$$g_{ij} = g(t_i, t_j) = g_{\mu\nu} x^\mu{}_{,i} x^\nu{}_{,j} \tag{2.8}$$

and in terms of the normal vectors as

$$g_{\alpha\beta} = g(n_\alpha, n_\beta) = g_{\mu\nu} n_\alpha{}^\mu n_\beta{}^\nu = n_\alpha{}^\mu n_{\beta\mu} \ . \tag{2.9}$$

Indices $a, b, \ldots$, $\alpha, \beta, \ldots$ and $i, j, \ldots$ are lowered with g_{ab}, $g_{\alpha\beta}$ and g_{ij} respectively.
The inverse metrics are denoted by g^{ab}, $g^{\alpha\beta}$ and g^{ij} and indices $a, b, \ldots$, $\alpha, \beta, \ldots$ and $i, j, \ldots$ are raised by them respectively. Obviously, $e^a{}_\mu e_b{}^\mu = \delta^a{}_b, n^\alpha{}_\mu n_\beta{}^\mu = \delta^\alpha{}_\beta, e_a{}^\mu n^\alpha{}_\mu = 0$. Moreover, the following equation of decomposition of identity is valid $\delta^\mu_\nu = e_a{}^\mu e^a{}_\nu + n_\alpha{}^\mu n^\alpha{}_\nu$. The inverse metrics can be expressed in the form

$$g^{ab} = g^{\mu\nu} e^a{}_\mu e^b{}_\nu \tag{2.10}$$

and

$$g^{\alpha\beta} = n^{\alpha\mu} n^{\beta}{}_{\mu} \ . \tag{2.11}$$

We can also express the coordinate basis $\{\partial_\mu\}$ in terms of the arbitrary parallel and normal vectors as

$$\partial_\mu = e^{a}{}_{\mu} e_a + n^{\alpha}{}_{\mu} n_\alpha \ . \tag{2.12}$$

The map ϕ:$\Sigma \longrightarrow M$ that embeds the submanifold $\phi[\Sigma]$ to the manifold M allows us to pull back the metric from M to Σ and define the induced metric $h = \phi^* g$ as

$$h_{\mu\nu} = g_{\mu\nu} - n_{\alpha\mu} n^{\alpha}{}_{\nu} \ , \tag{2.13}$$

from which we obtain that

$$h_{ij} = g_{ij} \ . \tag{2.14}$$

The fully contravariant and mixed forms of the induced metric are

$$h^{\mu\nu} = g^{\mu\nu} - n_{\alpha}{}^{\mu} n^{\alpha\nu} \ , \ h^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - n_{\alpha}{}^{\mu} n^{\alpha}{}_{\nu} \ . \tag{2.15}$$

The latter is a projection operator since

$$h^{\mu}{}_{\nu} h^{\nu}{}_{\kappa} = h^{\mu}{}_{\kappa} \, , \tag{2.16}$$

which can be easily seen from $(\delta^{\mu}_{\nu} - n_{\alpha}{}^{\mu} n^{\alpha}{}_{\nu})(\delta^{\nu}_{\kappa} - n_{\beta}{}^{\nu} n^{\beta}{}_{\kappa}) = \delta^{\mu}_{\kappa} - n_{\beta}{}^{\mu} n^{\beta}{}_{\kappa} - n_{\alpha}{}^{\mu} n^{\alpha}{}_{\kappa} + n_{\beta}{}^{\mu} n^{\beta}{}_{\kappa} = h^{\mu}{}_{\kappa}$.
Obviously, from the above decomposition of identity, the induced metric is also written as

$$h_{\mu\nu} = g_{ab} e^{a}{}_{\mu} e^{b}{}_{\nu} \, , \tag{2.17}$$

while its components in terms of the arbitrary parallel basis are

$$h_{ab} = h(e_a, e_b) = h_{\mu\nu} e_a{}^{\mu} e_b{}^{\nu} = g_{ab} \, . \tag{2.18}$$

Accordingly, the contravariant and mixed components are written as

$$h^{\mu\nu} = g^{ab} e_a{}^{\mu} e_b{}^{\nu} \ , \ h^{\mu}{}_{\nu} = e_a{}^{\mu} e^{a}{}_{\nu} \, . \tag{2.19}$$

In terms of the embedding fields x^μ the induced metric h takes a similar form

$$h^{\mu\nu} = g^{ij} x^{\mu}{}_{,i} x^{\nu}{}_{,j} \ . \tag{2.20}$$

The metric remaining to be defined is that of the normal space to $\phi[\Sigma]$ as

$$m_{\mu\nu} = g_{\mu\nu} - h_{\mu\nu} \tag{2.21}$$

with corresponding contravariant and mixed forms

$$m^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} \ , \ m^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - h^{\mu}_{\nu} \ . \tag{2.22}$$

The tensor $m^{\mu}{}_{\nu}$ is also a projection operator

$$m^{\mu}{}_{\nu} m^{\nu}{}_{\lambda} = m^{\mu}{}_{\lambda} \ , \tag{2.23}$$

since $(\delta^{\mu}_{\nu} - h^{\mu}{}_{\nu})(\delta^{\nu}_{\lambda} - h^{\nu}{}_{\lambda}) = \delta^{\mu}_{\lambda} - 2h^{\mu}{}_{\lambda} + h^{\mu}{}_{\nu} h^{\nu}{}_{\lambda} = \delta^{\mu}_{\lambda} - 2h^{\mu}{}_{\lambda} + h^{\mu}{}_{\lambda} = m^{\mu}{}_{\lambda}$. The components of the "normal" metric in terms of the normal basis takes the form

$$m_{\mu\nu} = g_{\alpha\beta} n^{\alpha}{}_{\mu} n^{\beta}{}_{\nu} = n_{\alpha\mu} n^{\alpha}{}_{\nu} \tag{2.24}$$

and similarly

$$m^{\mu\nu} = n_{\alpha}{}^{\mu} n^{\alpha\nu} \ , \ m^{\mu}{}_{\nu} = n_{\alpha}{}^{\mu} n^{\alpha}{}_{\nu} \ . \tag{2.25}$$

Of course, the inner product between the parallel and the normal metric is zero

$$h^{\mu}{}_{\lambda} m^{\lambda}{}_{\nu} = m^{\mu}{}_{\lambda} h^{\lambda}{}_{\nu} = 0 \ . \tag{2.26}$$

To conclude, the induced metric h acts as a projector for parallel vectors and annihilates the normal vectors, while the normal metric m annihilates the parallel vectors and acts as a projector for the tangent ones.

If we denote the Christoffel connection of g as $\bigtriangledown$ or ; and the Christoffel connection of h as $\bar{\bigtriangledown}$ or $|$, then in the basis $\{e_a, n_\alpha\}$ we have

$$\bigtriangledown_a e_b = \gamma^{c}{}_{ba} e_c - K^{\alpha}{}_{ab} n_\alpha = \gamma_{cba} e^c - K_{\alpha ab} n^\alpha \tag{2.27}$$

$$\bigtriangledown_a n_\alpha = K_{\alpha a}{}^{b} e_b + \varpi^{\beta}{}_{\alpha a} n_\beta = K_{\alpha ab} e^b + \varpi_{\beta\alpha a} n^\beta \ , \tag{2.28}$$

where

$$\gamma_{cba} = g(\bigtriangledown_a e_b, e_c) = e_{c\mu} \bigtriangledown_a e_b{}^{\mu} \tag{2.29}$$

are the surface connection coefficients,

$$K_{\alpha ab} = g(\bigtriangledown_a n_\alpha, e_b) = e_{b\mu} \bigtriangledown_a n_\alpha{}^{\mu} = -g(\bigtriangledown_a e_b, n_\alpha) = -n_{\alpha\mu} \bigtriangledown_a e_b{}^{\mu} = K_{\alpha ba} \tag{2.30}$$

are the extrinsic curvature coefficients and

$$\varpi_{\beta\alpha a} = g(\bigtriangledown_a n_\alpha, n_\beta) = n_{\beta\mu} \bigtriangledown_a n_\alpha{}^{\mu} \tag{2.31}$$

is the normal fundamental form or extrinsic twist potential of the surface. In the above notation $\bigtriangledown_a e_b{}^{\mu} \equiv (\bigtriangledown_a e_b)^{\mu}$ and $\bigtriangledown_a n_\alpha{}^{\mu} \equiv (\bigtriangledown_a n_\alpha)^{\mu}$. Of course, the above relations are valid for a coordinate basis as well, i.e. for $(a, b) \longrightarrow (i, j)$. Then, it holds that

$$K^{\alpha}{}_{ab} = n^{\alpha}{}_{a;b} = -e_{a}{}^{\alpha}{}_{;b} \tag{2.32}$$

since $K_{\alpha ab} = -g(\bigtriangledown_a e_b, n_\alpha) = -g(e_b{}^{\beta}{}_{;a} n_\beta + e_b{}^{c}{}_{;a} e_c, n_\alpha) = -e_{b\alpha;a}$ and also $K_{\alpha ab} = g(\bigtriangledown_a n_\alpha, e_b) = g(n_\alpha{}^{c}{}_{;a} e_c + n_\alpha{}^{\beta}{}_{;a} n_\beta, e_b) = n_{\alpha b;a}$.
For a discussion on the reduction of the formalism to the codimension-1 case see appendix.

