UNIVERSITY OF CRETE
Department of Mathematics and Applied Mathematics

Interacting Particle models for condensation

by Marios-Georgios Stamatakis

Supervisor: Michail Loulakis

Acknowledgements: The current research has been co-financed by the European Union (European Social Fund- ESD) and by national resources through the operational programme "Education and Lifelong Learning" of the National Strategic Research Frame (NSRF) - Financed Research Project: Herakleitos II. Investment in the society of knowledge through the European Social Fund.

European Union European Social Fund
OPERATIONAL PROGRAMME EDUCATION AND LIFELONG LEARNING investing in knowledge society
MINISTRY OF EDUCATION & RELIGIOUS AFFAIRS MANAGING AUTHORITY
Co-financed by Greece and the European Union
NSRF 2007-2013 programme for development EUROPEAN SOCIAL FUND

Contents

Acknowledgments

First of all I would like to express my gratitude to my supervisor Michail Loulakis for his support, the things that he taught me and all the useful conversations we shared during this research. What is more, I was always treated as a friend, and I hope we remain so even after the completion of this task. I am also grateful to Professor Athanasios Tzavaras in my advisory committee for his cooperation and the useful conversations that we had. I would also like to thank Claudio Landim, Inés Armendáriz and Stefan Grosskinsky for the interesting discussions that we 've had.

This is also a good place to thank the professors that influenced my progress in mathematics. So I would like to thank Apostolos Giannopoulos and Eirini Deliyanni for teaching me mathematics and Polychronis Strantzalos for teaching me to "read behind the lines". I would also like to thank my MSc advisor Souzana Papadopoulou for her support and faith at a critical stage of my "career".

I would also like to thank Eliza for her understanding... Finally, I would like to thank my family. My mother Roula and my sister Konstanti for their support and their faith in me which kept me going on, and especially my father Nikiforos who has been (and will always be) the brightest light in my life, without whom I would probably have nothing to do with science.

Frequently Used Notation

1. $\mathbb{N} = \{1, 2, \dots\}$.

2. $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$.

3. $\mathbb{R}_+ = [0, +\infty)$.

4. $\mathbb{R}^d$ is the d-dimensional Euclidean space and

$$\langle x, y \rangle := \sum_{i=1}^{d} x_i y_i, \quad x, y \in \mathbb{R}^d$$

the Euclidean inner product.

5. $\mathbb{Z}^d$ is the d-dimensional integer lattice.

6. $\mathbb{T}^d := {}^{\mathbb{R}^d}/_{\mathbb{Z}^d} \cong [0, 1)^d$ is the d-dimensional torus.

7. $\sharp E =$ the cardinal of the set E.

8. Given any sets $E \subseteq M$ we denote by $\mathbb{1}_E : M \longrightarrow \{0, 1\}$ the indicator function of E given by

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in M \setminus E. \end{cases}$$

9. For any function $f : X \longrightarrow (-\infty, \infty]$ we denote by $\mathcal{D}_f$ its proper domain:

$$\mathcal{D}_f = \{x \in X | f(x) < +\infty\}.$$

10. For any polish space M:

$\mathbb{P}M$ denotes the space of all Borel probability measures on M.

mM denotes the space of all measurable functions on M.

$B(M)$ denotes the space of all bounded Borel measurable functions on M.

$BC(M)$ denotes the space of all bounded continuous functions on M.

$D(\mathbb{R}_+, M)$ denotes the Skorohod space of all right continuous paths $x : \mathbb{R}_+ \longrightarrow M$ with left-hand limits.

11. Given any probability measure $\mu \in \mathbb{P}M$ and a measurable function $f : M \longrightarrow N$ we denote by $f_*\mu := \mu \circ f^{-1} \in \mathbb{P}N$ the push forward of μ through f or in other words the law of the random variable f under μ.

12. For any $a, b \geq 0$ we denote by $C^{a,b}(\mathbb{R}_+ \times \mathbb{T}^d)$ the space of all functions $f : \mathbb{R}_+ \times \mathbb{T}^d \longrightarrow \mathbb{R}$ that are C^a is time and C^b in space. Of course here when we say that a

function is C^a in a variable means that f is $[a]$ times continuously differentiable in that variable, where $[a]$ is the integer part of a, with its derivatives of maximal order being Holder continuous of exponent $a - [a]$.

Introduction-Overview

In this thesis we examine physical systems exhibiting phase transition via the emergence of condensation by using interacting particle models. The macroscopic description of the system is obtained by the interacting particles model in the hydrodynamic limit. Our study is centered around condensing Zero Range Processes, which are probably the simplest interacting particle models exhibiting condensation. In the hydrodynamic limit the evolution of condensing Zero Range Processes is expected to approach a saturated non-linear diffusion. The main results of the thesis are the proof of the hydrodynamic limit of condensing Zero Range Processes when starting from a sub-critical profile, in which case it is proved that no condensation will occur, and the validity of Fick's law at the macroscopic level as a limit of the microscopic conservation laws.

The main aim of the theory of hydrodynamic limits of stochastic interacting particle systems is to describe the macroscopic evolution of the thermodynamic characteristics (e.g. density, pressure, temperature, etc.) of the system in appropriate space and time scales, as solutions of a partial differential equation (PDE), the so called hydrodynamic equation. One way this is made precise is via the principle of conservation of local equilibrium: First, the equilibrium states of the system are characterized by the thermodynamic characteristics $\vec{\rho} = (\rho^1, \cdots, \rho^m) \in \mathbb{R}^m$, $m \in \mathbb{N}$, so that for each thermodynamic characteristic $\vec{\rho} \in \mathbb{R}^m$ we have a global equilibrium $\nu_{\vec{\rho}}$ corresponding to $\vec{\rho}$. A local equilibrium state of profile $\vec{\rho}$ is a state that near each macroscopic point u it is close to the global equilibrium state $\nu_{\vec{\rho}}$ that corresponds to the value $\vec{\rho}(u)$. According to the principle of local equilibrium one expects, and thus aims to prove, that starting from a local equilibrium of profile $\vec{\rho}_0$, at the appropriate space and time scales, at each later time $t > 0$ the system remains in local equilibrium of profile $\vec{\rho}_t$, where $\vec{\rho}_t(u) = \vec{\rho}(t, u)$ is the solution of a system of evolutionary partial differential equations (PDEs), the so called hydrodynamic equation, with initial condition $\vec{\rho}(0, \cdot) = \vec{\rho}_0$.

In this thesis we are interested in the hydrodynamic behavior of condensing Zero Range Processes (ZRPs). ZRPs are interacting particle systems such that each particle X jumps at an exponential rate $g(k)$ that depends only on the number k of particles that occupy the same site as particle X through some function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$, which is called the local jump rate. Particles that jump change position according to a translation invariant transition probability $p(x, y) = p(y - x)$. After their introduction by Spitzer in 1970, ZRPs have attracted a lot of attention, one reason being that for particular choices of local jump rate functions g they exhibit phase transition phenomena.

Since we are interested in the hydrodynamic limit of ZRPs we consider ZRPs whose particles evolve in the finite lattice $\mathbb{T}_N^d := \{0, \ldots, N-1\}^d$. Then *the ZRP with jump rate* $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ *and elementary step distribution* $p \in \mathbb{P}\mathbb{Z}^d$ is the Markov jump process on the state space

$$\mathbb{M}_N^d := \mathbb{Z}_+^{\mathbb{T}_N^d}$$

of configurations of particles on $\mathbb{T}_N^d$ with generator $L^N : D(L^N) \subseteq B(\mathbb{M}_N^d) \longrightarrow B(\mathbb{M}_N^d)$ given by

$$L^N f(\eta) = \sum_{x,y \in \mathbb{T}_N^d} \{f(\eta^{x,y}) - f(\eta)\} g(\eta_x) p_N(y-x),$$

where

$$\eta_z^{x,y} = \begin{cases} \eta_z, & \text{if } z \notin \{x,y\} \\ \eta_x - 1, & \text{if } z = x \\ \eta_y + 1, & \text{if } z = y \end{cases},$$

if $\eta_x \neq 0$ and, say, $\eta^{x,y} = \eta$ otherwise, and $p_N(z) = p(z + N\mathbb{Z}^d)$ for all $z \in \mathbb{T}_N^d$.

For particular decreasing local jump rate functions g there exists a critical value $\rho_c = \rho_c(g) < +\infty$ of the density such that there exist equilibrium states characterized by the density ρ iff $\rho \leq \rho_c$. In particular the translation invariant equilibrium distributions of the ZRP are the product measures $\bar{\nu}_\varphi^N \equiv \bar{\nu}_{\varphi,g}^N$ with common marginal $\bar{\nu}_\varphi^1 \in \mathbb{P}\mathbb{Z}_+$ given by

$$\bar{\nu}_\varphi^1\{k\} = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g!(k)}, \qquad k \in \mathbb{Z}_+$$

for all $\varphi \geq 0$ such that the series $Z(\varphi) = \sum_{k=0}^\infty \frac{\varphi^k}{g!(k)}$ converges. It is known that the mean density

$$R(\varphi) = \int \eta(0) d\bar{\nu}_\varphi^N$$

of the occupation variable under $\bar{\nu}_\varphi^1$ is smooth strictly increasing function of φ and so by reparametrizing the equilibrium distributions by its inverse $\Phi := R^{-1}$ we obtain the grand canonical ensemble

$$\nu_\rho^N = \bar{\nu}_{\Phi(\rho)}^N, \quad \text{for } \rho \geq 0 \text{ such that } Z(\Phi(\rho)) < +\infty.$$

We will refer to ZRPs for which the radius of convergence φ_c of the partition function Z is infinite as non-condensing. ZRPs such that

$$\varphi_c < +\infty \quad \text{and} \quad \rho_c := \sup_{\varphi < \varphi_c} R(\varphi) = +\infty$$

will be called weakly condensing, while ZRPs for which $\rho_c < +\infty$ will be called strictly condensing. The lack of equilibrium states corresponding to densities $\rho > \rho_c$ constitutes a main problem in the description of the hydrodynamic behavior of strictly condensing ZRPs, since the formulation via the notion of local equilibrium faces difficulty in observing densities higher than the critical density. Furthermore, even weakly condensing ZRPs exhibit pathological behavior.

So far two main methods have been developed for proving hydrodynamic limits of interacting particle systems. The Entropy method developed by Guo, Papanikolaou and Varadhan in [20] and the Relative Entropy method developed by H.T. Yau in [33]. Both methods have been applied to prove the hydrodynamic limit of ZRPs. The Entropy method has been applied to non-condensing ZRPs with super-linear jump rate function $g(k) \geq a_0 k$ for some $a_0 > 0$, while the Relative Entropy method has been applied to weakly condensing ZRPs which satisfy $\lim_{\varphi \to \varphi_c} Z(\varphi) = +\infty$, when started from initial distributions with sufficiently regular bounded profile. An exposition of the proofs is contained in chapters 5 and 6 of [25] respectively.

In this thesis (chapter 3) we extend the Relative Entropy method to all mean zero asymmetric condensing ZRPs with bounded jump rates, by interpreting the boundedness assumption on the initial profile as boundedness away from the critical density. This was achieved by extending the One-Block estimate, a main tool in all known approaches to the hydrodynamic limit of ZRPs, to condensing ZRPs. This extension of the One-Block estimate is made possible by the result in the equivalence of Ensembles contained in section 1.4, which has been originally proved in [19]. The hydrodynamic equation is the non-linear diffusion equation

$$\partial_t \rho = \Delta_x \Phi(\rho), \tag{1}$$

where $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is extended to be constantly equal to $\varphi_c = \Phi(\rho_c)$ densities $\rho \geq \rho_c$. This result has been accepted for publication in the Journal of Statistical Physics.

We expect that in order to describe the condensation phenomenon, the solutions of the hydrodynamic equation should be allowed to be measure-valued. The simple considerations in section 3.2 also point to this. This poses a major difficulty at the macroscopic level since one has to make sense of the equation (1) for degenerate (eventually constant) non-linear functions Φ and for measure-valued solutions $\mu : \mathbb{R}_+ \longrightarrow \mathcal{M}_+(\mathbb{T}^d)$. One such promising interpretation of this equation is as a gradient flow in the space of finite measures with fixed total mass equipped with L^2-Wasserstein metric. Due to the relevance of the continuity equation in the definition of gradient flows in Wasserstein spaces of probability measures, we proved that a sub-sequential hydrodynamic limit of condensing ZRPs can be given via the continuity equation

$$\partial_t \pi = \mathrm{div}_x W$$

where π is a finite non-negative measure and W is a vector valued measure. One advantage of the continuity equation is that it easily allows the consideration of measure-valued solutions.

Our approach on the deriving the continuity equation at the macroscopic level starts by following the Entropy method of Guo-Papanikolaou-Varadhan in considering the martingales associated to the ZRPs on the discrete toruses $\mathbb{T}_N^d$ by the martingale problem to prove the relative compactness of the laws of the image of the ZRPs under the empirical density

$$\pi^N := \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x) \delta_{\frac{x}{N}}$$

as $N \to \infty$. In the equation above defining the empirical density we have associated to each microscopic point $x \in \mathbb{T}_N^d$ the macroscopic point $\frac{x}{N} \in \mathbb{T}^d$. In these considerations we focus on the case of the nearest neighbor elementary step distribution. The real valued process

$$A_t^{N,G} := \langle G_t, \pi_t^N \rangle - \langle G_0, \pi_0^N \rangle - \int_0^t (\partial_s + N^2 L^N) \langle G_s, \pi^N \rangle (\eta_s^N) ds \tag{2}$$

defined on the filtered space $\big(D(\mathbb{R}_+, \mathbb{M}_N^d), (\mathcal{F}_t^N)_{t \geq 0}, \mathbb{P}^{\mu_0^N}\big)$ is a martingale, where $\mathbb{P}^{\mu_0^N}$ is the diffusively rescaled law of the ZRP, μ_0^N is a sequence of initial distributions which is allowed to have a condensate at some macroscopic point $u \in \mathbb{T}^d$, i.e.

$$\text{possibly} \quad \liminf_{N \to \infty} \frac{1}{N^d} \int \eta([Nu]) d\mu_0^N > 0 \quad \text{for some } u \in \mathbb{T}^d,$$

and $(\mathcal{F}_t^N)$ is the minimal right continuous filtration to which the ZRP is adapted. By Doob's inequality the martingale $A_t^{N,G}$ is asymptotically negligible:

$$\lim_{N \to \infty} \big\{ \sup_{0 \leq t \leq T} |A_t^{N,G}| \geq \delta \big\} = 0, \quad \forall\, \delta > 0.$$

Computing $L^N \langle G, \pi^N \rangle$, a discrete integration by parts gives

$$L^N \langle G, \pi^N \rangle = \frac{1}{N^d} \sum_{j=1}^d \sum_{x \in \mathbb{T}_N^d} \Big[G\Big(\frac{x+e_j}{N}\Big) - G\Big(\frac{x}{N}\Big) \Big] \big[g\big(\eta(x)\big) - g\big(\eta(x+e_j)\big) \big].$$

One more integration by parts gives

$$\begin{aligned} L^N \langle G, \pi^N \rangle &= \frac{1}{N^d} \sum_{j=1}^d \sum_{x \in \mathbb{T}_N^d} \Big[G\Big(\frac{x+e_j}{N}\Big) + G\Big(\frac{x-e_j}{N}\Big) - 2G\Big(\frac{x}{N}\Big) \Big] g\big(\eta(x)\big) \\ &= \frac{1}{N^{d+2}} \sum_{x \in \mathbb{T}_N^d} \Delta^N G\Big(\frac{x}{N}\Big) g\big(\eta(x)\big), \end{aligned}$$

where

$$\Delta^N G\Big(\frac{x}{N}\Big) = N^2 \sum_{j=1}^d \Big[G\Big(\frac{x+e_j}{N}\Big) + G\Big(\frac{x-e_j}{N}\Big) - 2G\Big(\frac{x}{N}\Big) \Big]$$

is the discrete Laplacian of G. By using a Taylor expansion of G around $\frac{x}{N}$ one gets for $G \in C^3(\mathbb{R}_+ \times \mathbb{T}^d)$ together with the asymptotic negligibility of the martingale A_t^G one gets that

$$\lim_{N \to \infty} \mathbb{P}^{\mu_0^N} \Big\{ \sup_{0 \leq t \leq T} \Big| \int_0^t \big[\langle \partial_s G_s, \pi_s^N \rangle + \langle \Delta G_s, \sigma_s^N \rangle \big] ds \Big| \geq \delta \Big\} = 0. \tag{3}$$

for all $G \in C_c^3([0,T] \times \mathbb{T}^d)$, where

$$\sigma_\eta^N = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} g(\eta_x) \delta_{\frac{x}{N}} \tag{4}$$

is the empirical jump rate and $\sigma_s^N = \sigma_{\eta_s}^N$. Equality (3) together with the Aldous criterion for relative compactness of laws on the Skorohod space yields the relative compactness of the laws of the ZRPs on the discrete toruses $\mathbb{T}_N^d$.

Next the application of the Entropy method proceeds by using estimates on the evolution of the entropy of the time marginals μ_t^N of the law of the ZRPs with respect to an invariant state to replace the integral term

$$\int_0^t \langle \Delta G_s, \sigma_s^N \rangle ds$$

by a function of the empirical measure in order to close the equation. This is the context of the replacement lemma in sections 5.1 to 5.5 in [25]. As described in [25] section 5.3 the Replacement Lemma breaks up in the proof of the One-Block and the Two-Block estimates. Although we have extended the One-Block estimate to condensing ZRPs the, we haven't managed to extend the Two-Block estimate because the cut-off of large densities in equation (5.1) of section 5.5 in [25] is not possible if the invariant distributions do not have full exponential moments, as in the case of condensing ZRPs. Furthermore due to the possible existence of a condensate in the initial distributions even in the case of a constant jump rate function the coupling techniques for attractive processes cannot be applied.

For the reasons above, we turn our attention to the two important, although non-conserved, quantities present in the microscopic level: the empirical jump rate σ^N and the empirical current

$$W^N = \frac{1}{N^{d-1}} \sum_{x \in \mathbb{T}_N^d} \sum_{j=1}^d \{g(\eta(x)) - g(\eta(x+e_j))\} e_j \delta_{\frac{x}{N}} =: -\frac{1}{N^{d-1}} \sum_{x \in \mathbb{T}_N^d} \nabla^N g(\eta(x)) \delta_{\frac{x}{N}}$$

and try to prove their relative compactness in the hydrodynamic limit in order to obtain the continuity equation. Taylor expansion of G together with the asymptotic negligibility of the martingale $A_t^{N,G}$ imply that also

$$\lim_{N\to\infty} \mathbb{P}^{\mu_0^N} \left\{ \sup_{0 \le t \le T} \left| \int_0^t \left[\langle \partial_s G_s, \pi_s^N \rangle + \langle \nabla G_s, W_s^N \rangle \right] ds \right| \ge \delta \right\} = 0 \tag{5}$$

for all $G \in C_c^3([0,T] \times \mathbb{T}^d)$. Once the required relative compactness of (the law of the images of the ZRPs through) the empirical current has been obtained, the limit above will lead to the continuity equation at the macroscopic level.

Due to the non-conserved character of the empirical jump rate and the empirical current, the Skorohod topology seems to be two strong to allow for the required relative compactness. For this reason we consider the empirical jump rate process. For this reason we consider these processes as random variables taking values in appropriate L^∞ spaces of Banach-valued functions equipped with their w^*-topology. For the empirical diffusion rate process σ^N we choose the state space

$$L^\infty_{w^*}(0,T; \mathcal{M}(\mathbb{T}^d)) \cong L^1(0,T; C(\mathbb{T}^d))^*,$$

for $T > 0$, and for the empirical current process W^N we choose the space

$$L^\infty_{w^*}(0,T; C^1(\mathbb{T}^d; \mathbb{R}^d)^*) \cong L^1(0,T; C^1(\mathbb{T}^d; \mathbb{R}^d))^*,$$

both equipped with their w^*-topologies. The results on L^p-spaces of Banach-valued functions that we require are contained in [10]. However in this way the limiting current that we get is at each time an element of the dual space $C^1(\mathbb{T}^d;\mathbb{R}^d)^*$ and further regularity estimates are required to prove that it is a measure. These regularity estimates are contained in section 4.5 and are in fact an adaptation of the estimates in section 5.7 of [25] to the case at hand. In this way we obtain the continuity equation

$$\partial_t \pi = -\mathrm{div}_x W = \Delta\sigma \tag{6}$$

as a sub-sequential hydrodynamic limit, which in addition expresses the macroscopic empirical current as the gradient of the macroscopic empirical jump rate.

We close this introduction with an overview of the material contained in this thesis. Chapter 1 contains the definition of ZRPs and preliminary material. In section 1.1 we define ZRPs and in section 1.2 we describe their equilibrium distributions. In section 1.3 we study some topological spaces that will be useful throughout the thesis. In particular subsection 1.3.2 contains a generalization of the results of section 2.1 of [25] in the weak convergence of probability measures on the space of configurations $\mathbb{Z}_+^{\mathbb{Z}^d}$ over the infinite lattice to the weak convergence with respect to functions with bounded polynomial growth. In section 1.4 we present the proof on the equivalence of ensembles given originally in [19] and in section 1.5 we describe some simple well-known examples of ZRPs. Finally in section 1.6 we extend the static large deviations principle for the empirical embeddings under the equilibrium distributions ν_ρ^∞, $\rho \in (0, \rho_c)$. This result generalizes a well-known result for non-condensing ZRPs, proved in p. 74 of [25], to condensing ZRPs. In contrast to the non-condensing case the large deviations functional takes finite values in measures that are not absolutely continuous with respect to the Lebesgue measure. This is one of the facts that leads to expect that measure-valued solutions to the hydrodynamic equation are required to describe the condensation phenomenon.

In chapter 2 we present various ways of formalizing the notion of local equilibrium and define hydrodynamic limits via the principle of local equilibrium. All these notions are well-known and contained in [25].

Chapter 3 contains our first main result, the proof of the hydrodynamic limit of condensing ZRPs with sub-critical initial profiles. In section 3.1 we generalize the One-Block estimate, in section 3.2 we see how one can easily apply the classic results in quasilinear linear parabolic equations obtained in [26] to eventually constant non-linearities in the case that the initial condition is sub-critical and finally in section 3.3 we apply the relative entropy method to prove the hydrodynamic limit.

Finally, in chapter 4 we present our results in the continuity equation. In section 4.1 we review the results on the relative compactness of the laws of the empirical density of the ZRP following section 5.1 in [25]. In section 4.2 following standard arguments of the Entropy method we prove the limits (3) and (5) showing that the equation $\partial_t \pi^N = -\mathrm{div} W^N = \Delta\sigma^N$ holds at the microscopic level. Then we proceed on describing the appropriate choice of state spaces for the empirical current and the empirical jump-rate processes that will allow us to prove relative compactness results to conclude that the continuity equation (6) remains valid at the macroscopic limit. Then in section 4.3 we give the definition of the continuity equation in the appropriate context that corresponds to the choice of the state spaces of the empirical current and jump rate processes and in section 4.4 we prove the relative compactness of the laws of the ZRP under the triples (π^N, W^N, σ^N) as well as that all limit points are concentrated in triples satisfying the continuity equation (6). Next in section 4.5 we prove regularity results for the limits curves W and σ that allow us to conclude that the limiting empirical currents W_t are in fact vector measures and that the empirical jump rates σ_t are in the Hilbert-Sobolev space $H^1(\mathbb{T}^d)$, and we finish this chapter in section 4.6 with a conjecture on what we believe the hydrodynamic equation should be when one starts from an initial profile that has a fully formed condensate at one site.

We close this introduction with a small overview of the Appendix. In chapter A.1 we review mostly without proofs some basic facts on Skorohod spaces that will be used extensively in chapter 4. In A.2 we review the martingales associated to Markov processes via the martingale problem in the context of Markov jump processes. These martingales e.g. (2) play an essential role in the Entropy method and are used to obtain estimates that yield the relative compactness of the empirical density and on exhibiting the validity of the continuity equation at the microscopic. In section A.3 we review the definition of relative entropy and some of its basic properties. Among others we prove give generalize the variational characterization of relative entropy to the case of general convex functionals h in place of the usual function $x \mapsto x \log x$, without assuming that h has superlinear growth. More general results have been proved for instance in [5], but the proof here uses elementary arguments and is based on a detailed description of the Legendre transform of real valued convex functions. These results are then used in the generalization of the static Large Deviations principle for the empirical embeddings of the ZRP to the condensing case given in section 1.6. Finally in A.4 we review classic results of topological measure theory that loosely speaking generalize the classic theory of probability measures on polish spaces to the class of Radon measures on completely regular topological spaces. These results are used in the relative compactness arguments for the empirical current and the empirical jump rate.

Chapter 1

Zero Range Processes

1.1 Basic Definitions

Zero range process (ZRPs) on a countable space M are a class of interacting particle systems (IPSs) for which the rate that a particle leaves a site $x \in M$ depends only on the number k of particles sharing the same site x through some appropriate function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$, called the local jump rate function, and after a particle leaves a site it chooses a destination according to a transition probability p. In this way, particles interact only with particles sharing the same site, whence comes the terminology ZRP.

In particular we will consider ZRPs in which the particles live in the discrete toruses

$$\mathbb{T}_N^d := \{0, 1, \dots, N-1\}^d \cong \left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right)^d, \quad N \in \mathbb{N},$$

jump with rate function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ and move according to a translation invariant transition probability $p(x, y) \equiv p(y-x)$, $x, y \in \mathbb{Z}^d$, $p \in \mathbb{P}\mathbb{Z}^d$, (i.e. we identify probability distributions $p \in \mathbb{P}\mathbb{Z}^d$ with transition probabilities $p : \mathbb{Z}^d \longrightarrow \mathbb{P}\mathbb{Z}^d$,) through the natural projections $[\cdot]_N : \mathbb{Z}^d \longrightarrow \mathbb{T}_N^d$,

$$[z]_N = z + N\mathbb{Z}^d = \{w \in \mathbb{Z}^d | w - z \in N\mathbb{Z}^d\},$$

i.e. their steps are governed by the translation invariant probability $p_N := [\cdot]_{N*} \circ p$ given for all $x, y \in \mathbb{T}_N^d$ by the formula

$$p_N(x, y) = p_N(y - x) = p(y - x + N\mathbb{Z}^d) = \sum_{z \in \mathbb{Z}^d} p(y - x + Nz). \tag{1.1}$$

We shall further assume that the support of the distribution $p \equiv p(0, \cdot) \in \mathbb{P}\mathbb{Z}^d$ spans $\mathbb{Z}^d$ over $\mathbb{Z}$, so that the transition probability $p_N : \mathbb{T}_N^d \longrightarrow \mathbb{P}\mathbb{T}_N^d$ is irreducible, i.e. for any $x, y \in \mathbb{T}_N^d$ there exist $m \in \mathbb{N}$ and $x_0, x_1, \dots, x_m \in \mathbb{T}_N^d$ such that $x_0 = x$, $x_m = y$ and

$$\prod_{i=1}^m p_N(x_k - x_{k-1}) > 0,$$

for all $N \in \mathbb{N}$, and that p has finite range, i.e. for some constant $A > 0$ we have $p(y) = 0$ for all $y \in \mathbb{Z}^d$ such that $|y| \geq A$.

Of course in such a ZRP the state space, i.e. the space of all possible configurations of particles is the space

$$\mathbb{M}_N^d := (\mathbb{Z}_+)^{\mathbb{T}_N^d},$$

i.e. the space of all functions $\eta : \mathbb{T}_N^d \longrightarrow \mathbb{Z}_+$ so that given a configuration $\eta \in \mathbb{M}_N^d$ and $x \in \mathbb{T}_N^d$, η_x is the number of particles at site x in the configuration η. We will denote by $\eta(x) : \mathbb{M}_N^d \longrightarrow \mathbb{Z}_+$, $x \in \mathbb{T}_N^d$ the natural projections

$$\eta(x)(\eta) = \eta_x.$$

Of course the product topology on $\mathbb{M}_N^d$ is the discrete topology and $\mathbb{M}_N^d$ is a Polish space. Also, we will denote by $|\cdot|_1 : \mathbb{Z}^{\mathbb{T}_N^d} \longrightarrow \mathbb{Z}_+$ the L^1-norm

$$|\eta|_1 = \sum_{x \in \mathbb{T}_N^d} |\eta_x|.$$

Of course for $\eta \in \mathbb{M}_N^d \subseteq \mathbb{Z}^{\mathbb{T}_N^d}$,

$$|\eta|_1 = \sum_{x \in \mathbb{T}_N^d} \eta_x$$

is the total number of particles in the configuration η.

In what follows given a configuration $\eta \in \mathbb{M}_N^d$ and $x \in \mathbb{T}_N^d$ such that $\eta_x \neq 0$ we denote by $\eta^{x,y}$ the configuration obtained by moving a particle from x to $y \in \mathbb{T}_N^d$. Of course if $x = y$ then $\eta^{x,y} = \eta$ and by convention if $\eta_x = 0$ we define $\eta^{x,y} = \eta$. We note that the transition $\eta \mapsto \eta^{x,y}$ preserves the total number of particles, i.e. $|\eta|_1 = |\eta^{x,y}|_1$ for all $\eta \in \mathbb{M}_N^d$ and all $x, y \in \mathbb{T}_N^d$. We will describe the ZRP as a Markov jump process on the countable space $\mathbb{M}_N^d$, so we will recall the basic definitions.

Definition 1.1.1 Let M be a countable measurable space with $\{x\}$ measurable for all $x \in M$. For any measurable strictly positive bounded function $\lambda : M \longrightarrow (0, +\infty)$ and any Markov kernel $p : M \longrightarrow \mathbb{P}M$ such that

$$p(x, M \setminus \{x\}) = 1 \;\; \forall\, x \in M \tag{1.2}$$

there exists a unique Markov kernel $P : M \longrightarrow \mathbb{P}D(\mathbb{R}_+, M)$ such that the canonical process $X = id_{D(\mathbb{R}_+,M)}$ is a Markov family on the state space M such that for all $x \in M$, with respect to P^x the following hold:

(a) The sequence $\{\tau_n\}_{n\in\mathbb{Z}_+}$ of stopping times defined inductively by

$$\tau_0 \equiv 0, \;\; \tau_n := \inf\left\{t > \tau_{n-1} \,\middle|\, X_t \neq X_{\tau_{n-1}}\right\}, \;\; n \in \mathbb{N} \tag{1.3}$$

is strictly increasing to $+\infty$, that is almost surely

$$\tau_n < \tau_{n+1} < +\infty \;\; \forall\, n \in \mathbb{Z}_+, \quad \text{and} \quad \lim_{n\uparrow+\infty} \tau_n = +\infty,$$

(b) The skeleton chain $(\xi_n)_{n\geq 0} := (X_{\tau_n})_{n\geq 0}$ is a discrete time Markov family with transition probability $p : M \longrightarrow \mathbb{P}M$ starting from x.

(c) The sequence $\tau'_n := \tau_{n+1} - \tau_n$, $n \in \mathbb{Z}_+$, is independent and exponentially distributed with parameters $\lambda(\xi_n)$, $n \in \mathbb{Z}_+$, that is

$$\big[(\tau'_n)_{n\geq 0}\big]_* P^x\big(\ \cdot\ \big|(\xi_n)_{n\geq 0}\big) = \bigotimes_{n\geq 0} \exp_{\lambda(\xi_n)}, \quad \xi_0 = x,$$

where for all $\lambda > 0$ we set

$$\exp_\lambda := \lambda e^{-\lambda t} \mathbb{1}_{[0,+\infty)}(t)dt$$

the exponential distribution of parameter λ.

This Markov process $(\{X_t\}_{t\geq 0}, (P^x)_{x\in M})$ is called the *Markov jump process with strictly positive bounded jump rate* $\lambda : M \longrightarrow (0,+\infty)$ *and transition probability* $p : M \longrightarrow \mathbb{P}M$, and its distribution $P = X_* \circ P : M \longrightarrow \mathbb{P}D(\mathbb{R}_+, M)$ is uniquely determined by the pair (p, λ).

By (c) it obviously follows that with respect to P^x the stopping time τ_1 is exponentially distributed with parameter $\lambda(x) = 1/\mathbb{E}^x\tau_1 > 0$ for all $x \in M$, that is $\tau_{1*}P^x = \exp_{\lambda(x)}$. One can easily extend the above definition for jump rate functions $\lambda : M \longrightarrow \mathbb{R}_+$ that are allowed to take the value 0 in the case that the pair (p, λ) in place of (1.2) satisfies

$$(a)\ \ p\big(x, \{\lambda \neq 0\} \setminus \{x\}\big) = 1 \quad \forall\, x \in \{\lambda \neq 0\}, \qquad (b)\ \ p_x = \delta_x \quad \forall\, x \in \{\lambda = 0\} \quad (1.4)$$

Indeed, intuitively, jump rate $\lambda(x) = 0$ at a site $x \in M$ means that a particle leaves the site x at rate 0, that is it never leaves x, and by (1.4) it follows that a particle moving according to p starting from $x \in \{\lambda \neq 0\}$ will always stay in $\{\lambda \neq 0\}$. Therefore $p^x\{\lambda \neq 0\} = 1$ for all $x \in \{\lambda \neq 0\}$ and thus the transition probability $p : M \longrightarrow \mathbb{P}M$ can be restricted to a transition probability $p_\lambda := p|_{\{\lambda\neq 0\}} : \{\lambda \neq 0\} \longrightarrow \mathbb{P}\{\lambda \neq 0\}$. So it makes sense to give the following definition, where for any continuous function $f : M \longrightarrow N$ between polish spaces we denote by $\bar{f} : D(\mathbb{R}_+, M) \longrightarrow D(\mathbb{R}_+, N)$ the continuous mapping induced on the Skorohod spaces associated to the polish spaces M, N by the formula

$$\bar{f}(\gamma)(t) = f\big(\gamma(t)\big), \quad \gamma \in D(\mathbb{R}_+, M).$$

Definition 1.1.2 Let $\lambda : M \longrightarrow \mathbb{R}_+$ be a measurable non-negative bounded function and let $p : M \longrightarrow \mathbb{P}M$ be a Markov kernel such that (1.4) holds. The Markov kernel $P : M \longrightarrow \mathbb{P}D(\mathbb{R}_+, M)$ given by

$$P^x = \begin{cases} \bar{i}_{\lambda *} P^x_\lambda & \text{if } \lambda(x) \neq 0 \\ \delta_{(x_t = x)_{t\in\mathbb{R}_+}} & \text{if } \lambda(x) = 0 \end{cases},$$

where $P_\lambda : \{\lambda \neq 0\} \longrightarrow \mathbb{P}D(\mathbb{R}_+, \{\lambda \neq 0\})$ is the Markov jump process on $\{\lambda \neq 0\}$ with transition probability p_λ and strictly positive jump rate function $\lambda|_{\{\lambda\neq 0\}}$ and

$$i_\lambda := i_{\{\lambda\neq 0\}} : \{\lambda \neq 0\} \hookrightarrow M$$

is the natural inclusion, is called the *Markov jump process with non-negative bounded jump rate* λ *and transition probability* p.

Of course with respect to Markov jump process $P : M \longrightarrow \mathbb{P}D(\mathbb{R}_+, M)$ with non-negative bounded jump rate λ and transition probability p the canonical process X on $D(\mathbb{R}_+, M)$ satisfies properties (a), (b) and (c) of definition 1.1.1 with respect to P^x for all $x \in \{\lambda \neq 0\}$, while for $x \in \{\lambda = 0\}$ we have that with respect to P^x,

(a') The sequence of stopping times $\{\tau_n\}_{n\in\mathbb{Z}_+}$ defined in (1.3) is almost surely $\tau_0 \equiv 0$, $\tau_n \equiv +\infty$, $n \in \mathbb{N}$.

(b') The *skeleton chain* defined inductively by

$$\xi_n := \begin{cases} X_{\tau_n} & \text{if } \tau_n < +\infty \\ \xi_{n-1} & \text{if } \tau_n = +\infty \end{cases} \tag{1.5}$$

is almost surely constant at x.

As we shall see next in the generator of Markov jump process with non-negative bounded jump rate $\lambda : M \longrightarrow \mathbb{R}_+$ is the bounded operator $L : B(M) \longrightarrow B(M)$ given by

$$Lf(x) = \sum_{y\in M} \big(f(y) - f(x)\big)\lambda(x)p(x, y).$$

Definition 1.1.3 Let $S_t : X \longrightarrow X$, $t \geq 0$, be a contraction semigroup on the Banach space X. The function $L : D_L \subseteq X \longrightarrow X$ defined on the set

$$D_L := \left\{ f \in X \,\middle|\, \text{the limit } \lim_{t\downarrow 0} \frac{(S_t - I)f}{t} \quad \text{exists in } X \right\}$$

by the formula

$$Lf = \lim_{t\downarrow 0} \frac{(S_t - I)f}{t}$$

is called *the generator of the contraction semigroup* $S = (S_t)_{t\geq 0}$.

Obviously D_L is a linear subspace of X and $L : D_L \longrightarrow X$ is a (not necessarily bounded) densely defined and closed linear operator. Now, to any Markov family

$$(X, P) = \big(X = id_{D(\mathbb{R}_+,M)}, \{P^x \in \mathbb{P}D(\mathbb{R}_+, M)\}_{x\in M}\big)$$

is associated the Markov semigroup $P_t : M \longrightarrow \mathbb{P}M$, $t \geq 0$, given by

$$P_t(x, y) := P^x\{X_t = y\}.$$

As a Markov semigroup, (P_t) induces a contraction semigroup $P_t : B(M) \longrightarrow B(M)$, $t \geq 0$, on the Banach space $B(M)$ and the generator L of this contraction semigroup is called *the generator of* (X, P) *in* $B(M)$.

Proposition 1.1.1 *Let* (X, P) *be a Markov jump process on the state space* M *with bounded non-negative jump rate* $\lambda : M \longrightarrow \mathbb{R}_+$ *and transition probability* $p : M \longrightarrow \mathbb{P}M$*. Then the generator of* (X, P) *is the bounded linear operator* $L : D_L = B(M) \longrightarrow B(M)$ *given by*

$$Lf(x) = \sum_{y\in M} \big(f(y) - f(x)\big)\lambda(x)p(x, y). \tag{1.6}$$

Proof We set $\bar{\lambda} := \sup_{x\in M} \lambda(x)$ the least upper bound of the jump rates suppose that $\bar{\lambda} > 0$, since otherwise we have nothing to prove. We note first that L is indeed bounded since if for any $f \in B(M)$ we have that

$$|Lf(x)| \leq 2\|f\|_u \sum_{y\in M} \lambda(x)p(x,y) \leq 2\bar{\lambda}\|f\|_u.$$

Therefore $\|Lf\|_u \leq 2\bar{\lambda}\|f\|_u$ for all $B(M)$ and thus L is bounded with $\|L\| \leq 2\bar{\lambda}$.

So, as a bounded operator, L induces the contraction semigroup $P_t := e^{tL}$, $t \geq 0$, on $B(M)$ and obviously the generator of $(P_t)_{t\geq 0}$ is L. Therefore it suffices to show that $(P_t)_{t\geq 0}$ coincides with Markov semigroup of (X, P), i.e. that

$$P_t f(x) = \mathbb{E}^x f(X_t)$$

for all $x \in M$, $t \geq 0$ and $f \in B(M)$. To do this we describe the Markov jump process (X, P) in an alternative way. We define the transition probability $\bar{p} : M \longrightarrow \mathbb{P}M$ by

$$\bar{p}(x,y) = \Big(1 - \frac{\lambda(x)}{\bar{\lambda}}\Big)\delta_x(y) + \frac{\lambda(x)}{\bar{\lambda}}p(x,y). \tag{1.7}$$

and let $\bar{P} : M \longrightarrow \mathbb{P}\big(M^{\mathbb{Z}_+} \times D(\mathbb{R}_+, \mathbb{Z}_+)\big)$ such that with respect to $\bar{P}^x$ the natural projections

$$\xi = (\xi_n)_{n\geq 0} : M^{\mathbb{Z}_+} \times D(\mathbb{R}_+, \mathbb{Z}_+) \longrightarrow M^{\mathbb{Z}_+},$$
$$N = (N_t)_{t\geq 0} : M^{\mathbb{Z}_+} \times D(\mathbb{R}_+, \mathbb{Z}_+) \longrightarrow D(\mathbb{R}_+, \mathbb{Z}_+)$$

are independent processes, ξ is a discrete time Markov chain with transition probability $\bar{p}$ starting from x and N is a Poisson process of parameter $\bar{\lambda} > 0$ (starting from 0), for all $x \in M$. Then with respect to $\bar{P}$ the process

$$Y : M^{\mathbb{Z}_+} \times D(\mathbb{R}_+, \mathbb{Z}_+) \longrightarrow D(\mathbb{R}_+, M)$$

defined by $Y_t := \xi_{N_t}$, $t \geq 0$, is a Markov jump process of parameters (p, λ), that is

$$Y \circ \bar{P} = X \circ P.$$

Note that by the definition of $\bar{p}$ we have that $L = \bar{\lambda}(\bar{p} - I)$ where $\bar{p} - I$ is the discrete generator of the chain ξ and therefore if we denote by $\bar{\mathbb{E}}^x$ the expectation with respect to $\bar{P}^x$ we have that

$$\begin{aligned}
\mathbb{E}^x f(X_t) &= \bar{\mathbb{E}}^x f(Y_t) = \sum_{n=0}^{\infty} \bar{\mathbb{E}}^x\big(f(\xi_n)\mathbb{1}_{\{N_t=n\}}\big) = \sum_{n=0}^{\infty} \bar{\mathbb{E}}^x f(\xi_n)\cdot \bar{P}^x\{N_t = n\} \\
&= e^{-\bar{\lambda}t}\sum_{n=0}^{\infty} \bar{p}^n f(x)\frac{(\bar{\lambda}t)^n}{n!} = e^{-\bar{\lambda}t}\sum_{n=0}^{\infty}\frac{1}{n!}(\bar{\lambda}t\bar{p})^n f(x) \\
&= = e^{-\bar{\lambda}t}e^{\bar{\lambda}t\bar{p}}f(x) = e^{tL}f(x) = P_t f(x)
\end{aligned}$$

for all $x \in M$, $t \geq 0$ and $f \in B(M)$ and completes the proof. □

According to the formula (1.6) the generator L of a Markov jump process with bounded jump rate λ and transition probability p on the state space M is given in matrix form by

$$\begin{aligned}\big(L(x,y)\big)_{x,y\in M} &:= \big(L\mathbb{1}_{\{y\}}(x)\big)_{x,y\in M} = \big(\lambda(x)p(x,y) - \lambda(x)\delta_x(y)\big)_{x,y\in M}\\ &= \big(\bar{\lambda}\bar{p}(x,y) - \bar{\lambda}\delta_x(y)\big)_{x,y\in M}.\end{aligned}$$

In matrix form the statement that the generator of (X,P) is the bounded operator $L : B(M) \longrightarrow B(M)$ given in (1.6) can be written as

$$\sup_{x\in M}\left|\frac{P_t(x,y) - \delta_{xy} - tL(x,y)}{t}\right| \overset{t\downarrow 0}{\longrightarrow} 0 \tag{1.8}$$

or in other words

$$P_t(x,y) = \delta_{xy} + tL(x,y) + o(t), \quad \text{uniformly over } x \in M.$$

We note that for each $x \in M$ the number $\lambda(x) = -L(x,x)$ is the rate at which particles leave the site x since

$$\frac{P^x\{X_t \neq x\}}{t} = \frac{1 - P^x\{X_t = x\}}{t} = -\frac{P_t(x,x) - 1}{t} \longrightarrow -L(x,x)$$

and that for $x, y \in M$, $x \neq y$, the number $\lambda(x)p(x,y) = L(x,y)$ is the rate at which the transition $x \mapsto y$ is made, since

$$\frac{P^x\{X_t = y\}}{t} = \frac{P_t(x,y) - \delta_{xy}}{t} \longrightarrow L(x,y).$$

Obviously any generator L of some Markov jump process with bounded jump rate λ and transition probability p satisfies the properties

$$(a)\;\; L(x,y) \geq 0 \quad \forall x \neq y, \quad (b)\;\; L(x,x) \leq 0 \quad \forall\, x \in M, \quad \text{and} \;\; (c)\;\; L\mathbb{1}_M = 0 \tag{1.9}$$

and the function $\lambda(x) = -L(x,x)$ is bounded. Furthermore, since we have required that with respect to p points $x \in M$ of zero jump rate $\lambda(x) = 0$ do not communicate with points of strictly positive jump rate we have by the matrix formula of L that

$$\lambda(x) \neq 0,\; \lambda(y) = 0 \quad \Longrightarrow \quad L(x,y) = 0 \tag{1.10}$$

for all $x, y \in M$. As we shall see next, any $M \times M$ matrix satisfying these properties is the matrix of the bounded generator of some Markov jump process with bounded rates.

Proposition 1.1.2 *Let $L = \big(L(x,y)\big)_{x,y\in M} \in \mathbb{R}^{M\times M}$ be an $M \times M$ matrix satisfying (1.9) and (1.10). Then L defines a bounded operator $L : B(M) \longrightarrow B(M)$ by the formula*

$$Lf(x) = \sum_{y\in M} L(x,y)f(y), \quad x \in M$$

iff the function $\lambda(x) = -L(x,x)$, $x \in M$, is bounded, and in this case L is the generator of a unique (up to distribution) Markov jump process with non-negative bounded jump rates, namely the Markov jump process with non-negative bounded jump rate function λ and transition probability $p : M \longrightarrow \mathbb{P}M$ given by

$$p(x,y) = \frac{L(x,y)}{\lambda(x)}\mathbb{1}_{\{\lambda\neq 0\}}(x)\mathbb{1}_{M\setminus\{x\}}(y) + \delta_x(y)\mathbb{1}_{\{\lambda=0\}}(x) \tag{1.11}$$

Proof Indeed, if L defines a bounded operator on $B(M)$ then we have that

$$|L(x,x)| = |L\mathbb{1}_{\{x\}}(x)| \leq \|L\mathbb{1}_{\{x\}}\|_u \leq \|L\| \cdot \|\mathbb{1}_{\{x\}}\|_u = \|L\|$$

and on the other hand if the function λ is bounded then for all $f \in B(M)$ we have that

$$Lf(x) = \sum_{y \in M: y \neq x} f(y)L(x,y) + f(x)L(x,x) \overset{(b)}{=} \sum_{y \in M: y \neq x} \big(f(y) - f(x)\big)L(x,y)$$

and therefore

$$|Lf(x)| \leq 2\|f\|_u \sum_{y: y \neq x} L(x,y) \overset{(c)}{=} 2\bar{\lambda}\|f\|_u$$

where again $\bar{\lambda} := \sum_{x \in M} \lambda(x)$.

We prove now the second claim. First we note that uniqueness is obvious since in general a contraction semigroup is uniquely determined by its generator. For the existence we note that the function p defined in (1.11) is indeed a transition probability by property (1.9c) of L that obviously satisfies (1.4b) and $p(x, M \setminus \{x\}) = 1$ for all $x \in \{\lambda \neq 0\}$. Now, due to assumption (1.10) we have that (1.4a) is also satisfied and therefore the requirement (1.4) in the definition of Markov jump process of non-negative bounded jump rates is satisfied. Therefore there exists a Markov jump process (X, P) with non-negative bounded jump rate function λ and transition probability p and it remains to show that the generator of (X, P) is L. So, let $\widetilde{L} : B(M) \longrightarrow B(M)$ be the generator of (X, P). Then by the matrix formula of $\widetilde{L}$ we have that

$$\begin{aligned}
\widetilde{L}(x,y) &= \lambda(x)p(x,y) - \lambda(x)\delta_{xy} \\
&= L(x,y)\mathbb{1}_{\{\lambda \neq 0\}}(x)\mathbb{1}_{M\setminus\{x\}}(y) + \lambda(x)\delta_{xy}\mathbb{1}_{\{\lambda = 0\}} - \lambda(x)\delta_{xy} \\
&= L(x,y)\mathbb{1}_{\{\lambda \neq 0\}}(x)\mathbb{1}_{M\setminus\{x\}}(y) - \lambda(x)\delta_{xy}
\end{aligned}$$

and for $y = x$ it follows that $\widetilde{L}(x,x) = -\lambda(x) = L(x,x)$ while for $y \neq x$ we have that

$$\widetilde{L}(x,y) = L(x,y)\mathbb{1}_{\{\lambda \neq 0\}}(x) = L(x,y)$$

since if $\lambda(x) = 0$ then $L(x,y) = 0$ for all $y \in M$, and the proof is complete. □

Remark: The hypothesis in proposition 1.1.2 that the matrix L satisfies (1.10) is needed only to ensure that the kernel p defined in (1.11) satisfies (1.4a) and not just

$$p(x,x) = 0 \quad \forall\, x \in \{\lambda \neq 0\} \tag{1.12}$$

since in the definition of Markov jump processes with non-negative jump rates we have assumed that points of zero jump rate cannot be reached by points of positive jump rate. However assumption (1.10) is not necessary in the definition of Markov jump processes with non-negative bounded jump rates and was made since it simplifies the presentation and is satisfied by the ZR process. In this more general definition of Markov jump processes with non-negative bounded jump rates assumption (1.4a) is replaced by assumption (1.12), Markov jump processes can reach points in $\{\lambda = 0\}$ from points in $\{\lambda \neq 0\}$ and from then on they remain constant, and proposition 1.1.2 holds without the need of assumption (1.10).

Definition 1.1.4 Let M be a countable measurable space with $\{x\}$ measurable for all $x \in M$. A matrix $\big(L(x,y)\big)_{x,y\in M} \in \mathbb{R}^{M\times M}$ such that (1.9) holds and such that the function $\lambda : M \longrightarrow \mathbb{R}_+$ given by $\lambda(x) = -L(x,x)$ is bounded will be called a *bounded Markov jump process generating matrix on M.*

According to proposition 1.1.2 and the subsequent remark, given any countable state space M, there is a bijective correspondence between the set (of distributions) of Markov jump processes with non-negative bounded jump rates on M and the set of bounded Markov jump process generating matrices on M. In other words, in order to define a Markov jump process with non-negative bounded jump rates one needs only specify *admissible transition rates for a Markov jump process with bounded jump rates* i.e. numbers $L(x,y) \geq 0$, $x, y \in M$, $y \neq x$, such that the function $\lambda : M \longrightarrow \mathbb{R}_+$ given by

$$\lambda(x) := \sum_{y:y\neq x} L(x,y) =: -L(x,x)$$

is bounded. Of course then this is the Markov jump process with jump rate function λ and transition probability p given by (1.11) and its generator is the bounded operator $L : B(M) \longrightarrow B(M)$ induced by L.

Moreover when M is equipped with an unbounded discrete metric $d : M\times M \longrightarrow \mathbb{R}_+$ and a distinguished point $o \in M$, a bounded Markov jump process generating matrix $L = \big(L(x,y)\big)_{x,y\in M}$ such that transition probability p associated to L by (1.11) satisfies

$$p|\cdot|^r(x) \equiv \int_M |y|^r dp^x(y) \leq |x|^r, \quad \forall\, x \in M \tag{1.13}$$

for some $r \geq 0$ induces bounded operators in the larger space $B_r(M) \equiv B_{o,r}(M,d)$ of all functions $f : M \longrightarrow \mathbb{R}$ such that there exists $C \geq 0$ such that $|f(x)| \leq C\big(1 + d(o,x)^r\big)$ equipped with the norm

$$\|f\|_{u,r} := \|f\|_{o,u,r} := \sup_{x\in M} \frac{|f(x)|}{1+|x|^r},$$

where for simplicity in notation we set $|x| := d(o,x)$, $|xy| := d(x,y)$, $x, y \in M$. Note that the space $B_r(M)$ as a set does not depend on the base point $o \in M$ while the norm $\|\cdot\|_{u,r}$ does. It is easy to see that the spaces $B_r(M)$ are Banach spaces and by (1.13) L induces a bounded linear operator $L = L_r : B_r(M) \longrightarrow B_r(M)$, since for all $x \in M$ we have that

$$\begin{aligned}
\frac{|Lf(x)|}{1+|x|^r} &\leq \bar{\lambda} \sum_{y\in M} \frac{|f(y)-f(x)|}{1+|x|^r} p(x,y) \leq \bar{\lambda}\|f\|_{u,r} \sum_{y\in M}\left(\frac{1+|y|^r}{1+|x|^r}+1\right)p(x,y) \\
&= \bar{\lambda}\|f\|_{u,r}\left(1+\frac{1}{1+|x|^r}\Big(1+\sum_{y\in M}|y|^r dp(x,y)\Big)\right) \leq 2\bar{\lambda}\|f\|_{u,r}
\end{aligned}$$

and so $\|Lf\|_{u,2} \leq 2\bar{\lambda}\|f\|_{u,2}$. As we shall see, when (1.13) is satisfied for some $r \geq 0$, the Markov semigroup $(P_t)_{t\geq 0}$ of the Markov jump process defined by the generating matrix L induces a contraction semigroup $P_t : B_r(M) \longrightarrow B_r(M)$ in the larger space $B_r(M)$ and the generator of $(P_t)_{t\geq 0}$ in $B_r(M)$ is the bounded operator $L : B_r(M) \longrightarrow B_r(M)$.

We note first that inequality (1.13) for some $r \geq 0$ characterizes the transition probabilities $p : M \longrightarrow \mathbb{P}M$ that induce either linear contractions $p : B_r(M) \longrightarrow B_r(M)$ or linear contractions $p : \mathbb{P}_r M \longrightarrow \mathbb{P}_r M$ with respect to the norm

$$\|\mu\|_{TV,r} := \int (1 + |x|^r) d|\mu|$$

on $\mathcal{M}_r(M)$.

Proposition 1.1.3 *Let $p : M \longrightarrow \mathbb{P}M$ be a transition probability in the polish space M. Then the following are equivalent.*
(a) *p satisfies (1.13) for $r \geq 0$.*
(b) *p induces a linear contraction $p : B_r(M) \longrightarrow B_r(M)$.*
(c) *p induces a linear contraction $p : \mathbb{P}_r M \longrightarrow \mathbb{P}_r M$.*

Proof(a)$\Longrightarrow$(b): Let $f \in B_r(M)$. Then

$$|pf(x)| = \int |f(y)| dp^x(y) \leq \|f\|_{u,r} \int (1 + |y|^r) dp^x(y) \leq \|f\|_{u,r}(1 + |x|^r)$$

for all $x \in M$. Therefore $pf \in B_r(M)$ with $\|pf\|_{u,r} \leq \|f\|_{u,r}$ and so p is a contraction.
(b)$\Longrightarrow$(c) Let $\mu \in \mathbb{P}_r M$. By (b) for any $f \in B_r(M)$ we have that $pf \in B_r(M)$ and so we can define an operator $p : \mathbb{P}_r M \longrightarrow \mathbb{P}M$ by

$$\int f d(\mu p) := \int p f d\mu, \quad \forall\, f \in B_r(M).$$

Then we obviously have that

$$\int |y|^r d(\mu p)(y) = \int p|\cdot|^r(x) d\mu(x) \leq \int |x|^r d\mu(x)$$

and therefore $\mu p \in \mathbb{P}_r M$ and $p : \mathbb{P}_r M \longrightarrow \mathbb{P}_r M$ is a contraction.
(c)$\Longrightarrow$(a) Since $p : \mathbb{P}_r M \longrightarrow \mathbb{P}_r M$ is a contraction for all $x \in M$ we have that

$$p|\cdot|^r(x) = \int |y|^r d(\delta_x p)(y) \leq |x|^r$$

for all $x \in M$ as required. □

Proposition 1.1.4 *Let (X, P) be a Markov jump process with bounded jump rate function $\lambda : M \longrightarrow \mathbb{R}_+$ and transition probability p satisfying (1.13) for $r \geq 0$. Then the semigroup of (X, P) induces a contraction semigroup $P_t : B_r(M) \longrightarrow B_r(M)$ and the generator of (X, P) in $B_r(M)$ i.e. the generator of $(P_t)_{t \geq 0}$ in $B_r(M)$ is the bounded operator $L : B_r(M) \longrightarrow B_r(M)$ given by (1.6).*

Proof We note first that the transition probability $\bar{p} : M \longrightarrow \mathbb{P}M$ defined by (1.7) satisfies (1.13). Indeed, for all $x \in M$ we have that

$$\bar{p}|\cdot|^r(x) = \int |y|^r d\bar{p}^x = \Big(1 - \frac{\lambda(x)}{\bar{\lambda}}\Big)|x|^r + \frac{\lambda(x)}{\bar{\lambda}} \int |y|^r p^x(y) \leq |x|^r$$

for all $x \in M$.

We show next that $\mathbb{E}|X_t|^r = P_t|\cdot|^r \in B_r(M)$. We describe again (X, P) as a the Markov jump process $(Y, \bar{P})$ where $\bar{P} : M \longrightarrow \mathbb{P}\big(M^{\mathbb{Z}_+} \times D(\mathbb{R}_+, \mathbb{Z}_+)\big)$ is a kernel such that with respect to $\bar{P}^x$ the natural projections

$$\xi = (\xi_n)_{n\geq 0} : M^{\mathbb{Z}_+} \times D(\mathbb{R}_+, \mathbb{Z}_+) \longrightarrow M^{\mathbb{Z}_+},$$

$$N = (N_t)_{t\geq 0} : M^{\mathbb{Z}_+} \times D(\mathbb{R}_+, \mathbb{Z}_+) \longrightarrow D(\mathbb{R}_+, \mathbb{Z}_+)$$

are independent processes, ξ is a discrete time markov chain with transition probability $\bar{p}$ starting from x and N is a Poisson process of parameter $\bar{\lambda} > 0$ (starting from 0), for all $x \in M$. Then obviously $\bar{p}^k|\cdot|^r \leq \bar{p}^{k-1}\bar{p}|\cdot|^r \leq \bar{p}^{k-1}|\cdot|^r \leq \cdots \leq |\cdot|^r$ and therefore

$$\begin{aligned}
\mathbb{E}|X_t|^r &= \bar{\mathbb{E}}|Y_t|^r = \sum_{k=0}^{\infty} \bar{\mathbb{E}}\big(|\xi_k|^r \mathbb{1}_{\{k\}}(N_t)\big) = \sum_{k=0}^{\infty} \bar{P}(N_t = k)\bar{\mathbb{E}}|\xi_k|^r \\
&= e^{-\bar{\lambda}t} \sum_{k=0}^{\infty} \frac{(\bar{\lambda}t)^k}{k!} \bar{p}^k |\cdot|^r \leq |\cdot|^r.
\end{aligned}$$

So for all $f \in B_r(M)$ all $x \in M$ we have that

$$|P_t f(x)| \leq \bar{\mathbb{E}}^x |f(Y_t)| \leq \|f\|_{u,r} \bar{\mathbb{E}}^x (1 + |Y_t|^r) \leq \|f\|_{u,r}(1 + |x|^r)$$

and thus $\|P_t f\|_{u,r} \leq \|f\|_{u,r}$ for all $f \in B_r(M)$. Therefore P_t is a contraction and $(P_t)_{t\geq 0}$ is a contraction semigroup in the Banach space $B_r(M)$.

We show finally that the generator of (P_t) is $L : B_r(M) \longrightarrow B_r(M)$. Since L is a bounded operator it induces the semigroup $(Q_t)_{t\geq 0} := (e^{tL})_{t\geq 0}$ on $B_r(M)$ and we have to prove that $(Q_t)_{t\geq 0} = (P_t)_{t\geq 0}$. But as we have seen $L = \bar{\lambda}(\bar{p} - I)$ and therefore for all $f \in B_r(M)$ and $x \in M$ we have that

$$P_t f(x) = \sum_{k=0}^{\infty} \bar{\mathbb{E}}^x f(\xi_k) P(N_t = k) = e^{-\bar{\lambda}t} \sum_{k=0}^{\infty} \frac{(\bar{\lambda}t)^k}{k!} \bar{p}^k f(x) = e^{tL} f(x) = Q_t f(x). \qquad \square$$

Finally it is useful to remark that if a kernel satisfies (1.13) for some $r > 0$ then it also satisfies (1.13) for all orders less than r.

Proposition 1.1.5 *Let $p : M \longrightarrow \mathbb{P}M$ be a transition kernel satisfying (1.13) for some $r > 0$. Then p satisfies (1.13) for all $0 \leq q < r$.*

Proof Indeed, as we know for probability spaces $(\Omega, \mathcal{F}, P)$ the function and measurable functions $f \in (\Omega, \mathcal{F}, P)$ the function

$$[0, +\infty) \ni r \mapsto \|f\|_{L^r(P)}$$

is an increasing function of r and therefore if p satisfies (1.13) then for all $0 \leq q \leq r$ and all $x \in M$ we have that

$$p|\cdot|^q(x) = \left(\int |y|^q dp_x(y)\right)^{\frac{q}{q}} \leq \left(\int |y|^r dp_x(y)\right)^{\frac{q}{r}} = \big(p|\cdot|^r(x)\big)^{\frac{q}{r}} \leq \big(|x|^r\big)^{\frac{q}{r}} = |x|^q,$$

that is (1.13) holds for all $0 \leq q \leq r$ as required. $\square$

We turn now to the definition of the Zero Range process.

Definition 1.1.5 Any function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ such that

$$g(0) = 0, \quad g(k) > 0, \quad \forall k \in \mathbb{N}, \tag{1.14}$$

$$\|g'\|_u := \sup_{k\in\mathbb{Z}_+} |g(k+1) - g(k)| < +\infty \tag{1.15}$$

will be called a *local jump rate function.*

Let now $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ be a local jump rate function and $p : \mathbb{Z}^d \longrightarrow \mathbb{P}\mathbb{Z}^d$ be a translation invariant and irreducible transition probability. As we have already stated, we would like to define the Zero Range processes on the lattice $\mathbb{T}_N^d$, $N \in \mathbb{N}$, with local jump rate function g and elementary step distribution p as a Markov jump process on the state space $\mathbb{M}_N^d$ of all possible configurations of particles on the discrete torus $\mathbb{T}_N^d$ in a way that it models a system of particles evolving according to the following stochastic dynamics: Given an initial configuration of particles $\eta_0 \in \mathbb{M}_N^d$ at time zero, an exponential alarm clock with parameter $g(\eta_{0,x})$ starts at each site $x \in \mathbb{T}_N^d$. At the first time τ_1 that an alarm clock rings at a non-empty site x, i.e. with $\eta_{0,x} \neq 0$, a particle leaves the site x at which the clock rang and moves instantly to the site $y \in \mathbb{T}_N^d$ with probability $p_N(x,y) = p(y - x + N\mathbb{Z}^d)$, and thus the new configuration $\eta_1 := \eta_0^{x,y}$ is created with probability $p_N(x,y)$. Then the parameters of the clocks are updated to the new values $g(\eta_{1,x})$, $x \in \mathbb{T}_N$, and the process goes on to yield a path

$$\mathbb{R}_+ \in t \mapsto \eta_t^N := \sum_{k=0}^{\infty} \eta_k \mathbb{1}_{[\tau_{k-1},\tau_k)}(t)$$

in the space of configurations $\mathbb{M}_N^d$. Of course due to the memory loss property of the exponential distribution one can assume if he likes that clocks are reset during the parameter update process. Note that in the stochastic dynamics described above preserve the total number of particles $|\eta_0|$ of the initial configuration.

According to the stochastic dynamics described above, the rate at which the transition $\eta \mapsto \eta^{x,y}$, $x, y \in \mathbb{T}_N^d$, $y \neq x$, is made is $L^N(\eta, \eta^{x,y}) = g(\eta_x)p(x,y)$ and a transition $\eta \mapsto \zeta$ cannot be made of ζ is not of the form $\zeta = \eta^{x,y}$ for some $x, y \in \mathbb{T}_N^d$. Therefore we would like to define the Zero Range process as a Markov jump process

$$(\eta^N, \mathbb{P}_N) := \big(\eta^N := id_{D(\mathbb{R}_+,\mathbb{M}_N^d)}, \{\mathbb{P}_N^{\eta_0} \in \mathbb{P}D(\mathbb{R}_+, \mathbb{M}_N^d)\}_{\eta_0\in\mathbb{M}_N^d}\big)$$

by specifying the transition rates

$$L^N(\eta,\zeta) = \begin{cases} g(\eta_x)p_N(y), & \zeta = \eta^{x,x+y},\ \eta_x \neq 0,\ y \neq 0 \\ 0, & \text{else} \end{cases}, \quad \eta \neq \zeta. \tag{1.16}$$

Then the jump rate function $\lambda : \mathbb{M}_N^d \longrightarrow \mathbb{R}_+$ of $(\eta^N, \mathbb{P}_N)$ would be given by

$$\lambda_N(\eta) := -L^N(\eta,\eta) := -\sum_{\zeta:\zeta\neq\eta} L(\eta,\zeta) = p_N(\mathbb{T}_N^d \setminus \{0\}) \sum_{x\in\mathbb{T}_N^d} g(\eta_x). \tag{1.17}$$

Note that since the support of p spans $\mathbb{Z}^d$ we have that $p_N(\mathbb{T}_N \setminus \{0\}) > 0$ for $N \geq 2$ and therefore $\lambda_N(\eta) = 0$ iff $\eta = \mathbf{0} \in \mathbb{M}_N^d$ is the configuration with no particles. Since p is assumed to have finite range $\leq A$ for $N > A$ we have that

$$p_N(\mathbb{T}_N^d \setminus \{0\}) = p(\mathbb{Z}^d \setminus N\mathbb{Z}^d) = p([-A, A]^d \setminus N\mathbb{Z}^d) = p([-A, A]^d) = 1$$

and since we are interested in large values of N we will omit the term $p_N(\mathbb{T}_N^d \setminus \{0\})$ in the formula for $L^N(\eta, \eta)$, $\eta \in \mathbb{M}_N^d$. The transition probability $\mathbb{p}_N : \mathbb{M}_N^d \longrightarrow \mathbb{P}\mathbb{M}_N^d$ of $(\eta^N, \mathbb{P}_N)$ would then be given by $\mathbb{p}_N(\mathbf{0}, \cdot) = \delta_{\mathbf{0}}$ and

$$\mathbb{p}_N(\eta, \zeta) = \begin{cases} \frac{L(\eta,\eta^{x,y})}{\lambda_N(\eta)} & \text{if } \zeta = \eta^{x,y},\ x \in \mathbb{T}_N^d,\ \eta_x \neq 0,\ y \neq x \\ 0 & \text{otherwise} \end{cases}. \tag{1.18}$$

for all $\eta \neq \mathbf{0}$.

However this does not fit exactly to the definition of Markov jump process with bounded jump rates since λ_N is not bounded, unless the local jump rate function g is bounded. Of course when g is bounded we have that

$$\bar{\lambda}_N := \sup_{\eta \in \mathbb{M}_N^d} \lambda_N(\eta) \leq N^d \|g\|_u < +\infty.$$

Nevertheless we can still define a Markov process with jump rate function λ_N and transition kernel $\mathbb{p}_N$ if we note that λ_N is bounded on the communication classes of the transition probability $\mathbb{p}_N$. Indeed, since the dynamics of the ZRP preserve the total number of particles we can easily see that the communication classes of the generator L^N, or equivalently of the transition probability $\mathbb{p}_N$, are the hyperplanes

$$\mathbb{M}_{N,K}^d := \big\{\eta \in \mathbb{M}_N^d \,\big|\, |\eta|_1 = K\big\}, \quad K \in \mathbb{Z}_+$$

consisting of configurations with a fixed number of particles and so λ_N is bounded on each hyperplane $\mathbb{M}_{N,K}^d$, since by assumption (1.15) we have that $g(k) \leq \|g'\|_u k$ for all $k \in \mathbb{Z}_+$ and therefore

$$\sup_{\eta \in \mathbb{M}_{N,K}^d} \lambda_N(\eta) = \sup_{\eta \in \mathbb{M}_{N,K}^d} \sum_{x \in \mathbb{T}_N^d} g(\eta_x) \leq \sup_{\eta \in \mathbb{M}_{N,K}^d} \|g'\|_u |\eta|_1 = \|g'\|_u K$$

for all $N \in \mathbb{N}$, $K \in \mathbb{Z}_+$. In this way L^N defines bounded generating matrices

$$\big(L^{N,K}(\eta, \zeta)\big)_{\eta,\zeta \in \mathbb{M}_{N,K}^d} := \big(L^N(\eta, \zeta)\big)_{\eta,\zeta \in \mathbb{M}_{N,K}^d} \in \mathbb{R}^{\mathbb{M}_{N,K}^d \times \mathbb{M}_{N,K}^d}$$

and therefore for all $K \geq 1$ we can define a Markov jump process $(\eta^{N,K}, \mathbb{P}_{N,K})$ separately on each communication class $\mathbb{M}_{N,K}^d$ of $\mathbb{p}^N$, with bounded positive jump rate $\lambda_{N,K} := \lambda_N|_{\mathbb{M}_{N,K}^d} : \mathbb{M}_{N,K}^d \longrightarrow (0, \infty)$ and transition kernel $\mathbb{p}_{N,K} : \mathbb{p}_N|_{\mathbb{M}_{N,K}^d} : \mathbb{M}_{N,K}^d \longrightarrow \mathbb{P}\mathbb{M}_{N,K}^d$, where $\mathbb{P}_{N,K} : \mathbb{M}_{N,K}^d \longrightarrow \mathbb{P}D(\mathbb{R}_+, \mathbb{M}_{N,K}^d)$ and $\eta^{N,K}$ is the canonical cadlag process, and consider the process $\mathbb{P}_N : \mathbb{M}_N^d \longrightarrow \mathbb{P}D(\mathbb{R}_+, \mathbb{M}_N^d)$ defined by

$$\mathbb{P}_N^\eta = \begin{cases} \delta_{(\eta_t \equiv \mathbf{0})_{t \geq 0}}, & \text{if } \eta = \mathbf{0} \\ \bar{i}_{K*} P_{N,K}^\eta & \text{if } \eta \in \mathbb{M}_{N,K},\ K \geq 1. \end{cases}$$

Then the process $\mathbb{P}_N$ defined this way is obviously a cadlag feller process and we consider it as a Markov jump process with unbounded rates, according to the following more general definition.

Definition 1.1.6 (a) Let $\lambda : M \longrightarrow \mathbb{R}_+$ be a measurable non-negative bounded function and let $p : M \longrightarrow \mathbb{P}M$ be a Markov kernel such that (1.4) holds and λ is bounded on the communication classes $\{M_j\}_{j\in\mathbb{N}}$ of p. The Markov kernel $P : M \longrightarrow \mathbb{P}D(\mathbb{R}_+, M)$ given by

$$P^x = \bar{i}_{j*}P_j^x \quad \text{if } x \in M_j,\ j \in \mathbb{N},$$

where $P_j : M_j \longrightarrow \mathbb{P}D(\mathbb{R}_+, M_j)$ is the Markov jump process on M_j with transition probability $p_j : M_j \longrightarrow \mathbb{P}M_j$ and strictly positive bounded jump rate function $\lambda_j := \lambda|_{M_j}$ and $i_j : M_j \hookrightarrow M$ is the natural inclusion, is called the *Markov jump process with partially bounded non-negative jump rate* λ *and transition probability* p.
(b) A matrix $\big(L(x,y)\big)_{x,y\in M} \in \mathbb{R}^{M\times M}$ such that (1.9) holds and such that the function $\lambda : M \longrightarrow \mathbb{R}_+$ given by $\lambda(x) = -L(x,x)$ is bounded on the communication classes of L will be called a *partially bounded Markov jump process generating matrix on* M.

Obviously, to any partially bounded Markov jump process generating matrix L on M corresponds a unique Markov jump process (X,P) with partially bounded non-negative jump rate $\lambda : M \longrightarrow \mathbb{R}_+$ and transition probability $p : M \longrightarrow \mathbb{P}M$ given by (1.11). Of course the uniqueness of (X,P) follows from the fact that (X,P) is uniquely determined on each of the communication classes M_j by proposition 1.1.2. We can give now the following definition for Zero Range Processes.

Definition 1.1.7 Let $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ be a local rate function and $p : \mathbb{Z}^d \longrightarrow \mathbb{P}\mathbb{Z}^d$ be a translation invariant and irreducible transition probability. The Markov jump process on the state space $\mathbb{M}_N^d$ defined by the partially bounded Markov jump process generating matrix L^N on $\mathbb{M}_N^d$ given by the transition rates (1.16) *is called the zero range process with elementary step distribution* p *and jump rate function* g*, or simply zero range process with parameters* (p,g).

Of course then the ZR process (X,P) is the Markov jump process on $\mathbb{M}_N^d$ with partially bounded jump rate function $\lambda_N : \mathbb{M}_N^d \longrightarrow \mathbb{R}_+$ given by (1.17) and transition probability $\mathbb{p}_N : \mathbb{M}_N^d \longrightarrow \mathbb{P}\mathbb{M}_N^d$ given by (1.18). However, in this case, unless λ_N is bounded, the generator of (X,P) in $B(\mathbb{M}_N^d)$ is not a bounded operator. In particular in general we have that $D_L \neq B(\mathbb{M}_N^d)$. However note that for each $K \in \mathbb{Z}_+$ the formula of the generator L^N defines bounded generators $L^N : B(\mathbb{M}_{N,K}^d) \longrightarrow B(\mathbb{M}_{N,K}^d)$.

In order to have as large as possible domain for the generator L^N of (X,P) we consider the generator in the spaces $B_r(\mathbb{M}_N^d)$, $r \in \mathbb{R}$. For this we note that the communication classes of (X,P) coincide with the spheres of the $|\cdot| \equiv |\cdot|_1$ norm on $\mathbb{M}_N^d$, and therefore for each $r \in \mathbb{R}$ we have that

$$\begin{aligned}
\mathbb{p}_N(1+|\cdot|_1^{|r|})^{\mathrm{sgn} r}(\eta) &= \sum_{\zeta\in\mathbb{M}_N^d}(1+|\zeta|_1^{|r|})^{\mathrm{sgn} r}\mathbb{p}_N(\eta,\zeta)\\
&= \sum_{\zeta\in\mathbb{M}_{N,|\eta|_1}^d}(1+|\zeta|_1^{|r|})^{\mathrm{sgn} r}\mathbb{p}_N(\eta,\zeta) = (1+|\eta|_1^{|r|})^{\mathrm{sgn} r}.
\end{aligned}$$

So (X,P) induces a transition semigroup on the Banach space $B_r(\mathbb{M}_N^d)$ and we can speak about the generator $L_r^N : D_{L_r^N} \longrightarrow B_r(\mathbb{M}_N^d)$ of (X,P) in $B_r(\mathbb{M}_N^d)$ for all $r \in \mathbb{R}$.

Proposition 1.1.6 *Let (X,P) be the ZR process with elementary step distribution p and jump rate function g on $\mathbb{M}_N^d$. Then for all $r \in \mathbb{R}$ the operator $L^N \equiv L_r^N$ in $B_r(\mathbb{M}_N^d)$ given by the formula*

$$L^N f(\eta) = \sum_{x,y\in\mathbb{T}_N^d} \big[f(\eta^{x,y}) - f(\eta)\big] g(\eta_x) p_N(x,y) \tag{1.19}$$

defines a bounded operator $L^N : B_{r-1}(\mathbb{M}_N^d) \longrightarrow B_r(\mathbb{M}_N^d)$ with $\|L^N\| \le 4\|g'\|_u$ and the generator L_r^N of (X,P) in $B_r(\mathbb{M}_N^d)$ is given in its domain $D_{L_r^N} \subseteq B_r(\mathbb{M}_N^d)$ by (1.19) and $e^{-|\cdot|}B_{r-2}(\mathbb{M}_N^d) \subseteq D_{L_r^N}$ for all $r \in \mathbb{R}$. Furthermore, the generator of (X,P) in $B_r(\mathbb{M}_N^d)$ with respect to the topology of uniform convergence on bounded subsets contains the operator $L_N : B_{r-1}(\mathbb{M}_N^d) \longrightarrow B_r(\mathbb{M}_N^d)$, that is

$$\frac{(P_t - I)f}{t} \xrightarrow{t\downarrow 0} L^N f$$

uniformly on bounded subsets of $\mathbb{M}_N^d$ for all $f \in B_{r-1}(\mathbb{M}_N^d)$.

Proof Let $r \in \mathbb{R}$. We show first that L^N defines a bounded operator from $B_{r-1}(\mathbb{M}_N^d)$ to $B_r(\mathbb{M}_N^d)$. So let $f \in B_{r-1}(\mathbb{M}_N^d)$. Then, we have that

$$\begin{aligned}
|L^N f(\eta)| &\le \sum_{x,y\in\mathbb{T}_N^d} \big|f(\eta^{x,x+y}) - f(\eta)\big| g(\eta_x) p_N(y) \\
&\le 2\|f\|_{u,r-1} \sum_{x,y\in\mathbb{T}_N^d} (1+|\eta|^{|r-1|})^{\mathrm{sgn(r-1)}} g(\eta_x) p_N(y) \\
&\le 2\|f\|_{u,r-1}\|g'\|_u (1+|\eta|^{|r-1|})^{\mathrm{sgn(r-1)}} |\eta|.
\end{aligned}$$

But if $r \ge 1$ then we have that

$$(1+|\eta|^{|r-1|})^{\mathrm{sgn(r-1)}}|\eta| = (1+|\eta|^{r-1})|\eta| \le 2(1+|\eta|^r) = 2(1+|\eta|_1^{|r|})^{\mathrm{sgnr}}$$

since $\sup_{K\in\mathbb{Z}_+} \frac{K+K^r}{1+K^r} \le 2$. On the other hand, if $0 \le r < 1$ then

$$(1+|\eta|^{|r-1|})^{\mathrm{sgn(r-1)}}|\eta| = \frac{|\eta|}{1+|\eta|^{1-r}} \le 1+|\eta|^r = (1+|\eta|_1^{|r|})^{\mathrm{sgnr}}$$

since $\sup_{K\in\mathbb{Z}_+} \frac{K}{1+K^r+K^{1-r}+K} \le 1$ while if $r < 0$ then

$$(1+|\eta|^{|r-1|})^{\mathrm{sgn(r-1)}}|\eta| = \frac{|\eta|}{1+|\eta|^{1-r}} \le 2\frac{1}{1+|\eta|^{-r}} = 2(1+|\eta|_1^{|r|})^{\mathrm{sgnr}}.$$

It follows that

$$|L^N f(\eta)| \le 4\|f\|_{u,r-1}\|g'\|_u (1+|\eta|_1^{|r|})^{\mathrm{sgnr}}$$

for all $\eta \in \mathbb{M}_N^d$ and therefore $L^N f \in B_r(\mathbb{M}_N^d)$ and formula (1.19) defines a bounded operator $L^N : B_{r-1}(\mathbb{M}_N^d) \longrightarrow B_r(\mathbb{M}_N^d)$ with $\|L_r^N\| \le 4\|g'\|_u$.

We prove next that $e^{-|\cdot|_1}B_{r-2}(\mathbb{M}_N^d) \subseteq D_{L_r^N}$ by showing that given $f \in e^{-|\cdot|_1}B_{r-2}(\mathbb{M}_N^d)$ we have that

$$\lim_{t\downarrow 0} \left\| \frac{(P_t - I)f}{t} - L^N f \right\|_{u,r} = 0.$$

Indeed, if $f \in e^{-|\cdot|_1} B_{r-2}(\mathbb{M}_N^d)$ then $e^{|\cdot|_1} f \in B_{r-2}(\mathbb{M}_N^d)$ and therefore there exists a constant $C \geq 0$ such that

$$|f(\eta)| \leq C(1+|\eta|^{|r-2|})^{\mathrm{sgn(r-2)}} e^{-|\eta|}$$

for all $\eta \in \mathbb{M}_N^d$, and so it follows that for all $\eta \in \mathbb{M}_N^d$ we have

$$\begin{aligned}
\left|\frac{(P_t - I)f(\eta) - tL^N f(\eta)}{t}\right| &= \left|\frac{\sum_{x,y\in\mathbb{T}_N^d}[f(\eta^{x,y}) - f(\eta)](P_t(\eta,\eta^{x,y}) - tL^N(\eta,\eta^{x,y}))}{t}\right| \\
&\leq \sum_{\eta_x\neq 0,\ y\neq x} \left|f(\eta^{x,y}) - f(\eta)\right| \left|\frac{P_t(\eta,\eta^{x,y}) - tL^N(\eta,\eta^{x,y})}{t}\right| \\
&\leq 2C\|f\|_{u,r-1}(1+|\eta|^{|r-2|})^{\mathrm{sgn(r-2)}} e^{-|\eta|} \times \\
&\quad \times \sum_{x,y\in\mathbb{T}_N^d:\eta_x\neq 0,\ y\neq x} \left|\frac{P_t(\eta,\eta^{x,y}) - tL^N(\eta,\eta^{x,y})}{t}\right|.
\end{aligned}$$

But on each communication class $\mathbb{M}_{N,K}^d$ we have that $P_t \equiv e^{tL^N}$ and therefore if we set $Q_t := \frac{1}{t}(P_t - tL^N)$, then for all $\eta \in \mathbb{M}_N^d$, for all $x, y \in \mathbb{T}_N^d$ such that $\eta_x \neq 0$, $x \neq y$,

$$\begin{aligned}
\left|Q_t(\eta,\eta^{x,y})\right| &= \left|\frac{P_t(\eta,\eta^{x,y}) - tL^N(\eta,\eta^{x,y})}{t}\right| = \frac{1}{t}\left|\sum_{k=2}^{\infty}\frac{t^k}{k!}(L^N)^k(\eta,\eta^{x,y})\right| \\
&\leq \frac{1}{t}\sum_{k=2}^{\infty}\frac{t^k}{k!}\sum_{\zeta_1,\dots,\zeta_{k-1}\in\mathbb{M}_{N,|\eta|}^d}\prod_{i=1}^{k}\left|L^N(\zeta_{i-1},\zeta_i)\right| \\
&= \frac{1}{t}\sum_{k=2}^{\infty}\frac{t^k}{k!}\sum_{\zeta_1,\dots,\zeta_{k-1}\in\mathbb{M}_{N,|\eta|}^d}\prod_{\substack{i=1,\\ \zeta_{i-1}=\zeta_i}}^{k}\lambda_N(\zeta_{i-1})\prod_{\substack{i=1,\\ \zeta_{i-1}\neq\zeta_i}}^{k}L^N(\zeta_{i-1},\zeta_i) \\
&= \frac{1}{t}\sum_{k=2}^{\infty}\frac{t^k}{k!}\sum_{\zeta_1,\dots,\zeta_{k-1}\in\mathbb{M}_{N,|\eta|}^d}\prod_{i=1}^{k}\lambda_N(\zeta_{i-1})\prod_{\substack{i=1,\\ \zeta_{i-1}\neq\zeta_i}}^{k}\mathbb{p}_N(\zeta_{i-1},\zeta_i) \\
&\leq \frac{1}{t}\sum_{k=2}^{\infty}\frac{(t\|g'\|_u|\eta|)^k}{k!}\sum_{\zeta_1,\dots,\zeta_{k-1}\in\mathbb{M}_{N,|\eta|}^d}\prod_{\substack{i=1,\\ \zeta_{i-1}\neq\zeta_i}}^{k}\mathbb{p}_N(\zeta_{i-1},\zeta_i) \\
&= \frac{1}{t}\sum_{k=2}^{\infty}\frac{(t\|g'\|_u|\eta|)^k}{k!}\mathbb{p}_N^k(\eta,\eta^{x,y})
\end{aligned}$$

where of course in the above formulas $\zeta_0 = \eta$ and $\zeta_k = \eta^{x,y}$. So we have that

$$\begin{aligned}
\sum_{\substack{x,y\in\mathbb{T}_N^d,\\ \eta_x\neq 0,\ y\neq x}}\left|Q_t(\eta,\eta^{x,y})\right| &\leq \sum_{\substack{x,y\in\mathbb{T}_N^d,\\ \eta_x\neq 0,\ y\neq x}}\frac{1}{t}\sum_{k=2}^{\infty}\frac{(t\|g'\|_u|\eta|)^k}{k!}\mathbb{p}_N^k(\eta,\eta^{x,y}) \\
&\leq \frac{1}{t}\sum_{k=2}^{\infty}\frac{(t\|g'\|_u|\eta|)^k}{k!} \leq \|g'\|_u|\eta|\sum_{k=1}^{\infty}\frac{(t\|g'\|_u|\eta|)^k}{k!} \\
&= \|g'\|_u|\eta|(e^{t\|g'\|_u|\eta|} - 1)
\end{aligned}$$

for all $\eta \in \mathbb{M}_N^d$.

It follows that for all $\eta \in \mathbb{M}_N^d$ we have the inequality

$$\left| \frac{(P_t - I)f(\eta) - tL^N f(\eta)}{t} \right| \leq 4C\|f\|_{u,r-1}\|g'\|_u (1 + |\eta|^{|r-1|})^{\mathrm{sgn(r-1)}} e^{-|\eta|} (e^{t\|g'\|_u|\eta|} - 1).$$

But then for all $t < \frac{1}{\|g'\|_u}$ we have that $-|\eta| < t\|g'\|_u|\eta|$ and therefore

$$\begin{aligned} \left| \frac{(P_t - I)f(\eta) - tL^N f(\eta)}{t} \right| &\leq 4C\|f\|_{u,r-1}\|g'\|_u (1 + |\eta|^{|r-1|})^{\mathrm{sgn(r-1)}} (1 - e^{-t\|g'\|_u|\eta|}) \\ &\leq 4tC\|f\|_{u,r-1}\|g'\|_u^2 (1 + |\eta|^{|r-1|})^{\mathrm{sgn(r-1)}} |\eta| \\ &\leq 8tC\|f\|_{u,r-1}\|g'\|_u^2 (1 + |\eta|^{|r|})^{\mathrm{sgnr}} \end{aligned}$$

for all $\eta \in \mathbb{M}_N^d$ and all $0 < t < \frac{1}{\|g'\|_u}$. Therefore, for all $t < \frac{1}{\|g'\|_u}$ we have that

$$\left\| \frac{(P_t - I)f}{t} - L^N f \right\|_{u,r} \leq 8tC\|f\|_{u,r-1}\|g'\|_u^2 \xrightarrow{t \downarrow 0} 0,$$

which proves that $e^{-|\cdot|}B_{r-2}(\mathbb{M}_N^d) \subseteq D_{L_r^N}$.

We prove next that the generator of (X, P) in $B_r(\mathbb{M}_N^d)$ with respect to uniform convergence on bounded subsets contains the operator $L^N : B_{r-1}(\mathbb{M}_N^d) \longrightarrow B_r(\mathbb{M}_N^d)$. Indeed, let $f \in B_r(\mathbb{M}_N^d)$ such that $L^N f \in B_r(\mathbb{M}_N^d)$. Then by the previous estimates we have that

$$\left| \frac{(P_t - I)f(\eta) - tL^N f(\eta)}{t} \right| \leq 4C\|f\|_{u,r}\|g'\|_u (1 + |\eta|^{|r|})^{\mathrm{sgnr}} (e^{t\|g'\|_u|\eta|} - 1)$$

and therefore if $A \subseteq \mathbb{M}_N^d$ is bounded, that is if $A \subseteq \{|\cdot|_1 \leq K\}$ for some $K \in \mathbb{Z}_+$, then

$$\sup_{\eta \in A} \left| \frac{(P_t - I)f(\eta) - tL^N f(\eta)}{t} \right| \leq 4C\|f\|_{u,r}\|g'\|_u (1 + K^{|r|})(e^{tK\|g'\|_u} - 1) \xrightarrow{t \downarrow 0} 0,$$

as required. Furthermore this implies that the generator L_r^N of (X, P) in $B_r(\mathbb{M}_N^d)$ is given on its domain $D_{L_r^N}$ by formula (1.19) since convergence in the $\|\cdot\|_{u,r}$-norm implies uniform convergence on bounded subsets. $\square$

By general Markov theory we have that for each $K \in \mathbb{N}$ there exists a unique extremal invariant distribution $\nu_{N,K}^d$ for L^N, $\nu_{N,K}^d L^N = 0$, supported by $\mathbb{M}_{N,K}^d$. Of equal interest is the monoparametric family of translation invariant equilibrium distributions defined on the whole space $\mathbb{M}_\infty^d := \mathbb{Z}_+^{\mathbb{Z}^d}$ which we discuss in the next section.

