Dynamics of Airy wavepackets and beams

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Abstract

We investigate the finite-energy Airy wave packets and beams appearing in applied optics and optical engineering, and we show that all of them can be represented as the convolution of the non-spreading, infinite-energy wave packet constructed by Berry and Balazs with an appropriated Gaussian function. Moreover, we show that the convolution with any square integrable function produces a finite-energy Airy beam, and therefore, in principle, there are infinitely many possibilities in the experimental construction of Airy beams.

Key Words. Schrödinger equation, Quantum mechanics, Gaussian wave packets, Airy wave packets, Airy beams, Dispersion, WKB method

Περίληψη

Στην εργασία αυτή μελετάμε τα κυματοπακέτα και τις δέσμες Airy που έχουν πεπερασμένη ενέργεια, και χρησιμοποιούνται στην εφαρμοσμένη οπτική. Αποδεικνύουμε ότι όλα αυτά τα κυματοπακέτα μπορούν να αναπαρασταθούν ως συνέλιξη του μη διασπειρόμενου κυματοπακέτου των Berry και Balazs, το οποίο έχει άπειρη ενέργεια, με μια κατάλληλη Γκαουσιανή συνάρτηση. Επιπλέον, δείχνουμε ότι η συνέλιξη του κυματοπακέτου των Berry και Balazs με οποιαδήποτε τετραγωνικά ολοκληρώσιμη συνάρτηση παράγει μια δέσμη Airy πεπερασμένης ενέργειας, και γι' αυτό τον λόγο, θεωρητικά, υπάρχουν άπειρες δυνατότητες στην πειραματική κατασκευή των δεσμών Airy.

Contents

CHAPTER 1

Introduction

1.1 Airy wave packets in quantum mechanics

In a remarkable development, Berry & Balazs [BB] constructed an Airy wave packet which has infinite energy and it propagates with constant acceleration and without spreading. They showed that the acceleration of the wave packet is not in conflict with *Ehrenfest's theorem*, which embodies Newton's second law for classical particles and it suggests that no wave packet can accelerate in free space, because the packet has infinite energy and ill-defined mean values. *Dispersion* in the Schrödinger equation, embodies the ability of classical particles to travel at different speeds, and it suggests that all wave packets with finite energy must change their shape as they propagate in free space. A concise explanation of the eventual spread of wave packets with finite energy, and the necessity of taking into account the appropriate mathematical structure, instead of using formal manipulations of unbounded operators, has been given by Klein [Kl] (see also [HS]).

Airy wave packets have been the subject of many interesting investigations from various perspectives. Greenberger [Gre1] has studied the behaviour of the Airy wave packet from the viewpoint of *equivalence principle* [Gre2], [Na]. More precisely he has interpreted the wave packet as a stationary state of Schrödinger's equation in the uniform gravitational field, and he used an extended Galilean transformation to the free fall system, to explain the acceleration. In this respect the Airy packet is no "less" trivial than the plane wave solution, which is the stationary solution for the force-free case.

From a different perspective, Unnikrishnan & Pau [UP] started from the definition of a nonspreading packet, as that having the modulus of its wave function invariant under time-dependent space translations, and they employed the evolution operator and Baker-Campbell-Hausdorff formula for non-commuting operators, in order to construct the Berry-Balazs wave packet in a "purely quantum mechanical way". More recently, Vyas [Vy] motivated by the shape preservation of the Airy wave packet, reconsidered this wave packet as a particular Perelomov coherent state. He showed that, in contrary to harmonic-oscillator coherent states, the Airy wave packet is the unique one that does not transverse the classical trajectories as it happens in systems without acceleration, but it moves along the accelerated caustic of the classical trajectories. Besieris, Shaarawi & Ziolkowksi [BSZ] have constructed a class of nonspreading solutions in three dimensions both for the free Schrödinger equation and the wave equation with constant speed. These solutions involve accelerating Airy envelopes and they are characterized by an asymmetric structure, in contrary to spherically symmetric packets moving with constant velocities.

However, Airy wave packets do not fit in the standard L^2 framework of quantum mechanics, and, besides their own mathematical interest, they can be considered only as building blocks of square integrable wave functions. Moreover, the experimental synthesis of wave packets carrying infinite energy is impossible, and the unavoidable truncation of amplitudes results to the realization of beams with small, but non zero, dispersion. Berry & Balazs [BB] theselves have proposed in the, seemingly forgotten, appendix of their pioneering paper, that any square-integrable wave function constructed from Airy wave packets must obey Ehrenfest's theorem. They suggested that such wave functions can be derived by convolving their wave packet with a time-independent Gaussian function, thus constructing an *Airy beam*, i.e. a superposition of Airy wave packets [1].

In the physics and optics literature, we find several approaches to construct finite energy Airy beams. Siviloglou & Christodoulides [SC] have modified the initial data by multiplying the Airy function with appropriately decay exponential function to ensure containment of the infinite Airy tail and thus enabled the experimental realization of such beams [SBDC]. Besieris and Shaarawi [BS] explained that the same solution can by obtained by complex scaling of the Berry-Balazs wave packet. They computed the mean values and the dispersion of the derived wave packet and they argued that it remains essentially dispersion-free for long time. Lekner [Lek] showed that a Galilean boost to a reference frame moving with complex velocity transforms Berry-Balazs wave packet into a square integrable wave packet. When the velocity is purely imaginary ones regains the Siviloglou-Christodoulides wave packet.

1.2 Scope and structure of the thesis

In this work we reconsider the various square-integrable wave packets appearing in physics' and optics' literature. We prove that the finite energy solutions $\psi(x,t)$ proposed by Siviloglou & Christodoulides [SC], Besieris & Shaarawi [BS] and Lekner [Lek], can be expressed as the convolution of the Berry-Balazs wave packet $\psi_{\text{BB}}(x,t)$ with an appropriate Gaussian function $g_\epsilon(x)$,

$$\psi(x,t) = \psi_{\text{BB}}(x,t) *_x g_\epsilon(x),$$

where

$$g_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon^2}}\, e^{-x^2/2\epsilon^2} \ ,$$

$\epsilon > 0$ being a parameter which is different for each particular solution. Therefore, all square integrable solutions can be expressed as weighted superpositions (Airy beams) of Berry-Balazs wave packets distributed along the whole x-axis (Airy beams). The destructive interference of the tails of the superposed Airy functions leads to a wave function with finite energy. Obviously, as $\epsilon \to 0$, $g_\epsilon(x) \to \delta(x)$, and therefore all solutions reduce to he Berry-Balazs wave packet.

In chapter 2 we present some basic elements of quantum mechanics in one dimension, in particular, the notions of mean value, dispersion and the Fourier integral representation of the solution of the free Schrödinger equation, and we make some comments on the dispersion effect and the delocalization of wave functions, and their relation with the group and phase velocity.

In chapter 3 we introduce the standard Gaussian wave packet. This is the simplest and most basic example of a wave packet with finite energy. This wave packet remains Gaussian as it evolves, but its width increases with time due to the dispersion, so that its shape eventually deteriorates and the packet becomes delocalized.

In chapter 4 we present the nonspreading Airy wave packet constructed by Berry and Balazs. This wave packet is the solution of the free Schrödinger equation with Airy initial data. Since the Airy function is not square integrable, this wave packet has infinite energy, and therefore is not physically realizable. From the theoretical point of view, it has very rich dynamics, the most interesting property being that, in a certain

[1] They also mentioned that the obtained wave function is Weyl's "eigendifferential", that is an eigenfuction belonging to the continuous spectrum of the Airy equation [Ke]. See also [Grein], Sec. 5.1, for a informal introduction to the notion of "eigendifferentials"

sense, the wave packet accelerates without external force. Moreover, like plane waves for the wave equation, it can be thought of as a building block of Airy beams. From the practical point of view it is exactly the last feature that makes the Berry-Balazs wave packet important in applied physics and optics.

In chapter 5 we present the Gaussian Berry-Balazs wave packet, and we explain how the solutions proposed by Siviloglou & Christodoulides, Besieris & Shaarawi, and Lekner, are expressed in the form of the Gaussian Berry-Balazs wave packet, by identifying the appropriate Gaussian functions.

CHAPTER 2

Elements of one-dimensional quantum mechanics

A quantum mechanical particle of mass m moving along the real line under the action of a potential field $V(x)$, is described by a complex wave function $\psi(x,t)$ (also referred to as the state of the particle), in the sense that the probability to find the particle in the interval $(x, x+dx)$ at the time moment $t \geq 0$ is given by $|\psi(x,t)|^2 dx$. In other words, the position of the particle is considered as a random variable with probability density $|\psi(x,t)|^2$. Since, for any fixed time t, we certainly (with probability one) find the particle somewhere on the axis, it follows that $\int_{\mathbb{R}} |\psi(x,t)|^2 dx = 1$ for any time $t \geq 0$, that is $\psi(x,t) \in L^2(\mathbb{R}, \mathbb{C}; dx)$.

The wave function is governed by the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \ , \ x \in \mathbb{R} \ , \ \ t > 0 \ , \tag{2.1}$$

and it satisfies a given initial condition $\psi(x, t=0) = \psi_0(x) \in L^2$, which defines the initial probability density. The parameter $\hbar$ is Planck's constant, which is appropriately scaled, so that equation (2.1) is dimensionless.