If Γ^A_{BC} $(A = a, \alpha)$ are the connection coefficients of $\bigtriangledown$ in the basis $E_A = \{e_a, n_\alpha\}$, i.e.

$$\bigtriangledown_A E_B = \Gamma^C_{BA} E_C \ , \tag{2.33}$$

then

$$\Gamma^{c}{}_{ba} = \gamma^{c}{}_{ba}, \qquad \Gamma_{\alpha ab} = -\Gamma_{a\alpha b} = -K_{\alpha ab}, \qquad \Gamma^{\beta}{}_{\alpha a} = \varpi^{\beta}{}_{\alpha a} \ , \tag{2.34}$$

since $\bigtriangledown_a e_b = \Gamma^{A}{}_{ba} E_A = \Gamma^{c}{}_{ba} e_c + \Gamma^{\alpha}{}_{ba} n_\alpha$ and $\bigtriangledown_a n_\alpha = \Gamma^A_{\alpha a} E_A = \Gamma^b_{\alpha a} e_b + \Gamma^\beta_{\alpha a} n_\beta$.
We also obtain

$$K^{\alpha}{}_{ab} = e_a{}^{\mu} e_b{}^{\nu} n^{\alpha}{}_{\mu;\nu} \ , \tag{2.35}$$

since $n^{\alpha}{}_{\mu;\nu} = n^{\alpha}{}_{\mu,\nu} - \Gamma^{\lambda}{}_{\mu\nu} n^{\alpha}{}_{\lambda} = n^{\alpha}{}_{\mu,\nu} - \big(E_A{}^{\lambda} E^B{}_{\mu} E^C{}_{\nu} \Gamma^A{}_{BC} + E^A{}_{\nu} E_B{}^{\lambda} E^B{}_{\mu,A}\big) n^{\alpha}{}_{\lambda}$ $= n^{\alpha}{}_{\mu,\nu} - E^B{}_{\mu} E^C{}_{\nu} \Gamma^{\alpha}{}_{BC} - E^A{}_{\nu} n^{\alpha}{}_{\mu,A} = n^{\alpha}{}_{\mu,\nu} - e^c{}_{\mu} e^d{}_{\nu} \Gamma^{\alpha}{}_{cd} - e^c{}_{\mu} n^{\gamma}{}_{\nu} \Gamma^{\alpha}{}_{c\gamma} - n^{\beta}{}_{\mu} e^c{}_{\nu} \Gamma^{\alpha}{}_{\beta c} - n^{\beta}{}_{\mu} n^{\gamma}{}_{\nu} \Gamma^{\alpha}{}_{\beta\gamma} - e^c{}_{\nu} n^{\alpha}{}_{\mu,c} - n^{\beta}{}_{\nu} n^{\alpha}{}_{\mu,\beta}$.
Another useful expression is the following

$$K^{\alpha}{}_{ab} = -n^{\alpha}{}_{\lambda}(e_{(a}{}^{\lambda}{}_{,b)} + \Gamma^{\lambda}{}_{\mu\nu} e_a{}^{\mu} e_b{}^{\nu}) \, , = -n^{\alpha}{}_{\lambda}(e_{(a}{}^{\lambda}{}_{;b)} + \Gamma^{\lambda}{}_{\mu\nu} e_a{}^{\mu} e_b{}^{\nu}) \tag{2.36}$$

where $f_{,a} = e_a{}^{\mu} f_{,\mu}$, and it is shown by using (2.35) and the orthogonality of $e_a{}^{\mu}$, $n_\alpha{}^{\mu}$.
Finally, the extrinsic curvature components can also be expressed in terms of the expansion tensor (one such tensor for the congruence of integral curves of any vector field n_α)

$$\Theta^{\alpha}{}_{\mu\nu} \equiv h^{\kappa}{}_{\mu} h^{\lambda}{}_{\nu} n^{\alpha}{}_{(\kappa;\lambda)} = h^{\kappa}{}_{\mu} h^{\lambda}{}_{\nu} n^{\alpha}{}_{\kappa;\lambda} = \Theta^{\alpha}{}_{\nu\mu} \tag{2.37}$$

as

$$K^{\alpha}{}_{ab} = \Theta^{\alpha}{}_{\mu\nu} e_a{}^{\mu} e_b{}^{\nu} \, . \tag{2.38}$$

Note that the symmetry of the μ, ν indices in $\Theta^{\alpha}{}_{\mu\nu}$ and of the a, b indices in $K^{\alpha}{}_{ab}$ is due to that $\phi[\Sigma]$ is a submanifold (surface). Formally, this is so, because

the so-called rotation tensor of the n_α lines defined as $\omega^\alpha{}_{\mu\nu} = h^\kappa{}_\mu h^\lambda{}_\nu n^\alpha{}_{[\kappa;\lambda]}$ vanishes since the vectors normal to $\{n_\alpha\}$ form an integrable subbundle.
The expansion tensor is a "parallel" tensor in the sense that

$$\Theta^\alpha{}_{\mu\nu} = \Theta^\alpha{}_{\kappa\lambda} h^\kappa{}_\mu h^\lambda{}_\nu \,, \tag{2.39}$$

or equivalently

$$\Theta^\alpha{}_{\mu\nu} n_\beta{}^\mu = 0\,. \tag{2.40}$$

We also have $\Theta^\alpha{}_{\mu\nu} e_a{}^\mu = K^\alpha{}_{ab} e^b{}_\nu$.
If additionally the vectors n_α are surface forming, $\Theta^\alpha{}_{\mu\nu}$ takes the form

$$\Theta^\alpha{}_{ij} = K^\alpha{}_{ij} = n^\alpha{}_{i;j}\ ,\ \Theta^\alpha{}_{\beta i} = \Theta^\alpha{}_{i\beta} = \Theta^\alpha{}_{\beta\gamma} = 0\ , \tag{2.41}$$

and moreover $K^\alpha{}_{ij}$ becomes

$$K^\alpha{}_{ij} = K^\alpha{}_{ab} e^a{}_i e^b{}_j = -n^\alpha{}_\lambda (x^\lambda{}_{;ij} + \Gamma^\lambda{}_{\mu\nu} x^\mu{}_{,i} x^\nu{}_{,j})\,. \tag{2.42}$$

Another useful relation concerns the contraction

$$g^{ij} K^\alpha{}_{ij} = -n^\alpha{}_\lambda (\Box_h x^\lambda + \Gamma^\lambda{}_{\mu\nu} h^{\mu\nu})\,, \tag{2.43}$$

which is derived by using (2.20), (2.42) and that $\Box_h = h^{AB}(E_A E_B - \gamma^C{}_{AB} E_C) = h^{AB}(E_A E_B - \Gamma^C_{AB} h^D{}_C E_D) = g^{ij}(\partial_i \partial_j - \Gamma^k{}_{ij}\partial_k) = \Box_{g_{ij}}$.
We also have

$$n_\alpha{}^\mu K^\alpha{}_{ij} = -(x^\mu{}_{;ij} + \Gamma^\mu{}_{\lambda\nu} x^\lambda{}_{,i} x^\nu{}_{,j})\,. \tag{2.44}$$

This is easily seen by using the following identity of the appendix

$$h^\nu{}_\mu (x^\mu{}_{;ij} + \Gamma^\mu{}_{\lambda\kappa} x^\lambda{}_{,i} x^\kappa{}_{,j}) = 0\,, \tag{2.45}$$

which means that the vector (with respect to the index μ) $x^\mu{}_{;ij} + \Gamma^\mu{}_{\lambda\kappa} x^\lambda{}_{,i} x^\kappa{}_{,j}$ is normal to the surface.
Another useful identity is

$$n_\alpha{}^\mu g^{ij} K^\alpha{}_{ij} = -(\Box_h x^\mu + \Gamma^\mu{}_{\lambda\nu} h^{\lambda\nu})\,, \tag{2.46}$$

which is found by using the previous equation for $g^{ij} K^\alpha{}_{ij}$ and the equation $h^\mu{}_\nu (\Box_h x^\nu + \Gamma^\nu{}_{\kappa\lambda} h^{\kappa\lambda}) = 0$.
Therefore, we have

$$g^{ij} K^\alpha{}_{ij} = 0 \Leftrightarrow \Box_h x^\mu + \Gamma^\mu{}_{\lambda\nu} h^{\lambda\nu} = 0\,, \tag{2.47}$$

which shows the equivalence between the definition of the minimal surface and of the Nambu-Goto equation of motion.
Additional assumptions of orthonormalization of $\{n_\alpha\}, \{e_a\}$ would imply additional symmetry conditions $\varpi_{\beta\alpha a} = -\varpi_{\alpha\beta a}, \gamma_{bac} = -\gamma_{abc}$ respectively.

If we make a local rotation of the normal frame

$$n^{'}_{\alpha} = (O^{-1})^{\beta}{}_{\alpha} n_\beta \ , \ n^{'\alpha} = O^{\alpha}{}_{\beta} n^{\beta} \ , \tag{2.48}$$

then

$$\gamma'^{c}{}_{ba} = \gamma^{c}{}_{ba} \ , \ K'^{\alpha}{}_{ab} = O^{\alpha}{}_{\beta} K^{\beta}{}_{ab} \ , \tag{2.49}$$

i.e. the extrinsic curvature coefficients transform homogeneously. On the contrary, the twist coefficient transforms as a connection (inhomogeneously)

$$\varpi'^{\beta}{}_{\alpha a} = O^{\beta}{}_{\gamma}(O^{-1})^{\delta}{}_{\alpha} \varpi^{\gamma}{}_{\delta a} + O^{\beta}{}_{\gamma}(O^{-1})^{\gamma}{}_{\alpha,a} \ . \tag{2.50}$$

With respect to this connection, the corresponding curvature

$$\Omega^{\beta}{}_{\alpha ab} = \varpi^{\beta}{}_{\alpha b,a} - \varpi^{\beta}{}_{\alpha a,b} + \varpi^{\gamma}{}_{\alpha b}\varpi^{\beta}{}_{\gamma a} - \varpi^{\gamma}{}_{\alpha a}\varpi^{\beta}{}_{\gamma b} - C^{c}{}_{ab}\varpi^{\beta}{}_{\alpha c} \tag{2.51}$$

transforms homogeneously

$$\Omega'^{\beta}{}_{\alpha ab} = O^{\beta}{}_{\gamma}(O^{-1})^{\delta}{}_{\alpha}\Omega^{\gamma}{}_{\delta ab} \ . \tag{2.52}$$

For fields Φ^{α}_{β} transforming as tensors under normal frame rotations

$$\Phi'^{\alpha}_{\beta} = O^{\alpha}{}_{\gamma}(O^{-1})^{\delta}{}_{\beta}\Phi^{\gamma}_{\delta} \ , \tag{2.53}$$

we can define a new worldsheet covariant derivative ! with respect to $\varpi^{\alpha}{}_{\beta a}$ as

$$\Phi^{\alpha}_{\beta!a} \equiv \Phi^{\alpha}_{\beta|a} + \varpi^{\alpha}{}_{\gamma a}\Phi^{\gamma}_{\beta} - \varpi^{\gamma}{}_{\beta a}\Phi^{\alpha}_{\gamma} \ , \tag{2.54}$$

where $|$ differentiation is meant on tangential indices $a, b, \ldots$ that Φ^{α}_{β} may possess.
The parallel and normal components of the metric tensor are seen to be covariantly flat with respect to ! differentiation

$$g_{ab!c} = g_{\alpha\beta!c} = 0 \ , \tag{2.55}$$

since $g_{ab!c} = g_{ab|c} = 0$ and $g_{\alpha\beta!c} = g_{\alpha\beta|c} - \varpi^{\gamma}{}_{\beta c}g_{\gamma\alpha} - \varpi^{\gamma}{}_{\alpha c}g_{\beta\gamma} = g_{\alpha\beta,c} - \Gamma^{\gamma}{}_{\beta c}g_{\gamma\alpha} - \Gamma^{\gamma}{}_{ac}g_{\beta\gamma} = g_{\beta\alpha,c} - \Gamma^{A}{}_{\beta c}g_{A\alpha} - \Gamma^{A}{}_{\alpha c}g_{\beta A} = g_{\alpha\beta;c} = 0$.