1.2 The extremal translation invariant equilibrium distributions

We set $\tau_x : \mathbb{M}_N^d \longrightarrow \mathbb{M}_N^d$, $x \in \mathbb{T}_N^d$, the translation operator given by $(\tau_x\eta)_y = \eta_{x+y}$ and recall that a probability measure $\mu \in \mathbb{P}\mathbb{M}_N^d$ is called translation invariant if $\tau_{x*}\mu = \mu$ for all $x \in \mathbb{T}_N^d$. Obviously any product measure with equal factors some common measure $\alpha \in \mathbb{P}\mathbb{Z}_+$ is translation invariant, since if, say $\nu = \alpha^{\otimes \mathbb{T}_N^d} \in \mathbb{P}\mathbb{M}_N^d$ and $x \in \mathbb{T}_N^d$, then

$$\tau_{x*}\nu(\eta) = \nu(\tau_x\eta) = \prod_{y \in \mathbb{T}_N^d} \alpha\big((\tau_x\eta)_y\big) = \prod_{y \in \mathbb{T}_N^d} \alpha(\eta_{x+y}) = \prod_{y \in \mathbb{T}_N^d} \alpha(\eta_y) = \nu(\eta).$$

We want to check whether there exists a product equilibrium distribution $\nu \in \mathbb{P}_1\mathbb{M}_N^d$ for the ZRP. Since the elementary transition probability is translation invariant we expect the marginals $\alpha_x := \eta(x)_*\nu \in \mathbb{P}_1\mathbb{Z}_+$, $x \in \mathbb{T}_N^d$, to be all equal to some distribution $\alpha \in \mathbb{P}_1\mathbb{Z}_+$, and of course then ν is translation invariant. So suppose that $\nu \in \mathbb{P}\mathbb{M}_N^d$ a translation invariant product equilibrium distribution for the ZRP with parameters (p, g), of the form $\nu = \alpha^{\otimes \mathbb{T}_N^d}$ for some $\alpha \in \mathbb{P}_1\mathbb{Z}_+$. Then $\nu L^N = 0$ and so for any $k \in \mathbb{Z}_+$,

$$\int L^N\big[\mathbb{1}_{\{k\}}\big(\eta(x)\big)\big]d\nu = \int \mathbb{1}_{\{\eta(x)=k\}} d\nu L^N = 0, \tag{1.20}$$

and for all $\eta \in \mathbb{M}_N^d$,

$$\begin{aligned}
L^N\big[\mathbb{1}_{\{k\}}\big(\eta(x)\big)\big](\eta) &= \sum_{y,z \in \mathbb{T}_N^d} \big[\mathbb{1}_{\{\eta(x)=k\}}(\eta^{y,y+z}) - \mathbb{1}_{\{\eta(x)=k\}}(\eta)\big] g(\eta_y) p_N(z) \\
&= \sum_{z \in \mathbb{T}_N^d} \big[\mathbb{1}_{\{\eta(x)=k\}}(\eta^{x,x+z}) - \mathbb{1}_{\{\eta(x)=k\}}(\eta)\big] g(\eta_x) p_N(z) \\
&\quad + \sum_{y \neq x} \big[\mathbb{1}_{\{\eta(x)=k\}}(\eta^{y,x}) - \mathbb{1}_{\{\eta(x)=k\}}(\eta)\big] g(\eta_y) p_N(x-y) \\
&= \sum_{z \in \mathbb{T}_N^d} \big[\mathbb{1}_{\{\eta(x)=k+1\}}(\eta) - \mathbb{1}_{\{\eta(x)=k\}}(\eta)\big] g(\eta_x) p_N(z) \\
&\quad + \sum_{y \neq x} \big[\mathbb{1}_{\{\eta(x)=k-1\}}(\eta) - \mathbb{1}_{\{\eta(x)=k\}}(\eta)\big] g(\eta_y) p_N(x-y) \\
&= \big(\mathbb{1}_{\{\eta(x)=k+1\}} - \mathbb{1}_{\{\eta(x)=k\}}\big)(\eta) \cdot g(\eta_x) \\
&\quad + \big(\mathbb{1}_{\{\eta(x)=k-1\}} - \mathbb{1}_{\{\eta(x)=k\}}\big)(\eta) \sum_{y \neq x} g(\eta_y) p_N(x-y).
\end{aligned} \tag{1.21}$$

Since ν is a translation invariant product measure, the r.v. $\eta(x)$, $x \in \mathbb{T}_N^d$, are identically distributed with common marginal $\alpha = \eta(0)_*\nu = \eta(x)_*\nu \in \mathbb{P}_1\mathbb{Z}_+$. Since we assume $\nu \in \mathbb{P}_1\mathbb{M}_N^d$ and $g(k) \leq \|g'\|_u k$ by assumption (1.15), we have that $g\big(\eta(x)\big) \leq \|g'\|_u \eta(x) \in L^1(\nu)$ for all $x \in \mathbb{T}_N^d$. So the quantities $\int g(\eta_x) d\nu(\eta)$ are non-negative real numbers independent of $x \in \mathbb{T}_N^d$, say

$$\int g\big(\eta(x)\big) d\nu = \int g d\alpha =: \varphi \in \mathbb{R}_+, \quad \forall\, x \in \mathbb{T}_N^d.$$

In addition, due to assumption (1.14) we can only have $\phi = 0$ when $\alpha = \delta_0 \in \mathbb{P}\mathbb{Z}_+$, in which case $\nu = \delta_{\mathbf{0}} \in \mathbb{P}\mathbb{M}_N^d$ is the Dirac distribution concentrated on the configuration $\mathbf{0}$

with no particles, the unique absorbing point of the ZRP. In this case ν is the trivial invariant Dirac distribution concentrated on the unique absorbing point of the ZRP, so let us assume that $\varphi > 0$. Then, integrating both sides of equality (1.21) with respect to ν and taking into account (1.20) and that the random variables $\eta(x)$, $x \in \mathbb{T}_N^d$, are i.i.d. we get that

$$\begin{aligned} 0 &= \int \big(\mathbb{1}_{\{\eta(x)=k+1\}} - \mathbb{1}_{\{\eta(x)=k\}}\big) \cdot g\big(\eta(x)\big) d\nu \\ &\quad + \sum_{y \neq x} \int \big(\mathbb{1}_{\{\eta(x)=k-1\}} - \mathbb{1}_{\{\eta(x)=k\}}\big) g\big(\eta(y)\big) d\nu \cdot p_N(x-y) \\ &= g(k+1)\nu\{\eta(x)=k+1\} - g(k)\nu\{\eta(x)=k\} \\ &\quad + \sum_{y \neq x} \big(\nu\{\eta(x)=k-1\} - \nu\{\eta(x)=k\}\big) \int g\big(\eta(y)\big) d\nu \cdot p_N(x-y) \\ &= g(k+1)\alpha(k+1) - g(k)\alpha(k) + \varphi \cdot \big(\alpha(k-1) - \alpha(k)\big). \end{aligned}$$

Therefore we get that if $\nu = \alpha^{\otimes \mathbb{T}_N^d}$ is to be an invariant distribution then it must satisfy

$$g(k+1)\alpha(k+1) - g(k)\alpha(k) = \varphi\alpha(k) - \varphi\alpha(k-1) \tag{1.22}$$

for all $k \in \mathbb{Z}_+$ where of course $\alpha(-1) = 0$. For $k = 0$, the above equation becomes $g(1)\alpha(1) = \varphi\alpha(0)$ and therefore

$$\alpha(1) = \alpha(0)\frac{\varphi}{g(1)}.$$

Then for $k = 2$ equation (1.22) becomes $g(2)\alpha(2) - g(1)\alpha(1) = \varphi\alpha(1) - \varphi\alpha(0)$, and since $g(1)\alpha(1) = \varphi\alpha(0)$ this implies that

$$\alpha(2) = \alpha(1)\frac{\varphi}{g(2)} = \alpha(0)\frac{\varphi^2}{g(1)g(2)}.$$

So it follows easily by induction that if $\nu = \alpha^{\otimes \mathbb{T}_N^d}$ is to be an invariant distribution then it must satisfy

$$\alpha(k) = \alpha(0)\frac{\varphi^k}{g!(k)}, \quad \forall\, k \in \mathbb{N}, \tag{1.23}$$

where for any function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ satisfying (1.14) we denote by $g! : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ the function given by

$$g!(k) = \prod_{i=1}^{k} g(i)$$

where of course the empty product is equal to 1. Finally, if a distribution $\alpha \in \mathbb{P}_1\mathbb{Z}_+ \setminus \{\delta_0\}$ is to satisfy (1.23) for all $k \geq 1$ we must necessarily have that $\alpha(0) > 0$, and since α is assumed a probability measure, by (1.23) we must necessarily have that

$$1 = \alpha(0) + \sum_{k=1}^{\infty} \alpha(k) = \alpha(0)\left(1 + \sum_{k=1}^{\infty} \frac{\varphi^k}{g!(k)}\right) = a(0)\sum_{k=0}^{\infty} \frac{\varphi^k}{g!(k)}.$$

In particular we have that the series $\sum_{k=0}^{\infty} \varphi^k/g!(k)$ must be convergent and that if an equilibrium distribution ν is to be of the form $\nu = \alpha^{\otimes \mathbb{T}_N^d}$ for some $\alpha \in \mathbb{P}_1\mathbb{Z}_+$, then the power series

$$Z(\varphi) \equiv Z_g(\varphi) := \sum_{k=0}^{\infty} \frac{\varphi^k}{g!(k)} \tag{1.24}$$

must be finite at $\varphi := \int g d\alpha$ and the one site marginal α must be given by the formula

$$\alpha(k) = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g!(k)}, \quad k \in \mathbb{Z}_+. \tag{1.25}$$

Definition 1.2.1 The function $Z \equiv Z_g : \mathbb{R}_+ \longrightarrow [1, \infty]$ defined by the power series in (1.24) is called the *normalizing partition function associated to* $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$. We will denote by $\mathcal{D}_Z = \{\varphi \in \mathbb{R}_+ | Z(\varphi) < +\infty\}$ the *proper domain* of the partition function Z.

According to the above, in order to have a chance of finding invariant distributions for the ZRP of the form $\nu = \alpha^{\otimes \mathbb{T}_N^d}$, $\alpha \in \mathbb{P}_1\mathbb{Z}_+$, the function g must be such that the partition function Z_g has non-trivial domain of convergence. By the root test, the radius of convergence $\varphi_c \equiv \varphi_g \equiv \varphi(g) := \sup \mathcal{D}_{Z_g}$ of Z_g is

$$\varphi_c = \frac{1}{\limsup_{k \to +\infty} \sqrt[k]{\frac{1}{g!(k)}}} = \liminf_{k \to +\infty} \sqrt[k]{g!(k)}, \tag{1.26}$$

and so along our basic assumptions (1.14) and (1.15) for g, we will also assume in what follows that the function g is such that

$$\liminf_{k \to +\infty} \sqrt[k]{g!(k)} > 0. \tag{1.27}$$

Of course whenever g is bounded from below by some positive number $\varepsilon > 0$, as is the case when g is increasing, condition (1.27) is satisfied since then $g!(k) \geq \varepsilon^k$ and therefore $\phi_g \geq \varepsilon > 0$. Another condition that guarantees that the normalizing partition function Z_g has non-trivial domain of convergence is the existence of constants $c, p > 0$ such that

$$\gamma := \liminf_{k \to +\infty} c^k k^p g!(k) > 0.$$

Indeed, then, if $0 < \varepsilon < \gamma$ there exists $k_0 \in \mathbb{N}$ such that $c^k k^p g!(k) \geq \varepsilon$ for all $k \geq k_0$ and so

$$\varphi_g = \liminf_{k \to +\infty} \sqrt[k]{g!(k)} \geq \frac{1}{c} \liminf_{k \to +\infty} \sqrt[k]{\frac{\varepsilon}{k^p}} = \frac{1}{c} > 0.$$

An example of a function g not satisfying assumption (1.27) is given by $g(k) = \frac{1}{k}$, $k \geq 1$, $g(0) = 0$, since then $g!(k) = \frac{1}{k!}$ and we can easily from the ratio test for sequences that

$$\lim_{k \to +\infty} \frac{c^k k^p}{k!} = 0.$$

Since mainly we will consider functions $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ satisfying the basic assumptions (1.14), (1.15) and (1.27), we will use the following terminology.

Definition 1.2.2 A function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ is called a *local rate function* if it satisfies assumptions (1.14), (1.15) and (1.27).

So, according to the discussion above, the partition function Z has non-trivial domain of convergence $\mathcal{D}_Z \subseteq [0, \varphi_c]$, $\varphi_c = \varphi_g > 0$, whenever g is a local rate function. Of course by our convention $g!(0) = 1$ we have that

$$Z(0) = \frac{1}{g!(0)} = 1 < Z(\varphi)$$

for all $\varphi \in \mathcal{D}_Z^o = (0, \varphi_c)$, and since Z is given by a power series it is analytic, and thus C^∞, on $[0, \varphi_c)$, with all of it's derivatives strictly positive, since for all $m \in \mathbb{Z}_+$ we have that

$$\frac{d^m}{d\varphi^m} Z(\varphi) = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} \frac{\varphi^{k-m}}{g!(k)} > 0,$$

for all $\varphi \in (0, \varphi_c)$. In particular the partition function Z is strictly increasing and strictly convex on $[0, \varphi_c)$. Recall that a convex function $Z : \mathbb{R} \longrightarrow (-\infty, \infty]$ is lower semicontinuous iff it is one-sidedly continuous at each endpoint of it's proper domain $\mathcal{D}_Z$ contained in $\mathcal{D}_Z$ and diverging to $+\infty$ at each endpoint of $\mathcal{D}_Z$ not belonging to $\mathcal{D}_Z$, and let us collect the basic properties of normalizing partitions functions in the following.

Proposition 1.2.1 *Let $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ be a local rate function and let $\varphi_c > 0$ be the radius of convergence of the partition function $Z_g : \mathbb{R}_+ \longrightarrow [1, \infty]$ defined by g. Then*

$$[0, \varphi_c) \subseteq \mathcal{D}_Z \subseteq [0, \varphi_c], \tag{1.28}$$

the partition function $Z_g : \mathbb{R}_+ \longrightarrow [1, +\infty]$ is a lower semicontinuous, strictly increasing and strictly convex function, C^∞ on $[0, \varphi_c)$ with all of it's derivatives strictly positive.

Proof Only the lower semi-continuity of Z remains to be proved. This follows from Abel's theorem on power series. Indeed, Z is increasing and so the limit $\lim_{\varphi \uparrow \varphi_c} Z(\varphi)$ exists in $[1, \infty]$, and since the coefficients $\frac{1}{g!(k)}$ of the power series defining Z_g are positive and $\varphi_c > 0$, the converse of Abel's theorem also holds, that is

$$Z(\varphi_c) < +\infty \quad \Longleftrightarrow \quad \lim_{\varphi \uparrow \varphi_c} Z(\varphi) < +\infty,$$

and in this case

$$Z(\varphi_c) = \lim_{\varphi \uparrow \varphi_c} Z(\varphi).$$

But because Z is convex, with proper domain $\mathcal{D}_Z$ satisfying (1.28), and left continuous at zero, this proves the lower semi-continuity of Z. $\square$

Furthermore, note that if we allow increasing limits to take the value $+\infty$, then we have that

$$Z(\varphi_c) = \lim_{\varphi \uparrow \varphi_c} Z(\varphi)$$

regardless of whether $\varphi_c \in \mathcal{D}_Z$ or not. Note, also, that even when $\varphi_c \in \mathcal{D}_Z$ the partition function may not be differentiable from the right at φ_c, i.e. we may well have $Z'(\varphi_c) = +\infty$. In fact, Abel's theorem applies to the behavior of the derivatives of Z at the critical point ϕ_c.

Proposition 1.2.2 *Let $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ be a local rate function and let $\varphi_c > 0$ be the radius of convergence of the partition function $Z_g : \mathbb{R}_+ \longrightarrow [1, \infty]$ defined by g. Then, for any $m \in \mathbb{Z}_+$ we have that*

$$Z_-^{(m)}(\varphi_c) := \sup_{0<\varphi<\varphi_c} Z^{(m)}(\varphi) = \lim_{\varphi \uparrow \varphi_c} Z^{(m)}(\varphi) = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} \frac{\varphi_c^{k-m}}{g!(k)},$$

where the above limits are limits in $[0, \infty]$, i.e. they are allowed to take the value $+\infty$.

Proof The proof follows from Abel's theorem as in the case $m = 0$ in the previous proposition and we will not repeat it. □

Next, we give a name to the one site marginal $\alpha \in \mathbb{P}\mathbb{Z}_+$ of our candidate for a product translation invariant equilibrium distribution of the ZRP.

Definition 1.2.3 Let $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ be a local rate function and let Z be the partition function associated to g. For any $\varphi \in \mathcal{D}_Z$, the distribution $\bar{\nu}_\varphi^1 \equiv \bar{\nu}_{\varphi,g}^1 \in \mathbb{P}\mathbb{Z}_+$ defined by

$$\bar{\nu}_\varphi^1\{k\} = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g!(k)}, \qquad k \in \mathbb{Z}_+ \tag{1.29}$$

will be called the *one-site zero range (ZR) distribution with rate g and parameter φ.*

A product distribution $\bar{\nu}_{\varphi,g}^N \in \mathbb{P}\mathbb{M}_N^d$ with common marginal $\bar{\nu}_\varphi^1 \in \mathbb{P}\mathbb{Z}_+$, $\varphi \in \mathcal{D}_{Z_g}$, will be called a *ZR distribution on the discrete torus $\mathbb{T}_N^d$ with rate g and parameter φ.*

In the physics literature, the parameter φ is known as the fugacity. Of course, according to the above definition the ZR distribution on the discrete torus $\mathbb{T}_N^d$ with local rate function g and parameter φ is given by the formula

$$\bar{\nu}_{\varphi,g}(\eta) = \prod_{x \in \mathbb{T}_N^d} \bar{\nu}_\varphi^1(\eta_x) = \frac{1}{Z(\varphi)^{N^d}} \prod_{x \in \mathbb{T}_N^d} \frac{\varphi^{\eta_x}}{g!(\eta_x)} = \frac{1}{Z(\varphi)^{N^d}} \frac{\varphi^{|\eta|_1}}{g!(\eta)},$$

for all $\eta \in \mathbb{M}_N^d$, where of course we have set $g!(\eta) := \prod_{x \in \mathbb{T}_N^d} g!(\eta_x)$.

Our first task is to prove that the translation invariant measures $\bar{\nu}_{\varphi,\gamma}$, $\varphi \in \mathcal{D}_{Z_g}$ that we have defined are indeed equilibrium distributions for the ZRP.

Proposition 1.2.3 *Let $Z_g : \mathbb{R}_+ \longrightarrow [1, +\infty]$ be the partition function associated to some local rate function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$. Then, for every $\varphi \in \mathcal{D}_{Z_g}$, the ZR distribution $\bar{\nu}_\varphi^N \in \mathbb{P}\mathbb{M}_N^d$ on the discrete torus $\mathbb{T}_N^d$ is an equilibrium distribution for the ZRP with parameters (p, g).*

Proof Of course if $\varphi = 0$ we have nothing to prove, so we assume that $\varphi \in (0, \varphi_c)$ and we will prove that $\bar{\nu}_\varphi^N L^N = 0$, i.e. that

$$\bar{\nu}_\varphi^N(\eta)\lambda_N(\eta) = \sum_{x,y \in \mathbb{T}_N^d,\, y \neq x} \bar{\nu}_\varphi^N(\eta^{x,y}) L^N(\eta^{x,y}, \eta),$$

for all $\eta \in \mathbb{M}_N^d$.

Obviously, by the formula of $\bar{\nu}_\varphi^N$, for all $\eta \in \mathbb{M}_N^d$, $x \in \mathrm{spt}(\eta) := \{x \in \mathbb{T}_N^d | \eta_x > 0\}$ and $y \in \mathbb{T}_N^d \setminus \{x\}$, we have that

$$\frac{\bar{\nu}_\varphi^N(\eta^{x,y})}{\bar{\nu}_\varphi^N(\eta)} = \frac{g!(\eta)}{g!(\eta^{x,y})} = \frac{g(\eta_x)}{g(\eta_y+1)} \tag{1.30}$$

and therefore it suffices to prove that

$$\lambda_N(\eta) = \sum_{x,y \in \mathbb{T}_N^d} \frac{g(\eta_x)}{g(\eta_y+1)} L^N(\eta^{x,y}, \eta)$$

for all $\eta \in \mathbb{M}_N^d$. But

$$L^N(\eta^{x,y}, \eta) = L^N\big(\eta^{x,y}, (\eta^{x,y})^{y,x}\big) = g(\eta_y+1) p_N(x-y) \tag{1.31}$$

for all $\eta \in \mathbb{M}_N^d$, $x \in \mathrm{spt}(\eta)$, $y \in \mathbb{T}_N^d \setminus \{x\}$, and therefore we have

$$\sum_{x,y \in \mathbb{T}_N^d} \frac{g(\eta_x)}{g(\eta_y+1)} L^N(\eta^{x,y}, \eta) = \sum_{x,y \in \mathbb{T}_N^d} g(\eta_x) p_N(x-y) = \lambda_N(\eta),$$

for all $\eta \in \mathbb{M}_N^d$, as required. □

So indeed there exists a monoparametric family of translation invariant product equilibrium distributions for the ZRP, the family $\{\bar{\nu}_{\varphi,g}^N\}_{\varphi \in \mathcal{D}_{Z_g}}$ of the ZR-distributions on the discrete torus. Of course by the discussion that led us to the formula of the distributions $\bar{\nu}_\varphi^1 \in \mathbb{P}\mathbb{Z}_+$, this family is the unique, up to reparametrization, family of translation invariant product equilibrium distributions of the ZRP with parameters (p, g). Furthermore, the adjoint process of the ZRP with respect to any of the measures $\bar{\nu}_{\varphi,g}^N$, $\varphi \in \mathcal{D}_{Z_g}$ is easily described.

Proposition 1.2.4 *Let $p \in \mathbb{P}\mathbb{Z}^d$ be an elementary step distribution and $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ be a local rate function. The adjoint process, with respect to any of the measures $\bar{\nu}_{\varphi,g}^N$, $\varphi \in \mathcal{D}_Z$, of a ZRP with parameters (p, g) is a ZRP with parameters $(\check{p}, g)$, where $\check{p} \in \mathbb{P}\mathbb{Z}^d$ is the reflection of $p \in \mathbb{P}\mathbb{Z}^d$ with respect to the origin, i.e. $\check{p}(z) = p(-z)$ for all $z \in \mathbb{Z}^d$.*

Consequently, if the elementary probability distribution p is symmetric with respect to the origin then the ZRP with parameters (p, g) is self-adjoint with respect to any of the measures $\bar{\nu}_{\varphi,g}^N$, $\varphi \in \mathcal{D}_{Z_g}$.

Proof Let $\varphi \in \mathcal{D}_{Z_g}$ be fixed. We have to prove that if L^N, $\check{L}^N$ are the generators of ZRPs on $\mathbb{T}_N^d$ with parameters (p, g) and $(\check{p}, g)$, respectively, then

$$\bar{\nu}_\varphi^N(\eta) \check{L}^N(\eta, \zeta) = \bar{\nu}_\varphi^N(\zeta) L^N(\zeta, \eta), \quad \forall\, \eta, \zeta \in \mathbb{M}_N^d.$$

But this is obvious, since we have only to consider the case where $\zeta = \eta^{x,y}$ for some $\eta \in \mathbb{M}_N^d$, $x \in \mathrm{spt}(\eta)$, $y \in \mathbb{T}_N^d \setminus \{x\}$, and by (1.30) and (1.31) we have that

$$\begin{aligned} \bar{\nu}_\varphi^N(\eta) \check{L}^N(\eta, \eta^{x,y}) &= \bar{\nu}_\varphi^N(\eta) g(\eta_x) \check{p}_N(y-x) = \bar{\nu}_\varphi(\eta) g(\eta_x) p_N(x-y) \\ &= \bar{\nu}_\varphi(\eta^{x,y}) g(\eta_y+1) p_N(x-y) = \bar{\nu}_\varphi(\eta^{x,y}) L^N(\eta^{x,y}, \eta). \end{aligned}$$ □

As defined, the distributions $\bar{\nu}^N_{\varphi,g}$, $\varphi \in \mathcal{D}_{Z_g}$ are not parametrized by the density of particles, since if

$$R(\varphi) := \int \eta(x) d\bar{\nu}^N_\varphi = \int k d\bar{\nu}^1_\varphi(k), \quad \varphi \in \mathcal{D}_Z,$$

the mean number of particles with respect to $\bar{\nu}_\varphi$ at site $x \in \mathbb{T}^d_N$, we have that

$$R(\varphi) = \frac{1}{Z(\varphi)} \sum_{k=0}^{\infty} k \frac{\varphi^k}{g!(k)} = \frac{\varphi Z'(\varphi)}{Z(\varphi)}, \qquad \forall\, 0 \leq \varphi < \varphi_c, \tag{1.32}$$

and therefore $R(\varphi) = \varphi$ iff $Z(\varphi) = Z'(\varphi)$. But $Z(0) = 1$ and so $Z = Z'$ iff $Z = \exp$, and therefore the distributions $\bar{\nu}_{\varphi,g}$ are parametrized by the density of particles iff $g = id_{\mathbb{Z}_+}$ in which of course $\bar{\nu}^N_\varphi$ is a Poisson distribution. Note that by equation (1.32) we have that

$$\int k d\nu_\varphi(k) \equiv R(\varphi) = \frac{\varphi Z'(\varphi)}{Z(\varphi)} < +\infty,$$

for all $\varphi \in [0, \varphi_c)$, and therefore $\nu_{\varphi,g}$ has first order moments for all $\varphi \in [0, \varphi_c)$. On the other hand, if $\varphi_c = \varphi_c(g) \in \mathcal{D}_{Z_g}$, in which case the distribution $\bar{\nu}^1_{\varphi_c,g}$ is defined for the critical value φ_c, it is not necessary that
$bar\nu^1_{\varphi_c}$ has finite first moment. In fact, in this case we have by Abel's theorem on power series that $R(\varphi_c) < +\infty$ iff $Z'(\varphi_c) < +\infty$, in which case $R(\varphi_c) = \varphi_c Z'(\varphi_c)/Z(\varphi_c)$. Indeed, if $Z(\varphi_c) < +\infty$ then $R(\varphi_c) < +\infty$ iff $Z(\varphi_c)R(\varphi_c) < +\infty$ and

$$Z(\varphi)R(\varphi) = \sum_{k=0}^{\infty} k \frac{\varphi^k}{g!(k)}$$

converges for all $\varphi \in [0, \varphi_c)$. Therefore, since the coefficients $\frac{k}{g!(k)}$ in the above power series are non-negative and $[0, \varphi_c) \ni \varphi \mapsto R(\varphi)Z(\varphi) = \varphi Z'(\varphi)$ is increasing, we have by Abel's theorem that

$$R(\varphi_c)Z(\varphi_c) \equiv \sum_{k=0}^{\infty} k \frac{\varphi_c^k}{g!(k)} < +\infty \quad \Longleftrightarrow \quad \lim_{\varphi \uparrow \varphi_c} R(\varphi)Z(\varphi) < +\infty,$$

in which case of course $R(\varphi_c)Z(\varphi_c) = \lim_{\varphi \uparrow \varphi_c} R(\varphi)Z(\varphi)$. Furthermore, since by the lower semi-continuity of Z we have that $\lim_{\varphi \uparrow \varphi_c} Z(\varphi) = Z(\varphi_c)$, if we allow limits to take the value $+\infty$, whenever $Z(\varphi_c) < +\infty$ we have that

$$R(\varphi_c) = \lim_{\varphi \uparrow \varphi_c} R(\varphi) = \int k d\alpha_{\varphi_c,g}(k) = \frac{\varphi_c Z'(\varphi_c)}{Z(\varphi_c)} \in [0, +\infty]. \tag{1.33}$$

The following proposition allows us to reparametrize the family $\{\bar{\nu}^1_\varphi\}_{\varphi \in \mathcal{D}_Z}$ so as to obtained a family that is parametrized by the density of particles

Proposition 1.2.5 *The density of particles $R : [0, \varphi_c) \longrightarrow \mathbb{R}_+$ defined in (1.32) is a smooth and strictly increasing function of the fugacity φ.*

Proof Obviously $R(0) = 0$ and R is smooth in $[0, \varphi_c)$ with first derivative

$$R'(\varphi) = -\frac{Z'(\varphi)}{Z(\varphi)^2} \sum_{k=0}^{\infty} \frac{k}{g!(k)} \varphi^k + \frac{1}{Z(\varphi)} \sum_{k=0}^{\infty} \frac{k^2}{g!(k)} \varphi^{k-1}.$$

Therefore $R'(\varphi) > 0$ iff

$$Z(\varphi) \sum_{k=0}^{\infty} \frac{k^2}{g!(k)} \varphi^{k-1} > Z'(\varphi) \sum_{k=0}^{\infty} \frac{k}{g!(k)} \varphi^k.$$

Multiplying the above inequality by $\varphi \in (0, \varphi_c)$ it follows that $R'(\varphi) > 0$ for $\varphi \in (0, \varphi_c)$ iff

$$\left(\sum_{k=0}^{\infty} \frac{\varphi^k}{g!(k)} \right) \left(\sum_{k=0}^{\infty} k^2 \frac{\phi^k}{g!(k)} \right) > \left(\sum_{k=0}^{\infty} k \frac{\varphi^k}{g!(k)} \right)^2 .$$

Furthermore, by multiplying by $\frac{1}{Z(\varphi)^2}$ the above inequality becomes

$$\left(\int id_{\mathbb{Z}_+} d\bar{\nu}_{\varphi}^1 \right)^2 < \left(\int id_{\mathbb{Z}_+}^2 d\bar{\nu}_{\varphi}^1 \right) \left(\int 1 d\bar{\nu}_{\varphi}^1 \right),$$

which is exactly the Cauchy-Schwartz inequality for the functions $id_{\mathbb{Z}_+}$, 1 in $L^2(\bar{\nu}_{\varphi}^1)$, and holds as a strict inequality since $\bar{\nu}_{\varphi}^1 \in \mathbb{P}_1\mathbb{Z}_+$ and $id_{\mathbb{Z}_+}^2$ and 1 are not $\bar{\nu}_{\varphi}^1$-almost surely collinear. □

Now, since the density of particles $R : [0, \varphi_c) \longrightarrow [0, \infty)$ is strictly increasing, it is injective with image the subinterval $[0, \rho_c)$ of $\mathbb{R}_+$, where

$$\rho_c := \lim_{\varphi \uparrow \varphi_c} R(\varphi) = \sup_{0 \le \varphi < \varphi_c} R(\varphi)$$

and it's inverse $\Phi := R^{-1} : [0, \rho_c) \longrightarrow [0, \phi_c)$ is well defined. So if we define the family $\{\nu_{\rho}^1\}_{0 \le \rho < \rho_c} \subseteq \mathbb{P}_1\mathbb{Z}_+$ by

$$\nu_{\rho}^1 := \bar{\nu}_{\Phi(\rho)}^1, \tag{1.34}$$

where $\{\bar{\nu}_{\varphi}^1\}_{0 \le \phi < \varphi_c}$ is the family of ZR distributions with local rate function g given in (1.29), then

$$\int k d\nu_{\rho}(k) = \int k d\bar{\nu}_{\Phi(\rho)}^1(k) = R\big(\Phi(\rho)\big) = \rho,$$

i.e. the family $\{\nu_{\rho,g}\}_{0 \le \rho < \rho_c} \subseteq \mathbb{P}_1\mathbb{Z}_+$ is parametrized by the density of particles. We will refer to a distribution $\nu_{\rho} = \bar{\nu}_{\Phi(\rho)}^1$, $0 \le \rho < \rho_c$, as a (normalized) one-site ZR distribution with density ρ. Likewise we will refer to a distribution $\nu_{\rho}^N \in \mathbb{PM}_N^d$ of the form $\nu_{\rho} := \prod_{x \in \mathbb{T}_N^d} \alpha_{\rho}$ as a *normalized g-ZR distribution with density ρ*.

It is worth to remark that by the definition of $\Phi = R^{-1}$ we have that $\Phi(\rho)$ is the mean local rate with respect to α_{ρ}, that is

$$\int g d\nu_{\rho} = \frac{1}{Z\big(\Phi(\rho)\big)} \sum_{k=1}^{\infty} g(k) \frac{\Phi(\rho)^k}{g!(k)} = \Phi(\rho). \tag{1.35}$$

Of course if $\rho_c = +\infty$ then there exists a normalized one-site ZR distribution with density ρ for all non-negative numbers $\rho \in \mathbb{R}_+$. On the other hand, if $\rho_c < +\infty$ then for all $\rho > \rho_c$ there does not exist a one-site ZR distribution with density ρ. However as we will see, if $\rho_c := \lim_{\varphi\uparrow\varphi_c} R(\varphi) < +\infty$, then $\varphi_c \in \mathcal{D}_{Z_g}$ and therefore by (1.33) there exists a normalized one-site ZR distribution with density equal to the critical density ρ_c, namely the one-site ZR distribution $\nu_{\rho_c} := \bar{\nu}^1_{\varphi_c}$ with fugacity φ_c.

Proposition 1.2.6 *Let $\{\bar{\nu}^1_{\varphi,g}\}_{0\leq\varphi<\varphi_c} \subseteq \mathbb{P}_1\mathbb{Z}_+$ be the family of one-site ZR distributions associated to the local rate function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ and let $R : [0, \varphi_c) \longrightarrow \mathbb{R}_+$ be the density of particles defined in (1.32). If*

$$\rho_c := \lim_{\varphi\uparrow\varphi_c} R(\varphi) < +\infty, \tag{1.36}$$

then $\varphi_c \in \mathcal{D}_{Z_g}$, Z is differentiable at ϕ_c from the left and

$$\rho_c = R(\varphi_c) = \frac{\varphi_c Z'(\varphi_c)}{Z(\varphi_c)} = \int k d\bar{\nu}^1_{\varphi_c,g}(k). \tag{1.37}$$

Proof As we have already seen, whenever $\varphi_c \in \mathcal{D}_Z$ we have that $R(\varphi_c) < +\infty$ iff $Z'_-(\varphi_c) < +\infty$ and (1.37) holds. So we have to prove the implication

$$\rho_c < +\infty \quad \Longrightarrow \quad \varphi_c \in \mathcal{D}_Z. \tag{1.38}$$

First we show that if $\rho_c < +\infty$ then $\varphi_c < +\infty$. Let's suppose to derive a contradiction that $\varphi_c = +\infty$. By (1.32) we obviously have that

$$R(\varphi) = \varphi \cdot \big(\log Z(\varphi)\big)', \qquad \forall\, 0 \leq \varphi < \varphi_c$$

and therefore since $\rho_c < +\infty$, there exists $M \in \mathbb{N}$ such that

$$\big(\log Z(\varphi)\big)' \leq \frac{M}{\varphi}, \quad \forall\, \varphi \in (0, \varphi_c). \tag{1.39}$$

So, for each $\varphi \geq 1$ we have that

$$\log Z(\varphi) - \log Z(1) = \int_1^\varphi \big(\log Z(\psi)\big)' d\psi \leq M \int_1^\varphi \frac{d\psi}{\psi} = M \log \varphi,$$

or equivalently that

$$Z(\varphi) \leq Z(1)\varphi^M, \quad \forall \varphi \geq 1.$$

But then,

$$R(\varphi) = \frac{\varphi Z'(\varphi)}{Z(\varphi)} \geq \frac{1}{Z(1)} \frac{Z'(\varphi)}{\varphi^{M-1}} \geq \frac{1}{Z(1)} \frac{M+1}{g!(M+1)} \varphi \overset{\varphi\to+\infty}{\longrightarrow} +\infty,$$

which contradicts the fact that $\rho_c < +\infty$. Therefore if $\rho_c < +\infty$ then we necessarily have that $\varphi_c < +\infty$. But for any $0 < \varepsilon < \varphi_c$ we have by (1.39) that

$$\log Z(\varphi) - \log Z(\varepsilon) = \int_\varepsilon^\varphi \big(\log Z(\psi)\big)' d\psi \leq M \log \phi - M \log \varepsilon,$$

for all $\varphi \in (\varepsilon, \varphi_c)$, or equivalently

$$Z(\varphi) \leq Z(\varepsilon)\Big(\frac{\varphi}{\varepsilon}\Big)^M, \quad \forall\, \varphi \in (\varepsilon, \varphi_c),$$

and therefore by the lower semicontinuity of Z we have that

$$Z(\varphi_c) = \lim_{\varphi \uparrow \varphi_c} Z(\varphi) \leq Z(\varepsilon)\frac{\varphi_c^M}{\varepsilon^M} < +\infty,$$

that is $\varphi_c \in \mathcal{D}_Z$ as required. □

As we have seen so far, the normalized ZR distributions α_ρ, $0 \leq \rho < \rho_c$, as well as the ZR distribution $\nu^1_{\rho_c}$ corresponding to the critical density ρ_c whenever defined, have always finite first order moments. According to the next proposition the normalized ZR distributions have finite moments of any order for all densities $\rho < \rho_c$, while at the critical density $\rho = \rho_c$ we do not necessarily have finite moments of all orders.

Proposition 1.2.7 *Let $\{\nu^1_\rho\}_{0 \leq \rho < \rho_c}$ be the family of normalized one-site ZR distributions associated to some local rate function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ with partition function $Z : \mathbb{R}_+ \longrightarrow [1, \infty]$. Then, for all $\rho \in [0, \rho_c)$ the one-site ZR distribution α_ρ has finite exponential moments. Furthermore, if $\rho_c < +\infty$, in which case $\nu^1_{\rho_c}$ is defined, and*

$$n_c := \sup\big\{m \in \mathbb{Z}_+ \,\big|\, Z^{(m)}(\varphi_c) < +\infty\big\},$$

where $Z^{(m)}(\varphi_c) := \sup_{0 \leq \varphi < \varphi_c} Z^{(m)}(\varphi)$, then $\nu^1_{\rho_c}$ has finite moments of order n_c, and infinite moments of order $\geq n_c + 1$.

Proof Let $\rho < \rho_c$. Then $\Phi(\rho) < \varphi_c$ and so there exists $\theta > 0$ such that $e^\theta \Phi(\rho) < \varphi_c$. But then, the Laplace transform $M_{\nu^1_\rho}$ of ν^1_ρ evaluated at θ is

$$M_{\alpha_\rho}(\theta) = \int e^{\theta k} d\alpha_\rho(k) = \frac{1}{Z\big(\Phi(\rho)\big)} \sum_{k=0}^\infty \frac{\big(\Phi(\rho)e^\theta\big)^k}{g!(k)} = \frac{Z\big(\Phi(\rho)e^\theta\big)}{Z\big(\Phi(\rho)\big)} < +\infty,$$

and therefore ν^1_ρ has finite exponential moments.

We consider now the case of $\nu^1_{\rho_c} = \bar{\nu}^1_{\varphi_c} \in \mathbb{P}_1\mathbb{Z}_+$, when it is defined. Note that

$$\sum_{k=0}^\infty k^m \frac{\varphi_c^k}{g!(k)} = Z(\varphi_c) \int k^m d\bar{\nu}^1_{\varphi_c}(k) \tag{1.40}$$

for all $m \in \mathbb{N}$. On the other hand, since Z is C^∞ on $[0, \varphi_c)$ we have that

$$\sum_{k=0}^\infty \frac{(k+m)!}{k!} \frac{\varphi^k}{g!(k+m)} = \sum_{k=m}^\infty \frac{k!}{(k-m)!} \frac{\varphi^{k-m}}{g!(k)} = Z^{(m)}(\varphi) < +\infty. \tag{1.41}$$

Also, for every $k, m \in \mathbb{N}$, $k \geq m$, we have that

$$k^m = \frac{k!}{(k-m)!} + q_{m-1}(k), \tag{1.42}$$

where $q_{m-1}(k) := \sum_{i=0}^{m-1} a_{i,m-1} k^i$ is a polynomial of order $m-1$. For each $m \in \mathbb{Z}_+$ now, we consider the function $g_m : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ given by the formula

$$g_m(k) = \begin{cases} \frac{g!(m+1)}{(m+1)!}, & k = 1 \\ k\frac{g(m+k)}{m+k}, & k \geq 2 \end{cases}.$$

Then obviously $g_0 \equiv g$ and

$$g_m!(k) = \frac{g!(m+1)}{(m+1)!} \cdot 2\frac{g(m+2)}{m+2} \cdot \ldots \cdot k\frac{g(m+k)}{m+k} = k!\frac{g!(m+k)}{(m+k)!}$$

for all $m, k \in \mathbb{Z}_+$. Furthermore since $\lim_{k\to+\infty}\left(\frac{k}{m+k}\right)^{1/k} = 1$ for all $m \in \mathbb{Z}_+$ we have that the functions g_m, $m \in \mathbb{Z}_+$, have all the same critical fugacity $\varphi_{g_m} = \varphi_c := \varphi(g)$. In particular, g_m is a local rate function for all $m \in \mathbb{Z}_+$ and by (1.41) we have that

$$Z_{g_m}(\varphi) = \sum_{k=0}^{\infty} \frac{\varphi^k}{g_m!(k)} = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{\varphi^k}{g!(k+m)} = Z^{(m)}(\varphi).$$

Now we can easily see that the set

$$A := \{m \in \mathbb{N} | Z^{(m)}(\varphi_c) < +\infty\} \tag{1.43}$$

is some subsegment $\{1, 2, \ldots, n_c\}$ of $\mathbb{N}$. Indeed, since $\rho_c < +\infty$ we have that $1 \in A$ and, as we will show, if $Z^{(m)}(\varphi_c) = +\infty$ for some $m \in \mathbb{N}$ then $Z^{(m+1)}(\varphi_c) = +\infty$. Indeed, suppose that $Z_{g_m}(\varphi_c) = Z^{(m)}(\varphi_c) = +\infty$. Then by propositions (1.2.5) and (1.2.6) we have that the function $R_m : [0, \varphi_c) \longrightarrow \mathbb{R}_+$ given by

$$R_m(\varphi) = \frac{\varphi Z^{(m+1)}(\varphi)}{Z^{(m)}(\varphi)}$$

is increasing to $+\infty$ as $\varphi \uparrow \varphi_c$, and therefore since $\lim_{\varphi\uparrow\varphi_c} Z^{(m)}(\varphi) = +\infty$ and $\varphi_c < +\infty$ we must necessarily have that $\lim_{\varphi\uparrow\varphi_c} Z^{(m+1)}(\varphi) = +\infty$.

We will prove now that $\nu^1_{\rho_c}$ has finite n_c-th moment. Of course as we have seen $\nu^1_{\rho_c}$ has finite first order moments. So it suffices to prove that if $\nu^1_{\rho_c}$ has finite m-th moments for some $m \leq n_c - 1$, then it also has $(m+1)$-th moments. But by (1.40) and (1.42) we have that

$$\begin{aligned} Z(\varphi_c)\int k^{m+1} d\nu^1_{\rho_c}(k) &= \sum_{k=0}^{m-1} k^{m+1}\frac{\varphi_c^k}{g!(k)} + \sum_{k=m+1}^{\infty} k^{m+1}\frac{\varphi_c^k}{g!(k)} \\ &= \sum_{k=0}^{m-1} k^{m+1}\frac{\varphi_c^k}{g!(k)} + \sum_{k=m+1}^{\infty} \frac{k!}{(k-m-1)!}\frac{\varphi_c^k}{g!(k)} \\ &\quad + \sum_{k=m+1}^{\infty} q_m(k)\frac{\varphi_c^k}{g!(k)} \\ &= \sum_{k=0}^{m} k^{m+1}\frac{\varphi_c^k}{g!(k)} + \varphi_c^{m+1} Z^{(m+1)}(\varphi_c) + \sum_{k=m+1}^{\infty} q_m(k)\frac{\varphi_c^k}{g!(k)}, \end{aligned}$$

which is finite since the first term in the last sum is obviously a finite number, the second term is finite since $m+1 \leq n_c$ and the set A in (1.43) is a segment, and the third term is finite since q_m has polynomial growth of order m and by the inductive hypotheses $\nu^1_{\rho_c} \in \mathbb{P}_m\mathbb{Z}_+$. This equation proves also that $\nu^1_{\rho_c}$ does not have finite n-th moments. $\square$

This proposition does not tell us what happens for moments of order $p \in (n_c, n_c+1)$. Also, note that by the above proof for all $\rho \in [0, \rho_c)$ the moment generating function $M_{\nu^1_\rho}$ of ν^1_ρ, given by

$$M_{\nu^1_\rho}(\theta) = \frac{Z\big(e^\theta \Phi(\rho)\big)}{Z\big(\Phi(\rho)\big)},$$

has proper domain $\mathcal{D}_{M_{\nu^1_\rho}}$ such that $(-\infty, b_\rho) \subseteq \mathcal{D}_{M_{\nu^1_\rho}} \subseteq (-\infty, b_\rho]$, where

$$b_\rho := \log \phi_c - \log \Phi(\rho) > 0.$$

In particular when $\varphi_c = +\infty$ then ν^1_ρ has full exponential moments for all $\rho \geq 0$, that is $M_{\nu^1_\rho}(\theta) = \int e^{\theta k} d\nu^1_\rho(k) < +\infty$ for all $\theta \in \mathbb{R}$, $\rho \geq 0$. On the other hand, for the critical density $\rho = \rho_c$ we have that $b_{\rho_c} = 0$ and $\nu^1_{\rho_c}$ does not have exponential moments. Next we will see some basic properties of the family the normalized one-site ZR distributions.

Proposition 1.2.8 *The family* $\{\nu^1_\rho\}_{\rho \in I_c} \subseteq \mathbb{P}_1\mathbb{Z}_+$, $I_c := [0, \rho_c] \cap \mathbb{R}$, *of normalized one-site ZR distributions defined through any local rate function* $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ *is* 1-*Wasserstein continuous (that is with respect to functions* $h : \mathbb{Z}_+ \longrightarrow \mathbb{R}$ *of linear growth,* $h(z) \leq C(1 + |z|)$*).*

Proof Since $\mathbb{Z}_+$ is countable, the family $\{\nu^1_\rho\}_{\rho \in I_c} \subseteq \mathbb{P}_1\mathbb{Z}_+$ is weakly continuous iff the function

$$I_c \ni \rho \overset{f_k}{\mapsto} \nu^1_\rho(k)$$

is continuous for all $k \in \mathbb{Z}_+$. But f_k is given by the formula

$$f_k(\rho) = \frac{1}{Z\big(\Phi(\rho)\big)} \frac{\Phi(\rho)^k}{g!(k)},$$

which is obviously continuous in $[0, \rho_c)$. Furthermore, if ρ_c, in which case $\varphi_c \in \mathcal{D}_Z$ and in particular $\varphi_c < +\infty$, since Z is continuous on $[0, \varphi_c]$ it suffices to prove that

$$\lim_{\rho \uparrow \rho_c} \Phi(\rho) = \varphi_c.$$

But this is obvious. The limit exists since Φ is strictly increasing, and if for some $\varepsilon > 0$ we had that $\sup_{0 \leq \rho \leq \rho_c} \Phi(\rho) \leq \varphi_c - \varepsilon$ then we would have that

$$\varphi_c - \varepsilon \geq \Phi\big(R(\varphi_c - \frac{\varepsilon}{2})\big) = \varphi_c - \frac{\varepsilon}{2},$$

which is absurd. This proves that $\{\nu^1_\rho\}_{\rho \in I_c}$ is weakly continuous and since the density function $I_c \ni \rho \mapsto \int k d\nu^1_\rho(k) = \rho$ is obviously continuous the family $\{\nu^1_\rho\}_{\rho \in I_c}$ is 1-Wasserstein continuous. $\square$

Proposition 1.2.9 *The family $\{\nu^1_\rho\}_{\rho\in I_c} \subseteq \mathbb{P}_1\mathbb{Z}_+$, $I_c := [0,\rho_c]\cap\mathbb{R}$, of normalized one-site ZR distributions defined through any local rate function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ is increasing.*

Proof We have to prove that for any bounded and increasing function $f : \mathbb{Z}_+ \longrightarrow \mathbb{R}$ we have that

$$\rho_1 \le \rho_2 \quad \Longrightarrow \quad \int f d\nu^1_{\rho_1} \le \int f d\nu^1_{\rho_2}. \tag{1.44}$$

Since any bounded increasing function $f : \mathbb{Z}_+ \longrightarrow \mathbb{R}$ can be written as uniformly bounded pointwise limit $f = \lim_{m\to+\infty} f_m$ of the increasing functions $f_m := f \wedge f(m)$, $m \in \mathbb{Z}_+$, it suffices to prove (1.44) for increasing functions $f \in B(\mathbb{Z}_+)$ that are in addition eventually constant. That is, it suffices to prove the claim for functions f of the form

$$f \equiv \sum_{i=1}^{m} x_{i-1}\mathbb{1}_{\{n_{i-1},n_{i-1}+1,\dots,n_i-1\}} + x_m\mathbb{1}_{\mathbb{Z}_+\setminus\{0,\dots,n_m-1\}}$$

for some $m \in \mathbb{N}$ and some increasing sequences $\{n_i\}_{i=0}^m \subseteq \mathbb{Z}_+$, $n_0 = 0$, and $\{x_i\}_{i=0}^m \subseteq \mathbb{R}$. Then if we set $x_{-1} := 0$, we have that

$$f = \sum_{i=0}^{m}(x_i - x_{i-1})\mathbb{1}_{\mathbb{Z}_+\setminus\{0,\dots,n_i-1\}},$$

and therefore it is obvious that it suffices to prove (1.44) for functions $f \in B(\mathbb{Z}_+)$ of the form $f = \mathbb{1}_{\mathbb{Z}_+\setminus\{0,\dots,m-1\}}$, $m \in \mathbb{Z}_+$, where of course $\{0,\dots,-1\} = \emptyset$. In other words, since $\Phi := R^{-1}$ is strictly increasing, it suffices to prove that for each $m \in \mathbb{Z}_+$, the function $F_m : (0,\varphi_c) \longrightarrow \mathbb{R}_+$ given by the formula

$$F_m(\varphi) = \bar{\nu}^1_\phi\big(\{m, m+1,\dots\}\big) = \frac{1}{Z(\phi)}\sum_{k=m}^{\infty}\frac{\varphi^k}{g!(k)}$$

is increasing. Obviously

$$F_m(\varphi) = F_{m+1}(\varphi) + \frac{1}{Z(\varphi)}\frac{\varphi^m}{g!(m)}$$

for all $m \in \mathbb{Z}_+$, $\varphi \in (0,\varphi_c)$ and therefore

$$\begin{aligned} F'_{m+1}(\varphi) &= F'_m(\varphi) + \frac{Z'(\varphi)}{Z(\varphi)^2}\frac{\varphi^m}{g!(m)} - \frac{1}{Z(\varphi)}\frac{m\varphi^{m-1}}{g!(m)} \\ &= F'_m(\varphi) + \frac{1}{Z(\varphi)}\frac{\varphi^{m-1}}{g!(m)}\big(R(\varphi) - m\big). \end{aligned}$$

for all $m \in \mathbb{Z}_+$, $\varphi \in (0,\varphi_c)$. In particular, for fixed $\varphi \in (0,\varphi_c)$ the sequence $\{F'_m(\varphi)\}_{m\in\mathbb{Z}_+}$ is increasing on the set $\{0,1,\dots,R(\varphi)+1\}$ and decreasing on the set $\{R(\varphi)+1,\dots\}$. Therefore, for all $\varphi \in (0,\varphi_c)$ and all $m \in \mathbb{Z}_+$ we have that

$$F'_m(\varphi) \ge \min\left\{F'_0(\varphi), \lim_{m\to+\infty} F'_m(\varphi)\right\} = \min\left\{0, \lim_{m\to+\infty} F'_m(\varphi)\right\} \tag{1.45}$$

But the derivative of F_m is

$$F'_m(\varphi) = -\frac{Z'(\varphi)}{Z(\varphi)^2}\sum_{k=m}^{\infty}\frac{\varphi^k}{g!(k)} + \frac{1}{Z(\varphi)}\sum_{k=m}^{\infty} k\frac{\varphi^{k-1}}{g!(k)}$$

and since the series $\sum_{k=0}^{\infty}\frac{\varphi^k}{g!(k)}$ and $\sum_{k=0}^{\infty} k\frac{\varphi^{k-1}}{g!(k)}$ converge we obviously have that

$$\lim_{m\to+\infty} F'_m(\varphi) = 0$$

for each fixed $\varphi \in (0, \varphi_c)$. Together with (1.45) this proves that $F'_m(\varphi) \geq 0$ for all $m \in \mathbb{Z}_+$, $\varphi \in (0, \varphi_c)$ as required. □

By the coupling characterization of the partial order of probability measures it follows that for all $\rho_1, \rho_2 \in I_c$ such that $\rho_1 \leq \rho_2$ there exists a *monotone plan* $\pi_{\rho_1,\rho_2} \in \Pi(\nu^1_{\rho_1}, \nu^1_{\rho_2})$, that is a probability measure $\pi_{\rho_1,\rho_2} \in \mathbb{P}_1(\mathbb{Z}_+ \times \mathbb{Z}_+)$ with first and second marginals the measures $\nu^1_{\rho_1}$ and $\nu^1_{\rho_2}$ respectively such that

$$\pi_{\rho_1,\rho_2}\Big\{(k,m) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \,\Big|\, k \leq m\Big\} = 1.$$

Here, given two Borel probability measures $\mu, \nu \in \mathbb{P}X$ in a polish space X the set $\Pi(\mu,\nu)$ denotes the set of all measures in $\mathbb{P}(X \times X)$ with first and second marginals equal to μ and ν respectively. For the following proposition we recall that the 1-Wasserstein metric on the set of Borel probability measures on a Polish space (X, d) is given by

$$W_1(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int d(x,y)d\pi(x,y).$$

The infimum is always attained, and any minimizer is called an optimal plan. For more a detailed description of the Wasserstein metrics the reader is referred to [31], chapter 7.

Proposition 1.2.10 *Let $\{\nu^1_\rho\}_{\rho\in I_c}$ be the family of normalized one-site ZR distribution associated to the local rate function g. Any monotone plan $\pi_{\rho_1,\rho_2} \in \Pi(\nu^1_{\rho_1}, \nu^1_{\rho_2})$, $\rho_1 \leq \rho_2$, is optimal and*

$$W_1(\nu^1_{\rho_1}, \nu^1_{\rho_2}) = |\rho_1 - \rho_2| \tag{1.46}$$

for all $\rho_1, \rho_2 \in I_c$. Furthermore, for any Lipschitz function $f : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ with Lipschitz constant L_f, the function $F : I_c \longrightarrow \mathbb{R}_+$ given by the formula

$$F(\rho) = \int f d\nu^1_\rho = \frac{1}{Z\big(\Phi(\rho)\big)}\sum_{k=0}^{\infty} f(k)\frac{\Phi(\rho)^k}{g!(k)}$$

is Lipschitz with Lipschitz constant $\leq L_f$.

Proof Let $\rho_1, \rho_2 \in I_c$ be such that $\rho_1 \leq \rho_2$ and let $\pi_{\rho_1,\rho_2} \in \Pi(\nu^1_{\rho_1}, \nu^1_{\rho_2})$ be a monotone plan from $\nu^1_{\rho_1}$ to $\nu^1_{\rho_2}$. Then,

$$\begin{aligned} W_1(\nu^1_{\rho_1}, \nu^1_{\rho_2}) &\leq \int |k-m| d\pi_{\rho_1,\rho_2}(k,m) = \int (m-k) d\pi_{\rho_1,\rho_2}(k,m) \\ &= \int m d\nu^1_{\rho_2}(m) - \int k\nu^1_{\rho_1}(k) = \rho_2 - \rho_1, \end{aligned}$$

and therefore

$$W_1(\nu^1_{\rho_1}, \nu^1_{\rho_2}) \leq |\rho_1 - \rho_2|$$

for all $\rho_1, \rho_2 \in I_c$. On the other hand, the barycentric projection $b : \mathbb{P}_1\mathbb{R} \longrightarrow \mathbb{R}$ given by $b(\mu) = \int_{\mathbb{R}} x d\mu(x)$ is a contraction with respect to the 1-Wasserstein distance and so if we denote by $i : \mathbb{Z}_+ \hookrightarrow \mathbb{R}$ the natural embedding, then

$$|\rho_1 - \rho_2| = |b(i_*\nu^1_{\rho_1}) - b(i_*\nu^1_{\rho_2})| \leq W_1(i_*\nu^1_{\rho_1}, i_*\nu^1_{\rho_2})$$

for all $\rho_1, \rho_2 \in I_c$. But if $i \times i : \mathbb{Z}_+ \times \mathbb{Z}_+ \longrightarrow \mathbb{R} \times \mathbb{R}$ is the product of the injection i with itself and $\pi_0 \in \Pi(\nu^1_{\rho_1}, \nu^1_{\rho_2})$ is an optimal plan then

$$(i \times i)_*\pi_0 \in \Pi(i_*\nu^1_{\rho_1}, i_*\nu^1_{\rho_2})$$

is a transport plan from $i_*\nu^1_{\rho_1}$ to $i_*\nu^1_{\rho_2}$ and therefore

$$W_1(i_*\nu^1_{\rho_1}, i_*\nu^1_{\rho_2}) \leq \int |x - y| d(i \times i)_*\pi_0(x, y) = \int |k - m| d\pi_0(k, m) = W_1(\nu^1_{\rho_1}, \nu^1_{\rho_2}).$$

This proves (1.46) and that π_{ρ_1,ρ_2} is optimal.

We prove now the second claim. Let L_f be the Lipschitz constant of f. Then, for all $\rho_1, \rho_2 \in I_c$, $\rho_1 \leq \rho_2$, if $\pi_{\rho_1,\rho_2} \in \Pi(\nu^1_{\rho_1}, \nu^1_{\rho_2})$ is a monotone plan, we have that

$$\begin{aligned} |F(\rho_1) - F(\rho_2)| &= \left| \int \big(f(k) - f(m)\big) d\pi_{\rho_1,\rho_2}(k, m) \right| \\ &\leq L_f \int |k - m| d\pi_{\rho_1,\rho_2}(k, m) = L_f(\rho_2 - \rho_1), \end{aligned}$$

and therefore F is Lipschitz with constant $\leq L_f$. □

The above properties of the one-site ZR distributions extend easily to the ZR distributions on the discrete toruses $\mathbb{T}^d_N$, $N \in \mathbb{N}$.

Proposition 1.2.11 *The family $\{\nu_\rho\}_{\rho \in I_c} \subseteq \mathbb{P}_1\mathbb{M}^d_N$ of normalized ZR distributions on the discrete torus $\mathbb{T}^d_N$ is increasing. Any monotone plan $\nu_{\rho_1,\rho_2} \in \Pi(\nu_{\rho_1}, \nu_{\rho_2})$ is optimal, and*

$$W_1(\nu_{\rho_1}, \nu_{\rho_2}) = N^d |\rho_1 - \rho_2|.$$

Furthermore, for any Lipschitz function $f : \mathbb{M}^d_N \longrightarrow \mathbb{R}$ with Lipschitz constant L_f the function $F : I_c \longrightarrow \mathbb{R}$ given by

$$F(\rho) = \int f(\eta) d\nu_\rho(\eta)$$

is Lipschitz with constant $N^d L_f$.

Proof The fact that $\{\nu_\rho\}_{\rho \in I_c}$ is increasing follows from the fact that the family of its common marginal $\{\nu^1_\rho\}$ is increasing. Indeed, let $\rho_1, \rho_2 \in I_c$ be such that $\rho_1 \leq \rho_2$ and let $\pi_{\rho_1,\rho_2} \in \Pi(\nu^1_{\rho_1}, \nu^1_{\rho_2})$ be a monotone plan from $\nu^1_{\rho_1}$ to $\nu^1_{\rho_2}$. But then the measure

$$\nu_{\rho_1,\rho_2} := \prod_{x \in \mathbb{T}^d_N} \pi_{\rho_1,\rho_2} \in \mathbb{P}_1(\mathbb{Z}_+ \times \mathbb{Z}_+)^{\mathbb{T}^d_N} \cong \mathbb{P}_1(\mathbb{M}^d_N \times \mathbb{M}^d_N). \tag{1.47}$$

is a transport plan from ν_{ρ_1} to ν_{ρ_2} through the identification

$$(\mathbb{Z}_+ \times \mathbb{Z}_+)^{\mathbb{T}_N^d} \ni \big(\eta_x, \zeta_x\big)_{x\in\mathbb{T}_N^d} \mapsto \big((\eta_x)_{x\in\mathbb{T}_N^d}, (\zeta_x)_{x\in\mathbb{T}_N^d}\big) \in \mathbb{M}_N^d \times \mathbb{M}_N^d, \tag{1.48}$$

and it is monotone, since

$$\begin{aligned}
\nu_{\rho_1,\rho_2}\Big\{(\eta,\zeta) \in \mathbb{M}_N^d \times \mathbb{M}_N^d \,\Big|\, \eta \le \zeta\Big\} &= \nu_{\rho_1,\rho_2}\Bigg(\bigcap_{x\in\mathbb{T}_N^d} \{(\eta,\zeta)|\eta_x \le \zeta_x\}\Bigg) \\
&= \prod_{x\in\mathbb{T}_N^d} \nu_{\rho_1,\rho_2}\{(\eta,\zeta)\,|\,\eta_x \le \zeta_x\} \\
&= \prod_{x\in\mathbb{T}_N^d} \pi_{\rho_1,\rho_2}\{(k,m) \in \mathbb{Z}_+^2 | k \le m\} = 1.
\end{aligned}$$

Furthermore, since the measure $\pi_{\rho_1,\rho_2} \in \Pi(\nu^1_{\rho_1}, \nu^1_{\rho_2})$ is optimal, as we know the product measure ν_{ρ_1,ρ_2} defined in (1.47) through the identification in (1.48) is an optimal transport plan from ν_{ρ_1} to ν_{ρ_2}, that is

$$\begin{aligned}
W_1(\nu_{\rho_1}, \nu_{\rho_2}) &= \int |\eta - \zeta|_1 d\nu_{\rho_1,\rho_2}(\eta,\zeta) = \sum_{x\in\mathbb{T}_N^d} \int |\eta(x) - \zeta(x)| d\nu_{\rho_1,\rho_2} \\
&= \sum_{x\in\mathbb{T}_N^d} \int |k-m| d\pi_{\rho_1,\rho_2}(k,m) = N^d W^1(\nu^1_{\rho_1}, \nu^1_{\rho_2}).
\end{aligned}$$

This proves the first claim. Next, if $f : \mathbb{M}_N^d \longrightarrow \mathbb{R}$ is Lipschitz with Lipschitz constant L_f, that is

$$|f(\eta) - f(\zeta)| \le L_f |\eta - \zeta|_1$$

for all $\eta, \zeta \in \mathbb{M}_N^d$, then for any $\rho_1, \rho_2 \in I_c$, $\rho_1 \le \rho_2$, we have that

$$|F(\rho_1) - F(\rho_2)| \le \int |f(\eta) - f(\zeta)| d\nu_{\rho_1,\rho_2}(\eta,\zeta) \le L_f W_1(\nu_{\rho_1,\rho_2}) \le N^d L_f |\rho_1 - \rho_2|.$$

This proves that F is Lipschitz with constant $\le N^d L_f$ and completes the proof. $\square$

As we have seen, when $\rho_* < \rho_c$, the distribution $\nu^1_{\rho_*} = \eta(x)_* \nu^N_{\rho_*}$ has finite exponential moments. We will see next the form that Cramer's theorem takes for the i.i.d. sequence $\{\eta(x)\}_{x\in\mathbb{Z}^d}$ of ν^1_ρ-distributed r.v., $\rho_* < \rho_c$.

Proposition 1.2.12 *Let $\nu_{\rho_*} \in \mathbb{PM}^d_\infty$ be the ZR-range distribution on the full lattice with density $\rho_* < \rho_c$. Then the i.i.d. sequence $\{\eta(x)\}_{x\in\mathbb{Z}^d}$ satisfies the large deviations principle with rate function*

$$I_{\rho_*}(\rho) := \Lambda^*_{\nu^1_{\rho_*}}(\rho) = \begin{cases} \rho \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)} - \log \frac{Z(\Phi(\rho\wedge\rho_c))}{Z(\Phi(\rho_*))}, & \rho \ge 0 \\ +\infty, & \rho < 0 \end{cases}.$$

Proof The logarithmic moment generating function $\Lambda_{\nu^1_{\rho_*}} : \mathbb{R} \longrightarrow (-\infty, \infty]$ is given by

$$\Lambda_{\nu^1_{\rho_*}}(\theta) = \log M_{\nu^1_{\rho_*}}(\theta) = \log \frac{Z\big(\Phi(\rho_*)e^\theta\big)}{Z\big(\Phi(\rho_*)\big)}.$$

By Cramer's theorem on $\mathbb{R}$ the i.i.d. sequence of ν_{ρ_*}-r.v. $\{\eta(x)\}_{x\in\mathbb{Z}^d}$ satisfies the large deviation principle with rate function I_{ρ_*} the Legendre transform

$$I_{\rho_*}(\rho) := \Lambda^*_{\nu^1_{\rho_*}}(\rho) = \sup_{\theta\in\mathbb{R}}\{\theta\rho - \Lambda_{\nu^1_{\rho_*}}(\theta)\}$$

of the logarithmic moment generating function of $\nu^1_{\rho_*}$. Obviously the logarithmic m.g.f. $\Lambda_{\nu^1_{\rho_*}}$ has proper domain $\mathcal{D}_{\Lambda_{\nu^1_{\rho_*}}}$ such that $(-\infty, b_{\rho_*}) \subseteq \mathcal{D}_{\Lambda_{\nu^1_{\rho_*}}} \subseteq (-\infty, b_{\rho_*}]$, where

$$b_{\rho_*} := \log\phi_c - \log\Phi(\rho_*) = \log\frac{\Phi(\rho_c)}{\Phi(\rho_*)},$$

and its derivative in $(-\infty, b_{\rho_*})$ is given by

$$\Lambda'_{\nu^1_{\rho_*}}(\theta) = \frac{e^\theta\Phi(\rho_*)Z'\big(e^\theta\Phi(\rho_*)\big)}{Z\big(e^\theta\Phi(\rho_*)\big)} = R\big(e^\theta\Phi(\rho_*)\big).$$

Obviously $\Lambda'_{\nu^1_{\rho_*}}(-\infty) := \lim_{\theta\downarrow-\infty}\Lambda'_{\nu^1_{\rho_*}}(\theta) = 0$ and $\Lambda'_{\nu^1_{\rho_*}}(b_{\rho_*}) := \lim_{\theta\uparrow b_{\rho_*}}\Lambda'_{\nu^1_{\rho_*}}(\theta) = \rho_c$, and as we know the Legendre transform $\Lambda^*_{\nu^1_{\rho_*}}$ is given on the interval

$$(0,\rho_c) = \big(\Lambda'_{\nu^1_{\rho_*}}(-\infty), \Lambda'_{\nu^1_{\rho_*}}(b_{\rho_*})\big) \subseteq \mathcal{D}_{\Lambda^*_{\nu^1_{\rho_*}}},$$

by the formula

$$\Lambda^*_{\nu^1_{\rho_*}}(\rho) = \rho(\Lambda'_{\nu^1_{\rho_*}})^{-1}(\rho) - \Lambda_{\nu^1_{\rho_*}}\big((\Lambda'_{\nu^1_{\rho_*}})^{-1}(\rho)\big)$$

Now since $\Phi = R^{-1}$ it is obvious that $(\Lambda'_{\nu^1_{\rho_*}})^{-1} : (0,\rho_c) \longrightarrow \mathbb{R}$ is given by the formula

$$(\Lambda'_{\nu^1_{\rho_*}})^{-1}(\rho) = \log\frac{\Phi(\rho)}{\Phi(\rho_*)}$$

and therefore

$$\Lambda^*_{\nu^1_{\rho_*}}(\rho) = \rho\log\frac{\Phi(\rho)}{\Phi(\rho_*)} - \log\frac{Z\big(\Phi(\rho)\big)}{Z\big(\Phi(\rho_*)\big)}, \quad \forall\, \rho\in(0,\rho_c).$$

On the other hand, whenever $\rho_c < +\infty$ we have that $Z(\varphi_c) < +\infty$ and thus

$$\Lambda_{\nu^1_{\rho_*}}(b_{\rho_*}) = \log\frac{Z\big(\Phi(\rho_*)e^{b_{\rho_*}}\big)}{Z\big(\Phi(\rho_*)\big)} = \log\frac{Z\big(\Phi(\rho_c)\big)}{Z\big(\Phi(\rho_*)\big)} < +\infty.$$

So $b_{\rho_*} \in \mathcal{D}_{\Lambda_{\nu^1_{\rho_*}}}$ and therefore $\Lambda^*_{\nu^1_{\rho_*}}$ is given on $[\rho_c, +\infty)$ by the formula

$$\Lambda^*_{\nu^1_{\rho_*}}(\rho) = b_{\rho_*}\cdot\rho - \Lambda_{\nu^1_{\rho_*}}(b_{\rho_*}) = \rho\log\frac{\Phi(\rho_c)}{\Phi(\rho_*)} - \log\frac{Z\big(\Phi(\rho_c)\big)}{Z\big(\Phi(\rho_*)\big)}.$$

Since $\Lambda^*_{\nu^1_{\rho_*}}(0) = -\log\nu^1_{\rho_*}(0)$ and $\Lambda^*_{\nu^1_{\rho_*}}(\rho) = +\infty$ for all $\rho < 0$, the claim is proved. □

Finally, we note that the knowledge that the normalized ZR distributions on the discrete toruses are invariant distributions for the ZRP allows us to obtain a formula for the extremal invariant distributions $\nu^d_{N,K} \in \mathbb{PM}^d_N$ concentrated on the communication classes $\mathbb{M}^d_{N,K}$ consisting of configurations with a fixed number of particles $K \in \mathbb{Z}_+$. Indeed, let $\{\nu^N_\rho\}_{\rho\in I_c}$ be the family of ZR distributions on the discrete torus $\mathbb{T}^d_N$ associated to the local rate function g. For any $\rho \in I_c$, the distribution

$$\nu^N_\rho\big|\mathbb{M}^d_{N,K} := \nu^N_\rho\big(\,\cdot\,\big|\{|\cdot|_1 = K\}\big) = \frac{\mathbb{1}_{\{K\}}(|\cdot|_1)}{\nu_\rho\{|\cdot| = K\}}d\nu_\rho$$

is an invariant distribution of the ZRP, since if $|\eta|_1 \neq K$ then

$$(\nu^N_\rho\big|\mathbb{M}^d_{N,K})L^N(\eta) = \sum_{\zeta\in\mathbb{M}^d_{N,|\eta|_1}} \nu^N_\rho\big|\mathbb{M}^d_{N,K}(\zeta)L^N(\zeta,\eta) = 0$$

since $\nu^N_\rho\big|\mathbb{M}^d_{N,K}$ is concentrated by definition on configurations with K particles, while if $|\eta|_1 = K$, then

$$\begin{aligned}
(\nu^N_\rho\big|\mathbb{M}^d_{N,K})L^N(\eta) &= \sum_{\zeta\in\mathbb{M}^d_{N,K}} \nu^N_\rho\big|\mathbb{M}^d_{N,K}(\zeta)L^N(\zeta,\eta) \\
&= \frac{1}{\nu^N_\rho\{|\eta|_1 = K\}} \sum_{\zeta\in\mathbb{M}^d_{N,K}} \nu^N_\rho(\zeta)L^N(\zeta,\eta) \\
&= \frac{1}{\nu^N_\rho\{|\eta|_1 = K\}} \sum_{\zeta\in\mathbb{M}^d_{N}} \nu^N_\rho(\zeta)L^N(\zeta,\eta) = 0.
\end{aligned}$$

So by the uniqueness of the extremal invariant distributions $\nu^d_{N,K}$ we have that for any $\rho \in I_c$,

$$\nu^d_{N,K}(\cdot) = \nu^N_\rho\big(\,\cdot\,\big|\{|\cdot|_1 = K\}\big).$$

Therefore for each $\eta \in \mathbb{M}^d_{N,K}$ we have that

$$\nu^d_{N,K}(\eta) = \frac{\nu_\rho(\eta)}{\sum_{\eta\in\mathbb{M}^d_{N,K}}\nu_\rho(\eta)} = \frac{\frac{1}{g!(\eta)}}{\sum_{\eta\in\mathbb{M}^d_{N,K}}\frac{1}{g!(\eta)}}.$$

The family $\{\nu_{N,K} \in \mathbb{P}\mathbb{M}^d_{N,K}\}$ is called the *canonical ensemble.*

1.3 Useful Topological Spaces

In the study of the hydrodynamic behavior of ZRPs we need to consider limits of distributions on the configuration spaces $\mathbb{M}_N^d$ as the inverse particle's distance N tends to infinity. We do so by embedding the configuration spaces $\mathbb{M}_N^d$ in the space $\mathbb{M}_\infty^d := \mathbb{Z}_+^{\mathbb{Z}^d}$ of configurations over the full lattice, via the periodic embeddings

$$\mathbb{M}_N^d \ni \eta = (\eta_x)_{x \in \mathbb{T}_N^d} \overset{e_N}{\mapsto} e_N(\eta) = (\eta_{[z]_N})_{z \in \mathbb{Z}^d} \in \mathbb{M}_\infty^d$$

induced by the the quotient mappings $[\cdot]_N : \mathbb{Z}^d \longrightarrow \mathbb{T}_N^d$. We will always suppress e_N from the notation identifying $e_N(\mathbb{M}_N^d)$ with $\mathbb{M}_N^d$.

To study the evolution of the particles density under the dynamics of the ZRPs we will also consider the empirical embeddings of configurations in the space $\mathcal{M}_+(\mathbb{T}^d)$ of positive measures on the macroscopic torus. These are defined by

$$\pi_\eta^N := \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_x \delta_{\frac{x}{N}} \in \mathcal{M}_+(\mathbb{T}^d), \quad \eta \in \mathbb{M}_N^d.$$

In this section we will describe the basic properties of the product topology on $\mathbb{M}_\infty^d$, the Wasserstein topologies on the spaces $\mathbb{P}_p \mathbb{M}_\infty^d$, $p \geq 0$, of probability measures with finite p-th moments on $\mathbb{M}_\infty^d$ and the weak topology on $\mathcal{M}_+(\mathbb{T}^d)$

Before proceeding with the description of these topologies we describe some relevant identifications that will be used throughout without being mentioned. For each $N \in \mathbb{N}$ we set

$$\mathbb{Z}_N := \left\{ -\left[\frac{N}{2}\right], -\left[\frac{N}{2}\right] + 1, \dots, \left[\frac{N-1}{2}\right] \right\}.$$

Obviously $\sharp \mathbb{Z}_N = N$ and the restriction of the quotient mapping $[\cdot]_N : \mathbb{Z}^d \longrightarrow \mathbb{T}_N^d$ given by $[z]_N = z + N\mathbb{Z}^d$ on the set $\mathbb{Z}_N^d$ is a bijection. It's inverse $j_N : \mathbb{T}_N^d \longrightarrow \mathbb{Z}_N^d \subseteq \mathbb{Z}^d$ defines an injection of the discrete torus $\mathbb{T}_N^d$ in the full lattice $\mathbb{T}_\infty^d := \mathbb{Z}^d$. We will consider the discrete toruses $\mathbb{T}_N^d$ embedded in the full lattice through the injections j_N, $N \in \mathbb{N}$. Through these embeddings, as $N \to +\infty$ the discrete toruses $\mathbb{T}_N^d$ converge as normed groups to the full lattice $\mathbb{T}_\infty^d := \mathbb{Z}^d$ in the pointed Hausdorff convergence, since if we denote by $|\cdot|_N : \mathbb{T}_N \longrightarrow \mathbb{R}_+$,

$$|x|_N := \min\{x, N - x\},$$

the translation invariant norm of the group $\mathbb{T}_N$ and by $|\cdot|_{N,\infty} : \mathbb{T}_N^d \longrightarrow \mathbb{R}_+$ the induced ℓ_∞-norm on $\mathbb{T}_N^d$,

$$|x|_{N,\infty} = \max_{i=1,\dots,d} |x_i|_N$$

we have that for all $N, R \in \mathbb{Z}_+$ such that $N \geq 4R$,

$$j_N\big(B_{N,\infty}(0,R)\big) = B_\infty(0,R)$$

where $B_{N,\infty}(0,R)$ is the ball of center 0 and radius R in $\mathbb{T}_N^d$ with respect to $|\cdot|_{N,\infty}$, and the ball $B_\infty(0,R) \subseteq \mathbb{Z}^d$ is with respect to the usual ℓ^∞ metric, $|x|_\infty := \max_{i=1,\dots,d} |x_i|$. Through the identifications $\mathbb{T}_N^d \cong \mathbb{Z}_N^d$ described above we can identify the space $\mathbb{M}_N^d$ with the space $\mathbb{Z}_+^{\mathbb{Z}_N^d}$ and then the natural projections $p^N : \mathbb{M}_\infty^d \longrightarrow \mathbb{Z}_+^{\mathbb{Z}_N^d}$ can be viewed as $p^N : \mathbb{M}_\infty^d \longrightarrow \mathbb{M}_N^d$ and obviously $p_N \circ e_N = id_{\mathbb{M}_N^d}$.

1.3.1 The Space $\mathbb{M}_\infty^d := \mathbb{Z}_+^{\mathbb{Z}^d}$

We will denote the $|\cdot|_\infty$-ball of radius ℓ in $\mathbb{Z}^d$ by $\Lambda_\ell^d := \{z \in \mathbb{Z}^d \big| |z| \leq \ell\} \cong \mathbb{T}_{(2\ell+1)}^d$.

Proposition 1.3.1 *The space of configurations $\mathbb{M}_\infty^d$ on the full lattice $\mathbb{Z}^d$ is a polish when equipped with the product topology. A polish metric inducing the product topology of $\mathbb{M}_\infty^d$ is $d : \mathbb{M}_\infty^d \times \mathbb{M}_\infty^d \longrightarrow [0,1]$ given by*

$$d(\eta, \zeta) = \sum_{x \in \mathbb{Z}^d} \frac{1}{h(|x|_\infty)} \frac{|\eta_x - \zeta_x|}{1 + |\eta_x - \zeta_x|}$$

where $h : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ is the mapping given by

$$h(\ell) = 2^{\ell+1} \sharp(\Lambda_\ell^d \setminus \Lambda_{\ell-1}^d),$$

with the convention that $\Lambda_{-1}^d = \emptyset$.

Proof It is obvious by the definition of d that

$$d(\eta, \zeta) = \sum_{\ell=0}^\infty \frac{1}{h(\ell)} \sum_{x \in \Lambda_\ell^d \setminus \Lambda_{\ell-1}^d} \frac{|\eta_x - \zeta_x|}{1 + |\eta_x - \zeta_x|} \leq \sum_{\ell=0}^\infty \frac{1}{2^{\ell+1}} = 1.$$

Is is also obvious that d is symmetric and that $d(\eta, \zeta) = 0$ iff $\eta = \zeta$. As usual with metrics of the form of d the triangle inequality follows by the fact that the function $t \mapsto \frac{t}{1+t}$, $t \geq 0$ is increasing and the elementary inequality $\frac{s+t}{1+s+t} \leq \frac{t}{1+t} + \frac{s}{1+s}$.

Next we verify that d metrized the product topology $\mathcal{T}$ of $\mathbb{M}_\infty^d$. By definition the product topology is weakest topology on $\mathbb{M}_\infty^d$ with respect to which the natural projections $\eta(x) : \mathbb{M}_\infty \longrightarrow \mathbb{Z}_+$, $x \in \mathbb{Z}^d$, given by $\eta(x)(\eta) = \eta_x$ are continuous. We will show first that the natural projections $\eta(x)$ are d-continuous. So let $\{\eta^N\}_{N=1}^\infty \subseteq \mathbb{M}_\infty^d$ be a sequence of configurations d-converging to $\eta \in \mathbb{M}_\infty^d$. Given $x \in \mathbb{Z}^d$ there exists then $N_x \in \mathbb{N}$ such that

$$N \geq N_x \quad \Longrightarrow \quad d(\eta^N, \eta) < \frac{1}{2h(|x|)}.$$

Then for every $N \geq N_x$ we have that

$$\frac{1}{h(|x|)} \frac{|\eta^N(x) - \eta(x)|}{1 + |\eta^N(x) - \eta(x)|} < d(\eta^N, \eta) < \frac{1}{2h(|x|)}.$$

Consequently $|\eta^N(x) - \eta(x)| < 1$ which since configurations $\eta \in \mathbb{M}_\infty^d$ take only integer values implies that $\eta^N(x) \equiv \eta(x)$ for all $N \geq N_x$. Since $x \in \mathbb{Z}^d$ was arbitrary this proves that the product topology $\mathcal{T}$ is contained in the topology $\mathcal{T}_d$ induced by the metric d. For the converse inclusion, suppose that $\{\eta^j\}_{j \in J} \subseteq \mathbb{M}_\infty^d$ is a net converging in product topology to $\eta \in \mathbb{M}_\infty^d$ and let $\varepsilon > 0$. There exists $\ell_\varepsilon \in \mathbb{N}$ such that $\sum_{\ell=\ell_\varepsilon+1}^\infty \frac{1}{2^{\ell+1}} < \varepsilon$ and since $\eta^j \longrightarrow \eta$ in the product topology there exists $j_0 \in J$ such that

$$x \in \Lambda_{\ell_\varepsilon}^d, \ j \geq j_0 \quad \eta^j(x) \equiv \eta(x).$$

But then for all $j \geq j_0$ we have that

$$d(\eta^j, \eta) = \sum_{\ell=\ell_\varepsilon+1} \frac{1}{h(\ell)} \sum_{x \in \Lambda_\ell^d \setminus \Lambda_{\ell-1}^d} \frac{|\eta^j(x) - \eta(x)|}{1 + |\eta^j(x) - \eta(x)|} \leq \sum_{\ell=\ell_\varepsilon+1}^\infty \frac{1}{2^{\ell+1}} < \varepsilon.$$

which proves that $d(\eta^j,\eta) \longrightarrow 0$ and so d metrizes the product topology on $\mathbb{M}^d_\infty$.

We prove next that $\mathbb{M}^d_\infty$ is a complete and separable metric space when equipped with the metric d. For the separability, a countable dense subset of $\mathbb{M}^d_\infty$ is the set

$$A := \bigcup_{\ell\in\mathbb{Z}_+} A_\ell, \quad \text{where } A_\ell := \big\{\eta \in \mathbb{M}^d_\infty \big| \eta|_{\mathbb{Z}^d\setminus\Lambda^d_\ell} \equiv 0\big\}.$$

Indeed, each $A_\ell \cong \mathbb{Z}_+^{\Lambda^d_\ell}$ is countable as a finite product of countable spaces and so A is countable as a countable union of countable sets. Is also easy to check that A is dense in $\mathbb{M}^d_\infty$. Indeed, given $\eta\in\mathbb{M}^d_\infty$ and $\varepsilon>0$ we pick $\ell_\varepsilon\in\mathbb{N}$ such that $\sum_{\ell=\ell_\varepsilon+1}^\infty \frac{1}{2^{\ell+1}} < \varepsilon$ and define $\zeta\in\mathbb{M}^d_\infty$ by $\zeta = \eta\mathbb{1}_{\Lambda^d_{\ell_\varepsilon}}$. Then $\zeta|_{\Lambda^d_{\ell_\varepsilon}} \equiv \eta|_{\Lambda^d_{\ell_\varepsilon}}$ and so

$$d(\eta,\zeta) = \sum_{\ell=\ell_\varepsilon+1} \frac{1}{h(\ell)} \sum_{x\in\Lambda^d_\ell\setminus\Lambda^d_{\ell-1}} \frac{|\eta(x)-\zeta(x)|}{1+|\eta(x)-\zeta(x)|} \leq \sum_{\ell=\ell_\varepsilon+1}^\infty \frac{1}{2^{\ell+1}} < \varepsilon$$

which proves that A is dense in $\mathbb{M}_\infty$.

We prove finally that d is a complete metric. So let $\{\eta^N\}\subseteq\mathbb{M}^d_\infty$ be d-Cauchy sequence. Then for each $x\in\mathbb{Z}^d$ there exists $N_x\in\mathbb{N}$ such that $d(\eta^N,\eta^M) < \frac{1}{2h(|x|)}$ for all $N,M\geq N_x$. Since configurations are integer valued this implies that

$$N,M\geq N_x \quad\Longrightarrow\quad \eta^N(x)=\eta^M(x).$$

Defining the configuration $\eta\in\mathbb{M}^d_\infty$ by $\eta_x = \eta^{N_x}_x$ it is easy to see that $d(\eta^N,\eta)\longrightarrow 0$ as $N\to\infty$. Indeed, given $\varepsilon>0$ we choose $\ell_\varepsilon\in\mathbb{N}$ such that $\sum_{\ell=\ell_\varepsilon+1}^\infty\frac{1}{2^{\ell+1}}<\varepsilon$ and set $N_0=\max_{x\in\Lambda^d_{\ell_\varepsilon}} N_x < +\infty$. Then obviously $\eta^N|_{\Lambda^d_{\ell_\varepsilon}} \equiv \eta|_{\Lambda^d_{\ell_\varepsilon}}$ for all $N\geq N_0$ and therefore $d(\eta^N,\eta)<\varepsilon$ which since $\varepsilon>0$ was arbitrary proves that d is a complete metric. □

Proposition 1.3.2 *Let $e_N:\mathbb{M}^d_N\longrightarrow\mathbb{M}^d_\infty$ denote the periodic embeddings $e_N(\eta)(x)=\eta(x+N\mathbb{Z}^d)$ and let $p_N:\mathbb{M}^d_\infty\longrightarrow\mathbb{M}^d_N$ be the natural projections, $N\in\mathbb{N}$. Then the functions $I_N := e_N\circ p_N:\mathbb{M}^d_\infty\longrightarrow\mathbb{M}^d_\infty$, $N\in\mathbb{N}$, converge uniformly on $\mathbb{M}^d_\infty$ to the identity function $id_{\mathbb{M}^d_\infty}:\mathbb{M}^d_\infty\longrightarrow\mathbb{M}^d_\infty$.*

Proof Let $\varepsilon>0$ and choose $\ell_0\in\mathbb{Z}_+$ such that $\sum_{\ell=\ell_0+1}^\infty\frac{1}{2^{\ell+1}}<\varepsilon$. Recalling the identification $\mathbb{T}^d_N\cong\mathbb{Z}^d_N\subseteq\mathbb{Z}^d$ we have that $I_N(\eta)|_{\mathbb{T}^d_N}\equiv\eta|_{\mathbb{T}^d_N}$ for all $N\in\mathbb{N}$ and all $\eta\in\mathbb{M}^d_\infty$ and so for all $N\geq(2\ell_0+1)$ we have that

$$d\big(I_N(\eta),\eta\big) \leq \sum_{\ell=\ell_0+1}^\infty \frac{1}{h(\ell)} \sum_{x\in\Lambda^d_\ell\setminus\Lambda^d_{\ell-1}} \frac{|I_N(\eta)(x)-\eta(x)|}{1+|I_N(\eta)(x)-\eta(x)|} \leq \sum_{\ell_0+1}^\infty \frac{1}{2^{\ell+1}} < \varepsilon,$$

for all $\eta\in\mathbb{M}^d_\infty$ which proves the the uniform convergence $I_N\longrightarrow id_{\mathbb{M}^d_\infty}$. □

Obviously the space $\mathbb{M}^d_\infty$ is not compact since $\mathbb{Z}_+$ is not. However the compact subsets of $\mathbb{M}^d_\infty$ can be easily characterized. This characterization can be stated in a nice form by considering the natural pointwise partial order:

$$\eta\leq\zeta \quad\Longleftrightarrow\quad \eta(x)\leq\zeta(x) \quad \text{for all } x\in\mathbb{Z}^d.$$

This partial order gives rise to the corresponding integrals

$$[\eta,\zeta] := \big\{\xi\in\mathbb{M}_\infty\big|\eta\leq\xi\leq\zeta\big\}.$$

Definition 1.3.1 A set $B \subseteq \mathbb{M}_\infty^d$ is called $\leq$-*bounded* if $B \subseteq [0,\zeta]$ for some $\zeta \in \mathbb{M}_\infty^d$.

Proposition 1.3.3 *A set $K \subseteq \mathbb{M}_\infty^d$ is compact iff it is closed and $\leq$-bounded.*

Proof We assume first that K is compact. Then K is closed as a compact subset of a Hausdorff space. Furthermore the functions $\eta(x) : \mathbb{M}_\infty \longrightarrow \mathbb{Z}_+$, $x \in \mathbb{Z}^d$, are continuous in the product topology and so by the compactness of K we have that

$$\zeta_x := \sup_{\eta \in K} \eta_x = \max_{\eta \in K} \eta_x < +\infty$$

and for the configuration $\zeta \in \mathbb{M}_\infty$ defined above we obviously have that $K \subseteq [0,\zeta]$. This proves that K is $\leq$-bounded and completes the proof of this implication.