2.1 Quantum mechanical operators and mean values

Let $\widehat{x}$ be the position operator, acting as the multiplication $\widehat{x} f(x) = x f(x)$, and $\widehat{p}$ the momentum operator acting by differentiation $\widehat{p} f(x) = -i\hbar \partial_x f(x)$ on reasonably smooth functions. By formal substitution of $\widehat{x}$ and $\widehat{p}$ in place of x and p, respectively, into Hamilton's function we get

$$H(\widehat{x}, \widehat{p}) = \frac{1}{2m} \big(-i\hbar \partial_x\big)^2 + V(x) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) =: \widehat{H} \ . \tag{2.2}$$

We observe that $\widehat{H}$ is the differential operator (Hamiltonian operator) in the r.h.s. of the Schrödinger equation (2.1). This formal calculation is the simplest way to map phase space functions to formally self-adjoint operators acting on functions of the position, and the simplest case of the so called quantization procedure.

Although $H(x,p)$ is the most important classical observable, in quantum mechanics we consider also phase functions $A(x,p)$ representing more complicated classical observables. Such functions are associated to more general self-adjoint operators $\widehat{A}$ by certain quantization procedures, but it is important to note that the formal rule $\widehat{A} = A(\widehat{x}, \widehat{p})$ is not, in general, correct. This fundamental fact becomes apparent when someone considers, for example, the function $A(x,p) = xp = px$ since the operators $\widehat{x}$ and $\widehat{p}$ do not commute.

For a time independent operator $\widehat{A}$, we define the *mean value* of the corresponding classical observable A, when the quantum particle is described by the wavefunction $\psi(x,t)$, as the real- valued function of time[1]

$$\langle A\rangle_t = \Big(\psi(\cdot,t), \widehat{A}\psi(\cdot,t)\Big)_{L^2} = \int_{-\infty}^{\infty} \psi^*(x,t)\widehat{A}\psi(x,t)dx \;, \tag{2.3}$$

where $\big(\cdot\;,\cdot\big)_{L^2}$ denotes the L^2-inner product. For notational simplicity we write $\langle A\rangle$ instead of $\langle A\rangle(t)$. Usually we normalize the wavefunction so that

$$\int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)dx = 1 \;, \tag{2.4}$$

and therefore the mean value of the identity operator is equal to one for any state.

In particular, the mean values of the position and momentum operators

$$\widehat{x} f(x) = x\, f(x) \;\; \text{(multiplication operator)} \;, \tag{2.5}$$

$$\widehat{p} f(x) = -i\hbar\partial_x\, f(x) \;\; \text{(differentiation operator)} \;, \tag{2.6}$$

are given by

$$\langle x\rangle_t = \int_{-\infty}^{\infty} \psi^*(x,t)x\psi(x,t)dx \;, \tag{2.7}$$

$$\langle p\rangle_t = \int_{-\infty}^{\infty} \psi^*(x,t)\big(-i\hbar\partial_x\big)\psi(x,t)dx \;. \tag{2.8}$$

Therefore, the position mean value $\langle x\rangle_t = \int_{-\infty}^{\infty} x|\psi(x,t)|^2 dx$ is the expected position of the quantum particle at time t. For a general function $h(x)$, the mean value is

$$\langle h(x)\rangle_t = \int_{-\infty}^{\infty} \psi^*(x,t)h(x)\psi(x,t)dx \;. \tag{2.9}$$

Moreover, the mean values of $\widehat{x}^2$ and $\widehat{p}^2$ are given by

$$\langle x^2\rangle_t = \int_{-\infty}^{\infty} \psi^*(x,t)x^2\psi(x,t)dx \;, \tag{2.10}$$

$$\langle p^2\rangle_t = \int_{-\infty}^{\infty} \psi^*(x,t)\big(-i\hbar\partial_x\big)^2\psi(x,t)dx \;. \tag{2.11}$$

Obviously $\langle x^2\rangle_t = \int_{-\infty}^{\infty} x^2|\psi(x,t)|^2 dx$ is the second moment of the position random variable. Then, the *dispersion* of the position and momentum random variables are given by

$$\Delta x_t = \sqrt{\langle x^2\rangle_t - \langle x\rangle_t^2} \;, \tag{2.12}$$

and

$$\Delta p_t = \sqrt{\langle p^2\rangle_t - \langle p\rangle_t^2} \;, \tag{2.13}$$

respectively. The dispersion Δx_t is referred also as the *width*, since it is a measure of the geometrical spreading of the wave function ψ due to the dispersion of the quantum mechanical wave. Applying Schwarz's inequality we derive the *uncertainty principle*

$$\Delta x_t\, \Delta p_t \geq \frac{\hbar}{2} \;. \tag{2.14}$$

[1] The subscript t means that $\langle A\rangle_t$ is a function of time

2.2 Ehrenfest's theorem

A fundamental idea of quantum mechanics is that in the so-called classical limit, that is, as the parameter $\hbar \to 0$, the quantum particle behaves classically, and its motion is described in the standard phase space $\mathbb{R}^2_{xp}$, x being the position and p being the momentum of the particle, by *Hamilton's equations*

$$\frac{dx}{dt} = \partial_p H(x,p) = \frac{p}{m}\,, \quad \frac{dp}{dt} = -\partial_x H(x,p) = -V'(x)\,, \tag{2.15}$$

where $H(x,p) = \frac{p^2}{2m} + V(x)$ is *Hamilton's function*, which is equal to the total mechanical energy of the particle. For a classical particle it is assumed that the initial position $x(0) = x_0$ and the initial momentum $p(t) = p_0$ are known.

It turns out that the mean values $\langle x\rangle$, $\langle p\rangle$ satisfy the so-called *Ehrenfest equations*, which are quite similar, but, in general, not identical, to Hamilton's equations, and under certain conditions they have exactly the same form. By differentiating the formula (2.3), we express the time derivative $\frac{d}{dt}\langle A\rangle$ in terms of the wavefunction, and using the Schrödinger equation and its complex conjugate, we derive the formula

$$i\hbar\frac{d}{dt}\langle A\rangle_t = \left(\psi\,, [\widehat{A},\widehat{H}]\psi\right)_{L^2}\,, \tag{2.16}$$

where $[\widehat{A},\widehat{H}] = \widehat{A}\widehat{H} - \widehat{H}\widehat{A}$ is the commutator of the operators $\widehat{A}, \widehat{H}$. Applying (2.16) for $\widehat{A} = \widehat{x}$ and $\widehat{A} = \widehat{p}$, we derive the system

$$\begin{aligned} \frac{d\langle x\rangle_t}{dt} &= \frac{\langle p\rangle_t}{m}\,, \\ \frac{d\langle p\rangle_t}{dt} &= \langle -V'(x)\rangle_t\,. \end{aligned} \tag{2.17}$$

This result is known as Ehrenfest's theorem. Since the mean value of the potential depends on the wave function $\psi(x,t)$, Ehrenfest's system (2.17) is not closed, and, in general, it cannot be solved for the mean values of the position and momentum.

However, using Taylor's expansion of $V'(x)$ near $\langle x\rangle_t$, we get the approximation

$$\langle V'(x)\rangle \approx V'(\langle x\rangle_t) + \frac{1}{2}(\Delta x_t)^2 V'''(\langle x\rangle_t). \tag{2.18}$$

For quadratic potentials, the second term of the above approximation vanishes identically, and in this case Ehrenfest's system has exactly the form of Hamilton's equations and it is closed, since for $t = 0$, the mean values $\langle x\rangle_0$, $\langle p\rangle_0$ can be computed from ψ_0. For well behaved general potentials, we can omit the second term in (2.18) when the width Δx_t is small, and then we can also get a closed system. The validity of this approximation is the main mathematical question in relation with the validity of the classical limit.

2.3 The free-particle Schrödinger equation

In the sequel we will deal with the construction of wavepacket solutions of the free-particle Schrödinger equation, i.e., the case when $V \equiv 0$. Such solutions can be constructed by using the Fourier transform. The derived Fourier integrals can be either computed explicitly for particular initial data like Gaussian and Airy functions, or, they can be approximated by the method of stationary phase, for more general initial data.

2.3.1 Solution of the free Schröedinger equation

Consider the free-particle Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\ ,\ x\in\mathbb{R}\ ,\ t>0\ , \tag{2.19}$$

with initial conditions

$$\psi(x,0) = \psi_0(x)\ . \tag{2.20}$$

Let

$$\widehat{\psi}(\xi,t) = \int_{-\infty}^{\infty} e^{-i\xi x}\psi(x,t)dx \tag{2.21}$$

be the Fourier transform of $\psi(x,t)$. By taking the Fourier transform of (2.19), we obtain the ordinary differential equation

$$\frac{\partial\widehat{\psi}(\xi,t)}{\partial t} = -\frac{i\hbar}{2m}\xi^2\widehat{\psi}(\xi,t)\ , \tag{2.22}$$

with initial conditions

$$\widehat{\psi}(\xi,0) = \widehat{\psi}_0(\xi)\ , \tag{2.23}$$

where $\widehat{\psi}_0$ is the Fourier transform of the initial data. The solution of the (2.22) is

$$\widehat{\psi}(\xi,t) = \widehat{\psi}_0(\xi)e^{-\frac{i\hbar}{2m}\xi^2 t}\ . \tag{2.24}$$

By using the inverse Fourier transform

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\psi}(\xi,t)e^{ix\xi}d\xi\ , \tag{2.25}$$

we derive the solution of the problem (2.19), (2.20) as the Fourier integral

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\psi}_0(\xi)e^{-i\left(\frac{\hbar}{2m}\xi^2 t-\xi x\right)}d\xi\ . \tag{2.26}$$

Remark. If the initial data have finite energy, that is $\psi_0\in L^2$, by Parseval's theorem $\widehat{\psi}_0(\xi)\in L^2$, and then by (2.24) we have that $\widehat{\psi}(\xi,t)\in L^2$ for any fixed t, which implies that $\psi(x,t)\in L^2$ (conservation of energy). This is the case, when, for example, the initial data is a Gaussian function (see Chapter 3).