If $\mathcal{R}^{\mu}{}_{\nu\kappa\lambda}$ and $R^{\mu}{}_{\nu\kappa\lambda}$ are the curvature tensors of $g_{\mu\nu}$ and $h_{\mu\nu}$ respectively, then the following geometric relations hold:
Gauss-Codazzi

$$\mathcal{R}_{abcd} = R_{abcd} + K^{\alpha}{}_{ad}K_{\alpha bc} - K^{\alpha}{}_{ac}K_{\alpha bd} \ , \tag{2.56}$$

Codazzi-Mainardi

$$\mathcal{R}^{\alpha}{}_{abc} = K^{\alpha}{}_{ab!c} - K^{\alpha}{}_{ac!b} \ , \tag{2.57}$$

Ricci

$$\mathcal{R}^{\beta}{}_{\alpha ab} = \Omega^{\beta}{}_{\alpha ab} + K_{\alpha a}{}^{c}K^{\beta}{}_{bc} - K_{\alpha b}{}^{c}K^{\beta}{}_{ac} \ . \tag{2.58}$$

These can be derived by computing the commutators $[\triangledown_a, \triangledown_b]e_c$ and $[\triangledown_a, \triangledown_b]n_\alpha$. Of course, as usually, indices flip from coordinate to tangent/normal by multiplying with the appropriate vierbein

$$\begin{gathered}\mathcal{R}_{abcd} = \mathcal{R}_{\mu\nu\kappa\lambda}e_a{}^{\mu}e_b{}^{\nu}e_c{}^{\kappa}e_d{}^{\lambda} \, , \, \mathcal{R}_{\alpha abc} = \mathcal{R}_{\mu\nu\kappa\lambda}n_\alpha{}^{\mu}e_a{}^{\nu}e_b{}^{\kappa}e_c{}^{\lambda} \, , \\ \mathcal{R}_{\beta\alpha ab} = \mathcal{R}_{\mu\nu\kappa\lambda}n_\beta{}^{\mu}n_\alpha{}^{\nu}e_a{}^{\kappa}e_b{}^{\lambda} \, .\end{gathered} \tag{2.59}$$

For codimension-1, the normal distribution is trivial and the Ricci identity reduces to a tautology.

Chapter 3

Brane action with Gauss-Bonnet in the bulk

We consider the total brane-bulk action

$$
\begin{aligned}
& S = S_{gr} + S_{mat} + S_c \\
& = \frac{1}{2\kappa_6^2}\int_M d^6x\sqrt{-|g|}\Big\{\mathcal{R} - 2\Lambda_6 + a_{GB}\Big(\mathcal{R}_{\mu\nu\kappa\lambda}\mathcal{R}^{\mu\nu\kappa\lambda} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2\Big)\Big\} \\
& + \int_\Sigma d^4x\sqrt{-|h|}(\frac{r_c^2}{2\kappa_6^2}R - \lambda) + \int_M d^6x\mathcal{L}_{mat} + \int_\Sigma d^4x L_{mat} \\
& + \int_\Sigma d^4x\sqrt{-|h|}\Big\{\lambda^{ij}(h_{ij} - g_{\mu\nu}x^\mu_{,i}x^\nu_{,j}) + \lambda^{\alpha i}n_{\alpha\mu}x^\mu_{,i} + \lambda^{\alpha\beta}(g_{\mu\nu}n^\mu_\alpha n^\nu_\beta - \delta_{\alpha\beta})\Big\} \quad (3.1)
\end{aligned}
$$

where $g_{\mu\nu}$ $(\mu,\nu = 1,\ldots,6)$ is the bulk metric tensor $h_{\mu\nu}$ is the induced metric on the brane $\mathcal{R}$ and R are the bulk and brane Ricci scalars, λ is the brane tension and r_c is the induced gravity crossover length scale and $\mathcal{L}_{mat}$, L_{mat} are the matter Langrangians of the bulk and of the brane respectively. It is convenient to consider ${n_\alpha}^\mu$ $(\alpha = 1,2)$ arbitrary unit normal vectors normal to the brane and to each other (of course, in the next chapter where we consider the axially symmetric ansatz for the bulk metric g they are surface forming). The relation of h_{ij} $(i,j = 1,\ldots,4)$ to $g_{\mu\nu}$ and the orthonormality of ${n_\alpha}^\mu$ imply the following constraints

$$h_{ij} = g_{\mu\nu}{x^\mu}_{,i}{x^\nu}_{,j} \quad (3.2)$$

$$g_{\mu\nu}{n_\alpha}^\mu{n_\beta}^\nu = \delta_{\alpha\beta} \quad (3.3)$$

$$n_{\alpha\mu}x^{\mu}{}_{,i} = 0\,. \tag{3.4}$$

It is obvious that the independent variables for the bulk/brane system are the bulk metric $g_{\mu\nu}$ and the embedding fields x^{μ}. Then, the induced metric and the normal space are uniquely determined. Besides, the variation $\delta g_{\mu\nu}$ of the bulk metric $g_{\mu\nu}$ away from the brane which is clear and gives the bulk equations of motion, in our consideration the interaction of the brane with the bulk is obtained by varying the action with respect to δx^{μ} and therefore at the brane position the variation

$$\delta g_{\mu\nu} = -\pounds_{\delta x} g_{\mu\nu} = -(g_{\mu\nu,\lambda}\delta x^{\lambda} + g_{\mu\lambda}\delta x^{\lambda}{}_{,\nu} + g_{\nu\lambda}\delta x^{\lambda}{}_{,\mu})\,. \tag{3.5}$$

Furthermore, the variation δh_{ij}, $\delta n_{\alpha}{}^{\mu}$ at the brane location could also be expressed in terms of δx^{μ}. However, we find it more convenient to vary h_{ij} and $n_{\alpha}{}^{\mu}$ independently by including corresponding Langrange multipliers λ^{ij}, $\lambda^{\alpha i}$, $\lambda^{\alpha\beta}$ and this is the meaning of the third line in the above action. Henceforth, we will vary this action with respect to $n_{\alpha}{}^{\mu}$, h_{ij} and $g_{\mu\nu}$, the last one both at the brane location and in the bulk. To proceed with these variations at the brane location we will specialize in the next chapter on the axially symmetric case for which we will derive explicitly the equations of motion for the brane where the impact from the bulk dynamics (back-reaction) will become manifest.
Variation with respect to the induced metric h_{ij} gives

$$\delta_h S = \int_{\Sigma} d^4x\sqrt{-|h|}\left\{\lambda^{ij} + \frac{1}{2}(T^{ij} - \lambda h^{ij}) - \frac{r_c^2}{2\kappa_6^2}G^{ij}\right\}\delta h_{ij}\,. \tag{3.6}$$

Variation with respect to $n_{\alpha\mu}$ gives

$$\delta_n S = \int_{\Sigma} d^4x\sqrt{-|h|}\left\{\lambda^{\alpha i}x^{\mu}{}_{,i} + 2\lambda^{\alpha\beta}n_{\beta}{}^{\mu}\right\}\delta n_{\alpha\mu}\,. \tag{3.7}$$

Using the standard relations:

$$\delta R^{\lambda}_{\mu\nu\kappa} = (\delta\Gamma^{\lambda}_{\mu\kappa})_{;\nu} - (\delta\Gamma^{\lambda}_{\mu\nu})_{;\kappa} \tag{3.8}$$

$$\delta R_{\mu\kappa} = (\delta\Gamma^{\lambda}_{\mu\kappa})_{;\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\kappa} \tag{3.9}$$

$$\delta_g\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\{\ (\delta g_{\rho\mu})_{;\nu} + (\delta g_{\rho\nu})_{;\mu} - (\delta g_{\mu\nu})_{;\rho}\} \tag{3.10}$$

$$\delta\sqrt{-|g|} = \frac{1}{2}\sqrt{-|g|}g^{\mu\nu}\delta g_{\mu\nu}\,, \tag{3.11}$$

variation with respect to the metric $g_{\mu\nu}$ gives

$$\delta_g S = \frac{1}{2\kappa_6^2}\int_M d^6x\sqrt{-|g|}\Bigg\{-\mathcal{G}^{\tau\rho} - \Lambda_6 g^{\tau\rho} + a_{GB}[g^{\tau\rho}(\frac{1}{2}\mathcal{R}^2 + \frac{1}{2}\mathcal{R}_{\mu\nu\kappa\lambda}\mathcal{R}^{\mu\nu\kappa\lambda}$$
$$-2\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}) - 2\mathcal{R}\mathcal{R}^{\tau\rho} - 2\mathcal{R}^{\rho\nu\kappa\lambda}\mathcal{R}^{\tau}{}_{\nu\kappa\lambda} + 4\mathcal{R}^{\rho\mu\tau\kappa}\mathcal{R}_{\mu\kappa} + 4\mathcal{R}^{\tau\lambda}\mathcal{R}^{\rho}{}_{\lambda}]\Bigg\}\delta g_{\tau\rho}$$
$$+\frac{1}{\kappa_6^2}\int_M d^6x\sqrt{-|g|}\frac{1}{2}\mathcal{T}^{\tau\rho}\delta g_{\tau\rho}$$
$$+\frac{1}{2\kappa_6^2}\int_M d^6x\sqrt{-|g|}\Bigg\{2g^{\tau[\kappa}g^{\lambda]\rho}(\delta g_{\rho\kappa})_{;\lambda\tau} + 4a_{GB}\Big(\mathcal{R}^{\tau\rho\kappa\lambda} + 2\mathcal{R}^{\rho[\kappa}g^{\lambda]\tau} - 2\mathcal{R}^{\tau[\kappa}g^{\lambda]\rho}$$
$$+\mathcal{R}g^{\tau[\kappa}g^{\lambda]\rho}\Big)(\delta g_{\rho\kappa})_{;\lambda\tau}\Bigg\} + \int_\Sigma d^4x\sqrt{-|h|}\Bigg\{\lambda^{\alpha\beta}n_\alpha{}^\mu n_\beta{}^\nu - \lambda^{ij}x^\mu{}_{,i}x^\nu{}_{;b}\Bigg\}\delta g_{\mu\nu}\,. \qquad (3.12)$$