We suppose for the converse that K is closed and that there exists $\zeta \in \mathbb{M}_\infty^d$ such that $K \subseteq [0,\zeta]$. By Tychonov's theorem the product space

$$[0,\zeta] = \prod_{x \in \mathbb{Z}^d} \{0,1,\ldots,\zeta_x\}$$

is compact. Now, the product topology of $[0,\zeta]$ coincides with the topology it inherits as a subspace of $\mathbb{M}_\infty^d$ and therefore K is relatively compact. Since it is also closed by assumption, it is compact. $\square$

We conclude this discussion on the topology of $\mathbb{M}_\infty^d$ with two more propositions to further illuminate its structure.

Proposition 1.3.4 *The space $\mathbb{M}_\infty^d$ contains no relatively compact neighborhoods.*

Proof The family $\mathcal{N}(\eta) = \{V(\eta,\ell) | \ell \in \mathbb{Z}_+\}$ where

$$V(\eta,\ell) := \left\{\zeta \in \mathbb{M}_\infty^d | \zeta|_{\Lambda_\ell^d} \equiv \eta_{\Lambda_\ell^d}\right\}$$

is a basis of neighborhoods around $\eta \in \mathbb{M}_\infty^d$ and no such neighborhood can be relatively compact. Indeed, given any such neighborhood $V(\eta,\ell)$, the sequence of configurations $\zeta^m := \eta \mathbb{1}_{\Lambda_\ell^d} + m\mathbb{1}_{\{(\ell+1)e_1\}}$ is contained in $V(\eta,\ell)$ and $\zeta^m((\ell+1)e_1) = m \longrightarrow \infty$ as $m \to \infty$ so that ζ^m can not have a convergent subsequence. $\square$

Proposition 1.3.5 *The space $\mathbb{M}_\infty^d$ is totally disconnected, i.e. the only continuous curves $\gamma : [0,1] \longrightarrow \mathbb{M}_\infty^d$ are the constant ones.*

Proof Let $\gamma : [0,1] \longrightarrow \mathbb{M}_\infty^d$ be a continuous curve. In order to prove that γ is constant it suffices to show that for each $\ell \geq 1$ we have

$$\gamma(t)|_{\Lambda_\ell^d} \equiv \gamma(0)|_{\Lambda_\ell^d} \tag{1.49}$$

for all $t \in [0,1]$. So let $\ell \in \mathbb{Z}_+$ be fixed. For each $t \in [0,1]$ the set $V(\gamma(t),\ell)$ is an open neighborhood of $\gamma(t)$ and so by the continuity of γ the family $\{V_t\}_{t\in[0,1]}$ where $V_t := \gamma^{-1}(V(\gamma(t),\ell))$ is an open cover of the compact set $[0,1]$. So by choosing a partition $\Delta = \{0 = s_0 < s_1 < \cdots < s_m = 1\}$ of $[0,1]$ with mesh $|\Delta| := \max_{1\leq j\leq m}(s_j - s_{j-1})$ less than the Lebesgue number of the covering $\{V_t\}_{t\in[0,1]}$ we have that there exist $t_1,\dots,t_m \in [0,1]$ such that $[s_{j-1},s_j] \subseteq V_{t_j}$ for all $j=1,\dots,m$. But then we obviously have that

$$\gamma([s_{j-1},s_j]) \subseteq V(\gamma(t),\ell) \quad \text{for all } j=1,\dots,m$$

which proves (1.49). By the definition of the neighborhoods $V(\eta,\ell)$ this implies that γ is constant on any of the intervals $[s_{j-1},s_j]$ and thus it is constant. □

1.3.2 The Spaces $\mathbb{P}_p\mathbb{M}^d_\infty$, $p \geq 0$

In this subsection we study topological spaces resulting from equipping the set $\mathbb{P}\mathbb{M}^d_\infty$ of all Borel probability measures on $\mathbb{M}^d_\infty$ with the topology of weak convergence with respect to the duality with bounded and continuous functions and more generally with respect to the duality with continuous functions of p-th polynomial growth.
Given any topology on $\mathbb{P}\mathbb{M}^d_\infty$ we can consider limits of sequences $\{\mu_N\}$ of distributions such that $\mu_N \in \mathbb{P}\mathbb{M}^d_N$, $N \in \mathbb{N}$ by using the periodic embeddings $e_N : \mathbb{M}^d_N \hookrightarrow \mathbb{M}^d_\infty$ to induce embeddings

$$\mathbb{P}\mathbb{M}^d_N \ni \mu_N \mapsto e_{N*}\mu_N \in \mathbb{P}\mathbb{M}^d_\infty,$$

via the push forward of measures. Here the measure $e_{N*}\mu_N$ is given by

$$e_{N*}\mu_N(A) = \mu_N\{\eta \in \mathbb{M}^d_N \,|\, \widetilde{\eta} \in A\}$$

for each Borel subset $A \subseteq \mathbb{M}^d_\infty$. In other words, the measure $e_{N*}\mu_N \in \mathbb{P}\mathbb{M}^d_\infty$ is characterized by the requirement that its projection on $\mathbb{P}\mathbb{M}^d_N$ equals μ_N through the identification $\mathbb{Z}^d_N \cong \mathbb{T}^d_N$, i.e. the requirement that for any $\zeta \in \mathbb{M}^d_N$ we have

$$e_{N*}\mu_N\Big\{\eta \in \mathbb{M}^d_\infty \,\Big|\, \eta_x = \zeta_{[x]_N},\ \forall\, x \in \mathbb{Z}^d_N\Big\} = \mu_N\{\zeta\},$$

and the requirement that the measure $e_{N*}\mu_N$ is periodic with period $N\mathbb{Z}^d$,

$$e_{N*}\mu_N\Big\{\eta \in \mathbb{M}^d_\infty \,\Big|\, x-y \in N\mathbb{Z}^d \implies \eta_x = \eta_y\Big\} = 1.$$

We will always omit the embeddings e_{N*} from the notation by identifying measures $\mu_N \in \mathbb{P}\mathbb{M}^d_N$ with measures $e_{N*}\mu^N \in \mathbb{P}\mathbb{M}^d_\infty$. So whenever we write the measures $\mu^N \in \mathbb{P}\mathbb{M}^d_N$ converge as $N \to \infty$ to the measure $\mu \in \mathbb{P}\mathbb{M}^d_\infty$ we will always mean the measures $e_{N*}\mu^N$ in place of μ_N.

We begin with some initial considerations on the weak topology and then we will proceed with a unified treatment of the weak topologies of all orders. As the space $\mathbb{M}^d_\infty$ is polish, by the general theory of weak convergence of probability measures we know that the weak topology on $\mathbb{P}\mathbb{M}^d_\infty$ is metrizable and that for any sequence $\{\mu^N\}_{N\in\mathbb{N}} \subseteq \mathbb{P}\mathbb{M}^d_\infty$,

$$\lim_{N\to\infty} \mu^N = \mu \quad \text{in } \mathbb{P}\mathbb{M}^d_\infty \qquad \iff \qquad \lim_{N\to\infty} \int f d\mu^N = \int f d\mu \quad \forall\, f \in BC(\mathbb{M}^d_\infty).$$

As we shall see, a basic property of this topology is that in order to characterize the weak convergence of probability measures in $\mathbb{P}\mathbb{M}^d_\infty$ the subspace of all cylinder functions $f : \mathbb{M}^d_\infty$ (i.e. functions depending only on a finite number of coordinates) suffices in place of the whole space $BC(\mathbb{M}^d_\infty)$. For any subset $J \subseteq \mathbb{Z}^d$ we will denote by $p_J : \mathbb{M}^d_\infty \longrightarrow \mathbb{Z}^J_+$ the natural projection on J.

Definition 1.3.2 A function $f : \mathbb{M}_\infty^d \longrightarrow \mathbb{R}$ is called a cylinder function if it is of the form $f = \widetilde{f} \circ p_J$ for some finite set $J \subseteq \mathbb{Z}^d$ and some function $\widetilde{f} : \mathbb{Z}_+^J \longrightarrow \mathbb{R}$. The set of all (bounded) cylinder functions on $\mathbb{M}_\infty^d$ will be denoted by $\mathrm{Cyl}(\mathbb{M}_\infty^d)$ ($B_{\mathrm{Cyl}}(\mathbb{M}_\infty^d)$ respectively).

We note that since for any finite set $J \subseteq \mathbb{Z}^d$ the space $\mathbb{Z}_+^J$ has the discrete topology and the natural projections are continuous by the definition of the topology of $\mathbb{M}_\infty^d$ any cylinder function is continuous, $\mathrm{Cyl}(\mathbb{M}_\infty^d) \subseteq C(\mathbb{M}_\infty)$. In fact any cylinder function is uniformly continuous. Indeed, note that for any $\ell \in \mathbb{Z}_+$,

$$d(\eta, \zeta) < \frac{1}{2h(\ell)} \quad \Longrightarrow \quad \eta|_{\Lambda_\ell^d} \equiv \zeta|_{\Lambda_\ell^d}.$$

Consequently if $f = \widetilde{f} \circ p_{\ell_0}$ for some function $\widetilde{f} : \mathbb{Z}_+^{\Lambda_{\ell+0}^d} \longrightarrow \mathbb{R}$, then choosing $\delta < \frac{1}{2h(\ell_0)}$ we have for all $\eta, \zeta \in \mathbb{M}_\infty^d$ that

$$|f(\eta) - f(\zeta)| = |\widetilde{f}(\eta|_{\Lambda_{\ell_0}^d}) - \widetilde{f}(\zeta|_{\Lambda_{\ell_0}^d})| = 0,$$

which proves the uniform continuity of f.

To explain the need of the following proposition **??** in approximating continuous functions by cylinders functions we describe an example of a continuous function on $\mathbb{M}_\infty^d$ that is not uniformly continuous, and thus can not be approximated uniformly by continuous functions. In dimension $d = 1$ let A_ℓ denote the set of all configurations that have exactly ℓ particles at each site $x \in \Lambda_\ell$ and set $A_\infty := \bigcup_{\ell=0}^\infty A_\ell$. We will show that the function $f := \mathbb{1}_{A_\infty} : \mathbb{M}_\infty^d \longrightarrow \{0, 1\}$ is not continuous but not uniformly continuous. For the continuity of f let first $\eta \in A_\infty$. Then $\eta \in A_\ell$ for some $\ell \in \mathbb{Z}_+$ and then $\eta|_{\Lambda_\ell} \equiv \ell$. But for every sequence $\eta_n \longrightarrow \eta$ we can choose $n_\ell \in \mathbb{N}$ such that $\eta_n \equiv \eta \equiv \ell$ on Λ_ℓ and so $\eta_n \in A_\ell \subseteq A_\infty$ which shows that $f(\eta_n) \longrightarrow f(\eta)$. Let on the other hand $\eta \notin A_\infty$ and let $\{\eta_n\}_{n\in\mathbb{N}}$ be a sequence converging to η. We choose $\ell_0 \geq \eta(0)$ and since $\eta_n \longrightarrow \eta$ we can choose $n_0 \in \mathbb{N}$ such that $\eta_n|_{\Lambda_{\ell_0}} \equiv \eta$. Then for any $n \geq n_0$ we have that $\eta_n \notin A_\infty$ since $\eta_n \notin \bigcup_{\ell=0}^{\ell_0}$ due to the fact that $\eta_n|_{\Lambda_{\ell_0}} \equiv \eta|_{\Lambda_{\ell_0}}$ and $\eta_n \notin \bigcup_{\ell_0+1}^\infty$ since $\eta(0) \leq \ell_0$. This proves the continuity of f. To prove that it is not uniformly continuous it suffices to show that for every $\delta > 0$ there exist $\eta, \zeta \in \mathbb{M}_\infty^d$ such that $d(\eta, \zeta) < \delta$ and $f(\eta) = 1 \neq 0 = f(\zeta)$. For this, given $\delta > 0$ one chooses $\ell \in \mathbb{Z}_+$ such that $\frac{1}{2h(\ell)} < \delta$ and $\eta := \ell\mathbb{1}_{\Lambda_\ell}$, $\zeta = \eta + \mathbb{1}_\ell$.

We proceed now with a unified study of the weak topologies of the spaces $\mathbb{P}_p\mathbb{M}_\infty^d$ consisting of measures with p-th moments. The most interesting cases for us will be the cases $p = 0$ that corresponds to the weak topology and $p = 1$ that is related to convergence of density, the conserved quantity. Of course since the metric d of $\mathbb{M}_\infty^d$ is bounded it can not be used to define the moments and the p-Wasserstein topologies on the spaces $\mathbb{P}_p\mathbb{M}_\infty^d$. Due to this, for every $p \geq 0$ we set

$$\mathbb{P}_p\mathbb{M}_\infty^d := \left\{ \mu \in \mathbb{P}\mathbb{M}_\infty^d \,\middle|\, \int \eta(x)^p d\mu < +\infty \quad \text{for all } x \in \mathbb{Z}^d \right\},$$

$$B_p(\mathbb{M}_\infty) := \bigcap_{\mu \in \mathbb{P}_p\mathbb{M}_\infty^d} \mathcal{L}^1(\mu)$$

and

$$B_pC(\mathbb{M}^d_\infty) := B_p(\mathbb{M}^d_\infty) \cap C(\mathbb{M}^d_\infty),$$

where as usual the calligraphic $\mathcal{L}$ denotes that we do not identify a.s. equal functions.

Proposition 1.3.6 *For every $p \geq 0$,*

$$B_p(\mathbb{M}^d_\infty) = \Big\{ f \in m\mathbb{M}^d_\infty \,\Big|\, \exists\, \ell \in \mathbb{Z}_+,\ A, B \geq 0 \text{ such that } |f| \leq A + B \sum_{x \in \Lambda^d_\ell} \eta(x)^p \Big\}.$$

Proof It is immediate by the definitions that the set in the right hand of the equality above is contained in $B_p(\mathbb{M}^d_\infty)$. For the converse inclusion we must prove that if a measurable function $f : \mathbb{M}^d_\infty \longrightarrow \mathbb{R}$ does not belong in the set in the right hand side then it does not belong in $B_p(\mathbb{M}^d_\infty)$, i.e. that there exists $\mu \in \mathbb{P}_p\mathbb{M}_\infty$ such that $f \notin \mathcal{L}^1(\mu)$. Indeed, since f does not belong in the right hand side, for each $\ell \in \mathbb{Z}_+$ there exists $\eta_\ell \in \mathbb{M}^d_\infty$ such that

$$|f(\eta_\ell)| > 2^\ell \Big(1 + \sum_{x \in \Lambda^d_\ell} \eta_\ell(x)^p \Big).$$

But then if we consider the probability measure

$$\mu := \frac{1}{c} \sum_{\ell=0}^{\infty} \frac{1}{2^\ell \sum_{\Lambda^d_\ell} \eta^p_\ell(x)} \delta_{\eta_\ell} \in \mathbb{P}\mathbb{M}^d_\infty,$$

where $c > 0$ is the appropriate renormalizing constant, it is easy to check that $\mu \in \mathbb{P}_p\mathbb{M}_\infty$ and $f \notin \mathcal{L}^1(\mu)$. □

It is obvious by the definition that $\mathbb{P}_0\mathbb{M}^d_\infty = \mathbb{P}\mathbb{M}^d_\infty$ and according to this last proposition it is obvious that $B_0C(\mathbb{M}^d_\infty) = BC(\mathbb{M}^d_\infty)$. Consequently the w_0-topology on $\mathbb{P}\mathbb{M}^d_\infty$ is exactly the topology of weak convergence of probability measures.

Definition 1.3.3 Let $p \geq 0$. The *p-th order Wasserstein topology w_p on $\mathbb{P}_p\mathbb{M}^d_\infty$* is the weak topology defined by the family of linear functionals $B_pC(\mathbb{M}^d_\infty)$ i.e. the weakest topology with respect to which all the functions

$$\mathbb{P}_p\mathbb{M}^d_\infty \ni \mu \mapsto \int f d\mu \in \mathbb{R}, \quad f \in B_pC(\mathbb{M}^d_\infty)$$

are continuous.

It is obvious that $\mathbb{P}_q\mathbb{M}_\infty \subseteq \mathbb{P}_p\mathbb{M}_\infty$ and $B_p(\mathbb{M}_\infty) \subseteq B_q(\mathbb{M}^d_\infty)$ whenever $p \leq q$ and therefore the restriction of the w_q topology is stronger than the restriction of the w_p topology on $\mathbb{P}_q\mathbb{M}^d_\infty$. We will show next that the sub-space

$$\mathrm{Cyl}^d_p := B_p(\mathbb{M}_\infty) \cap \mathrm{Cyl}(\mathbb{M}^d_\infty) \leq B_p(\mathbb{M}^d_\infty)$$

of all cylinder functions of p-th polynomial growth suffices for the description of the w_p topology on $\mathbb{P}_p\mathbb{M}^d_\infty$. It is easy to see that with the notation

$$B_p(\mathbb{Z}^J_+) := \Big\{ f \in m\mathbb{M}^d_\infty \,\Big|\, \exists\, A, B \geq 0 \text{ such that } |f| \leq A + B \sum_{x \in J} \eta(x)^p \Big\}$$

the cylinder functions of p-th polynomial growth are obviously

$$\mathrm{Cyl}_p^d = \big\{\Psi \in m\mathbb{M}_\infty^d \,\big|\, \Psi = \widetilde{\Psi}\circ p_J \ \text{ for some } \widetilde{\Psi} \in B_p(\mathbb{Z}_+^J),\ J \subseteq \mathbb{Z}^d \text{ finite}\big\}.$$

Definition 1.3.4 A sequence $\{f_n\} \subseteq B_p(\mathbb{M}_\infty^d)$ *converges B_p-pointwise to* $f \in B_p(\mathbb{M}_\infty^d)$ if f_n converges pointwise to f and it is *uniformly B_p-bounded* i.e. there exist constants $\ell \in \mathbb{Z}_+$ and $A, B \geq 0$ such that

$$\sup_{n\in\mathbb{N}} |f_n| \leq A + B \sum_{|x|\leq \ell} \eta(x)^p.$$

The B_p-*closure* of a set $F \subseteq B_p(\mathbb{M}_\infty^d)$ is the set

$$B_p\text{-cl}(F) := \big\{f \in B_p(\mathbb{M}_\infty^d) \,|\, \exists\, \{f_n\} \subseteq F \text{ such that } f_n \longrightarrow f \ B_p\text{-pointwise}\big\}.$$

Proposition 1.3.7 *For every $p \geq 0$,*

$$B_pC(\mathbb{M}_\infty^d) \subseteq B_p\text{-cl}(\mathrm{Cyl}_p^d).$$

If in addition the function $f \in B_pC(\mathbb{M}_\infty^d)$ is uniformly continuous then there exists a sequence $\{h_n\}_{n=1}^\infty \subseteq \mathrm{Cyl}_p^d$ such that $h_n \geq f$ and $h_n - f \in B(\mathbb{M}_\infty^d)$ for all $n \in \mathbb{N}$ and $h_n \longrightarrow f$ uniformly (and thus also B_p-pointwise).

Proof We prove first that $B_pC(\mathbb{M}_\infty^d)$ is contained in the B_p-pointwise closure of Cyl_p^d. So let $f \in B_pC(\mathbb{M}_\infty^d)$ and we will exhibit a sequence $\{f_\ell\}_{\ell\in\mathbb{N}} \subseteq \mathrm{Cyl}_p^d$ converging B_p-pointwise to f as $\ell \to \infty$. By proposition 1.3.2 we know that if

$$e_\ell : \mathbb{Z}_+^{\Lambda_\ell^d} \hookrightarrow \mathbb{M}_\infty^d \quad \text{and} \quad p_\ell : \mathbb{M}_\infty^d \longrightarrow \mathbb{Z}_+^{\Lambda_\ell^d}$$

denote the periodical embeddings $e_\ell(\eta)(x) = \eta(x+(2\ell+1)\mathbb{Z}^d)$ and the natural projections respectively then the functions $I_\ell : e_\ell \circ p_\ell : \mathbb{M}_\infty^d \longrightarrow \mathbb{M}_\infty^d$, $\ell \in \mathbb{Z}_+$ converge uniformly to $id_{\mathbb{M}_\infty^d}$. But then the functions $f_\ell := f \circ I_\ell = (f \circ e_\ell) \circ p_\ell$, $\ell \in \mathbb{Z}_+$, are cylinder functions and converge pointwise to f. So to complete the proof of the claimed inclusion it suffices to show that $f_\ell \in B_p(\mathbb{M}_\infty^d)$ for all $\ell \in \mathbb{Z}_+$ and that the sequence $\{f_\ell\}$ is uniformly B_p-bounded. Since $f \in B_p(\mathbb{M}_\infty^d)$ there exist $\ell_0 \in \mathbb{Z}_+$ and $A, B \geq 0$ such that

$$|f| \leq A + B \sum_{x\in\Lambda_{\ell_0}^d} \eta(x)^p$$

and then for every $\ell \geq \ell_0$ we have that

$$|f_\ell| = |f \circ I_\ell| \leq A + B \sum_{x\in\Lambda_{\ell_0}^d} \big(\eta(x)\circ I_\ell\big)^p = A + B \sum_{x\in\Lambda_{\ell_0}^d} \eta(x)^p.$$

Consequently we can exhibit $\{f_{\ell+\ell_0}\}_{\ell\in\mathbb{Z}_+} \subseteq \mathrm{Cyl}_p^d$ as the required B_p-pointwise converging to f sequence. Anyway, for $\ell < \ell_0$ we always have that

$$|f_\ell| \leq A + B\frac{\big(2\big(\ell_0 + (2\ell+1)\big)+1\big)^d}{(2\ell+1)^d} \sum_{x\in\Lambda_{\ell_0}^d} \eta(x)^p \leq A + B(2\ell_0+3)^d \sum_{x\in\Lambda_{\ell_0}^d} \eta(x)^p.$$

We suppose next that f is uniformly continuous and we will show how to obtain from the sequence $\{f_\ell\}$ a sequence that satisfies the required properties. First it is obvious that $\{f_\ell\}$ satisfies $f_\ell - f \in B(\mathbb{M}_\infty^d)$ and $\|f_\ell - f\|_u \longrightarrow 0$. Indeed, since f is uniformly continuous given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$d(\eta, \zeta) < \delta_\varepsilon \quad \Longrightarrow \quad |f(\eta) - f(\zeta)| < \varepsilon$$

and since $e_\ell \circ p_\ell$ converges uniformly to $id_{\mathbb{M}_\infty^d}$ there exists $\ell_\varepsilon \in \mathbb{N}$ such that

$$\ell \geq \ell_\varepsilon \quad \Longrightarrow \quad \sup_{\eta \in \mathbb{M}_\infty^d} d\big(\eta, e_\ell \circ p_\ell(\eta)\big) < \delta_\varepsilon$$

and then obviously for all $\ell \geq \ell_\varepsilon$ we have that

$$\|f_\ell - f\|_u = \sup_{\eta \in \mathbb{M}_\infty^d} \big|f\big(e_\ell \circ p_\ell(\eta)\big) - f(\eta)\big| < \varepsilon,$$

which shows that $f_\ell - f \in B(\mathbb{M}_\infty)$ and $\|f_\ell - f\|_u \longrightarrow 0$.

Finally, using once again the uniform continuity of f there exists for each $\ell \in \mathbb{N}$ a number $\delta_\ell > 0$ such that

$$d(\eta, \zeta) < \delta_\ell \quad \Longrightarrow \quad |f(\eta) - f(\zeta)| < \frac{1}{\ell}. \tag{1.50}$$

For each $\ell \in \mathbb{N}$ we choose $m_\ell \in \mathbb{N}$ such that $\sum_{k=m_\ell+1}^{\infty} \frac{1}{2^{k+1}} < \delta_\ell$ and define the sequence $h^\ell = f_{m_\ell} + \frac{1}{\ell}$, $\quad \ell \in \mathbb{Z}_+$, $\ell \in \mathbb{N}$. Obviously the sequence $\{h_\ell\}$ thus defined converges uniformly to f and we will show that $h^\ell \geq f$. Indeed, since $e_{m_\ell} \circ p_{m_\ell}(\eta)|_{\Lambda_{m_\ell}^d} \equiv \eta|_{\Lambda_{m_\ell}^d}$ for $\ell \in \mathbb{Z}_+$ and all $\eta \in \mathbb{M}_\infty^d$ we have that

$$d\big(e_{m_\ell} \circ p_{m_\ell}(\eta), \eta\big) \leq \sum_{k=m_\ell+1}^{\infty} \frac{1}{2^{k+1}} < \delta_\ell$$

for all $(\ell, \eta) \in \mathbb{Z}_+ \times \mathbb{Z}^{\mathbb{Z}^d}$ and therefore by (1.50) we obtain

$$\big|f\big(e_{m_\ell} \circ p_{m_\ell}(\eta)\big) - f(\eta)\big| < \frac{1}{\ell} \quad \text{for all } \ell \in \mathbb{Z}_+.$$

Consequently

$$h_\ell = f \circ e_{m_\ell} \circ p_{m_\ell} + \frac{1}{\ell} \geq f,$$

which completes the proof. □

Proposition 1.3.8 *For any $f \in B_pC(\mathbb{M}_\infty^d)$ the exists a sequence $\{h_\ell\}_{\ell \in \mathbb{Z}_+} \subseteq \mathrm{Cyl}_p^d$ such that $h_\ell \geq f$ for all $\ell \in \mathbb{Z}_+$ and $h_\ell \longrightarrow f$ B_p-pointwise.*

Proof Since $f \in B_p(\mathbb{M}_\infty^d)$ there exists $\ell_0 \in \mathbb{Z}_+$ and $A, B \geq 0$ such that

$$|f| \leq A + B \sum_{x \in \Lambda_{\ell_0+1}^d} \eta(x)^p.$$

Then the function

$$\bar{f} := \frac{f}{1 + \sum_{x \in \Lambda_{\ell_0+1}^d} \eta(x)^p}$$

is obviously in $B(\mathbb{M}^d_\infty)$ with $\|\bar f\|_u \le A \vee B$. Then by standard results in Moreau-Yosida approximations (see for instance display 5.1.4 in p.107 in [2]) the sequence $\{\bar f_\ell\}$ defined by

$$\bar f_\ell(\eta) = \sup_{\zeta\in\mathbb{M}^d_\infty}\{\bar f(\zeta) - \ell d(\eta,\zeta)\}$$

defines a sequence of bounded Lipschitz functions such that $\bar f_\ell \ge \bar f$ for all $\ell \in \mathbb{N}$ and $\bar f_\ell \downarrow \bar f$ B_0-pointwise as $\ell \to \infty$ with

$$-\|\bar f\|_u \le \bar f \le \bar f_\ell \le \|\bar f\|_u.$$

In particular each $\bar f_\ell$ is uniformly continuous and therefore by applying the previous proposition there exists for each $\ell \in \mathbb{N}$ a sequence $\{\bar f_{k,\ell}\}_{k\in\mathbb{N}} \subseteq B_{\mathrm{cyl}}(\mathbb{M}^d_\infty)$ such that $\bar f_{k,\ell} \ge f_\ell$ for all $k,\ell \in \mathbb{N}$ and $\lim_{k\to\infty}\|\bar f_{k,\ell} - \bar f_\ell\|_u = 0$. So for each $\ell \in \mathbb{N}$ we can choose $k_\ell \in \mathbb{N}$ such that $\|\bar f_{k_\ell,\ell} - \bar f_\ell\|_u < \frac{1}{\ell}$. We set $\bar h_\ell := \bar f_{k_\ell,\ell}$ and then obviously $\{\bar h_\ell\} \subseteq \mathrm{Cyl}^d_0$, $\bar h_\ell \ge \bar f$ for all $\ell \in \mathbb{N}$ and $\{\bar h_\ell\}$ is uniformly bounded by

$$\|\bar h_\ell\|_u = \|(\bar f_{k_\ell,\ell} - \bar f_\ell) + \bar f_\ell\|_u < \frac{1}{\ell} + \|\bar f_\ell\|_u \le 1 + \|\bar f\|_u.$$

Furthermore $\bar h_\ell \longrightarrow \bar f$ pointwise. Indeed, let $\eta \in \mathbb{M}^d_\infty$, $\varepsilon > 0$. We choose $\ell_1 \in \mathbb{N}$ such that $\frac{1}{\ell_1} < \frac{\varepsilon}{2}$ and since $\bar f_\ell \longrightarrow \bar f$ pointwise there exists $\ell_2 \ge \mathbb{N}$ such that $|\bar f_\ell(\eta) - \bar f(\eta)| < \frac{\varepsilon}{2}$ for all $\ell \ge \ell_2$. But then for all $\ell \ge \ell_1 \vee \ell_2$,

$$|\bar h_\ell(\eta) - \bar f(\eta)| \le \|\bar f_\ell - \bar f_\ell\|_u + |\bar f_\ell(\eta) - \bar f(\eta)| < \frac{1}{\ell} + \frac{\varepsilon}{2} < \varepsilon,$$

which proves that $\bar f_\ell \longrightarrow \bar f$ pointwise in $\mathbb{M}^d_\infty$.

Then if we set

$$h_\ell = \Big(1 + \sum_{x\in\Lambda^d_{\ell_0}} \eta(x)^p\Big)\bar h_\ell \ge \Big(1 + \sum_{x\in\Lambda^d_{\ell_0}} \eta(x)^p\Big)\bar f = f$$

we obviously have that $h_\ell \longrightarrow f$ pointwise and $\{h_\ell\}$ is uniformly B_p-bounded with

$$\sup_{\ell\in\mathbb{N}} |h_\ell| \le (1 + A \vee B)\Big(1 + \sum_{x\in\Lambda^d_{\ell_0}} \eta(x)^p\Big),$$

which completes the proof. □

Along the lines in the beginning section 5.1 in [2] one has the following

Lemma 1.3.1 *Let (M,d) be a metric space and let $\Xi_0 \subseteq B_pC(M)$ be such that*

$$\int f d\mu = \sup\Big\{\int h d\mu \,\Big|\, h \in \Xi_0,\ h \le f\Big\} \tag{1.51}$$

$$= \inf\Big\{\int h d\mu \,\Big|\, h \in \Xi_0,\ h \ge f\Big\} \tag{1.52}$$

for all $f \in B_pC(M)$, $\mu \in \mathbb{P}_pM$. Then the weak topologies induced on $\mathbb{P}_pM$ by the families Ξ_0 and $B_pC(M)$ coincide.

Proof Obviously, since $\Xi_0 \subseteq B_pC(\mathbb{M}_\infty^d)$, the topology induced on $\mathbb{P}_pM$ by Ξ_0 is weaker than the w_p topology on $\mathbb{P}_pM$. For the converse, suppose that $\int hd\mu_i \longrightarrow \int_X hd\mu$ for all $h \in \Xi_0$ and all nets $\{\mu_i\}_i$ in $\mathbb{P}_pM$ and let $f \in B_pC(\mathbb{M}_\infty^d)$. Then on one hand,

$$\sup_{i\in I}\inf_{j\geq i}\int fd\mu_i \geq \sup_{h\in\Xi_0,\, h\leq f}\ \sup_{i\in I}\inf_{j\geq i}\int hd\mu_i = \sup_{h\in\Xi_0,\, h\leq f}\int hd\mu = \int fd\mu,$$

while on the other hand

$$\inf_{i\in I}\sup_{j\geq i}\int fd\mu_i \leq \inf_{h\in\Xi_0,\, h\geq f}\ \inf_{i\in I}\sup_{j\geq i}\int hd\mu_i = \inf_{h\in\Xi_0,\, h\geq f}\int hd\mu = \int fd\mu.$$

Consequently

$$\limsup_i \int fd\mu_i \leq \int fd\mu \leq \liminf_i \int fd\mu_i,$$

which shows that $\lim_i \int fd\mu_i = \int fd\mu$. □

It is now easy to show that the cylinder functions are sufficient for the description of the w_p-topologies on $\mathbb{P}_p\mathbb{M}_\infty^d$.

Proposition 1.3.9 *For all $p \geq 0$ the families $\mathbb{P}_p\mathbb{M}_\infty^d$ and Cyl_p^d define the same weak topology on $\mathbb{P}_p\mathbb{M}_\infty^d$.*

Proof According to the previous lemma and since $\mathrm{Cyl}_p^d = -\mathrm{Cyl}_p^d$ it suffices to show that

$$\int fd\mu = \inf\left\{\int \Psi d\mu \,\middle|\, \Psi \in \mathrm{Cyl}_p^d,\ \Psi \geq f\right\}$$

for all $f \in B_pC(\mathbb{M}_\infty^d)$ and all $\mu \in \mathbb{P}_p\mathbb{M}_\infty^d$. So let $f \in B_pC(\mathbb{M}_\infty^d)$, $\mu \in \mathbb{P}_p\mathbb{M}_\infty^d$ be given. By the previous proposition there exists a sequence $\{\Psi_k\}_{k\in\mathbb{N}}$ such that

$$\Psi_k \downarrow f \quad B_p\text{-pointwise}$$

and therefore since $\mu \in \mathbb{P}_p\mathbb{M}_\infty^d$ we have by the dominated convergence theorem that

$$\int fd\mu = \lim_{k\to\infty}\int \Psi_k d\mu \geq \inf\left\{\int \Psi d\mu \,\middle|\, \Psi \in \mathrm{Cyl}_p^d,\ \Psi \geq f\right\}.$$

The converse inequality is obvious and therefore the proof is complete. □

For each $\ell \in \mathbb{Z}_+$ the space $\mathbb{Z}_+^{\Lambda_\ell^d}$ is finite dimensional and so by standard results for the Wasserstein metrics ([31], chapter 7) for each $p > 0$ the w_p-topology on the space $\mathbb{P}_p(\mathbb{Z}_+^{\Lambda_\ell^d}$, which is the weak topology defined by the family $B_p(\mathbb{M}_\infty^d) = B_pC(\mathbb{M}_\infty^d)$ is polish, with a complete and separable metric being the Wasserstein metric

$$W_{\ell,p}(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)}\left(\int |\eta-\zeta|_{\ell,p}^p d\pi(\eta,\zeta)\right)^{\frac{1}{p}} \wedge 1,$$

where $\Pi(\mu,\nu)$ denotes the set of all transport plans from μ to ν and $|\cdot|_p$ is the metric on $\mathbb{Z}_+^{\Lambda_\ell^d}$ given by

$$|\eta|_{\ell,p} = \left(\sum_{x\in\Lambda_\ell^d}\eta(x)^p\right)^{\frac{1}{p}}.$$

For the case $p = 0$ the weak topology on $\mathbb{P}\mathbb{Z}_+^{\Lambda_\ell^d}$ can be metrized ([31], chapter 7 again) by the metric

$$W_{\ell,0}(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \left(\int |\eta-\zeta|_{\ell,1} \wedge 1 d\pi(\eta,\zeta) \right).$$

Since cylinder functions are sufficient for the description of the w_p topologies we immediately have the following corollary.

Corollary 1.3.1 *Let $p \geq 0$. A net $\{\mu^i\}_{i\in I}$ in $\mathbb{P}_p\mathbb{M}_\infty^d$ converges to $\mu \in \mathbb{P}_p\mathbb{M}_\infty^d$ iff*

$$p_{\ell *}\mu_i \longrightarrow p_{\ell_*}\mu \in \mathbb{P}_p(\mathbb{Z}_+^{\Lambda_\ell^d})$$

in the w_p topology of $\mathbb{P}_p(\mathbb{Z}_+^{\Lambda_\ell^d})$.

Corollary 1.3.2 *Let $\mu \in \mathbb{P}_p\mathbb{M}_\infty^d$, $p \geq 0$, and let $\{\mu_i\}_{i\in I} \subseteq \mathbb{P}_p\mathbb{M}_\infty^d$ be a net. Then $\mu_i \longrightarrow \mu$ with respect the w_p topology on $\mathbb{P}_p\mathbb{M}_\infty^d$ iff $\mu_i \longrightarrow \mu$ weakly and*

$$\lim_i \int_{\mathbb{M}_\infty^d} \eta(x)^p d\mu_i = \int_{\mathbb{M}_\infty^d} \eta(x)^p d\mu, \quad \forall\, x \in \mathbb{Z}^d.$$

Corollary 1.3.3 *The w_p topology on $\mathbb{P}_p\mathbb{M}_\infty^d$, $p \geq 0$ has a countable base and in particular sequences are sufficient for the description of the w_p topology.*

Proof By standard results on the weak topology of measures (see e.g. section 5.1 in [2] again) there exists a countable family Ξ of bounded Lipschitz functions on $\mathbb{M}_\infty^d$ that defines the weak topology. But then by the previous corollary it follows that the countable family $\Xi \cup \{\eta(x) | x \in \mathbb{Z}^d\}$ defines the w_p topology and the claim is proved. □

As we will see next, the space $\mathbb{P}_p\mathbb{M}_\infty^d$, $p > 0$, is a polish space. The case $p = 0$ is immediate since the weak topology of probability measures on a polish space is always polish by standard results in the weak topology on probability measures. The same is also true for the Wasserstein weak topologies but here we do not exactly consider the Wasserstein topology corresponding to the metric d of $\mathbb{M}_\infty$ and this is the reason for the following proposition.

Proposition 1.3.10 *The space $(\mathbb{P}_p\mathbb{M}_\infty^d, w_p)$, $p > 0$, is a polish space. A complete and separable metric defining the topology w_p is the metric $W_p : \mathbb{P}_p\mathbb{M}_\infty^d \times \mathbb{P}_p\mathbb{M}_\infty^d \longrightarrow \mathbb{R}_+$ given by*

$$W_p(\mu,\nu) = \sum_{\ell=0}^{\infty} \frac{1}{2^{\ell+1}} \frac{W_{\ell,p}(p_{\ell *}\mu, p_{\ell *}\nu)}{1 + W_{\ell,p}(p_{\ell *}\mu, p_{\ell *}\nu)}$$

where $p_\ell : \mathbb{M}_\infty^d \longrightarrow \mathbb{Z}_+^{\Lambda_\ell^d}$, $\ell \in \mathbb{Z}_+$, denotes the natural projection.

Proof By standard arguments for metrics of this type that we have already described in proposition 1.3.1 it follows that W_p is indeed a metric. Furthermore convergence $W_p(\mu^N, \mu) \longrightarrow 0$ is equivalent to requiring that $W_{\ell,p}(p_{\ell *}\mu^N, p_{\ell *}\mu) \longrightarrow 0$ for all $\ell \in \mathbb{Z}_+$ and so according to corollary 1.3.1 this metric metrizes the w_p topology. So we have to prove that the w_p topology is separable and the metric W_p complete.

We prove first that the metric W_p is complete. So let $\{\mu^N\} \subseteq \mathbb{P}_p\mathbb{M}_\infty^d$ be a W_p-Cauchy sequence. For notational simplicity we set $\mu_\ell := p_{\ell *}\mu$ for each $\ell \in \mathbb{Z}_+$, $\mu \in \mathbb{P}\mathbb{M}_\infty^d$ and we will show that the sequence $\{\mu_\ell^N\}_{N\in\mathbb{N}}$ is a $W_{\ell,p}$-Cauchy sequence for each $\ell \in \mathbb{Z}_+$. Given $\varepsilon > 0$ we can choose $N_{\ell,\varepsilon} \in \mathbb{N}$ such that $W_p(\mu^N, \mu^K) < \frac{1}{2^{\ell+1}}\frac{\varepsilon}{1+\varepsilon}$ for all $N, K \geq N_{\ell,\varepsilon}$ and then

$$\frac{W_{\ell,p}(\mu^N\ell, \mu_\ell^K)}{1 + W_{\ell,p}(\mu^N\ell, \mu_\ell^K)} \leq 2^{\ell+1}W_p(\mu^N, \mu^K) < \frac{\varepsilon}{1+\varepsilon}$$

for all $N, K \geq N_{\ell,\varepsilon}$. But then since the function $\mathbb{R}_+ \ni t \mapsto \frac{t}{1+t}$ is strictly increasing we have that $W_{\ell,p}(\mu_\ell^N, \mu_\ell^K) < \varepsilon$ for all $N, K \geq N_{\ell,\varepsilon}$ which shows that the sequence $\{\mu_\ell^N\}_{N\in\mathbb{N}}$ is a Cauchy sequence for each $\ell \in \mathbb{Z}_+$.

Now, since $\{\mu_\ell^N\}_{N\in\mathbb{N}}$ is a Cauchy sequence there exists for each $\ell \in \mathbb{Z}_+$ a measure $\mu_\ell \in \mathbb{P}_p(\mathbb{Z}_+^{\Lambda_\ell^d})$ such that $\mu_\ell^N \longrightarrow \mu_\ell$ in the w_p topology of $\mathbb{P}_p(\mathbb{Z}_+^{\Lambda_\ell^d})$. We will show that the sequence $\{\mu_\ell\}_{\ell\in\mathbb{Z}_+}$ is a projective sequence of probability measures i.e. that

$$p_\ell^{\ell+1}\mu_{\ell+1} = \mu_\ell \quad \text{for all } \ell \in \mathbb{Z}_+,$$

where $p_\ell^{\ell+1} : \mathbb{Z}_+^{\Lambda_{\ell+1}^d} \longrightarrow \mathbb{Z}_+^{\Lambda_\ell^d}$ denotes the natural projection. Indeed, since $p_\ell = p_\ell^{\ell+1} \circ p_{\ell+1}$, for each $\ell \in \mathbb{Z}_+$ we have that $p_{\ell *}^{\ell+1}\mu_{\ell+1}^N = \mu_\ell^N$ for all $\ell \in \mathbb{Z}_+$, and for any $f \in B_p(\mathbb{Z}_+^{\Lambda_\ell^d})$ we have that $f \circ p_\ell^{\ell+1} \in B_p(\mathbb{Z}_+^{\Lambda_{\ell+1}^d})$. Therefore since $\mu_\ell^N \longrightarrow \mu_\ell$ for all $\ell \in \mathbb{Z}_+$ in the w_p topology we for any $f \in B_p(\mathbb{Z}_+^{\Lambda^d})$ that

$$\begin{aligned}\int f dp_{\ell *}^{\ell+1}\mu_{\ell+1} &= \int f \circ p_\ell^{\ell+1} d\mu_{\ell+1} = \lim_{N\to\infty} \int f \circ p_\ell^{\ell+1} d\mu_{\ell+1}^N = \lim_{N\to\infty} \int f d\mu_\ell^N \\ &= \int f d\mu_\ell,\end{aligned}$$

and since this is true for all $f \in B(\mathbb{Z}_+^{\Lambda_\ell^d})$ proves that the sequence $\{\mu_\ell\}_{\ell=1}^\infty$ is projective.

Now since $\{\mu_\ell\}_{\ell\in\mathbb{N}}$, by Kolmogorov's extension theorem (see e.g. [11] p.68, § 51) there exists $\mu \in \mathbb{P}\mathbb{M}_\infty^d$ such that $p_{\ell *}\mu = \mu_\ell \in \mathbb{P}_p\mathbb{M}_\infty^d$ for all $\ell \in \mathbb{Z}_+$. Then obviously $\mu \in \mathbb{P}_p\mathbb{M}_\infty^d$ and since by construction we have that

$$pi_{\ell *}^N = \mu_\ell^N \longrightarrow \mu_\ell = p_{\ell *}\mu$$

in the w_p topology it follows by corollary 1.3.1 that $\mu^N \longrightarrow \mu$ in the w_p topology of $\mathbb{P}_p\mathbb{M}_\infty^d$ which proves the completeness of W_p.

Likewise the separability of the w_p topology on $\mathbb{P}_p\mathbb{M}_\infty^d$ follows by the separability of the spaces $\mathbb{P}_p(\mathbb{Z}_+^{\Lambda_\ell^d})$, $\ell \in \mathbb{Z}_+$. Indeed, let $D_\ell \subseteq \mathbb{P}_p(\mathbb{Z}_+^{\Lambda_\ell^d})$ be the a countable dense subset of $\mathbb{P}_p(\mathbb{Z}_+^{\Lambda_\ell^d})$. Recalling that $e_\ell : \mathbb{Z}_+^{\Lambda_\ell^d} \longrightarrow \mathbb{M}_\infty$ denotes the periodic embedding we set

$$D := \bigcup_{\ell\in\mathbb{Z}_+} e_{\ell *}(D_\ell)$$

and we will show that D, which is obviously countable, is dense in $\mathbb{P}_p\mathbb{M}_\infty^d$. So let $\mu \in \mathbb{P}_p\mathbb{M}_\infty^d$ and let V be a neighborhood of μ. Since for any $f \in B_pC(\mathbb{M}_\infty^d)$ the sequence $\{f_\ell\} := \{f \circ e_\ell \circ p_\ell\}$ converges B_p-pointwise to f we have by the dominated convergence theorem that

$$\int f d(e_\ell \circ p_\ell)_* \mu = \int f_\ell d\mu \overset{\ell\to\infty}{\longrightarrow} \int f d\mu$$

for all $f \in B_pC(\mathbb{M}^d_\infty)$ and therefore $(e_\ell \circ p_\ell)_*\mu \longrightarrow \mu$ as $\ell \to \infty$ in the w_p-topology. Consequently for large enough $\ell_0 \in \mathbb{Z}_+$ we have that $(e_{\ell_0} \circ p_{\ell_0})_*\mu \in V$. But on the other hand, since

$$f \in B_pC(\mathbb{M}^d_\infty) \implies f \circ e_\ell \in B_pC(\mathbb{Z}_+^{\Lambda^d_\ell})$$

it follows that for each fixed $\ell \in \mathbb{Z}_+$ the function

$$e_{\ell *} : \mathbb{P}(\mathbb{Z}_+^{\Lambda^d_\ell}) \longrightarrow \mathbb{P}_p\mathbb{M}^d_\infty$$

is continuous. But then by the continuity of $e_{\ell_0 *}$ and since D_{ℓ_0} is dense in $\mathbb{P}_p(\mathbb{Z}_+^{\Lambda^d_{\ell_0}})$ we can choose $\mu_0 \in D_{\ell_0}$ close enough to $p_{\ell_0 *}\mu$ so that $e_{\ell_0 *}\mu_0 \in V$. □

Definition 1.3.5 A set $\mathcal{K} \subseteq \mathbb{P}\mathbb{M}^d_\infty$ is said to have *uniformly integrable p-th moments*, $p > 0$, if

$$\lim_{R\to\infty} \sup_{\mu\in\mathcal{K}} \int_{\{\sum_{|x|\le\ell} \eta(x)^p \ge R\}} \sum_{|x|\le\ell} \eta(x)^p d\mu = 0.$$

It is easy to see that a sequence $\{\mu^N\} \subseteq \mathbb{P}_p\mathbb{M}^d_\infty$ converges in the w_p topology iff it converges weakly and it has uniformly integrable p-th moments and that a set $\mathcal{K} \subseteq \mathbb{P}_p\mathbb{M}^d_\infty$ is relatively compact iff it is tight and has uniformly integrable p-th moments.

1.3.3 The Space $\mathcal{M}_+(\mathbb{T}^d)$

We denote by $\mathcal{M}_+(\mathbb{T}^d)$ the space of finite positive Borel measures on the torus $\mathbb{T}^d$ equipped with the topology of weak convergence of measures, according to which a sequence $\{\mu^N\} \subseteq \mathcal{M}_+(\mathbb{T}^d)$ converges to $\mu \in \mathcal{M}_+(\mathbb{T}^d)$ iff

$$\int f d\mu^N \longrightarrow \int f d\mu^N$$

for all $f \in C(\mathbb{T}^d)$. By the Riesz representation theorem (theorem 7.2 in [16]) the dual of the space $(C(\mathbb{T}^d), \|\cdot\|_u)$ where $\|\cdot\|_u$ is the uniform norm $\|f\|_u := \sup_{u\in\mathbb{T}^d} |f(u)|$ is exactly the space $\mathcal{M}(\mathbb{T}^d)$ of all finite Borel charges equipped with the total variation norm $\|\mu\| = \mu^+(\mathbb{T}^d) + \mu^-(\mathbb{T}^d)$ where $\mu = \mu^+ - \mu^-$ denotes the Hahn decomposition of μ. By definition the w^*-topology of $\mathcal{M}(\mathbb{T}^d) \cong C(\mathbb{T}^d)^*$ is the weakest topology on $\mathcal{M}(\mathbb{T}^d)$ that makes all the linear functionals $\ell_f : \mathcal{M}(\mathbb{T}^d) \longrightarrow \mathbb{R}$, $f \in C(\mathbb{T}^d)$, defined by

$$\ell_f(\mu) = \int f d\mu$$

continuous. It is obvious that the cone $\mathcal{M}_+(\mathbb{T}^d)$ is a w^*-closed subset of $\mathcal{M}(\mathbb{T}^d)$ and therefore the topology of weak convergence of non-negative measures on $\mathcal{M}_+(\mathbb{T}^d)$ is exactly the restriction of the w^*-topology of $\mathcal{M}(\mathbb{T}^d) \cong C(\mathbb{T}^d)^*$ in $\mathcal{M}_+(\mathbb{T}^d)$. As is known (see for instance theorem 3.25 in [7] and the remark following it) the w^*-topology on the dual of a Banach space X is never metrizable on the whole space X^*. It is metrizable on norm-bounded subsets of X^* iff X is separable. Nevertheless the weak topology on the cone $\mathcal{M}_+(\mathbb{T}^d)$ is metrizable. We will describe in this section two useful metrics inducing the topology of $\mathcal{M}_+(\mathbb{T}^d)$.

Since $\mathbb{T}^d$ is compact the space $C(\mathbb{T}^d)$ is separable and so there exists a sequence $\{f_k\}_{k=1}^\infty \subseteq C(\mathbb{T}^d)$ with $f_1 \equiv 1$ that is dense in $C(\mathbb{T}^d)$. Using this sequence we define a metric δ in $\mathcal{M}_+(\mathbb{T}^d)$ by the formula

$$\delta(\mu,\nu) = \sum_{k=1}^\infty \frac{1}{2^k} \cdot \frac{|\langle \mu-\nu, f_k\rangle|}{1+|\langle \mu-\nu, f_k\rangle|} \tag{1.53}$$

Proposition 1.3.11 *The function $\delta : \mathcal{M}_+(\mathbb{T}^d) \times \mathcal{M}_+(\mathbb{T}^d) \longrightarrow [0,1]$ defined in (1.53) is a complete and separable metric on $\mathcal{M}_+(\mathbb{T}^d)$ that metrizes the weak topology.*

Proof The proof that δ is a metric is standard. We prove next that δ metrizes the weak topology of $\mathcal{M}_+(\mathbb{T}^d)$. On one hand, if $\mu^N \longrightarrow \mu$ weakly, then $\langle \mu^N, f_k\rangle \longrightarrow \langle f_k, \mu\rangle$ for all $k \in \mathbb{N}$. So, given $\varepsilon > 0$, if we choose $k_0 \in \mathbb{N}$ such that $\sum_{k=k_0+1}^\infty \frac{1}{2^k} < \varepsilon/2$, we can then choose $N_0 \in \mathbb{N}$ such that

$$N \geq N_0 \quad \Longrightarrow \quad \max_{k=1,\dots,k_0} \left|\langle \mu^N - \mu, f_k\rangle\right| < \varepsilon/2,$$

and then we obviously have that $\delta(\mu^N,\mu) < \varepsilon$ for all $N \geq N_0$. Therefore weak convergence implies convergence in the δ metric.

On the other hand, if $\delta(\mu^N,\mu) \longrightarrow 0$ then $\mu^N \longrightarrow \mu$ weakly. Indeed, if $\delta(\mu^N,\mu) \longrightarrow 0$, then $\lim_{N\to\infty}\langle \mu^N - \mu, f_k\rangle = 0$ for all $k \in \mathbb{N}$. In particular, since $f_1 \equiv 1$ we have that $\mu^N(\mathbb{T}^d) \longrightarrow \mu(\mathbb{T}^d)$. Therefore, given $\varepsilon > 0$ and $f \in C(\mathbb{T}^d)$ we can choose $k_0 \in \mathbb{N}$ such that $\|f - f_{k_0}\|_u < \varepsilon/2[1+2\mu(\mathbb{T}^d)]$, and then choose $N_0 \in \mathbb{N}$ such that

$$N \geq N_0 \quad \Longrightarrow \quad |\langle \mu^N - \mu, f_{k_0}\rangle| \vee |\mu^N(\mathbb{T}^d) - \mu(\mathbb{T}^d)| < 1 \wedge (\varepsilon/2).$$

But then for all $N \geq N_0$ we have that

$$\begin{aligned}
\left|\langle \mu^N - \mu, f\rangle\right| &\leq \left|\langle \mu^N - \mu, f_{k_0}\rangle\right| + \langle |\mu^N - \mu|, |f - f_{k_0}|\rangle \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\frac{|\mu^N - \mu|(\mathbb{T}^d)}{1+2\mu(\mathbb{T}^d)} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\frac{\mu^N(\mathbb{T}^d) + \mu(\mathbb{T}^d)}{1+2\mu(\mathbb{T}^d)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\frac{1+\mu(\mathbb{T}^d)+\mu(\mathbb{T}^d)}{1+2\mu(\mathbb{T}^d)} = \varepsilon,
\end{aligned}$$

which since $\varepsilon > 0$ and $f \in C(\mathbb{T}^d)$ were arbitrary proves the weak convergence $\mu^N \longrightarrow \mu$.

Next, $\mathcal{M}_+(\mathbb{T}^d)$ is separable. A dense subset is the set

$$\left\{ \sum_{i=1}^n a_i \delta_{x_i} \,\middle|\, n \in \mathbb{N},\ a_i \in \mathbb{Q}_+,\ x_i \in D \right\}$$

consisting of all linear combinations of Dirac masses at points $x \in D$, where $D \subseteq \mathbb{T}^d$ is a countable dense subset of $\mathbb{T}^d$, with positive rational coefficients.

We prove next that the metric δ is complete. So let $\{\mu^N\}$ be a δ-Cauchy sequence.

Then for all $k \in \mathbb{N}$ the real sequence $\{\langle \mu^N, f_k \rangle\}_{N \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$, and therefore for all $k \in \mathbb{N}$ there exists the limit $\lim_{N\to\infty} \langle \mu^N, f_k \rangle =: I_k \in \mathbb{R}$. We define next a function $I : \{f_k | k \in \mathbb{N}\} \longrightarrow \mathbb{R}$ by $I(f_k) = I_k$. Since $f_1 \equiv 1$, we have that $I_1 = \lim_{N\to\infty} \mu^N(\mathbb{T}^d)$ and in particular $0 \leq \mu^N(\mathbb{T}^d) \leq C$ for all $N \in \mathbb{N}$ for some constant $C \geq 0$. Therefore for all $N, m, k \in \mathbb{N}$ we have that

$$\left|\langle \mu^N, f_k \rangle - \langle \mu^N, f_m \rangle\right| \leq C \|f_k - f_m\|_u$$

and so

$$\left|I(f_k) - I(f_m)\right| = \lim_{N\to\infty} \left|\langle \mu^N, f_k \rangle - \langle \mu^N, f_m \rangle\right| \leq C \|f_k - f_m\|_u$$

for all $m, k \in \mathbb{N}$. Therefore the function $I : \{f_k | k \in \mathbb{N}\} \longrightarrow \mathbb{R}$ is Lipschitz and as such has a Lipschitz extension $\bar{I} : C(\mathbb{T}^d) \longrightarrow \mathbb{R}$. We note next that $\bar{I}$ is linear. Indeed, let $a, b \in \mathbb{R}$ and $f, g \in C(\mathbb{T}^d)$. Since $\{f_k\}_{k\in\mathbb{N}}$ is dense in $C(\mathbb{T}^d)$ there exists a subsequence $\{h_k\}$ of $\{f_k\}$ such that $\|h_k - af - bg\|_u \longrightarrow 0$ and since the sequence $\{\mu^N(\mathbb{T}^d)\}$ is bounded above by $C \geq 0$, we have that

$$\sup_{N\in\mathbb{N}} \left|\langle \mu^N, af + bg \rangle - \langle \mu^N, h_k \rangle\right| \leq C \|h_k - af - bg\|_u,$$

that is $\langle \mu^N, h_k \rangle \longrightarrow \langle \mu^N, af + bg \rangle$ uniformly over $N \in \mathbb{N}$. Therefore we can exchange the order of limits and write

$$\bar{I}(af + bg) = \lim_{k\to\infty} \lim_{N\to\infty} \langle \mu^N, h_k \rangle = \lim_{N\to\infty} \lim_{k\to\infty} \langle \mu^N, h_k \rangle.$$

Also there exist subsequences $\{\bar{f}_k\}_{k\in\mathbb{N}}$ and $\{\bar{g}_k\}_{k\in\mathbb{N}}$ such that $\bar{f}_k \longrightarrow f$ and $\bar{g}_k \longrightarrow g$ in $C(\mathbb{T}^d)$ and then

$$\|h_k - a\bar{f}_k - b\bar{g}_k\|_u \leq \|h_k - af - bg\|_u + \|af + bg - a\bar{f}_k - b\bar{g}_k\|_u \overset{k\to\infty}{\longrightarrow} 0.$$

It follows that

$$\langle \mu^N, h_k \rangle = a\langle \mu^N, \bar{f}_k \rangle + b\langle \mu^N, \bar{g}_k \rangle + \langle \mu^N, h_k - a\bar{f}_k - b\bar{g}_k \rangle$$

and therefore

$$\lim_{N\to\infty} \lim_{k\to\infty} \langle \mu^N, h_k \rangle = a \lim_{N\to\infty} \lim_{k\to\infty} \langle \mu^N, \bar{f}_k \rangle + b \lim_{N\to\infty} \lim_{k\to\infty} \langle \mu^N, \bar{g}_k \rangle.$$

Then since the convergences $\langle \mu^N, \bar{f}_k \rangle \longrightarrow \langle \mu^N, f \rangle$ and $\langle \mu^N, \bar{g}_k \rangle \longrightarrow \langle \mu^N, g \rangle$ as $k \to \infty$ are uniform over N we can exchange the order of limits once more to obtain that

$$\bar{I}(af + bg) = a \lim_{k\to\infty} I(\bar{f}_k) + b \lim_{k\to\infty} I(\bar{g}_k) = a\bar{I}(f) + bI(g).$$

Now, since $\bar{I} : C(\mathbb{T}^d) \longrightarrow \mathbb{R}$ is a bounded linear functional on $C(\mathbb{T}^d)$, there exists by Riesz's theorem $\mu \in \mathcal{M}_+(\mathbb{T}^d)$ such that $\bar{I} \equiv \langle \mu, \cdot \rangle$, and since by the definition of I we have that

$$\lim_{N\to\infty} \langle \mu^N, f_k \rangle = I(f_k) = \langle \mu, f_k \rangle$$

for all $k \in \mathbb{N}$ and $\{f_k\}$ is dense in $C(\mathbb{T}^d)$ it follows that $\mu^N \longrightarrow \mu$ weakly as required.□

The weakly compact subsets of $\mathcal{M}_+(\mathbb{T}^d)$ are easily described.

Proposition 1.3.12 *A subset $\mathcal{K} \subseteq \mathcal{M}_+(\mathbb{T}^d)$ is relatively compact iff*

$$\sup_{\mu \in \mathcal{K}} \mu(\mathbb{T}^d) < +\infty. \tag{1.54}$$

Proof Indeed, the function $\mathcal{M}_+(\mathbb{T}^d) \ni \mu \mapsto \mu(\mathbb{T}^d) = \int 1 d\mu$ is weakly continuous by definition and therefore if $\mathcal{K}$ is compact, its supremum over $\mathcal{K}$ is finite. On the other hand, suppose that (1.54) is satisfied. Then since $\mathcal{K}$ consists of non-negative measures,

$$\sup_{\mu \in \mathcal{K}} \|\mu\|_{TV} = \sup_{\mu \in \mathcal{K}} \mu(\mathbb{T}^d) < +\infty.$$

Thus $\mathcal{K}$ is norm bounded in $C(\mathbb{T}^d)^*$ and therefore by Alaoglou's theorem according to which norm bounded subsets of a dual Banach space X^* are always w^*-relatively compact, we have that $\mathrm{cl}_{w^*}(B) \subseteq \mathrm{cl}_{w^*}(\mathcal{M}_+(\mathbb{T}^d))$ is compact. But $\mathcal{M}_+(\mathbb{T}^d)$ is w^*-closed, and therefore given any sequence $\{\mu_n\}$ in B we can extract a subsequence that converges to some $\mu \in \mathcal{M}_+(\mathbb{T}^d)$. This proves the relative compactness of B in $\mathcal{M}_+(\mathbb{T}^d)$ and completes the proof. $\square$

Another useful metric on $\mathcal{M}_+(\mathbb{T}^d)$ is given by the restriction of Dudley's norm $\|\cdot\|_{\mathcal{D}}$ which is defined on the whole space $\mathcal{M}(\mathbb{T}^d)$ of finite charges on $\mathbb{T}^d$ by

$$\|\mu\|_{\mathcal{D}} := \sup\left\{ \int f d\mu \,\middle|\, f \in \mathrm{Lip}(\mathbb{T}^d),\ \|f\|_{BL} \leq 1 \right\}, \tag{1.55}$$

where $\|\cdot\|_{BL} : \mathrm{Lip}(\mathbb{T}^d) \longrightarrow \mathbb{R}_+$ is the bounded-Lipschitz norm on $\mathrm{Lip}(\mathbb{T}^d)$ given by

$$\|f\|_{BL} = \|f\|_u + \|f\|_{\mathrm{Lip}}. \tag{1.56}$$

Examples:

1. For all $x \in \mathbb{T}^d$ we have that $\|\delta_x\|_{\mathcal{D}} = 1$.

Proof Since the constant function $c_1 \equiv 1$ has $\|c_1\|_{BL} = 1$ we obviously have that

$$\|\delta_x\|_{\mathcal{D}} \geq \langle c_1, \delta_x \rangle = 1.$$

On the other hand, for all $f \in \mathrm{Lip}(\mathbb{T}^d)$ such that $\|f\|_{BL} \leq 1$ we have that

$$\langle f, \delta_x \rangle = f(x) = f(x) - f(0) + f(0) \leq \|f\|_{\mathrm{Lip}} |x|_{\mathbb{T}^d} + \|f\|_u \leq \|f\|_{BL} \leq 1$$

and therefore

$$\|\delta_x\|_{\mathcal{D}} = \sup_{\|f\|_{BL} \leq 1} \langle f, \delta_x \rangle = 1.$$

2. For all $x, y \in \mathbb{T}^d$ we have that

$$\frac{2}{3}|x - y|_{\mathbb{T}^d} \leq \|\delta_x - \delta_y\|_{\mathcal{D}} \leq |x - y|_{\mathbb{T}^d}.$$

Proof On one hand, for all $f \in \mathrm{Lip}(\mathbb{T}^d)$ with $\|f\|_{BL} \leq 1$ we have that

$$\langle f, \delta_x - \delta_y \rangle = f(x) - f(y) \leq \|f\|_{\mathrm{Lip}} |x - y|_{\mathbb{T}^d} \leq |x - y|_{\mathbb{T}^d},$$

which proves the right hand inequality. On the other hand, the function $f_x : \mathbb{T}^d \longrightarrow [0,1]$ defined by $f_x(u) = -(2/3)|x-u|_{\mathbb{T}^d}$ satisfies $\|f_x\|_{BL} \leq 1$ and

$$\|\delta_x - \delta_y\|_{\mathcal{D}} \geq \langle f_x, \delta_x - \delta_y \rangle = \frac{2}{3}|x-y|_{\mathbb{T}^d}.$$

As we know, the restriction of Dudley's norm on $\mathbb{P}\mathbb{T}^d$ metrizes the weak convergence on $\mathbb{P}\mathbb{T}^d$.

Proposition 1.3.13 *The metric* $d_{\mathcal{D}} : \mathcal{M}_+(\mathbb{T}^d) \times \mathcal{M}_+(\mathbb{T}^d) \longrightarrow \mathbb{R}$ *on* $\mathcal{M}_+(\mathbb{T}^d)$ *defined by*

$$d_{\mathcal{D}}(\mu, \nu) := \|\mu - \nu\|_{\mathcal{D}}$$

is a metric on $\mathcal{M}_+(\mathbb{T}^d)$ *that metrizes the weak convergence of measures.*

Proof We prove first that if $d_{KR}(\mu^N, \mu) \longrightarrow 0$ then $\mu^N \longrightarrow \mu$ weakly. Indeed, for any $\phi \in C(\mathbb{T}^d)$ with $\mathrm{Lip}(\phi) \leq 1$ we have that

$$\left| \int \phi d(\mu^N - \mu) \right| = \int \phi d(\mu^N - \mu) \vee \int (-\phi) d(\mu^N - \mu) \leq d_{KR}(\mu^N, \mu) \longrightarrow 0$$

and therefore if for any non-constant Lipschitz function $\phi : \mathbb{T}^d \longrightarrow \mathbb{R}$ we set $\bar{\phi} := \mathrm{Lip}(\phi)^{-1}\phi$ we have that

$$\int \phi d\mu^N = \mathrm{Lip}(\phi) \int \bar{\phi} d\mu^N \longrightarrow \mathrm{Lip}(\phi) \int \bar{\phi} d\mu = \int \phi d\mu$$

for all Lipschitz functions $\phi : \mathbb{T}^d \longrightarrow \mathbb{R}$. As is well known ([2], chapter 5) this implies the weak convergence $\mu^N \longrightarrow \mu$.

We suppose next that $\mu^N \longrightarrow \mu$ weakly and we will prove that $d_{\mathcal{D}}(\mu^N, \mu) \longrightarrow 0$. Since $\mu^N \longrightarrow \mu$ weakly we have that $\mu^N(\mathbb{T}^d) \longrightarrow \mu(\mathbb{T}^d)$. Since d_{KR} is induced by a norm we have that

$$d_{\mathcal{D}}(\mu^N, \mu) = d_{\mathcal{D}}(\delta_0 + \mu^N, \delta_0 + \mu)$$

and so we can make the additional assumption that $\mu(\mathbb{T}^d) \wedge \mu^N(\mathbb{T}^d) > 0$ for all $N \in \mathbb{N}$. So we can set $\bar{\mu} := \mu(\mathbb{T}^d)^{-1}\mu$ and $\bar{\mu}^N := \mu^N(\mathbb{T}^d)^{-1}\mu^N$. Then $\bar{\mu}^N$, $N \in \mathbb{N}$, and $\bar{\mu}$ are probability measures and for all $f \in C(\mathbb{T}^d)$ we have that

$$\int f d\bar{\mu}^N = \frac{1}{\mu^N(\mathbb{T}^d)} \int f d\mu^N \longrightarrow \frac{1}{\mu(\mathbb{T}^d)} \int f d\mu = \int f d\bar{\mu}$$

and therefore $\bar{\mu}^N \longrightarrow \bar{\mu}$ weakly. Since as we know the $d_{\mathcal{D}}$ metric metrizes the weak convergence of measures on $\mathbb{P}\mathbb{T}^d$, it follows that

$$\lim_{N \to \infty} d_{KR}(\bar{\mu}^N, \bar{\mu}) = 0.$$

Therefore

$$\begin{aligned}
\|\mu^N - \mu\|_{\mathcal{D}} &= \|\mu^N(\mathbb{T}^d)\bar{\mu}^N - \mu(\mathbb{T}^d)\bar{\mu}\|_{\mathcal{D}} \\
&\leq \|\mu^N(\mathbb{T}^d)\bar{\mu}^N - \mu^N(\mathbb{T}^d)\bar{\mu}\|_{\mathcal{D}} + \|\mu^N(\mathbb{T}^d)\bar{\mu} - \mu(\mathbb{T}^d)\bar{\mu}\|_{\mathcal{D}} \\
&= \mu^N(\mathbb{T}^d) d_{\mathcal{D}}(\bar{\mu}^N, \bar{\mu}) + |\mu^N(\mathbb{T}^d) - \mu(\mathbb{T}^d)| \|\bar{\mu}\|_{\mathcal{D}} \overset{N \to \infty}{\longrightarrow} 0,
\end{aligned}$$

as required. $\square$

Sometimes it is useful to understand measures in $\mathcal{M}_+(\mathbb{T}^d)$ as equivalence classes of measures on $\mathbb{R}^d$. More generally, we denote by $\mathcal{M}(\mathbb{R}^d;\mathbb{R}^k)$ the set of all $\mathbb{R}^k$-valued measures with finite total variation and define an equivalence relation $\backsim$ on $\mathcal{M}(\mathbb{R}^d;\mathbb{R}^k)$ by

$$\mu \backsim \nu \quad \iff \quad \int f d\mu = \int f d\nu \qquad \forall\, f \in \widetilde{C}(\mathbb{R}^d)$$

where by $\widetilde{C}(\mathbb{R}^d)$ we denote all the continuous $\mathbb{Z}^d$-periodic functions on $\mathbb{R}^d$, i.e. $f \in \widetilde{C}(\mathbb{R}^d)$ iff $f(x) = f(y)$ whenever $x - y \in \mathbb{Z}^d$. We denote the equivalence classes of $\backsim$ by $[\mu]$. If we denote by $p : \mathbb{R}^d \longrightarrow \mathbb{T}^d \cong [0,1)^d$ the usual covering map defined by $p(x) = x - [x]$, where $[x] = ([x_1], \ldots, [x_d])$ denotes the integer part of $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, it is easy to see that $\mu \backsim \nu$ iff $p_*\mu = p_*\nu \in \mathcal{M}(\mathbb{T}^d;\mathbb{R}^k)$. Indeed, suppose that $\mu \backsim \nu$. Then for any function $f \in C(\mathbb{T}^d)$ we have that the function $f \circ p$ is in $\widetilde{C}(\mathbb{R}^d)$ and therefore

$$\int f dp_*\mu = \int f \circ p d\mu = \int f \circ p d\nu = \int f dp_*\nu,$$

which shows that $p_*\mu = p_*\nu$. On the other hand, if $f \in \widetilde{C}(\mathbb{R}^d)$ then obviously $f \equiv f \circ i \circ p$ where $i : \mathbb{T}^d \equiv [0,1)^d \hookrightarrow \mathbb{R}^d$ is the inclusion and so if $p_*\mu = p_*\nu$ then

$$\int f d\mu = \int (f \circ i) dp_*\mu = \int (f \circ i) dp_*\nu = \int f d\nu$$

which shows that $\mu \backsim \nu$.

It follows that the push forward $p_* : \mathcal{M}(\mathbb{R}^d;\mathbb{R}^k) \longrightarrow \mathcal{M}(\mathbb{T}^d;\mathbb{R}^k)$ induces a well defined bijection

$$\widetilde{p_*} : {}^{\mathcal{M}(\mathbb{R}^d;\mathbb{R}^k)}/_{\backsim} \longrightarrow \mathcal{M}(\mathbb{T}^d;\mathbb{R}^k)$$

on the quotient space by $\widetilde{p_*}([\mu]) = p_*\mu$. Finally, by the definition of the equivalence relation $\backsim$ it is obvious that equivalence classes $[\mu] \in {}^{\mathcal{M}(\mathbb{R}^d;\mathbb{R}^k)}/_{\backsim}$ can be used to integrate periodic functions $f \in \widetilde{C}(\mathbb{R}^d)$ by the formula

$$\int f d[\mu] = \int f d\mu = \int (f \circ i) d\widetilde{p_*}[\mu] = \int (f \circ i) dp_*\mu,$$

where of course μ is any representative of the equivalence class $[\mu]$. For simplicity we state the next proposition for the case of non-negative valued measures.

Proposition 1.3.14 *The mapping $p_* : \mathcal{M}_+(\mathbb{R}^d) \longrightarrow \mathcal{M}_+(\mathbb{T}^d)$ induced by the covering map $p : \mathbb{R}^d \longrightarrow \mathbb{T}^d$ induces a homeomorphism $\widetilde{p_*} : {}^{\mathcal{M}_+(\mathbb{R}^d)}/_{\backsim} \longrightarrow \mathcal{M}_+(\mathbb{T}^d)$ between the $\backsim$-quotient of the weak topology of $\mathcal{M}_+(\mathbb{R}^d)$ and the weak topology of $\mathcal{M}_+(\mathbb{T}^d)$.*

Proof Let $\pi : \mathcal{M}_+(\mathbb{R}^d) \longrightarrow {}^{\mathcal{M}_+(\mathbb{R}^d)}/_{\backsim}$ be the quotient mapping. Since $p_*\mu = p_*\nu$ whenever $\mu \backsim \nu$ and p_* is continuous it follows by the universal property of quotient mappings that $\widetilde{p_*} : {}^{\mathcal{M}_+(\mathbb{R}^d)}/_{\backsim} \longrightarrow \mathcal{M}_+(\mathbb{T}^d)$ is a continuous mapping. Furthermore $\widetilde{p_*}$ is obviously bijective, with inverse the function

$$\widetilde{p_*}^{-1}(\mu) = [i_*\mu], \qquad \mu \in \mathcal{M}_+(\mathbb{T}^d).$$

Now as we know, a set $\mathcal{A} \subseteq {}^{\mathcal{M}_+(\mathbb{R}^d)}/_{\backsim}$ is open in the quotient topology iff $\pi^{-1}(\mathcal{A})$ is open in the weak topology of $\mathcal{M}_+(\mathbb{R}^d)$. It follows that a sequence $\{[\mu_n]\}_{n\in\mathbb{N}} \subseteq {}^{\mathcal{M}_+(\mathbb{R}^d)}/_{\backsim}$ converges to some $[\mu] \in {}^{\mathcal{M}_+(\mathbb{R}^d)}/_{\backsim}$ iff there exist $\nu_n \in [\mu_n]$ for all $n \in \mathbb{N}$ and $\nu \in [\mu]$ such that $\nu_n \longrightarrow \nu$ in the weak topology of $\mathcal{M}_+(\mathbb{R}^d)$. In particular whenever $\{\mu_n\} \subseteq \mathcal{M}_+(\mathbb{T}^d)$ converges to $\mu \in \mathcal{M}_+(\mathbb{T}^d)$ we have that $i_*\mu_n \longrightarrow i_*\mu$ which shows that $\widetilde{p_*}^{-1}$ is continuous and thus $\widetilde{p_*}$ is a homeomorphism as claimed. □

Corollary 1.3.4 *Let $\{\mu_n\}_{n\in\mathbb{N}} \cup \{\mu\} \subseteq \mathcal{M}_+(\mathbb{R}^d)$. The following are equivalent:*
(a) *$p_*\mu_n \longrightarrow p_*\mu$ as $n \to \infty$ in the weak topology of $\mathcal{M}_+(\mathbb{T}^d)$.*
(b) *For all $u \in C^\infty(\mathbb{T}^d) \leq C^\infty(\mathbb{R}^d)$ it holds that*

$$\lim_{n\to\infty} \int u d\mu_n = \int u d\mu.$$

(c) *There exist $\nu_n \backsim \mu_n$ for all $n \in \mathbb{N}$ and $\nu \backsim \mu$ such that $\lim_{n\to\infty} \nu_n = \nu$ in $\mathcal{M}_+(\mathbb{R}^d)$.*

1.4 Equivalence of Ensembles

The main result of this section, theorem 1.4.1 on the equivalence of ensembles is taken from the article [19]. More refined results have been proved in [3, 4]. First, the definition ZR distributions on the discrete toruses $\mathbb{T}_N^d$, $N \in \mathbb{N}$, extends obviously to the full lattice $\mathbb{Z}^d$.

Definition 1.4.1 Let $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ be a local rate function and let I_c be the interval of the admissible densities for the one-site ZR distributions $\{\nu_\rho^1\}$, $\rho \in I_c$, associated to the local rate function g. For each $\rho \in I_c$ the distribution $\nu_\rho \equiv \nu_{\rho,g} \in \mathbb{P}_1\mathbb{M}_\infty^d$ given by

$$\nu_\rho^\infty := \prod_{z\in\mathbb{Z}^d} \nu_{\rho,g}^1$$

is called a *(normalized) ZR distribution on the full lattice* $\mathbb{Z}^d$ *with rate* g *and density* ρ.

Of course ν_ρ^∞ is the unique distribution on $\mathbb{M}_\infty^d$ that makes the natural projections

$$\eta(z) : \mathbb{M}_\infty^d \longrightarrow \mathbb{Z}_+, \quad z \in \mathbb{Z}^d,$$

i.i.d. random variables with common distribution the one-site ZR distribution ν_ρ^1. Also, through the identification $\mathbb{T}_N^d \cong \mathbb{Z}_N^d$, we have $\eta_*^N \nu_\rho^\infty = \nu_\rho^N, \quad N \in \mathbb{N}$.

Definition 1.4.2 The family of ZR distributions $\{\nu_\rho^\infty\}_{\rho\in I_c}$ associated to some local rate function g is known as the *grand canonical ensemble of the ZRP with rate function* g.

Definition 1.4.3 The family $\{\nu_{N,K}^d\}_{(N,K)\in\mathbb{N}\times\mathbb{Z}_+}$ of the extremal invariant distributions of the ZRP with local rate function g which are concentrated on the communication classes $\mathbb{M}_{N,K}^d$, given by

$$\nu_{N,K}^d = \frac{1}{Z(N^d, K)} \sum_{\eta\in\mathbb{M}_{N,K}^d} \frac{1}{g!(\eta)}\delta_\eta, \quad Z(N^d, K) = \sum_{\eta\in\mathbb{M}_{N,K}^d} \frac{1}{g!(\eta)}$$

is called the *canonical ensemble of the ZRP.*

As we have seen the grand canonical ensemble is translation invariant. The same is also true for the canonical ensemble.

Proposition 1.4.1 *Let* $\{\nu_{N,K} \in \mathbb{P}\mathbb{M}_N^d\}_{(N,K)\in\mathbb{N}\times\mathbb{Z}_+}$ *be the canonical ensemble of the ZRP with local rate function* g*. Then the distributions* $\nu_{N,K}$*,* $(N, K) \in \mathbb{N} \times \mathbb{Z}_+$ *are translation invariant, that is for all* $(N, K) \in \mathbb{N} \times \mathbb{Z}_+$ *we have that*

$$\tau_{x*}\nu_{N,K} = \nu_{N,K}, \quad \forall\, x \in \mathbb{T}_N^d.$$

Proof Obviously for each $(N, K) \in \mathbb{N} \times \mathbb{Z}_+$ and each $\eta \in \mathbb{M}_N$, $x \in \mathbb{T}_N^d$, we have that

$$|\tau_x\eta|_1 = \sum_{y\in\mathbb{T}_N^d} \tau_x\eta(y) = \sum_{y\in\mathbb{T}_N^d} \eta(x + y) = |\eta|_1,$$

and so $\tau_x(\mathbb{M}^d_{N,K}) = \mathbb{M}^d_{N,K}$ for all $(N,K) \in \mathbb{Z}_+$, $x \in \mathbb{T}^d_N$. So, for each $\eta \notin \mathbb{M}^d_{N,K}$ we have that

$$\nu_{N,K}(\eta) = 0 = \nu_{N,K}(\tau_{-x}\eta) = \tau_{x*}\nu_{N,K}(\eta).$$

On the other hand, for all $\eta \in \mathbb{M}^d_{N,K}$ and all $x \in \mathbb{T}^d_N$ we have that

$$g!(\tau_x\eta) = \prod_{y\in\mathbb{T}^d_N} g!(\tau_x\eta_y) = \prod_{y\in\mathbb{T}^d_N} g!(\eta_{x+y}) = \prod_{x\in\mathbb{T}^d_N} g!(\eta_x) = g!(\eta)$$

and therefore

$$\tau_{x*}\nu_{N,K}(\eta) = \nu_{N,K}(\tau_{-x}\eta) = \frac{1}{Z(N^d,K)}\frac{1}{g!(\tau_{-x}\eta)} = \frac{1}{Z(N^d,K)}\frac{1}{g!(\eta)} = \nu_{N,K}(\eta). \qquad \square$$

In what follows we will always consider the spaces of configurations $\mathbb{M}^d_N$ on the discrete toruses, and the respective spaces of probability measures $\mathbb{P}\mathbb{M}^d_N$ embedded in the space of configurations $\mathbb{M}^d_\infty$ on the full lattice and in the respective space of probability measures $\mathbb{P}\mathbb{M}^d_\infty$ via the periodic embeddings considered in the previous section. Furthermore, we let $\pi^L : \mathbb{M}^d_N \longrightarrow \mathbb{M}^d_L$, $N \geq L$, denote the natural projections and set $\nu^L_{N,K} := \pi^L_*\nu_{N,K}$.

Theorem 1.4.1 (Equivalence of Ensembles) *Let $\{\nu_{N,K}\}_{K\in\mathbb{Z}_+}$ and $\{\nu^N_\rho\}_{\rho\in[0,\rho_c]\cap\mathbb{R}_+}$ be the canonical and grand canonical ensemble of the ZRP. Then for fixed $L \in \mathbb{N}$, for all $\rho \geq 0$ it holds that*

$$\lim_{N\to+\infty} \mathcal{H}(\nu^L_{N,[\rho N^d]}|\nu^L_{\rho\wedge\rho_c}) = 0.$$

In particular by Pinsker's inequality (proposition A.3.5) $\|\nu^L_{N,[\rho N^d]} - \nu^L_{\rho\wedge\rho_c}\|_{TV} \longrightarrow 0$ and consequently $\nu^L_{N,[\rho N^d]} \longrightarrow \nu^L_{\rho\wedge\rho_c}$ weakly as $N \to \infty$.

Proof Let $\eta^\ell : \mathbb{M}^d_\infty \longrightarrow \mathbb{M}^d_\ell$, $\ell \in \mathbb{N}$, be the natural projections. Of course it suffices to prove that for each $\ell \in \mathbb{N}$

$$\lim_{N\to+\infty} \eta^\ell_*\nu_{N,[\rho N^d]} = \nu^\ell_\rho \tag{1.57}$$

weakly in $\mathbb{P}\mathbb{M}^d_\ell$. So we let $\ell \in \mathbb{N}$ and set $\nu^\ell_{N,K} := \eta^\ell_*\nu_{N,K}$ for all $(N,K) \in \mathbb{N}\times\mathbb{Z}_+$. By Pinsker's inequality, for any measurable space M, the total variation norm on $\mathbb{P}M$ is bounded by twice the relative entropy,

$$\|\nu - \mu\|^2_{TV} \leq 2\mathcal{H}(\nu|\mu), \quad \forall\, \mu,\nu \in \mathbb{P}M,$$

and therefore since convergence in total variation is stronger that weak convergence, it is obvious that it suffices to prove that

$$\lim_{N\to+\infty} \mathcal{H}(\nu^\ell_{N,[\rho N^d]}|\nu^\ell_{\rho\wedge\rho_c}) = 0.$$

Since by the definition of ν^N_ρ, $\rho \in I_c$, we have that

$$\nu^N_\rho(\eta) = \frac{1}{Z(\Phi(\rho))^{N^d}}\frac{\Phi(\rho)^{|\eta|_1}}{g!(\eta)} > 0,$$

for all $\eta \in \mathbb{M}_N^d$, it follows that $\nu_{N,K}$ is absolutely continuous with respect to ν_ρ^N with density

$$f_{N,K}(\eta) := \frac{d\nu_{N,K}}{d\nu_\rho^N}(\eta) = \frac{\nu_{N,K}(\eta)}{\nu_\rho^N(\eta)}\mathbb{1}_{\mathbb{M}_{N,K}^d}(\eta) = \frac{Z\big(\Phi(\rho)\big)^{N^d}}{Z(N^d,K)\Phi(\rho)^K}\mathbb{1}_{\mathbb{M}_{N,K}^d}(\eta).$$

Furthermore, for all $N \in \mathbb{N}$, $K \in \mathbb{Z}_+$, we obviously have

$$\nu_\rho^N(\mathbb{M}_{N,K}^d) = \frac{1}{Z\big(\Phi(\rho)\big)^{N^d}} \sum_{\eta\in\mathbb{M}_{N,K}^d} \frac{\Phi(\rho)^{|\eta|_1}}{g!(\eta)} = \frac{Z(N^d,K)\Phi(\rho)^K}{Z\big(\Phi(\rho)\big)^{N^d}} \tag{1.58}$$

and so

$$f_{N,\rho}(\eta) = \frac{1}{\nu_\rho^N(\mathbb{M}_{N,[\rho N^d]}^d)}\mathbb{1}_{\mathbb{M}_{N,[\rho N^d]}^d}(\eta).$$

Therefore the relative entropy of $\nu_{N,K}$ with respect to ν_ρ^N, $\rho \in I_c$, is given by

$$\begin{aligned}\mathcal{H}\big(\nu_{N,K}\big|\nu_\rho^N\big) &= \int \log f_{N,K} d\nu_{N,K} = \frac{1}{Z(N^d,K)} \sum_{\eta\in\mathbb{M}_{N,K}^d} \log f_{N,K}(\eta)\frac{1}{g!(\eta)} \\ &= -\log \nu_\rho^N(\mathbb{M}_{N,K}^d). \end{aligned}\tag{1.59}$$

By the super-additivity of the relative entropy, proposition A.3.6 in the appendix,

$$\mathcal{H}\big(\nu_{N,K}\big|\nu_\rho^N\big) \geq \sum_{x\in\mathbb{T}_N^d} \mathcal{H}\big(\eta(x)_*\nu_{N,K}\big|\nu_\rho^1\big).$$

Since $\nu_{N,K}$ is translation invariant, in particular it has equidistributed marginals, that is $\eta(x)_*\nu_{N,K} = \eta(0)_*\nu_{N,K}$ for all $x \in \mathbb{T}_N^d$, and therefore it follows that

$$\mathcal{H}\big(\eta(0)_*\nu_{N,K}\big|\nu_\rho^1\big) \leq \frac{1}{N^d}\mathcal{H}(\nu_{N,K}|\nu_\rho^1) = -\frac{1}{N^d}\log \nu_\rho^N(\mathbb{M}_{N,K}^d).$$

We prove next that more generally, for any $N \in \mathbb{N}$, $\Lambda \subseteq \mathbb{T}_N^d \subseteq \mathbb{Z}^d$ and $\rho \in I_c$ we have

$$\mathcal{H}\big(\nu_{N,K}^\Lambda \,\big|\, \nu_\rho^\Lambda\big) \leq -\frac{1}{\big[\frac{N^d}{\sharp\Lambda}\big]}\log \nu_\rho(\mathbb{M}_{N,K}^d), \tag{1.60}$$

where for all $(N,K) \in \mathbb{N}\times\mathbb{Z}_+$, $\rho \in I_c$, and $\Lambda \subseteq \mathbb{T}_N^d$ we have set

$$\nu_{N,K}^\Lambda := \big(\eta_\Lambda^{\mathbb{T}_N^d}\big)_*\nu_{N,K}, \quad \nu_\rho^\Lambda := \bigotimes_{x\in\Lambda}\nu_\rho^1,$$

and we denoted by

$$\eta_\Lambda^F : \mathbb{Z}_+^F \longrightarrow \mathbb{Z}_+^\Lambda$$

the natural projection for each $\Lambda \subseteq F \subseteq \mathbb{Z}^d$. To this end, we prove that the quantities

$$\mathcal{H}\big(\nu_{N,K}^\Lambda \,\big|\, \nu_\rho^\Lambda\big), \quad \Lambda \subseteq \mathbb{T}_N^d,$$

depend on the subset Λ only through its cardinality $\sharp\Lambda$, i.e. that

$$\Lambda, F \subseteq \mathbb{Z}^d,\ \sharp\Lambda = \sharp F \quad \Longrightarrow \quad \mathcal{H}\big(\nu_{N,K}^\Lambda \,\big|\, \nu_\rho^\Lambda\big) = \mathcal{H}\big(\nu_{N,K}^F \,\big|\, \nu_\rho^F\big) \tag{1.61}$$

Indeed, suppose that $\Lambda, F \subseteq \mathbb{T}_N^d$ are such that $\sharp\Lambda = \sharp F$. There exists then a bijection $\sigma : \Lambda \longrightarrow F$ which in turn induces the mapping $\sigma \equiv \sigma^* : \mathbb{Z}_+^F \longrightarrow \mathbb{Z}_+^\Lambda$ on the respective spaces of configurations given by

$$(\sigma\eta)_x = \eta_{\sigma(x)}, \quad \forall\, x \in \Lambda.$$

In its turn the transformation $\sigma = \sigma^*$ induces the mapping $\sigma_* : \mathbb{P}\mathbb{Z}_+^F \longrightarrow \mathbb{P}\mathbb{Z}_+^\Lambda$ through the push forward of measures on the spaces of distributions. Obviously for each $\rho \in I_c$ and each $\eta \in \mathbb{Z}_+^\Lambda$ we have that

$$\sigma_*\nu_\rho^F(\eta) = \nu_\rho^F(\sigma^{-1}\eta) = \prod_{y\in F}\nu_\rho^1(\eta_{s^{-1}(y)}) = \prod_{x\in\Lambda}\nu_\rho^1(\eta_x) = \nu_\rho^\Lambda(\eta). \tag{1.62}$$

We will prove that

$$\sigma_*\nu_{N,K}^F = \nu_{N,K}^\Lambda, \tag{1.63}$$

for then it will follow that $\mathcal{H}\big(\nu_{N,K}^\Lambda \,\big|\, \nu_\rho^\Lambda\big) = \mathcal{H}\big(\nu_{N,K}^F \,\big|\, \nu_\rho^F\big)$ as claimed, proving (1.61).