If however, we want to use more general data, we must assume that the initial data is a Schwartz distribution, $\psi_0\in\mathcal{S}'$, and use the distributional Fourier transform in order to construct $\psi(x,t)$ as a distribution, for any fixed t. Then, in many cases, by elaborating the Fourier integral, we are able to show that $\psi(x,t)$ is a smooth function. This is the case when the initial data is the Airy function (see Chapter 4).

Remark. Introducing the dispersion relation

$$\omega(\xi) = \frac{\hbar}{2m}\xi^2 t - \xi x\ , \tag{2.27}$$

we rewrite (2.26) in the typical form of

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\psi}_0(\xi)e^{-i\Phi(\xi;x,t)}d\xi\ ,\quad \Phi(\xi;x,t) = \omega(\xi)t-\xi x\ . \tag{2.28}$$

By the method of stationary phase, it follows that for any given $(x\ ,t)$, the main contribution in the integral (2.28) comes from the roots of the stationarity equation

$$\partial_\xi\Phi(\xi;x,t) = \omega'(\xi)t - x = 0\ , \tag{2.29}$$

that is from the points where the *group velocity* $\upsilon_g = \omega'(\xi)$ is equal to x/t. We observe that the *phase velocity* $\upsilon_{ph} = \omega(\xi)/\xi$ does not coincide with the group velocity. This is the very reason for the dispersion (i.e. the deterioration of the shape) of any localized wave function, which is a fundamental common feature of all dispersive waves [Whi].

2.3.2 Evolution of mean values

Integrating by parts the integral (2.26) we get

$$x\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x\widehat{\psi}_0(\xi)e^{-i(\frac{\hbar\xi^2 t}{2m}-x\xi)}d\xi$$

$$= \frac{i}{2\pi}\int_{-\infty}^{\infty}\widehat{\psi}_0(\xi)e^{-i(\frac{\hbar x i^2 t}{2m}-x\xi)}(\frac{-i\hbar\xi t}{m})d\xi + \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\partial\widehat{\psi}_0(\xi)}{\partial\xi}e^{-i(\frac{\hbar\xi^2 t}{2m}-x\xi)}d\xi \tag{2.30}$$

and using equation (2.7) we obtain

$$\langle x\rangle_t = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\psi}_0^*(\xi)\frac{\hbar\xi t}{m}\widehat{\psi}_0(\xi)d\xi + \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\psi}_0^*(\xi)i\frac{\partial\widehat{\psi}_0(\xi)}{\partial\xi}d\xi\ . \tag{2.31}$$

Differentiating (2.26) with respect to x we obtain that

$$\langle p\rangle_t = \int_{-\infty}^{\infty}\psi^*(x,t)(-i\hbar)\frac{\partial\psi(x,t)}{\partial x}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\psi}_0^*(\xi)\hbar\xi\widehat{\psi}_0(\xi)d\xi = \langle p\rangle(0)v\ . \tag{2.32}$$

Therefore,

$$\langle x\rangle_t = \frac{\langle p\rangle_0}{m}\,t + \langle x\rangle_0\ . \tag{2.33}$$

It must be emphasized that the fact that $\langle p\rangle_t = \langle p\rangle_0$ is anticipated by Ehrenfest's theorem since $V \equiv 0$. Note that when $\widehat{\psi}_0(\xi)$ is real, that is $\widehat{\psi}_0(\xi) = \widehat{\psi}_0^*(\xi)$, the second term

$$\langle x\rangle_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\frac{\partial\widehat{\psi}_0^2(\xi)}{\partial\xi}d\xi\ , \tag{2.34}$$

in the right hand side of (2.31) vanishes, if $\widehat{\psi}_0(\xi)$ vanishes as $\xi \to \pm\infty$, and the mean value reduces to

$$\langle x\rangle_t = \frac{\langle p\rangle_0}{m}\,t\ . \tag{2.35}$$

CHAPTER 3

The Gaussian wave packet

In this chapter we present some basic results about the dynamics of the Gaussian wave packet, which is the most basic example of spreading wave packet.

3.1 Construction of the Gaussian wave packet

3.1.1 Simple Gaussian initial data

The Gaussian wave packet is the solution of the free-particle Schrödinger equation (2.19),

$$i h \frac{\partial \psi}{\partial t} = -\frac{h^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \ ,$$

with Gaussian initial data ,

$$\psi(x,0) = \psi_0(x) = N e^{-a_0 x^2} \ , \ \ a_0 \in \mathbb{C} \ , \ \ \Re\, a_0 > 0 \ . \tag{3.1}$$

The constant N is chosen so that the energy of the packet is normalized, $\int_{-\infty}^{\infty} |\psi(x,0)|^2 dx = 1$, which implies

$$N = \Big(\frac{2\Re a_0}{\pi}\Big)^{1/4} \ .$$

By the standard Gaussian integration formula

$$\int_{-\infty}^{\infty} e^{-ax^2 + ibx} dx = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}} \ , \ \Re\, a > 0 \ , \tag{3.2}$$

we calculate the Fourier transform of the initial condition

$$\widehat{\psi}_0(\xi) = N \int_{-\infty}^{\infty} e^{-a_0 x^2 - ix\xi} dx = N \sqrt{\frac{\pi}{a_0}} e^{-\frac{\xi^2}{4a_0}} \ . \tag{3.3}$$

Then, from the integral representation (2.26), and using again (3.2), we obtain the solution

$$\psi(x,t) = \frac{N}{2\pi} \sqrt{\frac{\pi}{a_0}} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{4a_0}} e^{-i\frac{\hbar\xi^2}{2m}t} e^{i\xi x} d\xi = \frac{N}{\sqrt{1 + \frac{2i\hbar a_0 t}{m}}} \exp\Big(-\frac{x^2}{4(\frac{1}{a_0} + \frac{i\hbar t}{2m})}\Big) \ . \tag{3.4}$$

We introduce now the dynamic quantities

$$a_t = \frac{a_0}{1 + \frac{2i\hbar a_0 t}{m}} \;, \tag{3.5}$$

and

$$\gamma_t = \frac{i\hbar}{2} \ln\Big(1 + \frac{2i\hbar a_0 t}{m}\Big) \;, \tag{3.6}$$

and we rewrite (3.4) in the form

$$\psi(x,t) = Ne^{-a_t x^2 + i\gamma_t/\hbar} \tag{3.7}$$

Since clearly $\Re\, a > 0 \;, t \geq 0$, the solution (3.7) is a Gaussian function, that is, the initial Gaussian wave function propagated by the Schrödinger equation remains Gaussian for all time.

3.1.2 General Gaussian initial data. The Gaussian Ansatz

The wave packet (3.7) can be also constructed by a different technique, of asymptotic nature, which is applicable in the construction of an approximate wave packet solution also in the case when the potential is not zero. According to this technique one assumes the Ansatz

$$\psi_{\mathrm{G}}(x,t) = Ne^{-a_t(x-x_t)^2 + \frac{i}{\hbar}p_t(x-x_t) + \frac{i}{\hbar}\gamma_t} \tag{3.8}$$

where $x_t \;, p_t \;, a_t$ and γ_t are time-dependent quantities, which will be defined in order for (3.8) to satisfy (2.19) approximately.

Differentiating (3.8) we get the derivatives

$$\frac{\partial^2 \psi_{\mathrm{G}}(x,t)}{\partial x^2} = \left[\Big(-2a_t(x-x_t) + \frac{i}{h}p_t\Big)^2 - 2a_t\right]\psi_{\mathrm{G}}(x,t)$$

$$\frac{\partial \psi_{\mathrm{G}}(x,t)}{\partial t} = \left[-\dot{a}_t(x-x_t)^2 + 2a_t(x-x_t)\dot{x}_t + \frac{i}{h}(x-x_t)\dot{p}_t - \frac{i}{h}p_t\dot{x}_t + \frac{i}{h}\dot{\gamma}_t\right]\psi_{\mathrm{G}}(x,t) \;. \tag{3.9}$$

Then, by substituting these derivatives into (2.19) and equating the various powers of $(x - x_t)$ to zero, we derive the following equations for the unknown dynamical quantities entering the assumed Ansatz

$$\dot{x}_t = \frac{p_t}{m} \;, \tag{3.10}$$

$$\dot{p}_t = 0 \;, \tag{3.11}$$

$$\dot{a}_t = -\frac{2ih}{m}a_t^2 \;, \tag{3.12}$$

$$\dot{\gamma}_t = -\frac{p_t^2}{2m} + p_t\dot{x}_t - \frac{\hbar^2 a_t}{m} \;. \tag{3.13}$$

Note that (3.10)-(3.11) are Hamilton's equation for the free Hamiltonian $H(x\;,p) = \frac{p^2}{2m}$ corresponding to the free Schrödinger equation (2.19). [1]

Assuming that the initial position and momentum $x_{t=0} = x_0 \;, p_{t=0} = p_0$, we integrate the equations (3.10)-(3.13), and we obtain the following dynamical quantities that define the evolution of the wavepacket

$$p_t = p_0 \;, \tag{3.14}$$

[1] In the general case, where the potential $V(x)$ is not zero and smooth, the Hamitlonian is $H(x\;,p) = \frac{p^2}{2m} + V(x)$, and eq. (3.11), has the form $\dot{p}_t = -V'(x)$. In this case one needs to expand $V(x)$ near x_t, before equating powers of $(x - x_t)$ to zero.

$$x_t = x_0 + \frac{p_0}{m}t\,, \tag{3.15}$$

$$a_t = \frac{a_0}{1 + \frac{2i\hbar a_0 t}{m}}\,, \tag{3.16}$$

$$\gamma_t = \frac{p_0^2}{2m}t + \frac{i\hbar}{2}\ln\Big(1 + \frac{2i\hbar a_0 t}{m}\Big)\,. \tag{3.17}$$

Note that if $x_0 = p_0 = 0$, the solution (3.8) reduces to (3.7).