The first three lines of (3.12) give the well known bulk Einstein-Gauss-Bonnet equations of motion

$$\mathcal{G}_{\mu\nu} - \frac{a_{GB}}{2}\big(\mathcal{R}_{\kappa\lambda\sigma\tau}\mathcal{R}^{\kappa\lambda\sigma\tau} - 4\mathcal{R}_{\kappa\lambda}\mathcal{R}^{\kappa\lambda} + \mathcal{R}^2\big)g_{\mu\nu} + 2a_{GB}\big(\mathcal{R}\mathcal{R}_{\mu\nu} - 2\mathcal{R}_{\mu\kappa}\mathcal{R}_\nu{}^\kappa$$
$$-2\mathcal{R}_{\mu\nu\kappa\lambda}\mathcal{R}^{\kappa\lambda} + \mathcal{R}_{\mu\kappa\lambda\sigma}\mathcal{R}_\nu{}^{\kappa\lambda\sigma}\big) = \kappa_6^2\mathcal{T}^{\mu\nu} - \Lambda_6 g^{\mu\nu}\,, \qquad (3.13)$$

where we are mainly interested (chapter 6) in a bulk with a pure cosmological constant but for the moment we leave $\mathcal{T}^{\mu\nu}$ non-vanishing.
The tricky point however is how to handle the contribution on the brane from the various 6-dimensional terms in (3.12) and this will be done by focusing from now on to the axially symmetric case. As we will see, not all variations (3.6), (3.7), (3.12) are independent since contributions from (3.12) will modify the variation (3.6).

Chapter 4

Axial symmetry

For simplicity, we assume the axially symmetric bulk ansatz

$$ds_6^2 = dr^2 + L^2(x,r)d\theta^2 + h_{ij}(x,r)dx^i dx^j \,, \tag{4.1}$$

where $h_{ij}(x,0)$ is the braneworld metric which is assumed to be regular everywhere with the possible exception of isolated singular points and θ has the standard periodicity 2π. The defect is assumed to have the conical singularity structure with

$$\begin{aligned} L(x,r) &= rL'(x) + \frac{1}{2}r^2[L''(x)] + \frac{1}{3!}r^3[L'''(x)]\dots \\ &= r\beta_1(x) + \frac{1}{2}r^2\beta_2(x) + \frac{1}{6}r^3\beta_3(x) + \dots \,, \end{aligned} \tag{4.2}$$

where a $'$ denotes differentiation with respect to r, $L'(x) \equiv L'(x,0^+)$ and $[L''(x)]$ denotes the regular part of the general $L''(x) \equiv L''(x,0^+)$, which contains also distributional terms.
We consider two characteristic cases :
(i) pure cone, with

$$\begin{aligned} L'(x,0) &= 1 \\ L'(x,0^+) &= \beta_1(x) \\ h'_{ij}(x,0) &= h'_{ij}(x,0^+) \end{aligned} \tag{4.3}$$

where the extrinsic curvature is continuous, and
(ii) cone plus extrinsic curvature discontinuity, with

$$\begin{aligned} L'(x,0) &= 1 \\ L'(x,0^+) &= \beta_1(x) \\ h'_{ij}(x,0) = 0 &\neq h'_{\mu\nu}(x,0^+)\,. \end{aligned} \tag{4.4}$$

Both cases can be described by the conditions

$$\begin{aligned} L'(x,0) &= 1 \\ L'(x,0^+) &= \beta_1(x) \\ h'_{\mu\nu}(x,0) &= \eta h^{'}_{\mu\nu}(x,0^+) \end{aligned} \tag{4.5}$$

where $\eta = 1, 0$ for cases (i), (ii) respectively.
We now obtain the general $L''(x)$ at the position of the brane as

$$L''(x) = -\big(1-\beta_1(x)\big)\delta(r) + [L''(x)]\,. \tag{4.6}$$

We also get

$$h''_{ij}(x) = \big(1-\eta\big)h'_{ij}(x)\delta(r) + [h''_{ij}(x)]\,, \tag{4.7}$$

or in terms of the extrinsic curvature coefficients

$$K'_{ij}(x) = \big(1-\eta\big)K_{ij}(x)\delta(r) + [K'_{ij}(x)]\,. \tag{4.8}$$

The non-vanishing components of the Christoffel symbols, Riemann tensors, Ricci tensors and Ricci scalar of the metric (4.1), after a long calculation are found to be:

$$\begin{aligned} \Gamma^r{}_{\theta\theta} &= -LL'\,,\ \Gamma^r{}_{ij} = -\frac{1}{2}h'_{ij}\,,\ \Gamma^\theta{}_{r\theta} = \frac{L'}{L}\,,\ \Gamma^\theta{}_{\theta i} = \frac{L_{,i}}{L}\,,\ \Gamma^i{}_{\theta\theta} = -Lh^{ik}L_{,k} \\ \Gamma^i{}_{rj} &= \frac{1}{2}h^{ik}h'_{kj}\,,\ \Gamma^i{}_{jk} = \frac{1}{2}h^{i\mu}(h_{\mu j,k} + h_{\mu k,j} + h_{jk,\mu})\,, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \mathcal{R}_{rirj} &= -K_{ij} + K^k{}_i K_{jk}\,,\ \mathcal{R}_{\theta i\theta j} = -LL_{|i|j} + L^{'}K_{ij}\,,\ \mathcal{R}_{r\theta r\theta} = -LL'' \\ \mathcal{R}_{ijkl} &= R_{ijkl} + K_{ik}K_{jl} - K_{ik}K_{jl}\,,\ \mathcal{R}_{\theta r\theta i} = L(L_{|h}K^h{}_i - L'_{|i}) \\ \mathcal{R}_{ribk} &= K_{ij|k} - K_{ak|j}\,, \end{aligned} \tag{4.10}$$

$$\mathcal{R}_{rr} = -\frac{L''}{L} - K' - K^{ij}K_{ij}\,,\; \mathcal{R}_{\theta\theta} = -LL'' - L(\Box L + L^{'}K)\,,$$
$$\mathcal{R}_{ri} = \frac{L_{|j}}{L}K^{j}{}_{i} - \frac{L'_{|i}}{L} + K^{j}{}_{i|j} - K_{|i}\,,$$
$$\mathcal{R}_{ij} = R_{ij} - K'_{ij} + 2K^{k}{}_{i}K_{jk} - K_{ij}K - \frac{L_{|i|j}}{L} - \frac{L'}{L}K_{ij}\,, \qquad (4.11)$$

$$\mathcal{R} = R - 2\frac{L''}{L} - 2K' - K^{ij}K_{ij} - K^2 - \frac{2}{L}(\Box L + L'K)\,. \qquad (4.12)$$

$\Box$ is the Laplacian operator of the metric h_{ij}. The non-vanishing components of the Einstein tensor are:

$$\mathcal{G}_{ri} = \frac{L_{|j}}{L}K^{j}{}_{i} - \frac{L'}{L} + K^{j}{}_{i|j} - K_{|i}\,, \qquad (4.13)$$

$$\mathcal{G}_{rr} = -12R - \frac{1}{2}K^{jk}K_{jk} + \frac{1}{2}K^2 + \frac{\Box L}{L} + \frac{L'}{L}K\,, \qquad (4.14)$$

$$\mathcal{G}_{\phi\phi} = L^2\big(-\frac{1}{2}R + K' + \frac{1}{2}K^{jk}K_{jk} + \frac{1}{2}K^2\big)\,, \qquad (4.15)$$

$$\mathcal{G}_{ij} = G_{ij} - K'{}_{ij} + 2K_{i}{}^{k}K_{jk} - KK_{ij} - \frac{L_{|i|j}}{L} + \frac{L'}{L}K_{ij}$$
$$+ \; \frac{1}{2}h_{ij}\big(2\frac{L''}{L} + 2K^{'} + 4K^{jk}K_{jk} + K^2 + 2\frac{L'}{L}K + 2\frac{\Box L}{L}\big)\,. \qquad (4.16)$$

Since we are interested in the effective equations (all possible equations holding at the brane position), we will make an expansion of the metric $g_{\mu\nu}$ with respect to r (i. e. of $h_{ij}(x,r)$, $L(x,r)$), and therefore of all the quantities appearing in the previous components of the Riemanns, the Ricci and the Einstein tensors, and then we will take the limit $r \to 0$. Of course the arising brane equation will be exact and not approximate, as the expansions are used as a means to extract the correct terms. Plugging all these terms in the integral (3.12) as well as in the bulk equations (3.13) we obtain distributional terms with structure $\frac{\delta(r)}{r}$, singular terms diverging as $\frac{1}{r}$ as $r \to 0$ (denoted as $\mathcal{O}(\frac{1}{r})$), then regular terms with finite values at $r = 0$ (denoted as $\mathcal{O}(1)$) and finally a bunch of terms multiplied by powers of r, which of course do not contribute to the brane equations since they vanish for $r \to 0$. More precisely, two groups of equations at the brane will arise from the bulk equations (3.13), the first coming from the $\frac{1}{r}$ terms which have to cancel independently (assuming that the bulk matter $\mathcal{T}_{\mu\nu}$ is regular at $r = 0$, as e. g. a bulk cosmological constant), and the second from the regular terms. On the other hand, only the distributional terms $\frac{\delta(r)}{r}$ will contribute in (3.12), providing the new matching conditions. There are also milder distributional terms of the form $\delta(r)$ which however do not contribute.