We prove now equality (1.63). By definition, for all $N \in \mathbb{N}$, $\Lambda \subseteq \mathbb{T}_N^d$ and $\eta \in \mathbb{Z}_+^\Lambda$ we have that

$$\nu_{N,K}^\Lambda(\eta) = \nu_{N,K}\big\{\zeta \in \mathbb{M}_N^d \,\big|\, \zeta|_\Lambda = \eta\big\}.$$

Obviously, if $|\eta|_1 := |\eta|_{\Lambda,1} := \sum_{x\in\Lambda}\eta_x > K$ we have that $\nu_{N,K}^\Lambda(\eta) = 0$, while on the other hand for any $\zeta \in \mathbb{M}_N^d$ such that $\zeta|_\Lambda = \eta$ we have that

$$g!(\zeta) = \prod_{x\in\mathbb{T}_N^d} g!(\zeta_x) = \prod_{x\in\Lambda} g!(\eta_x)\prod_{x\in\mathbb{T}_N^d\setminus\Lambda} g!(\zeta_x) = g!(\eta)g!(\zeta|_{\mathbb{T}_N^d\setminus\Lambda}),$$

and so if $0 \le |\eta|_{\Lambda,1} \le K$ we have that

$$\nu_{N,K}^\Lambda(\eta) = \frac{1}{Z(N,K)}\frac{1}{g!(\eta)}\sum_{\zeta\in\mathbb{Z}_+^{\mathbb{T}_N^d\setminus\Lambda}\,:\,|\zeta|_1 = K-|\eta|_{\Lambda,1}}\frac{1}{g!(\zeta)}.$$

Therefore, if we set

$$Z(\Lambda,K) = \sum_{\eta\in\mathbb{Z}_+^\Lambda:|\eta|_1=K}\frac{1}{g!(\eta)},$$

for every subset $\Lambda \subseteq \mathbb{T}_N^d \subseteq \mathbb{Z}^d$ and any $K \in \mathbb{Z}_+$, then we can write that

$$\nu_{N,K}^\Lambda = \frac{1}{Z(N^d,K)}\sum_{\eta\in\mathbb{Z}_+^\Lambda:|\eta|_{\Lambda,1}\le K}\frac{\bar{Z}(\mathbb{T}_N^d\setminus\Lambda, K-|\eta|_{\Lambda,1})}{g!(\eta)}\delta_\eta.$$

Note that the quantity $Z(\Lambda,K) = Z(\sharp\Lambda,K)$ depends only the cardinality $\sharp\Lambda$ of Λ and according to this notation we have that $Z(N^d,K) = Z(\mathbb{T}_N^d,K)$ for all $(N,K) \in \mathbb{N}\times\mathbb{Z}_+$. So since for all $\eta \in \mathbb{Z}_+^F$ we have that

$$|\sigma\eta|_{\Lambda,1} = \sum_{x\in\Lambda}\eta_{\sigma(x)} = \sum_{y\in F}\eta_y = |\eta|_{F,1}$$

and similarly

$$g!(\sigma\eta) = \prod_{x\in\Lambda} g!(\eta_{\sigma(x)}) = \prod_{y\in F} g!(\eta_y) = g!(\eta),$$

it follows that indeed $\sigma_*\nu^F_{N,K}(\eta) = 0 = \nu^\Lambda_{N,K}(\eta)$ if $\eta \in \mathbb{Z}^\Lambda_+$, $|\eta|_1 > K$ and

$$\begin{aligned}
\sigma_*\nu^F_{N,K}(\eta) &= \nu^F_{N,K}(\sigma^{-1}\eta) = \frac{1}{Z(N^d,K)} \frac{Z(N^d - \sharp F, K - |\sigma^{-1}\eta|_{F,1})}{g!(\sigma^{-1}\eta)} \\
&= \frac{1}{Z(N^d,K)} \frac{Z(N^d - \sharp\Lambda, K - |\eta|_{\Lambda,1})}{g!(\eta)} = \nu^\Lambda_{N,K}(\eta),
\end{aligned}$$

for all $\eta \in \mathbb{Z}^\Lambda_+$ such that $|\eta|_1 \le K$, which proves (1.63).

Now indeed (1.61) follows easily by equalities (1.62) and (1.63), since

$$\frac{d\nu^\Lambda_{N,K}}{d\nu^\Lambda_\rho}(\eta) = \frac{d\sigma_*\nu^F_{N,K}}{d\sigma_*\nu^F_\rho}(\eta) = \frac{\nu^F_{N,K}(\sigma^{-1}\eta)}{\nu^F_\rho(\sigma^{-1}\eta)} = \frac{d\nu^F_{N,K}}{d\nu^F_\rho}(\sigma^{-1}\eta)$$

for all $\eta \in \mathbb{Z}^\Lambda_+$ and therefore since $\sigma^* : \mathbb{Z}^F_+ \longrightarrow \mathbb{Z}^\Lambda_+$ is a bijection we have that

$$\begin{aligned}
\mathcal{H}\big(\nu^\Lambda_{N,K} \,\big|\, \nu^\Lambda_\rho\big) &= \sum_{\eta\in\mathbb{Z}^\Lambda_+} \nu^\Lambda_{N,K}(\eta) \log \frac{d\nu^\Lambda_{N,K}}{d\nu^\Lambda_\rho}(\eta) = \sum_{\eta\in\mathbb{Z}^\Lambda_+} \nu^F_{N,K}(\sigma^{-1}\eta) \log \frac{d\nu^F_{N,K}}{d\nu^F_\rho}(\sigma^{-1}\eta) \\
&= \sum_{\eta\in\mathbb{Z}^F_+} \nu^F_{N,K}(\eta) \log \frac{d\nu^\Lambda_{N,K}}{d\nu^F_\rho}(\eta) = \mathcal{H}\big(\nu^F_{N,K} \,\big|\, \nu^F_\rho\big).
\end{aligned}$$

Now, by the implication (1.61) just proved and the super-additivity of the relative entropy it easily follows that (1.60) holds. Indeed, given any subset $\Lambda \subseteq \mathbb{T}^d_N$, there exist $[N^d/\sharp\Lambda] \in \mathbb{N}$ in number disjoint subsets $\Lambda_i \subseteq \mathbb{T}^d_N$, $i = 1, \dots, [N^d/\sharp\Lambda]$ such that $\sharp\Lambda_i = \sharp\Lambda$ for all $i = 1, \dots, [N^d/\sharp\Lambda]$. Then if we set

$$\Lambda_\upsilon := \mathbb{T}^d_N \setminus \bigcup_{i=1}^{[N^d/\sharp\Lambda]} \Lambda_i,$$

the discrete torus $\mathbb{T}^d_N$ is the disjoint union of the sets Λ_i, $i = 1, \dots, [N^d/\sharp\Lambda]$ and Λ_υ and so by the super-additivity the relative entropy

$$\begin{aligned}
\mathcal{H}(\nu_{N,K}|\nu^N_\rho) &\ge \sum_{i=1}^{[N^d/\sharp\Lambda]} \mathcal{H}(\nu^{\Lambda_i}_{N,K}|\nu^{\Lambda_i}_\rho) + \mathcal{H}(\nu^{\Lambda_\upsilon}_{N,K}|\nu^{\Lambda_\upsilon}_\rho) \ge \sum_{i=1}^{[N^d/\sharp\Lambda]} \mathcal{H}(\nu^{\Lambda_i}_{N,K}|\nu^{\Lambda_i}_\rho) \\
&= \Big[\frac{N^d}{\sharp\Lambda}\Big] \mathcal{H}(\nu^\Lambda_{N,K}|\nu^\Lambda_\rho),
\end{aligned}$$

for all $(N,K) \in \mathbb{N} \times \mathbb{Z}_+$ and all $\rho \in I_c$. Therefore, by (1.59) it follows that

$$\mathcal{H}(\nu^\Lambda_{N,K}|\nu^\Lambda_\rho) \le \frac{1}{[N^d/\sharp\Lambda]} \mathcal{H}(\nu_{N,K}|\nu^N_\rho) = -\frac{1}{[N^d/\sharp\Lambda]} \log \nu^N_\rho(\mathbb{M}^d_{N,K}),$$

which proves (1.60) as required. In particular it follows that for all $\rho \ge 0$ and all $N \in \mathbb{N}$, we have that

$$\mathcal{H}(\nu^\ell_{N,[\rho N^d]}|\nu^\ell_{\rho\wedge\rho_c}) \le -\frac{1}{[N^d/\ell^d]} \log \nu^N_{\rho\wedge\rho_c}(\mathbb{M}^d_{N,[\rho N^d]}), \tag{1.64}$$

where as in the beginning of the proof we use the notation $\nu^{\ell}_{N,K} := \nu^{\mathbb{T}^d_\ell}_{N,K}$.

Now, by (1.64) it is obvious that to complete the proof it suffices to prove that

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log \nu^N_{\rho\wedge\rho_c}(\mathbb{M}^d_{N,[\rho N^d]}) \geq 0, \tag{1.65}$$

for then we have that

$$\begin{aligned}
\limsup_{N\to+\infty} \mathcal{H}(\nu^{\ell}_{N,[\rho N^d]}|\nu^{\ell}_{\rho\wedge\rho_c}) &= \ell^d \cdot \limsup_{N\to+\infty} \frac{1}{N^d}\left[\frac{N^d}{\ell^d}\right]\mathcal{H}(\nu^{\ell}_{N,[\rho N^d]}|\nu^{\ell}_{\rho\wedge\rho_c}) \\
&\leq \ell^d \cdot \limsup_{N\to+\infty}\left(-\frac{1}{N^d}\log \nu^N_{\rho\wedge\rho_c}(\mathbb{M}^d_{N,[\rho N^d]})\right) \\
&= -\ell^d \cdot \liminf_{N\to+\infty} \frac{1}{N^d} \log \nu^N_{\rho\wedge\rho_c}(\mathbb{M}^d_{N,[\rho N^d]}) \leq 0,
\end{aligned}$$

as required. In the rest of the proof we separate cases on whether the density $\rho \geq 0$ is below, equal to, or above the critical density ρ_c.

Case 1: Subcritical density $\rho < \rho_c$. Since $\rho < \rho_c$, the distribution ν^1_ρ has finite moments of all orders, and it particular it has finite variance $\sigma^2_\rho > 0$. So by the local central limit theorem in the lattice case we have that

$$\lim_{N\to+\infty} \sup_{x\in\mathbb{Z}_+} \left| \sigma_\rho N^{\frac{d}{2}} \nu_\rho\{|\cdot|_{N,1} = x\} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\rho N^d)^2}{2\sigma^2_\rho N^d}} \right| = 0$$

and therefore

$$\lim_{N\to+\infty} N^{\frac{d}{2}} \left| \nu^N_\rho\big(\mathbb{M}^d_{N,[\rho N^d]}\big) - \frac{1}{\sigma_\rho\sqrt{2\pi N^d}} e^{-\frac{([\rho N^d]-\rho N^d)^2}{2\sigma^2_\rho N^d}} \right| = 0. \tag{1.66}$$

Now, obviously $([\rho N^d] - \rho N^d)^2/2\sigma^2_\rho N^d \longrightarrow 0$ as $N \to +\infty$, and so there exists $N_1 \in \mathbb{N}$ such that $\exp\big[-([\rho N^d]-\rho N^d)^2/2\sigma^2_\rho N^d\big] > \frac{3}{4}$ for all $N \geq N_1$, and by (1.66) there exists $N_2 \in \mathbb{N}$ such that

$$N \geq N_2 \quad \Longrightarrow \quad N^{\frac{d}{2}} \left| \nu^N_\rho\big(\mathbb{M}^d_{N,[\rho N^d]}\big) - \frac{1}{\sigma_\rho\sqrt{2\pi N^d}} e^{-\frac{([\rho N^d]-\rho N^d)^2}{2\sigma^2_\rho N^d}} \right| < \frac{1}{4\sigma_\rho\sqrt{2\pi}}.$$

But then for all $N \geq N_0 := N_1 \vee N_2$ we have that

$$\nu^N_\rho\big(\mathbb{M}^d_{N,[\rho N^d]}\big) > \frac{1}{\sigma_\rho\sqrt{2\pi N^d}}\left(e^{-\frac{([\rho N^d]-\rho N^d)^2}{2\sigma^2_\rho N^d}} - \frac{1}{4}\right) > \frac{1}{2\sigma_\rho\sqrt{2\pi N^d}} \tag{1.67}$$

and therefore

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log \nu^N_\rho\big(\mathbb{M}^d_{N,[\rho N^d]}\big) \geq \liminf_{N\to+\infty} \frac{1}{N^d} \log \frac{1}{2\sigma_\rho\sqrt{2\pi N^d}} = 0,$$

which proves (1.65) and completes the proof for the case of subcritical densities $\rho < \rho_c$.

Case 2: Critical density $\rho = \rho_c$. In this case $\nu^1_{\rho_c}$ does not have exponential moments. If $\sigma^2_\rho := \mathbb{V}(\nu^1_\rho) < +\infty$ then the previous estimate by the local central limit theorem remains in force. Nonetheless, even when the variance σ_ρ^2 of $\nu_{\rho_c}^1$ is infinite, by the more general local limit theorem for stable limits we get a bound of the form

$$\nu_\rho^N\big(\mathbb{M}_{N,[\rho N^d]}^d\big) > \frac{c}{N^d}$$

for some constant $c > 0$ and all N large enough, completing the critical case.
Case 3: Supercritical density $\rho > \rho_c$. In this case, for all $\rho > \rho_c$ we have that

$$\begin{aligned}\nu_{\rho_c}^N\big(\mathbb{M}_{N,[\rho N^d]}^d\big) &\geq \nu_{\rho_c}^N\Big\{|\eta|_{N-1,1} = [\rho_c(N-1)^d], \sum_{|x|_\infty = N}\eta(x) = [\rho N^d] - [\rho_c(N-1)^d]\Big\} \\ &= \nu_{\rho_c}^{N-1}\big(\mathbb{M}_{N,[\rho_c(N-1)^d]}^d\big)(\nu_{\rho_c}^1)^{*N^d-(N-1)^d}\big([\rho N^d] - [\rho_c(N-1)^d]\big),\end{aligned}$$

where $(\nu_{\rho_c}^1)^{*N}$ is the N-fold convolution product of $\nu_{\rho_c}^1$. Therefore, for all $N \in \mathbb{N}$ we have that

$$\begin{aligned}\liminf_{N\to+\infty}\frac{1}{N^d}\log\nu_{\rho_c}^N\big(\mathbb{M}_{N,[\rho N^d]}^d\big) &\geq \liminf_{N\to+\infty}\frac{1}{N^d}\log\nu_{\rho_c}^{N-1}\big(\mathbb{M}_{N,[\rho_c(N-1)^d]}^d\big) \\ &\quad + \liminf_{N\to+\infty}\frac{1}{N^d}\log(\nu_{\rho_c}^1)^{*N_\star^d}\big([\rho N^d] - [\rho_c(N-1)^d]\big),\end{aligned}$$

where we have set $N_\star^d := N^d - (N-1)^d$ for all $N \in \mathbb{N}$. Now the first term is obviously equal to zero by the critical case and so it suffices to prove that

$$\liminf_{N\to+\infty}\frac{1}{N^d}\log(\nu_{\rho_c}^1)^{*N_\star^d}\big([\rho N^d] - [\rho_c(N-1)^d]\big) \geq 0 \tag{1.68}$$

We will prove (1.68) first for dimension $d = 1$. Of course, if $d = 1$ then $N_\star^1 = 1$ for all $N \in \mathbb{N}$ and so (1.68) becomes

$$\liminf_{N\to+\infty}\frac{1}{N}\log\nu_{\rho_c}^1\big([\rho N] - [\rho_c(N-1)]\big) \geq 0.$$

Since the critical one-site ZR distribution $\nu_{\rho_c}^1$ does not have exponentials moments, with proper domain $\mathcal{D}_{M_{\nu_{\rho_c}^1}} = (-\infty, 0]$ for its moment generating function, it has heavy right tails. Since $\nu_{\rho_c}^1$ is supported on the lattice $\mathbb{Z}_+$, by a basic characterization of heavy tailed distributions on lattices this is equivalent to requiring its density with respect to the counting measure on $\mathbb{Z}_+$ to be heavy tailed, i.e. that

$$\limsup_{N\to+\infty} e^{\theta N}\nu_{\rho_c}^1(N), \qquad \forall\,\theta > 0,$$

and in its turn this is equivalent to

$$\lim_{N\to+\infty}\frac{1}{N}\log\nu_{\rho_c}^1(N) = 0.$$

In the particular case of the ZR distribution this can be seen directly since as we recall by (1.26) the critical fugacity φ_c is $\varphi_c = \liminf_{k\to+\infty}\sqrt[k]{g!(k)} > 0$ and

$$\frac{1}{N}\log\nu_{\rho_c}^1(N) = \frac{1}{N}\log\frac{1}{Z(\varphi_c)} + \frac{1}{N}\log\frac{\varphi_c^N}{g!(N)} = \log\varphi_c - \frac{1}{N}\log g!(N)$$

Therefore the limit

$$\lim_{N\to+\infty} \frac{1}{N} \log g!(N) = \lim_{N\to+\infty} \log \sqrt[N]{g!(N)} = \log \varphi_c > 0$$

exists and

$$\lim_{N\to+\infty} \frac{1}{N} \log \nu^1_{\rho_c}(N) = \log \varphi_c - \lim_{N\to+\infty} \log \sqrt[N]{g!(N)} = 0.$$

Moreover, as we will show for any bounded sequence $\{k_N\}_{N\in\mathbb{N}} \subseteq \mathbb{Z}_+$ of integers, say $\sup_{N\in\mathbb{N}} k_N \le M \in \mathbb{Z}_+$, and any $r > 0$ we have that

$$\lim_{N\to+\infty} \frac{1}{N} \log \nu^1_{\rho_c}([rN] + k_N) = 0. \tag{1.69}$$

Indeed, in this case we have that

$$\begin{aligned}\frac{1}{N} \log \nu^1_{\rho_c}([rN] + k_N) &= \frac{1}{N} \log \frac{1}{Z(\varphi_c)} + \frac{1}{N} \log \frac{\varphi_c^{[rN]+k_N}}{g!([rN] + k_N)} \\ &= \frac{1}{N} \log \frac{1}{Z(\varphi_c)} + \frac{[rN] + k_N}{N} \log \varphi_c - \log \sqrt[N]{g!([rN] + k_N)}\end{aligned}$$

and therefore, since g has a bounded discrete derivative $g'(k) := g(k+1) - g(k)$, $k \in \mathbb{Z}_+$, as a local rate function by (1.15), if $\varphi_c \ge 1$ we have that

$$\begin{aligned}\underline{\lim}_{N\uparrow\infty} \frac{1}{N} \log \nu^1_{\rho_c}([rN] + k_N) &= \log \varphi_c^r - \overline{\lim}_{N\to+\infty} \log \left(\sqrt[[rN]]{g!([rN] + k_N)} \right)^{\frac{[rN]}{N}} \\ &= \log \varphi_c^r \left[1 - \overline{\lim}_{N\uparrow\infty} \log \sqrt[N]{g([rN]+1) \cdot \ldots \cdot g([rN] + k_N)} \right] \\ &\ge \log \varphi_c^r \left[1 - \overline{\lim}_{N\uparrow\infty} \log \sqrt[N]{\|g'\|_\infty^{k_N} \prod_{i=1}^{k_N} ([rN] + i)} \right] \\ &= \log \varphi_c^r \left[1 - \overline{\lim}_{N\uparrow\infty} \log \sqrt[N]{\prod_{i=1}^{k_N} ([rN] + i)} \right] = 0,\end{aligned}$$

which implies that $\lim_{N\uparrow\infty} \frac{1}{N} \log \nu^1_{\rho_c}([rN] + k_N) = 0$ as required. Furthermore, as we shall see, by a simple rescaling we can always assume that $\varphi_c \ge 1$ which will complete the proof of (1.69). Indeed, for any local rate function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ and any $\lambda > 0$ the function λg is a local rate function with critical fugacity

$$\varphi_{\lambda g} = \liminf_{k\to+\infty} \sqrt[k]{(\lambda g)!(k)} = \liminf_{k\to+\infty} \sqrt[k]{\lambda^k g!(k)} = \lambda \varphi_g$$

and obviously their partition functions are related by $Z_{\lambda g}(\cdot) = Z_g(\cdot/\lambda)$ and therefore if we choose $\lambda = \frac{1}{\varphi_g}$ then $\nu^1_{\rho_c} = \bar{\nu}^1_{\varphi_g, g} = \bar{\nu}^1_{\varphi_{\lambda g}, \lambda g}$ can be considered as a ZR distribution corresponding to the critical fugacity $\varphi_{\lambda g} = 1$. In turn this proves (1.68) for the case of dimension $d = 1$, since for any $\rho > \rho_c$ we have that

$$[\rho N] - [\rho_c(N - 1)] = [(\rho - \rho_c)N] + k_N, \quad \forall\, N \in \mathbb{N},$$

for some bounded sequence $\{k_N\}_{N\in\mathbb{N}} \subseteq \mathbb{Z}_+$.

We prove next that (1.68) for general $d \geq 1$. Note that loosely speaking $N_\star^d$ is of order N^{d-1} for large $N \in \mathbb{N}$ while whenever $b > a > 0$ the term $bN^d - a(N-1)^d$ is of order N^d. For all $N, d \in \mathbb{N}$ and $\rho > 0$ we set

$$A^d_{N,\rho} := \left\{ (k_1, \ldots, k_{N_\star^d}) \in \mathbb{Z}_+^{N_\star^d} \;\Big|\; \sum_{i=1}^{N_\star^d} k_i = [\rho N^d] - [\rho_c (N-1)^d] \right\}$$

and then

$$(\nu^1_{\rho_c})^{*N_\star^d}\big([\rho N^d] - [\rho_c(N-1)^d]\big) = \sum_{(k_1,\ldots,k_{N_\star^d}) \in A^d_{N,\rho}} \nu^1_{\rho_c}(k_1) \cdot \ldots \cdot \nu^1_{\rho_c}(k_{N_\star^d}).$$

For fixed $\rho > \rho_c$ we set

$$m_{N,d} := \left[\frac{[\rho N^d] - [\rho_c(N-1)^d]}{N_\star^d} \right]$$

for all $N, d \in \mathbb{N}$. Then $[\rho N^d] - [\rho_c(N-1)^d] = m_{N,d} N_\star^d + \upsilon_{N,d}$ for some remain $0 \leq \upsilon_{N,d} < N_\star^d$ for all $N \in \mathbb{N}$ and

$$(\nu^1_{\rho_c})^{*N_\star^d}\big([\rho N^d] - [\rho_c(N-1)^d]\big) \geq \nu^1_{\rho_c}(m_{N,d})^{N_\star^d - 1} \nu^1_{\rho_c}(m_{N,d} + \upsilon_{N,d})$$

for all $N \in \mathbb{N}$. Therefore

$$\frac{1}{N^d} \log(\alpha_{\rho_c})^{*N_\star^d}\big([\rho N^d] - [\rho_c(N-1)^d]\big) \geq \frac{N_\star^d - 1}{N^d} \log \nu^1_{\rho_c}(m_{N,d}) + \frac{1}{N^d} \log \nu^1_{\rho_c}(m_{N,d} + \upsilon_{N,d}).$$

It is easy to see that

$$\sup_{N\in\mathbb{N}} \big| [(\rho - \rho_c)N] - m_{N,d} \big| < +\infty$$

and so since $\frac{N_\star^d - 1}{N^d}$ behaves like $\frac{1}{N}$ as $N \to +\infty$, i.e. $N \frac{N_\star^d - 1}{N^d} \longrightarrow 1$ as $N \to +\infty$, we have by (1.69) that

$$\liminf_{N\to+\infty} \frac{N_\star^d - 1}{N^d} \log \nu^1_{\rho_c}(m_{N,d}) = \liminf_{N\to+\infty} \frac{1}{N} \nu^1_{\rho_c}([(\rho - \rho_c)N] + k_N) = 0$$

with $k_N := m_{N,d} - [(\rho - \rho_c)N]$, $N \in \mathbb{N}$. It follows that

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log(\alpha_{\rho_c})^{*N_\star^d}\big([\rho N^d] - [\rho_c(N-1)^d]\big) \geq \liminf_{N\to+\infty} \frac{1}{N^d} \log \nu^1_{\rho_c}(m_{N,d} + \upsilon_{N,d})$$

and therefore it suffices to prove that

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log \nu^1_{\rho_c}(m_{N,d} + \upsilon_{N,d}) \geq 0.$$

But this follows by the following obvious generalization of (1.69): For any $d \in \mathbb{N}$, $r > 0$ and any sequence $\{k_N\}_{N\in\mathbb{N}} \subseteq \mathbb{Z}$ such that for some $M \in \mathbb{N}$,

$$-M \leq k_N \leq M + N^{d-1}, \quad \forall\, N \in \mathbb{N}$$

we have that

$$\lim_{N\to+\infty}\frac{1}{N^d}\log\nu^1_{\rho_c}([rN]+k_N)=0.$$

This completes the proof of the supercritical case, and the proof is complete. $\square$

The equivalence of ensembles gives us some useful limits. For instance, the equivalence of ensembles is equivalent to the existence for all $\Lambda\subseteq\mathbb{Z}^d$, $\rho\geq 0$ and $\eta\in\mathbb{Z}_+^\Lambda$, of the limit

$$\frac{Z(\mathbb{T}^d_N\setminus\Lambda,[\rho N^d]-|\eta|_1)}{Z(N^d,[\rho N^d])g!(\eta)}=\nu^\Lambda_{N,[\rho N^d]}(\eta)\overset{N\to+\infty}{\longrightarrow}\nu^\Lambda_{\rho\wedge\rho_c}(\eta)=\frac{\Phi(\rho\wedge\rho_c)^{|\eta|_1}}{Z\big(\Phi(\rho\wedge\rho_c)\big)^{\sharp\Lambda}g!(\eta)}.$$

Obviously this is equivalent to the existence for all $\ell,k\in\mathbb{Z}_+$ and all $\rho\geq 0$ of the limit

$$\lim_{N\to+\infty}\frac{Z(N^d-\ell,[\rho N^d]-k)}{Z(N^d,[\rho N^d])}=\frac{\Phi(\rho\wedge\rho_c)^k}{Z\big(\Phi(\rho\wedge\rho_c)\big)^\ell}. \tag{1.70}$$

Proposition 1.4.2 *Let $\{Z(N^d,K)\}_{(N,K)\in\mathbb{N}\times\mathbb{Z}_+}$ be the partition function of the canonical ensemble of the ZR process on the discrete toruses $\mathbb{T}^d_N$, $N\in\mathbb{N}$. Then*

$$\lim_{N\to+\infty}\frac{1}{N^d}\log Z(N^d,[\rho N^d])=\log Z\big(\Phi(\rho\wedge\rho_c)\big)-\rho\log\Phi(\rho\wedge\rho_c)$$

for all $\rho\geq 0$.

Proof By (1.58) we have that

$$\begin{aligned}\frac{1}{N^d}\log Z(N^d,[\rho N^d]) &= \frac{1}{N^d}\log\nu^N_{\rho\wedge\rho_c}(\mathbb{M}^d_{N,[\rho N^d]})\\ &\quad+\log Z\big(\Phi(\rho\wedge\rho_c)\big)-\frac{[\rho N^d]}{N^d}\log\Phi(\rho\wedge\rho_c)\end{aligned}$$

and the proof follows since in the course of the proof of the equivalence of ensembles we have proved the limit

$$\lim_{N\to+\infty}\frac{1}{N^d}\log\nu^N_{\rho\wedge\rho_c}(\mathbb{M}^d_{N,[\rho N^d]})=0.$$

$\square$

In appendix 1, corollary 1.7 in [25] a different version of the equivalence of ensembles is proved under the additional assumption that $Z(\varphi_c)=+\infty$: For each $\rho_0<+\infty$, for all cylinder functions (i.e. functions that depend on a finite number of coordinates) with finite second moment with respect to the measures ν^∞_ρ, $\rho\in[0,\rho_0]$, it holds that

$$\int f d\nu_{N,K}\longrightarrow\int f d\nu^\infty_\rho \qquad \text{as } N,K\to\infty \text{ and } K/N^d\to\rho$$

uniformly over all $\rho\in[0,\rho_0]$, where $\nu^\infty_\rho:=(\nu^1_\rho)^{\otimes\mathbb{Z}^d}\in\mathbb{PM}^d_\infty:=\mathbb{P}\mathbb{Z}_+^{\mathbb{Z}^d}$. An elegant extension of this result has been recently given in [18], where it is shown that for subcritical densities $\rho\leq\rho_c$ theorem 1.4.1 can be applied to yield weak convergence in duality with respect to functions $f \in L^p(\nu_\rho^\infty)$ for all $p > 1$. Of course this cannot be true for $\rho > \rho_c$ if $\rho_c < +\infty$ since even for the linear cylinder function $\eta(0)$

$$\int \eta(0) d\nu_{N,K} \longrightarrow \rho > \rho_c \qquad \text{as } N, K \to \infty \text{ and } K/N^d \to \rho.$$

In other words, at the thermodynamic limit we have a mean total loss of mass equal to $\rho - \rho_c$ at each site. As it has been proved, in many cases the excess mass of all the sites is concentrated on a single random site. We refer to [19, 3, 4] for a detailed description of this phase separation in the context of the Evans model. Evans' model was defined in [15]. Some particular cases of the Evans model are described briefly in the end of the next section.

If the local jump rate g is bounded, then the equivalence of ensembles yields that

$$\lim_{\substack{N,K\to\infty \\ K/N^d\to\infty}} \int g(\eta(0)) d\nu_{N,K} = \int g(\eta(0)) d\nu^\infty_{\rho\wedge\rho_c} = \Phi(\rho \wedge \rho_c),$$

for all $\rho \geq 0$. As noted in [19], this shows that for bounded local jump rate functions g the mean jump rate function Φ should be extended on all of $\mathbb{R}_+$ by

$$\Phi(\rho) \equiv \Phi(\rho \wedge \rho_c), \qquad \text{for all } \rho \geq 0. \tag{1.71}$$

It turns out that this choice of Φ is the right one in order to extend the one-block estimate to ZRPs with finite critical density. We will **always** consider Φ to be extended in this way for densities $\rho \geq \rho_c$.

1.5 Simple Examples of Zero Range Processes

In this section we study some simple examples of ZR processes.

System of i.i.d. Continuous Time Random Walks

The simplest Zero Range Process on $\mathbb{M}_{N,K}^d$ is the particle system comprised of K particles executing i.i.d. continuous time random walks in $\mathbb{T}_N^d$ with mean 1 exponential jump times according the modulo N projection $\mathbb{Z}^d \longrightarrow \mathbb{T}_N^d$ of an elementary step distribution $p \in \mathbb{P}\mathbb{Z}^d$. It's simplicity stems from the independence of the random walks describing the motion of the particles since due to the independence there is no interaction between the particles. It corresponds to the ZR process with local rate function $g = id_{\mathbb{Z}_+} : \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ and elementary step distribution p. In this case, of course $Z \equiv Z_g = \exp$, $\varphi_c = +\infty$, and thus also $\rho_c = +\infty$.

More generally, any local rate function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ having superlinear growth, i.e. any local rate function g for which there exists $a_0 > 0$ such that $g(k) \geq a_0 k$ gives rise to a grand canonical partition function Z with critical fugacity $\varphi_c = +\infty$ since in this case we have that

$$Z(\varphi) = \sum_{k=0}^{\infty} \frac{\varphi^k}{g!(k)} \leq \sum_{k=0}^{\infty} \frac{\varphi^k}{a_0^k k!} = e^{\frac{\varphi}{a_0}} < +\infty$$

for all $\varphi \geq 0$.

Returning to the case $g = id_{\mathbb{Z}_+}$ we obviously have that $R(\varphi) = \varphi$, and so for all $\varphi \geq 0$ the one site ZR distribution $\bar{\nu}_{g,\varphi}^1 \in \mathbb{P}\mathbb{Z}_+$ is parametrized by the density of the particles. In particular the one site ZR distribution $\nu_{g,\rho}^1 \in \mathbb{P}\mathbb{Z}_+$, $\rho \geq 0$, is the Poisson distribution with parameter ρ,

$$\nu_{g,\rho}^1 = e^{-\rho} \sum_{k=0}^{\infty} \frac{\rho^k}{k!} \delta_k.$$

The ZR range distributions on the torus $\mathbb{T}_N^d$ is the Poisson product distribution on the torus, that is

$$\nu_\rho^N = \bigotimes_{x \in \mathbb{T}_N^d} \nu_\rho^1 = e^{-\rho N^d} \sum_{\eta \in \mathbb{M}_N^d} \frac{\rho^{|\eta|_1}}{\eta!} \delta_\eta \in \mathbb{P}\mathbb{M}_N^d,$$

where

$$\eta! := \prod_{x \in \mathbb{T}_N^d} \eta_x! = \prod_{x \in \mathbb{T}_N^d} g!(\eta_x) = g!(\eta).$$

This describes the grand canonical ensemble of the system of i.i.d. random walks. The canonical ensemble is given in this case by

$$\nu_{N,K}^d(\eta) = \frac{1/\eta!}{\sum_{\eta \in \mathbb{M}_{N,K}^d} 1/\eta!} \mathbb{1}_{\mathbb{M}_{N,K}^d}(\eta) = N^{-Kd} \frac{K!}{\eta!} \mathbb{1}_{\mathbb{M}_{N,K}^d}(\eta),$$

since the canonical partition function $Z : (2^{\mathbb{Z}^d} \setminus \{\emptyset\}) \times \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ is given by the formula

$$Z(\Lambda, K) = \sum_{\eta \in \mathbb{Z}_+^\Lambda : |\eta|_1 = K} \frac{1}{\eta!} = \frac{1}{K!} \sum_{\eta \in \mathbb{Z}_+^\Lambda : |\eta|_1 = K} \frac{K!}{\eta!} = \frac{(\sharp\Lambda)^K}{K!}.$$

In this case, by (1.70) the equivalence of ensembles amounts to the existence of the limit

$$\lim_{N\to+\infty}\frac{Z(N^d-\ell,[\rho N^d]-k)}{Z(N^d,[\rho N^d])}=\lim_{N\to+\infty}\frac{[\rho N^d]!(N^d-\ell)^{[\rho N^d]-k}}{([\rho N^d]-k)!N^{d[\rho N^d]}}=e^{-\ell\rho}\rho^k$$

for all $\ell,k\in\mathbb{Z}_+$ and $\rho\geq 0$, which in this case can also be easily checked directly.

Systems of Queues

If $g=\mathbb{1}_{\mathbb{N}}$ then a ZRP on $\mathbb{T}_N^d$ with parameters (p,g) models a system of N queues with exponential service times in which whenever a customer is served in his queue he goes to another queue according to the transition probability $p_N\in\mathbb{P}\mathbb{T}_N^d$, where of course p_N is the modulo N projection of the elementary step distribution $p\in\mathbb{P}\mathbb{Z}^d$ given in (1.1).

In this case we obviously have that $g!(k)=1$ for all $k\in\mathbb{Z}_+$ and therefore the grand canonical partition function is given by

$$Z(\varphi)=\sum_{k=0}^{\infty}\varphi^k=\frac{1}{1-\varphi}.$$

In this case we obviously have that $\varphi_c=1$ with $Z(\varphi_c)=+\infty$ and thus $\rho_c=+\infty$ by proposition 1.2.6. The density function $R:[0,1)\longrightarrow\mathbb{R}_+$ is given by

$$R(\varphi)=\frac{\varphi Z'(\varphi)}{Z(\varphi)}=\frac{\varphi}{1-\varphi}$$

and the fugacity function $\Phi:=R^{-1}:\mathbb{R}_+\longrightarrow[0,1)$ is given by

$$\Phi(\rho)=\frac{\rho}{\rho+1}.$$

Therefore in this case the one-site ZR distribution is defined for all $\rho\geq 0$ by the formula

$$\nu_\rho^1=\frac{1}{\rho+1}\sum_{k=0}^{\infty}\Big(\frac{\rho}{\rho+1}\Big)^k\delta_k,$$

a geometric distribution with success probability $\frac{1}{\rho+1}$. The ZR distributions on the toruses $\mathbb{T}_N^d$ are given by

$$\nu_\rho^N=\frac{1}{(\rho+1)^{N^d}}\sum_{\eta\in\mathbb{M}_N^d}\Big(\frac{\rho}{\rho+1}\Big)^{|\eta|_1}\delta_\eta,$$

for all $N\in\mathbb{N}$. This describes the grand canonical ensemble.

Since $g!\equiv 1$ we have that $g!(\eta)=1$ for $\eta\in\mathbb{M}_N^d$ and therefore the canonical ensemble $\{\nu_{N,K}^d\}_{N,K\in\mathbb{N}\times\mathbb{Z}_+}$ consists of uniform distributions and the canonical partition function $Z:(2^{\mathbb{Z}^d}\setminus\{\emptyset\})\times\mathbb{Z}_+\longrightarrow\mathbb{R}_+$ is given by

$$Z(\Lambda,K)=\binom{\sharp\Lambda+K-1}{K}.$$

In this case by (1.70) the equivalence of ensembles amounts to the existence of the limit

$$\lim_{N\to+\infty} \frac{\binom{N^d-\ell+[\rho N^d]-k-1}{[\rho N^d]-k}}{\binom{N^d+[\rho N^d]-1}{[\rho N^d]}} = \frac{1}{(\rho+1)^\ell}\Big(\frac{\rho}{\rho+1}\Big)^k$$

for all $\ell, k \in \mathbb{Z}_+$ and all $\rho \geq 0$, which again can easily be checked directly.

ZR Processes with Finite Critical Density

In this subsection we will describe a monoparametric family $\eta_\beta^N = \{\eta_{\beta,t}^N\}_{t\geq 0}$, $\beta \in \mathbb{R}$, of ZR processes on the discrete torus $\mathbb{T}_N^d$ for which the critical density ρ_c is finite for appropriate values of the parameter β.

To begin with, we define for each $\beta \in \mathbb{R}$ the local rate function $g_\beta : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ by the formula

$$g_\beta(k) = \begin{cases} k & \text{if } k = 0, 1, \\ \big(\frac{k}{k-1}\big)^\beta & \text{if } k \geq 2. \end{cases}$$

Then for all $\beta \in \mathbb{R}$ and $k \geq 1$ we have that

$$g_\beta!(k) = g_\beta(1)\cdot\dots\cdot g_\beta(k) = 1\cdot 2^\beta\cdot\Big(\frac{3}{2}\Big)^\beta\cdot\dots\cdot\Big(\frac{k}{k-1}\Big)^\beta = k^\beta.$$

Therefore the partition function $Z_\beta := Z_{g_\beta}$ is given by the formula

$$Z_\beta(\varphi) = \sum_{k=0}^\infty \frac{\varphi^k}{g_\beta!(k)} = 1 + \sum_{k=1}^\infty \frac{\varphi^k}{k^\beta}$$

and obviously the critical fugacity $\varphi_\beta := \varphi_c(\beta) := \varphi_{g_\beta} = 1$ for all $\beta \in \mathbb{R}$. Note that for $\beta = 0$, $g_\beta = \mathbb{1}_\mathbb{N}$ is the local rate function studied in the previous example. As we know from proposition 1.2.6 the critical density $\rho_\beta := \rho_c(\beta) := \sup_{\varphi<1} R_\beta(\varphi)$, where $R_\beta : [0,1) \longrightarrow \mathbb{R}_+$ is the density function associated to g_β, is $< +\infty$ iff $\varphi_c = 1 \in \mathcal{D}_{Z'_\beta}$ and obviously

$$Z'_\beta(\varphi) = \sum_{k=1}^\infty \frac{\varphi^{k-1}}{k^{\beta-1}} = \frac{1}{\varphi}\sum_{k=1}^\infty \frac{\varphi^k}{k^{\beta-1}} = \frac{1}{\varphi}\big(Z_{\beta-1}(\varphi)-1\big).$$

Therefore $1 \in \mathcal{D}_{Z'_\beta}$ iff $1 \in \mathcal{D}_{Z_{\beta-1}}$ and since the series $\sum_{k=1}^\infty \frac{1}{k^\beta}$ diverges for $\beta \leq 1$ and converges for $\beta > 1$ we get by Abel's theorem on power series that $\rho_\beta < +\infty$ iff $\beta - 1 > 1$, that is

$$\rho_\beta < +\infty \iff \beta > 2.$$

The density function $R_\beta : [0,1) \longrightarrow \mathbb{R}_+$ is given by

$$R_\beta(\varphi) = \frac{\varphi Z'_\beta(\varphi)}{Z_\beta(\varphi)} = \frac{Z_{\beta-1}(\varphi)-1}{Z_\beta(\varphi)}$$

and the critical density is given as a function of the parameter $\beta > 2$ by the formula

$$\rho_c(\beta) = \rho_\beta = \frac{Z_{\beta-1}(1) - 1}{Z_\beta(1)} = \frac{\zeta(\beta-1)}{\zeta(\beta)+1}, \quad \beta > 2, \tag{1.72}$$

where $\zeta(\beta) := \sum_{k=1}^{\infty} \frac{1}{k^\beta}$, $\beta > 1$, is the zeta function.

Proposition 1.5.1 *The critical density function $\rho_c : (2, +\infty) \longrightarrow \mathbb{R}_+$ defined in (1.72) is strictly decreasing and*

$$\lim_{\beta \downarrow 2} \rho_c(\beta) = +\infty, \qquad \lim_{\beta \uparrow +\infty} \rho_c(\beta) = \frac{1}{2}.$$

Proof By standard theorems for interchanging the order of differentiation and integration we get that the derivative of the zeta function is given in $(1, +\infty)$ by the formula

$$\zeta'(s) = \left(\sum_{k=1}^{\infty} \frac{1}{k^s}\right)' \stackrel{(*)}{=} \sum_{k=1}^{\infty} \left(\frac{1}{k^s}\right)' = -\sum_{k=1}^{\infty} \frac{\log k}{k^s} < 0. \tag{1.73}$$

Indeed, let $\varepsilon > 0$ and set $h : \mathbb{N} \times (1+\varepsilon, \infty) :\longrightarrow \mathbb{R}_+$ the function given by $h(k,s) = \frac{1}{k^s}$. Then

$$\left|\frac{d}{ds} h(k,s)\right| = \frac{\log k}{k^s} \le \frac{\log k}{k^{1+\varepsilon}}, \quad \forall\, s \ge 1+\varepsilon,\ k \in \mathbb{N}$$

and a sufficient condition for the validity of the interchange of differentiation and integration in equality $(*)$ is the integrability of $k \mapsto \frac{\log k}{k^{1+\varepsilon}}$ with respect to counting measure i.e. that

$$\sum_{k=1}^{\infty} \frac{\log k}{k^{1+\varepsilon}} < +\infty.$$

But this is easily seen, since

$$\sum_{k=1}^{\infty} \frac{\log k}{k^{1+\varepsilon}} = \sum_{k=1}^{\infty} \frac{2}{\varepsilon k^{1+\frac{\varepsilon}{2}}} \frac{\log k^{\frac{\varepsilon}{2}}}{k^{\frac{\varepsilon}{2}}} \le \sum_{k=1}^{\infty} \frac{2}{\varepsilon k^{1+\frac{\varepsilon}{2}}} = \frac{2}{\varepsilon} \zeta\left(1 + \frac{\varepsilon}{2}\right) < +\infty.$$

This proves (1.73) for all $s > 1+\varepsilon$ and since $\varepsilon > 0$ was arbitrary it follows that (1.73) holds for all $s > 1$.

It follows that ρ_c is differentiable in $(2, +\infty)$ with

$$\rho_c'(\beta) = \frac{\zeta'(\beta-1)\big(\zeta(\beta)+1\big) - \zeta(\beta-1)\zeta'(\beta)}{\big(\zeta(\beta)+1\big)^2}$$

and therefore $\rho_c' < 0$ on $(2, +\infty)$ iff

$$\zeta'(\beta-1)\zeta(\beta) - \zeta(\beta-1)\zeta'(\beta) < -\zeta'(\beta-1)$$

for all $\beta > 2$. Since $-\zeta' < 0$ in $(1, +\infty)$ it suffices to prove that

$$\zeta'(\beta-1)\zeta(\beta) \le \zeta(\beta-1)\zeta'(\beta),$$

or equivalently that

$$\frac{\zeta'(\beta-1)}{\zeta(\beta-1)} \le \frac{\zeta'(\beta)}{\zeta(\beta)} \tag{1.74}$$

for all $\beta > 2$.

But inequality (1.74) will follow if the function

$$(1,\infty) \ni s \mapsto \frac{\zeta'(s)}{\zeta(s)}$$

is increasing, or equivalently if ζ is log-convex. But this is easily seen to be true since for all $s, t > 1$ and $p \in (0,1)$ we have that

$$\zeta\big(ps + (1-p)t\big) = \sum_{k=1}^{\infty} \frac{1}{k^{ps} k^{(1-p)t}}$$

and therefore by applying Holder's inequality with conjugate exponents $q := \frac{1}{p}$, $q^* = \frac{1}{1-p}$ we get that

$$\zeta\big(ps + (1-p)t\big) \leq \left(\sum_{k=1}^{\infty} \frac{1}{k^s}\right)^p \left(\sum_{k=1}^{\infty} \frac{1}{k^t}\right)^{1-p} = \zeta(s)^p \zeta(t)^{1-p}$$

for all $s, t > 1$ and $p \in (0,1)$, which proves the logarithmic convexity of ζ. Finally the required limits are obvious since $\lim_{s\downarrow 1} \zeta(s) = +\infty$ and $\lim_{s\uparrow +\infty} \zeta(s) = 1$. □

Another example with finite critical density, the Evans Model In [15] Evans introduces ZRPs with local jump rate function

$$g_b(k) = \mathbb{1}_{\{k\geq 1\}}\Big(1 + \frac{b}{k}\Big), \quad b \geq 0. \tag{1.75}$$

It is well known ([19]) that $\varphi_c = 1$ for all $b \geq 0$, $\varphi_c \notin \mathcal{D}_Z$ iff $b \in [0,1]$ and that for $b > 2$, the first moment of the grand canonical distribution $\nu^1_{\varphi_c}$ is finite, thus leading to a finite critical density $\rho_c < \infty$. A precursor of the Evans model was already studied in [13]. We refer to [15, 19] for a detailed description of the Evans model.

1.6 Large Deviations of the Empirical Embeddings

Let $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ be a local rate function and let $\{\nu_\rho^N\}_{\rho \in I_c} \subseteq \mathbb{P}\mathbb{M}_N^d$ be the family of the ZR distributions associated to the rate function g, where $I_c := [0, \rho_c] \cap \mathbb{R}_+$ is the interval of admissible densities. As usual, for each $N \in \mathbb{N}$ we denote by $\pi^N : \mathbb{M}_N^d \longrightarrow \mathcal{M}^+(\mathbb{T}^d)$ the empirical embeddings

$$\pi^N(\eta) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_x \delta_{\frac{x}{N}},$$

where $\mathcal{M}^+(\mathbb{T}^d)$ is the set of finite non-negative Borel measures on the torus $\mathbb{T}^d$. We fix $\rho_* \in (0, \rho_c)$ and set

$$\mu_N := \pi_*^N \nu_{\rho_*}^N \in \mathbb{P}\mathcal{M}^+(\mathbb{T}^d) \tag{1.76}$$

for all $N \in \mathbb{N}$.

Definition 1.6.1 Let X be a polish space and let $\{a_n\}_{n \in \mathbb{N}} \subseteq (0, +\infty)$ be a sequence such that $a_n \longrightarrow 0$. A sequence $\{\mu_n\} \subseteq \mathbb{P}X$ *satisfies the large deviations principle* (LDP for short) *with speed* $\{a_n\}$ if there exists a lower semi-continuous convex functional $I : X \longrightarrow \overline{\mathbb{R}}_+$ with compact sub-levels, called *the rate functional of the LDP* such that

$$\limsup_{n \to +\infty} a_n \log \mu_n(F) \leq -\inf_{x \in F} I(x)$$

for any closed $F \subseteq X$

$$\liminf_{n \to +\infty} a_n \log \mu_n(U) \geq -\inf_{x \in U} I(x)$$

for any open $U \subseteq X$.

A standard reference on the theory of large deviations is [12]. Our goal in this section is to obtain the large deviations principle for the sequence of measures $\{\mu_N\}_{N \in \mathbb{N}}$ with speed $\frac{1}{N^d}$.

Proposition 1.6.1 *The sequence* $\{\mu_N\}$ *is exponentially tight, that is for all* $c < +\infty$ *there exists a precompact set* $K_c \subseteq \mathcal{M}^+(\mathbb{T}^d)$ *such that*

$$\limsup_{N \to +\infty} \frac{1}{N^d} \log \mu_N\big(\mathcal{M}^+(\mathbb{T}^d) \setminus K_c\big) \leq -c. \tag{1.77}$$

Proof As we know, a set $K \subseteq \mathcal{M}^+(\mathbb{T}^d)$ is precompact iff

$$\sup_{\mu \in K} \langle \mu, 1 \rangle < +\infty.$$

By proposition 1.2.12 we know that the family $\{\eta(x)\}_{x \in \mathbb{Z}^d}$ satisfies the LDP on the probability space $(\mathbb{M}_\infty^d, \nu_{\rho_*})$ and in particular the sequence

$$q_N := \left[\frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x)\right]_* \nu_{\rho_*}, \quad N \in \mathbb{N}$$

is exponentially tight. Therefore, given $c < +\infty$ there exists $A_c > 0$ such that

$$\limsup_{N\to+\infty} \frac{1}{N^d} \log \nu_{\rho_*}^N \Big\{ \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \eta(x) > A_c \Big\} \le -c. \tag{1.78}$$

By the characterization of the precompact sets of $\mathcal{M}^+(\mathbb{T}^d)$ the set

$$K_c := \big\{ \mu \in \mathcal{M}^+(\mathbb{T}^d) \,\big|\, \langle \mu, 1\rangle \le A_c \big\}$$

is precompact in $\mathcal{M}^+(\mathbb{T}^d)$ and

$$\begin{aligned} \mu_N\big\{\mathcal{M}^+(\mathbb{T}^d) \setminus K_c\big\} &= \nu_{\rho_*}^N\{\pi^N \notin K_c\} = \nu_{\rho_*}^N\{\langle \pi^N, 1\rangle > A_c\} \\ &= \nu_{\rho_*}^N \Big\{ \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \eta(x) > A_c \Big\}. \end{aligned}$$

Therefore (1.77) follows from (1.78). □

We prove next the large deviations upper bound. We recall first the following general upper bound. Let $\mathcal{X}$ be a topological vector space with topological dual $\mathcal{X}^*$. We will denote by $\langle f, x\rangle = f(x)$, $(f,x) \in \mathcal{X}^* \times \mathcal{X}$ the duality between $\mathcal{X}$ and $\mathcal{X}^*$. Then for any sequence $\{a_N\}$ of positive numbers such that $a_N \longrightarrow 0$ any sequence of distributions $\{\mu_N\}_{N=1}^\infty \subseteq \mathbb{P}\mathcal{X}$ satisfies the weak large deviations upper bound with speeds a_N and rate function $I : \mathcal{X} \longrightarrow [0, +\infty]$ given by

$$I(x) = \sup_{f\in\mathcal{X}^*} \big\{ \langle f, x\rangle - \bar{\Lambda}(f) \big\}$$

where $\bar{\Lambda} : \mathcal{X}^* \longrightarrow [-\infty, +\infty]$ is given by

$$\bar{\Lambda}(f) := \limsup_{N\to+\infty} a_N \Lambda_{\mu_N}(f/a_N)$$

and $\Lambda_{\mu_N} : \mathcal{X} \longrightarrow (-\infty, \infty]$ is the logarithmic m.g.f. of μ_N given by

$$\Lambda_{\mu_N}(f) = \log \int e^{\langle f, x\rangle} d\mu_N(x).$$

Proposition 1.6.2 *The sequence $\{\mu_N\} \subseteq \mathbb{P}\mathcal{M}^+(\mathbb{T}^d)$ given in* (1.76) *satisfies the large deviations upper bound with speeds $a_N = \frac{1}{N^d}$ and rate function $I_{\rho_*} : \mathcal{M}^+(\mathbb{T}^d) \longrightarrow [0, \infty]$ given by*

$$I_{\rho_*}(\pi) = \sup_{f\in C(\mathbb{T}^d)} \Big\{ \int_{\mathbb{T}^d} f(u) d\pi(u) - \int_{\mathbb{T}^d} \Lambda_{\nu_{\rho_*}^1}\big(f(u)\big) \Big\}, \tag{1.79}$$

where $\Lambda_{\nu_{\rho_*}^1}$ is the logarithmic m.g.f. of the one site ZR distribution with density $\rho_* < \rho_c$.

Proof The space $\mathcal{M}(\mathbb{T}^d)$ of all finite signed measures is a topological vector space when equipped with the weak topology with topological dual the space of continuous functions on $\mathbb{T}^d$, i.e. $\mathcal{M}(\mathbb{T}^d)^* = C(\mathbb{T}^d)$ and in this case $\bar{\Lambda} : C(\mathbb{T}^d) \longrightarrow [-\infty, +\infty]$ is given by the formula

$$\bar{\Lambda}(f) = \limsup_{N\to+\infty} \frac{1}{N^d} \Lambda_{\mu_N}(N^d f).$$

Therefore since $\mathcal{M}^+(\mathbb{T}^d)$ is a closed subset of $\mathcal{M}(\mathbb{T}^d)$ we have by the exponential tightness of $\{\mu_N\}$ and the general weak large deviations upper bound that $\{\mu_N\}$ satisfies the large deviations upper bound with rate function

$$I_{\rho_*}(\pi) = \sup_{f\in C(\mathbb{T}^d)} \Big\{ \int_{\mathbb{T}^d} f(u)d\pi(u) - \bar{\Lambda}(f) \Big\}.$$

But for all $N \in \mathbb{N}$ we have that

$$\begin{aligned} \Lambda_{\mu_N}(N^d f) &= \log \int e^{N^d\langle f,\pi\rangle} d\mu_N(\pi) = \log \int e^{N^d\langle f,\pi^N\rangle} d\nu^N_{\rho_*} \\ &= \log \int e^{\sum_{x\in\mathbb{T}^d_N} \eta(x) f(\frac{x}{N})} d\nu^N_{\rho_*} = \log \int \prod_{x\in\mathbb{T}^d_N} e^{\eta(x) f(\frac{x}{N})} d\nu^N_{\rho_*} \\ &= \sum_{x\in\mathbb{T}^d_N} \log \int e^{kf(x/N)} d\nu^1_{\rho_*}(k) = \sum_{x\in\mathbb{T}^d_N} \Lambda_{\nu^1_{\rho_*}}\Big(f(\frac{x}{N})\Big) \end{aligned}$$

and so we obviously have that

$$\frac{1}{N^d}\Lambda_{\mu_N}(N^d f) = \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} \Lambda_{\nu^1_{\rho_*}}\Big(f(\frac{x}{N})\Big) = \int_{\mathbb{T}^d} \sum_{x\in\mathbb{T}^d_N} \Lambda_{\nu^1_{\rho_*}}\Big(f(\frac{x}{N})\Big) \mathbb{1}_{[\frac{x}{N},\frac{x+\bar{1}}{N})}(u)du$$

for all $f \in C(\mathbb{T}^d)$, where $\bar{1} := (1,\dots,1) \in \mathbb{R}^d$. As we know from proposition 1.2.12, the logarithmic m.g.f. $\Lambda_{\nu^1_{\rho_*}}$ is given by the formula

$$\Lambda_{\nu^1_{\rho_*}}(\theta) = \log Z\big(e^\theta \Phi(\rho_*)\big) - \log Z\big(\Phi(\rho_*)\big)$$

and has proper domain $\mathcal{D}_{\Lambda_{\nu^1_{\rho_*}}}$ such that

$$(-\infty, b_*) \subseteq \mathcal{D}_{\Lambda_{\nu^1_{\rho_*}}} \subseteq (-\infty, b_*],$$

where $b_* = \log\varphi_c - \log\Phi(\rho_*)$. In what follows we separate three cases on whether $b_* = +\infty$ or $b_* < +\infty$ and $b_* \in \mathcal{D}_{\Lambda_{\nu^1_{\rho_*}}}$ or $b_* < +\infty$ and $b_* \notin \mathcal{D}_{\Lambda_{\nu^1_{\rho_*}}}$ and we will show that for all $f \in C(\mathbb{T}^d)$,

$$\lim_{N\to+\infty} \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \Lambda_{\nu^1_{\rho_*}}\Big(f(\frac{x}{N})\Big) = \int_{\mathbb{T}^d} \Lambda_{\nu^1_{\rho_*}}\big(f(u)\big)du. \tag{1.80}$$

$b_* = +\infty$: Let $f \in C(\mathbb{T}^d)$. In this case $\Lambda_{\nu^1_{\rho_*}}$ is a smooth function on $\mathbb{R}$ and by the continuity of f we have that

$$\sum_{x\in\mathbb{T}^d_N} \Lambda_{\nu^1_{\rho_*}}\Big(f(\frac{x}{N})\Big) \mathbb{1}_{[\frac{x}{N},\frac{x+\bar{1}}{N})} \overset{N\to+\infty}{\longrightarrow} \Lambda_{\nu^1_{\rho_*}}(f) \tag{1.81}$$

pointwise on $\mathbb{T}^d$. Since $\mathbb{T}^d$ is compact, f is bounded, and so

$$\sup_{N\in\mathbb{N}} \Big| \sum_{x\in\mathbb{T}^d_N} \Lambda_{\nu^1_{\rho_*}}\Big(f(\frac{x}{N})\Big) \mathbb{1}_{[\frac{x}{N},\frac{x+\bar{1}}{N})} \Big| \leq \sup_{-\|f\|_u\leq\theta\leq\|f\|_u} \Lambda_{\nu^1_{\rho_*}}(\theta) < +\infty.$$

Therefore (1.80) follows from the bounded convergence theorem.
$b_* < +\infty$, $b_* \in \mathcal{D}_{\Lambda_{\nu^1_{\rho_*}}}$: Let $f \in C(\mathbb{T}^d)$ and suppose that $f(u) > b_*$ for some $u \in \mathbb{T}^d$. Then since f is continuous there exists an open neighborhood V_u of u such that $f(\upsilon) > b_*$ for all $\upsilon \in V_u$ and since $\Lambda_{\nu^1_{\rho_*}}$ is bounded from below by $-\log Z\big(\Phi(\rho_*)\big)$ and $\Lambda_{\nu^1_{\rho_*}} \equiv +\infty$ on (b_*, ∞) we have that

$$\int_{\mathbb{T}^d} \Lambda_{\nu^1_{\rho_*}}\big(f(u)\big)du \geq -\log Z\big(\Phi(\rho_*)\big)m_{\mathbb{T}^d}(\mathbb{T}^d \setminus V_u) + \int_{V_u} \Lambda_{\nu^1_{\rho_*}}(f)dm_{\mathbb{T}^d} = +\infty.$$

On the other hand, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ there exists $y_N \in \mathbb{T}^d_N$ such that $y_N/N \in V_u$ and therefore for all $N \geq N_0$ we have that

$$\frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} \Lambda_{\nu^1_{\rho_*}}\Big(f\big(\frac{x}{N}\big)\Big) = \frac{1}{N^d}\Lambda_{\nu^1_{\rho_*}}\Big(f\big(\frac{y_N}{N}\big)\Big) + \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N\setminus\{y_N\}} \Lambda_{\nu^1_{\rho_*}}\Big(f\big(\frac{x}{N}\big)\Big) = +\infty.$$

Therefore (1.80) holds in the case that $f(u) > b_*$ for some $u \in \mathbb{T}^d$. We consider next the case where $f(u) \leq b_*$ for all $u \in \mathbb{T}^d$. In this case by the continuity of $\Lambda_{\nu^1_{\rho_*}}$ on $(-\infty, b_*]$ and the continuity of f we have again that (1.81) holds pointwise on $\mathbb{T}^d$ and

$$\sup_{N\in\mathbb{N}}\bigg|\sum_{x\in\mathbb{T}^d_N} \Lambda_{\nu^1_{\rho_*}}\Big(f\big(\frac{x}{N}\big)\Big)\mathbb{1}_{[\frac{x}{N},\frac{x+\bar{1}}{N})}\bigg| \leq \sup_{-\|f\|_u \leq \theta \leq b_*} \Lambda_{\nu^1_{\rho_*}}(\theta) < +\infty,$$

and therefore (1.80) holds by the bounded convergence theorem.
$b_* < +\infty$, $b_* \notin \mathcal{D}_{\Lambda_{\nu^1_{\rho_*}}}$: As in the previous case, if $f(u) > b_*$ for some $u \in \mathbb{T}^d$ we have that

$$\lim_{N\to+\infty}\frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} \Lambda_{\nu^1_{\rho_*}}\Big(f\big(\frac{x}{N}\big)\Big) = +\infty = \int_{\mathbb{T}^d} \Lambda_{\nu^1_{\rho_*}}\big(f(u)\big)du.$$

In particular we have that

$$I_{\rho_*}(\pi) = \sup_{f\in C(\mathbb{T}^d),\ f\leq b_*}\bigg\{\int_{\mathbb{T}^d} f(u)d\pi(u) - \bar{\Lambda}(f)\bigg\}.$$

So let $f \in C(\mathbb{T}^d)$ such that $f(u) \leq b_*$ for all $u \in \mathbb{T}^d$. We set

$$\psi_N := \sum_{x\in\mathbb{T}^d_N} \Lambda_{\nu^1_{\rho_*}}\Big(f\big(\frac{x}{N}\big)\Big)\mathbb{1}_{[\frac{x}{N},\frac{x+\bar{1}}{N})}$$

and by the continuity of $\Lambda_{\rho_*} : (-\infty, b_*] \longrightarrow \big(-\infty, +\infty\big]$ and the continuity of f we have that $\psi_N \longrightarrow \Lambda_{\nu^1_{\rho_*}}(f)$ pointwise on $\mathbb{T}^d$. Therefore by Fatou's lemma we have that

$$\int \Lambda_{\nu^1_{\rho_*}}\big(f(u)\big)du \leq \liminf_{N\to+\infty}\int_{\mathbb{T}^d} \psi_N(u)du \leq \bar{\Lambda}(f)$$

and so we have that

$$I_{\rho_*}(\pi) \leq \sup_{f\in C(\mathbb{T}^d),\ f\leq b_*}\bigg\{\int_{\mathbb{T}^d} f(u)d\pi(u) - \int \Lambda_{\nu^1_{\rho_*}}\big(f(u)\big)du\bigg\}.$$

It remains to prove the converse inequality. First we note that for all $f \in C(\mathbb{T}^d)$ such that $f(u) < b_*$ for all $u \in \mathbb{T}^d$ we have by the compactness of $\mathbb{T}^d$ that $\sup_{u\in\mathbb{T}^d} f(u) < b_*$

and thus in this case $\bar{\Lambda}(f) = \int \Lambda_{\nu^1_{\rho_*}}(f)dm_{\mathbb{T}^d}$ by the bounded convergence theorem. Therefore for all $\varepsilon > 0$ we have that the supremum

$$\begin{aligned} I^\varepsilon_{\rho_*}(\pi) &:= \sup_{f\in C(\mathbb{T}^d),\, f\le b_*-\varepsilon} \left\{ \int_{\mathbb{T}^d} f(u)d\pi(u) - \int \Lambda_{\nu^1_{\rho_*}}\big(f(u)\big)du \right\} \\ &= \sup_{f\in C(\mathbb{T}^d),\, f\le b_*-\varepsilon} \left\{ \int_{\mathbb{T}^d} f(u)d\pi(u) - \bar{\Lambda}(f) \right\} \le I_{\rho_*}(\pi) \end{aligned}$$

and so it suffices to prove that

$$\lim_{\varepsilon\downarrow 0} I^\varepsilon_{\rho_*}(\pi) = \sup_{f\in C(\mathbb{T}^d),\, f\le b_*} \left\{ \int_{\mathbb{T}^d} f(u)d\pi(u) - \int \Lambda_{\nu^1_{\rho_*}}\big(f(u)\big)du \right\}.$$

Indeed, the limit $\lim_{\varepsilon\downarrow 0} I^\varepsilon_{\rho_*}(\pi)$ exists since $I^\varepsilon_{\rho*}(\pi)$ is increasing as ε decreases to 0 and obviously $\lim_{\varepsilon\downarrow 0} I^\varepsilon_{\rho_*}(\pi) \le I_{\rho_*}(\pi)$. On the other hand, given $\delta > 0$ let $f \in C(\mathbb{T}^d)$ such that $f(u) \le b_*$ for all $u \in \mathbb{T}^d$ and such that

$$\begin{aligned} &\int_{\mathbb{T}^d} f(u)d\pi(u) - \int \Lambda_{\nu^1_{\rho_*}}\big(f(u)\big)du \\ &\qquad \ge \sup_{f\in C(\mathbb{T}^d),\, f\le b_*} \left\{ \int_{\mathbb{T}^d} f(u)d\pi(u) - \int_{\mathbb{T}^d} \Lambda_{\nu^1_{\rho_*}}\big(f(u)\big)du \right\} - \delta. \end{aligned}$$

Then

$$I^\varepsilon_{\rho_*}(\pi) \ge \int_{\mathbb{T}^d} [f\wedge(b_*-\varepsilon)](u)d\pi(u) - \int_{\mathbb{T}^d} \Lambda_{\nu^1_{\rho_*}}\big([f\wedge(b_*-\varepsilon)](u)\big)du$$

and obviously

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{T}^d} [f\wedge(b_*-\varepsilon)](u)d\pi(u) = \int_{\mathbb{T}^d} f(u)d\pi(u)$$

by the bounded convergence theorem. Next, since $\Lambda_{\nu^1_{\rho_*}}$ is increasing the family

$$\{\Lambda_{\nu^1_{\rho_*}}\big([f\wedge(b_*-\varepsilon)]\big)\}_{\varepsilon>0}$$

is increasing as $\varepsilon \downarrow 0$ and thus since $\Lambda_{\nu^1_{\rho_*}}$ is bounded below we have by the monotone convergence theorem that

$$\lim_{\varepsilon\downarrow 0} \int \Lambda_{\nu^1_{\rho_*}}\big([f\wedge(b_*-\varepsilon)](u)\big)du = \int_{\mathbb{T}^d} \Lambda_{\nu^1_{\rho_*}}\big(f(u)\big)du.$$

Therefore we have that

$$\begin{aligned} \lim_{\varepsilon\downarrow 0} I^\varepsilon_{\rho_*}(\pi) &\ge \int_{\mathbb{T}^d} f(u)d\pi(u) - \int \Lambda_{\nu^1_{\rho_*}}\big(f(u)\big)du \\ &\ge \sup_{f\in C(\mathbb{T}^d),\, f\le b_*} \left\{ \int_{\mathbb{T}^d} f(u)d\pi(u) - \int_{\mathbb{T}^d} \Lambda_{\nu^1_{\rho_*}}\big(f(u)\big)du \right\} - \delta, \end{aligned}$$

which since $\delta > 0$ was arbitrary proves the claim. □

We prove next a large deviations lower bound.

Proposition 1.6.3 *Let $\{\nu^N_{\rho_*,g}\}_{N\in\mathbb{N}}$ be the family of ZR distributions with density $\rho_* > 0$ where g is a local rate function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ such that the* $\liminf$ *defining the critical fugacity $\varphi_c \in (0, +\infty]$ exists as a limit, i.e. such that*

$$0 < \varphi_c := \liminf_{k\to+\infty} \sqrt[k]{g!(k)} = \lim_{k\to+\infty} \sqrt[k]{g!(k)}.$$

Then the sequence $\{\mu_N\} \subseteq \mathbb{P}\mathcal{M}^+(\mathbb{T}^d)$ given in (1.76) satisfies the large deviations lower bound with speeds $a_N = \frac{1}{N^d}$ and rate function $I_{\rho_*} : \mathcal{M}^+(\mathbb{T}^d) \longrightarrow [0, \infty]$ given by

$$I_{\rho_*}(\pi) = \int_{\mathbb{T}^d} \Lambda^*_{\nu^1_{\rho_*}}\Big(\frac{\pi_{ac}}{dm_{\mathbb{T}^d}}(u)\Big)du + \pi_s(\mathbb{T}^d) \log \frac{\varphi_c}{\Phi(\rho_*)},$$

where $\Lambda^*_{\nu^1_{\rho_*}}$ is the Fenchel-Legendre transform of the logarithmic m.g.f. of $\nu^1_{\rho_*}$, $\rho_* < \rho_c$ and $\pi = \pi_{ac} + \pi_s$, $\pi_{ac} \ll m_{\mathbb{T}^d}$, $\pi_s \perp m_{\mathbb{T}^d}$ is the Radon-Nikodym decomposition of π with respect to $m_{\mathbb{T}^d}$.

Proof To prove the LDP lower bound it suffices to prove that for any $\pi \in \mathcal{M}^+(\mathbb{T}^d)$ we have that

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log \mu_N(U_\pi) \geq -I_{\rho_*}(\pi), \quad \forall\, U_\pi \in \mathcal{U}(\pi), \tag{1.82}$$

where $\mathcal{U}(\pi)$ is the set of all open neighborhoods of $\pi \in \mathcal{M}^+(\mathbb{T}^d)$ with respect to the weak topology. We consider first the case of absolutely continuous measures with respect to Lebesgue measure on the torus, that is we will prove (1.82) first for measures $\pi \in \mathcal{M}^+_{ac}(\mathbb{T}^d) := \{\pi \in \mathcal{M}^+(\mathbb{T}^d) \,|\, \pi \ll m_{\mathbb{T}^d}\}$. By a slight abuse of notation we will continue to denote by π the density of π with respect to the Lebesgue measure $m_{\mathbb{T}^d}$ on the torus. We will show first that we can make the additional assumption that the density $\pi \in L^1(\mathbb{T}^d)$ is strictly positive. Indeed, suppose that we have proved (1.82) for all measures $\pi = \pi dm_{\mathbb{T}^d} \in \mathcal{M}^+_{ac}(\mathbb{T}^d)$ with strictly positive density. Then given any $\pi = \pi dm_{\mathbb{T}^d} \in \mathcal{M}^+(\mathbb{T}^d)$ and an open neighborhood U_π of π we define the family $\{\pi_\varepsilon\}_{\varepsilon>0} \subseteq \mathcal{M}^+_{ac}(\mathbb{T}^d)$ given by $\pi_\varepsilon := [\pi \vee \varepsilon] dm_{\mathbb{T}^d}$. Then given $G \in C(\mathbb{T}^d)$ for all $\varepsilon \in (0, 1)$ we have that

$$|G \cdot (\pi \vee \varepsilon)| \leq \|G\|_u(\pi \vee \varepsilon) \leq \|G\|_u(\pi + 1) \in L^1(\mathbb{T}^d)$$

and $G \cdot (\pi \vee \varepsilon) \longrightarrow G \cdot \pi$ as $\varepsilon \to 0$. Therefore

$$\int G d\pi_\varepsilon = \int G[\pi \vee \varepsilon] dm_{\mathbb{T}^d} \longrightarrow \int G\pi dm_{\mathbb{T}^d} = \int G d\pi$$

by the dominated convergence theorem which gives as that $\pi_\varepsilon \longrightarrow \pi$ in the weak topology of $\mathcal{M}^+(\mathbb{T}^d)$. Then since U_π is open in the weak topology there exists $\varepsilon_0 \in (0, 1)$ such that $\pi_\varepsilon \in U_\pi$ for all $\varepsilon < \varepsilon_0$ and therefore $U_\pi \in \mathcal{U}(\pi_\varepsilon)$ for all $\varepsilon < \varepsilon_0$ and so by (1.82) we get that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log \mu_N(U_\pi) \geq -I(\pi_\varepsilon) \tag{1.83}$$

But $\Lambda^*_{\nu^1_{\rho_*}}$ is decreasing in $[0, \rho_*]$ and so the family $\{\Lambda^*_{\nu^1_{\rho_*}}(\pi \vee \varepsilon)\}_{\varepsilon\in(0,\varepsilon_0\wedge\rho_*)}$ is increasing as $\varepsilon > 0$ is decreasing to 0. Since $\Lambda^*_{\nu^1_{\rho_*}} \geq 0$ everywhere this gives us by the monotone convergence theorem that

$$I(\pi_\varepsilon) = \int \Lambda^*_{\nu^1_{\rho_*}}(\pi \vee \varepsilon) dm_{\mathbb{T}^d} \longrightarrow \int \Lambda^*_{\nu^1_{\rho_*}}(\pi) dm_{\mathbb{T}^d} = I(\pi)$$

and therefore taking the limit as $\varepsilon \downarrow 0$ in (1.83) we get that $\pi \in \mathcal{M}^+_{ac}(\mathbb{T}^d)$ satisfies (1.82) as required.

So in what follows we consider measures $\pi \in \mathcal{M}^+_{ac}(\mathbb{T}^d)$ with strictly positive density. We set $\bar\pi := \pi \wedge \rho_c$ and consider the measures $\nu^N_{\bar\pi(\cdot)} \in \mathbb{PM}^d_N$ with slowly varying parameter associated to the profile $\bar\pi : \mathbb{T}^d \longrightarrow \mathbb{R}_+$, i.e.

$$\nu^N_{\bar\pi(\cdot)} := \bigotimes_{x\in\mathbb{T}^d_N} \nu^1_{\bar\pi(\frac{x}{N})}.$$

Then for all $N \in \mathbb{N}$, $\eta \in \mathbb{M}^d_N$ we have that

$$\frac{d\nu^N_{\rho_*}}{d\nu^N_{\bar\pi(\cdot)}}(\eta) = \prod_{x\in\mathbb{T}^d_N} \frac{Z\big(\Phi(\bar\pi_{x/N})\big)\Phi(\rho_*)^{\eta_x}}{Z\big(\Phi(\rho_*)\big)\Phi\big(\bar\pi_{x/N}\big)^{\eta_x}}.$$

Therefore we have that

$$\begin{aligned}
\mu_N(U_\pi) &= \int_{\{\pi^N\in U_\pi\}} d\nu^N_{\rho_*} = \int_{\{\pi^N\in U_\pi\}} \frac{d\nu^N_{\rho_*}}{d\nu^N_{\bar\pi(\cdot)}} d\nu^N_{\bar\pi(\cdot)} \\
&= Z\big(\Phi(\rho_*)\big)^{-N^d} \prod_{x\in\mathbb{T}^d_N} Z\big(\Phi(\bar\pi_{x/N})\big) \int_{\{\pi^N\in U_\pi\}} \prod_{x\in\mathbb{T}^d_N} \left[\frac{\Phi(\rho_*)}{\Phi(\bar\pi_{x/N})}\right]^{\eta(x)} d\nu^N_{\bar\pi(\cdot)}
\end{aligned}$$

and so

$$\begin{aligned}
\frac{1}{N^d}\log\mu_N(U_\pi) &= -\log Z\big(\Phi(\rho_*)\big) + \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} \log Z\big(\Phi(\bar\pi_{x/N})\big) \\
&\quad + \frac{1}{N^d}\log\int_{\{\pi^N\in U_\pi\}} e^{-\sum_{x\in\mathbb{T}^d_N}\eta(x)\log\frac{\Phi(\bar\pi_{x/N})}{\Phi(\rho_*)}} d\nu^N_{\bar\pi(\cdot)} \\
&= -\log Z\big(\Phi(\rho_*)\big) + \int \sum_{x\in\mathbb{T}^d_N} \log Z\big(\Phi(\bar\pi_{x/N})\big) \mathbb{1}_{[\frac{x}{N},\frac{x+\bar 1}{N})} dm_{\mathbb{T}^d} \\
&\quad + \frac{1}{N^d}\log\int_{\{\pi^N\in U_\pi\}} e^{-N^d\int\log\frac{\Phi(\bar\pi)}{\Phi(\rho_*)}d\pi^N} d\nu^N_{\bar\pi(\cdot)}.
\end{aligned}$$

We make now the additional assumption that the density π is continuous. Then

$$\lim_{N\to+\infty}\sum_{x\in\mathbb{T}^d_N} \log Z\big(\Phi(\bar\pi_{x/N})\big)\mathbb{1}_{[\frac{x}{N},\frac{x+\bar 1}{N})} = \log Z\big(\Phi(\bar\pi)\big)$$

and therefore by Fatou's lemma we get that

$$\begin{aligned}
\liminf_{N\to+\infty}\frac{1}{N^d}\log\mu_N(U_\pi) &\geq -\log Z\big(\Phi(\rho_*)\big) + \int_{\mathbb{T}^d}\log Z\big(\Phi(\pi\wedge\rho_c)\big)dm_{\mathbb{T}^d} \\
&\qquad \liminf_{N\to+\infty}\frac{1}{N^d}\log\int_{\{\pi^N\in U_\pi\}} e^{-N^d\int\log\frac{\Phi(\bar\pi)}{\Phi(\rho_*)}d\pi^N} d\nu^N_{\bar\pi(\cdot)}.
\end{aligned}$$

Furthermore, the function $\log\frac{\Phi(\bar\pi)}{\Phi(\rho_*)}$ is continuous and therefore for any $\varepsilon > 0$ the set

$$A_\varepsilon := \left\{\mu \in \mathcal{M}^+(\mathbb{T}^d) \,\middle|\, \left|\int \log\frac{\Phi(\bar\pi)}{\Phi(\rho_*)} d(\mu-\pi)\right| < \varepsilon\right\}$$

is open in $\mathcal{M}^+(\mathbb{T}^d)$ and

$$\begin{aligned}\int_{\{\pi^N\in U_\pi\}} e^{-N^d\int \log\frac{\Phi(\bar\pi)}{\Phi(\rho_*)}d\pi^N} d\nu^N_{\bar\pi(\cdot)} &\geq \int_{\{\pi^N\in U_\pi\cap A_\varepsilon\}} e^{-N^d\int \log\frac{\Phi(\bar\pi)}{\Phi(\rho_*)}d\pi^N} d\nu^N_{\bar\pi(\cdot)}\\ &\geq e^{-N^d\varepsilon} e^{-N^d\int \log\frac{\Phi(\bar\pi)}{\Phi(\rho_*)}d\pi} \int_{\{\pi^N\in U_\pi\cap A_\varepsilon\}} d\nu^N_{\bar\pi(\cdot)}.\end{aligned}$$

It follows that

$$\begin{aligned}&\liminf_{N\to+\infty}\frac{1}{N^d}\log\int_{\{\pi^N\in U_\pi\}} e^{-N^d\int \log\frac{\Phi(\bar\pi)}{\Phi(\rho_*)}d\pi^N} d\nu^N_{\bar\pi(\cdot)}\\ &\qquad\geq -\varepsilon-\int\log\frac{\Phi(\pi\wedge\rho_c)}{\Phi(\rho_*)}d\pi+\liminf_{N\to+\infty}\frac{1}{N^d}\log\int_{\{\pi^N\in U_\pi\cap A_\varepsilon\}} d\nu^N_{\bar\pi(\cdot)}.\end{aligned}$$

and therefore if we show that

$$\liminf_{N\to+\infty}\frac{1}{N^d}\log\int_{\{\pi^N\in U_\pi\cap A_\varepsilon\}} d\nu^N_{\bar\pi(\cdot)}\geq 0 \tag{1.84}$$

for any $\varepsilon>0$, then we get that

$$\begin{aligned}\liminf_{N\to+\infty}\frac{1}{N^d}\log\mu_N(U_\pi) &\geq -\log Z\big(\Phi(\rho_*)\big)+\int_{\mathbb{T}^d}\log Z\big(\Phi(\pi\wedge\rho_c)\big)dm_{\mathbb{T}^d}\\ &\qquad -\int\log\frac{\Phi(\pi\wedge\rho_c)}{\Phi(\rho_*)}d\pi-\varepsilon\\ &= -\int\left[\pi\log\frac{\Phi(\pi\wedge\rho_c)}{\Phi(\rho_*)}-\log\frac{Z\big(\Phi(\pi\wedge\rho_c)\big)}{Z\big(\Phi(\rho_*)\big)}\right]dm_{\mathbb{T}^d}-\varepsilon\\ &= -\int_{\mathbb{T}^d}\Lambda^*_{\nu^1_{\rho_*}}\big(\pi(u)\big)du-\varepsilon=-I_{\rho_*}(\pi)-\varepsilon,\end{aligned}$$

which since $\varepsilon>0$ is arbitrary, proves the claim.

Now, we set $\bar\nu^N_\rho:=\nu_{\rho\wedge\rho_c}$ for all $\rho\geq 0$. Since the profile $\pi:\mathbb{T}^d\longrightarrow\mathbb{R}_+$ is continuous, the sequence $\{\nu^N_{\bar\pi(\cdot)}\}_{N\in\mathbb{N}}=\{\bar\nu^N_{\pi(\cdot)}\}_{N\in\mathbb{N}}$ of the measures with slowly varying parameter with respect to the profile $\pi\in C(\mathbb{T}^d)$ and the family $\{\bar\nu^N_\rho\}_{\rho\geq 0}$ is associated to the profile π, that is

$$\lim_{N\to+\infty}\nu_{\bar\pi(\cdot)}\left\{\left|\int fd\pi^N-\int f\pi dm_{\mathbb{T}^d}\right|\geq\varepsilon\right\}=0$$

for all $f\in C(\mathbb{T}^d)$ and all $\varepsilon>0$. On the other hand, since U_π is an open neighborhood of $\pi\in\mathcal{M}^+(\mathbb{T}^d)$ there exists continuous functions $f_1,\dots,f_k\in C(\mathbb{T}^d)$ and $\varepsilon_0>0$ such that

$$\bigcap_{i=1}^k\left\{\mu\in\mathcal{M}^+(\mathbb{T}^d)\,\middle|\,\left|\int f_i d(\mu-\pi)\right|<\varepsilon\right\}\subseteq U_\pi$$

for all $\varepsilon\in(0,\varepsilon_0)$. But then with $f_{k+1}:=\log\frac{\Phi(\pi\wedge\rho_c)}{\Phi(\rho_*)}$ we have that

$$U_\pi\cap A_\varepsilon\supseteq\bigcap_{i=1}^{k+1}\left\{\mu\in\mathcal{M}^+(\mathbb{T}^d)\,\middle|\,\left|\int f_i d(\mu-\pi)\right|<\varepsilon\right\}$$

for all $\varepsilon \in (0, \varepsilon_0)$, and therefore

$$\begin{aligned}\int_{\{\pi^N \in U_\pi \cap A_\varepsilon\}} d\nu^N_{\bar{\pi}(\cdot)} &\geq \nu^N_{\bar{\pi}(\cdot)}\Big(\bigcap_{i=1}^{k+1}\Big\{\Big|\int f_i d(\pi^N - \pi)\Big| < \varepsilon\Big\}\Big) \\ &\geq 1 - \sum_{i=1}^{k+1} \nu_{\bar{\pi}(\cdot)}\Big\{\Big|\int f_i d\pi^N - \int f_i \pi dm_{\mathbb{T}^d}\Big| \geq \varepsilon\Big\}.\end{aligned}$$

Therefore we have that

$$\liminf_{N\to+\infty} \int_{\{\pi^N \in U_\pi \cap A_\varepsilon\}} d\nu^N_{\bar{\pi}(\cdot)} \geq 1$$

which gives us (1.84), as required.

We will show now that we can remove our additional assumption that the density π is continuous. We do this in two steps. First we extend the lower bound to all measures $\pi \in \mathcal{M}^+_{ac}(\mathbb{T}^d)$ with bounded density with respect to the Lebesgue measure and then we remove the assumption of boundedness. For the first step we claim that it suffices to prove that for each bounded function $\pi \in B(\mathbb{T}^d)$ there exists a sequence $\{\pi_m\}_{m\in\mathbb{N}} \subseteq C(\mathbb{T}^d)$ such that $\pi_m dm_{\mathbb{T}^d} \longrightarrow \pi dm_{\mathbb{T}^d}$ in the weak topology of $\mathcal{M}^+(\mathbb{T}^d)$ and

$$\lim_{m\to+\infty} I_{\rho_*}(\pi_m dm_{\mathbb{T}^d}) = I_{\rho_*}(\pi) \tag{1.85}$$

Indeed, then given a measure $\pi \in \mathcal{M}^+_{ac}(\mathbb{T}^d)$ with bounded density $d\pi = \pi dm_{\mathbb{T}^d}$ there exists a sequence $\{\pi_m\}_{m\in\mathbb{N}} \subseteq C(\mathbb{T}^d)$ such that $\pi_m dm_{\mathbb{T}^d} \longrightarrow \pi$ in the weak topology of $\mathcal{M}^+(\mathbb{T}^d)$ and satisfying (1.85). Then, given an open neighborhood $U_\pi \subseteq \mathcal{M}^+(\mathbb{T}^d)$ of π there exists m_0 such that $\pi_m dm_{\mathbb{T}^d} \in U_\pi$ for all $m \geq m_0$. Then U_π is an open neighborhood of $\pi_m dm_{\mathbb{T}^d}$ for each $m \geq m_0$ and by the lower LDP bound for measures with continuous densities it follows that

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log \mu_N(U_\pi) \geq -I_{\rho_*}(\pi_m dm_{\mathbb{T}^d})$$

for all $m \geq m_0$, and by taking the limit as $m \to +\infty$ it follows by (1.85) that

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log \mu_N(U_\pi) \geq -I_{\rho_*}(\pi)$$

as required. But indeed, by Lusin's theorem, for each bounded function $\pi \in B(\mathbb{T}^d)$ there exists a sequence $\{\pi_m\}_{m\in\mathbb{N}} \subseteq C(\mathbb{T}^d)$ such that $\pi_m \longrightarrow \pi$ $m_{\mathbb{T}^d}$-a.s and $\sup_{m\in\mathbb{N}} \|\pi_m\|_u \leq \|\pi\|_u$. Then, for every function $G \in C(\mathbb{T}^d)$ we have that

$$\Big|\int G\pi_m dm_{\mathbb{T}^d} - \int G\pi dm_{\mathbb{T}^d}\Big| \leq \|G\|_u \int |\pi_m - \pi| dm_{\mathbb{T}^d} \overset{m\to+\infty}{\longrightarrow} 0$$

by the bounded convergence theorem which implies that $\pi_m dm_{\mathbb{T}^d} \longrightarrow \pi dm_{\mathbb{T}^d}$ weakly in $\mathcal{M}^+(\mathbb{T}^d)$ and since $\Lambda_{\nu^1_{\rho_*}}$ is continuous and $\sup_{m\in\mathbb{N}} \|\pi_m\|_u \leq \|\pi\|_u < +\infty$ we get by the bounded convergence theorem again that

$$I_{\rho_*}(\pi_m dm_{\mathbb{T}^d}) = \int_{\mathbb{T}^d} \Lambda^*_{\nu^1_{\rho_*}}\big(\pi_m(u)\big)du \longrightarrow \int_{\mathbb{T}^d} \Lambda^*_{\nu^1_{\rho_*}}\big(\pi(u)\big)du = I_{\rho_*}(\pi dm_{\mathbb{T}^d}),$$

that is (1.85) as required.