Remark. If we normalize the wave function (3.8) so that $\int_\infty^\infty |\psi(x,t)|^2 dx = 1$, we obtain

$$N = \Big(\frac{2\Re\, a_t}{\pi}\Big)^{1/4} e^{\Im\,\gamma_t/\hbar}\,. \tag{3.18}$$

It is straightforward to show that $\partial_t N = 0$, and thus $N = \Big(\frac{2\Re\, a_0}{\pi}\Big)^{1/4}$, since $\gamma_{t=0} = 0$.

3.2 Mean values of Gaussian wavepackets

By (2.7) with $\psi = \psi_G$, the mean value of the position is

$$\begin{aligned}\langle x\rangle_t = \int_{-\infty}^{\infty} x|\psi_G(x\,,t)|^2 dx &= N^2 \int_{-\infty}^{\infty} x e^{-(a_t+a_t^*)(x-x_t)^2 + \frac{i}{\hbar}(\gamma_t-\gamma_t^*)} dx \\ &= (\frac{2\Re\, a_t}{\pi})^{1/2} x_t \int_{-\infty}^{\infty} e^{-y^2(a_t+a_t^*)} dy = x_t\,,\end{aligned} \tag{3.19}$$

and by (2.8) the mean value of the momentum is

$$\langle p\rangle_t = p_t \tag{3.20}$$

Moreover, by (2.10), (2.11), and (2.12), (2.13, we get

$$\langle x^2\rangle_t = \frac{1}{4\Re\, a_t} + x_t^2\,, \tag{3.21}$$

$$\langle p^2\rangle_t = 2\hbar^2 a_t - \frac{2\hbar^2 a_t^2}{a_t + a_t^*} + p_t^2 = \frac{\hbar^2 |a_t|^2}{\Re(a_t)} + p_t^2\,, \tag{3.22}$$

and

$$\Delta x_t = \sqrt{\langle x^2\rangle_t - \langle x^2\rangle_t} = \sqrt{\frac{1}{4\Re\, a_t}}\,, \tag{3.23}$$

$$\Delta p_t = \sqrt{\langle p^2\rangle_t - \langle p\rangle_t^2} = \frac{\hbar|a_t|}{\sqrt{\Re\, a_t}}\,. \tag{3.24}$$

When a_0 is real, by (3.16) we get

$$\Re\, a_t = \frac{a_0}{1 + \frac{4\hbar^2 a_0^2 t^2}{m^2}}\,, \quad |a_t| = \frac{a_0}{\sqrt{1 + \frac{4\hbar^2 a_0^2 t^2}{m^2}}}\,, \tag{3.25}$$

and thus

$$\Delta x_t = \frac{1}{2}\sqrt{\frac{1 + 4\hbar^2 a_0^2 t^2/m^2}{a_0}}\,, \quad \Delta p_t = h\sqrt{a_0}\,. \tag{3.26}$$

Therefore, the initial Gaussian waveform deteriorates linearly with t as $t \to \infty$.This *spreading phenomenon* is common for any quantum mechanical wave packet with finite energy [Kl](see also [Bo], Ch. 3 & Ch. 10). As the wave packet propagates, its energy content is dispersed to larger space area, and eventually is delocalized. However, the uncertainty

$$\Delta x_t \, \Delta p_t = \frac{\hbar}{2}\sqrt{1 + \frac{4h^2 a_0^2 t^2}{m^2}} \geq \frac{\hbar}{2} \tag{3.27}$$

is minimum at $t = 0$, and it satisfies the uncertainty principle for all time.

CHAPTER 4

The Berry-Balaz's wavepacket

In this chapter we present the construction of the *Berry-Balaz's wavepacket* (in the sequel abbreviated as the *BB wave packet*), and we study some aspects of its geometrical optics. The BB wave packet is the solution of the free-particle Schrödinger equation (2.19) with initial data

$$\psi_0(x) = Ai\Big(\frac{Bx}{\hbar^{2/3}}\Big) \ , \tag{4.1}$$

where $Ai(x)$ is the Airy function, and $B > 0$ is a scaling parameter.

4.1 Construction of the BB wave packet

The Airy function is defined by the Fourier transform [Leb]

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi^3/3+i\xi x} d\xi \ , \tag{4.2}$$

which must be initially interpreted as the distributional Fourier transform of $e^{i\xi^3/3}$ in $\mathcal{S}'$. However, by appropriate use of Cauchy's theorem for complex integration, it can be shown that $Ai(x)$ is a C^∞ function [Horm]. Moreover, since $e^{i\xi^3/3} \notin L^2$, it follows that $Ai(x) \notin L^2$, an unpleasant fact which becomes clear also from the asymptotic behaviour of $Ai(x)$ as $x \to -\infty$.

Therefore, in order to construct the solution of the problem (2.19), (4.1), we must work with the distributional Fourier transform, although the formal calculus is quite similar with that for the L^2 case. The Fourier transform of ψ_0 is

$$\widehat{\psi}_0(\xi) = \frac{\hbar^{2/3}}{B} e^{i\frac{\hbar^2\xi^3}{3B^3}} \ , \tag{4.3}$$

and by (2.26) the BB wave packet is given by the Fourier integral

$$\psi_{\text{BB}}(x,t) = \frac{\hbar^{2/3}}{2\pi B}\int_{-\infty}^{\infty} e^{i\left(\xi x - \frac{\hbar\xi^2 t}{2m} + \frac{\hbar^2\xi^3}{3B^3}\right)} d\xi \ . \tag{4.4}$$

By elaborating the exponential term of the last integral, we express the BB wave packet in terms of a shifted Airy function, which is modulated by an oscillating exponential, as follows

$$\psi_{\text{BB}}(x,t) = Ai\Big(\frac{B}{\hbar^{2/3}}(x - \frac{B^3t^2}{4m^2})\Big) e^{\frac{iB^3t}{2m\hbar}\left(x - \frac{B^3t^2}{6m^2}\right)} \ . \tag{4.5}$$

We observe that since the exponential term has modulus one, ψ_{BB}, like its initial datum, is not square integrable. As a result of the infinite energy content of the BB wavepacket, $|\psi_{\text{BB}}(x,t)|$ propagates along the x-axis without distortion of its shape, and it also accelerates with acceleration

$$\frac{dx}{dt} = \frac{B^3t^2}{2m^2} \ .$$

4.2 Geometrical optics of the BB wavepacket

In order to make precise the meaning of the acceleration of the wavepacket, we will construct the *WKB expansion* of ψ_{BB} for $\hbar \ll 1$, and we will check that the derived phases and amplitudes satisfy the equations of geometrical optics, namely the Hamilton-Jacobi equation and the transport equation. Then, we will compute the rays and their caustic, and we will show that what really accelerates is the caustic of the WKB solution. The principal terms of asymptotic expansion for the function Ai(x) as $x \to \infty$ read as follows [Leb], [VS]

$$Ai(x) \approx \tfrac{1}{2}\pi^{-1/2}x^{-1/4}e^{-\frac{2x^{3/2}}{3}} \ , \tag{4.6}$$

$$Ai(-x) \approx \pi^{-1/2}x^{-1/4}\cos\Big(\tfrac{2}{3}x^{3/2} - \tfrac{\pi}{4}\Big) \ . \tag{4.7}$$