So, the above mentioned expansions are the following:

$$g_{\mu\nu}(x,r) = g_{\mu\nu}(x) + r g'_{\mu\nu}(x) + \frac{1}{2} r^2 g''_{\mu\nu} + \dots , \tag{4.17}$$

$$\begin{aligned} h_{ij}(x,r) &= h_{ij}(x) + r h'_{ij}(x) + \frac{1}{2} r^2 [h''_{ij}(x)] + \dots \\ &= h_{ij}(x) + 2r K_{ij}(x) + r^2 C_{ij}(x) + \dots \end{aligned} \tag{4.18}$$

where $2C_{ij}(x) = [h''_{ij}(x)] = 2K'_{ij}$.

$$\begin{aligned} {h_{ij}}'(x,r) &= h'_{ij}(x) + r[h''_{ij}(x)] + \dots \\ &= 2K_{ij}(x) + 2rK\prime_{ij(x)+\dots} \\ &= 2K_{ij}(x) + 2rC_{ij} + \dots , \end{aligned} \tag{4.19}$$

$$\begin{aligned} h^{ij}(x,r) &= h^{ij}(x) + r h'^{ij}(x) + \dots \\ &= h^{ij}(x) - 2r K^{ij}(x) + \dots , \end{aligned} \tag{4.20}$$

$$\begin{aligned} h'^{ij}(x,r) &= h'^{ij}(x) + r\rho^{ij}(x) + \dots \\ &= -2K^{ij}(x) + r(8K^{ik}(x)K^{j}{}_{k}(x) - 2C^{ij}(x)) + \dots , \end{aligned} \tag{4.21}$$

$$\begin{aligned} K_{ij}(x,r) &= K_{ij}(x) + r[K'_{ij}](x) + \dots \\ &= K_{ij}(x) + rC_{ij}(x) + \dots , \end{aligned} \tag{4.22}$$

$$K^{ij}(x,r) = K^{ij}(x) + r(C^{ij}(x) - 2K^{ik}(x)K^j_k(x) + \dots\,, \tag{4.23}$$

$$K^i{}_j(x,r) = K^i{}_j(x) + r(C^i{}_j(x) - 2K^{ik}(x)K_{kj}(x)) + \dots\,, \tag{4.24}$$

$$\begin{aligned}\Gamma^i{}_{jk}(x,r) &= \Gamma^i{}_{jk}(x) + r\Gamma'^i{}_{jk}(x) + \dots \\ &= \Gamma^i{}_{jk}(x) + r(K^i_{j|k}(x) + K^i_{k|j}(x) - K_{jk|}{}^i(x)) + \dots\,,\end{aligned} \tag{4.25}$$

$$\Gamma^r{}_{ij}(x,r) = -\frac{1}{2}h'_{ij}(x,r) = -K_{ij}(x,r) = -K_{ij}(x) - rC_{ij}(x) + \dots\,, \tag{4.26}$$

$$\frac{L_{,i}(x,r)}{L(x,r)} = \frac{\beta_{1,i}(x)}{\beta_1(x)} + r\Big(\frac{\beta_{2,i}(x)}{2\beta(x)} - \frac{\beta_2(x)\beta_{1,i}(x)}{2\beta_1^2(x)}\Big) + \dots\,, \tag{4.27}$$

$$\frac{L'_{,i}(x,r)}{L(x,r)} = \frac{1}{r}\frac{\beta_{1,i}(x)}{\beta_1(x)} + \frac{1}{\beta_1(x)}\Big(\beta_{2,i}(x) - \frac{\beta_{1,i}(x)\beta_2(x)}{2\beta_1(x)}\Big) - r\frac{\beta_{2,i}(x)\beta_2(x)}{2\beta_1^2(x)} + \dots\,, \tag{4.28}$$

$$\frac{L'(x,r)}{L(x,r)} = \frac{1}{r} + r\frac{\beta_2(x)}{2\beta_1(x)} + \dots\,, \tag{4.29}$$

$$\frac{L_{,i,j}(x,r)}{L(x,r)} = \frac{\beta_{1,i,j}(x)}{\beta_1(x)} + r\frac{1}{\beta_1(x)}\Big(\beta_{2,i,j}(x)\frac{1}{2} - \frac{\beta_2(x)}{2\beta_1(x)}\beta_{1,i,j}(x)\Big) + \dots \tag{4.30}$$

$$\frac{\Box L(x,r)}{L(x,r} = \frac{\Box L(x)}{L(x)} + \frac{1}{\beta_1(x)}\Big(\Box\beta(x)\frac{1}{2} - \frac{\beta_2(x)}{2\beta_1(x)}\Box\beta(x)\Big) + \dots\,, \tag{4.31}$$

$$\frac{[L''(x,r)]}{L(x,r)} = \frac{1}{r}\frac{\beta_2(x)}{\beta_1(x)} - \frac{\beta_2^2(x)}{2\beta_1(x)} + \beta_3 + \mathcal{O}(r)\,, \tag{4.32}$$

$$\frac{L'(x,r)}{L(x,r)}f(x,r) = \frac{1}{r}f(x) + \frac{\beta_2(x)}{2\beta_1(x)}f(x) + f'(x) + \dots \tag{4.33}$$

where $f(x,r)$ is any scalar function.

Chapter 5

Matching conditions

We will find here the matching conditions of the brane for the axially symmetric six-dimensional geometry. After the indices μ, ν in (3.12) are split into parallel i, j indices and normal ones, and taking into account the expansion described in chapter 4, we obtain regular terms, mild distributional terms $\delta(r)$ and essential distributional terms $\frac{\delta(r)}{r}$. Because of the axial symmetry, the angular dependence in the integral is trivially extracted out. The radial r integration is performed from 0 to ϵ in the limit $\epsilon \to 0$. For smooth $\delta g_{\mu\nu}$ variation, the regular and the $\delta(r)$ terms do not contribute as it is seen for the following two indicate terms

$$\int_M d^6x\sqrt{-|g|}K_{ij}\delta h^{ij}|_{brane} = \int_M \sqrt{-|h|}d\theta dr L(x,r)d^4x\frac{1}{2}h'_{ij}(x,r)\delta h^{ij} = 0 \tag{5.1}$$

$$\int_M d^6x\sqrt{-|g|}K'_{ij}\delta h^{ij}|_{brane} = \int_M \sqrt{-|h|}d\theta dr L(x,r)d^4x\bigg\{\big(1-\eta\big)K_{ij}(x)\delta(r) + [K'_{ij}(x,r)]\delta h^{ij}\bigg\} = 0\,. \tag{5.2}$$

It is obvious that $h'_{ij}(x,r)$, $[K'_{ij}(x,r)]$ are power series in r and therefore the radial integration vanishes as $\epsilon \to 0$. Also, the distributional term vanishes due to $r\delta(r) = 0$. The structures from chapter 4 that contribute in the integral

(3.112) are only

$$\frac{L''}{L} = -(1-\beta_1(x))\frac{\delta(r)}{\beta_1(x)r}\,, \tag{5.3}$$

$$\frac{{K_{ij}}'}{L} = K_{ij}(x)\frac{\delta(r)}{\beta_1(x)r}\,, \tag{5.4}$$

$$\frac{L'}{L}{K_{ij}}'(x) = (1-\eta)K_{ij}(x)\frac{\delta(r)}{r}\,, \tag{5.5}$$

for example the term

$$\begin{aligned}\int_M d^6x\sqrt{-|g|}\frac{L''}{L}h_{ij}\delta h^{ij}|_{brane} &= -\int_M d\theta dr L(x,r)d^4x\sqrt{-|h|}\frac{(1-\beta_1(x))}{L(x,r)}\delta(r)h_{ij}\delta h^{ij}\\ &= -2\pi\int_\Sigma d^4x\sqrt{-|h|}(1-\beta_1(x))h_{ij}(x)\delta h^{ij} \qquad (5.6)\end{aligned}$$

does contribute. The x-dependence of the quantities is suppressed from now on for notational convenience. The result for the total variation of the action is

$$\begin{aligned}\delta S\,|_{brane} = \delta_n S + \delta_h S + \delta_g S\,|_{brane} = \int_\Sigma d^4x\sqrt{-|h|}\Big(\lambda^{\alpha i}x^\mu{}_{,i} + 2\lambda^{\alpha\beta}n_\beta{}^\mu\Big)\delta n_{\alpha\mu}\\ + \int_\Sigma d^4x\sqrt{-|h|}\Bigg\{\lambda^{ij} + \frac{1}{2}(T^{ij}-\lambda h^{ij}) - \frac{r_c^2}{2\kappa_6^2}G^{ij} + \frac{\pi}{\kappa_6^2}(1-\beta_1(x))h^{ij}\\ + \frac{4\pi\alpha_{GB}}{\kappa_6^2}K^{ij}\Bigg\}\delta h_{ij} + \int_\Sigma d^4x\sqrt{-|h|}\Big(\lambda^{\alpha\beta}n_\alpha{}^\mu n_\beta{}^\nu - \lambda^{ij}x^\mu{}_{,i}x^\nu{}_{,j}\Big)\delta g_{\mu\nu}\,. \qquad (5.7)\end{aligned}$$

Since $\delta n_{\alpha\mu}$, δh_{ij}, are considered as independent, minimizing the action $\delta S\,|_{brane}=0$ gives the equations

$$\lambda^{\alpha i}x^\mu{}_{,i} + 2\lambda^{\alpha\beta}n_\beta{}^\mu = 0\,, \tag{5.8}$$

$$\lambda^{ij} + \frac{1}{2}(T^{ij}-\lambda h^{ij}) - \frac{r_c^2}{2\kappa_6^2}G^{ij} + \frac{\pi}{\kappa_6^2}(1-\beta_1)h^{ij} + \frac{4\pi\alpha_{GB}(\eta-\beta_1)}{\kappa_6^2}K^{ij} = 0\,, \tag{5.9}$$

$$\int_\Sigma d^4x\sqrt{-|h|}\Big(\lambda^{\alpha\beta}n_\alpha{}^\mu n_\beta{}^\nu - \lambda^{ij}x^\mu{}_{,i}x^\nu{}_{;j}\Big)\delta g_{\mu\nu} = 0\,. \tag{5.10}$$

Since the vectors $x^\mu{}_{,i}$ and $n_\alpha{}^\mu$ are normal (independent), eq.(5.8) implies

$$\lambda^{\alpha i} = 0 = \lambda^{\alpha\beta}\,. \tag{5.11}$$

The non-conventional δx^μ style of variation discussed before is done through $\delta g_{\mu\nu} = -\pounds_{\delta x} g_{\mu\nu}$ (eq. 3.5), thus eq. (5.10), using eq. (5.11) takes the form

$$\int_\Sigma d^4x\sqrt{-|h|}\lambda^{ij}x^\mu{}_{,i}x^\nu{}_{,j}\Big(g_{\mu\nu,\lambda}\delta x^\lambda + g_{\mu\lambda}\delta x^\lambda_{,\nu} + g_{\nu\lambda}\delta x^\lambda_{,\mu}\Big)\,. \tag{5.12}$$

After an integration by parts and imposing $\delta x^\mu|_{\partial\Sigma} = 0$ (5.12) becomes

$$2\int_\Sigma d^4x\sqrt{-|h|}g_{\mu\nu}\left(\lambda^{ij}{}_{|j}x^\mu{}_{,i} - \lambda^{ij}K^\alpha{}_{ij}n_\alpha{}^\mu\right)\delta x^\nu = 0\,, \tag{5.13}$$

where use of the equation (2.42) was made.
Due to the arbitrariness of δx^μ, the last equation is equivalent to

$$\lambda^{ij}{}_{|j}x^\mu{}_{,i} - \lambda^{ij}K^\alpha{}_{ij}n_\alpha{}^\mu = 0\,. \tag{5.14}$$