We remove finally the assumption of boundedness. Obviously, in order to do this it suffices to prove that for any measure $\pi = \pi dm_{\mathbb{T}^d} \in \mathcal{M}^+_{ac}(\mathbb{T}^d)$ there exists a sequence of bounded functions $\{\pi_M\}_{M\in\mathbb{N}} \subseteq B(\mathbb{T}^d)$ such that $\pi_M dm_{\mathbb{T}^d} \longrightarrow \pi$ in the weak topology of $\mathcal{M}^+(\mathbb{T}^d)$ and $I(\pi_M dm_{\mathbb{T}^d}) \longrightarrow I(\pi)$. Given $\pi = \pi dm_{\mathbb{T}^d} \in \mathcal{M}^+_{ac}(\mathbb{T}^d)$ such a sequence is given by $\pi_M := \pi \wedge M$, $M \in \mathbb{N}$. Indeed, thus defined, $\{\pi_M\}_{M\in\mathbb{N}}$ is increasing and for any $G \in C(\mathbb{T}^d)$ we have by the monotone convergence theorem that

$$\begin{aligned}\int G\pi_M dm_{\mathbb{T}^d} &= -\|G\|_u \int \pi_M dm_{\mathbb{T}^d} + \int (G + \|G\|_u)\pi_M dm_{\mathbb{T}^d} \\ &\overset{M\to+\infty}{\longrightarrow} -\|G\|_u \int \pi dm_{\mathbb{T}^d} + \int (G + \|G\|_u)\pi dm_{\mathbb{T}^d} = \int_{\mathbb{T}^d} G d\pi,\end{aligned}$$

and so $\pi_M dm_{\mathbb{T}^d}$ converges weakly to $\pi \in \mathcal{M}^+(\mathbb{T}^d)$. On the other hand, as we know $\Lambda^*_{\nu^1_{\rho_*}}$ is non negative and increasing on the interval $[\rho_*, \infty)$, which implies that the sequence $\{\Lambda^*_{\nu^1_{\rho_*}}(\pi_M)\}_{M\in\mathbb{N}}$ is increasing for large M and non-negative and therefore by the monotone convergence theorem again we get that

$$I_{\rho_*}(\pi_M dm_{\mathbb{T}^d}) = \int \Lambda^*_{\nu^1_{\rho_*}}(\pi_M) dm_{\mathbb{T}^d} \longrightarrow \int \Lambda^*_{\nu^1_{\rho_*}}(\pi) dm_{\mathbb{T}^d} = I(\pi),$$

as required and the proof is complete.

We prove next the lower bound for measures $\pi \in \mathcal{M}^+ \setminus \mathcal{M}^+_{ac}(\mathbb{T}^d)$. We consider first a measure of the form $\pi = r\delta_x + \rho dm_{\mathbb{T}^d}$, $x \in \mathbb{T}^d$, $a > 0$, for some strictly positive density $\rho \in C(\mathbb{T}^d)$ and a neighborhood U_π of π in $\mathcal{M}^+(\mathbb{T}^d)$. We have that

$$\mu_N(U_\pi) \geq \int_{\{\pi^N \in U_\pi\}} \mathbb{1}_{\{\eta([Nx]) = [rN^d]\}} d\nu^N_{\rho_*}$$

But by considering $\mathcal{M}^+(\mathbb{T}^d)$ as a subset of the topological vector space $\mathcal{M}(\mathbb{T}^d)$ of all finite Borel measures on the torus, there exist a neighborhood $U_\rho \subseteq \mathcal{M}(\mathbb{T}^d)$ of the absolutely continuous measure $\rho dm_{\mathbb{T}^d} \in \mathcal{M}(\mathbb{T}^d)$ and a neighborhood U_0 of the zero measure such that

$$U_\pi \supseteq (U_\rho + U_0 + r\delta_x) \cap \mathcal{M}^+(\mathbb{T}^d).$$

On the set $\{\eta([Nx]) = [rN^d]\}$ we have that

$$\pi^N = \frac{[rN^d]}{N^d}\delta_{\frac{[Nx]}{N}} + \frac{1}{N^d} \sum_{y\in\mathbb{T}^d_N,\, y\neq[Nx]} \eta(y)\delta_{\frac{y}{N}} =: \frac{[rN^d]}{N^d}\delta_{\frac{[Nx]}{N}} + \widetilde{\pi}^N.$$

Now obviously $\frac{[rN^d]}{N^d}\delta_{\frac{[Nx]}{N}} \longrightarrow r\delta_x$ weakly and therefore there exists $N_0 \in \mathbb{N}$ such that

$$N \geq N_0 \quad \Longrightarrow \quad \frac{[rN^d]}{N^d}\delta_{\frac{[Nx]}{N}} - r\delta_x \in U_0.$$

Then for all $N \geq N_0$ we have that

$$\{\eta([Nx]) = [rN^d]\} \cap \{\pi^N \in U_\pi\} \supseteq \{\eta([Nx]) = [rN^d]\} \cap \{\widetilde{\pi}^N \in U_\rho\}$$

since if $N \geq N_0$ and η belongs in the set in the right hand side we have that

$$\pi^N = \frac{[rN^d]}{N^d}\delta_{\frac{[Nx]}{N}} + \widetilde{\pi}^N \in (r\delta_x + U_0 + U_\rho) \cap \mathcal{M}^+(\mathbb{T}^d) \subseteq U_\pi$$

and therefore for all $N \geq N_0$ we have by the independence of $\mathbb{1}_{\{[rN^d]\}}\big(\eta([Nx])\big)$ and $\widetilde{\pi}^N$ that

$$\begin{aligned}
\frac{1}{N^d}\log \mu_N(U_\pi) &\geq \frac{1}{N^d}\log\int \mathbb{1}_{\{\eta([Nx])=[rN^d]\}}\mathbb{1}_{\{\widetilde{\pi}^N\in U_\rho\}}d\nu_{\rho_*}^N \\
&= \frac{1}{N^d}\log\int \mathbb{1}_{\{\eta([Nx])=[rN^d]\}}d\nu_{\rho_*}^N + \frac{1}{N^d}\log\int_{\{\widetilde{\pi}^N\in U_\rho\}} d\nu_{\rho(\cdot)}^N \\
&= \frac{1}{N^d}\log \nu_{\rho_*}^1([rN^d]) + \frac{1}{N^d}\log\int_{\{\widetilde{\pi}^N\in U_\rho\}} d\nu_{\rho_*}^N.
\end{aligned}$$

By the definition of the one-site ZR distributions we have that

$$\frac{1}{N^d}\log \nu_{\rho_*}^1([rN^d]) = \frac{1}{N^d}\log\frac{1}{Z\big(\Phi(\rho_*)\big)} + \frac{[rN^d]}{N^d}\log\Phi(\rho_*) - \frac{1}{N^d}\log g!([rN^d])$$

and therefore if we assume that the lim inf defining the critical fugacity exists as a limit, i.e. that

$$0 < \varphi_c := \liminf_{k\to+\infty}\sqrt[k]{g!(k)} = \lim_{k\to+\infty}\sqrt[k]{g!(k)}$$

it follows that

$$\begin{aligned}
\lim_{N\to+\infty}\frac{1}{N^d}\log \nu_{\rho_*}^1([rN^d]) &= r\log\Phi(\rho_*) - \lim_{N\to+\infty}\frac{[rN^d]}{N^d}\log\sqrt[{[rN^d]}]{g!([rN^d])} \\
&= r\log\frac{\Phi(\rho_*)}{\varphi_c}.
\end{aligned}$$

On the other hand it is easy to see that the sequence $\widetilde{\mu}_N := \widetilde{\pi}_*^N\nu_{\rho_*}^N$ satisfies the same large deviations lower bound as the sequence $\mu_N := \pi_*^N\nu_{\rho_*}^N$, at least for absolutely continuous measures, i.e. for any measure $\rho dm_{\mathbb{T}^d} \in \mathcal{M}_{ac}^+(\mathbb{T}^d)$ and any neighborhood U_ρ of $\rho dm_{\mathbb{T}^d}$ we have that

$$\liminf_{N\to+\infty}\frac{1}{N^d}\log\widetilde{\mu}_N(U_\rho) \geq -I_{\rho_*}(\rho) = -\int \Lambda_{\rho_*}^*(\rho)dm_{\mathbb{T}^d}. \tag{1.86}$$

Therefore we get by what we have proved for absolutely continuous measures that

$$\liminf_{N\to+\infty}\frac{1}{N^d}\log\mu_N(U_\pi) \geq -\left[r\log\frac{\varphi_c}{\Phi(\rho_*)} + \int_{\mathbb{T}^d}\Lambda_{\rho_*}^*(\rho(u))du\right] = -I_{\rho_*}(\pi).$$

Since we want to add any finite number of Dirac masses we need a slightly more general version of (1.86) which we prove now: Let $A = \{x_1,\ldots,x_m\} \subseteq \mathbb{T}^d$, $m \in \mathbb{N}$, be any finite set of points. For each $N \in \mathbb{N}$ we set $A_N := \{[Nx_1],\ldots,[Nx_m]\} \subseteq \mathbb{T}_N^d$ and we define the empirical embeddings

$$\widetilde{\pi}^N := \frac{1}{N^d}\sum_{y\in\mathbb{T}_N^d\setminus A_N}\eta(y)\delta_{\frac{y}{N}} : \mathbb{M}_N^d \longrightarrow \mathcal{M}^+(\mathbb{T}^d), \quad N \in \mathbb{N}.$$

Then the sequence $\widetilde{\mu}_N := \widetilde{\pi}_*^N \nu_{\rho_*}^N$ satisfies (1.86) for any absolutely continuous measure $\rho dm_{\mathbb{T}^d}$ and any neighborhood $U_\rho \subseteq \mathcal{M}^+(\mathbb{T}^d)$ of $\rho dm_{\mathbb{T}^d}$. As in the proof of the same lower bound for $\{\mu_N\}$ it suffices to consider the case where the density ρ is strictly positive and continuous. Then, making again the change of measure

$$d\nu_{\rho_*}^N = \frac{d\nu_{\rho_*}^N}{d\bar{\nu}_{\rho(\cdot)}^N} d\bar{\nu}_{\rho(\cdot)}^N,$$

where $\bar{\nu}_\rho^N := \nu_{\rho\wedge\rho_c}^N$ for all $\rho \geq 0$, we get that

$$\begin{aligned}\liminf_{N\to+\infty} \frac{1}{N^d} \log \widetilde{\mu}_N(U_\rho) \quad &\geq \quad -\log Z\big(\Phi(\rho_*)\big) + \int_{\mathbb{T}^d} \log Z\big(\Phi(\rho\wedge\rho_c)\big) dm_{\mathbb{T}^d} \\ &\qquad \liminf_{N\to+\infty} \frac{1}{N^d} \log \int_{\{\widetilde{\pi}^N \in U_\rho\}} e^{-N^d \int \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)} d\pi^N} d\bar{\nu}_{\rho(\cdot)}^N.\end{aligned}$$

Then if we set

$$q^N := \pi^N - \widetilde{\pi}^N = \frac{1}{N^d} \sum_{i=1}^m \eta([Nx_i]) \delta_{\frac{[Nx_i]}{N}}$$

for each $N \in \mathbb{N}$, the random variables $\widetilde{\pi}^N$ and q^N are independent and therefore

$$\begin{aligned}&\liminf_{N\to+\infty} \frac{1}{N^d} \log \int_{\{\widetilde{\pi}^N \in U_\rho\}} e^{-N^d \int \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)} d\pi^N} d\bar{\nu}_{\rho(\cdot)}^N \\ &\qquad \geq \liminf_{N\to+\infty} \frac{1}{N^d} \log \int_{\{\widetilde{\pi}^N \in U_\rho\}} e^{-N^d \int \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)} d\widetilde{\pi}^N} d\bar{\nu}_{\rho(\cdot)}^N \\ &\qquad\qquad + \liminf_{N\to+\infty} \frac{1}{N^d} \log \int e^{-N^d \int \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)} dq^N} d\bar{\nu}_{\rho(\cdot)}^N.\end{aligned} \tag{1.87}$$

For the first term we consider for each $\varepsilon > 0$ the open neighborhood

$$U_\varepsilon := \left\{ \mu \in \mathcal{M}^+(\mathbb{T}^d) \,\middle|\, \left| \int \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)} d(\mu - \rho) \right| < \varepsilon \right\}$$

and obviously

$$\begin{aligned}&\liminf_{N\to+\infty} \frac{1}{N^d} \log \int_{\{\widetilde{\pi}^N \in U_\rho\}} e^{-N^d \int \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)} d\widetilde{\pi}^N} d\bar{\nu}_{\rho(\cdot)}^N \\ &\qquad \geq -\varepsilon - \int \rho \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)} dm_{\mathbb{T}^d} + \liminf_{N\to+\infty} \frac{1}{N^d} \log \int_{\{\widetilde{\pi}^N \in U_\rho \cap U_\varepsilon\}} d\bar{\nu}_{\rho(\cdot)}^N\end{aligned}$$

for all $\varepsilon > 0$. Since $U_\rho \cap U_\varepsilon$ is open in the weak topology of $\mathcal{M}^+(\mathbb{T}^d)$ there exist $\varepsilon_0 > 0$ and functions $f_1, \dots, f_k, f_{k+1} \in C(\mathbb{T}^d)$, $k \in \mathbb{N}$, with $f_{k+1} = \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)}$ such that

$$U_\rho \cap A_\varepsilon \supseteq \bigcap_{i=1}^{k+1} \left\{ \mu \in \mathcal{M}^+(\mathbb{T}^d) \,\middle|\, \left| \int f_i d(\mu - \rho) \right| < \varepsilon \right\}$$

for all $0 < \varepsilon < \varepsilon_0$. But then, for all $\varepsilon \in (0, \varepsilon_0)$ we have that

$$\int_{\{\widetilde{\pi}^N \in U_\rho \cap U_\varepsilon\}} d\bar{\nu}_{\rho(\cdot)}^N \geq 1 - \sum_{i=1}^{k+1} \bar{\nu}_{\rho(\cdot)}^N \left\{ \left| \int f_i d\widetilde{\pi}^N - \int f_i \rho dm_{\mathbb{T}^d} \right| \geq \varepsilon \right\}$$

and so it suffices to prove that

$$\lim_{N\to+\infty} \bar{\nu}^N_{\rho(\cdot)}\Big\{\Big|\int f d\widetilde{\pi}^N - \int f\rho dm_{\mathbb{T}^d}\Big| \geq \varepsilon\Big\} = 0$$

for all $f \in C(\mathbb{T}^d)$ and all $\varepsilon > 0$. But this indeed holds, since for all $\varepsilon > 0$ we have that

$$\begin{aligned}\bar{\nu}^N_{\rho(\cdot)}\Big\{\Big|\int f d\widetilde{\pi}^N - \int f\rho dm_{\mathbb{T}^d}\Big| \geq \varepsilon\Big\} \quad &\leq \quad \bar{\nu}^N_{\rho(\cdot)}\Big\{\Big|\int f dq^N\Big| \geq \frac{\varepsilon}{2}\Big\} \\ &\quad + \bar{\nu}^N_{\rho(\cdot)}\Big\{\Big|\int f d\pi^N - \int f\rho dm_{\mathbb{T}^d}\Big| \geq \frac{\varepsilon}{2}\Big\},\end{aligned}$$

and the second term in the right hand side of the inequality above converges to zero since the family $\{\bar{\nu}^N_{\rho(\cdot)}\}$ is associated to the profile $\rho \in C(\mathbb{T}^d)$, while for the first term we note that for all $\varepsilon > 0$ we have that

$$\begin{aligned}\bar{\nu}^N_{\rho(\cdot)}\Big\{\Big|\int f dq^N\Big| \geq \varepsilon\Big\} \quad &\leq \quad \frac{1}{\varepsilon}\int\Big|\int f dq^N\Big| d\bar{\nu}^N_{\rho(\cdot)} \leq \frac{\|f\|_u}{\varepsilon N^d}\sum_{i=1}^m \int \eta([Nx_i]) d\bar{\nu}^N_{\rho(\cdot)} \\ &= \quad \frac{\|f\|_u}{\varepsilon N^d}\sum_{i=1}^m (\rho_c \wedge \rho)\Big(\frac{[Nx_i]}{N}\Big) \overset{N\to+\infty}{\longrightarrow} 0. \qquad (1.88)\end{aligned}$$

This proves that

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log \int_{\{\widetilde{\pi}^N \in U_\rho \cap U_\varepsilon\}} d\bar{\nu}^N_{\rho(\cdot)} \geq 0$$

for all $\varepsilon > 0$ and therefore the first term in the right hand side of (1.87) satisfies

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log \int_{\{\widetilde{\pi}^N \in U_\rho\}} e^{-N^d \int \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)} d\widetilde{\pi}^N} d\bar{\nu}^N_{\rho(\cdot)} \geq -\int \rho \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)} dm_{\mathbb{T}^d}.$$

It follows by (1.87) that

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log \widetilde{\mu}_N(U_\rho) \geq -I_{\rho_*}(\rho) + \liminf_{N\to+\infty} \frac{1}{N^d} \log \int e^{-N^d \int \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)} dq^N} d\bar{\nu}^N_{\rho(\cdot)}$$

and so to prove (1.86) as required, it suffices to show that

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log \int e^{-N^d \int \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)} dq^N} d\bar{\nu}^N_{\rho(\cdot)} \geq 0.$$

But this is indeed true since by Jensen's inequality we have that

$$\frac{1}{N^d} \log \int e^{-N^d \int \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)} dq^N} d\bar{\nu}^N_{\rho(\cdot)} \geq -\iint \log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)} dq^N d\bar{\nu}^N_{\rho(\cdot)}$$

for all $N \in \mathbb{N}$ and the term in the right hand side converges to 0 by (1.88) since $\log \frac{\Phi(\rho\wedge\rho_c)}{\Phi(\rho_*)}$ is a continuous function. This proves (1.86) for any finite number of Dirac masses.

By using this more general form of (1.86) and the same estimates as in the case of one Dirac mass one can easily prove the lower bound when $\pi = \rho dm_{\mathbb{T}^d} + \pi_s \in \mathcal{M}^+(\mathbb{T}^d)$ is the sum of an absolutely continuous measure $\pi_{ac} = \rho dm_{\mathbb{T}^d}$ and a finite sum of Dirac masses $\pi_s = \sum_{i=1}^k r_i \delta_{x_i}$, $x_i \in \mathbb{T}^d$, in which case we have that

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log \mu_N(U_\pi) \geq -\Big[\Big(\sum_{i=1}^k r_i\Big) \log \frac{\varphi_c}{\Phi(\rho_*)} + \int_{\mathbb{T}^d} \Lambda^*_{\rho_*}(\rho(u))du\Big] = -I_{\rho_*}(\pi).$$

For the general case let $\pi = \rho dm_{\mathbb{T}^d} + \pi_s$ be the Radon-Nikodym decomposition of an arbitrary measure $\pi \in \mathcal{M}^+(\mathbb{T}^d)$. Since the set

$$\left\{ \sum_{i=1}^{k} r_i \delta_{x_i}, \middle| r_i \geq 0,\ x_i \in \mathbb{T}^d,\ k \in \mathbb{N} \right\}$$

is dense in the weak topology of $\mathcal{M}^+(\mathbb{T}^d)$ there exists a sequence $\{\pi_{s,n}\}_{n\in\mathbb{N}}$ of the form $\pi_{s,n} = \sum_{i=1}^{k_n} r_{i,n}\delta_{x_{i,n}}$ for all $n \in \mathbb{N}$ such that $\pi_{s,n} \longrightarrow \pi_s$ as $n \to +\infty$. Now, obviously

$$\pi_n := \rho dm_{\mathbb{T}^d} + \pi_{s,n} \longrightarrow \rho dm_{\mathbb{T}^d} + \pi_s$$

weakly, and so given a neighborhood U_π of π there exists n_0 such that $\pi_n \in U_\pi$ for all $n \geq n_0$. Therefore, by the LDP lower bound (1.82) for measures $\pi \in \mathcal{M}^+(\mathbb{T}^d)$ with singular part equal to a finite sum of Dirac masses we get that for any neighborhood U_π of π there exists $n_0 \in \mathbb{N}$ such that

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log \mu_N(U_\pi) \geq -I_{\rho_*}(\pi_n)$$

for all $n \geq n_0$. But $\pi_{s,n}(\mathbb{T}^d) \longrightarrow \pi_s(\mathbb{T}^d)$ by the weak convergence $\pi_{s,n} \longrightarrow \pi_s$ and therefore

$$\lim_{n\to+\infty} I(\pi_n) = \lim_{n\to+\infty} \left[\pi_{s,n}(\mathbb{T}^d) \log \frac{\varphi_c}{\Phi(\rho_*)} + \int_{\mathbb{T}^d} \Lambda^*_{\rho_*}(\rho(u))du \right] = I(\pi).$$

It follows that

$$\liminf_{N\to+\infty} \frac{1}{N^d} \log \mu_N(U_\pi) \geq -\lim_{n\to+\infty} I_{\rho_*}(\pi_n) = -I_{\rho_*}(\pi)$$

as required, which completes the proof of the large deviations lower bound. □

As we will see next, the rate functions in the upper and lower LDP bounds proved in fact coincide and therefore the above large deviations bounds give us the full large deviations principle for the empirical embeddings. This is seen by the variational characterization of the generalized entropy functionals, according to which, given any lower semi-continuous convex function $h : \mathbb{R} \longrightarrow [0, +\infty]$, the h-entropy $H_h : \mathcal{M}^+(\mathbb{T}^d) \longrightarrow [0, +\infty]$ defined by

$$H_h(\pi) := H_h(\pi|m_{\mathbb{T}^d}) := \int h\left(\frac{d\pi_{ac}}{dm_{\mathbb{T}^d}}\right) dm_{\mathbb{T}^d} + \left[\lim_{t\uparrow+\infty} \frac{h(t)}{t}\right] \pi_s(\mathbb{T}^d),$$

where $\pi = \pi_{ac} + \pi_s$ is the Radon-Nikodym decomposition of π with respect to the Lebesgue measure on the torus $\mathbb{T}^d$, provides the solution of the variational problem

$$H_h(\pi) = \sup_{f\in C(\mathbb{T}^d)} \left\{ \int f d\pi - \int h^{\oplus}(f) d\mu \right\}$$

for all $\pi \in \mathcal{M}^+(\mathbb{T}^d)$, where $h^{\oplus} : \mathbb{R} \longrightarrow (-\infty, +\infty]$ is the right semi-Legendre transform of h given by

$$h^{\oplus}(s) = \sup_{t\geq 0} \left\{ st - h(t) \right\}.$$

Note that since π is a non-negative measure only the values of h on $[0,+\infty)$ are relevant for the definition of H_h and therefore if $\widetilde{h}$ is given by $\widetilde{h} = +\infty$ on $(-\infty, 0)$ and $\widetilde{h} = h$ on $[0,+\infty)$ then $H_{\widetilde{h}} = H_h$ and $h^{\oplus} = \widetilde{h}^*$, where $\widetilde{h}^*$ is the Legendre-Fenchel transform of the lower semicontinuous convex functional $\widetilde{h} : \mathbb{R} \longrightarrow [0,+\infty]$.

Proposition 1.6.4 *Let $\{\nu^N_{\rho_*,g}\}_{N\in\mathbb{N}}$ be the family of ZR distributions of density $\rho_* < \rho_c$, where g is a local rate function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ such that the* lim inf *defining the critical fugacity $\varphi_c \in (0,+\infty]$ exists as a limit, i.e. such that*

$$0 < \varphi_c := \liminf_{k\to+\infty} \sqrt[k]{g!(k)} = \lim_{k\to+\infty} \sqrt[k]{g!(k)},$$

and let Λ_{ρ_*} be the logarithmic m.g.f. of the one site ZR distribution with density ρ_*. Then the sequence $\{\mu_N\} \subseteq \mathbb{P}\mathcal{M}^+(\mathbb{T}^d)$ of the empirical embeddings of $\nu^N_{\rho_*}$ given by (1.76) *satisfies the large deviations principle with speeds $\frac{1}{N^d}$ and rate function $H_{\Lambda^*_{\rho_*}}$, given by*

$$H_{\Lambda^*_{\rho_*}}(\pi) = \int_{\mathbb{T}^d} \Lambda^*_{\rho_*}(\pi_{ac}(u))du + \left[\lim_{\rho\uparrow+\infty} \frac{\Lambda^*_{\rho_*}(\rho)}{\rho}\right]\pi_s(\mathbb{T}^d),$$

where $\pi = \pi_{ac} dm_{\mathbb{T}^d} + \pi_s$, $\pi_s \perp m_{\mathbb{T}^d}$ is the Radon-Nikodym decomposition of π with respect to $m_{\mathbb{T}^d}$.

Proof By the formula of $\Lambda^*_{\rho_*}$ we know that $\widetilde{\Lambda^*_{\rho_*}} = \Lambda^*_{\rho_*}$ and so

$$\Lambda^{*\oplus}_{\rho_*} = \widetilde{\Lambda^*_{\rho_*}}^* = \Lambda^{**}_{\rho_*} = \Lambda_{\rho_*}.$$

Therefore, by the variational characterization of the generalized entropy functionals it follows that

$$H_{\Lambda^*_{\rho_*}}(\pi) = \sup_{f\in C(\mathbb{T}^d)} \left\{ \int f d\nu - \int \Lambda_{\rho_*}(f) dm_{\mathbb{T}^d} \right\},$$

and thus $H_{\Lambda^*_{\rho_*}}$ coincides with the rate function given in the LDP upper bound for $\{\mu_N\}$. It remains to prove that $H_{\Lambda^*_{\rho_*}}$ coincides also with the LDP lower bound, and for this it suffices to prove that

$$\lim_{\rho\uparrow+\infty} \frac{\Lambda^*_{\rho_*}(\rho)}{\rho} = \log \frac{\varphi_c}{\Phi(\rho_*)}. \tag{1.89}$$

By the variational characterization of the h-relative entropy functionals we know that for any lower semicontinuous and convex function $h : \mathbb{R} \longrightarrow [0,+\infty]$ we have

$$\lim_{t\to+\infty} \frac{h(t)}{t} = \begin{cases} +\infty, & \mathcal{D}_{h^\oplus} = \mathbb{R}, \\ \sup_{t\in\mathcal{D}_h} h'_-(t), & \mathcal{D}_{h^\oplus} \neq \mathbb{R} \end{cases}.$$

In our case $h = \Lambda^*_{\rho_*}$ and therefore $\mathcal{D}_{h^\oplus} = \mathcal{D}_{\Lambda_{\rho_*}}$. But as we know

$$(-\infty, b_*) \subseteq \mathcal{D}_{\Lambda_{\rho_*}} \subseteq (-\infty, b_*]$$

where $b_* := \log \frac{\varphi_c}{\Phi(\rho_*)}$ and therefore $\mathcal{D}_{h^\oplus} = \mathcal{D}_{\Lambda_{\rho_*}} = \mathbb{R}$ iff $\varphi_c = +\infty$, and in this case we trivially have that

$$\lim_{\rho\uparrow+\infty} \frac{\Lambda^*_{\rho_*}(\rho)}{\rho} = +\infty = \log \frac{\varphi_c}{\Phi(\rho_*)}.$$

On the other hand, $\mathcal{D}_{h^\oplus} = \mathcal{D}_{\Lambda_{\rho_*}} \neq \mathbb{R}$ iff $\varphi_c < +\infty$ and since in any case we have that $(0, \infty) \subseteq \mathcal{D}_{\Lambda_{\rho_*}}$, we have in this case that

$$\lim_{\rho\uparrow+\infty} \frac{\Lambda^*_{\rho_*}(\rho)}{\rho} = \lim_{\rho\uparrow+\infty} (\Lambda^*_{\rho_*})'_-(\rho).$$

Now, we know that if $\rho_c = +\infty$ then

$$\Lambda^*_{\rho_*}(\rho) = \rho\frac{\Phi(\rho)}{\Phi(\rho_*)} - \log\frac{Z\big(\Phi(\rho)\big)}{Z\big(\Phi(\rho_*)\big)}$$

for all $\rho \geq 0$, and therefore we have in this case that

$$\begin{aligned}
(\Lambda^*_{\rho_*})'(\rho) &= \log\frac{\Phi(\rho)}{\Phi(\rho_*)} + \rho\frac{\Phi'(\rho)}{\Phi(\rho)} - \frac{Z'\big(\Phi(\rho)\big)\Phi'(\rho)}{Z\big(\Phi(\rho)\big)} \\
&= \log\frac{\Phi(\rho)}{\Phi(\rho_*)} + \rho\frac{\Phi'(\rho)}{\Phi(\rho)} - R\big(\Phi(\rho)\big)\frac{\Phi'(\rho)}{\Phi(\rho)} \\
&= \log\frac{\Phi(\rho)}{\Phi(\rho_*)} \overset{\rho\to+\infty}{\longrightarrow} \log\frac{\varphi_c}{\Phi(\rho)}.
\end{aligned}$$

Finally, if $\rho_c < +\infty$, then as we know for all $\rho \geq \rho_*$ we have that

$$\Lambda^*_{\rho_*}(\rho) = \rho\log\frac{\varphi_c}{\Phi(\rho_*)} - \log\frac{Z(\varphi_c)}{Z\big(\Phi(\rho_*)\big)}.$$

and therefore in this case we have that $(\Lambda^*_{\rho_*})'(\rho) \equiv \log\frac{\varphi_c}{\Phi(\rho_*)}$ for all $\rho \geq \rho_c$. This proves that in any case (1.89) holds and so the rate functions for the upper and lower LDP bounds coincide and the required large deviation principle is proved. □

Chapter 2

Formulations of the Notion of Local Equilibrium and Hydrodynamic Limits

The notion of local equilibrium is central in describing the hydrodynamic limit of interacting particle systems. Loosely speaking a local equilibrium is a state that locally, i.e. near each macroscopic point, looks like a an equilibrium state. In this section we describe various ways to formulate the notion of local equilibrium.

2.1 Strong Local equilibrium

Recall that we denote by $\{\nu_{\rho\wedge\rho_c}\}_{\rho\in\mathbb{R}_+}$ the family of the product and translation invariant equilibrium distributions of the ZRP on the discrete torus $\mathbb{T}_N^d$. Its marginal distributions are given by

$$\eta(x)_*\nu_\rho)(k) = \frac{1}{Z(\Phi(\rho))}\frac{\varphi^k}{g!(k)}, \quad x \in \mathbb{T}_N^d.$$

Furthermore, given $\rho \in [0, \rho_c] \cap \mathbb{R}_+$ we denote by $\nu_\rho^\infty \in \mathbb{PM}_\infty^d$ the corresponding distribution on the space of configurations supported by the infinite lattice $\mathbb{Z}^d$.

Definition 2.1.1 A sequence $\{\mu_N \in \mathbb{PM}_N^d\}$ is called a *strong local equilibrium* of profile $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ if

$$\tau_{[Nu]*}\mu_N \longrightarrow \nu_{\rho(u)}^\infty$$

in the weak topology of $\mathbb{PM}_N^d$ for all continuity points $u \in \mathbb{T}^d$ of the function ρ.

A particular example of a strong local equilibrium is given by the so-called families of slowly varying profile with respect to a function $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$.

Definition 2.1.2 Let $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ be a function. The family $\{\nu_{\rho(\cdot)}^N\}_{N\in\mathbb{N}}$ given by

$$\nu_{\rho(\cdot)}^N := \prod_{x\in\mathbb{T}_N^d} \nu^1_{\rho(\frac{x}{N})} \in \mathbb{PM}_M^d$$

is called the {family of distributions with slowly varying parameter of profile} ρ.

Proposition 2.1.1 *Given any function $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$, the family $\{\nu^N_{\rho(\cdot)}\}$ of distributions with slowly varying parameter of profile ρ is a strong local equilibrium.*

Proof Let $u \in \mathbb{T}^d$ be a continuity point of ρ. Since $\nu^N_{\rho(\cdot)}$ is a product distribution it suffices to check that for each $z \in \mathbb{Z}^d$,

$$\eta(z + N\mathbb{Z}^d)_*\big(\tau_{[Nu]_*}\nu^N_{\rho(\cdot)}\big) = \nu^1_{\rho(\frac{z+[Nu]+N\mathbb{Z}^d}{N})} \stackrel{N\to\infty}{\longrightarrow} \nu^1_{\rho(u)} \in \mathbb{P}\mathbb{Z}_+.$$

But this is obvious since $\frac{z+[Nu]+N\mathbb{Z}^d}{N} \longrightarrow u$ as $N \to \infty$ and the family $\nu^1\rho \wedge \rho_{c_{\rho\in\mathbb{R}_+}}$ is continuous with respect to the parameter ρ. □

Once we have a notion of local equilibrium we can speak of the conservation of local equilibrium and define the notion of a hydrodynamic limit. We do so in the context of ZRPs.

Definition 2.1.3 Let $\{\eta^N\}$ denote the family of ZRPs on the discrete toruses with jump rate $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ and elementary transition probability $p \in \mathbb{P}\mathbb{Z}^d$ and let $\{Q^N_t\}_{t\in\mathbb{R}_+}$ denote the transition probability of η^N. We say that a strong local equilibrium $\{\mu_N\}$ of profile $\rho_0 : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ is conserved by the evolution of the ZRP in the time scale $\{\theta_N\}_{N\in\mathbb{N}}$ if there exists a measurable function $\rho : \mathbb{R}_+ \times \mathbb{T}^d \longrightarrow \mathbb{R}_+$, called *the hydrodynamic limit*, such that for each $t \in \mathbb{R}_+$ the sequence $\{Q_{t\theta_N*}\}_{N\in\mathbb{N}}$ is a strong local equilibrium of profile ρ_t.

Usually we ask of the function $\rho : \mathbb{R}_+ \times \mathbb{T}^d \longrightarrow \mathbb{R}_+$ to be the solution of an evolutionary Cauchy problem with initial condition ρ_0. The equation satisfied by ρ is then called the hydrodynamic equation of the ZRP. It is easy to see that if $\rho_t : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ is continuous for each fixed $t > 0$ then the hydrodynamic limit is unique. Indeed, let $\{\mu_N\}$ be an initial strong local equilibrium and let $\rho, \bar\rho : \mathbb{R}_+ \times \mathbb{T}^d \longrightarrow \mathbb{R}_+$ be two hydrodynamic limits in the same timescale $\{\theta_N\}_{N\in\mathbb{N}}$ of the ZRP starting from $\{\mu_N\}$. Then we have that

$$\nu^\infty_{\rho(u)} = \lim_{N\to\infty} \tau_{[Nu]*}Q^N_{t\theta_N*}\mu_N = \nu^\infty_{\bar\rho(u)}$$

for each $(t, u) \in (0, \infty) \times \mathbb{T}^d$ which since the family ν^∞_ρ is parametrized by the density implies that

$$\rho_t(u) = \int \eta(0)d\nu_{\rho(u)} = \int \eta(0)d\nu_{\bar\rho(u)} = \bar\rho(u).$$

2.2 Weak Local Equilibrium

Usually one works with weaker notions of local equilibrium in order to describe the hydrodynamic behavior of interacting particle systems. One way this is done is by replacing the convergence $\tau_{[Nu]*}\mu_N \longrightarrow \nu_{\rho(u)}$ for each continuity point of the profile $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ by the convergence of the spatial mean

$$\bar\mu_N := \int_{\mathbb{T}^d} \tau_{[Nu]*}\mu_N du = \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} \tau_{x*}\mu_N \in \mathbb{P}\mathbb{M}^d_N$$

to the spatial mean

$$\bar{\nu}^{\infty}_{\rho(\cdot)} := \int_{\mathbb{T}^d} \nu^{\infty}_{\rho(u)} du \in \mathbb{PM}^d_{\infty}.$$

It is easy to see that at least for a.s. continuous profiles such a notion of local equilibrium is weaker than the strong local equilibrium. We will refer to this motion of local equilibrium by saying that $\{\mu_N\}$ is a itlocal equilibrium of profile ρ is spatial mean.

Proposition 2.2.1 *Let $\{\mu_N\}$ be a strong local equilibrium with respect to an a.s. continuous profile $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$. Then with the notation above $\bar{\mu}_N \longrightarrow \bar{\nu}_{\rho(\cdot)}$.*

Proof Since weak convergence in $\mathbb{PM}^d_{\infty}$ is determined by the duality with respect to bounded cylindrical functions it suffices to show that

$$\int \Psi d\bar{\mu}_N \longrightarrow \int \Psi d\bar{\nu}_{\rho(\cdot)}$$

for all bounded cylindrical functions $\Psi : \mathbb{M}^d_{\infty} \longrightarrow \mathbb{R}$. So let $\Psi \in B_{\mathrm{cyl}}(\mathbb{M}^d_{\infty})$. Since ρ is a.s. continuous and $\{\mu_N\}$ is a strong local equilibrium we have that

$$h_N(u) := \int \Psi d\tau_{[Nu]*}\mu_N \longrightarrow \int \Psi d\nu_{\rho(\cdot)}$$

for almost all $u \in \mathbb{T}^d$. But $\{h_N\}$ is obviously uniformly bounded by $\|\Psi\|_{\infty}$ and so the bounded convergence theorem yields

$$\int \Psi d\bar{\mu}_N = \int h_N(u) du \longrightarrow \int_{\mathbb{T}^d} \int \Psi d\nu_{\rho(u)} du = \int \Psi d\bar{\nu}_{\rho(\cdot)}$$

as required. □

For any cylinder function $\Psi : \mathbb{M}^d_{\infty} \longrightarrow \mathbb{R}$ we will denote by $\widetilde{\Psi} : \mathbb{R}_+ \longrightarrow \mathbb{R}$ the function defined by

$$\widetilde{\Psi}(\rho) = \int \Psi d\nu_{\rho \wedge \rho_c}.$$

Since the family $\{\nu_{\rho \wedge \rho_c}\}_{\rho \in \mathbb{R}_+}$ is weakly continuous the function $\widetilde{\Psi}$ is continuous for any cylinder function Ψ, and obviously if Ψ is bounded then so is $\widetilde{\Psi}$ with $\|\widetilde{\Psi}\|_{\infty} \leq \|\Psi\|_{\infty}$. Of course with this notation we have that

$$\int \Psi d\bar{\nu}_{\rho(\cdot)} = \int_{\mathbb{T}^d} \widetilde{\Psi}(\rho(u)) du.$$

Furthermore, given any cylinder function $\Psi : \mathbb{M}^d_{\infty} \longrightarrow \mathbb{R}$ and any $z \in \mathbb{Z}^d$ we define its translation $\tau_z \Psi$ by z via the formula

$$\tau_z \Psi(\eta) = \Psi(\tau_z \eta).$$

In this way we have that

$$\int \Psi d\bar{\mu}_N = \int \Big(\frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} \tau_x \Psi \Big) d\mu_N$$

where as usual we identify $\mathbb{T}_N^d$ with the subset $\mathbb{Z}_N^d := \{-[\frac{N}{2}], \cdots, [\frac{N}{2}]-1\}^d$. In this way the requirement of the weak convergence of the spatial means $\{\bar{\mu}_N\}$ to the spatial mean $\bar{\nu}_{\rho(\cdot)}$ can equivalently rephrased by requiring the stochastic means with respect to $\{\mu_N\}$ of the spatial means

$$\bar{\Psi}^N := \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x \Psi$$

to converge to the spatial mean $\int_{\mathbb{T}^d} \widetilde{\Psi}(\rho(u))du$ for all cylinder functions $\Psi : \mathbb{M}_\infty^d \longrightarrow \mathbb{R}$, i.e. by requiring that

$$\lim_{N\to\infty} \mathbb{E}_{\mu_N} \bar{\Psi}^N = \int \widetilde{\Psi}(\rho(u))du \tag{2.1}$$

for all cylinder functions $\Psi : \mathbb{M}_\infty^d \longrightarrow \mathbb{R}$. In the definition of weak local equilibrium, instead of the convergence in (2.1) one imposes the stronger requirement that the spatial mean $\bar{\Psi}^N$ converges in probability with respect to the sequence $\{\mu_N\}$ to the constant $\int \widetilde{\Psi}(\rho(u))du$ for all cylinder functions $\Psi : \mathbb{M}_\infty^d \longrightarrow \mathbb{R}$.

Definition 2.2.1 Let $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ be a measurable profile. A sequence $\{\mu_N \in \mathbb{P}\mathbb{M}_N^d\}_{N\in\mathbb{N}}$ is called a weak local equilibrium of profile ρ if

$$\lim_{N\to\infty} \mu_N \Big\{ \Big| \bar{\Psi}^N - \int_{\mathbb{T}^d} \widetilde{\Psi}(\rho(u))du \Big| > \delta \Big\} = 0$$

for all $\Psi \in B_{\text{cyl}}(\mathbb{M}_\infty^d)$ and all $\delta > 0$.

Proposition 2.2.2 *If $\{\mu_N\}$ is a weak local equilibrium of profile $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ then it is also a local equilibrium of profile ρ in the spatial mean, i.e. $\bar{\mu}_N \longrightarrow \bar{\nu}_{\rho()}^\infty$.*

Proof Let $\Psi \in B_{\text{cyl}}(\mathbb{M}_\infty^d)$ and let $\delta > 0$. We set $\theta := \int_{\mathbb{T}}^d \widetilde{\Psi}(\rho(u))du$. Since $\{\mu_N\}$ is weak local equilibrium there exists $N_0 \in \mathbb{N}$ such that

$$N \geq N_0 \quad \Longrightarrow \quad \mu_N\{|\bar{\Psi}^N - \theta| > \delta/2\} \leq \frac{\delta}{2(\|\Psi\|_\infty + \theta + 1)}.$$

Then for each $N \geq N_0$ we have that

$$\begin{aligned} \Big| \int \bar{\Psi}^N d\mu_N - \theta \Big| &\leq \int |\bar{\Psi}^N - \theta| d\mu_N \leq \int_{\{|\bar{\Psi}^N - \theta| > \delta/2\}} |\bar{\Psi}^N - \theta| d\mu_N + \frac{\delta}{2} \\ &\leq (\|\Psi\|_\infty + \theta)\mu_N\{|\bar{\Psi}^N - \theta| > \delta/2\} + \frac{\delta}{2} < \delta, \end{aligned}$$

as required, since we obviously have that $\|\bar{\Psi}^N\|_\infty \leq \|\Psi\|_\infty$ for all $N \in \mathbb{N}$. $\square$

Following [25] we prove next that the notion of weak local equilibrium is indeed weaker than the notion of strong local equilibrium, at least for almost surely continuous profiles.

Proposition 2.2.3 *Let $\{\mu_N\}$ be a strong local equilibrium with respect to an a.s. continuous profile $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$. Then $\{\mu_N\}$ is a weak local equilibrium of profile ρ.*

Proof Let $\Psi \in B_{\text{cyl}}(\mathbb{M}^d_\infty)$ be a cylinder function. By Chebyshev's inequality it suffices to prove that

$$\lim_{N\to\infty} \int \Big| \bar{\Psi}^N - \int_{\mathbb{T}^d} \widetilde{\Psi}(\rho(u))du \Big| d\mu_N = 0.$$

A generally useful remark is that for each $\ell \in \mathbb{Z}_+$ the global spatial mean

$$\frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \tau_x \bar{\Psi}^{N,\ell}$$

of the local spatial mean $\bar{\Psi}^\ell$ of Ψ is equal to the global spatial mean. Indeed, by changing the order of summation,

$$\begin{aligned}
\frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \tau_x \bar{\Psi}^{N,\ell} &= \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \frac{1}{(2\ell+1)^d} \sum_{y\in x+\Lambda^d_\ell} \tau_y \Psi \\
&= \frac{1}{N^d} \sum_{y\in\mathbb{T}^d_N} \frac{1}{(2\ell+1)^d} \sum_{x\in y+\Lambda^d_\ell} \tau_y \Psi = \bar{\Psi}^N. \qquad (2.2)
\end{aligned}$$

Consequently, in order to prove that the sequence $\{\mu_N\}$ is a weal local equilibrium it suffices to show that

$$\liminf_{\ell\to\infty} \lim_{N\to\infty} \int \Big| \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \tau_x \bar{\Psi}^\ell - \int_{\mathbb{T}^d} \widetilde{\Psi}(\rho(u))du \Big| d\mu_N = 0.$$

Furthermore, since $\widetilde{\Psi}$ is continuous due to the weak continuity of the family $\{\nu_{\rho\wedge\rho_c}\}_{\rho\in\mathbb{R}_+}$ and since ρ is an a.s. continuous function we have that

$$\lim_{N\to\infty} \sum_{x\in\mathbb{T}^d_N} \mathbb{1}_{[\frac{x}{N},\frac{x}{N})}(u)\Psi\Big(\rho(\frac{x}{N})\Big) = \widetilde{\Psi}\big(\rho(u)\big)$$

for a.s. all $u \in \mathbb{T}^d$, and so by the bounded convergence theorem

$$\frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \Psi\Big(\rho(\frac{x}{N})\Big) = \int \sum_{x\in\mathbb{T}^d_N} \mathbb{1}_{[\frac{x}{N},\frac{x}{N})}(u)\Psi\Big(\rho(\frac{x}{N})\Big) \overset{N\to\infty}{\longrightarrow} \int_{\mathbb{T}^d} \widetilde{\Psi}\big(\rho(u)\big)du.$$

So in order to prove the claim it suffices to show that

$$\liminf_{\ell\to\infty} \lim_{N\to\infty} \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \int \big|\tau_x \bar{\Psi}^\ell - \widetilde{\Psi}\big(\rho(\frac{x}{N})\big)\big| d\mu_N = 0. \qquad (2.3)$$

Next, denoting for each $N \in \mathbb{N}$ and $\ell \in \mathbb{Z}_+$ by $h_{N,\ell} : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ the function defined by

$$\begin{aligned}
h_{N,\ell}(u) &= \sum_{x\in\mathbb{T}^d_N} \int \big|\tau_x \bar{\Psi}^\ell - \widetilde{\Psi}\big(\rho(\frac{x}{N})\big)\big| d\mu_N \cdot \mathbb{1}_{[\frac{x}{N},\frac{x+1}{N})}(u) \\
&= \int \big|\bar{\Psi}^\ell - \widetilde{\Psi}\big(\rho(\frac{[Nu]}{N})\big)\big| d\tau_{[Nu]_*}\mu_N
\end{aligned}$$

the limit in (2.3) can be rewritten as

$$\liminf_{\ell\to\infty}\lim_{N\to\infty}\int_{\mathbb{T}^d} h_{N,\ell}(u)du.$$

But since $\{\mu_N\}$ is a strong local equilibrium, for every continuity point $u\in\mathbb{T}^d$ of ρ we have that

$$\lim_{N\to\infty} h_{N,\ell}(u)=\int\big|\bar{\Psi}^\ell-\widetilde{\Psi}\big(\rho(u)\big)\big|d\nu^\infty_{\rho(u)}$$

and therefore since the family $\{h_{N,\ell}\}_{N,\ell}$ is uniformly bounded by $2\|\Psi\|_\infty$, by the bounded convergence theorem we can pass the limit as $N\to\infty$ inside the integral to obtain that in order to complete the proof it suffices to show that

$$\liminf_{\ell\to\infty}\int_{\mathbb{T}^d}\int\big|\bar{\Psi}^\ell-\widetilde{\Psi}\big(\rho(u)\big)\big|d\nu^\infty_{\rho(u)}du=0. \tag{2.4}$$

By the bounded convergence theorem again, it follows that in order to prove (2.4) it suffices to show that for all $u\in\mathbb{T}^d$,

$$\liminf_{\ell\to\infty}\int\bigg|\frac{1}{(2\ell+1)^d}\sum_{|y|\le\ell}\tau_y\Psi-\widetilde{\Psi}\big(\rho(u)\big)\bigg|d\nu^\infty_{\rho(u)}=0. \tag{2.5}$$

As we will see next this follows by the law of large numbers. Of course since $\nu^\infty_{\rho(u)}$ is translation invariant, the functions $\{\tau_y\Psi\}_{y\in\mathbb{Z}^d}$ are equidistributed under $\nu^\infty_{\rho(u)}$ with common mean value

$$\mathbb{E}_{\nu^\infty_{\rho(u)}}\big(\tau_x\Psi\big)=\mathbb{E}_{\nu^\infty_{\rho(u)}}\Psi=\widetilde{\Psi}\big(\rho(u)\big),$$

and so if Ψ was a function of one coordinate, i.e. if it was of the form $\Psi(\eta)=\psi(\eta(z_0))$ for some $z_0\in\mathbb{Z}^d$ the limit above would be exactly the law of the large numbers. In general, as a cylinder function, Ψ will depend on a finite number of coordinates and so for $y_1,y_2\in\mathbb{Z}^d$ that are a sufficiently large distance apart the functions $\tau_{y_1}\Psi$ and $\tau_{y_2}\Psi$ will be independent. It is well known that one can use this fact to apply the law of large numbers in this more general case, as follows. Since Ψ is a cylinder function, there exists $\ell_0\in\mathbb{Z}_+$ such that Ψ is of the form $\Psi=\psi\circ\pi^{\ell_0}$, where for each $\ell\in\mathbb{Z}_+$ we denote by $\pi^\ell:\mathbb{Z}^d\longrightarrow\mathbb{Z}_+^{\Lambda^d_\ell}\cong\mathbb{T}^d_{2\ell+1}$ the natural projection. Then whenever $|x-y|>2\ell_0+1$ the functions $\tau_x\Psi$ and $\tau_y\Psi$ are independent. For each $\ell\in\mathbb{Z}_+$ we set $\ell_\star:=2\ell+1$ and with this notation in order to complete the proof it suffices to show that for all $u\in\mathbb{T}^d$,

$$\lim_{k\to\infty}\frac{1}{\big(\ell_0+k(2\ell_0+1)\big)^d_\star}\sum_{|y|\le\ell_0+k(2\ell_0+1)}\tau_y\Psi=\widetilde{\Psi}\big(\rho(u)\big)\quad\text{in }\ L^1(\nu^\infty_{\rho(u)}).$$

Note that

$$\big(\ell_0+k(2\ell_0+1)\big)^d_\star=(\ell_0)^d_\star k^d_\star \tag{2.6}$$

and so the ball $\Lambda^d_{\ell_0+k(2\ell_0+1)}$ is the union of $(2k+1)^d$ balls of radius ℓ_0. So if we set

$$L^k_y:=\{z\in\Lambda^d_{\ell_0+k(2\ell_0+1)}|z-y\in(2\ell_0+1)\mathbb{Z}^d\}$$

for each $y \in \Lambda^d_{\ell_0}$, then $\sharp L^k_y = k^d_\star$ for each $y \in \Lambda^d_{\ell_0}$ and

$$\Lambda^d_{\ell_0+k(2\ell_0+1)} = \coprod_{y\in\Lambda^d_{\ell_0}} L^k_y.$$

So the sum of all the $\tau_y\Psi$'s over all $y \in \Lambda^d_{\ell_0+k(2\ell_0+1)}$ can be written as

$$\sum_{|y|\leq \ell_0+k(2\ell_0+1)} \tau_y\Psi = \sum_{|y|\leq\ell_0}\sum_{z\in L^k_y} \tau_z\Psi.$$

By definition L^k_y has $(2k+1)^d$ elements and the family $\{\tau_z\Psi\}_{z\in L^k_y}$ is obviously i.i.d. and therefore by the L^2-weak law of large numbers and (2.6) we get

$$\frac{1}{(\ell_0+k(2\ell_0+1))^d_\star}\sum_{|y|\leq \ell_0+k(2\ell_0+1)} \tau_y\Psi = \frac{1}{(\ell_0)^d_\star}\sum_{|y|\leq\ell_0}\frac{1}{k^d_\star}\sum_{z\in L^k_y} \tau_z\Psi$$

which converges in $L^2(\nu_{\rho(u)})$ as $k\to\infty$ to

$$\frac{1}{(\ell_0)^d_\star}\sum_{|y|\leq\ell_0}\widetilde{\Psi}\big(\rho(u)\big) = \widetilde{\Psi}\big(\rho(u)\big).$$

This proves (2.5) as required and completes the proof. □

In the literature the notion of weak local equilibrium is usually defined is a seemingly stronger (but in fact equivalent) form. To state it in a more compact way will use the following notation. For any bounded cylinder function $\psi : \mathbb{M}^d_\infty \longrightarrow \mathbb{R}$ we denote by $\pi^{N,\Psi}$ the random measure $\pi^{N,\Psi} : \mathbb{M}^d_N \longrightarrow \mathcal{M}_+(\mathbb{T}^d)$ given by $\pi^{N,\Psi} = \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}\tau_x\Psi\delta_{\frac{x}{N}}$. The following equivalent characterization of weak local equilibrium holds.

Proposition 2.2.4 *Let $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ be a measurable function. A sequence $\mu^N \in \mathbb{PM}^d_N$, $N\in\mathbb{N}$ is weak local equilibrium of profile ρ iff for all bounded cylinder functions $\Psi : \mathbb{M}^d_\infty \longrightarrow \mathbb{R}$ and all $G \in C(\mathbb{T}^d)$ it holds that*

$$\lim_{N\to\infty}\mu^N\Big\{\Big|\int G(u)d\pi^{N,\Psi}(u) - \int_{\mathbb{T}^d} G(u)\widetilde{\Psi}\big(\rho(u)\big)du\Big| > \delta\Big\} = 0, \quad \forall\,\delta>0.$$

Proof The one implication is obvious, the condition of this proposition of the constant function $G\equiv 1$ is exactly the condition in the definition of weak local equilibrium. For the other implication let $G \in C(\mathbb{T}^d)$, $\Psi \in B_{\rm cyl}(\mathbb{M}^d_\infty)$ and $\delta > 0$. Of course, if G is identically equal to zero we have nothing to show and so we assume that $\|G\|_\infty > 0$. Note that

$$\langle G, \pi^{N,\Psi}\rangle := \int G(u)d\pi^{N,\Psi}(u) = \int_{\mathbb{T}^d} G\Big(\frac{[Nu]}{N}\Big)\tau_{[Nu]}\Psi du$$

for any $G\in C(\mathbb{T}^d)$ and any $\Psi \in B_{\rm cyl}(\mathbb{M}^d_\infty)$ and therefore by adding and subtracting the function $u \mapsto G(u)\tau_{[Nu]}\Psi$ inside the integrand we get

$$\begin{aligned}\big|\langle G,\pi^{N,\Psi}\rangle - \langle G, \widetilde{\Psi}(\rho)dm_{\mathbb{T}^d}\rangle\big| &= \Big|\int_{\mathbb{T}^d}\Big[G\Big(\frac{[Nu]}{N}\Big)\tau_{[Nu]}\Psi - G(u)\widetilde{\Psi}\big(\rho(u)\big)\Big]du\Big| \\ &\leq \|\Psi\|_\infty\int_{\mathbb{T}^d}\Big|G\Big(\frac{[Nu]}{N}\Big) - G(u)\Big|du \\ &\quad +\|G\|_\infty\Big|\bar{\Psi}^N - \int_{\mathbb{T}^d}\widetilde{\Psi}\big(\rho(u)\big)du\Big|.\end{aligned}$$

By the continuity of G and the bounded convergence theorem we have that

$$\lim_{N\to\infty}\int_{\mathbb{T}^d}\Big|G\Big(\frac{[Nu]}{N}\Big)-G(u)\Big|du=0.$$

Furthermore, this term is deterministic, i.e. independent of $\eta\in\mathbb{M}^d_\infty$, and therefore for large enough $N\in\mathbb{N}$ we have that

$$\mu_N\big\{\big|\langle G,\pi^{N,\Psi}\rangle-\langle G,\widetilde{\Psi}(\rho)dm_{\mathbb{T}^d}\rangle\big|>\delta\big\}\le\mu_N\bigg\{\bigg|\bar{\Psi}^N-\int_{\mathbb{T}^d}\widetilde{\Psi}\big(\rho(u)\big)du\bigg|>\frac{\delta}{2\|G\|_\infty}\bigg\}$$

which tends to zero as $N\to\infty$ since $\{\mu_N\}$ is a weak local equilibrium. □

It is useful to know to what extent the profile of a weak local equilibrium is unique.

Proposition 2.2.5 *Let $\rho_1,\rho_2\in L^1_+(\mathbb{T}^d)$ be profiles and let $\mu^N\in\mathbb{PM}^d_N$, $N\in\mathbb{N}$. If the sequence $\{\mu^N\}$ is both a weak local equilibrium of profile ρ_1 and of profile ρ_2, then*

$$\rho_1\wedge\rho_c=\rho_2\wedge\rho_c\quad a.s.\ \ in\ \mathbb{T}^d.$$

Proof Let $G\in C(\mathbb{T}^d)$ and for each $K\in\mathbb{N}$ set $\Psi_K:=\eta(0)\wedge K$. Since $\{\mu^N\}$ is a weak local equilibrium of profile ρ_i, $i=1,2$, for each $\delta>0$ there exists $N_\delta=N_\delta(G)\in\mathbb{N}$ such that

$$N\ge N_\delta\quad\Longrightarrow\quad\mu_N\Big\{\big|\langle G,\pi^{N,\Psi_K}\rangle-\langle G,\widetilde{\Psi_K}(\rho_i)dm_{\mathbb{T}^d}\rangle\big|>\frac{\delta}{2}\Big\}<\frac{1}{4},$$

for $i=1,2$. Then,

$$\begin{aligned}\mu_{N_m}&\Big\{\big|\langle G,\widetilde{\Psi_K}(\rho_1)dm_{\mathbb{T}^d}\rangle-\langle G,\widetilde{\Psi_K}(\rho_2)dm_{\mathbb{T}^d}\rangle\big|>\delta\Big\}\\&\le\sum_{i=1,2}\mu_{N_m}\Big\{\big|\langle G,\pi^{N,\Psi_K}\rangle-\langle G,\widetilde{\Psi_K}(\rho_i)dm_{\mathbb{T}^d}\rangle\big|>\frac{\delta}{2}\Big\}<\frac{1}{2}.\end{aligned}$$

Since the event $\{|\langle G,\widetilde{\Psi_K}(\rho_1)dm_{\mathbb{T}^d}\rangle-\langle G,\widetilde{\Psi_K}(\rho_2)dm_{\mathbb{T}^d}\rangle|>\delta\}$ is deterministic, this implies that $|\langle G,\widetilde{\Psi_K}(\rho_1)dm_{\mathbb{T}^d}\rangle-\langle G,\widetilde{\Psi_K}(\rho_2)dm_{\mathbb{T}^d}\rangle|\le\delta$ and since $\delta>0$ was arbitrary we have in fact that

$$\langle G,\widetilde{\Psi_K}(\rho_1)dm_{\mathbb{T}^d}\rangle=\langle G,\widetilde{\Psi_K}(\rho_2)dm_{\mathbb{T}^d}\rangle.$$

Since $G\in C(\mathbb{T}^d)$ was also arbitrary this holds for all $G\in C(\mathbb{T}^d)$, which implies that

$$\widetilde{\Psi_K}(\rho_1)=\widetilde{\Psi_K}(\rho_2)\quad\text{a.s. in}\ \ \mathbb{T}^d$$

for all $K\in\mathbb{N}$. So the set

$$E:=\bigcap_{K=1}^\infty\big\{\widetilde{\Psi_K}(\rho_1)=\widetilde{\Psi_K}(\rho_2)\big\}$$

is of full measure in $\mathbb{T}^d$, i.e. $m(E)=1$, and

$$\widetilde{\Psi_K}\big(\rho_1(u)\big)=\widetilde{\Psi_K}\big(\rho_2(u)\big)\quad\text{for all}\ u\in E,\ K\in\mathbb{N}.\tag{2.7}$$

But for each $\rho\in[0,\infty)$ the monotone convergence theorem that

$$\lim_{K\to\infty}\widetilde{\Psi_K}(\rho)=\lim_{K\to\infty}\int\eta(0)\wedge K d\nu^1_{\rho\wedge\rho_c}=\int\eta(0)d\nu^1_{\rho\wedge\rho_c}=\rho\wedge\rho_c,$$

and so taking the limit as $K\to\infty$ in (2.7) yields that $\rho_1(u)\wedge\rho_c=\rho_2(u)\wedge\rho_c$ for all $u\in E$, which completes the proof since E is of full measure in $\mathbb{T}^d$. □

2.3 Local Equilibrium in the Entropy Sense

Definition 2.3.1 Let $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}$ be a measurable function. A sequence $\mu^N \in \mathbb{PM}_N^d$ is an *entropy local equilibrium of profile* ρ if

$$\lim_{N\to\infty} \frac{1}{N^d} H(\mu^N | \nu^N_{\rho(\cdot)}) = 0.$$

According to the proof of corollary 1.3 in [25] the notion of entropy local equilibrium is stronger than the notion of weak local equilibrium.

Proposition 2.3.1 *Let $\mu^N \in \mathbb{PM}_N^d$ be a sequence of probability measures. If $\{\mu^N\}$ is a local equilibrium with respect to a continuous profile $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ in the entropy sense then it is also a weak local equilibrium of profile ρ.*

Proof Let $H \in C(\mathbb{T}^d)$. By Chebyshev's inequality it suffices to prove that

$$\lim_{N\to\infty} \int \Big| \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} H\Big(\frac{x}{N}\Big)\tau_x\Psi - \int H(u)\widetilde{\Psi}\big(\rho(u)\big)du \Big| d\mu_N = 0 \tag{2.8}$$

for all $\Psi \in \mathrm{Cyl}_b(\mathbb{M}^d_\infty)$. Since ρ is assumed continuous, we have that

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} H\Big(\frac{x}{N}\Big)\widetilde{\Psi}\Big(\rho\Big(\frac{x}{N}\Big)\Big) = \int_{\mathbb{T}^d} H(u)\widetilde{\Psi}\big(\rho(u)\big)du$$

and therefore in order to prove (2.8) it suffices to prove that

$$\limsup_{N\to\infty} \int \Big| \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} H\Big(\frac{x}{N}\Big)\Big[\tau_x\Psi - \widetilde{\Psi}\Big(\rho\Big(\frac{x}{N}\Big)\Big)\Big]\Big| d\mu_N = 0 \tag{2.9}$$

for all $\Psi \in \mathrm{Cyl}_b(\mathbb{M}^d_\infty)$. We recall that for each $\Psi \in \mathrm{Cyl}_b(\mathbb{M}^d_\infty)$ and each $\ell \in \mathbb{Z}_+$ we denote by $\bar{\Psi}^\ell$ the local spatial mean of Ψ of radius ℓ. Then obviously for each $x \in \mathbb{T}_N^d$ we have that $\overline{(\tau_x\Psi)}^\ell = \tau_x(\bar{\Psi}^\ell)$. It is easy to see by a summation by parts that for all $\ell \in \mathbb{Z}_+$ we have

$$\limsup_{N\to\infty} \int \Big| \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} H\Big(\frac{x}{N}\Big)\big[\tau_x\Psi - \tau_x\Psi^\ell\big]\Big| d\mu_N = 0$$

and therefore in order to show the claim it suffices to prove that

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty} \int \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \Big|\tau_x\Psi^\ell - \widetilde{\Psi}\Big(\rho\Big(\frac{x}{N}\Big)\Big)\Big| d\mu_N = 0. \tag{2.10}$$

and $\Psi \in \mathrm{Cyl}_b(\mathbb{M}^d_\infty)$. By the relative entropy inequality, for every $\gamma > 0$ the integral in (2.8) is bounded above by

$$\frac{1}{\gamma N^d} H(\mu_N|\nu^N_{\rho(\cdot)}) + \frac{1}{\gamma N^d}\log \int e^{\gamma \sum_{x\in\mathbb{T}_N^d} |\tau_x\Psi^\ell - \widetilde{\Psi}(\rho(x/N))|} d\nu^N_{\rho(\cdot)}.$$

The first term tends to zero as $N \to \infty$ since $\{\mu_N\}$ is a local equilibrium in the entropy sense. So we have to show that for each $\gamma > 0$, the second term in the sum above which we will denote by $J_\gamma(N,\ell)$, vanishes as $N \to \infty$ and then $\ell \to \infty$.

For simplicity we assume in what follows that $\ell_\Psi = 0$, i.e. that $\Psi(\eta) = \overline{\Psi}(\eta(0))$ for some bounded function $\overline{\Psi} : \mathbb{Z}_+ \longrightarrow \mathbb{R}$. Then Ψ^ℓ is a function of the random vector $\big(\eta(x)\big)_{x\in\Lambda_\ell^d}$ and so the random variables $\tau_x\Psi^\ell$ and $\tau_y\Psi^\ell$ are independent whenever $|x-y| \geq 2\ell+1$. We will take advantage of this fact to decompose the expectation in the term $J^\gamma(N,\ell)$ to simpler terms. We assume without loss of generality that $2\ell+1$ divides N and set

$$G(u,\lambda) \equiv:= \big|\lambda - \widetilde{\Psi}\big(\rho(u)\big)\big|, \qquad (u,\lambda) \in \mathbb{T}^d \times \mathbb{R}_+.$$

Then the sum in the exponential can be written as

$$\sum_{x\in\mathbb{T}_N^d} G\Big(\frac{x}{N}, \tau_x\Psi^\ell\Big) = \sum_{x\in\Lambda_\ell^d}\sum_{y\in E_x^{N,\ell}} G\Big(\frac{x+(2\ell+1)y}{N}, \tau_{x+(2\ell+1)y}\Psi^\ell\Big)$$

where we have set $E_x^{N,\ell} := \{y \in \mathbb{T}_N^d : x + (2\ell+1)y \in \mathbb{T}_N^d\}$. Then by the generalized Holder inequality and the independence for each fixed $x \in \mathbb{T}_N^d$ of the random variables $\tau_{x+(2\ell+1)y}\Psi^\ell$, $y \in E_x^{N,\ell}$, we have that

$$\begin{aligned}
J^\gamma(N,\ell) &= \frac{1}{\gamma N^d}\log\int\prod_{x\in\Lambda_\ell^d} e^{\gamma\sum_{y\in E_x^{N,\ell}} G(\frac{x+(2\ell+1)y}{N},\tau_{x+(2\ell+1)y}\Psi^\ell)}d\nu_{\rho(\cdot)}^N\\
&\leq \frac{1}{\gamma N^d}\log\prod_{x\in\Lambda_\ell^d}\bigg(\int e^{\gamma(2\ell+1)^d\sum_{y\in E_x^{N,\ell}} G(\frac{x+(2\ell+1)y}{N},\tau_{x+(2\ell+1)y}\Psi^\ell)}d\nu_{\rho(\cdot)}^N\bigg)^{\frac{1}{(2\ell+1)^d}}\\
&= \frac{1}{\gamma(2\ell+1)^dN^d}\sum_{x\in\Lambda_\ell^d}\log\int e^{\gamma(2\ell+1)^d\sum_{y\in E_x^{N,\ell}} G(\frac{x+(2\ell+1)y}{N},\tau_{x+(2\ell+1)y}\Psi^\ell)}d\nu_{\rho(\cdot)}^N\\
&= \frac{1}{\gamma(2\ell+1)^dN^d}\sum_{x\in\Lambda_\ell^d}\sum_{y\in E_x^{N,\ell}}\log\int e^{\gamma(2\ell+1)^d G(\frac{x+(2\ell+1)y}{N},\tau_{x+(2\ell+1)y}\Psi^\ell)}d\nu_{\rho(\cdot)}^N\\
&= \frac{1}{\gamma(2\ell+1)^dN^d}\sum_{x\in\mathbb{T}_N^d}\log\int e^{\gamma(2\ell+1)^d|\tau_x\Psi^\ell-\widetilde{\Psi}(\rho(x/N))|}d\nu_{\rho(\cdot)}^N\\
&= \frac{1}{\gamma(2\ell+1)^dN^d}\sum_{x\in\mathbb{T}_N^d}\log\int e^{\gamma(2\ell+1)^d|\Psi^\ell-\widetilde{\Psi}(\rho(x/N))|}d\tau_x\nu_{\rho(\cdot)}^N\\
&= \frac{1}{\gamma(2\ell+1)^d}\int_{\mathbb{T}^d}\log\int e^{\gamma(2\ell+1)^d|\Psi^\ell-\widetilde{\Psi}(\rho([Nu]/N))|}d\tau_{[Nu]}\nu_{\rho(\cdot)}^N du.
\end{aligned}$$

Now, since the profile ρ is continuous, the sequence $\nu_{\rho(\cdot)}^N$ is a local equilibrium of profile ρ, that is for each macroscopic point $u \in \mathbb{T}^d$ we have that $\tau_{[Nu]}\nu_{\rho(\cdot)}^N \longrightarrow \nu_{\rho(u)}^\infty$ weakly in $\mathbb{PM}_\infty^d$, and therefore the upper bound for $J_\gamma(N,\ell)$ given above converges as $N \to \infty$ to

$$\frac{1}{\gamma(2\ell+1)^d}\int_{\mathbb{T}^d}\log\int e^{\gamma(2\ell+1)^d|\Psi^\ell-\widetilde{\Psi}(\rho(u))|}d\nu_{\rho(u)}^\infty du.$$

Now, since Ψ is bounded, by the elementary inequality $e^x \leq 1 + x + \frac{1}{2}x^2e^{|x|}$ we get

$$e^{\gamma\ell_\star^d|\Psi^\ell-\widetilde{\Psi}(\rho(u))|} \leq 1 + \gamma\ell_\star^d|\Psi^\ell - \widetilde{\Psi}(\rho(u))| + 2\gamma^2\ell_\star^{2d}\|\Psi\|_\infty^2 e^{2\gamma\ell_\star^d\|\Psi\|_\infty},$$

where as usual we have set $\ell_\star = 2\ell+1$. Then by the elementary inequality $\log(1+x) \le x$ we get

$$\log \int e^{\gamma \ell_\star^d |\Psi^\ell - \widetilde{\Psi}(\rho(u))|} d\nu_{\rho(u)}^\infty \le \gamma \ell_\star^d \int |\Psi^\ell - \widetilde{\Psi}(\rho(u))| d\nu_{\rho(u)}^\infty + 2\gamma^2 \ell_\star^{2d} \|\Psi\|_\infty^2 e^{2\gamma \ell_\star^d \|\Psi\|_\infty}.$$

It follows that for all $\gamma > 0$ and all $\ell \in \mathbb{Z}_+$,

$$\limsup_{N\to\infty} J_\gamma(N,\ell) \le \int_{\mathbb{T}^d} \int |\Psi^\ell - \widetilde{\Psi}(\rho(u))| d\nu_{\rho(u)}^\infty du + 2\gamma \ell_\star^d \|\Psi\|_\infty^2 e^{2\gamma \ell_\star^d \|\Psi\|_\infty}.$$

Applying this for each $\ell \in \mathbb{Z}_+$ with $\gamma = \theta(2\ell+1)^{-d}$, for arbitrary $\theta > 0$, we get that

$$\limsup_{N\to\infty} J_\gamma(N,\ell) \le \int_{\mathbb{T}^d} \int |\Psi^\ell - \widetilde{\Psi}(\rho(u))| d\nu_{\rho(u)}^\infty du + 2\theta \|\Psi\|_\infty^2 e^{2\theta \|\Psi\|_\infty}.$$

Then by the law of the large numbers and the bounded convergence theorem it follows that

$$\limsup_{\ell\to\infty} \limsup_{N\to\infty} J_\gamma(N,\ell) \le 2\theta \|\Psi\|_\infty^2 e^{2\theta \|\Psi\|_\infty},$$

which since θ can be taken arbitrarily close to 0, proves that

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} J_\gamma(N,\ell) = 0$$

as required and completes the proof. □

2.4 Weak Local Equilibrium of First Order

The notions of strong and weak local equilibrium can be also considered by using the w_p topologies in the spaces $\mathbb{P}_p\mathbb{M}_\infty^d$ and the space of cylinder functions of polynomial growth considered in subsection 1.3.2 respectively. We do so in this chapter in the case $p = 1$ which is the most interesting due to its connection with the conserved quantity, the density.

In all of this section $\{\nu_\rho^N\}_{\rho\in\mathbb{R}_+} \equiv \{\nu_{\rho\wedge\rho_c}^N\}_{\rho\in\mathbb{R}_+}$ will be the grand canonical ensemble of a ZRP on the discrete torus $\mathbb{T}_N^d$, $N \in \mathbb{N}$. Is we have already seen, the family $\{\nu_\rho^N\}_{\rho\in\mathbb{R}_+}$ is weakly continuous and since it is parametrized by the density,

$$\int \eta(x) d\nu_{\rho\wedge\rho_c} d\nu_\rho^N = \rho \wedge \rho_c$$

for all $\rho \in \mathbb{R}_+$. Consequently the family $\{\nu_\rho^N\}_{\rho\in\mathbb{R}_+}$ is always w_1-continuous.

Definition 2.4.1 Let $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ be any function. A sequence $\{\mu^N \in \mathbb{P}_1\mathbb{M}_N^d\}$ is called a *w_1-strong local equilibrium* if

$$\tau_{[Nu]}\mu^N \overset{N\to\infty}{\longrightarrow} \nu^\infty_{\rho(u)\wedge\rho_c}$$

in the w_1 topology of $\mathbb{P}_1\mathbb{M}_\infty^d$ for every continuity point $u \in \mathbb{T}^d$ of ρ.

In complete analogy to proposition 2.1.1 we have the following

Proposition 2.4.1 *Given any function $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$, the family $\{\nu^N_{\rho(\cdot)}\}$ of distributions with slowly varying parameter of profile ρ is a w_1-strong local equilibrium.*

Definition 2.4.2 Let $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ measurable. A sequence $\{\mu^N \in \mathbb{P}_1\mathbb{M}^d_N\}$ is *a state of w_1-local equilibrium of profile ρ in spatial mean* if

$$\frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}\tau_{x*}\mu^N = \int_{\mathbb{T}^d}\tau_{[Nu]}\mu^N du \longrightarrow \int_{\mathbb{T}^d}\nu^\infty_{\rho(u)}du$$

in the w_1 topology of $\mathbb{P}_1\mathbb{M}^d_\infty$.

We note that for any integrable profile $\rho \in L^1(\mathbb{T}^d)$ we have for each $x \in \mathbb{Z}^d$ that

$$\Big\langle \eta(x), \int_{\mathbb{T}^d}\nu^\infty_{\rho(u)}du\Big\rangle = \int \mathbb{E}_{\nu^\infty_{\rho(u)}}(\eta(x))du = \int_{\mathbb{T}^d}\rho(u)\wedge\rho_c du < +\infty$$

and thus

$$\bar{\nu}_{\rho(\cdot)} := \int_{\mathbb{T}^d}\nu^\infty_{\rho(u)}du \in \mathbb{P}_1\mathbb{M}^d_\infty.$$

As in the case of bounded cylinder functions given any cylinder function $\Psi \in \mathrm{Cyl}^d_1$ we will denote by

$$\widetilde{\Psi}(\rho) := \int \Psi d\nu^\infty_{\rho\wedge\rho_c}$$

the expected value of Ψ over $\nu^\infty_{\rho\wedge\rho_c}$, $\rho \in \mathbb{R}_+$. Since Ψ is in Cyl^d_1 there exist $\ell_0 \in \mathbb{Z}_+$ and a constant $C \geq 0$ such that $|\Psi| \leq C\big(1 + \sum_{|x|\leq\ell_0}\eta(x)\big)$ and therefore

$$|\widetilde{\Psi}| \leq \int |\Psi| d\nu^\infty_\rho \leq C + C\sum_{|x|\leq\ell_0}\int \eta(x)d\nu^\infty_\rho = C + C(2\ell_0+1)^d\rho\wedge\rho_c. \quad (2.11)$$

So $\widetilde{\Psi} \in B_1C(\mathbb{R}_+)$ if $\rho_c = +\infty$ and $\widetilde{\Psi} \in BC(\mathbb{R}_+)$ if $\rho_c < +\infty$.

Proposition 2.4.2 *Let $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ be an a.s. continuous and integrable profile. If $\{\mu^N\}$ is a w_1-strong local equilibrium of profile ρ and satisfies in addition the uniform integrability condition*

$$\lim_{M\to\infty}\lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N:\mathbb{E}_{\mu^N}[\eta(x)]>M}\int \eta(x)d\mu^N = 0, \quad (2.12)$$

then $\{\mu^N\}$ is w_1-local equilibrium of profile ρ in spatial mean.

Proof By proposition 2.2.1 we know that

$$\bar{\mu}^N := \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}\tau_{x*}\mu^N \longrightarrow \bar{\nu}_{\rho(\cdot)}$$

in the weak topology. Taking into account that the measures $\bar{\mu}^N$ and $\bar{\nu}_{\rho(\cdot)}$ both have equidistributed one-site marginals in order to prove that the convergence above holds in the w_1 topology it suffices to show that

$$\lim_{N\to\infty}\int \eta(0)d\bar{\mu}^N = \int \eta(0)d\bar{\nu}_{\rho(\cdot)} \equiv \int_{\mathbb{T}^d}\rho(u)\wedge\rho_c du.$$

Considering the function $h^N : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ given by

$$h^N(u) := \int \eta([Nu])d\mu^N$$

we have

$$\int \eta(0)d\bar{\mu}^N = \int_{\mathbb{T}^d} h^N(u)du$$

and by the assumption that $\{\mu^N\}$ is a w_1-strong local equilibrium and the profile ρ a.s. continuous the function h^N converges a.s. to

$$\int \eta(0)d\nu^{\infty}_{\rho(u)} = \rho(u) \wedge \rho_c.$$

Furthermore, since

$$\{h^N > M\} = \bigcup_{x\in\mathbb{T}^d_N:\mathbb{E}_{\mu^N}[\eta(x)]>M} \Big[\frac{x}{N}, \frac{x+1}{N}\Big)$$

we also have that

$$\int_{\{h^N>M\}} h^N(u)d\mu^N = \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N:\mathbb{E}_{\mu^N}[\eta(x)]>M} \int \eta(x)d\mu^N$$

and so the uniform integrability condition (2.12) is exactly the uniform integrability of the sequence $\{h^N\}_{N\in\mathbb{N}}$. Consequently the claim follows by the Vitalli convergence theorem. □

It is worth noting that in the case that $\{\mu^N\}$ is the sequence of distributions with slowly varying parameter of some profile ρ then the uniform integrability condition (2.12) reduces to the uniform integrability in $L^1(\mathbb{T}^d)$ of the sequence $\{\rho([N\cdot]/N)\}_{N\in\mathbb{N}}$ is automatically satisfied whenever the $\{\mu^N\}$ is the sequence $\{\nu^N_{\rho(\cdot)}\}$ of distributions with slowly varying parameter of some a.s. continuous and bounded profile.

Definition 2.4.3 Let $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ be measurable. A sequence $\{\mu^N \in \mathbb{P})p\mathbb{M}^d_N\}$ is a w_1-weak local equilibrium of profile ρ if for any $\Psi \in \mathrm{Cyl}^d_1$ and any $\delta > 0$ we have that

$$\lim_{N\to\infty} \mu^N \bigg\{ \bigg| \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \tau_x\Psi - \int_{\mathbb{T}^d} \widetilde{\Psi}\big(\rho(u)\big)du \bigg| > \delta \bigg\} = 0.$$

As in the case of weak local equilibrium, the notion of w_1-weak local equilibrium can be stated in the following seemingly stronger way.

Proposition 2.4.3 *A sequence $\{\mu^N\}$ is a w_1-weak local equilibrium of measurable profile ρ iff for any $G \in C(\mathbb{T}^d)$, $\Psi \in \mathrm{Cyl}^d_1$ and $\delta > 0$,*

$$\lim_{N\to\infty} \bigg\{ \bigg| \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} G\Big(\frac{x}{N}\Big)\tau_x\Psi - \int_{\mathbb{T}^d} G(u)\widetilde{\Psi}\big(\rho(u)\big)du \bigg| \bigg\} = 0.$$

Proof The idea is similar to the one in the proof of proposition 2.2.4 and we omit the details. □

Proposition 2.4.4 *Let $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ be an a.s. continuous profile such that the sequence $\{h^N\} := \{\rho(\frac{[N\cdot]}{N}) \wedge \rho_c\}_{N\in\mathbb{N}}$ is uniformly integrable in $L^1(\mathbb{T}^d)$. If a sequence $\{\mu^N\}$ is w_1-strong local equilibrium of profile ρ_0 and satisfies the uniform integrability condition (2.12) then $\{\mu^N\}$ is also a w_1-weak local equilibrium of profile ρ.*

Proof Setting $\overline{\Psi}^N := \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x\Psi$, by Chebyshev's inequality it suffices to prove that

$$\lim_{N\to\infty}\int\Big|\overline{\Psi}^N - \int_{\mathbb{T}^d}\widetilde{\Psi}\big(\rho(u)\big)du\Big| = 0$$

for all $\Psi \in \mathrm{Cyl}_1^d$. So let $\Psi \in \mathrm{Cyl}_1^d$ be given. Since ρ is assumed a.s. continuous and uniformly integrable by Vitalli's convergence theorem we have that $h^N \longrightarrow \rho\wedge\rho_c$ in $L^1(\mathbb{T}^d)$. By (2.11) there exists a constant $C = C(\Psi) \geq 0$ such that

$$|\widetilde{\Psi}| \leq C(1+\rho\wedge\rho_c)$$

and so if we set $h_\Psi^N := \widetilde{\Psi}\big(\rho([Nu]/N)\big)$, $u \in \mathbb{T}^d$, $N \in \mathbb{N}$, then h_Ψ^N is dominated by $C(1+h^N)$ which is uniformly integrable. So $\{h_\Psi^N\}_{N\in\mathbb{N}}$ is uniformly integrable and since obviously $h_\Psi^N \longrightarrow \wr\Psi(\rho)$ we have by that $h_\Psi^N \longrightarrow \widetilde{\Psi}(\rho)$ in $L^1(\mathbb{T}^d)$. In particular

$$\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\widetilde{\Psi}\Big(\rho\Big(\frac{x}{N}\Big)\Big) = \int_{\mathbb{T}^d}h_\Psi^N(u)du \overset{N\to\infty}{\int}_{\mathbb{T}^d}\widetilde{\Psi}(\rho),$$

and as in proposition 2.2.3, by (2.2) to complete the proof it suffices to prove that

$$\liminf_{\ell\to\infty}\lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\int\Big|\tau\Psi^\ell - \widetilde{\Psi}\Big(\rho\Big(\frac{x}{N}\Big)\Big)\Big|d\mu^N = 0,$$

where $\Psi^\ell := \frac{1}{(2\ell+1)^d}\sum_{|y|\leq\ell}\tau_y\Psi$.

For this we consider the function $h^{N,\ell} : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ given by

$$g^{N,\ell}(u) = \sum_{x\in\mathbb{T}_N^d}\int\Big|\tau\Psi^\ell - \widetilde{\Psi}\Big(\rho\Big(\frac{x}{N}\Big)\Big)\Big|d\mu^N\cdot\mathbb{1}_{[\frac{x}{N},\frac{x+1}{N})}(u).$$

Since $\{\mu^N\}$ is a w_1-strong local equilibrium, $\Psi^\ell \in \mathrm{Cyl}_1^d$ and $h_\Psi^N \longrightarrow \widetilde{\Psi}(\rho)$ a.s. we have that

$$g^{N,\ell}(u) = \int\Big|\Psi^\ell - \widetilde{\Psi}\Big(\rho\Big(\frac{[Nu]}{N}\Big)\Big)\Big|d\tau_{[Nu]*}\mu^N \longrightarrow \int\big|\Psi^\ell - \widetilde{\Psi}\big(\rho(u)\big)\big|d\nu_{\rho(u)}^\infty$$

for almost all $u \in \mathbb{T}^d$. So if we show that $\{g^{N,\ell}\}_{N\in\mathbb{N}}$ is uniformly integrable for each fixed $\ell \in \mathbb{Z}_+$ we will obtain that the above convergence is in $L^1(\mathbb{T}^d)$, and in particular that

$$\int_{\mathbb{T}^d}g^{N,\ell} = \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\int\big|\tau_x\Psi^\ell - \widetilde{\Psi}\big(\rho(u)\big)\big|d\mu^N \overset{N\to\infty}{\longrightarrow}\int_{\mathbb{T}^d}\int\big|\Psi^\ell - \widetilde{\Psi}\big(\rho(u)\big)\big|d\nu_{\rho(u)}^\infty du.$$

But
$$\liminf_{\ell\to\infty}\int_{\mathbb{T}^d}\int\big|\Psi^\ell-\widetilde{\Psi}\big(\rho(u)\big)\big|d\nu^\infty_{\rho(u)}du$$
by the law of large numbers, the fact that the function $u\mapsto\int\big|\Psi^\ell-\widetilde{\Psi}\big(\rho(u)\big)\big|d\nu^\infty_{\rho(u)}$ is dominated by the function $2\widetilde{\Psi}(\rho)\in L^1(\mathbb{T}^d)$ and the dominated convergence theorem.

So in order to complete the proof it remains to show that $\{g^{N,\ell}\}_{N\in\mathbb{N}}$ is uniformly integrable. As we will see this follows by the uniform integrability condition (2.12). First, $g^{N,\ell}$ is bounded above by
$$\int|\Psi^\ell|d\tau_{[Nu]*}\mu^N+\widetilde{\Psi}\Big(\rho\Big(\frac{[Nu]}{N}\Big)\Big),$$
the second function in the right hand side is uniformly integrable and the first function in the right hand side is bounded above by
$$\begin{aligned}|\Psi^\ell| &\le \frac{1}{(2\ell+1)^d}\sum_{|y|\le\ell}\tau_y|\Psi|\le\frac{C}{(2\ell+1)^d}\sum_{|y|\le\ell}\Big(1+\sum_{|x|\le\ell_0}\eta(x+y)\Big)\\ &\le C_\ell\Big(1+\frac{1}{(2m_\ell+1)^d}\sum_{|z|\le m_\ell}\eta(z)\Big)\end{aligned}$$
for some constants $C_\ell=C_\ell(\Psi)$ and $m_\ell=m_\ell(\Psi)$ depending on $\ell\in\mathbb{Z}_+$ and Ψ. Consequently, in order to complete the proof it suffices to show that the sequence $\{f^{N,\ell}\}_{N\in\mathbb{N}}$ defined by
$$f^{N,\ell}(u)=\frac{1}{(2m_\ell+1)^d}\sum_{|y|\le m_\ell}\int\eta([Nu]+y)d\mu^N$$
is uniformly integrable for each fixed $\ell\in\mathbb{Z}_+$.

For simplicity in the notation and since ℓ is fixed we will write $f^N=f^{N,\ell}$ and $m=m_\ell$ in the proof of the uniform integrability of $\{f^{N,\ell}\}_{N\in\mathbb{N}}$. We also set
$$S_N(M):=\Big\{x\in\mathbb{T}^d_N\,\Big|\,\frac{1}{(2m+1)^d}\sum_{|y|\le m}\int\eta(x+y)d\mu^N>M\Big\}$$
and
$$E_N(M):=\Big\{x\in\mathbb{T}^d_N\,\Big|\,\exists\,y\in\Lambda^d_m\ \text{ such that }\int\eta(x+y)d\mu^N>M\Big\}.$$
Then obviously $S_N(M)\subseteq E_N(M)$ and
$$\begin{aligned}\int_{\{f^N>M\}}f^N(u)du &= \frac{1}{N^d}\sum_{x\in S_N(M)}\frac{1}{(2m+1)^d}\sum_{|y|\le m}\int\eta(x+y)d\mu^N\\ &\le \frac{1}{N^d}\sum_{x\in E_N(M)}\frac{1}{(2m+1)^d}\sum_{|y|\le m}\int\eta(x+y)d\mu^N.\end{aligned}$$
Then if we set
$$I_{N,x}(M):=\Big\{|z|\le m\,\Big|\,\int\eta(x+z)d\mu^N>M\Big\}$$

we have for every $|y| \leq m$ that

$$\sum_{x \in E_N(M)} \int \eta(x+y) d\mu^N \leq \sum_{x \in \mathbb{T}_N^d} \sum_{z \in I_{N,x}(M)} \int \eta(x+y) d\mu^N$$

and if $|y| \leq m$ is such that $\int \eta(x+y) d\mu^N \leq M$, then

$$\sum_{z \in I_{N,x}(M)} \int \eta(x+y) d\mu^N \leq M \cdot \sharp I_{N,x}(M) \leq \sum_{z \in I_{N,x}(M)} \int \eta(x+z) d\mu^N,$$

while if $|y| \leq m$ is such that $\int \eta(x+y) d\mu^N > M$ then

$$\sum_{z \in I_{N,x}(M)} \int \eta(x+y) d\mu^N \leq (2m+1)^d \sum_{z \in I_{N,x}(M)} \int \eta(x+z) d\mu^N. \tag{2.13}$$

In any case (2.13) holds for all $|y| \leq \ell$. Now, given $\varepsilon > 0$, using the uniform integrability condition (2.12) we choose $M > 0$ and $N_0 \in \mathbb{N}$ such that

$$N \geq N_0 \quad \Longrightarrow \quad \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d : \int \eta(x) d\mu^N > M} \int \eta(x) d\mu^N < \frac{\varepsilon}{(2m+1)^{2d}}$$

and then

$$\begin{aligned}
\int_{\mathbb{T}^d} f^N \mathbb{1}_{\{f^N > M\}} &\leq \frac{1}{N^d (2m+1)^d} \sum_{|y| \leq m} \sum_{x \in E_N(M)} \int \eta(x+y) d\mu^N \\
&\leq \frac{1}{N^d (2m+1)^d} \sum_{|y| \leq m} \sum_{x \in \mathbb{T}_N^d} \sum_{z \in I_{N,x}(M)} \int \eta(x+y) d\mu^N \\
&\leq \frac{1}{N^d} \sum_{|y| \leq m} \sum_{x \in \mathbb{T}_N^d} \sum_{z \in I_{N,x}(M)} \int \eta(x+z) d\mu^N \\
&= \frac{(2m+1)^d}{N^d} \sum_{x \in \mathbb{T}_N^d} \sum_{z \in I_{N,x}(M)} \int \eta(x+z) d\mu^N \\
&= \frac{(2m+1)^d}{N^d} \sum_{|z| \leq m} \sum_{x \in \mathbb{T}_N^d : \int \eta(x+y) d\mu^N > m} \int \eta(x+y) d\mu^N \\
&= \frac{(2m+1)^d}{N^d} \sum_{|z| \leq m} \sum_{x \in \mathbb{T}_N^d : \int \eta(x) d\mu^N > m} \int \eta(x) d\mu^N \\
&= \frac{(2m+1)^{2d}}{N^d} \sum_{x \in \mathbb{T}_N^d : \int \eta(x) d\mu^N > m} \int \eta(x) d\mu^N < \varepsilon,
\end{aligned}$$

which proves the uniform integrability of $\{f^{N,\ell}\}_{N \in \mathbb{N}}$ and completes the proof. $\square$

Corollary 2.4.1 *Let $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ be an a.s. continuous profile such that the sequence $\{\rho([N\cdot]/N) \wedge \rho_c\}_{N \in \mathbb{N}}$ is uniformly integrable in $L^1(\mathbb{T}^d)$. Then the sequence of distributions with slowly varying parameter of profile ρ is a w_1-weak local equilibrium.*

2.5 Association to a Profile

In the case that $\rho_c = +\infty$ the association to a profile is the variant of the notion of weak local equilibrium that results from the notion of weak local equilibrium by replacing the arbitrary bounded cylinder function $\Psi : \mathbb{M}_\infty^d \longrightarrow \mathbb{R}$ in its definition by only one cylinder function, the unbounded cylinder function $\eta(0) : \mathbb{M}_\infty^d \longrightarrow \mathbb{R}$. Of course since the family $\{\nu_\rho^\infty\}_{\rho\in\mathbb{R}_+}$ is parametrized by the density we have that $\widetilde{\eta(0)}(\rho) = \rho$ for all $\rho \in \mathbb{R}_+$. So the precise definition is as follows.