Thus the initial datum (4.1) is exponentially small as $x \to +\infty$, while as $x \to -\infty$ it is highly oscillating with algebraically decaying amplitude. The BB wavepacket (4.5) has similar behaviour for any fixed time t. Then, for $x < 0$, the initial datum has the expansion

$$\begin{aligned}\psi_0(x) &= Ai\Big(-\frac{B}{h^{2/3}} \mid x \mid\Big) \approx \frac{h^{1/6}}{\pi^{1/2}B^{1/4}} \mid x \mid^{-1/4} cos\Big(\frac{2}{3}\frac{B^{3/2}}{h} \mid x \mid^{3/2} - \pi/4\Big) \\ &= C\Big(A_+(x)e^{\frac{i}{h}S_+(x)} + A_-(x)e^{\frac{i}{h}S_-(x)}\Big) \ , \end{aligned} \tag{4.8}$$

where $S_\pm(x)$ are the phases

$$S_\pm(x) = \pm\frac{2}{3}(-Bx)^{3/2} \ , \tag{4.9}$$

$A_\pm(x)$ are the amplitudes

$$A_+(x) = \Big(\frac{\partial^2 S_+(x)}{\partial x^2}\Big)^{1/2} = \frac{B}{\sqrt{2}}(-Bx)^{-1/4} \ , \ \ A_-(x) = \Big(\frac{\partial^2 S_-(x)}{\partial x^2}\Big)^{1/2} = i\frac{B}{\sqrt{2}}(-Bx)^{-1/4} \ , \tag{4.10}$$

and

$$C = \frac{e^{-\frac{i\pi}{4}}h^{1/6}}{\sqrt{2\pi}B} \ . \tag{4.11}$$

Now we rewrite (4.5) in the form

$$\psi_{\text{BB}}(x,t) = Ai\Big(\frac{F(x,t)}{h^{2/3}}\Big)e^{\frac{i}{h}\Phi(x,t)} \ , \tag{4.12}$$

where

$$F(x,t) = B\Big(x - \frac{B^3t^2}{4m^2}\Big) \ , \tag{4.13}$$

and

$$\Phi(x,t) = \frac{B^3t}{2m}\Big(x - \frac{B^3t^2}{6m^2}\Big) \ . \tag{4.14}$$

Then, by the asymptotic formulas (4.6) , we find that in the illuminated zone $F > 0$, the Airy term has the approximation

$$Ai(-\frac{|F(x,t)|}{h^{2/3}}) \approx \frac{h^{1/6}e^{-i\frac{\pi}{4}}}{2\sqrt{\pi}|F(x,t)|^{1/4}} \left(e^{\frac{i}{h}\frac{2}{3}|F(x,t)|^{3/2}} + e^{-\frac{i}{h}\frac{2}{3}|F(x,t)|^{3/2}}e^{i\frac{\pi}{2}}\right) , \tag{4.15}$$

and therefore the BB has the following WKB expansion

$$\psi_{\text{BB}}(x,t) \approx C\Big(A_+(x,t)e^{\frac{i}{h}S_+(x,t)} + A_-(x,t)e^{\frac{i}{h}S_-(x,t)}\Big) , \tag{4.16}$$

where

$$S_\pm(x,t) = \Phi(x,t) \pm \frac{2}{3}(-F(x,t))^{3/2} , \tag{4.17}$$

and

$$A_+(x,t) = \frac{B}{\sqrt{2}(-F(x,t))^{1/4}} , \tag{4.18}$$

$$A_-(x,t) = i\frac{B}{\sqrt{2}(-F(x,t))^{1/4}} . \tag{4.19}$$

Obviously, the phases and the amplitudes satisfy their initial conditions $S_\pm(x,0) = S_\pm(x)$ and $A_\pm(x,0) = A_\pm(x)$. By direct calculation we can chek that $S_\pm(x,t)$ satisfy the Hamilton-Jacobi equation

$$\partial_t S + \frac{1}{2m}\Big(\partial_x S\Big)^2 = 0 , \tag{4.20}$$

and that the amplitudes $A_\pm(x,t)$ satisfy the transport equation

$$\partial_t A^2 + \frac{1}{m}\partial_x\Big(A^2\partial_x S\Big) = 0 . \tag{4.21}$$

Let

$$\frac{dx}{dt} = \frac{p}{m} , \quad x(0) = x_0 \tag{4.22}$$

$$\frac{dp}{dt} = 0 , \quad p(0) = \frac{\partial S_\pm(x_0)}{\partial x} = \mp(-B^3x_0)^{1/2} =: p_0 \tag{4.23}$$

the Hamiltonian system for the characteristics of the Hamilton-Jacobi equation (4.20). The solutions of this system are

$$x_\pm(t;x_0) = x_0 \mp (-B^3x_0)^{1/2}t/m , \tag{4.24}$$

$$p_\pm(t:x_0) = \mp(-B^3x_0)^{1/2} , \tag{4.25}$$

and $x = x_\pm(t;x_0)$ are the rays (projections of the characteristics onto the x-axis) emanating from the initial point x_0 at time $t = 0$. By the solutions $x_0 = x_0(x,t)$ of $x = x_\pm(t;x_0)$, for a given $(x\ ,t)$, it is easy to check that along these rays the momentums (slopes of the ray) satisfy

$$p_\pm(t;x_0) = \frac{\partial S_\pm(x\ ,t)}{\partial x} . \tag{4.26}$$

The caustic is the envelope of the rays on the space-time plane $x - t$ and it is defined as the set of point where the Jacobian of the transformation $x_0 \mapsto x_\pm(t;x_0)$ vanishes, that is

$$J(t;x_0) = \frac{\partial x_\pm(t;x_0)}{\partial x_0} = 0 \tag{4.27}$$

Eliminating the parameter x_0 from the equations (4.24) and (4.27), we obtain the equation of the caustic

$$x(t) = \frac{B^3 t^2}{4m^2} \,. \tag{4.28}$$

Obviously the caustic is accelerating with acceleration

$$\frac{dx}{dt} = \frac{B^3 t}{4m^2} \,.$$

It is interesting to observe that on the caustic the differentials of (4.13) and (4.14) vanish, and therefore, on the caustic $d\psi_{\text{BB}}(x,t) = 0$. This means that the boundary between the illuminated and the shadow zone of the BB wave packet moves along the caustic. Thus, the freely moving BB wave packet ($V \equiv 0$) follows a curved trajectory, which is incombatible with its acceleration. This counter intuitive situation is a consequence of the infinite energy of the wave. Balazs and Berry [BB] have explained by simple geometrical arguments that their nonspeading wave packet is the only one that has these properties. However, every physically realisable wave packet must have finite energy, and therefore it eventually spreads and its initial shape deteriorates as it is dictated by the dispersion of the quantum mechanical system [Kl].

CHAPTER 5

Airy-type wave packets with finite energy

In this chapter we explain that all Airy-type wave packets with finite energy, which have been proposed in Physics' and Optics' literature, can be represented as the convolution of an appropriate Gaussian function with the BB wave packet.

5.1 The Gaussian Berry-Balasz wavepacket

In the appendix of their paper, Berry and Balazs [BB] constructed an Airy-type wave packet with finite energy by convolving the BB wave packet with a Gaussian function. This modified wave packet is a weighted superposition of BB wave packets, and in this sense we think of it as an *Airy beam*. Following this idea we consider the *Gaussian Berry-Balasz wave packet* (in the sequel abbreviated as GBB wave packet)

$$\begin{aligned} \psi_{\text{GBB}}(x,t) &= \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/2\sigma^2} *_x \psi_{BB}(x,t) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(x-x')^2/2\sigma^2} \psi_{BB}(x',t)dx' \ , \ \ \sigma > 0 \ . \end{aligned} \tag{5.1}$$

The considered wave packet is slightly different than that proposed by Berry and Balazs [BB], because in the sequel we want to examine the limit $\sigma \to 0$, and for this reason we are obliged to admit that the energy of the wave packet (see (5.5) below) depends singularly on σ.

It is easy to check that $\psi_{\text{GBB}}(x,t)$ satisfies the Schrödinger equation. By differentiating the convolution with respect to x and t, we get

$$\partial_t \psi_{\text{GBB}}(x,t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} *_x \partial_t \psi(x,t) \ , \tag{5.2}$$

$$\partial_x^2 \psi_{\text{GBB}}(x,t) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/2\sigma^2} *_x \partial_x^2 \psi(x,t) \ , \tag{5.3}$$

and thus we have

$$\begin{aligned} & i\hbar\partial_t \psi_{\text{GBB}}(x,t) + \tfrac{\hbar^2}{2m} \partial_x^2 \psi_{\text{GBB}}(x,t) \\ &= \tfrac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/2\sigma^2} *_x \Big(i\hbar\partial_t \psi_{BB}(x,t) + \tfrac{\hbar^2}{2m} \partial_x^2 \psi_{BB}(x,t) \Big) = 0 \ , \end{aligned} \tag{5.4}$$

since $\psi_{\text{BB}}(x,t)$ satisfies the Schrödinger equation.

By a long but straightforward calculation we find that

$$\int_{-\infty}^{\infty} |\psi_{\text{GBB}}(x\ ,0)|^2 dx = \frac{\hbar^{1/3}}{2\pi^{1/2}B^2}\frac{\hbar}{\sigma}\ , \tag{5.5}$$

thus $\psi_{\text{GBB}} \in L^2$ for any $\sigma > 0$. The mean value of position is given by

$$\langle x\rangle_t = \int_{-\infty}^{\infty} x|\psi_{\text{GBB}}(x,t)|^2 dx = -\frac{\hbar^{1/3}}{4\pi^{1/2}B^5}\frac{\hbar^3}{\sigma^3}\ , \tag{5.6}$$

and it is time independent, therefore the Gaussian BB wave packet obeys Ehrenfest's theorem. We observe that the mean position is consistently negative, because the Airy function carries larger mass at $x < 0$ than at $x > 0$, and the Gaussian function is an even function.