It follows that

$$\lambda^{ij}{}_{|j} = 0 \tag{5.15}$$

and

$$\lambda^{ij}K^\alpha{}_{ij} = 0\,, \tag{5.16}$$

where λ^{ij} is given by (5.9). Finally, by differentiating eq.(5.9) and using eq.(5.15) we get

$$T^{ij}_{|j} = \frac{2\pi}{\kappa_6^2}\beta_{1,j}(h^{ij} - 4a_{GB}G^{ij}) - \frac{8\pi a_{GB}}{\kappa_6^2}\Bigg\{(\eta-\beta_1)\Big(-\frac{1}{2}h^{ij} \Big(K^\alpha K_\alpha - K^{\alpha kl}K_{\alpha kl}\Big) + K^\alpha K_\alpha{}^{ij} - K^{\alpha mi}K_\alpha{}^j{}_m\Big)\Bigg\}_{|j}\,, \tag{5.17}$$

which is the first matching condition and $K^\alpha = K^{\alpha i}{}_i$. This equation expresses a non-conservation equation of the brane energy-momentum tensor, where this energy exchange between the brane and the bulk is due to the variability along the brane of both the deficit angle and its intrinsic and extrinsic geometry. In the absence of the Gauss-Bonnet term, it reduces to the non-conservation equation found in [20], where the exchange is only due to the deficit angle.

Eq.(5.16) becomes

$$\Big\{-\frac{1}{2}h^{ij}\big(K^{\beta}K_{\beta}-K^{\beta kl}K_{\beta kl}\big)+K^{\beta}K^{ij}_{\beta}-K^{\beta ki}K_{\beta}{}^{j}{}_{k}$$
$$-\ \frac{1-\beta_1}{\eta-\beta_1}\Big(1+\frac{r_c^2}{8\pi\alpha_{GB}(1-\beta_1)}\Big)G^{ij}+\frac{\kappa_6^2}{8\pi\alpha_{GB}[\eta-\beta_1]}T^{ij}$$
$$-\ \frac{\kappa_6^2\lambda-2\pi(1-\beta_1)}{8\pi\alpha_{GB}(\eta-\beta_1)}h^{ij}\Big\}K^{\alpha}_{ij}=0\,, \tag{5.18}$$

which is the second matching condition. Roughly speaking, this is a cubic equation in the extrinsic curvature, contrary to the matching condition discussed in [8], [18] which is quadratic. In the absence of the Gauss-Bonnet term, it reduces to the matching condition of [20], which is linear in extrinsic curvature; additionally, the special case of a probe brane of tension λ ($T_{ij}=0, r_c=0$) reduces the equation to the Nambu-Goto equation of motion (2.47) as expected.

We will now examine the singular $\mathcal{O}(\frac{1}{r})$ terms of the bulk equations (3.13) and see what are the corresponding induced equations on the brane. From the ri components we get the equation

$$\mathcal{R}^{\alpha l}{}_{il}K_{\alpha}-\mathcal{R}^{\alpha}{}_{jil}K_{\alpha}{}^{kl}-\mathcal{R}^{\alpha l}{}_{dl}K_{\alpha i}{}^{d}=\frac{\beta_{1|k}(x)}{\beta_1(x)}\Bigg(-\frac{1}{4a_{GB}}\delta^k_i+G_i{}^k-K^{\alpha}K_{\alpha i}{}^{k}$$
$$+K^{\alpha j}_{i}K_{\alpha k}{}^{j}-\frac{1}{2}\big(K^{\alpha kl}K_{\alpha kl}-K^{\alpha}K_{\alpha}\big)\delta^k_i\Bigg)\,. \tag{5.19}$$

Differentiating this equation and using the Codazzi-Mainardi identity, we can get another form for the mathcing condition (5.17)

$$T^{ij}{}_{|j}=\eta\frac{\beta_{1|j}}{\beta_1}\frac{8\pi a_{GB}}{\kappa_6^2}\Bigg\{\frac{1}{4a_{GB}}h^{ij}-\frac{1}{2}h^{ij}\Big(K^{\alpha}K_{\alpha}-K^{\alpha kl}K_{\alpha kl}\Big)-G^{ij}$$
$$+\ K^{\alpha}K_{\alpha}{}^{ij}-K_{\alpha}{}^{ik}K_{\alpha k}{}^{j}\Bigg\}\,. \tag{5.20}$$

From the $\mathcal{O}(\frac{1}{r})$ of the rr components we get

$$\left\{\frac{1}{4a_{GB}}h_{ij}-G_{ij}-K^{\beta l}{}_{i}K_{\beta jl}+K^{\beta}K_{\beta ij}-\frac{1}{2}h_{ij}\Big(K^{\beta}K_{\beta}-K^{\beta lm}K_{\beta lm}\Big)\right\}K^{\alpha ij}=0\,. \tag{5.21}$$

Combining this equation with the matching condition (5.18), we see that (5.21) is equivalent to the equation

$$\big(\sigma_1 G_{ij}+\sigma_2 h_{ij}-T_{ij}\big)K^{\alpha ij}=0\,, \tag{5.22}$$

where

$$\sigma_1=\frac{r_c^2}{\kappa_6^2}+\frac{8\pi a_{GB}(1-\eta)}{\kappa_6^2}\ \ ,\ \sigma_2=\lambda-\frac{2\pi(1-\eta)}{\kappa_6^2}\,. \tag{5.23}$$

The equations coming from the $\mathcal{O}(\frac{1}{r})$ terms of the ij components are more complicated and we do not give their general expression here. Similarly, the general form of the regular equations is even more complicated. For the case of cosmology we will write all these equations and check their consistency.

Chapter 6

Cosmology

Consider the bulk cosmological metric

$$ds_6^2 = dr^2 + L^2(x,r)d\theta^2 - n^2(t,r)dt^2 + a^2(t,r)\gamma_{\hat{i}\hat{j}}(x)dx^{\hat{i}}dx^{\hat{j}} \,, \tag{6.1}$$

where $\gamma_{\hat{i}\hat{j}}$ is a maximally symmetric 3-dimensional metric characterised by its spatial curvature $k = -1, 0, 1$ and $\hat{i} = 1, 2, 3$. The energy momentum tensor on the brane is assumed to be that of a perfect fluid with diagonal elements (ρ, p, p, p). For the scalars $n(t,r)$, $a(t,r)$ we have the expansion

$$a(t,r) = a(t) + ra(t)A(t) + \frac{1}{2}r^2 a(t)A_2(t) + \dots \,, \tag{6.2}$$

$$n(t,r) = 1 + rN(t) + \frac{1}{2}r^2 N_2(t) + \dots \,, \tag{6.3}$$

where

$$A = \frac{a'}{a} \,,\ N = \frac{n'}{n} \tag{6.4}$$

We also define the quantities

$$\begin{aligned} X = H^2 + \frac{k}{a^2} \,,\ Y = \frac{\dot{H}}{n} + H^2 \,, H = \frac{\dot{a}}{na} \\ \mathcal{X} = X - A^2 + \frac{1}{12a_{GB}} \,,\ \mathcal{Y} = Y - AN + \frac{1}{12a_{GB}} \end{aligned} \tag{6.5}$$

and begin the same procedure followed in the previous chapter.
For this metric, the first matching condition found becomes

$$\dot{\rho} + 3nH(\rho + p)$$
$$= \frac{8\pi a_{GB}}{\kappa_6^2}\left(-3\mathcal{X}\dot{\beta}_1 - \big(\eta - \beta_1\big)\big(6A\dot{A} - 6nNH + 6HnA^2\big)\right), \qquad (6.6)$$

while the second matching condition becomes

$$A^3 + 3A^2N - \frac{1-\beta_1}{\eta - \beta_1}\big(1 + \frac{r_c^2}{8\pi\alpha_{GB}(1-\beta_1)}\big)\big(X(A+N) + 2YA\big)$$
$$- \frac{\kappa_6^2}{24\pi\alpha_{GB}(\eta - \beta_1)}\big(3Ap - N\rho\big) + \frac{\kappa_6^2\lambda - 2\pi(1-\beta_1)}{24\pi\alpha_{GB}(\eta-\beta_1)}\big(3A + N\big) = 0\,. \qquad (6.7)$$

The $\mathcal{O}(\frac{1}{r})$ part of the rt component (5.19), takes the form

$$2A\Big(\frac{\dot{A}}{n} - HN + AH\Big) = \frac{\dot{\beta}_1}{n\beta_1}\mathcal{X} \qquad (6.8)$$

and the $\mathcal{O}(\frac{1}{r})$ part of the rr components, (5.21), becomes

$$\mathcal{X}\big(1 + \frac{N}{A}\big) + 2\mathcal{Y} = 0\,. \qquad (6.9)$$

Applying the cosmological metric straightforwardly to (5.20) and (5.22), or alternatively combining (6.6) with (6.8) and (6.7) with (6.9) we get

$$\dot{\rho} + 3nH(\rho + p) = -\eta\frac{\dot{\beta}_1}{\beta_1}\frac{24\pi a_{GB}\mathcal{X}}{\kappa_6^2}\,, \qquad (6.10)$$

which is a non-conservation equation for ρ, and

$$N = fA\,, \qquad (6.11)$$

where

$$f = 3\frac{-p + \sigma_2 - \sigma_1(X + 2Y)}{-\rho - \sigma_2 + 3\sigma_1 X}\,. \qquad (6.12)$$

The terms $\mathcal{O}(\frac{1}{r})$ of the ij components contain the tt equation

$$2\frac{[a]''}{a} - \frac{\beta_2}{\beta_1 A}\mathcal{X} - \mathcal{X} - \frac{1}{6a_{GB}} = 0 \qquad (6.13)$$

and the $\hat{i}\hat{j}$ one which with use of (6.7), (6.9), (6.12) and (6.8) becomes

$$\frac{[n]''}{n} + f\frac{[a]''}{a} - \frac{1+f}{12a_{GB}} - \frac{\beta_2}{A\beta_1}\mathcal{Y} + \mathcal{X} + (\frac{\beta_1}{n\beta_1})^2\frac{\mathcal{X}}{A^2} = 0\,. \tag{6.14}$$