Definition 2.5.1 Let $\{\mu^N \in \mathbb{P}_1\mathbb{M}_N^d\}_{N\in\mathbb{N}}$ be a sequence of distributions. We say that the sequence $\{\mu^N\}_{N\in\mathbb{N}}$ *is associated to the macroscopic profile* $\rho \in L^1(\mathbb{T}^d)$ if

$$\lim_{N\to\infty} \mu^N\big\{|\langle G, \pi^N\rangle - \langle G, \rho dm_{\mathbb{T}^d}\rangle| > \varepsilon\big\} = 0 \tag{2.14}$$

for all $G \in C(\mathbb{T}^d)$ and all $\varepsilon > 0$.

In condition (2.14) above the dependence on the grand canonical ensemble $\{\nu_\rho^\infty\}_{\rho\in\mathbb{R}_+}$ disappears. In this sense one can then interpret condition (2.14) as saying that the empirical distributions $\pi^N := \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} \delta_{\frac{x}{N}}$ converge in probability to the measure $\rho dm_{\mathbb{T}^d}$. Adopting this viewpoint one can then adapt the definition above for any measure $\mu \in \mathcal{M}_+(\mathbb{T}^d)$ even in the cases that $\rho_c < +\infty$, as follows

Definition 2.5.2 Let $\{\mu^N \in \mathbb{P}_1\mathbb{M}_N^d\}_{N\in\mathbb{N}}$ be a sequence of distributions. We say that the sequence $\{\mu^N\}_{N\in\mathbb{N}}$ *is associated to the macroscopic profile* $\mu \in \mathcal{M}_+(\mathbb{T}^d)$ if

$$\lim_{N\to\infty} \mu^N\big\{|\langle G, \pi^N\rangle - \langle G, \mu\rangle| > \varepsilon\big\} = 0 \tag{2.15}$$

for all $G \in C(\mathbb{T}^d)$ and all $\varepsilon > 0$.

Definition 2.5.3 We say that the sequence $\{\mu^N \in \mathbb{P}_1\mathbb{M}_N^d\}_{N\in\mathbb{N}}$ is *associated to the macroscopic profile* $\mu \in \mathcal{M}_+(\mathbb{T}^d)$ *in the Kantorovich-Rubinstein sense* if

$$\lim_{N\to\infty} W_1(\pi_*^N\mu^N, \delta_\mu) = 0$$

where W_1 denotes the Wasserstein metric on $\mathbb{P}_1\mathcal{M}_+(\mathbb{T}^d)$ induced Dudley's metric $d_{\mathcal{D}}$ on $\mathcal{M}_+(\mathbb{T}^d)$.

Proposition 2.5.1 *If* $\{\mu^N\}$ *is associated to the macroscopic profile* $\mu \in \mathcal{M}_+(\mathbb{T}^d)$ *in the Kantorovich-Rubinstein sense then it is associated to the profile* μ.

Proof Since the only transport plan between a probability measure and a Dirac mass is the product measure, we have that

$$W^1(\pi_*^N\mu^N, \delta_\mu) = \int \|\pi^N - \mu\|_{\mathcal{D}}\, d\mu^N$$

and therefore by Chebyshev's inequality we have that

$$\mu^N\{\|\pi^N - \mu\|_{\mathcal{D}} > \varepsilon\} \leq \frac{1}{\varepsilon}\int \|\pi^N - \mu\|_{\mathcal{D}}\, d\mu^N = \frac{1}{\varepsilon}W^1(\pi_*^N\mu^N, \delta_\mu) \longrightarrow 0$$

for all $\varepsilon > 0$. Then, for all $G \in \text{Lip}(\mathbb{T}^d)$ we have that

$$|\langle G, \pi^N \rangle - \langle G, \mu \rangle| \leq \|G\|_{BL} \|\pi^N - \mu\|_{\mathcal{D}}$$

and therefore we have that

$$\mu^N \big\{ |\langle G, \pi^N \rangle - \langle G, \mu \rangle| > \varepsilon \big\} \leq \mu^N \big\{ \|\pi^N - \mu\|_{\mathcal{D}} > \varepsilon / (\|G\|_{BL} + 1) \big\} \longrightarrow 0$$

for all $G \in \text{Lip}(\mathbb{T}^d)$ and all $\varepsilon > 0$.

Let now $G \in C(\mathbb{T}^d)$. For each $k \in \mathbb{N}$ we pick $G_k \in \text{Lip}(\mathbb{T}^d)$ so that $\|G - G_k\|_u < \frac{1}{k}$. Then

$$\mu^N \big\{ |\langle G, \pi^N - \mu \rangle| > \varepsilon \big\} \leq \mu^N \big\{ |\langle G - G_k, \pi^N - \mu \rangle| > \varepsilon/2 \big\} + \mu^N \big\{ |\langle G_k, \pi^N - \mu \rangle| > \varepsilon/2 \big\}$$

for all $k, N \in \mathbb{N}$ and all $\varepsilon > 0$ and therefore

$$\limsup_{N \to \infty} \mu^N \big\{ |\langle G, \pi^N - \mu \rangle| > \varepsilon \big\} \leq \limsup_{N \to \infty} \mu^N \big\{ |\langle G - G_k, \pi^N - \mu \rangle| > \varepsilon/2 \big\}$$

for all $k \in \mathbb{N}$ and all $\varepsilon > 0$.

It remains to prove that whenever $\{G_k\} \subseteq C(\mathbb{T}^d)$ is such that $\lim_{k \to \infty} \|G_k\|_u = 0$, then

$$\lim_{k \to \infty} \limsup_{N \to \infty} \mu^N \big\{ |\langle G_k, \pi^N - \mu \rangle| > \varepsilon \big\} = 0 \tag{2.16}$$

for all $\varepsilon > 0$. So let $\gamma > 0$ be arbitrary. First, since $\{G_k\}$ converges uniformly to zero there exists a constant $C_1 > 0$ such that $\|G_k\|_u \leq C_1$ for all $k \in \mathbb{N}$ and since $1 \in \text{Lip}(\mathbb{T}^d)$ there exists $N_1 \in \mathbb{N}$ such that

$$N \geq N_1 \quad \Longrightarrow \quad \mu^N \big\{ |\langle 1, \pi^N - \mu \rangle| > \varepsilon / 2C_1 \big\} < \gamma.$$

We set $A_N := \{ |\langle 1, \pi^N - \mu \rangle| \leq \varepsilon / 2C_1 \}$ for all $N \geq N_1$. Then,

$$\mu^N \big\{ |\langle G_k, \pi^N - \mu \rangle| > \varepsilon \big\} \leq \mu^N \big\{ |\langle G_k, \pi^N \rangle| > \frac{\varepsilon}{2} \big\} + \mu^N \big\{ |\langle G_k, \mu \rangle| > \frac{\varepsilon}{2} \big\}$$

for all $k \in \mathbb{N}$, $N \geq N_1$, and if we pick $k_1 \in \mathbb{N}$ such that $\|G_k\|_u < \varepsilon / 2\mu(\mathbb{T}^d)$ for all $k \in k_1$ then we have for all $k \geq k_1$, $N \geq N_1$ that

$$\begin{aligned}
\mu^N \big\{ |\langle G_k, \pi^N - \mu \rangle| > \varepsilon \big\} &\leq \mu^N \big\{ |\langle G_k, \pi^N \rangle| > \frac{\varepsilon}{2} \big\} \\
&\leq \mu^N \Big(A_N \cap \Big\{ |\langle G_k, \pi^N \rangle| > \frac{\varepsilon}{2} \Big\} \Big) + \mu^N (A_N^c) \\
&\leq \mu^N \Big(A_N \cap \Big\{ \pi^N(\mathbb{T}^d) > \frac{\varepsilon}{2\|G_k\|_u} \Big\} \Big) + \gamma.
\end{aligned}$$

Now, since

$$\sup_{N \geq N_1} \sup_{\eta \in A_N} \pi^N_\eta(\mathbb{T}^d) \leq \mu(\mathbb{T}^d) + \frac{\varepsilon}{2C_1} < +\infty,$$

if we choose $k_2 \geq k_1$ such that

$$k \geq k_2 \quad \Longrightarrow \quad \frac{\varepsilon}{2\|G_k\|_u} > \mu(\mathbb{T}^d) + \frac{\varepsilon}{2C_1}$$

then we have that

$$\mu^N\big\{|\langle G_k,\pi^N-\mu\rangle|>\varepsilon\big\}\leq\gamma,\qquad\forall\, N\geq N_1,\;\forall\, k\geq k_2.$$

Therefore

$$k\geq k_2\quad\Longrightarrow\quad\limsup_{N\to\infty}\mu^N\big\{|\langle G_k,\pi^N-\mu\rangle|>\varepsilon\big\}\leq\gamma,$$

which since $\gamma>0$ was arbitrary proves (2.16) and completes the proof. $\square$

Examples

1. Let $\{\nu_\rho^N\}_{\rho\geq0}\equiv\{\nu_{\rho\wedge\rho_c}^N\}_{\rho\geq0}$ be the sequence of normalized invariant distribution of a ZRP with critical density $\rho_c<+\infty$. For each $\rho\geq0$ and each $u\in\mathbb{T}^d$ we consider the configuration $\eta_{\rho,u}^N\in\mathbb{M}_N^d$ given by

$$\eta_{\rho,u}^N=[\rho N^d]\mathbb{1}_{\{[Nu]\}}.$$

Then for all $\rho\geq0$, $u\in\mathbb{T}^d$ the sequence $\{\mu_{\rho,u}^N\}_{N\in\mathbb{N}}$ defined by

$$\mu_{\rho,u}^N:=\delta_{\eta_{\rho,u}^N}\in\mathbb{P}\mathbb{M}_N^d$$

is associated in the Kantorovich-Rubinstein sense to the macroscopic profile

$$\mu_{\rho,u}:=\rho\delta_u\in\mathcal{M}_+(\mathbb{T}^d),$$

since we have that

$$\begin{aligned}W_1(\pi_*^N\mu_{\rho,u}^N,\delta_{\mu_{\rho,u}})&=\int\|\pi_\eta^N-\mu_{\rho,u}\|_{\mathcal{D}}d\mu_{\rho,u}^N(\eta)=\|\pi_{\eta_{\rho,u}^N}^N-\mu_{\rho,u}\|_{\mathcal{D}}\\&=\Big\|\frac{[\rho N^d]}{N^d}\delta_{\frac{[Nu]}{N}}-\rho\delta_u\Big\|_{\mathcal{D}}\overset{N\to\infty}{\longrightarrow}0.\end{aligned}$$

2. Let $\rho_0:\mathbb{T}^d\longrightarrow\mathbb{R}_+$ be a bounded and a.s. continuous function. Then the sequence of $\{\nu_{\rho_0(\cdot)}^N\in\mathbb{P}\mathbb{M}_N^d\}_{N\in\mathbb{N}}$ of product measures with regularly varying parameter associated to the profile ρ_0 is associated to the macroscopic profile $\mu:=\rho_0\wedge\rho_c dm_{\mathbb{T}^d}$.

Proof. This is immediate from corollary 2.4.1. Indeed, the sequence $\{\rho_0([N\cdot]/N)\}_{N\in\mathbb{N}}$ is uniformly bounded and thus uniformly integrable and since the uniform integrability condition (2.12) reduces in the case of the sequence $\{\nu_{\rho_0(\cdot)}^N\}$ to the uniform integrability of $\{\rho_0([N\cdot]/N)\}_{N\in\mathbb{N}}$ the conditions of corollary 2.4.1 are satisfied. So $\{\nu_{\rho_0(\cdot)}^N\}_{N\in\mathbb{N}}$ is w_1-weak local equilibrium and applying this to the cylinder function $\eta(0)$ the claim follows.

3. Let $\rho_0\in L_+^1(\mathbb{T}^d;[0,\rho_c])$ be an integrable profile and let $u\in\mathbb{T}^d$ be a macroscopic point of continuity of ρ_0. Then the sequence $\{\nu_{\rho_0(\cdot),\rho,u}^N\}_{N\in\mathbb{N}}$ defined by

$$\nu_{\rho_0(\cdot),\rho,u}^N=\delta_{[\rho N^d]}\otimes\bigotimes_{x\in\mathbb{T}_N^d\setminus\{[Nu]\}}\nu_{\rho_0(\frac{x}{N})}=:\delta_{[\rho N^d]}\otimes\mu_{\rho_0(\cdot),\rho}^{N,u}\in\mathbb{P}(\mathbb{Z}_+\times\mathbb{Z}_+^{\mathbb{T}_N^d\setminus\{[Nu]\}})\cong\mathbb{P}\mathbb{M}_N^d,$$

i.e. $\nu^N_{\rho_0(\cdot),\rho,u}$ is the probability measure given by

$$\nu^N_{\rho_0(\cdot),\rho,u}(\eta) = \delta_{[\rho N^d]}(\eta_{[Nu]}) \prod_{x \neq [Nu]} \nu_{\rho_0(\frac{x}{N})}(\eta_x),$$

is associated to the macroscopic profile

$$\mu := \rho_0 dm_{\mathbb{T}^d} + \rho\delta_u \in \mathcal{M}_+(\mathbb{T}^d).$$

Proof. For every function $G \in C(\mathbb{T}^d)$, $\varepsilon > 0$, we have that

$$\begin{aligned} \nu^N_{\rho_0(\cdot),\rho,u}\{\left|\langle G, \pi^N - \mu\rangle\right| > \varepsilon\} \quad &\leq \quad \nu^N_{\rho_0(\cdot),\rho,u}\Big\{\Big|G\Big(\frac{[Nu]}{N}\Big)\frac{\eta([Nu])}{N^d} - \rho G(u)\Big| > \frac{\varepsilon}{2}\Big\} \\ &\quad + \nu^N_{\rho_0(\cdot),\rho,u}\Big\{\Big|\sum_{x\neq[Nu]} G\Big(\frac{x}{N}\Big)\frac{\eta(x)}{N^d} - \int_{\mathbb{T}^d} G\rho_0\Big| > \frac{\varepsilon}{2}\Big\} \\ &\leq \quad \frac{2}{\varepsilon}\Big|G\Big(\frac{[Nu]}{N}\Big)\frac{[\rho N^d]}{N^d} - \rho G(u)\Big| \\ &\quad + \nu^N_{\rho_0(\cdot),\rho,u}\Big\{\Big|\sum_{x\neq[Nu]} G\Big(\frac{x}{N}\Big)\frac{\eta(x)}{N^d} - \int_{\mathbb{T}^d} G\rho_0\Big| > \frac{\varepsilon}{2}\Big\} \end{aligned}$$

Now the first term obviously converges to zero. For the second term, since the event under consideration does not depend on the number of particles at the site $[Nu]$, we have that it is equal to

$$\nu^N_{\rho_0(\cdot)}\Big\{\Big|\sum_{x\neq[Nu]} G\Big(\frac{x}{N}\Big)\frac{\eta(x)}{N^d} - \int_{\mathbb{T}^d} G\rho_0\Big| > \frac{\varepsilon}{2}\Big\}$$

which is bounded above by

$$\nu^N_{\rho_0(\cdot)}\Big\{\Big|\sum_{x\in\mathbb{T}^d_N} G\Big(\frac{x}{N}\Big)\frac{\eta(x)}{N^d} - \int_{\mathbb{T}^d} G\rho_0\Big| > \frac{\varepsilon}{4}\Big\} + \nu^N_{\rho_0(\cdot)}\Big\{\Big|G\Big(\frac{[Nu]}{N}\Big)\frac{\eta([Nu])}{N^d}\Big| > \frac{\varepsilon}{4}\Big\}.$$

Now, the first term in the sum above converges to zero as $N \to \infty$ by the previous example while by Chebyshev's inequality the second term is bounded above by

$$\frac{4}{\varepsilon N^d}\Big|G\Big(\frac{[Nu]}{N}\Big)\Big|\int \eta([Nu])d\nu^N_{\rho_0(\cdot)} = \frac{4}{\varepsilon N^d}\Big|G\Big(\frac{[Nu]}{N}\Big)\Big|\rho_0\Big(\frac{[Nu]}{N}\Big),$$

which tends to zero since $u \in \mathbb{T}^d$ is a continuity point of ρ_0 and $G \in C(\mathbb{T}^d)$.

2.6 The $O(N^d)$-Entropy assumption

In practice the initial local equilibrium from which we start the ZRP is required to have entropy of order $O(N^d)$ with respect to the equilibrium distributions $\{\nu_{\rho_*}\}_{N\in\mathbb{N}}$ for some $\rho_* \in (0, \rho_c)$.

Definition 2.6.1 A sequence $\{\mu_0^N \in \mathbb{PM}_N^d\}$ satisfies the $O(N^d)$-entropy assumption

$$C(\rho_*) := \limsup_{N\to\infty} \frac{1}{N^d}\mathcal{H}(\mu_0^N|\nu_{\rho_*}^N) < +\infty \quad \text{for some } \rho_* \in (0, \rho_c). \tag{2.17}$$

Of course if $\{\mu_0^N\}$ satisfies the $O(N^d)$-entropy assumption for $\rho_* \in (0, \rho_c)$ then

$$\sup_{N\in\mathbb{N}} \frac{1}{N^d}\mathcal{H}(\mu_0^N|\nu_{\rho_*}) < +\infty.$$

Indeed, since the $O(N^d)$-entropy assumption is satisfied there exists $N_0 \in \mathbb{N}$ such that

$$N \geq N_0 \quad \Longrightarrow \quad \frac{1}{N^d}\mathcal{H}(\mu_0^N|\nu_{\rho_*}^N) \leq C(\rho_*) + 1$$

and then obviously

$$\sup_{N\in\mathbb{N}} \frac{1}{N^d}\mathcal{H}(\mu_0^N|\nu_{\rho_*}^N) \leq [C(\rho_*) + 1] \vee \max_{N=1,\dots,N_0-1} \frac{1}{N^d}\mathcal{H}(\mu_0^N|\nu_{\rho*}^N) < +\infty.$$

By an application of the relative entropy inequality it easy to see that if the relative entropy assumption is satisfied for some $\rho_* \in (0, \rho_c)$ then it is satisfied for all $\rho \in (0, \rho_c)$.

Proposition 2.6.1 *If the sequence $\{\mu_0^N \in \mathbb{PM}_N^d\}$ satisfies (2.17) for some $\rho_* \in (0, \rho_c)$ then (2.17) is satisfied for all $\rho \in (0, \rho_*)$.*

Proof Indeed, for all $\rho^*, \rho \in (0, \rho_c)$ and all $\theta > 0$ we have by the entropy inequality that

$$\begin{aligned}
H(\mu^N|\nu_\rho^N) &= \int \log \frac{d\mu^N}{d\nu_\rho^N} d\mu^N = H(\mu^N|\nu_{\rho_*}^N) + \int \log \frac{d\nu_{\rho_*}^N}{d\nu_\rho^N} d\mu^N \\
&\leq H(\mu^N|\nu_{\rho_*}^N) + \frac{1}{\theta}\Big\{ \log \int \Big(\frac{d\nu_{\rho_*}^N}{d\nu_\rho^N}\Big)^\theta d\nu_{\rho_*}^N + H(\mu^N|\nu_{\rho_*}^N)\Big\} \\
&= \frac{\theta+1}{\theta} H(\mu^N|\nu_{\rho_*}^N) + \frac{1}{\theta} \log \int \Big(\frac{d\nu_{\rho_*}^N}{d\nu_\rho^N}\Big)^\theta d\nu_{\rho_*}^N
\end{aligned}$$

and therefore if (2.17) holds for ρ_* and we set $K_* := \sup_{N\in\mathbb{N}} \frac{1}{N^d} H(\mu^N|\nu_{\rho_*}^N) < +\infty$ then

$$\sup_{N\in\mathbb{N}} \frac{1}{N^d} H(\mu^N|\nu_\rho^N) \leq \frac{\theta+1}{\theta} K_* + \frac{1}{\theta} \sup_{N\in\mathbb{N}} \log \Big(\int \Big(\frac{d\nu_{\rho_*}^N}{d\nu_\rho^N}\Big)^\theta d\nu_{\rho_*}^N \Big)^{\frac{1}{N^d}}.$$

So in order to prove that (2.17) holds for ρ it suffices to prove that

$$\sup_{N\in\mathbb{N}} \log \Big(\int \Big(\frac{d\nu_{\rho_*}^N}{d\nu_\rho^N}\Big)^\theta d\nu_{\rho_*}^N \Big)^{\frac{1}{N^d}} < +\infty$$

for some appropriately chosen constant $\theta > 0$. But for all $\eta \in \mathbb{M}_N^d$, $N \in \mathbb{N}$ we have that

$$\frac{d\nu^N_{\rho_*}}{d\nu^N_\rho}(\eta) = \left(\frac{Z\circ\Phi(\rho)}{Z\circ\Phi(\rho_*)}\right)^{N^d}\left(\frac{\Phi(\rho_*)}{\Phi(\rho)}\right)^{|\eta|}$$

and therefore

$$\begin{aligned}\left(\int\left(\frac{d\nu^N_{\rho_*}}{d\nu^N_\rho}\right)^\theta d\nu^N_{\rho_*}\right)^{\frac{1}{N^d}} &= \left(\frac{Z\circ\Phi(\rho)}{Z\circ\Phi(\rho_*)}\right)^\theta\left(\int\left(\frac{\Phi(\rho_*)}{\Phi(\rho)}\right)^{\theta|\eta|} d\nu^N_{\rho_*}(\eta)\right)^{\frac{1}{N^d}}\\ &= \left(\frac{Z\circ\Phi(\rho)}{Z\circ\Phi(\rho_*)}\right)^\theta\int\left(\frac{\Phi(\rho_*)}{\Phi(\rho)}\right)^{\theta k} d\nu^1_{\rho_*}(k)\\ &= \left(\frac{Z\circ\Phi(\rho)}{Z\circ\Phi(\rho_*)}\right)^\theta\int e^{k\theta\log\frac{\Phi(\rho_*)}{\Phi(\rho)}} d\nu^1_{\rho_*}(k)\end{aligned}$$

It follows that

$$\frac{1}{\theta}\sup_{N\in\mathbb{N}}\log\left(\int\left(\frac{d\nu^N_{\rho_*}}{d\nu^N_\rho}\right)^\theta d\nu^N_{\rho_*}\right)^{\frac{1}{N^d}} = \log\frac{Z\circ\Phi(\rho)}{Z\circ\Phi(\rho_*)} + \frac{1}{\theta}\Lambda_{\rho_*}\Big(\theta\log\frac{\Phi(\rho_*)}{\Phi(\rho)}\Big)$$

where Λ_{ρ_*} is logarithmic m.g.f. of the one site ZR distribution $\nu^1_{\rho_*}$. Now obviously the first term on the sum in the right hand side of the equality above is finite for all values of $\rho \in I_c$ while the second term is finite iff

$$\theta\log\frac{\Phi(\rho_*)}{\Phi(\rho)} \in \mathcal{D}_{\Lambda_{\rho_*}}, \tag{2.18}$$

and as we know

$$\mathcal{D}_{\Lambda_{\rho_*}} = \begin{cases}(-\infty, b_{\rho_*}) & \text{if } Z(\varphi_c) = +\infty\\ (-\infty, b_{\rho_*}] & \text{if } Z(\varphi_c) < +\infty\end{cases}, \qquad b_{\rho_*} := \log\frac{\varphi_c}{\Phi(\rho_*)}.$$

Obviously $b_{\rho_*} > 0$ since we assume $\rho_* < \rho_c$. So, since Φ is increasing, if $\rho \in [\rho_*, \rho_c] \cap \mathbb{R}$ we have that $\log[\Phi(\rho_*)/\Phi(\rho)] \leq 0$ and so in this case we have that (2.18) holds for all $\theta > 0$, while on the other hand if $\rho \in (0, \rho_*)$ we have that $\log[\Phi(\rho_*)/\Phi(\rho)] > 0$ and in order for (2.18) to hold it suffices to choose

$$0 < \theta < \frac{\log\frac{\varphi_c}{\Phi(\rho_*)}}{\log\frac{\Phi(\rho_*)}{\Phi(\rho)}}.$$

Note that according to what we have proved, we have for all $\rho_*, \rho \in (0, \rho_c)$ that

$$\begin{aligned}K_\rho &\leq \log\frac{Z\circ\Phi(\rho)}{Z\circ\Phi(\rho_*)} + \inf_\theta\left\{\frac{\theta+1}{\theta}K_{\rho_*} + \frac{1}{\theta}\Lambda_{\rho_*}\Big(\theta\log\frac{\Phi(\rho_*)}{\Phi(\rho)}\Big)\right\}\\ &= K_{\rho_*} + \log\frac{Z\circ\Phi(\rho)}{Z\circ\Phi(\rho_*)} + \inf_\theta\left\{\frac{1}{\theta}\Big[K_{\rho_*} + \Lambda_{\rho_*}\Big(\theta\log\frac{\Phi(\rho_*)}{\Phi(\rho)}\Big)\Big]\right\}\end{aligned}$$

where the infimum is taken over all $\theta > 0$ satisfying (2.18). □

By similar computations one can show that any local equilibrium of continuous profile $\rho_0\mathbb{T}^d \longrightarrow [0, \rho_c)$ in the entropy sense satisfies the $O(N^d)$-entropy assumption.

Proposition 2.6.2 *Let $\rho_0 : \mathbb{T}^d \longrightarrow [0, \rho_c)$ be a continuous profile. Any sequence $\{\mu_0^N\}_{N\in\mathbb{N}}$ of initial distributions that is a local equilibrium in the entropy sense of profile ρ_0 satisfies the $O(N^d)$-entropy assumption.*

Proof For convenience we pick $a \in (\rho_c - \varepsilon, \rho_c)$ where $\rho_c - \varepsilon$ is an upper bound on the initial profile ρ_0. Then by the relative entropy inequality we have that

$$\begin{aligned} H(\mu_0^N|\nu_a^N) &= \int \log \frac{d\mu_0^N}{d\nu_a^N} d\mu_0^N = H(\mu_0^N|\nu_{\rho_0(\cdot)}^N) + \int \log \frac{d\nu_{\rho_0(\cdot)}^N}{d\nu_a^N} d\mu_0^N \\ &\leq \Big(1 + \frac{1}{\gamma}\Big) H(\mu_0^N|\nu_{\rho_0(\cdot)}^N) + \frac{1}{\gamma} \log \int \Big(\frac{d\nu_{\rho_0(\cdot)}^N}{d\nu_a^N}\Big)^\gamma d\nu_{\rho_0(\cdot)}^N \end{aligned} \tag{2.19}$$

for every $\gamma > 0$ and the first term $(1 + \frac{1}{\gamma}) H(\mu_0^N|\nu_{\rho_0(\cdot)}^N)$ in the right hand side of the inequality above is of order $o(N^d)$ by assumption. For the second term, we compute first the Radon-Nikodym derivative $\frac{d\nu_{\rho_0(\cdot)}^N}{\nu_a^N}$. For all $\eta \in \mathbb{M}_N^d$,

$$\frac{d\nu_{\rho_0(\cdot)}^N}{\nu_a^N}(\eta) = \prod_{x\in\mathbb{T}_N^d} \frac{Z(\Phi(a))\Phi(\rho_0(x/N))^{\eta_x}}{Z(\Phi(\rho_0(x/N)))\Phi(a)^{\eta_x}} = \prod_{x\in\mathbb{T}_N^d} \frac{\Phi_a(\rho_0(x/N))^{\eta_x}}{Z_a(\rho_0(x/N))}$$

and thus since $\nu_{\rho_0(\cdot)}^N$ is a product measure we have that

$$\begin{aligned} \int \Big(\frac{d\nu_{\rho_0(\cdot)}^N}{d\nu_a^N}\Big)^\gamma d\nu_{\rho_0(\cdot)}^N &= \prod_{x\in\mathbb{T}_N^d} \int \frac{\Phi_a(\rho_0(x/N))^{\gamma\eta_x}}{Z_a(\rho_0(x/N))^\gamma} d\nu_{\rho_0(x/N)}^1(\eta_x) \\ &= \prod_{x\in\mathbb{T}_N^d} \frac{1}{Z_a(\rho_0(x/N))^\gamma} \int e^{\log \gamma k \Phi_a(\rho_0(x/N))} d\nu_{\rho_0(x/N)}^1(k). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{N^d} \log \int \Big(\frac{d\nu_{\rho_0(\cdot)}^N}{d\nu_a^N}\Big)^\gamma d\nu_{\rho_0(\cdot)}^N &= \frac{\gamma}{N^d} \sum_{x\in\mathbb{T}_N^d} \log \frac{1}{Z_a(\rho_0(x/N))} \\ &\quad + \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \Lambda_{\rho_0(x/N)}\big(\gamma \log \Phi_a(\rho_0(x/N))\big) \end{aligned}$$

where for each $\rho \in [0, \rho_c)$, $\Lambda_\rho : \mathbb{R} \longrightarrow (-\infty, +\infty]$ is the logarithmic moment generating function of $\nu_\rho^1 \in \mathbb{P}\mathbb{Z}_+$:

$$\Lambda_\rho(r) := \log \int e^{rk} d\nu_\rho^1(k) = \log \frac{Z(e^r \Phi(\rho))}{Z(\Phi(\rho))}. \tag{2.20}$$

We recall that for each $\rho \in (0, \rho_c)$ the logarithmic moment generating function Λ_ρ has proper domain $\mathcal{D}_{\Lambda_\rho}$ that satisfies $(-\infty, b_\rho) \subseteq \mathcal{D}_{\Lambda_\rho} \subseteq (-\infty, b_\rho]$ with

$$b_\rho := \log \frac{\varphi_c}{\Phi(\rho)} \in (0, \infty).$$

Now, since we have assumed that $a \in (\rho_c - \varepsilon, \rho_c)$ and $\sup_{u\in\mathbb{T}^d} \rho_0(u) < \rho_c - \varepsilon$ and Φ is increasing, we have that $\Phi_a(\rho_0(x/N)) = \frac{\Phi(\rho_0(x/N))}{\Phi(a)} \leq 1$ for all $x \in \mathbb{T}_N^d$. Consequently, we have that $\gamma \log \Phi_a(\rho_0(x/N)) \in (-\infty, 0] \subseteq \mathcal{D}_{\Lambda_{\rho_0(x/N)}}$ for all $\gamma > 0$. So for any fixed $\gamma > 0$, since the finite real valued functions

$$\mathbb{T}^d \ni u \mapsto Z\big(\Phi\big(\rho_0(u)\big)\big), \qquad \mathbb{T}^d \ni u \mapsto \Lambda_{\rho_0(u)}\big(\gamma \log \Phi_a\big(\rho_0(u)\big)\big)$$

are continuous, taking lim sup as $N \to \infty$ we get

$$\limsup_{N\to\infty} \frac{1}{N^d} H(\mu_0^N|\nu_a^N) \le \int_{\mathbb{T}^d} \log \frac{1}{Z_a(\rho_0(u))} du + \frac{1}{\gamma} \int_{\mathbb{T}^d} \Lambda_{\rho_0(u)}\big(\gamma \log \Phi_a(\rho_0(u))\big) du < \infty$$

as required. □

As the following examples show the sequences of product measures with slowly varying parameter associated to a profile satisfy the $O(N^d)$-entropy assumption.

Proposition 2.6.3 *Let $\{\nu^N_{\rho_0(\cdot)}\}_{N\in\mathbb{N}}$ be the sequence of product measures with slowly varying parameter associated to some bounded and a.s. continuous profile $\rho_0 \in B(\mathbb{T}^d)$. Then*

$$\lim_{N\to+\infty} \frac{1}{N^d} H(\nu^N_{\rho_0(\cdot)}|\nu^N_{\rho_*}) = H_{\Lambda_{\rho_*}}(\rho_0 \wedge \rho_c | m_{\mathbb{T}^d}) < +\infty \tag{2.21}$$

for all $\rho_* \in (0, \rho_c)$ and so the sequence $\{\nu^N_{\rho_0(\cdot)}\}$ satisfies the $O(N^d)$-entropy assumption.

Proof Since the measures are product, by proposition A.3.6 we have for all $N \in \mathbb{N}$ that

$$H(\nu^N_{\rho_0(\cdot)}|\nu^N_{\rho_*}) = \sum_{x\in\mathbb{T}^d_N} H(\nu_{\rho_0(x/N)\wedge\rho_c}|\nu_{\rho_*}).$$

Now for all $\rho \in [0, \rho_c] \cap \mathbb{R}$ we obviously have that

$$\begin{aligned} H(\nu_\rho|\nu_{\rho_*}) &= \int \log \frac{\nu_\rho(k)}{\nu_{\rho_*}(k)} d\nu_\rho(k) = \int \log \frac{Z(\Phi_{\rho_*})\Phi_\rho^k}{Z(\Phi_\rho)\Phi_{\rho_*}^k} d\nu_\rho(k) \\ &= \log \frac{Z(\Phi_{\rho_*})}{Z(\Phi_\rho)} + \rho \log \frac{\Phi_\rho}{\Phi_{\rho_*}} = \Lambda^*_{\nu_{\rho_*}}(\rho) \end{aligned}$$

and therefore

$$\frac{1}{N^d} H(\nu^N_{\rho_0(\cdot)}|\nu^N_{\rho_*}) = \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \Lambda^*_{\rho_*}\Big(\rho_0\Big(\frac{x}{N}\Big) \wedge \rho_c\Big) = \int_{\mathbb{T}^d} \Lambda^*_{\rho_*}\big(\rho_0\big([Nu]/N\big) \wedge \rho_c\big) du.$$

As we know by proposition 1.2.12 the function $\Lambda^*_{\nu_{\rho_*}}$ is always finite and smooth on all of $\mathbb{R}_+$, and therefore since we assume the profile ρ_0 to be bounded and almost surely continuous, the required limit in (2.21) follows by the bounded convergence theorem. □

According to he following proposition the $O(N^d)$-entropy assumption is satisfied even by initial distributions that can have a condensate at some macroscopic point $u \in \mathbb{T}^d$.

Proposition 2.6.4 *Let $\{\nu^N_{x,\rho,\rho_0(\cdot)}\}_{N\in\mathbb{N}}$ be the sequence of product measures with slowly varying parameter associated to some bounded and a.s. continuous profile $\rho_0 \in B(\mathbb{T}^d)$ and a Dirac mass ρ at $x \in \mathbb{T}^d$, i.e.*

$$\nu^N_{\rho_0,x,\rho} = \delta_{[\rho N^d]} \otimes \bigotimes_{y\in\mathbb{T}^d_N\setminus\{[Nx]\}} \nu_{\rho_0(\frac{y}{N})} =: \delta_{[\rho N^d]} \otimes \nu^{N,x}_{\rho_0,\rho} \in \mathbb{P}(\mathbb{Z}_+ \times \mathbb{Z}_+^{\mathbb{T}^d_N\setminus\{[Nx]\}}) \cong \mathbb{P}\mathbb{M}^d_N,$$

and let μ_0 denote the measure

$$\mu_0 = \rho\delta_x + (\rho_0 \wedge \rho_c)dm_{\mathbb{T}^d}.$$

Then

$$\lim_{N\to+\infty} \frac{1}{N^d} H(\nu^N_{\rho_0,x,\rho}|\nu^N_{\rho_*}) = H_{\Lambda_{\rho_*}}(\mu_0|m_{\mathbb{T}^d}) \tag{2.22}$$

for all $\rho_* \in (0,\rho_c)$. In particular, whenever $\phi_c < +\infty$ we have that $H_{\Lambda_{\rho_*}}(\mu_0|m_{\mathbb{T}^d}) < +\infty$ and therefore the sequence $\{\nu^N_{\rho_0,x,\rho}\}$ has entropy of order $O(N^d)$.

Proof For all $N \in \mathbb{N}$ we have that

$$H(\nu^N_{\rho_0,x,\rho}|\nu^N_{\rho_*}) = H(\delta_{[\rho N^d]}|\nu_{\rho_*}) + H(\nu^{N,x}_{\rho_0,\rho}|\nu_{\rho^*}^{\mathbb{T}^d_N\setminus\{[Nx]\}}). \tag{2.23}$$

Now with the usual convention $0\log 0 = 0$ in the definition of relative entropy we obviously have for any $K \in \mathbb{Z}_+$ that

$$\begin{aligned}
H(\delta_K|\nu_{\rho_*}) &= \int \frac{\delta_K(k)}{\nu_{\rho_*}(k)} \log \frac{\delta_K(k)}{\nu_{\rho_*}(k)} d\nu_{\rho_*}(k) = -\log \nu_{\rho_*}(K) \\
&= -\log\left(\frac{1}{Z\big(\Phi(\rho_*)\big)} \frac{\Phi(\rho_*)^K}{g!(K)}\right) = \log Z\big(\Phi(\rho_*)\big) + \log \frac{g!(K)}{\Phi(\rho_*)^K} \\
&= \log Z\big(\Phi(\rho_*)\big) + K \log \frac{\sqrt[K]{g!(K)}}{\Phi(\rho_*)}.
\end{aligned}$$

for all $\rho \in [0,\rho_c] \cap \mathbb{R}$ and therefore

$$\begin{aligned}
\frac{1}{N^d} H(\delta_{[\rho N^d]}|\nu_{\rho_*}) &= \frac{1}{N^d} \log Z\big(\Phi(\rho_*)\big) + \frac{[\rho N^d]}{N^d} \log \frac{\sqrt[[\rho N^d]]{g!([\rho N^d])}}{\Phi(\rho_*)} \\
&\overset{N\to\infty}{\longrightarrow} \rho \log \frac{\phi_c}{\Phi(\rho_*)}.
\end{aligned} \tag{2.24}$$

Furthermore, we obviously have that

$$\begin{aligned}
H(\nu^{N,x}_{\rho_0,\rho}|\nu_{\rho^*}^{\mathbb{T}^d_N\setminus\{[Nx]\}}) &= H(\nu^N_{\rho_0(\cdot)}|\nu^N_{\rho_*}) - H(\nu_{\rho_0([Nx]/N)}|\nu_{\rho_*}) \\
&= H(\nu^N_{\rho_0(\cdot)}|\nu^N_{\rho_*}) - \Lambda^*_{\nu_{\rho_*}}\Big(\rho_0\Big(\frac{[Nx]}{N}\Big) \wedge \rho_c\Big).
\end{aligned}$$

Since the profile $\rho_0 : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ is assumed bounded and $\Lambda^*_{\nu_{\rho_*}}$ is continuous and finite on $\mathbb{R}_+$ we obviously have that

$$\lim_{N\to\infty} \frac{1}{N^d} \Lambda^*_{\nu_{\rho_*}}\Big(\rho_0\Big(\frac{[Nx]}{N}\Big) \wedge \rho_c\Big) = 0$$

and therefore by (2.24), (2.23) and the previous proposition it follows that

$$\lim_{N\to+\infty}\frac{1}{N^d}H(\nu^N_{\rho_0,x,\rho}|\nu^N_{\rho_*})=\int_{\mathbb{T}^d}\Lambda^*_{\nu_{\rho_*}}\big(\rho_0(u)\wedge\rho_c\big)du+\rho\log\frac{\phi_c}{\Phi(\rho_*)}=H_{\Lambda_{\nu_{\rho_*}}}(\mu_0|m_{\mathbb{T}^d}),$$

and obviously $H_{\Lambda_{\nu_{\rho_*}}}(\mu_0|m_{\mathbb{T}^d})<+\infty$ whenever $\phi_c<+\infty$. □

We close this section with another interesting example of distributions that satisfy the $O(N^d)$-entropy assumption. For simplicity we consider the 1-dimensional case $d=1$ and the subsequence of the squares of the scaling parameter N.

Proposition 2.6.5 *For each N we set*

$$A_N:=\{kN|k=0,\dots,N-1\}\subseteq\mathbb{T}_{N^2}.$$

Let $\rho:\mathbb{T}\longrightarrow R(\mathcal{D}_Z)\subseteq[0,\rho_c]$ be a continuous profile and consider the distributions

$$\nu_{N^2}:=\Big(\bigotimes_{x\in A_N}\delta_{[aN]}\Big)\otimes\Big(\bigotimes_{x\in\mathbb{T}_{N^2}\setminus A_N}\nu^1_{\rho(x/N^2)}\Big)\in\mathbb{PM}_{N^2},\quad N\in\mathbb{N}.$$

Then for any $\rho_*\in(0,\rho_c)$,

$$\lim_{N\to\infty}\frac{1}{N^2}\mathcal{H}(\nu_{N^2}|\nu^{N^2}_{\rho_*})=H_{\Lambda^*_{\rho_*}}(\rho|m_{\mathbb{T}^d})+a\log\frac{\varphi_c}{\Phi(\rho_*)}$$

and in particular $\nu_{N^2}\in\mathbb{PM}_{N^2}$ satisfies the $O(N^d)$-entropy assumption if $\varphi_c<+\infty$.

Proof Since the measures are product measures,

$$\begin{aligned}\frac{1}{N^2}\mathcal{H}(\nu_{N^2}|\nu^{N^2}_{\rho_*})&=\frac{1}{N^2}\sum_{x\in A_N}\mathcal{H}(\delta_{[aN]}|\nu^1_{\rho_*})+\frac{1}{N^2}\sum_{x\in\mathbb{T}_{N^2}\setminus A_N}\mathcal{H}(\nu^1_{\rho(x/N^2)}|\nu^1_{\rho_*})\\&=\frac{1}{N}\mathcal{H}(\delta_{[aN]}|\nu^1_{\rho_*})+\frac{1}{N^2}\sum_{x\in\mathbb{T}_{N^2}\setminus A_N}\mathcal{H}(\nu^1_{\rho(x/N^2)}|\nu^1_{\rho_*})\end{aligned}$$

From the previous example we have for the first term that

$$\lim_{N\to\infty}\frac{1}{N}\mathcal{H}(\delta_{[aN]}|\nu^1_{\rho_*})=a\log\frac{\varphi_c}{\Phi(\rho_*)}.$$

For the second term, as in the previous example we write

$$\frac{1}{N^2}\sum_{x\in\mathbb{T}_{N^2}\setminus A_N}\mathcal{H}(\nu^1_{\rho(x/N^2)}|\nu^1_{\rho_*})=\frac{1}{N^2}\sum_{x\in\mathbb{T}_{N^2}}\mathcal{H}(\nu^1_{\rho(x/N^2)}|\nu^1_{\rho_*})-\frac{1}{N^2}\sum_{x\in A_N}\mathcal{H}(\nu^1_{\rho(x/N^2)}|\nu^1_{\rho_*}).$$

As we have also seen in the previous examples, the first term in the right hand side above converges to

$$\int_{\mathbb{T}}\Lambda^*_{\rho_*}\big(\rho(u)\big)du$$

as $N\to\infty$ while for the second term we have

$$\frac{1}{N^2}\sum_{x\in A_N}\mathcal{H}(\nu^1_{\rho(x/N^2)}|\nu^1_{\rho_*})=\frac{1}{N^2}\sum_{k=0}^{N-1}\Lambda_{\rho_*}\Big(\rho\Big(\frac{k}{N}\Big)\Big)$$

which converges to 0 as $N \to \infty$ since

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \Lambda_{\rho_*}\Big(\rho\Big(\frac{k}{N}\Big)\Big) = \int_{\mathbb{T}} \Lambda^*_{\rho_*}(\rho(u))du.$$

This completes the proof. □

The term $a \log \frac{\varphi_c}{\Phi(\rho_*)}$ implies that the entropy density "sees" the mass a of the singular part of the measures ν_{N^2}, $N \in \mathbb{N}$. On the other hand the measures ν^{N_2} are associated to the absolutely continuous profile $\mu := (\rho + a)dm_{\mathbb{T}^d}$. Indeed, let

$$p : \mathbb{M}_{N^2} \longrightarrow \mathbb{Z}_+^{A_N}, \quad q : \mathbb{M}_{N^2} \longrightarrow \mathbb{Z}_+^{\mathbb{T}_{N^2}\setminus A_N}$$

denote the natural projections. Then, given $\delta > 0$, $G \in C(\mathbb{T}^d)$ and $c \in \mathbb{R}$,

$$\begin{aligned}\Big\{\eta \in \mathbb{M}_{N^2} \,\Big|\, \Big| \frac{1}{N^2} \sum_{x\in A_N} G\Big(\frac{x}{N}\Big)\eta(x) - c\Big| > \delta\Big\} \\ = p^{-1}\Big\{\zeta \in \mathbb{Z}_+^{A_N} \,\Big|\, \Big| \frac{1}{N^2} \sum_{x\in A_N} G\Big(\frac{x}{N^2}\Big)\zeta(x) - c\Big| > \delta\Big\}\end{aligned}$$

and likewise

$$\begin{aligned}\Big\{\eta \in \mathbb{M}_{N^2} \,\Big|\, \Big| \frac{1}{N^2} \sum_{x\in A_N} G\Big(\frac{x}{N}\Big)\eta(x) - c\Big| > \delta\Big\} \\ = q^{-1}\Big\{\xi \in \mathbb{Z}_+^{\mathbb{T}_{N^2}\setminus A_N} \,\Big|\, \Big| \frac{1}{N^2} \sum_{x\notin A_N} G\Big(\frac{x}{N^2}\Big)\xi(x) - c\Big| > \delta\Big\}.\end{aligned}$$

Therefore we can write

$$\begin{aligned}\nu_{N^2}\big\{\big|\langle G, \pi^{N^2} - \mu\rangle\big| > \delta\big\} \quad &\leq \quad \nu_{N^2}\Big\{\Big| \frac{1}{N^2} \sum_{x\in A_N} G\Big(\frac{x}{N^2}\Big)\eta(x) - a\int_{\mathbb{T}} G(u)du\Big| > \frac{\delta}{2}\Big\} \\ &\quad + \nu_{N^2}\Big\{\Big| \frac{1}{N^2} \sum_{x\notin A_N} G\Big(\frac{x}{N^2}\Big)\eta(x) - \int_{\mathbb{T}} G(u)\rho(u)du\Big| > \frac{\delta}{2}\Big\} \\ &= \delta_{[aN]}^{\otimes A_N}\Big\{\Big| \frac{1}{N^2} \sum_{x\in A_N} G\Big(\frac{x}{N^2}\Big)\zeta(x) - a\int_{\mathbb{T}} G(u)du\Big| > \frac{\delta}{2}\Big\} \\ &\quad + q_*\nu_{N^2}\Big\{\Big| \sum_{x\notin A_N} G\Big(\frac{x}{N^2}\Big)\frac{\xi(x)}{N^2} - \int_{\mathbb{T}} G(u)\rho(u)du\Big| > \frac{\delta}{2}\Big\}.\end{aligned}$$

By Chebyshev's inequality the first term is bounded above by

$$\frac{2}{\delta}\Big| \frac{1}{N^2} \sum_{x\in A_N} G\Big(\frac{x}{N^2}\Big)[aN] - a\int_{\mathbb{T}} G(u)du\Big| = \frac{2}{\delta}\Big| \frac{1}{N} \sum_{k=0}^{N-1} G\Big(\frac{k}{N}\Big)\frac{[aN]}{N} - a\int_{\mathbb{T}} G(u)du\Big|$$

which obviously converges to zero as $N \to \infty$. Since the event in the second term does not depend on the coordinates in A_N, we can write the second term as

$$\nu_{\rho(\cdot)}^{N^2}\Big\{\Big| \sum_{x\notin A_N} G\Big(\frac{x}{N^2}\Big)\frac{\eta(x)}{N^2} - \int_{\mathbb{T}} G(u)\rho(u)du\Big| > \frac{\delta}{2}\Big\},$$

which is bounded above by

$$\nu^{N^2}_{\rho(\cdot)}\Big\{\Big|\sum_{x\in T_{N^2}} G\Big(\frac{x}{N^2}\Big)\frac{\eta(x)}{N^2} - \int_{\mathbb{T}} G(u)\rho(u)du\Big| > \frac{\delta}{4}\Big\} + \nu^{N^2}_{\rho(\cdot)}\Big\{\Big|\sum_{x\in A_N} G\Big(\frac{x}{N^2}\Big)\frac{\eta(x)}{N^2}\Big| > \frac{\delta}{4}\Big\}.$$

Now the first term above obviously tends to zero since $\{\nu^N_{\rho(\cdot)}\}$ is a w_1-weak local equilibrium and thus associated to the measures $\rho dm_{\mathbb{T}^d}$, while the second term is bounded above according to Chebyshev's inequality by

$$\frac{4}{\delta N^2}\sum_{x\in A_N}\Big|G\Big(\frac{x}{N^2}\Big)\Big|\int \eta(x)d\nu^{N^2}_{\rho(\cdot)} \quad = \quad \frac{4}{\delta N^2}\sum_{k=0}^{N-1}\Big|G\Big(\frac{k}{N}\Big)\Big|\rho\Big(\frac{k}{N}\Big)$$

which converges to zero $\frac{1}{N}\sum_{k=0}^{N-1}\big|G\big(\frac{k}{N}\big)\big|\rho\big(\frac{k}{N}\big) \longrightarrow \int_{\mathbb{T}^d} G\rho < +\infty$ as $N\to\infty$ due to the fact that G and ρ are continuous functions.

These considerations show that this family of initial distributions is in some sense pathological since it is perceived different by the entropy and different by the notion of association to a profile.

Chapter 3

The Relative Entropy Method

In this chapter we apply the Relative Entropy method of H.T. Yau to prove the hydrodynamic behavior of condensing ZRPs starting from a weak local equilibrium of sub-critical profile.

Theorem 3.0.1 (Hydrodynamic Limit) *Suppose that the local jump rate function g of the ZRP is bounded and let Φ be the mean jump rate function associated to g. Then any initial entropy local equilibrium $\mu_0^N \in \mathbb{PM}_N^d$, $N \in \mathbb{N}$, of profile $\rho_0 \in C^{2+\theta}\big(\mathbb{T}^d;(0,\rho_c)\big)$ for some $\theta > 0$ is conserved in the diffusive timescale along the unique solution $\rho : \mathbb{R}_+ \times \mathbb{T}^d \longrightarrow (0,\rho_c)$ of the initial value problem*

$$\begin{cases}\partial_t\rho = \Delta_\Sigma\Phi(\rho) \quad \text{in} \quad (0,\infty)\times\mathbb{T}^d \\ \rho(0,\cdot) = \rho_0.\end{cases} \tag{3.1}$$

In other words, if $\mathcal{H}(\mu_0^N|\nu^N_{\rho_0(\cdot)}) = o(N^d)$ then $\mathcal{H}(\mu_t^N|\nu^N_{\rho_t(\cdot)}) = o(N^d)$ for all $t > 0$, where $\mu_t^N := \mu_0^N S_{tN^2}$, $\rho_t(\cdot) \equiv \rho(t,\cdot)$, and in particular

$$\lim_{N\to\infty}\mu_N\Big\{\Big|\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x\Psi - \int_{\mathbb{T}^d}\widetilde{\Psi}(\rho(u))du\Big| > \delta\Big\} = 0$$

for all $\Psi \in B_{\text{cyl}}(\mathbb{M}_\infty^d)$, all $G \in C(\mathbb{T}^d)$ and all $t,\varepsilon > 0$.

Remark 3.0.1 As will be seen in the proof, one can assume the initial profile ρ_0 to be only of class $C(\mathbb{T}^d;[0,\rho_c))$, provided that the unique classical solution ρ of the hydrodynamic equation (3.1) with initial condition $\rho(0,\cdot) = \rho_0$ is such that the functions

$$(a)\ \ t \mapsto \log\Phi\big(m_t\big) \quad \text{and} \quad (b)\ \ t \mapsto \frac{\|\Delta_\Sigma\Phi(\rho_t)\|_\infty + \|D^2[\Phi(\rho_t)]\|_\theta}{\Phi(m_t)} \tag{3.2}$$

belong in $L^2_{\text{loc}}(\mathbb{R}_+)$, where $m_t := \min_{u\in\mathbb{T}^d}\rho_t(u)$, $\|D^2f\|_\theta := \max_{|\alpha|=2}|\partial^\alpha f|_\theta$, $\theta \in (0,1]$, and $|f|_\theta$ is the θ-Hölder semi-norm of the function f. By the properties of the solutions of the hydrodynamic equation (3.1) given in proposition 3.2.4 of the next section, the functions in (3.2) belong in $L^\infty_{\text{loc}}(\mathbb{R}_+)$ whenever the initial profile is of class $C^{2+\theta}(\mathbb{T}^d;(0,\rho_c))$.

3.1 The One Block Estimate

For each configuration $\eta \in \mathbb{M}_N^d$ we denote by η^ℓ the spatial mean of η over microscopic boxes of radius ℓ given by

$$\eta^\ell(x) = \frac{1}{(2\ell+1)^d} \sum_{y \in x+\Lambda_\ell^d} \eta(y) = \frac{1}{(2\ell+1)^d} \sum_{y \in \mathbb{T}_N^d : |y-x| \leq \ell} \eta(y)$$

and obviously $\eta^\ell(x) = \tau_x[\eta^\ell(0)]$. Our goal in this section is to investigate conditions on the jump rate $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ of a nearest neighbor ZRP and its sequence of initial distributions $\{\mu_0^N \in \mathbb{P}\mathbb{M}_N^d\}$ that allows the replacement in probability and in duality with respect to functions in $L^1(0,T;C(\mathbb{T}^d))$ of the empirical jump rate process

$$\sigma_t^N := \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} g\big(\eta_t(x)\big), \quad t \in I := [0,T] \tag{3.3}$$

by the process

$$\sigma_t^{N,\ell,\Phi} := \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \Phi\big(\eta_t^\ell(x)\big) \delta_{\frac{x}{N}}, \quad t \in I, \tag{3.4}$$

over large microscopic boxes, i.e. as $\ell \to \infty$, in the sense that

$$\lim_{\ell \to \infty} \limsup_{N \to \infty} P^N \left\{ \left| \int_0^T \langle f_s, \sigma_s^N - \sigma_s^{N,\ell,\Phi} \rangle ds \right| > \varepsilon \right\} = 0 \tag{3.5}$$

for all $f \in L^1(I;C(\mathbb{T}^d))$ and all $\varepsilon > 0$, where here, and in all of this section, $P^N \in \mathbb{P}D(I;\mathbb{M}_N^d)$ denotes the diffusively rescaled distribution of the nearest neighbor ZRP with jump rate g, starting from $\mu_0^N \in \mathbb{P}\mathbb{M}_N^d$ and the mean jump rate function Φ will be always considered extended according to (1.71), as suggested by the equivalence of ensembles.

Definition 3.1.1 Let g be a local jump rate function and let $\mathcal{C} \subseteq L^1(I;C(\mathbb{T}^d))$. We say that g *satisfies the time dependent one block estimate in $\mathcal{C}$ with respect to a sequence of initial distributions* $\{\mu_0^N\}$ if the limit (3.5) holds for all $f \in \mathcal{C}$ and all $\varepsilon > 0$.

The main result proved in this section is the validity of the One-Block estimate for all bounded local jump rate functions g with respect to any sequence $\{\mu_0^N\}$ of initial distributions satisfying the $O(N^d)$-entropy assumption:

$$\limsup_{N \to \infty} \mathcal{H}(\mu_0^N | \nu_a^N) =: C(a) < +\infty,$$

for some, and thus for all, $a \in (0, \rho_c)$.

Proposition 3.1.1 *Let g be a bounded local jump rate function. Then the One-Block estimate in $L^1(0,T;C(\mathbb{T}^d))$ is satisfied for any sequence $\{\mu_0^N \in \mathbb{P}_1\mathbb{M}_N^d\}_{N \in \mathbb{N}}$ of initial distributions with finite first order moments that satisfies the $O(N^d)$-entropy assumption.*

The proof of this result is the content of this section. The One-Block estimate had been proved so far only for non-strictly condensing ZRPs, i.e. ZRPs satisfying the assumption

$$\lim_{\varphi\uparrow\varphi_c} Z(\varphi) = +\infty.$$

Here we manage to remove this assumption by using the results on the equivalence of ensembles proved originally in [19] and reviewed here in section 1.4.

We begin by proving that in the case of bounded local jump rates g the time dependent one block estimate in $L^1(I; C(\mathbb{T}^d))$ is equivalent to the time dependent one block estimate in any subset $\mathcal{C} \subseteq L^1(I; C(\mathbb{T}^d))$ that is dense in $L^1(I; C(\mathbb{T}^d))$.

Proposition 3.1.2 *Let g be a bounded local jump rate function, let $\{\mu_0^N\}$ be a sequence of initial distributions and let $\mathcal{C}$ be dense in $L^1(I; C(\mathbb{T}^d))$. If g satisfies the time dependent one block estimate in $\mathcal{C}$ with respect to $\{\mu_0^N\}$, then g also satisfies the time dependent one block estimate in $L^1(I; C(\mathbb{T}^d))$ with respect to $\{\mu_0^N\}$.*

Proof Indeed, let $f \in L^1(I; C(\mathbb{T}^d))$ and $\varepsilon > 0$. Since $\mathcal{C}$ is dense in $L^1(I; C(\mathbb{T}^d))$ there exists $h \in \mathcal{C}$ such that

$$\|f - h\|_{L^1(I;C(\mathbb{T}^d))} = \int_0^T \|f_t - h_t\|_u dt < \frac{\varepsilon}{2(\|g\|_u + \phi_c)}.$$

Then since $\|\sigma^N\|_{TV} \leq \|g\|_u$ and $\|\sigma^{N,\ell,\Phi}\|_{TV} \leq \phi_c$ we have that

$$\left| \int_0^T \langle f_s, \sigma_s^N - \sigma_s^{N,\ell,\Phi} \rangle ds \right| < \left| \int_0^T \langle h_s, \sigma_s^N - \sigma_s^{N,\ell,\Phi} \rangle ds \right| + \frac{\varepsilon}{2}$$

and therefore

$$P^N\left\{ \left| \int_0^T \langle f_s, \sigma_s^N - \sigma_s^{N,\ell,\Phi} \rangle ds \right| > \varepsilon \right\} \leq P^N\left\{ \left| \int_0^T \langle h_s, \sigma_s^N - \sigma_s^{N,\ell,\Phi} \rangle ds \right| > \frac{\varepsilon}{2} \right\}.$$

Since g satisfies the time dependent one block estimate in $\mathcal{C}$ and $h \in \mathcal{C}$ it follows by this inequality that

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} P^N\left\{ \left| \int_0^T \langle f_s, \sigma_s^N - \sigma_s^{N,\ell,\Phi} \rangle ds \right| > \varepsilon \right\} = 0.$$

Since $f \in L^1(I; C(\mathbb{T}^d))$ and $\varepsilon > 0$ were arbitrary this proves that g satisfies the time dependent one block estimate in $L^1(I; C(\mathbb{T}^d))$.

3.1.1 Replacement by Spatial Averages

We prove in this section that we can replace the empirical diffusion-rate process σ^N by its ℓ-spatial mean process

$$\sigma_t^{N,\ell} := \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} (g \circ \eta_t)^\ell(x) \delta_{\frac{x}{N}} \tag{3.6}$$

where for each $\eta \in \mathbb{M}_N^d$ we denote by $(g\circ\eta)^\ell$ the ℓ-spatial mean of the function $g\circ\eta \in \mathbb{R}_+^{\mathbb{T}_N^d}$ given by

$$(g\circ\eta)^\ell(x) = \frac{1}{(2\ell+1)^d}\sum_{y\in x+\Lambda_\ell^d} g(\eta_y),$$

in the sense that

$$\lim_{\ell\to\infty}\limsup_{N\to\infty} P^N\Big\{\Big|\int_0^T \langle f_t, \sigma_t^N - \sigma_t^{N,\ell}\rangle dt\Big| > \varepsilon\Big\} = 0 \tag{3.7}$$

for all $f \in L^1(I; C(\mathbb{T}^d))$ and all $\varepsilon > 0$.

For all $f \in C(\mathbb{T}^d)$ we have that

$$\begin{aligned}
\langle f, \sigma^N - \sigma^{N,\ell}\rangle &= \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} f\Big(\frac{x}{N}\Big)\Big(g\big(\eta(x)\big) - \frac{1}{(2\ell+1)^d}\sum_{y\in x+\Lambda_\ell^d} g\big(\eta(y)\big)\Big)\\
&= \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\Big(f\Big(\frac{x}{N}\Big) - \frac{1}{(2\ell+1)^d}\sum_{y\in x+\Lambda_\ell^d} f\Big(\frac{y}{N}\Big)\Big)g\big(\eta(x)\big)\\
&= \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\frac{1}{(2\ell+1)^d}\sum_{y\in x+\Lambda_\ell^d}\Big[f\Big(\frac{x}{N}\Big) - f\Big(\frac{y}{N}\Big)\Big]g\big(\eta(x)\big)
\end{aligned}$$

and therefore if the jump rate g is Lipschitz then for all $f \in L^1(I; C(\mathbb{T}^d))$ and measurable subsets $E \subseteq I$ we have that

$$\Big|\int_E \langle f_t, \sigma_t^N - \sigma_t^{N,\ell}\rangle dt\Big| \le \frac{\|g'\|_u}{(2N\ell+N)^d}\sum_{\substack{x\in\mathbb{T}_N^d\\ y\in x+\Lambda_\ell^d}}\int_E\Big|f_t\big(\frac{x}{N}\big) - f_t\big(\frac{y}{N}\big)\Big|\eta_t(x)dt. \tag{3.8}$$

Lemma 3.1.1 *For all $f \in L^1(I; C(\mathbb{T}^d))$ and all $\varepsilon > 0$ there exists a measurable and a.s. strictly positive function $\bar\delta = \bar\delta(\varepsilon, f) \in L_+^\infty(I)$ such that implication*

$$x, y \in \mathbb{T}^d,\ |x-y| < \bar\delta_t \quad\Longrightarrow\quad |f_t(x) - f_t(y)| < \varepsilon \tag{3.9}$$

holds for all $t \in I$.

Proof We consider the function $\delta = \delta(\varepsilon) : C(\mathbb{T}^d) \longrightarrow [0,1]$ given by $\delta_f := \sup A_f$ where

$$A_f := \big\{\delta \in [0,1]\,\big|\, x, y \in \mathbb{T}_N^d,\ |x-y| < \delta \quad\Longrightarrow\quad |f(x) - f(y)| < \varepsilon\big\}$$

for all $f \in C(\mathbb{T}^d)$. We note that A_f is obviously an interval with $[0, \delta_f) \subseteq A_t \subseteq [0, \delta_f]$. In fact $A_f = [0, \delta_f]$, since if $x, y \in \mathbb{T}_N^d$ are such that $|x-y| < \delta_f$, there exists $\delta_0 > 0$ such that $|x-y| < \delta_0 < \delta_f$ and then $\delta_0 \in A_f$ and therefore $|f(x) - f(y)| < \varepsilon$, which proves that $\delta_f \in A_f$. Furthermore, every $f \in C(\mathbb{T}^d)$ is uniformly continuous which shows that $\delta_f > 0$ for all $f \in C(\mathbb{T}^d)$ and since $\delta_f \in A_f$ we have that

$$x, y \in \mathbb{T}^d,\ |x-y| < \delta_f \quad\Longrightarrow\quad |f(x) - f(y)| < \varepsilon$$

for all $f \in C(\mathbb{T}^d)$.

Let now $f \in L^1(I; C(\mathbb{T}^d))$. We consider a measurable representative of f which we continue to denote by $f : I \longrightarrow C(\mathbb{T}^d)$ and we define the function $\bar{\delta} \equiv \bar{\delta}(\varepsilon, f)$ by $\bar{\delta} = \delta \circ f$. By the previous paragraph it is obvious that the function $\bar{\delta}$ satisfies the required properties and it remains to prove that it is measurable.

Since f is strongly measurable, for the measurability of $\bar{\delta} : I \longrightarrow [0,1]$ it suffices to prove that the function $\delta : C(\mathbb{T}^d) \longrightarrow [0,1]$ is lower semicontinuous with respect to the uniform norm on $C(\mathbb{T}^d)$. So let $\{f_n\} \subseteq C(\mathbb{T}^d)$ be a sequence such that $\|f_n - f\|_u \longrightarrow 0$ for some $f \in C(\mathbb{T}^d)$ and set

$$\underline{\delta} := \liminf_{n\to\infty} \delta_{f_n}.$$

Let $\theta \in (1,\infty)$ be arbitrary and let $\delta > \theta\underline{\delta}$. There exists then a subsequence of $\{f_{k_n}\}$ such that $\theta\delta_{f_{k_n}} < \delta$ for all $n \in \mathbb{N}$ and therefore for each $n \in \mathbb{N}$ we can choose $x_n, y_n \in \mathbb{T}^d_N$ such that

$$|x_n - y_n| < \frac{\delta}{\theta} \quad \text{and} \quad |f_{k_n}(x_n) - f_{k_n}(y_n)| \geq \varepsilon.$$

Then since $\mathbb{T}^d$ is compact there exists $x, y \in \mathbb{T}^d$ and subsequences $\{x_{m_n}\}$ and $\{y_{m_n}\}$ such that $x_{m_n} \longrightarrow x$ and $y_{m_n} \longrightarrow y$ as $n \to \infty$ and since $f_n \longrightarrow f$ uniformly it follows that $\lim_{n\to\infty} f_{k_{m_n}}(x_{m_n}) = f(x)$ and $\lim_{n\to\infty} f_{k_{m_n}}(y_{m_n}) = f(y)$. Then for those $x, y \in \mathbb{T}^d$ we have that

$$|x - y| \leq \frac{\delta}{\theta} < \delta \quad \text{and} \quad |f(x) - f(y)| \geq \varepsilon,$$

which proves that $\delta > \delta_f$. Since $\delta > \theta\underline{\delta}$ was arbitrary, this proves that $\delta_f \leq \theta\underline{\delta}$, and letting $\theta \in (1,\infty)$ tend to 1 it finally follows that

$$\delta_f \leq \underline{\delta} = \liminf_{n\to\infty} f_n.$$

Thus the lower semicontinuity of δ is proved and the proof is complete. □

Lemma 3.1.2 *Let $\{\mu_0^N \in \mathbb{P}\mathbb{M}^d_N\}$ be a sequence of initial distributions associated to a macroscopic profile $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$. Then*

$$\lim_{A\to+\infty} \limsup_{N\uparrow\infty} \mu_0^N \big\{ \langle \pi^N, 1\rangle > A \big\} = 0. \tag{3.10}$$

If in addition the distributions μ_0^N have finite first order moments, i.e. $\mu_0^N \in \mathbb{P}_1\mathbb{M}^d_N$ for all $N \in \mathbb{N}$, then

$$\lim_{A\to+\infty} \sup_{N\in\mathbb{N}} \mu_0^N \big\{ \langle \pi^N, 1\rangle > A \big\} = 0. \tag{3.11}$$

Proof Since $\{\mu^N\}$ is associated to μ_0 we have that

$$\lim_{N\uparrow+\infty} \mu^N \big\{ |\langle \pi^N, 1\rangle - \mu_0(\mathbb{T}^d)| > \delta \big\} = 0$$

for all $\delta > 0$. In particular, given $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$N \geq N_0 \quad \Longrightarrow \quad \mu^N \big\{ |\langle \pi^N, 1\rangle - \mu_0(\mathbb{T}^d)| > 1 \big\} \leq \varepsilon.$$

Then for all $N \geq N_0$ and all $A > \mu_0(\mathbb{T}^d)+1$ we have that

$$\mu_0^N\big\{\langle \pi^N, 1\rangle > A\big\} \leq \mu_0^N\big\{\langle \pi^N, 1\rangle > \mu_0(\mathbb{T}^d)+1\big\} \leq \mu_0^N\big\{|\langle \pi^N, 1\rangle - \mu_0(\mathbb{T}^d)| > 1\big\} \leq \varepsilon,$$

which proves the (3.10). Next, if in addition the measures μ_0^N have finite first order moments, then for all $N = 1, \ldots, N_0 - 1$ and all $A > 0$ we have that

$$\mu_0^N\big\{\langle \pi^N, 1\rangle > A\big\} \leq \frac{1}{A}\int \langle \pi^N, 1\rangle d\mu_0^N = \frac{1}{AN^d}\int |\eta| d\mu_0^N(\eta) \stackrel{A\uparrow\infty}{\longrightarrow} 0.$$

Therefore for all $N = 1, \ldots, N_0-1$ there exists $A_N > 0$ such that $\mu_0^N\big\{\langle \pi^N, 1\rangle > A_N\big\} \leq \varepsilon$ and if we set $A_0 := A_1 \vee \ldots \vee A_{N_0-1} \vee (\mu_0(\mathbb{T}^d)+1)$ we obviously have that

$$A > A_0 \quad \Longrightarrow \quad \sup_{N\in\mathbb{N}} \mu_0^N\big\{\langle \pi^N, 1\rangle > A\big\} \leq \varepsilon$$

which proves (3.11) and completes the proof. □

Using these lemmas and (3.8) we can prove the following.

Proposition 3.1.3 *Let g be a Lipschitz jump rate and let $\{\mu_0^N \in \mathbb{PM}_N^d\}$ be a sequence of initial distributions associated to some macroscopic profile $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$. Then for all $f \in L^1(I; C(\mathbb{T}^d))$ and all $\varepsilon > 0$ it holds that*

$$\lim_{\ell\to\infty}\limsup_{N\to\infty} P^N\bigg\{\bigg|\int_0^T \langle f_t, \sigma_t^N - \sigma_t^{N,\ell}\rangle dt\bigg| > \varepsilon\bigg\} = 0.$$

Proof Indeed, let $f \in L^1(I; C(\mathbb{T}^d))$ and $\varepsilon > 0$ be arbitrary. Since $\|f\|_u \in L^1(I)$, for each $\ell \in \mathbb{N}$ there exists $\delta_\ell > 0$ such that

$$E \in \mathcal{B}_I,\ \ m(E) < \delta_\ell \quad \Longrightarrow \quad \int_E \|f_t\|_u dt < \frac{1}{4\|g'\|_u(2\ell+1)^d}.$$

We consider also the function $\bar\delta_\ell = \bar\delta\big(\frac{1}{2T\|g'\|_u(2\ell+1)^d}, f\big)$ given by the previous lemma. Since the set

$$\{\bar\delta_\ell = 0\} = \bigcap_{k\in\mathbb{N}}\{\bar\delta_\ell < \frac{1}{k}\}$$

is a null set it follows that for all $\ell \in \mathbb{N}$ there exists $k_\ell \in \mathbb{N}$ such that $m\{\bar\delta_\ell < \frac{1}{k_\ell}\} < \delta_\ell$. Then, by (3.8) we have for all $N, \ell \in \mathbb{N}$ that

$$\begin{aligned}
\bigg|\int_{\{\bar\delta_\ell < \frac{1}{k_\ell}\}} \langle f_t, \pi_t^{N,g} - \pi_t^{N,\ell,g}\rangle dt\bigg| &\leq \frac{2\|g'\|_u}{(2N\ell+N)^d}\sum_{\substack{x\in\mathbb{T}_N^d\\ y\in x+\Lambda_\ell^d}}\int_{\{\bar\delta_\ell < \frac{1}{k_\ell}\}} \|f_t\|_u \eta_t(x) dt \\
&= \frac{2\|g'\|_u}{N^d}\int_{\{\bar\delta_\ell < \frac{1}{k_\ell}\}} \|f_t\|_u |\eta_t| dt \\
&\stackrel{P^n\text{-}a.s.}{=} 2\|g'\|\langle \pi_0^N, 1\rangle \int_{\{\bar\delta_\ell < \frac{1}{k_\ell}\}} \|f_t\|_u dt \\
&\leq \frac{1}{2}(2\ell+1)^{-d}\langle \pi^N, 1\rangle.
\end{aligned}$$

On the other hand, for each $\ell \in \mathbb{N}$ there exists $N_\ell \in \mathbb{N}$ such that

$$N \geq N_\ell \quad \Longrightarrow \quad \frac{\ell}{N} < \frac{1}{k_\ell}.$$

Then, for all $\ell \in \mathbb{N}$ and all $N \geq N_\ell$ we have that

$$\Big| f_t\Big(\frac{x}{N}\Big) - f_t\Big(\frac{y}{N}\Big)\Big| < \frac{1}{2T\|g'\|_u(2\ell+1)^d}$$

for all $t \in \{\bar{\delta}_\ell \geq \frac{1}{k_\ell}\}$ and all $x, y \in \mathbb{T}_N^d$ such that $|x-y| \leq \ell$, and therefore

$$\Big(\sum_{y\in x+\Lambda_\ell^d} \Big| f_t\Big(\frac{x}{N}\Big) - f_t\Big(\frac{y}{N}\Big)\Big|\Big)\mathbb{1}_{\{\bar{\delta}^\ell \geq \frac{1}{k_\ell}\}} < \frac{1}{2T\|g'\|_u}.$$

It follows that for all $\ell \in \mathbb{N}$ and all $N \geq N_\ell$ we have

$$\begin{aligned}
\Big|\int_{\{\bar{\delta}^\ell \geq \frac{1}{k_\ell}\}} \langle f_t, \sigma_t^N - \sigma_t^{N,\ell}\rangle dt\Big| &\leq \frac{1}{2T(2N\ell+N)^d}\int_{\{\bar{\delta}^\ell \geq \frac{1}{k_\ell}\}} \sum_{x\in\mathbb{T}_N^d} \eta_t(x)dt \\
&\overset{P^N\text{-}a.s.}{=} \frac{1}{2T}(2\ell+1)^{-d}\langle \pi_0^N, 1\rangle \int_{\{\bar{\delta}^\ell \geq \frac{1}{k_\ell}\}} dt \\
&\leq \frac{1}{2}(2\ell+1)^{-d}\langle \pi_0^N, 1\rangle.
\end{aligned}$$

Therefore, for all $\ell \in \mathbb{N}$ and all $N \geq N_\ell$ we have that

$$\Big|\int_0^T \langle f_t, \pi_t^{N,g} - \pi_t^{N,\ell,g}\rangle dt\Big| \leq (2\ell+1)^{-d}\langle \pi_0^N, 1\rangle, \qquad P^N\text{-a.s.}.$$

It follows that for all $\ell \in \mathbb{N}$ and all $N \geq N_\ell$ we have that

$$\begin{aligned}
P^N\Big\{\Big|\int_0^T \langle f_t, \sigma_t^N - \sigma_t^{N,\ell}\rangle dt\Big| > \varepsilon\Big\} &\leq P^N\big\{\langle \pi_0^N, 1\rangle > \varepsilon(2\ell+1)^d\big\} \\
&= \mu_0^N\big\{\langle \pi^N, 1\rangle > \varepsilon(2\ell+1)^d\big\}
\end{aligned}$$

and

$$\limsup_{N\to\infty} P^N\Big\{\Big|\int_0^T \langle f_t, \sigma_t^N - \sigma_t^{N,\ell}\rangle dt\Big| > \varepsilon\Big\} \leq \limsup_{n\to\infty} \mu_0^N\big\{\langle \pi^N, 1\rangle > \varepsilon(2\ell+1)^d\big\}$$

But as we have seen, since $\{\mu_0^N \in \mathbb{P}_1\mathbb{M}_N^d\}$ is associated to macroscopic profile $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$ we have that

$$\lim_{\ell\to\infty}\limsup_{N\to\infty} \mu_0^N\big\{\langle \pi^N, 1\rangle > \varepsilon(2\ell+1)^d\big\} = 0,$$

and the proof is complete. $\square$

3.1.2 Entropy Production and The Dirichlet Form

The following estimates play a central role in the proof of the One-Block estimate.

Proposition 3.1.4 *Let g be a local rate function and let $\{\mu_0^N \in \mathbb{P}_1\mathbb{M}_N^d\}$ be a sequence of initial distributions satisfying the entropy assumption for some $\rho_* \in (0,\rho_c)$ and some constant $C_{\rho_*} \geq 0$. Let $(S_t^N)_{t\in\mathbb{R}_+}$ denote the transition semigroup of the diffusively rescaled nearest neighbor ZR process with local rate function g on the discrete torus $\mathbb{T}_N^d$. We set $\mu_t^N := \mu_0^N S_t^N$ for all $t \in \mathbb{R}_+$. Then $\mu_t^N \ll \nu_{\rho_*}^N$ for all $t \in \mathbb{R}_+$ and if we denote by $f_t^N \equiv f_t^{N,\rho_*}$ the density of μ_t^N with respect to $\nu_{\rho_*}^N$ we have for all $t \in \mathbb{R}_+$ that*

$$\mathcal{H}\bigg(\frac{1}{t}\int_0^t f_s^N ds\Big|\nu_{\rho_*}^N\bigg) \leq C_{\rho_*}N^d, \qquad D_N\bigg(\frac{1}{t}\int_0^t f_s^N ds\bigg) \leq \frac{C_{\rho_*}}{2t}N^{d-2},$$

where $D_N := \mathfrak{D}_N(\sqrt{\cdot}) : L^1_{+,1} \longrightarrow \mathbb{R}_+$ and $\mathfrak{D}_N$ is the Dirichlet form of the ZRP on $\mathbb{T}_N^d$.

Proof We prove first that $\mu_t^N \ll \nu_{\rho_*}^N$ for all $t \geq 0$. Since the sequence $\{\mu_0^N\}$ of the initial distributions satisfies the entropy assumption we have that

$$\mathcal{H}(\mu_0^N|\nu_{\rho_*}^N) \leq C_{\rho_*}N^d < \infty$$

which by proposition A.3.7 implies that $\mu_t^N \ll \nu_{\rho_*}^N$ for all $t \geq 0$ with density given by

$$f_t^N := \frac{d\mu_t^N}{d\nu_{\rho_*}^N} = S_t^N\frac{d\mu_0^N}{d\nu_{\rho_*}}$$

since S_t^N is self-adjoint in $L^2(\nu_{\rho_*}^N)$ due to the fact that the n.n. ZRP is symmetric. Since by the same proposition the microscopic entropy $\mathcal{H}(\mu_t^N|\nu_{\rho_*}^N)$ is a non-increasing function of time we have that

$$\mathcal{H}(\mu_t^N|\nu_{\rho_*}^N) \leq \mathcal{H}(\mu_0^N|\nu_{\rho_*}^N) \leq C_{\rho_*}N^d$$

and therefore by the linear convexity of the relative entropy we have that

$$\mathcal{H}\bigg(\frac{1}{t}\int_0^t f_s^N ds\Big|\nu_{\rho_*}^N\bigg) = \mathcal{H}\bigg(\frac{1}{t}\int_0^t \mu_s^N ds\Big|\nu_{\rho_*}^N\bigg) \leq \frac{1}{t}\int_0^t \mathcal{H}(\mu_s^N|\nu_{\rho_*}^N)ds \leq C_{\rho_*}N^d.$$

Furthermore, by (A.84) we have that

$$2N^2\int_0^t D_N(f_s)ds \leq \mathcal{H}(\mu_t^N|\nu_{\rho_*}^N) + 2N^2\int_0^t D_N(f_s)ds \leq \mathcal{H}(\mu_0^N|\nu_{\rho_*}^N) \leq C_{\rho_*}N^d$$

and therefore by the convexity of the functional D_N we have that

$$D_N\bigg(\frac{1}{t}\int_0^t f_s^N ds\bigg) \leq \frac{1}{t}\int_0^t D_N(f_s^N)ds \leq \frac{C_{\rho_*}}{2t}N^{d-2}. \qquad \square$$

3.1.3 Reduction to a Static Problem

In this section we will see how by using the estimates of the previous subsection one can reduce the time dependent problem of whether a Lipschitz local jump rate g satisfies the time dependent one block estimate to a static one, i.e. one that does not depend on time.

We begin by noting that by proposition 3.1.3 in order to prove that a Lipschitz local jump rate function g satisfies the time dependent one block estimate in $C(I \times \mathbb{T}^d) \subseteq L^1(I; C(\mathbb{T}^d))$, $I := [0, T]$, $T > 0$, it suffices to prove that

$$\lim_{\ell \to \infty} \limsup_{N \to \infty} P^N \left\{ \left| \int_0^T \langle h_t, \pi_t^{N,\ell,g} - \pi_t^{N,\ell,\Phi} \rangle dt \right| > \varepsilon \right\} = 0$$

for all $h \in C(I \times \mathbb{T}^d)$ and all $\varepsilon > 0$. For each $\ell \in \mathbb{N}$ we consider the cylinder function $V^\ell : \mathbb{M}_N^d \longrightarrow \mathbb{R}_+$ given by

$$V^\ell = \left| (g \circ \eta)^\ell(0) - \Phi\big(\eta^\ell(0)\big) \right| = \left| \frac{1}{(2\ell+1)^d} \sum_{y \in \Lambda_\ell^d} g\big(\eta(y)\big) - \Phi\big(\eta^\ell(0)\big) \right|$$

and as usual we write $V_t^\ell(\eta) = V^\ell(\eta_t)$ for all $\eta \in D(\mathbb{R}_+; \mathbb{M}_N^d)$.

Let $f \in C(I \times \mathbb{T}^d)$. Then

$$\left| \int_0^T \langle h_t, \pi_t^{N,\ell,g} - \pi_t^{N,\ell,\Phi} \rangle dt \right| \leq \|f\|_{C(I \times \mathbb{T}^d)} \int_0^T \|\pi_t^{N,\ell,g} - \pi_t^{N,\ell,\Phi}\|_{TV} dt.$$

We denote by $m^N := \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \delta_{\frac{x}{N}}$ the normalized counting measure on $\mathbb{T}_N^d$ and set $L^1(\mathbb{T}_N^d) := L^1(m^N)$. Then the measures $\sigma^{N,\ell}$ and $\pi^{N,\ell,\Phi}$ are absolutely continuous with respect to m^N and

$$\|\sigma^{N,\ell} - \pi^{N,\ell,\Phi}\|_{TV} = \left\| \frac{d\sigma^{N,\ell}}{dm^N} - \frac{d\pi^{N,\ell,\Phi}}{dm^N} \right\|_{L^1(\mathbb{T}_N^d)} = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x V^\ell.$$

Therefore

$$\left| \int_0^T \langle h_t, \pi_t^{N,\ell,g} - \pi_t^{N,\ell,\Phi} \rangle dt \right| \leq \|h\|_{C(I \times \mathbb{T}^d)} \int_0^T \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x V_t^\ell dt.$$

Consequently, for all $h \in C(I \times \mathbb{T}^d)$ and all $\varepsilon > 0$ we have that

$$\begin{aligned}
P^N \left\{ \left| \int_0^T \langle h_t, \pi_t^{N,\ell,g} - \pi_t^{N,\ell,\Phi} \rangle dt \right| > \varepsilon \right\} &\leq P^N \left\{ \int_0^T \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x V_t^\ell dt > \frac{\varepsilon}{\|h\|_u} \right\} \\
&\leq \frac{\|h\|_u}{\varepsilon} \iint_0^T \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x V_t^\ell dt dP^N \\
&= \frac{\|h\|_u}{\varepsilon} \int_0^T \int \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x V_t^\ell dP^N dt \\
&= \frac{\|h\|_u}{\varepsilon} \int_0^T \int \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x V^\ell d\mu_t^N dt,
\end{aligned}$$

and therefore in order to prove that the local jump rate g satisfies the time-dependent one block estimate it suffices to prove that

$$\lim_{\ell \to \infty} \limsup_{N \to \infty} \int_0^T \int \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x V^\ell d\mu_t^N dt \leq 0. \tag{3.12}$$

Now, the limit above can be easily reduced to a static one by considering the time average

$$\bar{\mu}_T^N := \frac{1}{T}\int_0^T \mu_t^N dt$$

of $\{\mu_t^N\}_{t\in I}$. Since $\mu_t^N \ll \nu_{\rho_*}^N$ for all $t \in \mathbb{R}_+$, the time average $\bar{\mu}_T^N$ is obviously absolutely continuous with respect to $\nu_{\rho_*}^N$ with density

$$\bar{f}_T^N := \frac{\bar{\mu}_T^N}{d\nu_{\rho_*}^N} = \frac{1}{T}\int f_t^N dt,$$

and so in this notation the double integral in (3.12) can be written as

$$\int_0^T \int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} \tau_x V^\ell d\mu_t^N dt = \int \frac{T}{N^d}\sum_{x\in\mathbb{T}_N^d} \tau_x V^\ell d\bar{\mu}_T^N = \int \frac{T}{N^d}\sum_{x\in\mathbb{T}_N^d} (\tau_x V^\ell)\bar{f}_T^N d\nu_{\rho_*}^N,$$

and therefore in order to prove (3.12) it suffices to prove that

$$\lim_{\ell\to\infty}\limsup_{N\to\infty}\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}(\tau_x V^\ell)\bar{f}_T^N d\nu_{\rho_*}^N \leq 0. \tag{3.13}$$

By the estimates of proposition 3.1.4 for the time averaged density $\bar{f}_T^N$ we know that there exists a constant $C_0 > 0$, say $C_0 := C_{\rho_*} \vee \frac{C_{\rho_*}}{2T}$, such that

$$\mathcal{H}(\bar{f}_T^N|\nu_{\rho_*}^N) \leq C_0 N^d \quad \text{and} \quad D_N(\bar{f}_T^N) \leq C_0 N^{d-2},$$

where for each positive density $f \in L^1_{+,1}(\nu_{\rho_*}^N) \subseteq \mathbb{PM}_N^d$ we abbreviate by

$$H_N(f) \equiv H_{N,\rho_*}(f) := \mathcal{H}(f|\nu_{\rho_*}^N) \equiv \mathcal{H}(f d\nu_{\rho_*}^N|\nu_{\rho_*}^N)$$

its relative entropy with respect to $\nu_{\rho_*}^N$. Therefore in order to prove (3.13) it suffices to prove that for all finite constants $C_0 > 0$ we have that

$$\lim_{\ell\to\infty}\limsup_{N\to\infty}\sup_{\substack{H_N(f)\leq C_0N^d\\ D_N(f)\leq C_0 N^{d-2}}}\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}(\tau_x V^\ell) f d\nu_{\rho_*}^N \leq 0, \tag{3.14}$$

where the supremum is taken among densities $f \in L^1_{+,1}(\nu_{\rho_*}^N)$.

Definition 3.1.2 Let g be a local jump rate function. We say that g *satisfies the static one block estimate* if the limit (3.14) holds for some $\rho_* \in (0, \rho_c)$ and all positive constants $C_0 > 0$.

In this terminology we can state the results proves in this section as follows.

Proposition 3.1.5 *Let g be a Lipschitz local jump rate function. If g satisfies the static one block estimate then it satisfies the time dependent one block estimate in $C(I \times \mathbb{T}^d)$ with respect to any sequence of initial distributions $\{\mu_0^N\}$ that satisfies the $O(N^d)$-entropy assumption.*

3.1.4 Proof of the Static One Block Estimate

In this section we prove the static one block estimate.