Moreover, the dispersion is given by

$$(\Delta x_t)^2 = \langle x^2\rangle_t - \langle x\rangle_t^2 = \kappa + \lambda t^2\ , \tag{5.7}$$

where

$$\kappa = \frac{\hbar^{4/3}}{8\sqrt{\pi}\sigma B^2}\left(\sigma^2 + \frac{\hbar^{4/3}}{\sigma^4 B^6}\Big(3 - \frac{\hbar^4}{2\sqrt{\pi}\sigma B^2}\Big)\right)\ , \quad \lambda = \frac{1}{2m^2\sigma^2\hbar^{2/3}}\ ,$$

are constants depending on the various parameters. It follows that the dispersion of the GBB wave packet, increases linearly with time for large time, which is qualitatively similar to the behaviour of the Gaussian wave packet.

Remark. Since

$$g_\sigma(x) = \frac{1}{(2\pi\sigma^2)^{1/2}}e^{-x^2/2\sigma^2} \to \delta(x)\ ,\ \text{ as } \sigma \to 0\ , \tag{5.8}$$

distributionally, it follows from (5.1) that

$$\psi_{\text{GBB}}(x,t) \to \psi_{\text{BB}}(x,t)\ ,\ \text{as } \sigma \to 0\ , \tag{5.9}$$

for any fixed time $t > 0$, which is intuitively correct. Moreover, as $\sigma \to 0$, the energy (5.5) of ψ_{GBB} tends to infinity, which is also correct since ψ_{BB} has infinite energy. Finally, the mean position (5.6) moves to infinity, since most of the infinite energy of ψ_{BB} "accumulates" there.

5.2 The Siviloglou-Christodoulides' (SC) solution

Siviloglou and Christodoulides [SC] have constructed an Airy wave packet with finite energy by using special initial data of finite energy. More precisely, by putting $\hbar = 1$, $m = 1$ and $B = 1$, for simplifying the calculation, they solved the free-particle Schrödinger equation (2.19)-(2.20),

$$i\frac{\partial\psi_{\text{SC}}}{\partial t} + \frac{1}{2}\frac{\partial^2\psi_{\text{SC}}}{\partial x^2} = 0\ ,\ \ x \in \mathbb{R}\ ,\ t > 0\ , \tag{5.10}$$

with initial condition

$$\psi_{\text{SC}}(x,0) := e^{bx}\, Ai(x)\ ,\ \ b > 0\ , \tag{5.11}$$

where the exponential factor ensures the containment of slowly decaying Airy tail for negative x. In fact, by standard calculation, we get

$$\int_{-\infty}^{\infty} |\psi_{\text{SC}}(x,0)|^2 dx = \frac{1}{\sqrt{8\pi a}}e^{\frac{2}{3}b^3}\ , \tag{5.12}$$

which implies $\psi_{\text{SC}}(x,0) \in L^2$ for any $b > 0$. The Fourier transform of $\psi_{\text{SC}}(x,0)$ is

$$\widehat{\psi}_{\text{SC}}(\xi) = e^{-b\xi^2} e^{\frac{i}{3}(\xi^3 - 3b^2\xi)} e^{b^3/3} , \tag{5.13}$$

so by (2.26) we obtain the following integral representation of the SC solution

$$\psi_{\text{SC}}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-b\xi^2 + \frac{b^3}{3} + \frac{i}{3}(\xi^3 - 3b\xi^2)} e^{-i(\xi^2 t - x\xi)} d\xi . \tag{5.14}$$

By writing the exponential in terms of the cubics $(b - i\xi)^3$ and $(\frac{t}{2} - y)^3$, $y = \xi + bi$, and using the definition of the Airy function, we obtain the SC solution in the explicit form

$$\psi_{\text{SC}}(x,t) = Ai\Big(x - t^2/4 + ibt\Big) e^{bx - \frac{bt^2}{2} + i(b^2 t/2 + xt/2 - t^3/12)} . \tag{5.15}$$

It is simple to check that for $b = 0$ the SC solution coincides with the BB wave packet (compare with (4.5) for $\hbar = 1$ and $m = 1$), as it should, because for $b = 0$, ${}_{\text{SC}}(x,0) = \psi_{\text{BB}}(x,0)$.

Here the following question arises naturally. Is there a relation between the initial data proposed by Siviloglou and Christodoulides and $\psi_{\text{GBB}}(x,0)$, or are they essentially independent of each other? By either using the Airy transform[VS], p.84, or by straightforward calculation involving certain interchanges of Fourier integrals, we derive that

$$\psi_{\text{GBB}}(x,0) = \exp\Big(\frac{1}{4\alpha^3}\big(z + \frac{1}{24\alpha^3}\big)\Big)\, Ai\Big(\frac{z}{\alpha} + \frac{1}{16\alpha^4}\Big) , \tag{5.16}$$

where

$$z = \frac{x}{\sigma\sqrt{2}} \quad \text{and} \quad \alpha = \frac{h^{2/3}}{B\sigma\sqrt{2}} . \tag{5.17}$$

Then, we make the change of variable $X = \frac{z}{\alpha} + \frac{1}{16\alpha^4}$ and we rewrite the last formula in the form

$$\psi_{\text{GBB}}(x,0) = e^{-\frac{1}{192\alpha^6}}\, e^{\frac{X}{4\alpha^2}} Ai(X) , \quad X = \frac{Bx}{h^{2/3}} + \frac{B^4\sigma^4}{4h^{8/3}} , \tag{5.18}$$

which is essentially a translation along the x-axis of the the initial data proposed by Siviloglou & Christodoulides. In this sense the SC solution is a particular case of the GBB wave packet.

In the sequel, we will explain that $\psi_{\text{SC}}(x,0)$ can be represented as the convolution of the Airy function with a certain function $g(x)$, and thus the SC solution can be also represented as the convolution of g with $\psi_{\text{GBB}}(x,t)$. Let us consider the relation

$$\phi(x)Ai(x) = g(x) *_x Ai(x) , \tag{5.19}$$

for a given function ϕ, such that $\phi(x)Ai(x)$ is square integrable. This assumption implies that $g(x)$ must also be square integrable. We will show that g is expressed as the following integral transform of ϕ

$$g(x) = \int_{-\infty}^{\infty} Ai(z)Ai(z - x)\phi(z)dz . \tag{5.20}$$

Note that $g(x) = Ai(x)\phi(x) *_x Ai(x)$ is the Airy transform of $Ai(x)\phi(x)$.

We first apply the Fourier transform on the relation (5.19), and we get

$$\widehat{g}(\xi) = \left(\frac{1}{2\pi}\, \widehat{\phi}(\xi) *_\xi e^{i\frac{\xi^3}{3}} \right) e^{-i\frac{\xi^3}{3}} ,$$

which implies the representation

$$g(x) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{i\xi x} \int_{-\infty}^{\infty} \widehat{\phi}(\xi - \sigma) e^{i\frac{\sigma^3 - \xi^3}{3}} d\sigma\, d\xi \,. \tag{5.21}$$

Then, by the change of variables $u = \xi - \sigma$, $v = \xi + \sigma$ we transform g as follows

$$\begin{aligned} g(x) &= \tfrac{1}{2(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\frac{xv}{2}} e^{i\frac{ux}{2} - \frac{uv^2}{4} - i\frac{u^3}{12}} \widehat{\phi}(u) dv du \\ &= \tfrac{1}{2(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\frac{xv}{2}} e^{i\frac{ux}{2} - \frac{uv^2}{4} - i\frac{u^3}{12}} \Big(\int_{-\infty}^{\infty} e^{-izu} \phi(z) dz \Big) dv du \,, \end{aligned}$$

and we write it as the integral tranformation

$$g(x) = \int_{-\infty}^{\infty} K(x,z)\phi(z)dz \,, \tag{5.22}$$

with kernel

$$K(x,z) = \frac{1}{2(2\pi)^2} \int_{-\infty}^{\infty} e^{i\frac{xv}{2}} \Big(\int_{-\infty}^{\infty} e^{-i\frac{u^3}{12} + iu\big(x/2 - z - v^2/4\big)} du \big) \Big) dv \,. \tag{5.23}$$

By the definition of the Airy function and simple changes of variables, we may write K as a single integral

$$\begin{aligned} K(x,z) &= \frac{2^{2/3}}{2(2\pi)} \int_{-\infty}^{\infty} e^{ixv/2} Ai\big(2^{2/3}(v^2/4 + z - x/2)\big) dv \\ &= \frac{2^{2/3}}{2\pi} \int_{-\infty}^{\infty} e^{ixt} Ai\big(2^{2/3}(t^2 + z - x/2)\big) dt \,. \end{aligned} \tag{5.24}$$

Now, by applying the formula for the product of Airy functions ([VS], eq. (2.151))

$$\int_{-\infty}^{\infty} Ai\Big(2^{2/3}(t^2 + \frac{u+v}{2})\Big) e^{i(u-v)t} dt = 2^{1/3}\pi Ai(u)Ai(v) \,, \tag{5.25}$$

and $u = z$, $v = z - x$, we express the kernel K in the explicit form

$$K(x,z) = Ai(z)Ai(z-x) \,. \tag{5.26}$$

Substituting (5.26) into (5.22) we complete the derivation of (5.20).