The regular terms of the $\theta\theta$ component with use of (6.8), (6.11) and (6.12) become

$$\frac{[n]''}{n} - f\frac{[a]''}{a} - \mathcal{Y} - \frac{1-f}{12a_{GB}} - \frac{1}{2}(\frac{\dot{\beta}_1}{n\beta_1})^2\frac{\mathcal{X}}{A^2} - \frac{1}{12a_{GB}\mathcal{X}}(\kappa_6^2\mathcal{T}_\theta^\theta - \Lambda_6 - \frac{5}{12a_{GB}}) = 0\,. \tag{6.15}$$

The three last equations (6.13), (6.14), (6.15) form an algebraic system for $\frac{[a]''}{a}$, $\frac{[n]''}{n}$ and $\frac{\beta_2}{\beta_1 A}$ which can be also solved in terms of $[A]'$ and $[N]'$

$$\frac{\beta_2}{\beta_1 A}\big(1+3f\big) = -(1+f) - \frac{3}{A^2}(\frac{\dot{\beta}_1}{n\beta_1})^2 - \frac{1}{6a_{GB}\mathcal{X}^2}\Big(\kappa_6^2\mathcal{T}_\theta^\theta - \Lambda_6 - \frac{5}{12a_{GB}}\Big)\,, \tag{6.16}$$

$$\begin{aligned}[A]\,'\big(1+3f\big) &= f\mathcal{X} - \frac{3}{2A^2}(\frac{\dot{\beta}_1}{n\beta_1})^2\mathcal{X} - \frac{1}{12a_{GB}\mathcal{X}}\Big(\kappa_6^2\mathcal{T}_\theta^\theta - \Lambda_6 - \frac{5}{12a_{GB}}\Big)\\ &+\ \big(1+3f\big)\Big(\frac{1}{12a_{GB}} - A^2\Big)\,,\end{aligned} \tag{6.17}$$

$$\begin{aligned}[N]'\big(1+3f\big) &= (\frac{\dot{\beta}_1}{n\beta_1})^2\frac{\mathcal{X}}{2A^2} + \frac{1+2f}{12a_{GB}\mathcal{X}}\Big(\kappa_6^2\mathcal{T}_\theta^\theta - \Lambda_6 - \frac{5}{12a_{GB}}\\ &+ \big(1+3f\big)\Big(\frac{1}{12a_{GB}} - N^2\Big) + \frac{-f^2-4f-1}{2}\mathcal{X}\,.\end{aligned} \tag{6.18}$$

From these equations we take also

$$\begin{aligned}f\ [A]' - [N]' &= \frac{f-1}{12a_{GB}} + f\big(f-1\big)[A]^2 - (\frac{\dot{\beta}_1}{n\beta_1})^2\frac{\mathcal{X}}{2A^2} + \frac{\mathcal{X}}{2}\big(1+f\big)\\ &- \frac{1}{12a_{GB}\mathcal{X}}\Big(\kappa_6^2\mathcal{T}_\theta^\theta - \Lambda_6 - \frac{5}{12a_{GB}}\Big)\,.\end{aligned} \tag{6.19}$$

The regular part of the rr component is

$$
\begin{aligned}
&[A]' \big(X+2Y\big)+[N]'X+\frac{3[A]'+[N]'}{12a_{GB}}+X'\big(A+N\big)+2AY'-12A^4\\
&-\;\; 6A^3N-6AN^3+3[A]'\big(-A^2-2AN\big)-3A^2[N]'-\frac{\mathcal{X}+\mathcal{Y}}{6a_{GB}}-\mathcal{X}\mathcal{Y}\\
&+\Big(\frac{\dot\beta_1}{\beta_1}\frac{\dot n}{n^3}-\frac{\dddot\beta_1}{\beta_1}\frac{1}{n^2}\Big)\mathcal{X}+\frac{\dot\beta_1}{\beta_1}\frac{H}{n}\big(-\mathcal{X}-2\mathcal{Y}\big)\\
&-\;\;\frac{1}{12a_{GB}}\Big(\kappa_6^2\mathcal{T}_r^r-\Lambda_6-\frac{5}{12a_{GB}}\Big)=0\,,
\end{aligned}
\tag{6.20}
$$

where all the quantities are at $r=0$ [1]. Rearranging (6.14) with repeated use of (6.18) and equation

$$
\begin{aligned}
&\Big(X+2Y+\frac{1}{4a_{GB}}\Big)[A]'+\Big(X+\frac{1}{12a_{GB}}\Big)[N]'=(\frac{\dot\beta_1}{n\beta_1})^2\frac{\mathcal{X}}{2A^2}\big(\mathcal{X}-2A^2\big)\\
&+\;\;\frac{1-f}{12a_{GB}}\mathcal{X}-\frac{1+f}{2}\mathcal{X}^2-\frac{1+2f^2-3f}{2}A^2\mathcal{X}-\left(A^2+fA^2-\frac{1}{12a_{GB}}\right)^2\\
&+\;\;\frac{1}{12a_{GB}}\Big(\kappa_6^2\mathcal{T}_\theta^\theta-\Lambda_6-\frac{1}{3a_{GB}}\Big)
\end{aligned}
\tag{6.21}
$$

which is proven in the appendix, we get the following useful equation

$$
\begin{aligned}
&[A]\,'\big(X+2Y\big)+[N]'X+\frac{3[A]'+[N]'}{12a_{GB}}-12A^4-6A^3N-6AN^3+3[A]'\\
&\big(\;-A^2-2AN\big)-3A^2[N]'=(\frac{\dot\beta_1}{n\beta_1})^2\frac{\mathcal{X}}{2A^2}\big(\mathcal{X}+4a^2\big)+\frac{\mathcal{X}}{12a_{GB}}\big(1-f\big)\\
&-\;\;\big(1+f\big)\frac{\mathcal{X}^2}{2}+A^2\mathcal{X}\big(1-f^2\big)+2A^4\big(1+f\big)^2-\frac{1+f}{3a_{GB}}A^2\\
&+\;\;\frac{1}{12a_{GB}}\Big(\kappa_6^2\mathcal{T}_\theta^\theta-\Lambda_6-\frac{5}{12a_{GB}}\Big)\,.
\end{aligned}
\tag{6.22}
$$

Combining the previous equation, the regular part of the rr component and differentiating the X, Y quantities from the previous equations of the algebraic system we find

$$
-12A^4-6A^3N-6AN^3+\frac{1}{12a_{GB}}\Big(\kappa_6^2\mathcal{T}_\phi^\phi-\kappa_6^2\mathcal{T}_\rho^\rho\Big)=0\,.
\tag{6.23}
$$

[1] The regular part of the ij components contain $[A]''$, $[N]''$, β_3 and therefore, these equations are not significant for the consistency of the problem

Because we are interested in a bulk with a cosmological constant

$$12A^4 + 6A^4 f + 6A^4 f^3 = 0\,. \tag{6.24}$$

with solution

$$f = -1 \tag{6.25}$$

We can eliminate the quantities N, A^2, A^3 from the matching condition (6.7) using (6.8), (6.9) to get

$$\frac{1}{6a_{GB}} - \frac{\rho - 2\sigma_2 + 3p}{3\sigma_1} = -\frac{1-\beta_1}{\eta - \beta_1}\big(1 + \frac{r_c^2}{8\pi a_{GB}(1-\beta_1)}\big)\Big(\frac{\rho - 2\sigma_2 + 3p}{3\sigma_1}\Big) + \frac{\kappa_6^2}{24\pi a_{GB}(\eta-\beta_1)}\big(3p+\rho\big) - \frac{\kappa_6^2\lambda - 2\pi(1-\beta_1)}{12\pi a_{GB}(\eta-\beta_1)}\,, \tag{6.26}$$

which with substitution of the parameters σ_1, σ_2 gives $0 = 0$. We now focus on the simpler and most interesting case of a spatially flat universe (k=0, and from (6.20) and $f = -1$, we get a differential equation for the Hubble parameter

$$\frac{\dot{H}}{n} + H^2 - \frac{\sigma_2}{3\sigma_1} + \frac{\rho}{6\sigma_1}(1+3w) = 0\,. \tag{6.27}$$

Using

$$\frac{\dot{H}}{n} = \frac{dH}{da}\frac{da}{dt}\frac{1}{n} = \frac{dH}{da}\frac{naH}{n} = \frac{1}{2}\frac{d(H^2)}{d(lna)}\,, \tag{6.28}$$

rewrite the previous equations as

$$\frac{d(H^2)}{d(lna)} + 2H^2 - \frac{2\sigma_2}{3\sigma_1} + \frac{\rho}{3\sigma_1}(1+3w) = 0\,. \tag{6.29}$$

We analyse the two characteristic cases $\eta = 0$, $\eta = 1$.

• case (i): η =0 .

The non-conservation equation becomes the standard conservation equation

$$\dot{\rho} + 3(p+\rho)nH = 0\,, \tag{6.30}$$

which for $p = w\rho$ has the standard solution

$$\rho = c_1 a^{-3(1+w)}\,, \tag{6.31}$$

with c_1 an integration constant.

Substitute this result in the differential equation for the Hubble parameter which is integrated to

$$H^2(a) = \frac{\sigma_2}{3\sigma_1} + \frac{c_1}{3\sigma_1(1+w)}\frac{1}{a^3} + \frac{c_2}{a^2}\,, \tag{6.32}$$

where c_2 is an integration constant and the parameters σ_1, σ_2 now take the values

$$\sigma_1 = \frac{r_c^2}{\kappa_6^2} + \frac{8\pi a_{GB}}{\kappa_6^2} \qquad \sigma_2 = \lambda - \frac{2\pi}{\kappa_6^2}\,. \tag{6.33}$$

Finally

$$H^2(a) = \frac{\kappa_6^2\lambda - 2\pi}{3(8\pi a_{GB} + r_c^2)} + \frac{c_1\kappa_6^2}{3(8\pi a_{GB} + r_c^2)a^{3(1+w)}} + \frac{c_2}{a^2}\,. \tag{6.34}$$

The equation implies a late time modification of FLRW cosmology with an extra term a^{-2}, which mimics non-vanishing curvature (or cosmic string networks).

• case (ii): η =1 .