Cutting off large densities

We begin with a technical lemma that allows us to cut off large densities in the integral appearing in the defining limit of the static one block estimate.

Lemma 3.1.3 *For all finite constants $C_0 > 0$ and all $\rho_* \in (0, \rho_c)$ there exists a finite constant $C_1 = C_1(C_0, \rho_*) > 0$ such that*

$$\sup_{N\in\mathbb{N}} \sup_{H_N(f)\leq C_0 N^d} \int \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \eta(x) f d\nu_{\rho_*}^N \leq C_1 < \infty.$$

Proof By the entropy inequality we have that for all $f \in L^1_{+,1}(\nu_{\rho_*}^N)$ and all $\rho > 0$,

$$\begin{aligned}\int \langle \pi^N, 1\rangle f d\nu_{\rho_*}^N &\leq \frac{1}{\rho N^d}\left\{\log \int e^{\rho\sum_{x\in\mathbb{T}_N^d}\eta(x)} d\nu_{\rho_*}^N + H_N(f)\right\}\\ &= \frac{1}{\gamma N^d}\log\int \prod_{x\in\mathbb{T}_N^d} e^{\rho\eta(x)} d\nu_{\rho_*}^N + \frac{H_N(f)}{\rho N^d}\\ &\leq \frac{1}{\rho}\log\int e^{\rho\eta(0)} d\nu_{\rho_*} + \frac{C_0}{\rho} = \frac{\Lambda_{\rho_*}(\rho) + C_0}{\rho},\end{aligned}$$

where $\Lambda_{\rho_*} = \Lambda_{\nu_{\rho_*}}$ is the logarithmic m.g.f. of $\nu_{\rho_*} \in \mathbb{P}\mathbb{Z}_+$. But since $\rho_* < \rho_c$, as we know ν_{ρ_*} has exponential moments with $[0, \log\frac{\phi_c}{\Phi(\rho_*)}) \subseteq \mathcal{D}_{\Lambda_{\rho_*}}$ and therefore we can take as $C_1 = C_1(C_0, \rho_*)$ the constant

$$C_1 := \inf_{\rho>0} \frac{\Lambda_{\nu_{\rho_*}}(\rho) + C_0}{\rho} = \inf_{\rho\in\mathcal{D}_{\Lambda_{\rho_*}}} \frac{\Lambda_{\rho_*}(\rho) + C_0}{\rho} < +\infty.$$

Corollary 3.1.1 *Let g be a local rate function. If for some $\rho_* \in (0, \rho_c)$ and all finite constants $a, C_0 > 0$ we have*

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty} \sup_{D_N(f)\leq C_0 N^{d-2}} \int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\left[\tau_x V^\ell - a\eta^\ell(x)\right] f d\nu_{\rho_*}^N \leq 0 \tag{3.15}$$

then g satisfies the static one block estimate.

Proof Let $C_0 > 0$ be a finite constant and let $\rho_* \in (0, \rho_c)$ such that (3.15) holds. By the previous lemma there exists a constant $C_1 = C_1(C_0, \rho_*) > 0$ such that

$$\sup_{N\in\mathbb{N}} \sup_{H_N(f)\leq C_0 N^d} \int \langle \pi^N, 1\rangle f d\nu_{\rho_*}^N \leq C_1 < \infty.$$

Then for all $a > 0$ and all $f \in L^1_{+,1}(\nu_{\rho_*}^N)$ such that $H_N(f) \leq C_0 N^d$ and $D_N(f) \leq C_0 N^{d-2}$ we have that

$$\begin{aligned}\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}(\tau_x V^\ell) f d\nu_{\rho_*}^N &= a\int \langle \pi^N, 1\rangle f d\nu_{\rho_*}^N + \int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\left[\tau_x V^\ell - a\eta^\ell(x)\right] f d\nu_{\rho_*}^N\\ &\leq aC_1 + \sup_{D_N(f)\leq C_0 N^{d-2}} \int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\left[\tau_x V^\ell - a\eta^\ell(x)\right] f d\nu_{\rho_*}^N\end{aligned}$$

for all $N, \ell \in \mathbb{N}$. Therefore

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty}\sup_{\substack{H_N(f)\le C_0N^d\\ D_N(f)\le C_0N^{d-2}}}\int\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}(\tau_xV^\ell)fd\nu_{\rho_*}^N\le aC_1,$$

which since $a>0$ is arbitrary proves that g satisfies the static one block estimate. □

These results allows us to restrict the integral in the static one block estimate to configurations with mean spatial density bounded by some finite constant. Indeed, since g is assumed bounded, it has in particular sub-linear growth rate, i.e.

$$\limsup_{k\to\infty}\frac{g(k)}{k}=0.$$

Therefore, for each $a>0$ there exists a constant $C_2(a)\ge 0$ such that

$$g(k)\le C_2(a)+ak$$

for all $k\in\mathbb{Z}_+$. Consequently for all $b>0$,

$$\Phi(\rho)=\int g(k)d\nu_\rho(k)\le C_2(a)+a\cdot(\rho\wedge\rho_c)$$

for all $\rho\ge 0$, and therefore for all $a>0$ V^ℓ is bounded above by

$$V^\ell\le\frac{1}{(2\ell+1)^d}\sum_{y\in\Lambda_\ell^d}g\big(\eta(y)\big)+\Phi\big(\eta^\ell(0)\big)\le 2C_2\big(\frac{a}{4}\big)+\frac{a}{2}\eta^\ell(0).$$

It follows that the function

$$V^\ell-a\eta^\ell(0)\le 2C_2\big(\frac{a}{4}\big)-\frac{a}{2}\eta^\ell(0)$$

is negative on configurations $\eta\in\mathbb{M}_N^d$ satisfying

$$\eta_0^\ell>C_3(a):=\frac{4}{a}C_2\big(\frac{a}{4}\big),$$

and therefore we can bound the function $V^\ell-a\eta^\ell(0)$ above by

$$V^\ell-a\eta^\ell(0)\le\big[V^\ell-a\eta^\ell(0)\big]\mathbb{1}_{\{\eta^\ell(0)\le C_3(a)\}}\le V^\ell\mathbb{1}_{\{\eta^\ell(0)\le C_3(a)\}}.$$

Therefore for each $a>0$, we can bound the integral in (3.15) above by

$$\int\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\big[\tau_xV^\ell-a\eta^\ell(x)\big]fd\nu_{\rho_*}^N\le\int\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_xV^\ell\mathbb{1}_{\{\eta^\ell(x)\le C_3(a)\}}fd\nu_{\rho_*}^N.$$

It follows that if

$$\lim_{\ell\to\infty}\limsup_{N\to\infty}\sup_{D_N(f)\le C_0N^{d-2}}\int\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_xV^\ell\mathbb{1}_{\{\eta^\ell(x)\le C_3\}}fd\nu_{\rho_*}^N\le 0,\tag{3.16}$$

for some $\rho_*\in(0,\rho_c)$ and all finite constants $C_0,C_3>0$ then (3.15) holds for all $a>0$ and thus the static one block estimate is satisfied. In other words we have proved

Proposition 3.1.6 *Let g be a local jump rate function with sublinear growth rate. If (3.16) holds for some $\rho_*\in(0,\rho_c)$ and all finite constants $C_0,C_3>0$ then g satisfies the static one block estimate.*

Reduction to microscopic cubes

Notice that by definition the cylinder function $V^\ell \mathbb{1}_{\{\eta^\ell(0)\leq C_3\}}$ is a function of the natural projection

$$p^\ell : \mathbb{M}^d_\infty := \mathbb{Z}_+^{\mathbb{Z}^d} \longrightarrow \mathbb{Z}_+^{\Lambda_\ell^d} =: \mathbb{M}^d_{(2\ell+1)}, \quad \ell \in \mathbb{Z}_+.$$

This step of the proof consists in taking advantage of this fact to project the density f over a configuration space that does not depend on the scale parameter N.

In the rest of the proof we fix an arbitrary $\rho_* \in (0, \rho_c)$ and prove that (3.16) holds. Since the measure $\nu^N_{\rho_*}$ is translation invariant we can write the integral in (3.16) as

$$\int \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \tau_x V^\ell \mathbb{1}_{\{\eta^\ell(x)\leq C_3\}} f d\nu^N_{\rho_*} = \int V^\ell \mathbb{1}_{\{\eta^\ell(0)\leq C_3\}} \bar{f} d\nu^N_{\rho_*}, \tag{3.17}$$

where here $\bar{f} \equiv \bar{f}^N$ denotes the average of all translations of f,

$$\bar{f} := \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \tau_x f.$$

For each density $f \in L^1_{+,1}(\nu^N_{\rho_*})$ we denote by $f|_\ell$ the quotient conditional expectation with respect to the probability measure $\nu^N_{\rho_*}$ of f given the projection p^ℓ, that is the $\pi^\ell_* \nu^N_{\rho_*}$-a.s. uniquely defined function $h : \mathbb{M}^d_{(2\ell+1)^d} \longrightarrow \mathbb{R}$ with the property that

$$h \circ p^\ell = \mathbb{E}_{\nu^N_{\rho_*}}(f|p^\ell),$$

where here $\mathbb{E}_{\nu^N_{\rho_*}}$ denotes the expectation with respect to $\nu^N_{\rho_*}$. Then since $V^\ell \mathbb{1}_{\{\eta^\ell(0)\leq C_3\}}$ is a function of p^ℓ, and thus $\sigma(p^\ell)$-measurable, by the definition of conditional expectation we can replace $\bar{f}$ by $\bar{f}|_\ell(p^\ell)$ in the right hand side of (3.17) and write

$$\int V^\ell \mathbb{1}_{\{\eta^\ell(0)\leq C_3\}} \bar{f} d\nu^N_{\rho_*} = \int V^\ell \mathbb{1}_{\{\eta^\ell(0)\leq C_3\}} \bar{f}|_\ell(p^\ell) d\nu^N_{\rho_*} = \int V^\ell \mathbb{1}_{\{\eta^\ell(0)\leq C_3\}} \bar{f}|_\ell dp^\ell_* \nu^N_{\rho_*},$$

where of course in the right hand side, with a slight abuse of notation $V^\ell \mathbb{1}_{\{\eta^\ell(0)\leq C_3\}}$ is identified with its restriction on $\mathbb{M}^d_{(2\ell+1)^d}$. Obviously the measure

$$p^\ell_* \nu^N_{\rho_*} = \nu^{\Lambda^d_\ell}_{\rho_*} = \nu^{(2\ell+1)}_{\rho_*} \in \mathbb{P}\mathbb{M}^d_{(2\ell+1)}$$

does not depend on the scaling parameter N. In what follows we set $\ell_\star := 2\ell + 1$ for all $\ell \in \mathbb{Z}_+$ and we write $\nu^{\ell_\star}_{\rho_*}$ for $p^\ell_* \nu^N_{\rho_*}$. In this notation, for all constants $C_0, C_3 > 0$ we can rewrite inequality (3.16) as

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} \sup_{D_N(f)\leq C_0 N^{d-2}} \int V^\ell \mathbb{1}_{\{\eta^\ell(0)\leq C_3\}} \bar{f}|_\ell d\nu^{\ell_\star}_{\rho_*} \leq 0. \tag{3.18}$$

Consequently, in order to prove that g satisfies the static one block estimate it suffices to show that (3.18) holds for all constants $C_0, C_3 > 0$.

Before proceeding further it will be good to have an explicit expression for $f|_\ell$ for each $f \in L^1_{+,1}(\nu^N_{\rho_*})$. Such an explicit expression is given by the following lemma.

Lemma 3.1.4 *We let* $\{(\Omega_i, \mathcal{F}_i, P_i)\}_{i\in I}$ *be a family of probability spaces and set*

$$(\Omega^J, \mathcal{F}^J, P^J) := \bigotimes_{i\in J}(\Omega_i, \mathcal{F}_i, P_i)$$

for all $J \subseteq I$*. For notational simplicity we set* $(\Omega, \mathcal{F}, P) := (\Omega^I, \mathcal{F}^I, P^I)$ *and for each* $J \subseteq I$ *we denote by* $\pi_J : \Omega \longrightarrow \Omega_J$ *the natural projection. Then for each random variable* $X \in L^1(\Omega, \mathcal{F}, P)$ *and all* $J \subseteq I$ *we have that*

$$\mathbb{E}(X|\pi_J) = X^J \circ \pi_J$$

where $X^J : \Omega^J \longrightarrow \mathbb{R}$ *is the function defined by* $X^J(\omega_J) = \int X(\omega_J, \theta)dP_{I\setminus J}(\theta)$*.*

Proof Let $\pi_J^{-1} : \Omega^J \longrightarrow \mathbb{P}\Omega$ denote the disintegration of P with respect to $\pi_{J_*}P = P_J$. Since P is a product measure, π_J^{-1} is given by the formula

$$\pi_{J,\omega_J}^{-1} = \delta_{\omega_J} \times P_{I\setminus J} \in \mathbb{P}\Omega, \qquad \omega_J \in \Omega_J,$$

and let $\delta_X : \Omega \longrightarrow \mathbb{P}\mathbb{R}$ denote the kernel given by $\delta_X(\omega) = \delta_{X(\omega)}$. We claim that the Markovian kernel $\delta_X \circ \pi_J^{-1} : \Omega^J \longrightarrow \mathbb{P}\mathbb{R}$ is the quotient conditional distribution of X given π_J, i.e. that for all functions $F \in B(\Omega^J \times \mathbb{R})$ we have that

$$\int Fd(\pi_J, X)_*P = \iint F(\omega_J, a)d[\delta_X \circ \pi_J^{-1}]_{\omega_J}(a)dP_J(\omega_J).$$

Indeed, the term in the right hand side in equality above is equal to

$$\begin{aligned}\iint F\big(\omega_J, X(\omega)\big)d\pi_{J,\omega_J}^{-1}(\omega)dP_J(\omega_J) &= \iint F\big(\omega_J, X(\omega_J, \omega_{I\setminus J})\big)dP_{I\setminus J}(\omega_{I\setminus J})dP_J(\omega_J)\\ &= \int F\big(\pi_J(\omega), X(\omega)\big)dP(\omega)\\ &= \int Fd(\pi_J, X)_*P,\end{aligned}$$

as required. It follows that the quotient conditional expectation $\widetilde{\mathbb{E}}(X|\pi_J)$ of X given π_J is given by the formula

$$\begin{aligned}\widetilde{\mathbb{E}}(X|\pi_J)(\omega_J) &= [\delta_X \circ \pi_J^{-1}](id_{\mathbb{R}})(\omega_J) = \int id_{\mathbb{R}}d[\delta_X \circ \pi_J^{-1}]_{\omega_J}\\ &= \iint id_{\mathbb{R}}d\delta_{X(\omega)}d\pi_{J,\omega_J}^{-1}(\omega) = \int X(\omega_J, \omega_{I\setminus J})dP_{I\setminus J}(\omega_{I\setminus J}),\end{aligned}$$

which proves the claim. □

According to this lemma, the quotient conditional expectation $f|_\ell \in L^1_{+,1}(\nu_{\rho_*}^{\ell_\star})$ of a density $f \in L^1_{+,1}(\nu_{\rho_*}^N)$ is given by the formula

$$f|_\ell(\xi) = \int f(\xi, \zeta)d\nu_{\rho_*}^{\mathbb{T}_N^d\setminus\Lambda_\ell^d}(\zeta).$$

Furthermore, since in this particular case we have that $\nu_{\rho_*}^{\ell_\star}(\xi) > 0$ for all $\xi \in \mathbb{M}_{\ell_\star}^d$, we can rewrite the formula of $f|_\ell$ as

$$f|_\ell(\xi) = \frac{1}{\nu_{\rho_*}^{\ell_\star}(\xi)}\int \mathbb{1}_{\{\xi\}}(p^\ell)fd\nu_{\rho_*}^N. \tag{3.19}$$

Indeed, for all $\xi \in \mathbb{M}^d_{\ell_\star}$ we have that

$$\begin{aligned}\int \mathbb{1}_{\{\xi\}}(p^\ell) f d\nu^N_{\rho_*} &= \iint \mathbb{1}_{\{\xi\}}\big(p^\ell(\zeta_1,\zeta_2)\big) f(\zeta_1,\zeta_2) d\nu^{\ell_\star}_{\rho_*}(\zeta_1) d\nu^{\mathbb{T}^d_N\setminus\Lambda^d_\ell}_{\rho_*}(\zeta_2) \\ &= \nu^{\ell_\star}_{\rho_*}(\xi)\int f(\xi,\zeta_2) d\nu^{\mathbb{T}^d_N\setminus\Lambda^d_\ell}_{\rho_*}(\zeta_2) = \nu^{\ell_\star}_{\rho_*}(\xi) f|_\ell(\xi),\end{aligned}$$

and (3.19) is proved.

Estimates on the Dirichlet form of $\bar{f}|_\ell$

The third step of the proof consists in obtaining information regarding the density $\bar{f}|_\ell$ from the estimate $D_N(f) \leq C_0 N^{d-2}$ on the Dirichlet form of f. To this end, for each pair of sites $x, y \in \mathbb{T}^d_N$ we denote by $L_{x,y}$ the part of the generator corresponding to jumps across the bond $\{x,y\}$, that is

$$L_{x,y} f(\eta) := \{f(\eta^{x,y}) - f(\eta)\} g(\eta_x) p(y-x) + \{f(\eta^{y,x}) - f(\eta)\} g(\eta_y) p(x-y).$$

We denote furthermore by $D^{x,y}_N$ the part of the (normalized) Dirichlet form D_N corresponding to jumps over the bond $\{x,y\}$, that is for all $f \in L^1_{+,1}(\nu^N_{\rho_*})$ we define:

$$D^{x,y}_N(f) := -\langle L_{x,y}\sqrt{f}, \sqrt{f}\rangle_{L^2(\nu^N_{\rho_*})} = \int \Big\{\sqrt{f(\eta^{x,y})} - \sqrt{f(\eta)}\Big\}^2 g(\eta_x) p^s(y-x) d\nu^N_{\rho_*}(\eta),$$

where $p^s(z) := \frac{p(z)+p(-z)}{2}$ is the symmetrization of the elementary step distribution p. To verify this explicit formula for the piece of the Dirichlet form let $L^*_{x,y}$ denote the adjoint of $L_{x,y}$ in $L^2(\nu^N_{\rho_*})$. Then, since $L_{x,y}(\eta,\zeta)\nu^N_{\rho_*}(\eta) = L^*_{x,y}(\zeta,\eta)\nu^N_{\rho_*}(\zeta)$ for all $\eta,\zeta \in \mathbb{M}^d_N$ we have for all functions $f \in L^2(\nu^N_{\rho_*})$ that

$$\begin{aligned}2\langle f, L_{x,y} f\rangle_{L^2(\nu^N_{\rho_*})} &= \langle f, L_{x,y} f\rangle_{L^2(\nu^N_{\rho_*})} + \langle L^*_{x,y} f, f\rangle_{L^2(\nu^N_{\rho_*})} \\ &= \sum_{\eta,\zeta\in\mathbb{M}^d_N} f(\eta)[f(\zeta) - f(\eta)] L_{x,y}(\eta,\zeta)\nu^N_{\rho_*}(\eta) \\ &\quad + \sum_{\eta,\zeta\in\mathbb{M}^d_N} f(\zeta)[f(\eta) - f(\zeta)] L_{x,y}(\zeta,\eta)\nu^N_{\rho_*}(\zeta) \\ &= -\sum_{\eta,\zeta\in\mathbb{M}^d_N} [f(\zeta) - f(\eta)]^2 \nu^N_{\rho_*}(\eta) L_{x,y}(\eta,\zeta) \\ &= -\sum_{\eta\in\mathbb{M}^d_N} [f(\eta^{x,y}) - f(\eta)]^2 \nu^N_{\rho_*}(\eta) L_{x,y}(\eta,\eta^{x,y}) \\ &\quad - \sum_{\eta\in\mathbb{M}^d_N} [f(\eta^{y,x}) - f(\eta)]^2 \nu^N_{\rho_*}(\eta) L_{x,y}(\eta,\eta^{y,x}).\end{aligned}$$

But by the change of variables $\zeta := \eta^{y,x}$ the second term in the sum above becomes

$$\begin{aligned}\sum_{\eta\in\mathbb{M}^d_N} [f(\eta^{y,x}) - f(\eta)]^2 \nu^N_{\rho_*}(\eta) L_{x,y}(\eta,\eta^{y,x}) &= \sum_{\zeta\in\mathbb{M}^d_N} [f(\zeta) - f(\zeta^{x,y})]^2 \nu^N_{\rho_*}(\zeta^{x,y}) L_{x,y}(\zeta^{x,y},\zeta) \\ &= \sum_{\zeta\in\mathbb{M}^d_N} [f(\zeta) - f(\zeta^{x,y})]^2 \nu^N_{\rho_*}(\zeta) L^*_{x,y}(\zeta,\zeta^{x,y}).\end{aligned}$$

So denoting by $L^s_{x,y} := \frac{L_{x,y}+L^*_{x,y}}{2}$ the symmetrization of $L_{x,y}$ we have that

$$D^{x,y}_N(f) = -\langle \sqrt{f}, L_{x,y}\sqrt{f}\rangle_{L^2(\nu^N_{\rho_*})} = \sum_{\eta\in\mathbb{M}^d_N} \left\{\sqrt{f(\eta^{x,y})} - \sqrt{f(\eta)}\right\}^2 \nu^N_{\rho_*}(\eta) L^s_{x,y}(\eta,\eta^{x,y}).$$

But $L^*_{x,y}(\eta,\eta^{x,y}) = g(\eta_x)p(x-y)$ and so $L^s_{x,y}(\eta,\eta^{x,y}) = g(\eta_x)p^s(y-x)$ which proves the claimed formula for the Dirichlet form.

We denote next by $\mathfrak{B}_N$ the set of all admissible non-oriented bonds in $\mathbb{T}^d_N$, i.e.

$$\mathfrak{B}_N := \left\{\{x,y\} \subseteq \mathbb{T}^d_N \,\middle|\, p^s(y-x) > 0\right\}.$$

With this notation we have that

$$L = \sum_{\{x,y\}\in\mathfrak{B}_N} L_{x,y} \qquad \text{and} \qquad D_N = \sum_{\{x,y\}\in\mathfrak{B}_N} D^{x,y}_N.$$

Note that the Dirichlet form is translation invariant, i.e.

$$D_N(\tau_z f) = D_N(f)$$

for all $z \in \mathbb{T}^d_N$ and all $f \in L^1_{+,1}(\nu^N_{\rho_*})$. Indeed, it is easy to see that $\tau_z(\eta^{x,y}) = (\tau_z\eta)^{x-z,y-z}$ and therefore since $\nu^N_{\rho_*}$ is translation invariant and the Dirichlet form is given by a sum over all bonds we have that

$$\begin{aligned}
D_N(\tau_z f) &= \sum_{\{x,y\}\in\mathfrak{B}_N} D^{x,y}_N(\tau_z f) \\
&= \sum_{\{x,y\}\in\mathfrak{B}_N} p^s(y-x) \sum_{\eta\in\mathbb{M}^d_N} g(\eta_x)\left[\sqrt{\tau_z f(\eta^{x,y})} - \sqrt{\tau_z f(\eta)}\,\right]^2 \nu^N_{\rho_*}(\eta) \\
&= \sum_{\{x,y\}\in\mathfrak{B}_N} p^s(y-x) \sum_{\eta\in\mathbb{M}^d_N} g(\eta_{x-z})\left[\sqrt{f(\eta^{x-z,y-z})} - \sqrt{f(\eta)}\,\right]^2 \nu^N_{\rho_*}(\eta) \\
&= \sum_{\{x,y\}\in\mathfrak{B}_N} p^s(y-x) \sum_{\eta\in\mathbb{M}^d_N} g(\eta_x)\left[\sqrt{f(\eta^{x,y})} - \sqrt{f(\eta)}\,\right]^2 \nu^N_{\rho_*}(\eta) = D_N(f).
\end{aligned}$$

It follows by the convexity of the Dirichlet form that for every density $f \in L^1_{+,1}(\nu^N_{\rho_*})$,

$$D_N(\bar{f}) = D_N\left(\frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} \tau_x f\right) \le \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} D_N(\tau_x f) = D_N(f).$$

Taking advantage again of the convexity of the Dirichlet form and the translation invariance of $\bar{f}$, we prove next a bound of order N^{-2} for the Dirichlet form restricted to bonds in Λ^d_ℓ of $\bar{f}|_\ell$. Since the Dirichlet form is convex and since conditional expectation is an average, we have that

$$D^{x,y}_{\ell_\star}(\bar{f}|_\ell) = D^{x,y}_N(\bar{f}|_\ell \circ p^\ell) \le D^{x,y}_N(\bar{f})$$

for all bonds $\{x,y\} \in \mathfrak{B}_{\ell_\star}$, where we identify Λ^d_ℓ with $\mathbb{T}^d_{\ell_\star}$. By this inequality it follows that

$$D_{\ell_\star}(\bar{f}|_\ell) \le \sum_{\{x,y\}\in\mathfrak{B}_{\ell_\star}} D^{x,y}_N(\bar{f}).$$

On the other hand, by the translation invariance of $\bar{f}$, p^s, and $\nu^N_{\rho_*}$, we have for all $z \in \mathbb{T}^d_N$ that

$$\begin{aligned} D_N^{x+z,y+z}(\bar{f}) &= \sum_{\eta\in\mathbb{M}^d_N} \left[\sqrt{\bar{f}(\eta^{x+z,y+z})} - \sqrt{\bar{f}(\eta)}\,\right]^2 g(\eta_{x+z}) p^s(x+z,y+z)\nu^N_{\rho_*}(\eta) \\ &= \sum_{\eta\in\mathbb{M}^d_N} \left[\sqrt{\tau_z\bar{f}(\eta^{x+z,y+z})} - \sqrt{\tau_z\bar{f}(\eta)}\,\right]^2 g\big((\tau_z\eta)_x\big) p^s(y-x)\nu^N_{\rho_*}(\tau_z\eta) \\ &= \sum_{\eta\in\mathbb{M}^d_N} \left[\sqrt{\bar{f}\big((\tau_z\eta)^{x,y}\big)} - \sqrt{\bar{f}(\tau_z\eta)}\,\right]^2 g\big((\tau_z\eta)_x\big) p^s(y-x)\nu^N_{\rho_*}(\tau_z\eta) \\ &= \sum_{\eta\in\mathbb{M}^d_N} \left[\sqrt{\bar{f}(\eta^{x,y})} - \sqrt{\bar{f}(\eta)}\,\right]^2 g(\eta_x) p^s(y-x)\nu^N_{\rho_*}(\eta) = D_N^{x,y}(\bar{f}). \end{aligned}$$

Therefore,

$$D_N(\bar{f}) = \sum_{\{x,y\}\in\mathfrak{B}_N} D_N^{x,y}(\bar{f}) = \sum_{x\in\mathbb{T}^d_N}\sum_{z:p^s(z)>0} D_N^{x,x+z}(\bar{f}) = N^d \sum_{p^s(z)>0} D_N^{0,z}(\bar{f})$$

and thus

$$\begin{aligned} D^\ell(\bar{f}|_\ell) &\leq \sum_{p^s(z)>0}\sum_{x\in\Lambda^d_\ell\cap(\Lambda^d_\ell - z)} D_N^{x,x+z}(\bar{f}) \leq (2\ell+1)^d \sum_{p^s(z)>0} D_N^{0,z}(\bar{f}) \\ &= (2\ell+1)^d N^{-d} D_N(\bar{f}). \end{aligned}$$

Consequently, for every density $f \in L^1_{+,1}(\nu^N_{\rho_*})$ with Dirichlet form bounded by $C_0 N^{d-2}$ we have that

$$D^\ell(\bar{f}|_\ell) \leq C_0(2\ell+1)^d N^{-2} = C_4(C_0,\ell)N^{-2}. \tag{3.20}$$

It follows that

$$\sup_{D_N(f)\leq C_0 N^{d-2}} \int V^\ell \mathbb{1}_{\{\eta^\ell(0)\leq C_3\}} \bar{f}|_\ell d\nu^{\ell_\star}_{\rho_*} \leq \sup_{D^\ell(\bar{f}|_\ell)\leq C_4(C_0,\ell)N^{-2}} \int V^\ell \mathbb{1}_{\{\eta^\ell(0)\leq C_3\}} \bar{f}|_\ell d\nu^{\ell_\star}_{\rho_*},$$

and therefore in order to prove (3.18), and thus the one block estimate, it suffices to prove that

$$\lim_{\ell\to\infty}\limsup_{N\to\infty} \sup_{D^\ell(f)\leq C_4(C_0,\ell)N^{-2}} \int V^\ell \mathbb{1}_{\{\eta^\ell(0)\leq C_3\}} f d\nu^{\ell_\star}_{\rho_*} \leq 0, \tag{3.21}$$

where here the supremum is taken among all densities $f \in L^1_{+,1}(\nu^{\ell_\star}_{\rho_*})$.

The Limit as $N \to \infty$

The next step consists in examining the behavior of the supremum in (3.21) as $N \to \infty$. Relying on the relative compactness provided by the indicator function $\mathbb{1}_{\{\eta^\ell(0)\leq C_1\}}$ and on the lower semicontinuity of the Dirichlet form we can bound the lim sup as $N \to \infty$ of this last supremum by the supremum over all densities $f \in L^1_{+,1}(\nu^{\ell_\star}_{\rho_*})$ with vanishing

Dirichlet form $D^\ell(f) = 0$.

From the presence of the indicator function and since V^ℓ is positive, we can restrict last supremum to densities $f \in L^1_{+,1}(\nu^{\ell_\star}_{\rho_*})$ supported by the set $\{\xi \in \mathbb{M}^d_{\ell_\star} | \xi^\ell(0) \leq C_3\}$. Now, the set $\{\xi \in \mathbb{M}^d_{\ell_\star} | \xi^\ell(0) \leq C_3\}$ is a compact subset of $\mathbb{M}^d_{\ell_\star}$ and thus the set of all densities supported by this set is also compact in the weak topology of probability measures. Therefore, for each fixed $N \in \mathbb{N}$ there exists a density $f^N \in L^1_{+,1}(\nu^{\ell_\star}_{\rho_*})$ with $D^\ell(f^N) \leq C_4(C_0,\ell)N^{-2}$, supported by the set $\{\xi \in \mathbb{M}^d_{\ell_\star} | \xi^\ell(0) \leq C_3\}$ that reaches the supremum:

$$\int V^\ell \mathbb{1}_{\{\xi^\ell(0)\leq C_3\}} f^N d\nu^{\ell_\star}_{\rho_*} = \sup_{D^\ell(f)\leq C_4(C_0,\ell)N^{-2}} \int V^\ell \mathbb{1}_{\{\xi^\ell(0)\leq C_3\}} f d\nu^{\ell_\star}_{\rho_*}.$$

We can choose next a subsequence $\{f^{k_N}\}$ of $\{f^N\}$ such that

$$\lim_{N\to\infty} \int V^\ell \mathbb{1}_{\{\xi^\ell(0)\leq C_3\}} f^{k_N} d\nu^{\ell_\star}_{\rho_*} = \limsup_{N\to\infty} \int V^\ell \mathbb{1}_{\{\xi^\ell(0)\leq C_3\}} f^N d\nu^{\ell_\star}_{\rho_*}.$$

Since all densities f^N are supported by the set $\{\xi \in \mathbb{M}^d_{\ell_\star} | \xi^\ell(0) \leq C_3\}$, the sequence $\{f^{k_N}\}$ is relatively compact and so we can choose a further subsequence $\{f^{m_{k_N}}\}$ converging weakly to some density $f^\infty \in L^1_{+,1}(\nu^{\ell_\star}_{\rho_*})$ supported by the set $\{\xi \in \mathbb{M}^d_{\ell_\star} | \xi^\ell(0) \leq C_3\}$. By the lower semicontinuity of the Dirichlet form it follows that

$$D^\ell(f^\infty) \leq \liminf_{N\to\infty} D^\ell(f^{m_{k_N}}) \leq \liminf_{N\to\infty} C_4(C_0,\ell)N^{-2} = 0$$

and since the function $V^\ell \mathbb{1}_{\{\xi^\ell(0)\leq C_3\}} : \mathbb{M}^d_{\ell_\star} \longrightarrow \mathbb{R}_+$ is continuous it follows by the weak convergence $f^{m_{k_N}} \longrightarrow f^\infty$ that

$$\lim_{N\to\infty} \int V^\ell \mathbb{1}_{\{\xi^\ell(0)\leq C_3\}} f^{m_{k_N}} d\nu^{\ell_\star}_{\rho_*} = \int V^\ell \mathbb{1}_{\{\xi^\ell(0)\leq C_3\}} f^\infty d\nu^{\ell_\star}_{\rho_*}.$$

Consequently,

$$\limsup_{N\to\infty} \sup_{D^\ell(f)\leq C_4(C_0,\ell)N^{-2}} \int V^\ell \mathbb{1}_{\{\xi^\ell(0)\leq C_3\}} f d\nu^{\ell_\star}_{\rho_*} = \int V^\ell \mathbb{1}_{\{\xi^\ell(0)\leq C_3\}} f^\infty d\nu^{\ell_\star}_{\rho_*},$$

and therefore in order to prove (3.21), and thus also the one block estimate, it suffices to prove that

$$\lim_{\ell\to\infty} \sup_{D^\ell(f)=0} \int V^\ell \mathbb{1}_{\{\xi^\ell(0)\leq C_3\}} f d\nu^{\ell_\star}_{\rho_*} = 0. \tag{3.22}$$

Decomposition along hyperplanes with a fixed number of particles

By proposition A.3.10 it is obvious that any probability density $f \in L^1_{+,1}(\nu^{\ell_\star}_{\rho_*})$ with Dirichlet form $D^\ell(f) = 0$ is constant on each hyperplane with a fixed number of particles. It is convenient therefore to decompose each density f along these hyperplanes with particles density bounded above by C_3. To this end, for each density $f \in L^1_{+,1}(\nu^{\ell_\star}_{\rho_*})$ with Dirichlet form $D^\ell(f) = 0$ we denote by $C_K(f)$ the constant value of f on the hyperplane consisting of configurations with K particles.

Recall that we denote by $\{\nu_{N,K}\}_{(N,K)\in\mathbb{N}\times\mathbb{Z}_+}$ the canonical ensemble of the ZRP given by

$$\nu_{N,K}=\nu^N_{\rho_*}\Big(\,\cdot\,\Big|\sum_{x\in\mathbb{T}^d_N}\eta(x)=K\Big)=\frac{1}{Z(N^d,K)}\sum_{\eta\in\mathbb{M}^d_N}\frac{1}{g!(\eta)}\delta_\eta$$

for all $\rho_*\in(0,\rho_c)$. Then obviously for all $\rho_*\in(0,\rho_c)$ we have that

$$\nu^N_{\rho_*}=\sum_{K=0}^{\infty}\nu^N_{\rho_*}\Big(\eta\Big|\sum_{x\in\mathbb{T}^d_N}\eta(x)=K\Big)\cdot\nu_{N,K}. \tag{3.23}$$

Identifying Λ^d_ℓ with $\mathbb{T}^d_{\ell_\star}$, where as always $\ell_\star:=2\ell+1$ we consider the distribution $\nu_{\ell_\star,K}$. Then, with this notation we can write

$$\begin{aligned}\int V^\ell\mathbb{1}_{\{\xi^\ell(0)\le C_3\}}fd\nu^{\ell_\star}_{\rho_*}&=\sum_{K=0}^{\infty}\nu^{\ell_\star}_{\rho_*}\Big(\sum_{x\in\Lambda^d_\ell}\xi(x)=K\Big)\int V^\ell\mathbb{1}_{\{\xi^\ell(0)\le C_3\}}fd\nu_{\ell_\star,K}\\&=\sum_{K=0}^{(2\ell+1)^dC_3}C_K(f)\nu^{\ell_\star}_{\rho_*}\Big(\sum_{x\in\Lambda^d_\ell}\xi(x)=K\Big)\int V^\ell d\nu_{\ell_\star,K}\\&=:\sum_{K=0}^{(2\ell+1)^dC_3}\bar{C}_K(f)\int V^\ell d\nu_{\ell_\star,K}.\end{aligned}$$

Now obviously we have that

$$\sum_{K=0}^{\infty}\bar{C}_K(f)=\sum_{K=0}^{\infty}\int f\mathbb{1}_{\{\sum_{x\in\Lambda^d_\ell}\xi(x)=K\}}d\nu^{\ell_\star}_{\rho_*}=\int fd\nu^{\ell_\star}_{\rho_*}=1$$

and therefore by applying the bounded convergence theorem on the probability measure $\{C_k(f)\}_{K\in\mathbb{Z}_+}$ and the sequence of functions $b^\ell:\mathbb{Z}_+\longrightarrow\mathbb{R}_+$ given by $b^\ell(K)=\mathbb{1}_{\{0,\dots,(2\ell+1)^dC_3\}}(K)\int V^\ell d\nu_{\ell_\star,K}$ it follows that in order to complete the proof of the one block estimate it suffices to show

$$\limsup_{\ell\to\infty}\sup_{K\le(2\ell+1)^dC_3}\int V^\ell(\xi)d\nu_{\ell_\star,K}(\xi)=0. \tag{3.24}$$

An application of the Equivalence of Ensembles

The final step in the proof of the one block estimate consists in applying the equivalence of ensembles to prove (3.24). Since the measure $\nu_{\ell_\star,K}$ is concentrated on configurations with K particles, the integral appearing in (3.24) is equal to

$$\begin{aligned}\int V^\ell d\nu_{\ell_\star,K}&=\int\Big|\frac{1}{(2\ell+1)^d}\sum_{|x|\le\ell}g\big(\xi(x)\big)-\Phi\big(\xi^\ell(0)\big)\Big|d\nu_{\ell_\star,K}\\&=\int\Big|\frac{1}{\sharp\Lambda^d_\ell}\sum_{|x|\le\ell}g\big(\xi(x)\big)-\Phi\Big(\frac{K}{(2\ell+1)^d}\Big)\Big|d\nu_{\ell_\star,K}.\end{aligned}$$

We fix now a positive integer k that shall increase to infinity after ℓ and decompose the set Λ^d_ℓ in cubes of side-length $2k+1$ as follows: We consider the set $A:=[(2k+1)\mathbb{Z}^d]\cap\Lambda^d_{\ell-k}$

and enumerate its elements, $A = \{x_1, \ldots, x_q\}$, so that $|x_i| \le |x_j|$ whenever $i \le j$. For $1 \le i \le q$ let $B_i := x_i + \Lambda_k^d$. Then by construction we have that $B_i \cap B_j = \emptyset$ for $i \ne j$ and $\bigcup_{i=1}^q B_i \subseteq \Lambda_\ell^d$. Then if we set $B_0 := \Lambda_\ell^d \setminus \bigcup_{i=1}^q B_i$ we have that $\sharp B_0 \le C_d k \ell^{d-1}$ for some universal constant depending only on the dimension d. Indeed, by definition $q = [\frac{2\ell+1}{2k+1}]^d$, and

$$\begin{aligned}
\sharp B_0 &= (2\ell+1)^d - q(2k+1)^d \le (2\ell+1)^d - 2^d(\ell-k)^d \\
&= (2\ell)^d \Big[\Big(1 + \frac{1}{2\ell}\Big)^d - \Big(1 - \frac{k}{\ell}\Big)^d \Big] = (2\ell)^d \frac{2k+1}{2\ell} \sum_{m=0}^{d-1} \Big(1 + \frac{1}{2\ell}\Big)^{d-m-1} \Big(1 - \frac{k}{\ell}\Big)^m \\
&\le (2\ell)^{d-1}(2k+1) \sum_{m=0}^{d-1} \Big(1 + \frac{1}{2}\Big)^{d-m-1} \le \ell^{d-1}(2k+1)3^{d-1}d \le (3^d d)k\ell^{d-1},
\end{aligned}$$

where in the calculations above using the fact that k tends to infinity after ℓ we assume that $1 \le k \le \ell$. Consequently, inequality $\sharp B_0 \le C_d k \ell^{d-1}$ holds with constant $C_d := 3^d d$.

Through the decomposition $\Lambda_\ell^d = \bigcup_{i=0}^q B_i$ we can write

$$\begin{aligned}
\int V^\ell d\nu_{\ell_\star,K} &= \int \Big| \frac{1}{\sharp\Lambda_\ell^d} \sum_{i=0}^q \sum_{x\in B_i} g(\xi(x)) - \Phi\Big(\frac{K}{(2\ell+1)^d}\Big) \Big| d\nu_{\ell_\star,K} \\
&= \int \Big| \frac{1}{\sharp\Lambda_\ell^d} \sum_{i=0}^q \sum_{x\in B_i} \Big[g(\xi(x)) - \Phi\Big(\frac{K}{(2\ell+1)^d}\Big) \Big] \Big| d\nu_{\ell_\star,K} \\
&\le \sum_{i=0}^q \frac{1}{\sharp\Lambda_\ell^d} \int \Big| \sum_{x\in B_i} \Big[g(\xi(x)) - \Phi\Big(\frac{K}{(2\ell+1)^d}\Big) \Big] \Big| d\nu_{\ell_\star,K} \\
&= \sum_{i=0}^q \frac{1}{\sharp\Lambda_\ell^d} \int \Big| \sum_{x\in B_i} g(\xi(x)) - (\sharp B_i)\Phi\Big(\frac{K}{(2\ell+1)^d}\Big) \Big| d\nu_{\ell_\star,K} \\
&= \sum_{i=0}^q \frac{\sharp B_i}{\sharp\Lambda_\ell^d} \int \Big| \frac{1}{\sharp B_i} \sum_{x\in B_i} g(\xi(x)) - \Phi\Big(\frac{K}{(2\ell+1)^d}\Big) \Big| d\nu_{\ell_\star,K}.
\end{aligned}$$

Assuming the local jump rate g to be bounded, since $\sharp B_0 \le C_d k \ell^{d-1}$ we can bound above the term in the last sum corresponding to $i = 0$ by

$$\frac{C_d k \ell^{d-1}}{(2\ell+1)^d} \big(\|g\|_u + \Phi(C_1 \wedge \rho_c) \big) =: \overline{C} \frac{k\ell^{d-1}}{(2\ell+1)^d}.$$

Therefore the integral $\int V^\ell d\nu_{\ell_\star,K}$ can be bounded above by

$$\int V^\ell d\nu_{\ell_\star,K} \le \frac{\sharp\Lambda_k^d}{\sharp\Lambda_\ell^d} \sum_{i=0}^q \int \Big| \frac{1}{\sharp\Lambda_k^d} \sum_{x\in B_i} g(\xi(x)) - \Phi\Big(\frac{K}{(2\ell+1)^d}\Big) \Big| d\nu_{\ell_\star,K} + \overline{C} \frac{k\ell^{d-1}}{(2\ell+1)^d}.$$

Since the distribution of the random vector $\big((\xi(x))_{x\in B_i}, \nu^d_{\ell_\star,K}\big)$ does not depend on i and since

$$q = \Big[\frac{2\ell+1}{2k+1}\Big]^d \le \frac{(2\ell+1)^d}{(2k+1)^d}$$

it follows that

$$\int V^\ell d\nu_{\ell_\star,K} \le \int \Big| \frac{1}{(2k+1)^d} \sum_{|x|\le k} g(\xi(x)) - \Phi\Big(\frac{K}{(2\ell+1)^d}\Big) \Big| d\nu_{\ell_\star,K} + \overline{C} \frac{k\ell^{d-1}}{(2\ell+1)^d}.$$

Therefore, since

$$\limsup_{\ell\to\infty} \sup_{K\le(2\ell+1)^d C_3} \overline{C}\frac{k\ell^{d-1}}{(2\ell+1)^d} = 0,$$

in order to prove (3.24) and complete the proof of the one block estimate with respect to continuous functions in the case that the local jump rate g is bounded, it suffices to prove that

$$\lim_{k\to\infty}\lim_{m\to\infty} \sup_{\substack{\ell\ge m\\ K\le(2\ell+1)^d C_3}} \int \left|\frac{1}{(2k+1)^d}\sum_{|x|\le k} g\big(\xi(x)\big) - \Phi\Big(\frac{K}{(2\ell+1)^d}\Big)\right| d\nu_{\ell_\star,K} = 0.$$

For notational simplicity we denote the supremum above by

$$S(m,k) := \sup_{\substack{\ell\ge m\\ K\le(2\ell+1)^d C_3}} \int \left|\frac{1}{(2k+1)^d}\sum_{|x|\le k} g\big(\xi(x)\big) - \Phi\Big(\frac{K}{(2\ell+1)^d}\Big)\right| d\nu_{\ell_\star,K} = 0.$$

For each fixed $(m,k)\in\mathbb{N}\times\mathbb{N}$ we can pick a sequence $\{(\ell_n^{m,k}, K_n^{m,k})\}_{n\in\mathbb{N}}$ such that $\ell_n^{m,k}\ge m$ and $K_n^{m,k}\le(2\ell_n^{m,k}+1)^d C_3$ for all $n\in\mathbb{N}$ that achieves the supremum, i.e. such that

$$S(m,k) = \lim_{n\to\infty}\int \left|\frac{1}{(2k+1)^d}\sum_{|x|\le k} g\big(\xi(x)\big) - \Phi\Big(\frac{K_n^{m,k}}{(2\ell_n^{m,k}+1)^d}\Big)\right| d\nu_{(\ell_n^{m,k})_\star,K_n^{m,k}}.$$

Since the sequence $\{\rho_n^{m,k}\}_{n\in\mathbb{N}}$ defined by

$$\rho_n^{m,k} := \frac{K_n^{m,k}}{(2\ell_n^{m,k}+1)^d}, \qquad n\in\mathbb{N},$$

is contained in the compact interval $[0,C_3]$, for each fixed $(m,k)\in\mathbb{N}\times\mathbb{N}$ we can pick a sequence $\{n_j\}_{j\in\mathbb{N}}\equiv\{n_j^{m,k}\}$ such that $\rho_{n_j}^{m,k}$ converges to some $\rho^{m,k}\in[0,C_3]$ as $j\to\infty$. Then since g is assumed bounded it follows by the equivalence of ensembles that

$$S(m,k) = \int \left|\frac{1}{(2k+1)^d}\sum_{|x|\le k} g\big(\xi(x)\big) - \Phi\big(\rho^{m,k}\big)\right| d\nu^{k_\star}_{\rho^{m,k}}.$$

Furthermore, for each fixed $k\in\mathbb{N}$, the sequence $\{\rho^{m,k}\}_{m\in\mathbb{N}}$ is also contained in $[0,C_3]$ and thus we can choose a sequence $\{m_j\}_{j\in\mathbb{N}}=\{m_j^{(k)}\}$ such that $\rho^{m_j,k}$ converges to some $\rho^k\in[0,C_3]$, and therefore by the weak continuity of the grand canonical ensemble,

$$\lim_{m\to\infty} S(m,k) = \int \left|\frac{1}{(2k+1)^d}\sum_{|x|\le k} g\big(\xi(x)\big) - \Phi\big(\rho^{k}\big)\right| d\nu^{k_\star}_{\rho^{k}}.$$

Consequently, in order to complete the proof of the one block estimate it suffices to prove that

$$\lim_{k\to\infty}\int \left|\frac{1}{(2k+1)^d}\sum_{|x|\le k} g\big(\xi(x)\big) - \Phi\big(\rho^{k}\big)\right| d\nu^{\infty}_{\rho^{k}} = 0, \tag{3.25}$$

where since $\{\rho^k\}_{k\in\mathbb{N}}\subseteq[0,C_3]$, $\Phi(\cdot)=\Phi(\cdot\wedge\rho_c)$ and $\nu_\cdot=\nu_{\cdot\wedge\rho_c}$, we can assume of course that $\{\rho^k\}$ converges to some $\rho\in[0,\rho_c]$ as $k\to\infty$. Now, to prove (3.25) it suffices to show that the law of large numbers holds in $L^1(\nu_\rho^\infty)$ uniformly over all parameters $\rho \in [0, \rho_c]$, i.e. that

$$\lim_{N\to\infty} \sup_{\rho\in[0,\rho_c]} \int \left| \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} g\big(\eta(x)\big) - \Phi\big(\rho\big) \right| d\nu_\rho^\infty = 0. \tag{3.26}$$

But since the random variables $g\big(\eta(x)\big)$, $x \in \mathbb{Z}^d$, are uniformly bounded by $\|g\|_u$ and i.i.d. with respect to ν_ρ^∞ for all $\rho \in [0, \rho_c]$, this holds by the following simple uniform L^2-weak law of large numbers, and thus the static one block-estimate for bounded local jump rate functions g is proved.

Lemma 3.1.5 *Let $(\Omega, \mathcal{F})$ be an arbitrary probability space and let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of random variables on $(\Omega, \mathcal{F})$ uniformly bounded by some constant $M \geq 0$. Let also $\mathcal{P} \subseteq \mathbb{P}(\Omega, \mathcal{F})$ be any family of probability measures such that the X_i's are independent and identically distributed with respect to any $P \in \mathcal{P}$ and denote by $\mu_P := \mathbb{E}_P X_1$ the common mean of the X_i's with respect to $P \in \mathcal{P}$. Then*

$$\lim_{n\to\infty} \sup_{P\in\mathcal{P}} \mathbb{E}_P \left| \frac{1}{n} \sum_{i=0}^{n} X_i - \mu_P \right|^2 = 0.$$

Proof Indeed, since the X_i's are are uniformly bounded by M, for each $P \in \mathcal{P}$ we have that $|\mu_P| \leq M$ and thus

$$\mathbb{V}_P(X_1) = \mathbb{E}_P|X_1 - \mu_P|^2 \leq 2\mathbb{E}_P(X_1^2 + \mu_P^2) \leq 4M^2.$$

Therefore, for each $P \in \mathcal{P}$ we have that

$$\mathbb{E}_P \left| \frac{1}{n} \sum_{i=0}^{n} X_i - \mu_P \right|^2 = \mathbb{V}_P \left(\frac{1}{n} \sum_{i=1}^{n} X_i \right) = \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{V}_P(X_i) = \frac{1}{n} \mathbb{V}_P(X_1) \leq \frac{4M^2}{n}$$

which shows that

$$\sup_{P\in\mathcal{P}} \mathbb{E}_P \left| \frac{1}{n} \sum_{i=0}^{n} X_i - \mu_P \right|^2 \leq \frac{4M^2}{n} \overset{n\to\infty}{\longrightarrow} 0,$$

as required. □

3.2 A Degenerate Non-Linear Diffusion

In this section, using the sub-criticality of the initial profile we apply the classic results in uniformly parabolic equations (e.g. [26]) to degenerate non-linear parabolic equations of the form $\partial_t\rho = \Delta_\Sigma\Phi(\rho)$ with increasing non-linearities Φ that may be finally constant, where

$$\Delta_\Sigma := \sum_{i,j=1}^{d} \sigma_{ij}\partial_{ij}^2$$

and $\Sigma \in \mathbb{R}^{d\times d}$ is a symmetric positive definite matrix. To be more precise, for each $\rho_c \in [0,\infty)$, $k \geq 2$, we denote by $\mathcal{C}_{\rho_c}^k$ the set of all functions $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

(a) $\Phi(0) = 0$,

(b) Φ is C^k on the interval $[0,\rho_c)$,

(c) Φ is C^1 from the left on the interval $(0,\rho_c]$,

(d) $\Phi'(\rho) > 0$ for all $0 \leq \rho < \rho_c$, and

(e) $\Phi(\rho) = \Phi(\rho \wedge \rho_c)$ for all $\rho \geq 0$.

We begin by considering first the notion of classical solutions.

Definition 3.2.1 Let $\rho_0 : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ be a measurable function. We say that a measurable function $\rho : \mathbb{R}_+ \times \mathbb{T}^d \longrightarrow \mathbb{R}_+$ is a *classical solution of the initial value problem*

$$\begin{cases} \partial_t\rho = \Delta_\Sigma\Phi(\rho), \\ \rho(0,\cdot) = \rho_0, \end{cases} \tag{3.27}$$

if

(a) the set $\{\rho_t > \rho_c\}$ is open for all $t \geq 0$,

(b) ρ is C^1 in time for each fixed $u \in \mathbb{T}^d$,

(c) $\Phi(\rho_t) \in C^2(\mathbb{T}^d)$ for all $t \in \mathbb{R}_+$, and

(d) ρ satisfies (3.27).

Since the function $\Phi(\rho_t)$ is C^2, and Φ is continuous on $[0,\rho_c]$ and C^k for $k \geq 2$ on $[0,\rho_c)$, it follows that $\rho_t \wedge \rho_c$ is continuous on $\mathbb{T}^d$ and C^2 on the set $\{\rho_t < \rho_c\}$. Note that since $\rho_t \wedge \rho_c$ is continuous, the fluid phase set $\{\rho_t < \rho_c\} = \{\rho_t \wedge \rho_c < \rho_c\}$ is open for each time $t \geq 0$.

Proposition 3.2.1 *Let ρ be a solution of the Cauchy problem (3.27) starting from any measurable profile $\rho_0 : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ and let $u \in \mathbb{T}^d$. If there exists $t \in \mathbb{R}_+$ such that $\rho_t(u) > \rho_c$ then the function $t \mapsto \rho^u(t) := \rho(t,u)$ is constant.*

Proof Indeed, let $t \in \mathbb{R}_+$ such that $\rho_t(u) > \rho_c$. Since the set $\{\rho_t > \rho_c\}$ is assumed open and ρ is continuous in time there exists $\varepsilon > 0$ such that $\rho_s(u) > \rho_c$ for all $s \in (t-\varepsilon, t+\varepsilon)$. Then, given $s \in (t-\varepsilon, t+\varepsilon)$, since $\{\rho_s > \rho_c\}$ is open there exists $\delta_{s,u} > 0$ such that $D(u, \delta_{s,u}) \subseteq \{\rho_s > \rho_c\}$. But then $\Phi(\rho_s)|_{D(u,\delta_{s,u})} \equiv \rho_c$ and therefore $\Delta_\Sigma \Phi(\rho_s)(u) = 0$ for all $s \in (t-\varepsilon, t+\varepsilon)$. So since ρ solves (3.27) we have that $\partial_s \rho(u) = 0$ for all $s \in (t-\varepsilon, t+\varepsilon)$ and so ρ^u is constant in $(t-\varepsilon, t+\varepsilon)$.

Then the number

$$s_+(t, u) := \sup\{s \geq t | \rho_s(u) = \rho_t(u)\}$$

is obviously $s_+(t, u) \geq t + \varepsilon$ and it is easy to see that in fact $s_+(t, u) = +\infty$. Indeed, suppose to derive a contradiction that $s_+ := s_+(t, u) < +\infty$. Then since ρ^u is continuous in time we have that $\rho_{s_+}(u) = \rho_t(u)$, and so repeating the argument applied initially to the pair (t, u) for the pair (s_+, u), we get that there exists $\varepsilon' > 0$ such that $\rho_s(u) = \rho_{s_+}(u)$ for all $s \in (s_+ - \varepsilon', s_+ + \varepsilon')$ which contradicts the definition of s_+. Similarly one sees that

$$s_+(t, u) := \inf\{s \in [0, t] | \rho_s(u) = \rho_t(0)\} = 0. \qquad \square$$

According to this proposition, in the case of initial conditions $\rho_0 : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ taking supercritical values, i.e. such that $\{\rho_0 > \rho_c\} \neq \emptyset$, the problem (3.27) can be considered as the initial and boundary value problem

$$\begin{cases} \partial_t \rho = \Delta_\Sigma \Phi(\rho) & \text{in } (0, \infty) \times \{\rho_0 \leq \rho_c\}, \\ \rho(0, \cdot) = \rho_0 & \text{in } \{\rho_0 \leq \rho_c\}, \\ \rho \equiv \rho_c & \text{in } \mathbb{R}_+ \times \partial\{\rho_0 > \rho_c\} \end{cases}$$

where in this problem we allow as solutions only functions $\rho : \mathbb{T}^d \longrightarrow [0, \rho_c]$ that take values in the interval $[0, \rho_c]$ and ∂ denotes the topological boundary operator.

So at a first stage we consider the problem (3.27) only for lower semi-continuous initial conditions ρ_0 satisfying $\rho_0 \leq \rho_c$. The following is obvious from proposition (3.2.1).

Corollary 3.2.1 *Let $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ be a solution of problem (3.27) starting from some measurable profile $\rho_0 : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ such that $\rho_0 \leq \rho_c$. Then $\rho \leq \rho_c$ in $\mathbb{R}_+ \times \mathbb{T}^d$.*

Given any solution $\rho : \mathbb{R}_+ \times \mathbb{T}^d \longrightarrow \mathbb{R}_+$ of problem (3.27) starting from some lower semi-continuous initial profile $\rho_0 : \mathbb{T}^d \longrightarrow [0, \rho_c]$, the set $\{\rho = \rho_c\} \subseteq \mathbb{R}_+ \times \mathbb{T}^d$ is the degenerate region, which we interpret as the region of space containing the condensed phase and $\{\rho_t = \rho_c\} \subseteq \mathbb{T}^d$ is the degenerate region at time t. Of course

$$\{\rho = \rho_c\} = \bigcup_{t \in \mathbb{R}_+} \{t\} \times \{\rho_t = \rho_c\}.$$

We denote by

$$B_t \equiv B_t(\rho) := \partial\{\rho_t = \rho_c\}$$

denotes the boundary of the condensed phase region. We will refer to the open set $A_t := A_t(\rho) = \text{Int}\{\rho_t = \rho_c\} = \{\rho_t = \rho_c\} \setminus B_t$ as the strictly condensed region at time t. Finally, when need arises to simplify the notation we will write $F_t := \{\rho_t < \rho_c\}$ for the open fluid phase region at time t.

As we have seen, for each solution ρ of problem (3.27), ρ_t is C^2 in $\{\rho_t < \rho_c\}$. On the other hand it is obviously constantly equal to ρ_c in the strictly condensed region $A_t = \text{Int}\{\rho_t = \rho_c\}$, and so the only possible non differentiability points of ρ_t are points in the boundary B_t of the condensed phase. For points at the boundary $B_t(\rho)$ along some classical solution ρ of problem (3.27) we consider the following time-dependent gradients along this solution.

Definition 3.2.2 Let ρ be a classical solution of problem (3.27) Let $f : \mathbb{T}^d \longrightarrow \mathbb{R}$ be any function. The *upper j-th partial derivative of f at $u \in B_t(\rho)$ from the fluid phase* $\{\rho_t < \rho_c\}$ is the upper limit

$$\overline{\partial}^{\uparrow}_{j;t} f(u) := \limsup_{\substack{h\to 0\\ u+he_j\in\{\rho_t<\rho_c\}}} \frac{f(u+he_j)-f(u)}{h}.$$

Likewise, the *lower j-th partial derivative of f at $u \in B_t(\rho)$ from the fluid phase* $\{\rho_t < \rho_c\}$ is the lower limit

$$\underline{\partial}^{\uparrow}_{j;t} f(u) := \liminf_{\substack{h\to 0\\ u+he_j\in\{\rho_t<\rho_c\}}} \frac{f(u+he_j)-f(u)}{h}.$$

If

$$\overline{\partial}^{\uparrow}_{j;t} f(u) = \underline{\partial}^{\uparrow}_{j;t} f(u),$$

we say denote their common value by $\partial^{\uparrow}_{j;t} f(u)$ and say that the *j-th partial derivative $\partial^{\uparrow}_{j;t} f(u)$ of f at $u \in B_t(\rho)$ from the fluid phase exists.*

We say that f *is differentiable at time t from the fluid phase at the point $u \in B_t$* if there exists $w \in \mathbb{R}^d$ such that

$$\lim_{\substack{h\to 0\\ u+h\in\{\rho_t<\rho_c\}}} \frac{|f(u+h)-f(u)-\langle w,h\rangle|}{\|h\|} = 0. \tag{3.28}$$

If f is differentiable at time t from the fluid phase at $u \in B_t$ then there exists a unique vector $w \in \mathbb{R}^d$ satisfying (3.28), denoted by $\nabla^{\uparrow}_t f(u)$.

If f is differentiable at time t from the fluid phase at $u \in B_t$ then the j-th partial derivative at time t from the fluid phase exists at u and

$$\partial^{\uparrow}_{j;t} f(u) = \langle \nabla^{\uparrow}_t f(u), e_j\rangle, \quad j = 1,\dots,d.$$

Also if f is differentiable at $u \in B_t$ then it is also differentiable at time t from the fluid phase at $u \in B_t$ with $\nabla^{\uparrow}_t f(u) = \nabla f(u)$. Note, that since we assume $\Phi(\rho_t)$ to be C^2 we have that

$$\nabla^{\uparrow}_t \Phi(\rho_t)(u) = \nabla\Phi(\rho_t)(u) = 0$$

at each point $u \in B_t$, since each such point $u \in B_t$ is a point of a maximum of $\Phi(\rho_t)$.

Lemma 3.2.1 *If the solution ρ of problem (3.27) is differentiable from the fluid phase at the point $u \in B_t$ at time t we have that*

$$\nabla^{\uparrow}_t \Phi(\rho_t)(u) = \Phi'_-(\rho_c)\nabla^{\uparrow}_t \rho_t(u).$$

Proof Indeed, for $u \in B_t$, we set

$$D_h\Phi(\rho_t)(u) := \Phi(\rho_t(u+h)) - \varphi_c - \Phi'_-(\rho_c)\langle \nabla_t^{\uparrow}\rho_t(u), h\rangle,$$

and we have that

$$\begin{aligned} |D_h\Phi(\rho_t)(u)| &\leq |\Phi(\rho_t(u+h)) - \varphi_c - \Phi'_-(\rho_c)(\rho_t(u+h) - \rho_c)| \\ &\quad + |\Phi'_-(\rho_c)||\rho_t(u+h) - \rho_c - \langle \nabla_t^{\uparrow}\rho_t(u), h\rangle|. \end{aligned}$$

Let $\varepsilon > 0$. By the definition of the gradient from the fluid phase the second term is of order $o(h)$ as $h \to 0$ from inside the fluid phase $\{\rho_t < \rho_c\}$, and therefore there exists $\delta_1 > 0$ such that

$$\|h\| < \delta_1,\ u+h \in \{\rho_t < \rho_c\} \quad \Longrightarrow \quad \frac{|\rho_t(u+h) - \rho_c - \langle \nabla_t^{\uparrow}\rho_t(u), h\rangle|}{\|h\|} \leq \frac{\varepsilon}{2(1+\Phi'_-(\rho_c))}.$$

For the first term we note that by the differentiability of Φ from the left at ρ_c, given any $\varepsilon > 0$ there exists there exists $\delta_2 > 0$ such that

$$0 < \rho_c - r < \delta_2 \quad \Longrightarrow \quad |\Phi(r) - \varphi_c - \Phi'_-(\rho_c)(r - \rho_c)| < \frac{\varepsilon}{2}.$$

Also by the differentiability of ρ_t at time t from the fluid phase at the point $u \in B_t$ there exists $\delta_3 > 0$ such that

$$\|h\| < \delta_3,\ u+h \in \{\rho_t < \rho_c\} \quad \Longrightarrow \quad |\rho_t(u+h) - \rho_c| \leq (1 + \|\nabla_t^{\uparrow}\rho_t(u)\|)\|h\|.$$

Consequently, if we choose $\delta_0 > 0$ small enough so that $\delta_0 < \delta_1 \wedge \delta_3 \wedge [\delta_2(1+\|\nabla_t^{\uparrow}\rho_t(u)\|)^{-1}]$ then for all $h \in -u + \{\rho_t < \rho_c\}$ with $\|h\| < \delta_0$ we have that

$$\frac{|D_h\Phi(\rho_t)(u)|}{\|h\|} < \varepsilon$$

and the proof is complete.

Lemma 3.2.2 *Let $\rho : \mathbb{R}_+ \times \mathbb{T}^d \longrightarrow [0, \rho_c]$ be a solution to problem (3.27) with respect to some lower-semicontinuous initial profile ρ_0, and let $u \in B_t$, $t \geq 0$. Then ρ_t is differentiable at u iff it is differentiable from the fluid phase at u at time t with*

$$\nabla_t^{\uparrow}\rho_t(u) = 0.$$

Proof One implication is obvious. For the converse we note that if $u + h \in \{\rho_t = \rho_c\}$ then

$$\frac{\rho_c - \rho_t(u+h)}{\|h\|} = 0$$

and therefore

$$\sup_{\|h\|<\delta} \frac{\rho_c - \rho_t(u+h)}{\|h\|} = \sup_{\substack{\|h\|<\delta \\ u+h \in \{\rho_t < \rho_c\}}} \frac{\rho_c - \rho_t(u+h)}{\|h\|}$$

which tends to zero as $\delta \to 0$ since ρ_t is differentiable from the fluid phase at u at time t with $\nabla_t^{\uparrow}\rho_t(u) = 0$. This proves that ρ_t is differentiable at $u \in B_t$ with $\nabla\rho_t(u) = 0$. □

Proposition 3.2.2 *Suppose that* $\Phi \in \mathcal{C}^k_{\rho_c}$ *satisfies* $\Phi'_-(\rho_c) > 0$ *and let* $\rho : \mathbb{R}_+\mathbb{T}^d \longrightarrow [0, \rho_c]$ *be a solution to problem (3.27) with respect to some lower-semicontinuous initial profile* ρ_0*. Then* $\rho_t \in C^1(\mathbb{T}^d)$ *for all* $t > 0$*.*

Proof Let $u \in B_t$. We will prove first that ρ_t is differentiable from the fluid phase at u at time t. Since Φ is differentiable from the left at ρ_c with $\Phi'_-(\rho_c) > 0$ it follows that its inverse $R := \Phi^{-1}$ is differentiable from the left at $\varphi_c := \Phi(\rho_c)$ with

$$R'_-(\varphi_c) = \frac{1}{\Phi'_-(\rho_c)},$$

since the inverse function theorem holds for one-sided derivatives. So if we set $\phi_t := \Phi(\rho_t)$ for all $t \geq 0$, then by the chain rule for differentiating from the fluid phase we have for all $u \in B_t$ that

$$\nabla_t^{\uparrow}\rho_t(u) = \nabla_t^{\uparrow}R(\phi_t)(u) = R'_-(\varphi_c)\nabla\phi_t(u) = 0,$$

which according to the previous lemma shows that ρ_t is differentiable at u. Furthermore since we assume Φ to be C^1 from the left on $(0, \rho_c]$ it is easy to see that ρ_t is C^1. □

Having seen some basic properties of solutions of problem (3.27), we apply next the classical results in quasi-linear uniformly parabolic equations by Ladyženskaya et al in [26]. In divergence form these are equations of the form

$$\partial_t\rho(t, u) = \mathrm{div}_u\mathcal{A}(t, u, \rho_t(u), \nabla\rho_t(u)), \tag{3.29}$$

where $\mathcal{A} = (\mathcal{A}^1, \ldots, \mathcal{A}^d) : \mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a C^2 vector valued function

$$(0, +\infty) \times \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d \ni (t, u, r, v) \mapsto \mathcal{A}(t, u, r, v) \in \mathbb{R}^d,$$

such that it satisfies linear growth condition of the form

$$|\mathcal{A}(t, u, r, v)| \leq A + Br + C|v|$$

for some constants $A, B, C \geq 0$ and such that it satisfies the uniform parabolicity condition, that is there exists constants $c_2 > c_1 > 0$ such that for all $(t, u, r, v) \in \mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d$,

$$c_1|\xi|^2 \leq \sum_{i,j=1}^{d} \partial_{v_j}\mathcal{A}^i(t, u, r, v)\xi_i\xi_j \leq c_2|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^d \setminus \{0\}.$$

According to the chain rule,

$$\begin{aligned}
\mathrm{div}_u[\mathcal{A}(t, u, \rho_t(u), \nabla\rho_t(u))] \quad = \quad & \sum_{i=1}^{d}[\partial_{u_i}\mathcal{A}^i](t, u, \rho_t(u), \nabla\rho_t(u)) \\
& + \sum_{i=1}^{d} \partial_r\mathcal{A}^i(t, u, \rho_t(u), \nabla\rho_t(u))\partial_i\rho_t(u) \\
& + \sum_{i,j=1}^{d} \partial_{v_j}\mathcal{A}^i(t, u, \rho_t(u), \nabla\rho_t(u))\partial_{ij}\rho_t(u)
\end{aligned}$$

and so the problem (3.29) can be written in the general form as

$$\partial_t\rho(t,u) = \sum_{i,j=1}^{d} a_{ij}\big(t,u,\rho_t(u),\nabla\rho_t(u)\big)\partial_{ij}\rho_t(u) - a\big(t,u,\rho_t(u),\nabla\rho_t(u)\big),$$

where

$$a_{ij}(t,u,r,v) := \partial_{v_j}\mathcal{A}^i(t,u,r,v), \quad i,j=1,\dots,d$$

and

$$a(t,u,r,v) := -\mathrm{div}_u\mathcal{A}(t,u,r,v) - \langle\partial_r\mathcal{A}(t,u,r,v),v\rangle.$$

In particular, we are interested in time-independent and space-homogeneous quasi-linear equations in which case the vector-valued function $\mathcal{A}$ is a function only of the variables $(r,v)\in\mathbb{R}\times\mathbb{R}^d$, that is $\mathcal{A}(t,u,r,v)\equiv\mathcal{A}(r,v)$, in which case the problem (3.29) becomes

$$\begin{aligned}\partial_t\rho(t,u) &= \mathrm{div}_u\mathcal{A}\big(\rho_t(u),\nabla\rho_t(u)\big)\\ &= \sum_{i,j=1}^{d}\partial_{v_j}\mathcal{A}^i(\rho_t(u),\nabla\rho_t(u))\partial_{ij}\rho_t(u) + \langle\partial_r\mathcal{A}(\rho_t(u),\nabla\rho_t(u)),\nabla\rho_t(u)\rangle\end{aligned}$$

In the uniformly parabolic case the according to the basic theory of classical solutions (see for instance [26, 30, 23]) we have the following result.

Proposition 3.2.3 *Let $\rho\in C(\mathbb{R}^d)$. The uniformly problem (3.29) with a C^k, $k\geq 0$, vector field $\mathcal{A}$ admits a unique classical solution $\rho\in C(\mathbb{R}_+\times\mathbb{R}^d)$. Furthermore the solution ρ is C^k in $(0,\infty)\times\mathbb{R}^d$ and a strong comparison principle holds:*

Given two classical solutions $\rho_1,\rho_2\in C([0,T]\times\mathbb{R}^d)$, $T>0$, of the same equation of the the form (3.29), if $\rho_1(0,\cdot)\leq\rho_2(0,\cdot)$ on $\mathbb{R}^d$ then either $\rho_1\equiv\rho_2$ in $[0,T]\times\mathbb{R}^d$, or $\rho_1<\rho_2$ everywhere in $(0,T)\times\mathbb{R}^d$.

In our particular case of equations of the form $\partial_t\rho=\Delta_\Sigma\Phi(\rho)$ this result obviously carries over to the torus $\mathbb{T}^d$ in place of $\mathbb{R}^d$, since functions on the torus can be considered as periodic functions on Euclidean space and due to the particular form of the equation, translations of solutions are also solutions, which implies that a solution starting from a periodic initial condition remains periodic at all times.

We will first make sure that in the case of continuous sub-critical initial data the initial value problem (3.27) with non-uniformly parabolic non-linearity $\Phi\in\mathcal{C}^\infty_{\rho_c}$, $\rho_c\in(0,\infty]$ admits classical solutions. This is done by using the sub-criticality of the initial data, the comparison principle and the following lemma to avoid the degeneracy of Φ at ρ_c. The idea of the argument is already present on [30] but it is not described there how to make arbitrarily smooth changes to Φ as in the following lemma.

Lemma 3.2.3 *Let $\Phi:(0,b]\longrightarrow(0,\infty)$, $b\in(0,\infty)$, be a strictly positive C^k function, $k\in\mathbb{N}$. There exists then large enough $M\equiv M(k)\geq 0$ such that the function*

$$\widetilde{\Phi}_{M,k}(\rho) = \begin{cases}\Phi(\rho), & 0<\rho\leq b\\ T^{b;k}_M\Phi(\rho-b), & \rho\geq b\end{cases},$$

where $T_M^{b;k}\Phi$ *is the Taylor-type polynomial*

$$T_M^{b;k}\Phi(r) = \frac{M}{(k+1)!}r^{k+1} + \sum_{m=1}^{k}\frac{\Phi^{(m)}(b)}{m!}r^m + \Phi(b),$$

is a strictly positive C^k *extension* $\widetilde{\Phi} : (0,\infty) \longrightarrow (0,\infty)$ *of* Φ.

Proof The lemma follows by induction on k. For $k = 1$ it is obvious that for each $M \geq 0$ the function $\widetilde{\Phi}_{M,1}$ is a C^1-extension of Φ. Furthermore, since $\Phi(b) > 0$ it is obvious that if $\Phi'(b) \geq 0$ then the function $\widetilde{\Phi}_{M,1}$ is strictly positive for all $M \geq 0$, and in particular for $\rho > b$ we can choose the linear extension $T_0^{b;1}\Phi$ of Φ. On the other hand, we can always choose $M > \frac{1}{2}\Phi'(b)^2/\Phi(b)$ and then the discriminant of $T_M^{b;1}\Phi$ becomes negative, which since $T_M^{b;1}\Phi(0) = \Phi(b) > 0$ implies that

$$\inf_{r\in\mathbb{R}} T_M^{b;1}\Phi(r) > 0.$$

Consequently, we can can choose large enough $M > 0$ so that the function $\widetilde{\Phi}_{M,1}$ is the required strictly positive C^1 extension of Φ.

We suppose next that the claim holds for $k \in \mathbb{N}$ and we prove that it is also true for $k+1$. Since Φ is C^{k+1}, $\widetilde{\Phi}_{M,k+1}$ is a C^{k+1} extension of Φ. Since Φ is also C^k, by the inductive hypothesis there exists $M_k \geq 0$ such that

$$\theta_0 := \min_{r\geq 0} T_{M_k}^{b,k}\Phi(r) = \min_{r\geq 0}\Big\{\frac{M_k}{(k+1)!}r^{k+1} + \sum_{m=1}^{k}\frac{\Phi^{(m)}(b)}{m!}r^m + \Phi(b)\Big\} > 0.$$

We can then bound $T_M^{b;k+1}\Phi$ from below by

$$T_M^{b;k+1}\Phi(r) \geq \frac{M}{(k+2)!}r^{k+2} + \frac{\Phi^{(k+1)}(b) - M_k}{(k+1)!}r^{k+1} + \theta_0.$$

Setting $c_k := \Phi^{(k+1)}(b) - M_k$, it suffices to find $M > 0$ such that

$$\frac{M}{(k+2)!}r^{k+2} + \frac{c_k}{(k+1)!}r^{k+1} + \theta_0 > 0 \quad \text{for all } r > 0. \tag{3.30}$$

But this inequality can be rewritten as

$$Mr + c_k(k+2) > -\frac{\theta_0(k+2)!}{r^{k+1}}, \qquad r > 0,$$

from where it becomes obvious that there exists M large enough such that (3.30) holds.□

Proposition 3.2.4 *Let* $\Phi \in \mathcal{C}_{\rho_c}$, $\rho_c \in (0,\infty]$ *and let* $\rho_0 : \mathbb{T}^d \longrightarrow [0,\rho_c)$ *be a continuous initial profile. There exists then a unique classical solution* $\rho \in C(\mathbb{R}_+ \times \mathbb{T}^d)$ *of the initial value problem (3.1). Furthermore,* ρ *is* C^∞ *on* $(0,\infty)\times\mathbb{T}^d$ *and if the initial profile is of class* $C^{2+\theta}$ *for some* $\theta \in (0,1]$, *then* $\rho \in C^{1+\theta,2+\theta}(\mathbb{R}_+ \times \mathbb{T}^d)$. *Moreover,*

$$\max_{u\in\mathbb{T}^d}\rho_t(u) < \max_{u\in\mathbb{T}^d}\rho_0(u) < \rho_c, \tag{3.31}$$

for all $t > 0$. *Finally, if* ρ_0 *is not constant, then for all* $t > 0$ *we have that*

$$\min_{u\in\mathbb{T}^d}\rho_t(u) > \min_{u\in\mathbb{T}^d}\rho_0(u) \geq 0. \tag{3.32}$$

Proof Since ρ_0 is continuous and takes values in the interval $[0, \rho_c)$ it follows by the compactness of $\mathbb{T}^d$ that there exists $\varepsilon > 0$ such that $\max_{u \in \mathbb{T}^d} \rho_0(u) < \rho_c - \varepsilon$. Then, since $\Phi'(0) \wedge \Phi'(\rho_c - \varepsilon) > 0$, for any fixed $k \geq 1$ there exists by lemma 3.2.3 a strictly positive C^k extension $\Psi : \mathbb{R} \longrightarrow \mathbb{R}$ of $\Phi'|_{[0,\rho_c-\varepsilon]}$ of the form

$$\Psi(\rho) = \begin{cases} T_M^{0;k}(\Phi')(-\rho), & \rho \leq 0 \\ \Phi'(\rho), & 0 \leq \rho \leq b \\ T_M^{b;k}(\Phi')(\rho - b), & \rho \geq b \end{cases}$$

for some $M > 0$. Then obviously $\lim_{|\rho| \to \infty} \Psi(\rho) = +\infty$ and therefore $c := \inf_{\rho \in \mathbb{R}} \Psi(\rho) > 0$. We set $B := \max_{\rho \in [0,\rho_c - \varepsilon]} \Psi(\rho)$, we choose a smooth function $\chi : \mathbb{R}_+ \longrightarrow [0, B+1]$ such that $\chi(y) = y$ for $0 \leq y \leq B$ and $\chi(y) = B + 1$ for $y \geq B + 1$ and consider the function $\widetilde{\Psi} := \chi \circ \Psi : \mathbb{R} \longrightarrow (0, \infty)$. Then its anti-derivative $\widetilde{\Phi}(\rho) = \int_0^\rho \widetilde{\Psi}(r) dr$, $\rho \in \mathbb{R}$, is a C^{k+1} extension of the restriction $\Phi|_{[0,\rho_c-\varepsilon]}$ satisfying $c \leq \widetilde{\Phi}'(\rho) \leq B + 1$ for all $\rho \in \mathbb{R}$. The claim then follows by applying the results on uniformly parabolic quasilinear equations obtained in [26] (see also section 3.1.1 in [30] for a review of these results) to the initial value problem $\partial_t \rho = \Delta_\Sigma \widetilde{\Phi}(\rho)$ with initial condition $\rho(0, \cdot) \equiv \rho_0$, for each $k \geq 1$. Finally the strict inequalities in the left hand sides of (3.31) and (3.32) are justified by the strong comparison principle for uniformly parabolic quasilinear equations as stated in section 3.1.1, p. 31 of [30]. ☐

3.3 Application of the Relative Entropy Method

Let $\{\mu_0^N \in \mathbb{PM}_N^d\}$ be an initial entropy local equilibrium of profile ρ_0 and let $\mu_t^N := \mu_0^N S_{tN^2}^N \in \mathbb{PM}_N^d$, $t \in I$, denote the evolution of the initial distribution μ_0^N under the diffusively rescaled transition semigroup of the ZRP. Before proceeding the proof we introduce some notation that will be used in throughout the proof and give a simple bound on the entropy production $\partial_t H(\mu_t^N|\nu_{\rho_t(\cdot)}^N)$. For each $a \in (0, \rho_c)$ and $\varepsilon > 0$ we denote by $\psi_t^{N;a}$ the Radon-Nikodym derivative of $\nu_{\rho_t(\cdot)}^N$ with respect to ν_a^N, that is we set

$$\psi_t^{N;a} := \frac{d\nu_{\rho_t(\cdot)}^N}{d\nu_a^N}.$$

Since the measures involved are product measures we can easily obtain an explicit expression for $\psi_t^{N;a}$. We have

$$\psi_t^{N;a}(\eta) = \prod_{x \in \mathbb{T}_N^d} \frac{d\nu_{\rho_t(x/N)}^1}{d\nu_a^1}(\eta_x) = \prod_{x \in \mathbb{T}_N^d} \frac{Z(\Phi(a))\Phi(\rho_t(x/N))^{\eta_x}}{Z(\Phi(\rho_t(x/N)))\Phi(a)^{\eta_x}}.$$

Setting

$$\Phi_a(\beta) := \frac{\Phi(\beta)}{\Phi(a)} \qquad \text{and} \qquad Z_a(\beta) := \frac{Z(\Phi(\beta))}{Z(\Phi(a))},$$

we can write $\psi_t^{N,\varepsilon;a}$ as

$$\psi_t^N = \prod_{x \in \mathbb{T}_N^d} \frac{\Phi_a(\rho_t(x/N))^{\eta_x}}{Z_a(\rho_t(x/N))} = \exp\Big\{ \sum_{x \in \mathbb{T}_N^d} \Big[\eta_x \log \Phi_a\big(\rho_t(x/N)\big) - \log Z_a\big(\rho_t(x/N)\big)\Big]\Big\}.$$

Next we set $f_t^N \equiv f_t^{N;a}$ the Radon-Nikodym derivative of μ_t^N with respect to the reference measure ν_a^N:

$$f_t^N := \frac{d\mu_t^N}{d\nu_a^N}.$$

Then

$$\frac{d\mu_t^N}{d\nu_{\rho_t(\cdot)}^N} = \frac{f_t^N}{\psi_t^N}$$

for every $a \in (0, \rho_c)$ and so if we denote by $H_N(t)$ the relative entropy of μ_t^N with respect to $\nu_{\rho_t(\cdot)}^N$, that is

$$H_N(t) := H(\mu_t^N|\nu_{\rho_t(\cdot)}^N),$$

then we have that

$$H_N(t) = \int \frac{f_t^N}{\psi_t^N} \log \frac{f_t^N}{\psi_t^N} d\nu_{\rho_t(\cdot)}^N = \int f_t^N \log \frac{f_t^N}{\psi_t^N} d\nu_a^N. \tag{3.33}$$

As we know the density f_t^N is a solution of the Kolmogorov equation $\partial_t f_t^N = N^2 L_N^* f_t^N$, and as such it is a pointwise C^1 in time function. Since the function ψ_t^N is also obviously C^1 in time and does not vanish, the integrand in the right hand side of (3.33) is C^1 in time and therefore we can pass the differentiation inside the integral to get the following upper bound on the entropy production:

Lemma 3.3.1 *Let $a \in (0,\rho_c)$ be fixed. For every $t \geq 0$ we have*

$$\begin{aligned}\partial_t H_N(t) &= \int N^2 f_t^N L_N \log\frac{f_t^N}{\psi_t^N} d\nu_a^N - \int \frac{\partial_t \psi_t^N}{\psi_t^N} f_t^N d\nu_a^N \\ &\leq \int \frac{1}{\psi_t^N}\{N^2 L_N^* \psi_t^N - \partial_t \psi_t^N\} d\mu_t^N,\end{aligned}$$

where L_N^* denotes the adjoint of the generator L_N in $L^2(\nu_a^N)$.

ProofProof Since f_t^N is the solution of the Kolmogorov forward equation

$$\partial_t f_t^N = N^2 L_N^* f_t^N$$

f_t^N is sufficiently smooth we can pass the differentiation inside the integral in the equation (3.33) expressing the entropy to get

$$\begin{aligned}\partial_t H_N(t) &= \int \partial_t\Big(f_t^N \log\frac{f_t^N}{\psi_t^N}\Big) d\nu_a^N = \int \partial_t f_t^N \log\frac{f_t^N}{\psi_t^N} d\nu_a^N + \int f_t^N \partial_t \log\frac{f_t^N}{\psi_t^N} d\nu_a^N \\ &= \int N^2 (L_N^* f_t^N) \log\frac{f_t^N}{\psi_t^N} d\nu_a^N + \int \psi_t^N \partial_t\Big(\frac{f_t^N}{\psi_t^N}\Big) d\nu_a^N \\ &= \int N^2 (L_N^* f_t^N) \log\frac{f_t^N}{\psi_t^N} d\nu_a^N + \int \frac{\psi_t^N \partial_t f_t^N - f_t^N \partial_t \psi_t^N}{\psi_t^N} d\nu_a^N \\ &= \int N^2 (L_N^* f_t^N) \log\frac{f_t^N}{\psi_t^N} d\nu_a^N + \int \partial_t f_t^N d\nu_a^N - \int \frac{\partial_t \psi^N}{\psi_t^N} f_t^N d\nu_a^N.\end{aligned}$$

Now, the term $\int \partial_t f_t^N d\nu_a^N$ vanishes since $\partial_t f_t^N = N^2 L_N^* f_t^N$ and ν_a^N is an invariant distribution and therefore

$$\partial_t H_N(t) = \int N^2 f_t^N L_N \log\frac{f_t^N}{\psi_t^N} d\nu_a^N - \int \frac{\partial_t \psi_t^N}{\psi_t^N} f_t^N d\nu_a^N$$

which proves the equality stated in the lemma.

Next, by the elementary inequality

$$a(\log b - \log a) \leq b - a, \quad a, b > 0$$

which remains valid even with $b = 0$ under the convention $\log 0 = -\infty$, it follows that

$$h L_N \log h \leq L_N h$$

for any non-negative function $h : \mathbb{M}_N^d \longrightarrow [0,\infty)$. Using this inequality, we get

$$\begin{aligned}\int f_t^N L_N \log\frac{f_t^N}{\psi_t^N} d\nu_a^N &= \int \psi_t^N \frac{f_t^N}{\psi_t^N} L_N \log\frac{f_t^N}{\psi_t^N} d\nu_a^N \leq \int \psi_t^N L_N \frac{f_t^N}{\psi_t^N} d\nu_a^N \\ &= \int (L_N^* \psi_t^N) \frac{f_t^N}{\psi_t^N} d\nu_a^N = \int \frac{L_N^* \psi_t^N}{\psi_t^N} d\mu_t^N.\end{aligned}$$

Combining this inequality with the already proved equality the lemma is proved. □

Denoting by

$$H(t) := \limsup_{N\to\infty} \frac{1}{N^d} H_N(t), \quad t \in \mathbb{R}_+, \tag{3.34}$$

the limiting entropy density, the main step in the application of the relative entropy method is to use this upper bound on $\partial_t H_N(t)$ to get an inequality of the form

$$H(t) \leq H(0) + \int_0^t H(s)\beta(s)ds \tag{3.35}$$

for a non-negative function β. Since $H(0) = 0$ by assumption, this implies by Gronwall's inequality that $H(t) = 0$ for all $t \in \mathbb{R}_+$ as required. Of course in order for Gronwall's inequality to be applicable, the function $s \mapsto H(s)\beta(s)$ must belong at least in $L^1_{\text{loc}}(\mathbb{R}_+)$.