Remark. If we abandon the assumption of square integrability, for $\phi(x) \equiv 1$ into (5.20), and the orthogonality relation for the Airy functions ([VS], eq. (3.108)),

$$\int_{-\infty}^{\infty} Ai(u-x)Ai(v-x)dx = \delta(u-v) \,,$$

we get

$$g(x) = \int_{-\infty}^{\infty} Ai(z)Ai(z-x)dz = \delta(x) \,. \tag{5.27}$$

Then, eq. (5.19) is satisfied distributionally since

$$Ai(x) = \delta(x) *_x Ai(x) \,. \tag{5.28}$$

5.3 The Besieris-Shaarawi's (BS) solution

Besieris and Shaarawi [BS] have observed that the SC solution can be derived from the BB wave packet by the transformation

$$x \to x - b^2\ ,\ t \to t - i\,2b\ ,\ \ b > 0$$

More precisely, they observed that

$$\psi_{\rm BS}(x,t) := \psi_{\rm BB}(x - b^2, t - i\,2b) = e^{-\frac{b^3}{3}}\,\psi_{\rm SC}(x,t)\ . \tag{5.29}$$

We will show that the BS solution is also a GBB wave packet, since it can be represented as a convolution

$$\psi_{\rm BS}(x,t) = g_{\rm b}(x) *_x \psi_{\rm BB}(x,t)\ , \tag{5.30}$$

with the b-dependent Gaussian function

$$g_{\rm b}(x) = \frac{1}{\sqrt{4b\pi}} e^{-\frac{(x-b^2)^2}{4b}}\ . \tag{5.31}$$

By taking the Fourier transform with respect to x of the equation (5.29), and using the standard properties of the Fourier transform of the convolution and of shifted functions, we get

$$\widehat{\psi}_{\rm BS}(\xi,t) = e^{-ib^2\xi}\widehat{\psi}_{\rm BB}(\xi, t - i2b)\ . \tag{5.32}$$

Then, by appropriately using (2.24) and (4.3), we get

$$\widehat{\psi}_{\rm BB}(\xi, t - i2b) = e^{-b\xi^2}\,\widehat{\psi}_{\rm BB}(\xi,t)\ . \tag{5.33}$$

We rewrite the last equation in the form

$$\widehat{\psi}_{\rm BS}(\xi,t) = \widehat{\psi}_{\rm BB}(\xi,t)\widehat{g}_b(\xi)\ , \tag{5.34}$$

where

$$\widehat{g}_b(\xi) = e^{-b\xi^2 - i\xi b^2}\ . \tag{5.35}$$

By inverting the Fourier transform $\widehat{\psi}_{\rm BS}(\xi,t)$ we get the convolution formula (5.30), where $g_{\rm b}$ is the inverse Fourier transform of $\widehat{g}_{\rm b}(\xi)$,

$$g_{\rm b}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x - b\xi^2 - i\xi b^2} d\xi = \frac{1}{\sqrt{4b\pi}} e^{-\frac{(x-b^2)^2}{4b}}\ . \tag{5.36}$$

From the above Fourier integral representation of $g_{\rm b}$ it follows that as $b \to 0$, $g_{\rm b}(x) \to \delta(x)$ and therefore we get

$$\psi_{\rm BS}(x,t) \to \psi_{\rm BB}(x,t)\ ,\ \text{as}\ \ b \to 0\ , \tag{5.37}$$

which is consistent with (5.29) for $b = 0$.

5.4 Lekner's (L) solution

Lekner [Lek], inspired by the fundamental role of coordinate transformations in quantum mechanis [Bal], has investigated the relation between the BB wave packet and the SC solution by employing a Galilean transformation with complex velocity.

First, by rewriting the BB wave packet in the form

$$\psi_{\rm BB}(x,t) = Ai\Big(q(x-\frac{1}{2}at^2)\Big)\exp\Big(i\frac{mat}{\hbar}(x-\frac{1}{3}at^2)\Big)\ ,$$

where $q = \frac{B}{\hbar^2/3}$ can be thought of a *real wave number*, and $a = \frac{B^3}{2m^2}$ is the *acceleration* of the wave packet, Lekner considered the wavepacket

$$\psi_{\rm L}(x,t) = \psi_{\rm BB}(x-ct,t)e^{i\frac{mc}{\hbar}(x-\frac{1}{2}ct)}\ . \tag{5.38}$$

This wave packet is derived by applying the *real Galilean transformation*

$$x \to x-ct\ ,\ \ t\to t\ ,\ \ c\in\mathbb{R}\ , \tag{5.39}$$

on $\psi_{\rm BB}$, and it is then augmented by the phase factor

$$e^{i\frac{mc}{\hbar}(x-\frac{1}{2}ct)}\ . \tag{5.40}$$

Obviously, this wave packet, like $\psi_{\rm BB}$, has infinite energy since the phase factor has modulus one.

Then, he observed that if the velocity of the moving frame is complex, $c = u - iv\ , v > 0$ the phase factor contributes the real factor

$$e^{\frac{mv}{\hbar}x}\ , \tag{5.41}$$

which ensures the containment of slowly decaying Airy tail for negative x. In the special case $u = 0$, $\psi_{\rm L}$, for $\hbar = m = B = 1$, reduces to the SC solution.

In terms of the components of the complex velocity, Lekner's wavepacket (5.38) reads as follows

$$\psi_{\rm L}(x,t) = Ai\Big(q(x-ut+ivt-\frac{1}{2}at^2)\Big)e^{i\frac{mat}{\hbar}\left(x-ut-\frac{1}{3}at^2\right)}e^{\frac{mv}{\hbar}\left(x-ut+\frac{i}{2}vt-at^2\right)}e^{\frac{mu}{\hbar}\left(x-\frac{1}{2}ut\right)}\ . \tag{5.42}$$

The energy of the wave packet is finite, and is given by

$$\int_{-\infty}^{\infty}|\psi_{\rm L}(x,t)|^2dx = \int_{-\infty}^{\infty}|\psi_{\rm L}(x,0)|^2dx = \int_{-\infty}^{\infty}Ai^2(qx)\,e^{\frac{2mvx}{\hbar}}\,dx = \frac{\sqrt{\hbar}}{2^{3/2}\sqrt{\pi mvq}}e^{\frac{2(mv)^3}{3(\hbar q)^3}}\ . \tag{5.43}$$

It is interesting to observe that only the imaginary part $v = \Im(c)$ of the velocity conrtibutes to the energy of the wave packet, and as $v \to 0$ the energy becomes infinite since the factor (5.41) disappears.

We will show now that Lekner's wave packet is also a GBB wave packet, since it can be represented a s convolution

$$\psi_{\rm L}(x,t) = g_{\rm c}(x) *_x \psi_{\rm BB}(x,t)\ , \tag{5.44}$$

with the c-dependent Gaussian function

$$g_{\rm c}(x) = \frac{B^{3/2}}{\sqrt{4\pi\hbar mci}}e^{-i\frac{m^3c^3}{3\hbar B^3}}\exp\left(-\frac{B^3\big(x+\frac{m^2c^2}{B^3}\big)^2}{4\hbar m\,ic}\right)\ , \tag{5.45}$$

which encodes the *effect of the complex Galilean transformation*

By taking the Fourier transform of (5.38) with respect to x we get

$$\widehat{\psi}_{\rm L}(\xi,t) = e^{i\frac{mc^2t}{2h}-i\xi ct}\widehat{\psi}_{\rm BB}(\xi-mc/h,t)\ , \tag{5.46}$$

and using the Fourier transform of $\psi_{\rm BB}$ we have

$$\widehat{\psi}_{\rm L}(\xi\ ,t) = \frac{h^{2/3}}{B} e^{-i\left(\frac{ht}{2m}(\xi - mc/h)^2 - \frac{h^2}{3B^3}(\xi - mc/h)^3\right)}$$

$$= \widehat{\psi}_{\rm BB}(\xi, t)\widehat{g}_{\rm c}(\xi)\ , \tag{5.47}$$

where

$$\widehat{g}_{\rm c}(\xi) = e^{-i\left(\frac{hmc}{B^3}\xi^2 - \frac{m^2c^2}{B^3}\xi + \frac{m^3c^3}{3hB^3}\right)}\ . \tag{5.48}$$

Finally, by inverting the Fourier transform (5.47) we obtain (5.44). The inverse Fourier transform of $\widehat{g}_{\rm c}$ is given by

$$g_{\rm c}(x) = \frac{e^{-i\frac{m^3c^3}{3hB^3}}}{2\pi} \int_{-\infty}^{\infty} e^{-i\frac{hmc}{B^3}\xi^2 + i(x + \frac{m^2c^2}{B^3})\xi} d\xi$$

$$= \frac{B^{3/2}}{\sqrt{4\pi\hbar mci}} e^{-i\frac{m^3c^3}{3\hbar B^3}} \exp\left(-\frac{B^3\left(x + \frac{m^2c^2}{B^3}\right)^2}{4\hbar m\, ic}\right)\ .$$

The branch of the complex square root is chosen so that $\Re(ic) > 0$. From the Fourier integral representation of $g_{\rm c}$ it follows that as $c \to 0$, $g_{\rm c}(x) \to \delta(x)$ and therefore we get

$$\psi_{\rm L}(x,t) \to \psi_{\rm BB}(x,t)\ ,\ \text{as} c \to 0\ , \tag{5.49}$$

which is consistent with (5.42) for $c = 0$. The mean values of position and momentum of $\psi_{\rm L}$ are given by

$$\langle x \rangle_t = \int_{-\infty}^{\infty} x|\psi_L(x,t)|^2 dx = \frac{v^2}{2a} - \frac{\hbar}{4mv} + ut\ , \tag{5.50}$$

and

$$\langle p \rangle_t = -i\hbar \int_{-\infty}^{\infty} \psi_L^*(x,t)\partial_x \psi_L(x,t) dx = mu\ . \tag{5.51}$$