The non-conservation equation becomes

$$\dot{\rho} + 3nH\rho(1+w) = -\frac{\dot{\beta}_1}{\beta_1}\frac{24\pi a_{GB}}{\kappa_6^2}\left(2H^2 + \frac{\dot{H}}{n} + \frac{1}{6a_{GB}}\right). \tag{6.35}$$

In this case, we do not have a close system of equations on the brane due to arbitrariness of β_1, so we cannot uniquely specify the Hubble evolution. To get an indication here, we close the system by specifying the evolution of one quantity and solve for the remaining. For example, assuming the energy density to have a power law $\rho(a) = c_4 a^p$, we find from (6.29)

$$H^2(a) = \frac{\lambda\kappa_6^2 - 2\pi}{3(8\pi a_{GB} + r_c^2)} - \frac{(1+3w)c_4\kappa_6^2}{3(2+p)(8\pi a_{GB} + r_c^2)}a^p + \frac{c_3}{a^2}\,. \tag{6.36}$$

Considering the three cases $w = 0, \frac{1}{3}, -1$ we get

• $w = 0$

$$H^2(a) = \frac{\lambda\kappa_6^2 - 2\pi}{3(8\pi a_{GB} + r_c^2)} - \frac{c_4\kappa_6^2}{3(2+p)(8\pi a_{GB} + r_c^2)}a^p + \frac{c_3}{a^2} \tag{6.37}$$

• $w = \frac{1}{3}$

$$H^2(a) = \frac{\lambda\kappa_6^2 - 2\pi}{3(8\pi a_{GB} + r_c^2)} - \frac{2c_4\kappa_6^2}{3(2+p)(8\pi a_{GB} + r_c^2)}a^p + \frac{c_3}{a^2} \tag{6.38}$$

- $w = -1$

$$H^2(a) = \frac{\lambda\kappa_6^2 - 2\pi}{3(8\pi a_{GB} + r_c^2)} + \frac{2c_4\kappa_6^2}{3(2+p)(8\pi a_{GB} + r_c^2)} a^p + \frac{c_3}{a^2} \,. \tag{6.39}$$

These cosmologies have to be analysed more with respect to their phenomenological implications.

Chapter 7

Conclusion

We have adopted a non-conventional style of variation with respect to the embedding fields for a six-dimensional Einstein-Gauss-Bonnet theory and derived new matching conditions taking into account the gravitational back-reaction of the brane to the bulk. Besides these equations there are also other equations valid at the position of the brane, characterized as regular and singular depending on the behaviour of the relevant six-dimensional terms as we approach the brane. The energy-momentum tensor of the brane can either be conserved or not depending on the conditions imposed on the geometry at the conical singularity. For a cosmological metric ansatz we have shown the consistency of the problem and derived the four-dimensional cosmological evolution. In the case of combined cone singularity with extrinsic curvature discontinuity, this cosmology is unique and for a spatially flat universe it gives a late-time modification of FRW of the form a^{-2} (curvature-like). In the other case of pure cone singularity, the effective equations do not form a closed system and characteristic situations were studied with similar results. In general, one can study further the phenomenological implications of these cosmologies. Due to the above IR sort of modifications at the level of cosmology, we speculate that in the spherically symmetric solutions large distance corrections of gravity will appear. However, as usually when one works from the braneworld viewpoint, the bulk extensions of the derived four-dimensional solutions have not been studied and so the physical viability of the solutions is not certain.

Chapter 8

Appendix

• The structure constants of an arbitrary basis $\{E_A\}$ are defined by

$$[E_A, E_B] = C^C{}_{AB} E_C\,. \tag{8.1}$$

If the basis is coordinate there exist x^μ such that $E_\mu = \frac{\partial}{\partial x^\mu}$, then the dual basis is $E^\mu = dx^\mu$ and therefore

$$dE^\mu = d^2 x^\mu = 0\,. \tag{8.2}$$

Since

$$dE^\mu = -\frac{1}{2} C^\mu{}_{\nu\lambda} E^\nu \wedge E^\lambda \Rightarrow C^\mu{}_{\nu\lambda} = 0\,. \tag{8.3}$$

Therefore if the torsion $T^\mu{}_{\nu\lambda} = \Gamma^\mu{}_{\lambda\nu} - \Gamma^\mu{}_{\nu\lambda} - C^\mu{}_{\nu\lambda}$ vanishes, then in a coordinate basis the Christoffel connection is symmetric in its lower indices.

• For a hypersurface, i.e. codimension-1, the projection tensor reduces to the expression

$$h_{\mu\nu} = g_{\mu\nu} - \varepsilon n_\mu n_\nu\,, \tag{8.4}$$

where n^μ is the unit normal and $\varepsilon = g(n,n) = \pm 1$ its sign. It projects any vector V^μ tangent to the hypersurface as $h_{\mu\nu} V^\nu = g_{\mu\nu} V^\nu - \varepsilon n_\mu n_\nu V^\nu = V_\mu - 0 = V_\mu$. Acting more than once it produces the same result as acting only once, i.e. $h^\mu_\kappa h^\kappa_\nu = (\delta^\mu_\kappa - \varepsilon n^\mu n_\kappa)(\delta^\kappa_\nu - \varepsilon n^\kappa n_\nu) = \delta^\mu_\nu - \varepsilon n^\mu n_\nu - \varepsilon n^\mu n_\nu + \varepsilon^2 n^\mu n_\nu = \delta^\mu_\nu - \varepsilon n^\mu n_\nu = h^\mu_\nu$. Acting on two vectors V^μ, W^ν tangent to $\phi[\Sigma]$, the projection tensor acts like the metric, i.e. $h_{\mu\nu} V^\mu W^\nu = g_{\mu\nu} V^\mu W^\nu - \varepsilon n_\mu n_\nu V^\mu W^\nu = g_{\mu\nu} V^\mu W^\nu$.

The dual bases are $E_A = (e_a, e_0 = n)$, $E^A = (e^a, e^0 = \varepsilon\mathbf{n})$ and the relation $E^A(E_B) = \delta^A_B$ gives

$$\mathbf{n}\big(n\big) = \varepsilon\,. \tag{8.5}$$

The index notation is simplified for the normal vector as

$$n_\alpha = n \Leftrightarrow n_\alpha{}^\mu = n^\mu \Leftrightarrow n_{\alpha\mu} = n_\mu \tag{8.6}$$

and for its dual as

$$n^\alpha = \varepsilon\mathbf{n} \Leftrightarrow n^\alpha{}_\mu = \varepsilon n_\mu = \varepsilon\mathbf{n}_\mu \Leftrightarrow n^{\alpha\mu} = \varepsilon n^\mu = \varepsilon\mathbf{n}^\mu\,. \tag{8.7}$$

The correspondence for the extrinsic curvature coefficients is

$$K_\alpha{}^a{}_b = K_n{}^a{}_b = K^a{}_b \tag{8.8}$$

and for the expansion tensor

$$\Theta^\alpha{}_{\mu\nu} = \Theta_{\mu\nu} = n_{\mu;\nu} = n_{\nu;\mu}\,. \tag{8.9}$$

The known expression holds

$$K_{ab} = n_{a;b}\,. \tag{8.10}$$

• We here prove equation (2.45).
If $\sigma^\mu{}_{ij} \equiv t_i{}^\mu{}_{;j} + \Gamma^\mu{}_{\lambda\nu} t_i{}^\lambda t_j{}^\nu$, then $\sigma^\mu{}_{ij} t_{l\mu} + \sigma^\mu{}_{lj} t_{i\mu} = t_i{}^\mu{}_{;j} t_{l\mu} + t_l{}^\mu{}_{;j} t_{i\mu} + g_{\rho\lambda,\mu} t_l{}^\lambda t_j{}^\mu t_i{}^\rho$.
Since $t_i{}^\mu t_l{}^\nu g_{\mu\nu} = g_{il} \Rightarrow t_i{}^\mu{}_{,j} t_{l\mu} + t_{i\mu} t_l{}^\mu{}_{,j} + t_i{}^\mu t_l{}^\nu g_{\mu\nu,\rho} t_j{}^\rho = g_{il,j} \Rightarrow t_i{}^\mu{}_{;j} t_{l\mu} + t_{i\mu} t_l{}^\mu{}_{;j} + t_i{}^\mu t_l{}^\nu g_{\mu\nu,\rho} t_j{}^\rho = 0$.
Therefore $\sigma^\mu{}_{ij} t_{l\mu} + \sigma^\mu{}_{lj} t_{i\mu} = 0$.
Changing the indices (ijl) to (lij) and once more from (ijl) to (jli) we take another two equations and the combination of these three equations gives $\sigma^\mu{}_{ij} t_{l\mu} = 0$, which is the wanted.

• The proof of eq. (6.21) is given here.
The $\mathcal{O}(\frac{1}{r})$ part of the rr components becomes

$$X + 2Y + \frac{1}{4a_{GB}} = -f\mathcal{X} + A^2 + 2fA^2. \tag{8.11}$$

The expression of $\mathcal{X}$ gets the form

$$X + \frac{1}{12a_{GB}} = \mathcal{X} + A^2\,. \tag{8.12}$$

Using these and equation (6.17) we get

$$
\begin{aligned}
&\left(X + 2Y\frac{1}{4a_{GB}}\right)[A]' + \left(X + \frac{1}{12a_{GB}}\right)[N]' \\
&= -\mathcal{X}\big(f[A]' - [N]'\big) + A^2\big([N]' + (1+2f)[A]'\big) \\
&= (\frac{\dot{\beta}_1}{n\beta_1})^2\frac{1}{2A^2}\big(\mathcal{X}^2 - 2\mathcal{X}A^2\big) + \mathcal{X}\frac{1-f}{12a_{GB}} - \frac{\mathcal{X}^2}{2}(1+f) - A^2\mathcal{X}\frac{1+2f^2-3f}{2} \\
&- \big(A^2(1{+}f) - \frac{1}{12a_{GB}}\big)^2 + \frac{1}{144a_{GB}} + \frac{1}{12a_{GB}}\big(\kappa_6^2\mathcal{T}_\theta^\theta - \Lambda_6 - \frac{5}{12a_{GB}}\big) \\
&= (\frac{\dot{\beta}_1}{n\beta_1})^2\frac{1}{2A^2}\mathcal{X}\big(\mathcal{X} - 2A^2\big) + \frac{1-f}{12a_{GB}}\mathcal{X} - \frac{1+f}{2}\mathcal{X}^2 - \frac{1+2f^2-3f}{2}A^2\mathcal{X} \\
&- \Big((1+f)A^2 - \frac{1}{12a_{GB}}\Big)^2 + \frac{1}{12a_{GB}}\Big(\kappa_6^2\mathcal{T}_\theta^\theta - \Lambda_6 - \frac{1}{3a_{GB}}\Big). \qquad (8.13)
\end{aligned}
$$

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