Obviously, the L solution obeys Ehrenfest's theorem. In particular, for $u = 0$, the mean position of the L solution is constant, but it is negative only if $v < B\hbar^{1/3}/2^{2/3}m$, and the mean momentum is zero, thus showing a behaviour similar to that of the GBB wave packet. Moreover, we have

$$\langle x^2 \rangle_t = \int_{-\infty}^{\infty} x^2|\psi_L(x,t)|^2 dx = \frac{v^2}{4a} + \frac{v^2ut}{a} + (ut)^2 - \frac{hut}{2mv} + \frac{\hbar at^2}{2mv} + \frac{3\hbar^2}{16(mv)^2} + \frac{hv}{4ma}\ , \tag{5.52}$$

and thus the dispersion of the position is given by

$$(\Delta x_t)^2 = \langle x^2 \rangle_t - \langle x \rangle_t^2 = \frac{\hbar^2}{8(mv)^2} + \frac{\hbar v}{2ma} + \frac{\hbar at^2}{2mv}\ . \tag{5.53}$$

It follows that the dispersion of the L solution, like that one of the GBB wave packet, increases linearly with time for large time.

CHAPTER 6

Discussion of the results

We have shown that all the Airy-type wave packets (Airy beams) with finite energy which appear in applied physics and optics literature can be expressed as the convolution of the non square integrable Berry-Balazs wave packet $\psi_{\text{BB}}(x,t)$ with an appropriate Gaussian function $g_\epsilon(x)$,

$$\psi(x,t) = \psi_{\text{BB}}(x,t) *_x g_\epsilon(x),$$

where

$$g_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon^2}}\, e^{-x^2/2\epsilon^2} ,$$

and $\epsilon > 0$ is a parameter depending on the particular solution (GBB, SC, BS or L). The underlying reason is that the convolution

$$f(x) = g(x) *_x Ai(x)$$

defines a square integrable function f when $g \in L^2$. In fact, by taking the Fourier transform of the last equation,

$$\widehat{f}(\xi) = \widehat{g}(\xi)\, \exp\big(i\xi^3/3\big) ,$$

and using Parseval's theorem, it follows $\widehat{f} \in L^2$, since $\widehat{g} \in L^2$. Therefore, *any* function $g \in L^2$ can be used to generate by convolution a square integrable Airy beam.

Recall now that in the analysis of the SC solution we have seen that (see eqs. (5.19),(5.20)),

$$\phi(x)Ai(x) = g(x) *_x Ai(x) ,$$

implies

$$g(x) = \int_{-\infty}^{\infty} Ai(z)Ai(z-x)\phi(z)dz , \tag{6.1}$$

when $\phi(x)Ai(x) \in L^2$. Thus, the following natural question arises. Given $g \in L^2$, can we find ϕ such that the above pair of relations hold? In other words, can we always choose initial data in the SC way, which, in a sense, implies that the SC solution and the Gaussian BB wave packet are equivalent?

By the definition and the L^2 properties of the Airy transform ([VS], Sec. 4.2.1), we can prove that for any $g \in L^2$, $\phi(x)Ai(x) \in L^2$. Since $Ai(x)$ has zeros on the negative real axis, ϕ is not necessarily smooth. Although the properties of ϕ itself have some mathematical interest, the initial data of the SC solution is $\phi(x)Ai(x)$. Therefore, there is a one-to-one correspondence between g and the SC initial data, and in this sense *the GBB wave packet and the SC solution are equivalent*. In an analogous way we can show that *the BS solution and the L solution are equivalent to a SC solution with appropriate initial data which can be constructed as above from the corresponding function* g.

It is a remarkable property of all finite energy Airy wave packets that the dispersion has the form $\Delta x_t = \sqrt{\alpha + \beta t^2}$, the constants $\alpha\ ,\beta$ depending on the various parameters entering each particular solution, and, for large time, it increases linearly with time, thus complying with the corresponding behaviour of the Gaussian wave packet.

A final remark concerns the semiclassical scaling of the GBB, and it related to the *the open problem of the semiclassical limit* of Airy beams. Recall that the L^2 norm of ψ_{BB} is $\hbar$-dependent and it is given by eq. (5.5)

$$\int_{-\infty}^{\infty} |\psi_{\text{GBB}}(x\ ,0)|^2 dx = \frac{\hbar^{1/3}}{2\pi^{1/2}B^2}\frac{\hbar}{\sigma}\ .$$

It follows that in order to retain the natural Airy scale in the initial data

$$\psi_0(x) = Ai\Big(\frac{Bx}{\hbar^{2/3}}\Big)\ ,$$

and at the same time the energy of the wave packet be finite as $\hbar \to 0$ (semiclassical limit), we must choose $\sigma = \hbar^{4/3}$. Thus, the Gaussian function (5.8) involved in construction of ψ_{GBB}, is required to be

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{\hbar^{4/3}}\ \exp\Bigg(-\frac{1}{2}\Big(\frac{x}{\hbar^{4/3}}\Big)^2\Bigg)\ .$$

It follows that the necessary width of the Gaussian function is $\mathcal{O}(\hbar^{4/3})$, and it is much smaller that the Airy scale $\mathcal{O}(\hbar^{2/3})$, which implies that the Airy beams are semiclassically very narrow.

Bibliography

[Bal] L.E. Ballentine, *Quantum Mechanics: A Modern Development*, World Scientific, Singapore, 1998.

[BB] M.V. Berry & N.L.Balazs, *Nonspreading wave packets*, Am. J. Phys. 47(3), 1979, pp. 264-267.

[BLP] A . Bensoussan, J.-L. Lions & G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam, 1987.

[Bo] D. Bohm, *Quantum Theory*, Dover Publ. Inc., New York, 1989.

[BS] I. M. Besieris & A. M. Shaarawi *A note on an accelerating finite energy Airy beam* , Opt. Lett. 32(16), 2007, PP. 2447-2449

[BSZ] I.M. Besieris, A. Shaarawi & R.W. Ziolkowski, *Nondispersive accelerating wave packets*, Am. J. Phys. 62(6), 1994, pp. 519-521.

[GR] Gradshteyn I.S. & Ryzhik I. M., *Table of Integrals, Series, and Products, 7th ed.*, Academic Press, New York, 2007.

[Gre1] D.M. Greenberger, *Comment on "Nonspreading wave packets*, Am. J. Phys. 48, 1980, p. 256.

[Gre2] D.M. Greenberger, *The Inconsistency of the Usual Galilean Transformation in Quantum Mechnaics and How To Fix It*, Z. Naturforsch 56a, 2001, pp. 7-75.

[Grein] W. Greiner, *Quantum Mechanics. An Introduction*, Springer-Verlag, Berlin, 2001.

[HS] B.R. Holstein & A.R. Swift, *S*preading Wave Packets-A Cautionary Note, Am. J. Phys.40, 1972, pp. 829-832.

[Horm] L. Hörmander, *The Analysis of linear partial differential operators*, Vol. I, Springer-Verlag, Berlin Heidelberg, 1983.

[Karn] A. Karnezis, *Quantum hydrodynamics near caustics*, Diploma Thesis, Dep. Math. & Appl. Math., Univ. Crete, 2018.

[Ke] E.C. Kemble, *The fundamental principles of quantum mechanics*, McGraw-Hill, New York, 1937.

[Kl] J.R. Klein, *Do free quantum-mechanical wave packets always spread?*, Am. J. Phys. 48(12), 1980, pp. 1035-1037.

[Leb] N. N. Lebedev, *Special functions and their applications*, Prentice- Hall, Inc., Englewood Cliffs, N. J., 1965

[Lek] J. Lekner *Airy wavepacket solutions of the Schrodinger equation*, Eur. J. Phys. 30, 2009, pp. L43-L46.

[Na] M. Nauenberg, *E*instein's equivalence principle in quantum mechnaics revisited, Am. J. Phys 84(11), 2016, pp. 879-882.

[SC] G. A. Siviloglou & D. N. Christodoulides *Accelerating finite energy Airy beams*, Opt. Lett. 32(8), 2007, pp. 979-981.

[SBDC] G.A. Siviloglou, A. Dogariou & D. N. Christodoulides, *Observation of accelerating Airy beams*, Phys. Rev. Let. 99, 2007, pp. 213901-213904.

[Tan] D. Tannor, *Introduction to quantum mechanics. A time dependent perspective*, University Science Books, Sausalito, California, 2007.

[UP] K. Unnikrishnan & A.R.P. Pau, *Uniqueness of the Airy packet in quantum mechanics*, Am. J. Phys. 64(8), 1996, pp. 1034-1035.

[VS] O. Vallee & M. Soares, *Airy Functions and Applications in Physics*, Imperial College Press, London, 2004.

[Vy] V.M. Vyas, *Airy wavepackets are Perelomov coherent states*, Am. J. Phys. 86(10, 2018, pp.750-754.

[Whi] G. B. Whitham, *Linear and Nonlinear Waves*, John Wiley & Sons, Canada,1999.