Lemma 3.3.2 *Let $\rho : \mathbb{R}_+ \times \mathbb{T}^d \longrightarrow [0, \rho_c)$ be a continuous function such that the function in (3.2a) is in $L^2_{\text{loc}}(\mathbb{R}_+)$. If a sequence of initial distributions $\{\mu_0^N\}$ has relative entropy of order $o(N^d)$ with respect to $\nu^N_{\rho_0(\cdot)}$, then the upper entropy $\overline{H}$ belongs in $L^2_{\text{loc}}(\mathbb{R}_+)$, where*

$$\overline{H}(t) := \sup_{N\in\mathbb{N}} \frac{1}{N^d} H(\mu_t^N | \nu^N_{\rho_t(\cdot)}), \quad t \in \mathbb{R}_+.$$

Proof According to proposition 2.6.2, the relative entropy inequality shows that the sequence $\{\mu_0^N \in \mathbb{PM}^d_N\}$ satisfies the $O(N^d)$-entropy assumption. Using the relative entropy inequality once again we prove that $\overline{H} \in L^2_{\text{loc}}(\mathbb{R}_+)$. Indeed, given $T > 0$ we pick $\varepsilon > 0$ such that $\rho_c - \varepsilon$ is an upper bound of the set $\rho([0,T] \times \mathbb{T}^d)$ and fix $a \in (\rho_c - \varepsilon, \rho_c)$. By the relative entropy inequality and proposition A.1.9.1 of [25], according to which the function $t \mapsto \mathcal{H}(\mu_t^N | \nu_a^N)$ is non-increasing,

$$H_N(t) \leq \Big(1 + \frac{1}{\gamma}\Big)\mathcal{H}(\mu_0^N | \nu_a^N) + \frac{1}{\gamma} \log \int \Big(\frac{d\nu_a^N}{d\nu^N_{\rho_t(\cdot)}}\Big)^\gamma d\nu_a^N \tag{3.36}$$

for all $t \geq 0$ and all $\gamma > 0$. By similar computations in the previous step,

$$\log \int \Big(\frac{d\nu_a^N}{d\nu^N_{\rho_t(\cdot)}}\Big)^\gamma d\nu_a^N \quad = \quad \sum_{x\in\mathbb{T}^d_N} \Big\{\gamma \log \frac{Z\big(\Phi(\rho_t(x/N))\big)}{Z(\Phi(a))} + \Lambda_a\Big(\gamma \log \frac{\Phi(a)}{\Phi(\rho_t(x/N))}\Big)\Big\}.$$

So if for each $t > 0$ we set

$$\gamma(t) := \frac{1}{2} \frac{\log \frac{\varphi_c}{\Phi(a)}}{\log \frac{\Phi(a)}{\Phi(m_t)}},$$

where $m_t := \min_{u\in\mathbb{T}^d} \rho_t(u)$, then $\gamma(t) \log \frac{\Phi(a)}{\Phi(\rho_t(x/N))} \leq \frac{1}{2} \log \frac{\varphi_c}{\Phi(a)}$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^d_N$, and by (3.36) for all $t \in [0, T]$

$$\overline{H}(t) \leq \Big(1 + \frac{1}{\gamma(t)}\Big) C(a) + \log \frac{Z(\Phi(\rho_c - \varepsilon))}{Z(\Phi(a))} + \frac{1}{\gamma(t)} \log Z\big(\sqrt{\varphi_c \Phi(a)}\,\big).$$

Since the function in (3.2a) is in $L^2_{\text{loc}}(\mathbb{R}_+)$, the right hand side above is in $L^2([0, T])$. □

The bound of lemma 3.3.1 on the entropy production can be estimated using the explicit formula of ψ_t^N. A simple computation of the action of the adjoint of the generator L_N on ψ_t^N shows that

$$\frac{L_N^* \psi_t^N}{\psi_t^N} = \sum_{x,y \in \mathbb{T}_N^d} \Big[\frac{\Phi\big(\rho(t,y/N)\big)}{\Phi\big(\rho(t,x/N)\big)} - 1\Big] g\big(\eta(x)\big)p(x-y).$$

This is well defined since ρ_t is strictly positive for all $t>0$ even if the initial profile takes the value zero not identically. On the other hand, the sum

$$\sum_{x,y \in \mathbb{T}_N^d} \Big[\frac{\Phi\big(\rho(t,y/N)\big)}{\Phi\big(\rho(t,x/N)\big)} - 1\Big] \Phi\big(\rho(x/N)\big)p(x-y)$$

obviously vanishes and therefore we can write that

$$\frac{N^2 L_N \psi_t^N}{\psi_t^N} = N^2 \sum_{x,y \in \mathbb{T}_N^d} \Big[\frac{\Phi\big(\rho(t,y/N)\big)}{\Phi\big(\rho(t,x/N)\big)} - 1\Big] \big[g\big(\eta(x)\big) - \Phi\big(\rho(t,\frac{x}{N})\big)\big]p(x-y). \quad (3.37)$$

Proposition 3.3.1 (Taylor's Theorem) *Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a $C^{k+\theta}$ function for some $k \in \mathbb{N}$, $0 < \theta \leq 1$. Then for all $x, h \in \mathbb{R}^d$ we have that*

$$\begin{aligned} R_k(x,h) &:= f(x+h) - \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \partial^\alpha f(x) h^\alpha \\ &= k \int_0^1 \sum_{|\alpha|=k} \frac{1}{\alpha!} [\partial^\alpha f(x+sh) - \partial^\alpha f(x+h)] h^\alpha (1-s)^{k-1} ds \\ &\quad + \sum_{|\alpha|=k} \frac{1}{\alpha!} [\partial^\alpha f(x+h) - \partial^\alpha f(x)] h^\alpha. \end{aligned}$$

Furthermore

$$\sup_{x \in \mathbb{R}^d} |R_k(x,h)| \leq 2\sqrt{d} \frac{\|D^k f\|_\theta}{k!} \|h\|^{k+\theta},$$

where $\|D^k f\|_\theta := \max_{|\alpha|=k} \|\partial^\alpha f\|_\theta$ and $\|\cdot\|_\theta$ is the θ-Holder semi-norm. In particular

$$\sup_{x \in \mathbb{R}^d} |R_k(x,h)| \leq O(|h|^{k+\theta}) \leq o(|h|^k).$$

Since $\Phi(\rho_t)$ is $C^{2+\theta}$ for some $\theta > 0$ we have by this Taylor expansion that

$$\begin{aligned} \Phi\Big(\rho_t\big(\frac{y}{N}\big)\Big) - \Phi\Big(\rho_t\big(\frac{x}{N}\big)\Big) &= \frac{1}{N} \langle \nabla[\Phi(\rho_t)]\big(\frac{x}{N}\big), y-x \rangle + \frac{1}{2N^2} H^2[\Phi(\rho_t)]\big(\frac{x}{N}\big)(y-x) \\ &\quad + R_t\big(\frac{x}{N}, \frac{y-x}{N}\big), \end{aligned}$$

where H^2 is the Hessian form and the remainder R_t satisfies

$$\sup_{u \in \mathbb{T}^d} |R_t(u,h)| \leq \sqrt{d} \|D^2[\Phi(\rho_t)]\|_{C^{0,\theta}} \|h\|^{2+\theta}.$$

Since $\Phi(\rho_t)$ is smooth and since the elementary step distribution p has mean zero, we have by the second order Taylor expansion for stated above that

$$\sum_{y\in\mathbb{T}_N^d}\Big[\Phi\Big(\rho_t(\frac{y}{N})\Big)-\Phi\Big(\rho_t(\frac{x}{N})\Big)\Big]p(y-x) \quad = \quad \frac{1}{N^2}\Delta_\Sigma[\Phi(\rho_t)](\frac{x}{N})+R_t(\frac{x}{N})$$

for all $x\in\mathbb{T}_N^d$, where the remainder R_t satisfies $\|R_t\|_\infty\le CN^{-(2+\theta)}\big\|D^2[\Phi(\rho_t)]\big\|_\theta$ for some constant $C>0$ depending only on $p\in\mathbb{P}\mathbb{Z}^d$ and the dimension d:

$$\begin{aligned}\big|\bar{R}_t(\frac{x}{N})\big| &\le \sum_{y\in\mathbb{T}_N^d}\big|R_t(\frac{x}{N},\frac{y-x}{N})\big|p(y-x)\\ &\le \sqrt{d}\|D^2[\Phi(\rho_t)]\|_\theta\frac{1}{N^{2+\theta}}\sum_{y\in\mathbb{Z}^d}\|y-x\|^{2+\theta}p(y-x)\\ &=: CN^{-(2+\theta)}\big\|D^2[\Phi(\rho_t)]\big\|_\theta.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{N^2[L_N]^*\psi_t^N}{\psi_t^N} &= N^2\sum_{x,y\in\mathbb{T}_N^d}\Big[\Phi\big(\rho_t(y/N)\big)-\Phi\big(\rho_t(x/N)\big)\Big]\Big[\frac{g\big(\eta(x)\big)}{\Phi\big(\rho(t,x/N)\big)}-1\Big]p(x-y)\\ &= \sum_{x\in\mathbb{T}_N^d}\Big[\Delta_\Sigma[\Phi(\rho_t)](\frac{x}{N})+N^2\bar{R}_t(\frac{x}{N})\Big]\Big[\frac{g\big(\eta(x)\big)}{\Phi\big(\rho(t,x/N)\big)}-1\Big]\\ &= \sum_{x\in\mathbb{T}_N^d}\Big(\frac{\Delta_\Sigma[\Phi(\rho_t)]}{\Phi(\rho_t)}\Big)(\frac{x}{N})\big[g(\eta(x))-\Phi(\rho_t(x/N))\big]\\ &\quad+\sum_{x\in\mathbb{T}_N^d}\frac{N^2\bar{R}_t(x/N)}{\Phi\big(\rho(t,x/N)\big)}\Big[g\big(\eta(x)\big)-\Phi\big(\rho(t,x/N)\big)\Big],\end{aligned}$$

and for the remainder

$$r_N(t):=\sum_{x\in\mathbb{T}_N^d}\frac{N^2\bar{R}_t(x/N)}{\Phi\big(\rho(t,x/N)\big)}\Big[g\big(\eta(x)\big)-\Phi\big(\rho(t,x/N)\big)\Big]$$

we have that

$$\begin{aligned}|r_N(t)| &\le \sum_{x\in\mathbb{T}_N^d}\Big|\frac{N^2\bar{R}_t(x/N)}{\Phi\big(\rho(t,x/N)\big)}\Big|\Big|g\big(\eta(x)\big)-\Phi\big(\rho(t,x/N)\big)\Big|\\ &\le N^2\frac{\|g\|_\infty+\Phi(\rho_c-\varepsilon)}{\Phi(m_t)}\sum_{x\in\mathbb{T}_N^d}\big|\bar{R}_t(x/N)\big|\\ &\le C\cdot\big(\|g\|_\infty+\Phi(\rho_c-\varepsilon)\big)N^{d-\theta}\frac{\big\|D^2[\Phi(\rho_t)]\big\|_\theta}{\Phi(m_t)}.\end{aligned}$$

Therefore we can write (3.37) as

$$N^2\frac{L_N^*\psi_t^N}{\psi_t^N}=\sum\nolimits_{x\in\mathbb{T}_N^d}\Big(\frac{\Delta_\Sigma[\Phi(\rho_t)]}{\Phi(\rho_t)}\Big)(\frac{x}{N})\big[g(\eta(x))-\Phi(\rho_t(x/N))\big]+r_N(t)$$

where for the remainder term $r_N(t)$ we have

$$|r_N(t)| \le \bar{C} N^{d-\theta} \frac{\big\| D^2[\Phi(\rho_t)] \big\|_\theta}{\Phi(m_t)}, \tag{3.38}$$

with $\bar{C} := C(\|g\|_\infty + \Phi(\rho_c - \varepsilon))$.

Furthermore, using the fact that ρ is a solution of $\partial_t \rho = \Delta_x \Phi(\rho)$ and the equality $R(\varphi) = \frac{\varphi Z'(\varphi)}{Z(\varphi)}$, $0 \le \varphi \le \varphi_c$, we get that

$$\begin{aligned} \frac{\partial_t \psi_t^N}{\psi_t^N} &= \partial_t (\log \psi_t^N) = \partial_t \sum_{x \in \mathbb{T}_N^d} \big[\eta(x) \log \Phi_a\big(\rho_t(x/N)\big) - \log Z_a\big(\rho_t(x/N)\big) \big] \\ &= \sum_{x \in \mathbb{T}_N^d} \frac{\Delta_\Sigma[\Phi(\rho_t)]}{\Phi(\rho_t)} \big(\frac{x}{N}\big) \Phi'\Big(\rho_t\big(\frac{x}{N}\big)\Big) \Big[\eta(x) - \rho_t\big(\frac{x}{N}\big)\Big] \end{aligned}$$

and therefore by the calculations made so far, by lemma 3.3.1 and the one block estimate we have the following upper bound for the entropy.

Lemma 3.3.3 *For all $t > 0$ we have*

$$H_N(t) \le H_N(0) + \int_0^t \int \sum_{x \in \mathbb{T}_N^d} \Big(\frac{\Delta[\Phi(\rho_s)]}{\Phi(\rho_s)}\Big)\big(\frac{x}{N}\big) M\big(\eta_s^\ell(x), \rho_s(x/N)\big) d\mu_s^N ds + o_\ell(N^d),$$

where $M : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is the function given by the formula

$$M(\lambda, \rho) = \Phi(\lambda) - \Phi(\rho) - \Phi'(\rho)(\lambda - \rho)$$

and the term $o_\ell(N^d)$ satisfies

$$\limsup_{\ell \to \infty} \limsup_{N \to \infty} \frac{1}{N^d} o_\ell(N^d) = 0.$$

Proof By lemma 3.3.1 and the calculations performed so far we have that

$$\begin{aligned} H_N(t) \quad \le \quad & H_N(0) + \int_0^t r_N(s) ds \\ & + \int_0^t \int \sum_{x \in \mathbb{T}_N^d} \frac{\Delta[\Phi(\rho_s)]}{\Phi(\rho_s)} \big(\frac{x}{N}\big) \Big\{ g(\eta(x)) - \Phi(\rho_s(x/N)) \\ & \qquad - \Phi'\Big(\rho_s\big(\frac{x}{N}\big)\Big) \Big[\eta(x) - \rho_s\big(\frac{x}{N}\big)\Big] \Big\} d\mu_s^N ds. \end{aligned} \tag{3.39}$$

By the bound (3.38) on the remainder and the $L^2_{\text{loc}}(\mathbb{R}_+)$-integrability of the function defined in (3.2b) it follows that $\int_0^t \int r_N(s) d\mu_s^N ds = o(N^d)$ for each $t > 0$. So we deal next with the last term in the right hand side of inequality (3.39). This term is equal to

$$\begin{aligned} & \int_0^t \int \sum_{x \in \mathbb{T}_N^d} \frac{\Delta_\Sigma[\Phi(\rho_s)]}{\Phi(\rho_s)} \big(\frac{x}{N}\big) \Big\{ g(\eta(x)) - \Phi(\rho_s(x/N)) \Big\} d\mu_s^N ds \\ & - \int_0^t \int \sum_{x \in \mathbb{T}_N^d} \frac{\Delta_\Sigma[\Phi(\rho_s)]}{\Phi(\rho_s)} \big(\frac{x}{N}\big) \Phi'\big(\rho_s\big(\frac{x}{N}\big)\big) \big[\eta(x) - \rho_s\big(\frac{x}{N}\big)\big] d\mu_s^N ds \end{aligned}$$

Since $\frac{\Delta_\Sigma[\Phi(\rho)]}{\Phi(\rho)} \in L^2(0,T;C(\mathbb{T}^d))$, $T>0$, by (3.2b) we can apply the one block estimate with respect to functions in $L^1(0,T;C(\mathbb{T}^d))$ to write the first term as

$$\int_0^t \int \sum_{x\in\mathbb{T}_N^d} \frac{\Delta_\Sigma[\Phi(\rho_s)]}{\Phi(\rho_s)}\big(\frac{x}{N}\big)\Big\{\Phi(\eta^\ell(x)) - \Phi(\rho_s(x/N))\Big\} d\mu_s^N ds + o_\ell(N^d).$$

On the other hand a simple change of variables shows that we can write the second term as

$$\int_0^t \int \sum_{x\in\mathbb{T}_N^d} \frac{\Delta_\Sigma[\Phi(\rho_s)]}{\Phi(\rho_s)}\big(\frac{x}{N}\big)\Phi'\big(\rho_s\big(\frac{x}{N}\big)\big)\big[\eta^\ell(x) - \rho_s\big(\frac{x}{N}\big)\big] d\mu_s^N ds + o(N^d),$$

and the proof is complete. □

In this way, the macroscopic Taylor expansion of $\Phi(\rho_t)$ around the point $\frac{x}{N}$ and the one block estimate give a microscopic Taylor expansion of $\Phi(\eta_t^\ell(x))$ around the point $\rho_t(x/N)$.

To simplify the notation, we set $G_t : \mathbb{T}^d \times \mathbb{R}_+ \longrightarrow \mathbb{R}$, $t \geq 0$, the function defined by

$$G_t(u,\lambda) = \frac{\Delta_\Sigma[\Phi(\rho_t)]}{\Phi(\rho_t)}(u) M\big(\lambda, \rho_t(u)\big). \tag{3.40}$$

Then we can rewrite the bound of the last lemma as

$$H_N(t) \leq \int_0^t \int \sum_{x\in\mathbb{T}_N^d} G_s\Big(\frac{x}{N}, \eta^\ell(x)\Big) d\mu_s^N ds + o_\ell(N^d),$$

where we have also taken into account the fact the $H_N(0) = o(N^d)$ by assumption. By the relative entropy inequality we get that

$$\int \sum_{x\in\mathbb{T}_N^d} G_s\Big(\frac{x}{N}, \eta^\ell(x)\Big) d\mu_s^N \quad \leq \quad \frac{1}{\gamma_s} H_N(s) + \frac{1}{\gamma_s} \log \int e^{\gamma_s \sum_{x\in\mathbb{T}_N^d} G_s(\frac{x}{N}, \eta^\ell(x))} d\nu_{\rho_s(\cdot)}^N$$

for any positive measurable function $(0,\infty) \ni s \mapsto \gamma_s \in (0,\infty)$ and each $s > 0$. Dividing by N^d and taking lim sup first as $N \to \infty$ and then as $\ell \to \infty$ we get that

$$\begin{aligned} H(t) \quad \leq \quad & \limsup_{N\to\infty} \frac{1}{\gamma_s N^d} \int_0^t H_N(s) ds \\ & + \limsup_{\ell\to\infty} \limsup_{N\to\infty} \frac{1}{\gamma_s N^d} \int_0^t \log \int e^{\gamma_s \sum_{x\in\mathbb{T}_N^d} G_s(\frac{x}{N}, \eta^\ell(x))} d\nu_{\rho_s(\cdot)}^N ds, \end{aligned}$$

Then if the function γ can be chosen so that $\beta := 1/\gamma \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$, we can use lemma 3.3.2 to pass the lim sup as $N \to \infty$ inside the time integral of $s \mapsto H_N(s)\beta(s)/N^d$ to get Gronwall's inequality (3.35) but with the term

$$\limsup_{\ell\to\infty} \limsup_{N\to\infty} \frac{1}{N^d} \int_0^t \frac{1}{\gamma_s} \log \int e^{\gamma_s \sum_{x\in\mathbb{T}_N^d} G_s(\frac{x}{N}, \eta^\ell(x))} d\nu_{\rho_s(\cdot)}^N ds \tag{3.41}$$

added to its right hand side.

So the rest of the proof is devoted to proving that the function $\beta \equiv 1/\gamma \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$

can be chosen so that for each time $t > 0$ the term in (3.41) is non-positive. We begin by noting that the function $G : \mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ defined in (3.40) satisfies the inequality

$$\sup_{u\in\mathbb{T}^d} |G_t(u,\lambda)| \le C \cdot C_t \cdot (1+\lambda) \quad \text{for all } t, \lambda > 0 \tag{3.42}$$

where

$$C = \big\{\Phi(\rho_c - \varepsilon) + \max_{r\in[0,\rho_c-\varepsilon]} r\Phi'(r)\big\} \vee 2\|g'\|_\infty < +\infty,$$

and

$$C_t := \Big\| \frac{\Delta_\Sigma \Phi(\rho_t)}{\Phi(\rho_t)} \Big\|_\infty. \tag{3.43}$$

For each $K > 1$ we denote by $\gamma^K : (0,\infty) \longrightarrow (0,\infty)$ the function $\gamma^K_t := \frac{1}{KCC_t} \log \frac{\varphi_c}{\Phi(\rho_c-\varepsilon)}$. Since the function in (3.2b) is in $L^2_{\text{loc}}(\mathbb{R}_+)$, the function $\beta^K := 1/\gamma^K$ belongs in $L^2_{\text{loc}}(\mathbb{R}_+)$. Using inequality (3.42) and the $L^2_{\text{loc}}(\mathbb{R}_+)$-integrability of β^K it is straightforward to check that the family $\{h^{N,\ell}_K\}_{(N,\ell)\in\mathbb{N}}$ of the functions

$$h^{N,\ell}_K(t) = \frac{1}{\gamma^K_t N^d} \log \int e^{\gamma^K_t \sum_{x\in\mathbb{T}^d_N} G_t(\frac{x}{N},\eta^\ell(x))} d\nu^N_{\rho_t(\cdot)}, \quad t \ge 0$$

is dominated by an $L^2_{\text{loc}}(\mathbb{R}_+)$-function for each $K > 1$. Indeed, on one hand we have by Jensen's inequality that for all $t, \gamma \ge 0$ and all $N \in \mathbb{N}$, $\ell \in \mathbb{Z}_+$ that

$$\begin{aligned}
h^{N,\ell}_K(t) &\ge \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \int G_t(\frac{x}{N},\eta^\ell(x))\nu^N_{\rho_t(\cdot)} \ge -\frac{C\cdot C_t}{N^d} \sum_{x\in\mathbb{T}^d_N} \int \big(1+\eta^\ell(x)\big) d\nu^N_{\rho_t(\cdot)} \\
&= -C\cdot C_t - \frac{C\cdot C_t}{N^d} \int \sum_{x\in\mathbb{T}^d_N} \eta(x) d\nu^N_{\rho_t(\cdot)} = -C\cdot C_t - \frac{C\cdot C_t}{N^d} \int \sum_{x\in\mathbb{T}^d_N} \rho_t(x/N) \\
&\ge -C\cdot C_t(1+\rho_c-\varepsilon),
\end{aligned}$$

which shows that $h^{N,\ell}_K$ is bounded below by an $L^2_{\text{loc}}(\mathbb{R}_+)$-function uniformly over all $N, \ell \in \mathbb{N}$. On the other hand, for all $N \in \mathbb{N}$, $\ell \in \mathbb{Z}_+$ we have that

$$\begin{aligned}
h^{N,\ell}_K(t) &\le \frac{1}{\gamma^K_t N^d} \log \int e^{\gamma^K_t \sum_{x\in\mathbb{T}^d_N} C\cdot C_t(1+\eta^\ell(x))} d\nu^N_{\rho_t(\cdot)} \\
&= C\cdot C_t + \frac{1}{\gamma^K_t N^d} \log \int e^{\gamma^K_t C\cdot C_t \sum_{x\in\mathbb{T}^d_N} \eta(x)} d\nu^N_{\rho_t(\cdot)} \\
&= C\cdot C_t + \frac{\beta^K_t}{N^d} \sum_{x\in\mathbb{T}^d_N} \Lambda_{\rho_t(\frac{x}{N})}\Big(\frac{1}{K} \log \frac{\varphi_c}{\Phi(\rho_c-\varepsilon)}\Big).
\end{aligned}$$

As we have seen, the logarithmic moment generating function $\Lambda_{\rho_t(x/N)}$ has proper domain $\mathcal{D}_{\Lambda_{\rho_t(x/N)}} \supseteq (-\infty, b_{\rho_t(x/N)})$ where $b_\rho := \log \frac{\varphi_c}{\Phi(\rho)}$ for all $\rho \in (0,\rho_c)$, and so since $\rho(t,u) \le \rho_c - \varepsilon$ for all $(t,u) \in \mathbb{R}_+ \times \mathbb{T}^d$ we have that

$$(-\infty, b_{\rho_c-\varepsilon}) \subseteq \mathcal{D}_{\Lambda_{\rho_t(\frac{x}{N})}}, \qquad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{T}^d_N.$$

Therefore, for any $K > 1$ we have that $\Lambda_{\rho_t(x/N)}(\frac{1}{K}\log\frac{\varphi_c}{\Phi(\rho_c-\varepsilon)}) < +\infty$ for all $(t,x) \in \mathbb{T}_N^d$ and in particular,

$$\begin{aligned}\Lambda_{\rho_t(x/N)}\Big(\frac{1}{K}\log\frac{\varphi_c}{\Phi(\rho_c-\varepsilon)}\Big) &= \log\frac{Z\big(\Phi(\rho_t(x/N))e^{\frac{1}{K}\log\frac{\varphi_c}{\Phi(\rho_c-\varepsilon)}}\big)}{Z(\Phi(\rho_t(x/N)))}\\ &\leq \log Z(\Phi(\rho_c-\varepsilon)^{1-\frac{1}{K}}\varphi_c^{\frac{1}{K}}) =: A_K < +\infty\end{aligned}$$

for all $(t,x) \in [0,T]\times\mathbb{T}^d$. It follows that

$$h_K^{N,\ell}(t) \leq C\cdot C_t + A_K\beta_t^K$$

for all $t \in [0,T]$, $N \in \mathbb{N}$, $\ell \in \mathbb{Z}_+$, and so $\{h_K^{N,\ell}\}$ is also bounded above by an L^2_{loc}-function uniformly over all $N, \ell \in \mathbb{N}$. This permits to pass the superior limits as $N \to \infty$ and then $\ell \to \infty$ inside the time integral in (3.41) for each $K > 1$. Consequently, in order to complete the proof it suffices to show that we can choose $K > 1$ so that for each $t > 0$,

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty}\frac{1}{N^d}\log\int e^{\gamma_t^K\sum_{x\in\mathbb{T}_N^d}G_t(\frac{x}{N},\eta^\ell(x))}d\nu^N_{\rho_t(\cdot)} \leq 0. \tag{3.44}$$

This inequality follows from the estimate of the following lemma, which is a generalization of lemma 6.1.8 in [25] to the case of finite critical density $\rho_c < +\infty$. It is a consequence of Cramer's theorem for the occupation variables $\eta(x)$, $x \in \mathbb{Z}^d$, under the grand canonical ensemble $\nu_\rho^\infty \in \mathbb{PM}_\infty^d$, $\rho \in [0,\rho_c]$, and the Laplace-Varadhan principle.

Lemma 3.3.4 *Let $\rho : \mathbb{T}^d \longrightarrow (0, \rho_c - \varepsilon)$, $\varepsilon \in (0,\rho_c)$, be a continuous profile and let $G : \mathbb{T}^d \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a continuous function such that*

$$\sup_{u\in\mathbb{T}^d}|G(u,\lambda)| \leq C_0 + C_1\lambda \quad \textit{for all } \lambda \in \mathbb{R}_+$$

for some constants $C_0 \geq 0$ and $C_1 \in [0, \frac{1}{2}\log\frac{\varphi_c}{\Phi(\rho_c-\varepsilon)})$. Then

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty}\frac{1}{N^d}\log\int e^{\sum_{x\in\mathbb{T}_N^d}G(\frac{x}{N},\eta^\ell(x))}d\nu^N_{\rho(\cdot)} \leq \int_{\mathbb{T}^d}\sup_{\lambda\geq 0}\Big\{G(u,\lambda)-\frac{1}{2}\Lambda^*_{\rho(u)}(\lambda)\Big\}du.$$

Proof For fixed $\ell \in \mathbb{Z}_+$, the mean $\eta^\ell(0)$ is a function of the random vector $\big(\eta(x)|x \in \Lambda_\ell^d\big)$. So the random variables $\eta(x)^\ell$ and $\eta(y)^\ell$ are independent whenever $|x-y|_{\mathbb{T}_N^d} \geq 2\ell+1$. We will take advantage of this fact to decompose the expectation with respect to the product measure $\nu^N_{\rho(\cdot)}$ to a product of simpler terms. We will first take the limit as $N \to \infty$. So we fix $\ell \in \mathbb{Z}_+$ and use Euclidean division to write each $N \in \mathbb{N}$ uniquely as

$$N = q_{N,\ell}(2\ell+1) + r_{N,\ell}, \qquad r_{N,\ell} \in \{0,1,\dots,2\ell\}.$$

Furthermore, we set $N_\ell := N - r_{N,\ell} = q_{N,\ell}(2\ell+1)$ the largest integer less or equal to N that is divided by $(2\ell+1)$ and we first decompose the sum in the exponential as

$$\sum_{x\in\mathbb{T}_N^d}G\Big(\frac{x}{N},\eta^\ell(x)\Big) = \sum_{x\in\mathbb{T}_{N_\ell}^d}G\Big(\frac{x}{N},\eta^\ell(x)\Big) + \sum_{x\in\mathbb{T}_N^d\setminus\mathbb{T}_{N_\ell}^d}G\Big(\frac{x}{N},\eta^\ell(x)\Big).$$

Since $C_1 < \log \frac{\varphi_c}{\Phi(\rho_c-\varepsilon)}$ we can choose $p > 1$ close enough to 1 so that $pC_1 < \log \frac{\varphi_c}{\Phi(\rho_c-\varepsilon)}$. Then by the Cauchy-Schwartz inequality we have that

$$\begin{aligned} J(N,\ell) &:= \frac{1}{N^d} \log \int e^{\sum_{x\in\mathbb{T}_N^d} G(\frac{x}{N},\eta^\ell(x))} d\nu^N_{\rho(\cdot)} \\ &\le \frac{1}{2N^d} \log \int e^{2'\sum_{x\in\mathbb{T}_{N_\ell}^d} G(\frac{x}{N},\eta^\ell(x))} d\nu^N_{\rho(\cdot)} \\ &\quad + \frac{1}{2N^d} \log \int e^{2\sum_{x\in\mathbb{T}_N^d\setminus\mathbb{T}_{N_\ell}^d} G(\frac{x}{N},\eta^\ell(x))} d\nu^N_{\rho(\cdot)}. \end{aligned}$$

We will show first that the second summand in the right hand side is non-positive in the limit as $N \to \infty$. Indeed, we have that

$$\begin{aligned} \sum_{x\in\mathbb{T}_N^d\setminus\mathbb{T}_{N_\ell}^d} G\Big(\frac{x}{N},\eta^\ell(x)\Big) &\le \sum_{x\in\mathbb{T}_N^d\setminus\mathbb{T}_{N_\ell}^d} [C_0 + C_1\eta^\ell(x)] \\ &= (N^d - N_\ell^d)C_0 + C_1 \sum_{x\in\mathbb{T}_N^d} \eta(x) - C_1 \sum_{x\in\mathbb{T}_{N_\ell}^d} \sum_{z\in\Lambda_\ell^d} \eta(x+z) \\ &\le (N^d - N_\ell^d)C_0 + C_1 \sum_{x\in\mathbb{T}_N^d\setminus\mathbb{T}_{N_\ell-2\ell}^d} \eta(x), \end{aligned}$$

where obviously

$$N^d - N_\ell^d = O(N^{d-1}) \quad \text{and} \quad \sharp\mathbb{T}_N^d \setminus \mathbb{T}_{N_\ell-2\ell}^d = N^d - (N_\ell - 2\ell)^d = O(N^{d-1}).$$

Therefore

$$\begin{aligned} J_2(N,\ell) &:= \frac{1}{2N^d} \log \int e^{2\sum_{x\in\mathbb{T}_N^d\setminus\mathbb{T}_{N_\ell}^d} G(\frac{x}{N},\eta^\ell(x))} d\nu^N_{\rho(\cdot)} \\ &\le \frac{N^d - N_\ell^d}{N^d} C_0 + \frac{1}{2N^d} \sum_{x\in\mathbb{T}_N^d\setminus\mathbb{T}_{N_\ell-2\ell}^d} \Lambda_{\rho(x/N)}(2C_1) \\ &\le \frac{N^d - N_\ell^d}{N^d} C_0 + \frac{N^d - (N_\ell - 2\ell)^d}{2N^d} \log Z\big(\Phi(\rho_c-\varepsilon)2C_1\big), \end{aligned}$$

where $\log Z\big(\Phi(\rho_c-\varepsilon)2C_1\big) < +\infty$ since $C_1 < \frac{1}{2}\log\frac{\varphi_c}{\Phi(\rho_c-\varepsilon)}$. It follows that

$$\limsup_{N\to\infty} J_2(N,\ell) \le 0.$$

For the first term

$$J_1(N,\ell) := \frac{1}{2N^d} \log \int e^{2\sum_{x\in\mathbb{T}_{N_\ell}^d} G(\frac{x}{N},\eta^\ell(x))} d\nu^N_{\rho(\cdot)}$$

we decompose the sum in the exponential as

$$\sum_{x\in\mathbb{T}_{N_\ell}^d} G\Big(\frac{x}{N},\eta^\ell(x)\Big) = \sum_{x\in\Lambda_\ell^d} \sum_{y\in E_x^{N,\ell}} G\Big(\frac{x+(2\ell+1)y}{N}, \eta^\ell\big(x+(2\ell+1)y\big)\Big),$$

where we have set $E_x^{N,\ell} := \{y \in \mathbb{T}_N^d : x + (2\ell+1)y \in \mathbb{T}_{N_\ell}^d\}$. Then by the generalized Holder inequality and the independence of the random variables $\{\eta^\ell(x+(2\ell+1)y) | y \in E_x^{N,\ell}\}$, we have that

$$\begin{aligned}
J_1(N,\ell) &\le \frac{1}{2N^d} \log \prod_{x \in \Lambda_\ell^d} \left(\int e^{\sum_{y \in E_x^{N,\ell}} 2(2\ell+1)^d G(\frac{x+(2\ell+1)y}{N}, \eta^\ell(x+(2\ell+1)y))} d\nu_{\rho(\cdot)}^N \right)^{\frac{1}{(2\ell+1)^d}} \\
&= \frac{1}{2(2\ell+1)^d N^d} \sum_{x \in \Lambda_\ell^d} \log \int e^{\sum_{y \in E_x^{N,\ell}} 2(2\ell+1)^d G(\frac{x+(2\ell+1)y}{N}, \eta^\ell(x+(2\ell+1)y))} d\nu_{\rho(\cdot)}^N \\
&= \frac{1}{2(2\ell+1)^d N^d} \sum_{x \in \Lambda_\ell^d} \sum_{y \in E_x^{N,\ell}} \log \int e^{2(2\ell+1)^d G(\frac{x+(2\ell+1)y}{N}, \eta^\ell(x+(2\ell+1)y))} d\nu_{\rho(\cdot)}^N \\
&= \frac{1}{2(2\ell+1)^d N^d} \sum_{x \in \mathbb{T}_N^d} \log \int e^{2(2\ell+1)^d G(\frac{x}{N}, \eta^\ell(x))} d\nu_{\rho(\cdot)}^N \\
&\quad - \frac{1}{2(2\ell+1)^d N^d} \sum_{x \in \mathbb{T}_N^d \setminus \mathbb{T}_{N_\ell}^d} \log \int e^{2(2\ell+1)^d G(\frac{x}{N}, \eta^\ell(x))} d\nu_{\rho(\cdot)}^N
\end{aligned}$$

Now, since the profile ρ is continuous, the sequence $\nu_{\rho(\cdot)}^N$ is a local equilibrium of profile ρ, that is for each macroscopic point $u \in \mathbb{T}^d$ we have that $\tau_{[Nu]} \nu_{\rho(\cdot)}^N \longrightarrow \nu_{\rho_t(u)}^\infty$ weakly in $\mathbb{PM}_\infty^d$, and therefore on one hand the first term in the upper bound for $J_1(N,\ell)$ given above converges as $N \to \infty$ to

$$\frac{1}{2(2\ell+1)^d} \int_{\mathbb{T}^d} \log \int e^{2(2\ell+1)^d G(u, \eta^\ell(0))} d\nu_{\rho(u)}^\infty du.$$

On the other hand, as we will show the lim sup of the second term as $N \to \infty$ is non-positive. Indeed,

$$\begin{aligned}
J_{12}(N,\ell) &:= \frac{1}{2(2\ell+1)^d N^d} \sum_{x \in \mathbb{T}_N^d \setminus \mathbb{T}_{N_\ell}^d} \log \int e^{2(2\ell+1)^d G(\frac{x}{N}, \eta^\ell(x))} d\nu_{\rho(\cdot)}^N \\
&\le \frac{1}{2(2\ell+1)^d N^d} \sum_{x \in \mathbb{T}_N^d \setminus \mathbb{T}_{N_\ell}^d} \log \int e^{2(2\ell+1)^d (C_0 + C_1 \eta^\ell(x))} d\nu_{\rho(\cdot)}^N \\
&= \frac{1}{2(2\ell+1)^d N^d} \sum_{x \in \mathbb{T}_N^d \setminus \mathbb{T}_{N_\ell}^d} \log \int e^{2(2\ell+1)^d C_1 \eta^\ell(x)} d\nu_{\rho(\cdot)}^N + C_0 \frac{N^d - N_\ell^d}{N^d} \\
&= \frac{1}{2(2\ell+1)^d N^d} \sum_{x \in \mathbb{T}_N^d \setminus \mathbb{T}_{N_\ell}^d} \sum_{y : |y-x| \le \ell} \Lambda_{\rho(\frac{y}{N})}(2C_1) + C_0 \frac{N^d - N_\ell^d}{N^d} \\
&\le \left(C_0 + \frac{1}{2} \log Z(e^{2C_1} \Phi(\rho_c - \varepsilon)) \right) \frac{N^d - N_\ell^d}{N^d},
\end{aligned}$$

which obviously tends to zero since $\log Z(e^{2C_1} \Phi(\rho_c - \varepsilon))$ is finite by our assumption on the constant C_1.

It follows by the proof so far that

$$\limsup_{N\to\infty}\frac{1}{N^d}\log\int e^{\sum_{x\in\mathbb{T}_N^d}G(\frac{x}{N},\eta^\ell(x))}d\nu^N_{\rho(\cdot)}$$
$$\leq \frac{1}{2(2\ell+1)^d}\int_{\mathbb{T}^d}\log\int e^{2(2\ell+1)^dG(u,\eta^\ell(0))}d\nu^\infty_{\rho(u)}du. \tag{3.45}$$

We will first calculate the limit as $\ell\to\infty$ of the integrand for each fixed $u\in\mathbb{T}^d$ in this last term and then we will show that we can exchange the limit with integral. According to the Laplace-Varadhan theorem, if we could show that $2G(u,\cdot)$ is exponentially uniformly integrable with respect to the sequence $\{\eta^\ell(0)_*\nu^\infty_{\rho(u)}\}_{\ell\in\mathbb{Z}_+}$, i.e. that

$$\lim_{M\to\infty}\lim_{\ell\to\infty}\frac{1}{(2\ell+1)^d}\log\int e^{(2\ell+1)^d2G(u,\eta^\ell(0))}\mathbb{1}_{\{G(u,\eta^\ell(0))\geq M\}}d\nu^\infty_{\rho(u)}=-\infty,$$

it would follow by the Laplace-Varadhan theorem that for each fixed $u\in\mathbb{T}^d$,

$$\lim_{\ell\to\infty}\frac{1}{(2\ell+1)^d}\log\int e^{2(2\ell+1)^dG(u,\eta^\ell(0))}d\nu^\infty_{\rho(u)}=\sup_{\lambda\geq 0}\big\{2G(u,\lambda)-\Lambda^*_{\rho(u)}(\lambda)\big\}.$$

For all $M>0$ we obviously have that

$$e^{(2\ell+1)^d2G(u,\eta^\ell(0))}\mathbb{1}_{\{G(u,\eta^\ell(0))\geq M\}}\leq e^{2(2\ell+1)^d(C_0+C_1\eta^\ell(0))}\mathbb{1}_{\{C_0+C_1\eta^\ell(0)\geq M\}}$$

and therefore setting $C_M:=\frac{M-C_0}{C_1}$ and $\ell_\star:=2\ell+1$ by our assumption on the function G and Holder's inequality we have that for all $p>1$,

$$\begin{aligned}\int e^{\ell_\star^d2G(u,\eta^\ell(0))}\mathbb{1}_{\{G(u,\eta^\ell(0))\geq M\}}d\nu^\infty_{\rho(u)} &\leq \int e^{2\ell_\star^d(C_0+C_1\eta^\ell(0))}\mathbb{1}_{\{\eta^\ell(0)\geq C_M\}}d\nu^\infty_{\rho(u)}\\ &\leq e^{\ell_\star^d2C_0}\nu^\infty_{\rho(u)}(\eta^\ell(0)\geq C_M)^{\frac{1}{q}}\times\\ &\qquad\times\Big(\int e^{\ell_\star^d2pC_1\eta^\ell(0)}d\nu^\infty_{\rho(u)}\Big)^{\frac{1}{p}},\end{aligned}$$

where q is the conjugate exponent of p. But we obviously have that

$$\log\int e^{\ell_\star^d2pC_1\eta^\ell(0)}d\nu^\infty_{\rho(u)}=\sum_{|y|\leq\ell}\log\int e^{2pC_1\eta(y)}d\nu^\infty_{\rho(u)}=(2\ell+1)^d\Lambda_{\rho(u)}(2pC_1),$$

and by the assumption on the constant C_1 we can choose p close enough to 1 so that $2pC_1<\log\frac{\varphi_c}{\Phi(\rho_c-\varepsilon)}$ in which case $\Lambda_{\rho(u)}(2pC_1)<+\infty$. It follows that

$$\limsup_{\ell\to\infty}\frac{1}{(2\ell+1)^d}\log\int e^{(2\ell+1)^d2G(u,\eta^\ell(0))}\mathbb{1}_{\{G(u,\eta^\ell(0))\geq M\}}d\nu^\infty_{\rho(u)}$$
$$\leq 2C_0+\frac{1}{p}\Lambda_{\rho(u)}(2pC_1)+\frac{1}{q}\limsup_{\ell\to\infty}\frac{1}{(2\ell+1)^d}\log\nu^\infty_{\rho(u)}(\eta^\ell(0)\geq C_M),$$

where the first two terms in the sum in the right hand side are finite, and so in order to prove the required exponential uniform integrability it suffices to show that

$$\lim_{M\to\infty}\limsup_{\ell\to\infty}\frac{1}{(2\ell+1)^d}\log\nu^\infty_{\rho(u)}(\eta^\ell(0)\geq C_M)=-\infty.$$

But this is seen easily since on one hand we have that $C_M \longrightarrow \infty$ as $M \to \infty$ and on the other hand by the large deviations principle for the family $\{\eta(x)\}_{x\in\mathbb{Z}^d}$ we have that

$$\limsup_{\ell\to\infty} \frac{1}{(2\ell+1)^d} \log \nu^\infty_{\rho(u)}(\eta^\ell(0) \geq C_M) \leq -\inf_{\lambda\geq C_M} \Lambda^*_{\rho(u)}(\lambda) \overset{M\to+\infty}{\longrightarrow} -\infty,$$

since the function $\Lambda_{\rho(u)}$ is increasing on $[\rho(u),\infty)$ and tends to ∞ as $\lambda\to\infty$.

It remains to show that we can exchange the limit with the integral. So we denote by $h_\ell : \mathbb{T}^d \longrightarrow \mathbb{R}$ the function given by

$$h_\ell(u) = \frac{1}{(2\ell+1)^d} \log \int e^{2(2\ell+1)^d G(u,\eta^\ell(0))} d\nu^\infty_{\rho(u)},$$

which as we have shown converges pointwise to the function $\sup_{\lambda\geq 0}\{2G(\cdot,\lambda) - \Lambda_{\rho(\cdot)}(\lambda)\}$. By our assumption on the function G we have on one hand that

$$\begin{aligned}
h_\ell(u) &\leq \frac{1}{(2\ell+1)^d} \log \int e^{2(2\ell+1)^d[C_0 + C_1\eta^\ell(0)]} d\nu^\infty_{\rho(u)} \\
&= 2C_0 + \frac{1}{(2\ell+1)^d} \log \int e^{2C_1 \sum_{|y|\leq\ell}\eta(y)} d\nu^\infty_{\rho(u)} \\
&= 2C_0 + \frac{1}{(2\ell+1)^d} \sum_{|y|\leq\ell} \log \int e^{2C_1\eta(y)} d\nu^\infty_{\rho(u)} = 2C_0 + \Lambda_{\rho(u)}(2C_1) \\
&\leq 2C_0 + \log Z\big(\Phi(\rho_c-\varepsilon)e^{2C_1}\big) < +\infty,
\end{aligned}$$

while on the other hand we have by Jensen's inequality that

$$\begin{aligned}
h_\ell(u) &\geq \int 2G(u,\eta^\ell(0)) d\nu^\infty_{\rho(u)} \geq -2C_0 - 2C_1 \int \eta^\ell(0) d\nu^\infty_{\rho(u)} = -2C_0 - 2C_1\rho(u) \\
&\geq -2C_0 - 2C_1\cdot(\rho_c-\varepsilon).
\end{aligned}$$

Consequently we have a bound for the sequence $\{h_\ell\}$ uniform over $\ell\in\mathbb{Z}_+$ and $u\in\mathbb{T}^d$ and an application of the bounded convergence theorem, completes the proof. □

We recall that the function $G : [0,T]\times\mathbb{T}^d\times\mathbb{R}_+ \longrightarrow \mathbb{R}$ defined in (3.40) satisfies the bound (3.42). Therefore if we choose $K > 2$ then the function $\gamma^K_t G_t$ satisfies the assumptions of lemma 3.3.3 for each fixed $t > 0$, and so for each $K > 2$ the term in (3.44) is bounded above by

$$\int_{\mathbb{T}^d} \sup_{\lambda>0} \Big\{\gamma^K_t G_t(u,\lambda) - \frac{1}{2}\Lambda^*_{\rho_t(u)}(\lambda)\Big\} du.$$

To complete the application of the relative entropy method it remains to show that, by enlarging $K > 2$ if necessary, this last term is non-positive for all $t > 0$.

We note that this would follow if we had a bound of the form

$$K(\varepsilon) := \sup_{\substack{\rho\in(0,\rho_c-\varepsilon]\\ \lambda\geq 0}} \frac{|M(\lambda,\rho)|}{\Lambda^*_\rho(\lambda)} < +\infty. \tag{3.46}$$

Indeed, since for $(\lambda,\rho)\in\mathbb{R}+\times(0,\rho_c-\varepsilon]$ we have that $\Lambda^*_\rho(\lambda) = 0$ iff $\lambda = \rho$ iff $M(\lambda,\rho) = 0$, we would then have that

$$|M(\lambda,\rho)| \leq K(\varepsilon)\Lambda^*_\rho(\lambda),$$

for all $(\lambda, \rho) \in \mathbb{R}_+ \times (0, \rho_c - \varepsilon]$, and so we would have that

$$\gamma_t^K G_t(u,\lambda) \le \gamma_t^K C_t |M(\lambda, \rho_t(u))| \le \gamma_t^K C_t K(\varepsilon) \Lambda^*_{\rho_t(u)}(\lambda) = \frac{K(\varepsilon)}{KC} \log \frac{\varphi_c}{\Phi(\rho_c - \varepsilon)} \Lambda^*_{\rho_t(u)}(\lambda)$$

for all $(u, \lambda) \in \mathbb{T}^d \times \mathbb{R}_+$. Then by choosing $K > 2$ large enough so that in addition $\frac{K(\varepsilon)}{KC} \log \frac{\varphi_c}{\Phi(\rho_c - \varepsilon)} < \frac{1}{2}$, it would follow that the right hand side in the inequality of the above corollary is non-positive, as required. The bound (3.46) is proved in the following lemma which is a generalization of lemma 6.1.10 in [25].

Lemma 3.3.5 *For every $\varepsilon > 0$,*

$$\sup_{(\lambda,\rho)\in(0,\infty)\times(0,\rho_c-\varepsilon]} \frac{|M(\lambda, \rho)|}{\Lambda^*_\rho(\lambda)} < +\infty.$$

Proof We first choose $\delta \in (0, \frac{\varepsilon}{2})$. We decompose the set $(0, \infty) \times (0, \rho_c - \varepsilon]$ in two disjoint subsets ($\lambda \backsim \rho$ and $\lambda \gg \rho$) and prove the claim on each. We start with the region $\lambda \backsim \rho$:

$$\mathcal{E}_1 := \big\{(\lambda, \rho) \in \mathbb{R}_+ \times (0, \rho_c - \varepsilon] \,\big|\, 0 < \lambda \le \rho_c - \varepsilon + \delta\big\},$$

where we recall that if $\rho_c = +\infty$, $\rho_c - \varepsilon$ is to be interpreted as $1/\varepsilon$. By the Taylor expansion of Φ around the point $\rho \in (0, \rho_c)$, we have $M(\lambda, \rho) = \int_\rho^\lambda \Phi''(r)(\lambda - r)dr$ for all $\lambda, \rho \in (0, \rho_c)$. So since $\{\lambda | (\lambda, \rho) \in \mathcal{E}_1 \text{ for some } \rho \in (0, \rho_c - \varepsilon]\} \subseteq (0, \rho_c)$,

$$|M(\lambda, \rho)| \le \frac{A_1}{2}(\lambda - \rho)^2 \quad \text{for all } (\lambda, \rho) \in \mathcal{E}_1$$

where $A_1 := \sup_{0 \le r \le \rho_c - \varepsilon + \delta} |\Phi''(r)| < +\infty$. For the denominator we note that the rate functional Λ^*_ρ is C^1 on $(0, \infty)$ and C^2 on $(0, \rho_c)$ with

$$\frac{d}{d\lambda}\Lambda^*_\rho(\lambda) = \log \frac{\Phi(\lambda \wedge \rho_c)}{\Phi(\rho)}, \quad \lambda > 0, \qquad \frac{d^2}{d\lambda^2}\Lambda^*_\rho(\lambda) = \frac{\Phi'(\lambda)}{\Phi(\lambda)}, \quad \lambda \in (0, \rho_c).$$

Since Λ^*_ρ and its derivative vanish at ρ, by the Taylor expansion of Λ^*_ρ around $\rho \in (0, \rho_c)$ we have that $\Lambda^*_\rho(\lambda) = \int_\rho^\lambda (\Lambda^*_\rho)''(r)(\lambda - r)dr$ for all $\lambda \in (0, \rho_c)$ and therefore

$$\Lambda^*_\rho(\lambda) \ge \frac{B_1}{2}(\lambda - \rho)^2 \quad \text{for all } (\lambda, \rho) \in \mathcal{E}_1,$$

where $B_1 := \inf_{0 < r \le \rho_c - \varepsilon + \delta} (\Lambda^*_\rho)''(r) > 0$. Combining these estimates, we get the required bound on the region $\mathcal{E}_1$.

We turn now to the set

$$\mathcal{E}_2 = \big\{(\lambda, \rho) \in (0, \infty) \times (0, \rho_c - \varepsilon] \,\big|\, \lambda > \rho_c - \varepsilon + \delta\big\}.$$

Note that for all $(\lambda, \rho) \in \mathcal{E}_2$ we have that $\lambda > \rho + \delta$. Recalling that Φ is Lipschitz with Lipschitz constant $\le \|g'\|_\infty$, we get an upper bound for the numerator

$$|M(\lambda, \rho)| \le 2\|g'\|_\infty \lambda \quad \text{for all } (\lambda, \rho) \in \mathcal{E}_2.$$

Since Λ_ρ^* is convex as the supremum of linear functions we also have

$$\Lambda_\rho^*(\lambda) \geq A_2 + B_2 \cdot (\lambda - \rho_c - \varepsilon + \delta) \quad \text{for all } (\lambda, \rho) \in \mathcal{E}_2,$$

where $A_2 = \inf_{\rho \in (0, \rho_c - \varepsilon]} \Lambda_\rho^*(\rho_c - \varepsilon + \delta) > 0$ and $B_2 = \inf_{\rho \in (0, \rho_c - \varepsilon]} (\Lambda_\rho^*)'(\rho_c - \varepsilon + \delta) > 0$. The last two displays together imply the required bound on the region $\mathcal{E}_2$. This completes the proof of the lemma and the application of the relative entropy method. □

Chapter 4

A Macroscopic Continuity Equation for the Symmetric Zero Range Process

In this section we mainly concentrate in the case of the symmetric nearest neighbor ZRPs with bounded jump rates and we will prove that the laws of the triples (π^N, W^N, σ^N) of the empirical density, the empirical current and the empirical jump rate respectively are concentrated on paths (π, W, σ) satisfying the continuity equation

$$\partial_t \pi = -\mathrm{div} W = \Delta \sigma$$

in the sense of distributions where for (almost) all $t \geq 0$ π_t is finite non-negative measure, W_t is a vector-valued measure absolutely continuous with respect to Lebesgue measure and $\sigma_t \in H^1(\mathbb{T}^d)$. More precisely we prove the following

Proposition 4.0.2 *Suppose the local jump rate g is bounded and let $\mu_0^N \in \mathbb{P}_3\mathbb{M}_N^d$ be a sequence of initial distributions associated to the macroscopic profile $\mu_0 \in \mathcal{M}_+$. Set*

$$\Omega := D(0,T;\mathcal{M}_+(\mathbb{T}^d)) \times L^\infty_{w^*}(0,T;C^1(\mathbb{T}^d;\mathbb{R}^d)^*) \times L^\infty_{w^*}(0,T;\mathcal{M}(\mathbb{T}^d))$$

and consider the image $R^{\mu_0^N} \in \mathbb{P}\Omega$ of the law of the diffusively rescaled ZRP starting from μ_0^N via the triple (π^N, W^N, σ^N). Then the sequence $\{R^{\mu_0^N}\}_{N\in\mathbb{N}} \subseteq \mathbb{P}\Omega$ is sequentially relatively compact in the weak topology of $\mathbb{P}\Omega$. Furthermore, any limit point R^∞ of the sequence $\{R^{\mu_0^N}\}$ is concentrated on trajectories (π, W, σ) such that:
(a) $\pi \in C(\mathbb{R}_+;\mathcal{M}_+)$ *and* $\pi_0 = \mu_0$.
(b) $\sigma_t \ll m_{\mathbb{T}^d}$, $\|\sigma_t\|_{L^\infty(\mathbb{T}^d)} \leq \varphi_c$ *a.s. for all* $0 \leq t \leq T$.
(c) $W_t \in \mathcal{M}(\mathbb{T}^d;\mathbb{R}^d)$ *and* $W_t \ll m_{\mathbb{T}^d}$ *for a.s. all* $t \geq 0$, *and*
(d) *The continuity equation*

$$\partial_t \pi = -\mathrm{div} W_t = \Delta_x \sigma \tag{4.1}$$

holds in the sense of distributions.

The fact that $W_t \in \mathcal{M}(\mathbb{T}^d;\mathbb{R}^d)$ and $W_t \ll m_{\mathbb{T}^d}$ for a.s. all $t \geq 0$ follows from the following regularity result which is worth stating in its own right.

Proposition 4.0.3 *Let $\{\mu_0^N \in \mathbb{P}_1\mathbb{M}_N^d\}$ be a sequence of initial distributions satisfying the $O(N^d)$-entropy assumption. The third marginal Q^g of $R^{\mu_0^N}$ on $L^\infty_{w^*}(0,T;\mathcal{M}_+)$ is concentrated on paths σ such that there exist $L^2((0,T)\times\mathbb{T}^d)$ functions denoted by $\partial_j\sigma$, $j=1,\dots,d$, satisfying*

$$\int_0^T\int_{\mathbb{T}^d}\partial_j H_t(x)\sigma(t,x)dxdt = -\int_0^T\int_{\mathbb{T}^d}H_t(x)\partial_j\sigma(t,x)dxdt$$

and

$$\int_0^T\int_{\mathbb{T}^d}\frac{\|\nabla\sigma(t,x)\|^2}{\sigma(t,x)}dxdt < +\infty. \tag{4.2}$$

In particular $Q^g\big\{\sigma\big|\sigma_t \in H^1(\mathbb{T}^d)$ a.s. $\forall\, t\in[0,T]\big\} = 1$.

This regularity result is an adaptation of the results in section 5.7 of [25].

4.1 The Law of Large Numbers

This section contains the proof of the relative compactness of the distribution of the empirical density. The arguments rely on the analysis of the martingales associated to the speeded up ZR process $(\eta^N_{tN^2})$ through the martingale problem and appropriately chosen functions on $\mathbb{R}_+\times\mathbb{M}_N^d$. The description of these martingales is contained in chapter A.2 in the appendix for Markov jump processes. We recall here that given any polish space M the space of all cadlag (i.e. right continuous and with left hand limits) paths is denoted by $D(\mathbb{R}_+;M)$. The set $D(\mathbb{R}_+;M)$ when equipped with the Skorohod metric becomes a polish space. Loosely speaking the Skorohod metric metrizes a kind of uniform convergence, but with the difference that it allows for time reparametrizations that converge to the identity to account of the possible jump discontinuities of the paths. We review the basic facts on the Skorohod space in section A.1 in the appendix, mostly without proofs. A detailed exposition of the Skorohod topology as well of criteria for relative compactness of sets of laws on Skorohod spaces is contained in sections 3.5 to 3.10 of [14]. A more concise treatment of this subject, which contains almost all the results on the Skorohod topology than we will use is contained in section 4.1 of [25]. Let $\eta^N := id_{D(\mathbb{R}_+,\mathbb{M}_N^d)} : \Omega^N \longrightarrow D(\mathbb{R}_+,\mathbb{M}_N^d)$, $N\in\mathbb{N}$, be the sequence of the natural cadlag symmetric ZR processes on the discrete toruses $\mathbb{T}_N^d$, with common local rate function $g:\mathbb{Z}_+\longrightarrow\mathbb{R}_+$ and symmetric elementary transition probability $p\in\mathbb{P}\mathbb{Z}^d$, defined on the probability kernel $\Omega^N = \big(D(\mathbb{R}_+,\mathbb{M}_N^d),\mathcal{B}_{D(\mathbb{R}_+,\mathbb{M}_N^d)},(\mathbb{P}_N^{\eta_0})_{\eta_0\in\mathbb{M}_N^d}\big)$. In this way, for fixed $N\in\mathbb{N}$ the ZR process $\eta^N = \{\eta_t^N\}_{t\geq 0}$ consists of the natural evaluation maps $\eta_t^N : D(\mathbb{R}_+,\mathbb{M}_N^d)\longrightarrow\mathbb{M}_N^d$,

$$\eta_t^N(\eta) = \eta_t, \quad \eta = (\eta_t)_{t\geq 0}\in D(\mathbb{R}_+,\mathbb{M}_N^d),$$

and given any initial distribution $\mu\in\mathbb{P}\mathbb{M}_N^d$ the distribution of the process η^N starting from μ is

$$\mathbb{P}_N^\mu := \int\mathbb{P}_N^{\eta_0}d\mu(\eta_0)\in\mathbb{P}D(\mathbb{R}_+,\mathbb{M}_N^d).$$

Recalling that $\eta(x) : \mathbb{M}_N^d \longrightarrow \mathbb{Z}_+$ denotes the natural projection at the site $x \in \mathbb{T}_N^d$, given by

$$\eta(x)(\eta) = \eta_x, \quad \eta = (\eta_x)_{x\in\mathbb{T}_N^d} \in \mathbb{M}_N^d$$

we set $\eta_t^N(x) = \eta(x) \circ \eta_t^N$ for all $x \in \mathbb{T}_N^d$, $N \in \mathbb{N}$.

In the hydrodynamic description of the ZR process we are interested in the empirical distribution of the ZR process induced by the empirical embeddings $\pi^N : \mathbb{M}_N^d \longrightarrow \mathcal{M}^+(\mathbb{T}^d)$ given by

$$\pi_\eta^N = \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \eta_x^N \delta_{\frac{x}{N}}.$$

To be more precise, since the empirical embeddings π^N are continuous and injective they induce a continuous injective map

$$\bar{\pi}^N : D(\mathbb{R}_+, \mathbb{M}_N^d) \longrightarrow D(\mathbb{R}_+, \mathcal{M}_+^d)$$

between the respective Skorohod spaces. In its turn, $\bar{\pi}^N$ induces a map

$$\bar{\pi}_*^N : \mathbb{P}D(\mathbb{R}_+, \mathbb{M}_N^d) \longrightarrow \mathbb{P}D(\mathbb{R}_+, \mathcal{M}_+^d)$$

between the respective spaces of distributions, and using this map we can consider the distribution $\mathbb{P}_N^\mu := \int \mathbb{P}_N^{\eta_0} d\mu(\eta_0) \in \mathbb{P}D(\mathbb{R}_+, \mathbb{M}_N^d)$ of η^N starting from any initial distribution $\mu^N \in \mathbb{P}\mathbb{M}_N^d$ embedded in $D(\mathbb{R}_+, \mathcal{M}_+^d)$ as the distribution

$$\bar{\mathbb{P}}_N^{\mu^N} := \bar{\pi}_*^N \mathbb{P}_N^{\mu^N} = \bar{\pi}_*^N \int \mathbb{P}_N^{\eta_0} d\mu(\eta_0) = \int \bar{\pi}_*^N \mathbb{P}_N^{\eta_0} d\mu(\eta_0) \in \mathbb{P}D(\mathbb{R}_+, \mathcal{M}_+^d).$$

The distribution $\bar{\mathbb{P}}_N^{\mu^N}$ is the empirical distribution of the ZR process η^N starting from $\mu^N \in \mathbb{P}\mathbb{M}_N^d$. In this way, given any sequence $\{\mu^N \in \mathbb{P}\mathbb{M}_N^d\}_{N\in\mathbb{N}}$ we can regard the sequence

$$\mathbb{P}_N^{\mu^N} \in \mathbb{P}D(\mathbb{R}_+, \mathbb{M}_N^d), \quad N \in \mathbb{N},$$

as living in a single space, namely in $\mathbb{P}D(\mathbb{R}_+, \mathcal{M}_+^d)$ by working with the sequence

$$\bar{\mathbb{P}}_N^{\mu^N} := \bar{\pi}_*^N \mathbb{P}_N^{\mu^N} \in \mathbb{P}D(\mathbb{R}_+, \mathcal{M}_+^d), \quad N \in \mathbb{N}.$$

We notice next that the empirical distribution $\bar{\mathbb{P}}_N^{\mu^N} \in \mathbb{P}D(\mathbb{R}_+, \mathcal{M}_+^d)$ of η^N starting from μ^N is in fact the distribution of the *empirical process*

$$\bar{\pi}^N : (D(\mathbb{R}_+, \mathbb{M}_N^d), \mathbb{P}_N^{\mu^N}) \longrightarrow D(\mathbb{R}_+, \mathcal{M}^+(\mathbb{T}^d)).$$

It is preferable to consider the empirical process as a Markov family, namely as

$$\bar{\pi}^N : (D(\mathbb{R}_+, \mathbb{M}_N^d), \{\mathbb{P}_N^{\eta_0}\}_{\eta_0\in\mathbb{M}_N^d}) \longrightarrow D(\mathbb{R}_+, \mathcal{M}^+(\mathbb{T}^d)).$$

Of course, the empirical process as defined above is in fact a Markov family on the subspace $\pi^N(\mathbb{M}_N^d)$ and not on the full state space $\mathcal{M}_+^d$ since any path of the empirical process must obviously start from some measure in $\pi^N(\mathbb{M}_N^d)$ and remain in $\pi^N(\mathbb{M}_N^d)$ for all times $t > 0$. Nevertheless it obviously has cadlag paths, it satisfies

$$\mathbb{P}_N^{\eta_0}\{\bar{\pi}_0^N = \pi_{\eta_0}^N\} = \mathbb{P}_N^{\eta_0}\{\eta_0^N = \eta_0\} = 1, \quad \forall \eta_0 \in \mathbb{M}_N^d,\ N \in \mathbb{N}$$

and inherits the Markov property from η^N for all $N \in \mathbb{N}$, that is

$$\mathbb{P}^{\eta_0}\{\bar{\pi}^N_t = \pi^N_\eta | \bar{\mathcal{F}}^N_s\} = \mathbb{P}_N^{\eta^N_s}\{\bar{\pi}^N_{t-s} = \pi^N_\eta\}, \quad \mathbb{P}^{\eta_0}_N \ \ a.s.$$

for all $t \geq s \geq 0$ and all $\eta, \eta_0 \in \mathbb{M}^d_N$, where $(\bar{\mathcal{F}}^N_s)_{s\geq 0}$ is the natural filtration of $\bar{\pi}^N$. Indeed, if $(\mathcal{F}^N_s)_{s\geq 0}$ is the natural filtration then by the Markov property of η^N we have that

$$\mathbb{P}^{\eta_0}_N\{\eta^N_t = \eta | \mathcal{F}^N_s\} = \mathbb{P}_N^{\eta^N_s}\{\eta^N_{t-s} = \eta\}, \quad \mathbb{P}^{\eta_0}_N \ \ a.s..$$

But the empirical embedding π^N is injective and so we have that $(\bar{\mathcal{F}}^N_s)_{s\geq 0} = (\mathcal{F}_s)_{s\geq 0}$ and $\{\eta^N_t = \eta\} = \{\bar{\pi}^N_t = \pi^N_\eta\}$ for all $\eta \in \mathbb{M}^d_N$ and all $t \geq 0$, which gives the Markov property of the empirical process $\bar{\pi}^N$.

Let now $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$ be an initial profile and fix $\{\mu^N \in \mathbb{P}\mathbb{M}^d_N\}_{N\in\mathbb{N}}$ be a sequence associated to the profile μ_0, i.e. such that

$$\mu^N\Big\{\big|\langle G, \pi^N\rangle - \langle G, \rho_0 dm_{\mathbb{T}^d}\rangle\big| > \varepsilon\Big\} = 0$$

for all $G \in C(\mathbb{T}^d)$ and all $\varepsilon > 0$. For the rest of this section, for each $N \in \mathbb{N}$ we set $P^N \in \mathbb{P}D(\mathbb{R}_+, \mathbb{M}^d_N)$ to be the distribution of η^N speeded up by N^2 and starting from μ^N, that is

$$P^N := \big[(\eta^N_{tN^2})_{t\geq 0}\big]_* \mathbb{P}_N^{\mu^N}, \quad N \in \mathbb{N}, \tag{4.3}$$

where of course $\eta^N = id_{D(\mathbb{R}_+, \mathbb{M}^d_N)} = (\eta^N_t)_{t\geq 0}$ is the original ZR process, and we set

$$Q^N := \bar{\pi}^N_* P^N = \big[(\pi^N_{tN^2})_{t\geq 0}\big]_* \mathbb{P}_N^{\mu^N} \in \mathbb{P}D(\mathbb{R}_+, \mathcal{M}^d_+), \quad N \in \mathbb{N}. \tag{4.4}$$

Our main goal in this section is to find conditions on the local rate function g and the initial sequence $\{\mu^N \in \mathbb{P}\mathbb{M}^d_N\}_{N\in\mathbb{N}}$ which ensure the relative compactness of the sequence $\{Q^N\}_{N\in\mathbb{N}} \subseteq \mathbb{P}D(\mathbb{R}_+, \mathcal{M}^d_+)$. The arguments rely on the analysis of the martingales associated to the ZRP via the results of section A.2.

Proposition 4.1.1 *Let $(\eta^N, \mathbb{P}_N)$ be a ZR process with generator L^N. Then for any initial distribution $\mu^N \in \mathbb{P}_2\mathbb{M}^d_N$ ($\mu^N \in \mathbb{P}_1\mathbb{M}^d_N$ if the jump rate function of η^N is bounded) and any $G \in BC^1_{\mathbb{R}_+}(\mathbb{T}^d)$, the real process*

$$A^{N,G}_t := \langle G_t, \pi^N_t\rangle - \langle G_0, \pi^N_0\rangle - \int_0^t (\partial_s + N^2 L^N)\langle G_s, \pi^N\rangle(\eta^N_s)ds, \quad t \geq 0,$$

defined on the filtered probability space $\big(D(\mathbb{R}_+, \mathbb{M}^d_N), (\mathcal{F}^N_t)_{t\geq 0}, P^N\big)$ is a martingale, where $(\mathcal{F}^N_t)$ is the minimal right continuous filtration to which η^N is adapted and P^N is the distribution of η^N starting from μ^N speeded up by N^2.

Proof By the previous proposition we know that for any function $F \in B_1C^1_{\mathbb{R}_+}(\mathbb{M}^d_N)$ the real process

$$M^{\eta^N, F}_t := F_t(\eta^N_t) - F_0(\eta^N_0) - \int_0^t (\partial_s + N^2 L^N)F_s(\eta^N_s)ds, \quad t \geq 0,$$

is a $(\mathcal{F}_t^N, P^N)$- martingale, where of course in the above formula the generator L^N of η^N is multiplied by N^2 since we are working with speeded up process by a factor N^2. Now, for any function $G \in B(\mathbb{R}_+ \times \mathbb{T}^d)$ we define the function $F_G \equiv \langle G, \pi^N \rangle : \mathbb{R}_+ \times \mathbb{M}_N^d \longrightarrow \mathbb{R}$ by the formula

$$F_G(t, \cdot) = (F_G)_t(\cdot) = \langle G_t, \pi^N \rangle = \int_{\mathbb{T}^d} G_t d\pi^N, \tag{4.5}$$

and since $\pi_t^N = \pi^N \circ \eta_t^N : D(\mathbb{R}_+, \mathbb{M}_N^d) \longrightarrow \mathcal{M}_+^d$ we obviously have that

$$F_G(t, \eta_t^N) = \langle G_t, \pi^N \rangle \circ \eta_t^N = \langle G_t, \pi_t^N \rangle.$$

Therefore we have that $M^{\eta^N, F_G} = A^{N,G}$ for all $G \in B(\mathbb{R}_+ \times \mathbb{T}^d)$ and thus the claim follows if we show that

$$G \in BC^1_{\mathbb{R}_+}(\mathbb{T}^d) \quad \Longrightarrow \quad F_G \in B_1 C^1_{\mathbb{R}_+}(\mathbb{M}_N^d). \tag{4.6}$$

So let $G \in BC^1_{\mathbb{R}_+}(\mathbb{T}^d)$. Then for all $(t, \eta) \in \mathbb{R}_+ \times \mathbb{M}_N^d$ we have that

$$|F_G(t, \eta)| \leq \int |G_t| d\pi_\eta^N \leq \frac{\|G\|_{B(\mathbb{R}_+ \times \mathbb{T}^d)}}{N^d} |\eta|$$

and therefore $F_G \in B_1 C_{\mathbb{R}_+}(\mathbb{M}_N^d)$. Hence it remains to prove that

$$\lim_{h \downarrow 0} \frac{(F_G)_{t+h} - (F_G)_t}{h} = \partial_t (F_G)_t \quad \text{in} \;\; B_1(\mathbb{M}_N^d)$$

for all $t \geq 0$. Since $G \in BC^1_{\mathbb{R}_+}(\mathbb{T}^d)$ we have that $\frac{G_{t+h} - G_t}{h} \longrightarrow \partial_t G_t$ uniformly in $\mathbb{T}^d$ and therefore since π^N is a (random) finite measure we have that

$$\partial_t (F_G)_t = \lim_{h \downarrow 0} \frac{(F_G)_{t+h} - (F_G)_t}{h} = \lim_{h \downarrow 0} \left\langle \frac{G_{t+h} - G_t}{h}, \pi^N \right\rangle = \langle \partial_t G_t, \pi^N \rangle = (F_{\partial G})_t$$

pointwise in $\mathbb{M}_N^d$ for all $t \geq 0$, i.e. $\partial F_G = F_{\partial G}$. Therefore, for all $\eta \in \mathbb{M}_N^d$, $t \geq 0$, we have that

$$\begin{aligned} \left| \frac{(F_G)_{t+h} - (F_G)_t}{h} - \partial_t (F_G)_t \right| &= \left| \frac{(F_G)_{t+h} - (F_G)_t}{h} - (F_{\partial G})_t \right| \\ &= \left| \left\langle \frac{G_{t+h} - G_t - h \partial_t G_t}{h}, \pi^N \right\rangle \right| \\ &\leq \frac{1}{N^d} \left\| \frac{G_{t+h} - G_t}{h} - \partial_t G_t \right\|_u |\cdot| \end{aligned}$$

and therefore

$$\begin{aligned} \left\| \frac{(F_G)_{t+h} - (F_G)_t}{h} - \partial_t (F_G)_t \right\|_{u,1} &= \sup_{\eta \in \mathbb{M}_N^d} \left| \frac{(F_G)_{t+h}(\eta) - (F_G)_t(\eta) - h \partial_t (F_G)_t(\eta)}{h \cdot (1 + |\eta|)} \right| \\ &\leq \frac{1}{N^d} \left\| \frac{G_{t+h} - G_t}{h} - \partial_t G_t \right\|_u \stackrel{h \downarrow 0}{\longrightarrow} 0, \end{aligned}$$

which proves that $F \in B_1 C^1_{\mathbb{R}_+}(\mathbb{M}_N^d)$. Therefore (4.6) holds and the proof is complete.□

Remark: Note that by proposition A.2.1 and the remark following it, we have that when $M = \mathbb{T}^d$ then

$$C_c^{2,0}(\mathbb{R}_+ \times \mathbb{T}^d) \subseteq BC^1_{\mathbb{R}_+}(\mathbb{T}^d),$$

where $C_c^{2,0}(\mathbb{R}_+ \times \mathbb{T}^d)$ denotes the space of all continuous functions $G : \mathbb{R}_+ \times \mathbb{T}^d \longrightarrow \mathbb{R}$ with compact support that are C^2 with respect to time pointwise in $\mathbb{T}^d$. Thus the real process $A^{G,N}$ of proposition 4.1.1 is a $(\mathcal{F}_t^N, \mathbb{P}_N^{\mu^N})$-martingale for all $G \in C^{2,0}(\mathbb{R}_+ \times M)$ and all $\mu^N \in \mathbb{P}_2\mathbb{M}_N^d$ ($\mu^N \in \mathbb{P}_1\mathbb{M}_N^d$ if the local rate function of the ZRP is bounded).

The quadratic variation of the martingale $A^{N,G}$ is given in the following

Proposition 4.1.2 *Let $\eta^N = (\eta, \mathbb{P}_N)$ be a ZR process with parameters (p, g) and L^N denote the generator of η^N. Then for any initial distribution $\mu^N \in \mathbb{P}_4\mathbb{M}_N^d$ ($\mu^N \in \mathbb{P}_3\mathbb{M}_N^d$ if the jump rate function of η^N is bounded) and any $G \in C_c^{2,0}(\mathbb{R}_+ \times M)$ the $\mathbb{P}_N^\mu$-martingale $A^{N,G}$ associated to $(\eta, \mathbb{P}_N)$ by proposition 4.1.1 is square integrable and its quadratic variation is given by*

$$\begin{aligned}
\langle A^{N,G}\rangle_t &= N^2 \int_0^t \Big\{ L(\langle G_s, \pi^N\rangle^2)(\eta_s) - 2\langle G_s, \pi_s^N\rangle L\langle G_s, \pi^N\rangle(\eta_s)\Big\} ds \\
&= \frac{1}{N^{2d-2}} \int_0^t \sum_{x,y\in\mathbb{T}_N^d} \Big[G_s\Big(\frac{y}{N}\Big) - G_s\Big(\frac{x}{N}\Big)\Big]^2 g(\eta_s(x)) p_N(x,y) ds.
\end{aligned}$$

Proof Indeed, as we have seen for any $G \in C_c^{2,0}(\mathbb{R}_+ \times \mathbb{T}^d)$ the function $F_G = \langle G, \pi^N\rangle$: $\mathbb{R}_+ \times \mathbb{M}_N^d \longrightarrow \mathbb{R}$ defined in (4.5) is in $BC^1_{\mathbb{R}_+}(\mathbb{M}_N^d)$ and since $A^{N,G} = M^{\eta^N, F_G}$ we have by proposition A.2.3 that $A^{N,G}$ is a square integrable $\mathbb{P}_N^\mu$-martingale with quadratic variation

$$\langle A^{N,G}\rangle_t = N^2 \int_0^t \Big\{ L(\langle G_s, \pi^N\rangle^2)(\eta_s) - 2\langle G_s, \pi_s^N\rangle L\langle G_s, \pi^N\rangle(\eta_s)\Big\} ds$$

where the term N^2 appears of course due to the fact that for a scaled up Markov jump process the generator and the jump rate function of the scaled process are multiplied by the scale parameter. But then since for all $\eta \in \mathbb{M}_N^d$ and all $x \in \{\eta \neq 0\}$ we have that

$$\pi^N_{\eta^{x,y}} - \pi^N_\eta = \frac{1}{N^d}(\delta_{\frac{y}{N}} - \delta_{\frac{x}{N}})$$

it follows by proposition A.2.4 that

$$\begin{aligned}
\langle A^{N,G}\rangle_t &= N^2 \int_0^t \sum_{\zeta\in\mathbb{M}_N^d} \big[\langle G_s, \pi_\zeta^N\rangle - \langle G_s, \pi_{\eta_s}^N\rangle\big]^2 \lambda_N(\eta_s) \mathrm{p}_N(\eta_s, \zeta) ds \\
&= N^2 \int_0^t \sum_{x,y\in\mathbb{T}_N^d} \langle G_s, \pi^N_{\eta_s^{x,y}} - \pi^N_{\eta_s}\rangle^2 g(\eta_s(x)) p_N(y-x) ds \\
&= \frac{1}{N^{2d-2}} \int_0^t \sum_{x,y\in\mathbb{T}_N^d} \Big[G_s\Big(\frac{y}{N}\Big) - G_s\Big(\frac{x}{N}\Big)\Big]^2 g(\eta_s(x)) p_N(y-x) ds. \qquad \Box
\end{aligned}$$

Furthermore, for any function $G \in C(\mathbb{T}^d)$ we have that

$$\begin{aligned} L^N\langle G, \pi^N\rangle(\eta) &= \sum_{x,z\in\mathbb{T}_N^d} \langle G, \pi^N_{\eta^{x,x+z}} - \pi^N_\eta\rangle g\big(\eta(x)\big)p_N(z) \\ &= \frac{1}{N^d}\sum_{x,z\in\mathbb{T}_N^d}\Big[G\Big(\frac{x+z}{N}\Big) - G\Big(\frac{x}{N}\Big)\Big]g\big(\eta(x)\big)p_N(z), \end{aligned}$$

and using the fact that p is symmetric we can write the action of the generator on the function $\langle G, \pi^N\rangle$ as

$$\begin{aligned} L^N\langle G, \pi^N\rangle(\eta) &= \frac{1}{2N^d}\sum_{x,z\in\mathbb{T}_N^d}\Big[G\Big(\frac{x+z}{N}\Big) - G\Big(\frac{x}{N}\Big)\Big]g\big(\eta(x)\big)p_N(z) \\ &\quad + \frac{1}{2N^d}\sum_{x,z\in\mathbb{T}_N^d}\Big[G\Big(\frac{x+z}{N}\Big) - G\Big(\frac{x}{N}\Big)\Big]g\big(\eta(x)\big)p_N(-z) \\ &= \frac{1}{2N^d}\sum_{x,z\in\mathbb{T}_N^d}\Big[G\Big(\frac{x+z}{N}\Big) + G\Big(\frac{x-z}{N}\Big) - 2G\Big(\frac{x}{N}\Big)\Big]g\big(\eta(x)\big)p_N(z). \end{aligned}$$

In particular, in the case that p is the elementary step distribution we have that

$$L^N\langle G, \pi^N\rangle(\eta) = \frac{1}{N^d}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\Big[G\Big(\frac{x+e_j}{N}\Big) + G\Big(\frac{x-e_j}{N}\Big) - 2G\Big(\frac{x}{N}\Big)\Big]g\big(\eta(x)\big)p_N(e_j)$$

So, since as we have seen $\partial F_G = F_{\partial G}$, the martingale $A^{N,G}$ can be written in more detail as

$$\begin{aligned} A^{N,G}_t &= \langle G_t, \pi^N_t\rangle - \langle G_0, \pi^N_0\rangle - \int_0^t \langle \partial_s G_s, \pi^N_s\rangle ds \\ &\quad - \frac{1}{2N^{d-2}}\int_0^t \sum_{x,z\in\mathbb{T}_N^d} \Delta_p^N G_s G_s\Big(\frac{x}{N}\Big)g\big(\eta_s(x)\big)p_N(z)ds, \end{aligned}$$

where for any function $G : \mathbb{T}^d \longrightarrow \mathbb{R}$ we denote by $\Delta_p^N G : \mathbb{T}^d \longrightarrow \mathbb{R}$ the discrete Laplacian associated to the elementary step distribution $p \in \mathbb{P}\mathbb{Z}^d$ defined by

$$\Delta_p^N G(u) := N^2\sum_{z\in\mathbb{T}_N^d}\Big[G\Big(u+\frac{z}{N}\Big) + G\Big(u-\frac{z}{N}\Big) - 2G(u)\Big]p(z), \quad u \in \mathbb{T}^d.$$

Therefore if we denote by

$$\sigma^N := \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} g\big(\eta(x)\big)\delta_{\frac{x}{N}}$$

the empirical jump rate and by $\sigma^N_s := \sigma^N_{\eta_s}$, $s \geq 0$, the associated empirical process we can write the martingale $A^{N,G}$ as

$$A^{N,G}_t = \langle G_t, \pi^N_t\rangle - \langle G_0, \pi^N_0\rangle - \int_0^t \Big[\langle \partial_s G_s, \pi^N_s\rangle + \frac{1}{2}\langle \Delta_p^N G_s, \sigma^N_s\rangle\Big]ds. \tag{4.7}$$

We will see next that at least for C^3 functions we can replace, modulo some terms that will be shown to converge to zero, the discrete Laplacian Δ_p^N by its continuous analogue Δ_Σ defined by

$$\Delta_\Sigma = \sum_{i,j=1}^{d} \sigma_{ij}\partial_{ij}^2,$$

where

$$\Sigma := (\sigma_{ij})_{1\le i,j\le d}, \quad \sigma_{ij} := \sum_{z\in\mathbb{Z}^d} z_i z_j p(z),$$

is the covariance matrix of the elementary distribution p. In the case that $p \in \mathbb{P}\mathbb{Z}^d$ is the nearest neighbor step distribution then

$$\sigma_{ij} = \sum_{k=1}^{d} \langle e_k, e_i\rangle\langle e_k, e_j\rangle[p(e_k) + p(-e_k)] = \frac{1}{d}\delta_{ij},$$

and so in this case $\Delta_\Sigma = \frac{1}{d}\Delta$ where $\Delta = \sum_{j=1}^d \partial_{jj}^2$ is the Laplacian. In order to make the dimension disappear from the coefficient of the Laplacian sometimes the nearest neighbor distribution is renormalized to have a total probability of d, or $2d$. Of course this amounts to rescaling the generator of the process by the same factor.

Proposition 4.1.3 *Let $G \in C_c^3(\mathbb{R}_+ \times \mathbb{T}^d)$ and let $A^{N,G}$ be the martingale associated by proposition 4.1.1 to the ZRP of parameters (p, g) on the discrete torus $\mathbb{T}_N^d$. Then there exists a constant $C = C(G, p, g) \ge 0$ such that*

$$\left|\langle G_t, \pi_t^N\rangle - \langle G_0, \pi_0^N\rangle - \int_0^t \langle \partial_s G_s, \pi_s^N\rangle + \frac{1}{2}\langle \Delta_\Sigma G_s, \sigma_s^N\rangle ds - A_t^{N,G}\right| \le \frac{C}{N^{d+1}}\int_0^t \langle 1, \pi_s^N\rangle ds,$$

for all $t \ge 0$, where $(\pi_t^N)_{t\ge0}$ is the empirical ZRP and Σ is the covariance matrix of p.

Proof Denoting by $B^{N,G} = (B_t^{N,G})_{t\ge0}$ the process defined by

$$B_t^{N,G} := \langle G_t, \pi_t^N\rangle - \langle G_0, \pi_0^N\rangle - \int_0^t \Big(\langle \partial_s G_s, \pi_s^N\rangle + \frac{1}{2}\langle \Delta_\Sigma G_s, \sigma_s^N\rangle\Big) ds, \tag{4.8}$$

we have to prove that

$$|B_t^{N,G} - A_t^{N,G}| = \frac{1}{2}\left|\int_0^t \langle \Delta_\Sigma G_s - \Delta_p^N G_s, \sigma_s^N\rangle\right| ds \le \frac{C}{N^{d+1}}\int_0^t |\eta_s| ds.$$

By Taylor's theorem, for all $G \in C_c^3(\mathbb{R}_+ \times \mathbb{T}^d)$ we have that

$$G_s(x+h) - G_s(x) = \langle \nabla G_s(x), h\rangle + \frac{1}{2}H^2 G_s(x)(h) + R_s^G(x, h) \tag{4.9}$$

for all $x, h \in \mathbb{R}^d$, where $H^2 G_s$ is the (spatial) Hessian form of G at time s and

$$R_s^G(x, h) = \sum_{k,l,m=1}^{d} \int_0^1 \frac{(1-t)^2}{2}\partial_{k\ell m}^3 G_s(x + th) h_k h_l h_m dt$$

is the remainder in integral form. Since G is C^3 with compact support we have that

$$\|D^3 G\|_u := \sup_{s\in\mathbb{R}_+} \sup_{k,l,m=1,\dots,d} \|\partial^3_{klm} G_s\|_u < +\infty$$

and therefore

$$\Big|R_s^G\Big(\frac{x}{N}, \frac{z}{N}\Big)\Big| \le \frac{\|D^3 G\|_u}{6N^3} \sum_{k,l,m=1}^d z_k z_l z_m$$

for all $N \in \mathbb{N}$, $x, z \in \mathbb{T}^d_N$. Therefore since $H^2 G_s(x)(\lambda h) = \lambda^2 H^2 G_s(x)(h)$ for all $x, h \in \mathbb{R}^d$ and all $\lambda \in \mathbb{R}$ it follows that for all $x \in \mathbb{T}^d$ and all $z \in \mathbb{T}^d_N$, $N \in \mathbb{N}$, we have

$$\begin{aligned} G_s\Big(x + \frac{z}{N}\Big) + G_s\Big(x - \frac{z}{N}\Big) - 2G_s(x) &= G_s\Big(x + \frac{z}{N}\Big) - G_s(x) + G\Big(x - \frac{z}{N}\Big) - G_s(x) \\ &= \frac{1}{N^2} H^2 G_s(x)(z) + \bar{R}_s\Big(x, \frac{z}{N}\Big), \end{aligned}$$

where we have set $\bar{R}_s(x,h) := R_s^G(x,h) + R_s^G(x,-h)$ for all $x, h \in \mathbb{R}^d$. Therefore,

$$\begin{aligned} \Delta_p^N G_s(x) &= \sum_{z\in\mathbb{T}^d_N} H^2 G_s(x)(z) p(z) + N^2 \sum_{z\in\mathbb{T}^d_N} \bar{R}_s\Big(x, \frac{z}{N}\Big) p(z) \\ &= \sum_{z\in\mathbb{T}^d_N} \sum_{i,j=1}^d \partial^2_{ij} G_s(x) z_i z_j p(z) + N^2 \sum_{z\in\mathbb{T}^d_N} \bar{R}_s\Big(x, \frac{z}{N}\Big) p(z) \\ &= \sum_{i,j=1}^d \partial^2_{ij} G_s(x) \sum_{z\in\mathbb{T}^d_N} z_i z_j p(z) + N^2 \sum_{z\in\mathbb{T}^d_N} \bar{R}_s\Big(x, \frac{z}{N}\Big) p(z) \\ &= \Delta_\Sigma G_s(x) + N^2 \sum_{z\in\mathbb{T}^d_N} \bar{R}_s\Big(x, \frac{z}{N}\Big) p(z). \end{aligned}$$

since for all N large enough we have that $\sum_{z\in\mathbb{T}^d_N} z_i z_j p(z) = \sum_{z\in\mathbb{Z}^d} z_i z_j p(z) = \sigma_{ij}$. But obviously

$$\Big|\bar{R}_s\Big(x, \frac{z}{N}\Big)\Big| \le \frac{\|D^3 G\|_u}{3N^3} \sum_{i,j,k=1}^d z_i z_j z_k$$

and therefore it follows that

$$|\Delta_p^N G_s(x) - \Delta_\Sigma G_s(x)| \le N^2 \sum_{z\in\mathbb{T}^d_N} \Big|\bar{R}_s\Big(x, \frac{z}{N}\Big)\Big| p(z) \le \frac{\|D^3 G\|_u R^3}{3N}.$$

Therefore for all $x \in \mathbb{T}^d$ we have that

$$\big|\langle \Delta_p^N G_s - \Delta_\Sigma G_s, \sigma_s^N\rangle\big| \le \frac{\|D^3 G\|_u R^3}{3N^{d+1}} \sum_{x\in\mathbb{T}^d_N} g\big(\eta_s(x)\big) \le \frac{\|D^3 G\|_u \|g'\|_u R^3}{3N^{d+1}} \langle 1, \pi_s^N\rangle$$

Therefore for all $N \in \mathbb{N}$ and all $t \ge 0$ we have that

$$|B_t^{N,G} - A_t^{N,G}| \le \frac{\|D^3 G\|_u \|g'\|_u R^3}{6N^{d+1}} \int_0^t \langle 1, \pi^N\rangle ds$$

which proves the claim with constant $C = \frac{1}{6}\|D^3G\|_u\|g'\|_u R^3$. $\square$

We prove next that under suitable assumptions on the sequence $\{\mu^N \in \mathbb{P}\mathbb{M}_N^d\}$ of initial distributions the sequence of the empirical distributions $Q^N \in \mathbb{P}D(\mathbb{R}_+, \mathcal{M}_+^d)$ of the speeded up by N^2 ZRP defined in (4.4) is relatively compact in $\mathbb{P}D(\mathbb{R}_+, \mathcal{M}_+^d)$. We will assume that the sequence $\{\mu_N\}$ of initial distributions is associated to some macroscopic profile $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$. Furthermore in order for the martingale $A^{N,G}$ associated to the ZRP to be square integrable we will assume that $\mu_0^N \in \mathbb{P}_4\mathbb{M}_N^d$ ($\mu_0^N \in \mathbb{P}_1\mathbb{M}_N^d$ if g is bounded) and finally we will assume that $\{\mu_0^N\}_{N\in\mathbb{N}}$ satisfies the $O(N^d)$-entropy assumption. Recall that according to proposition 2.6.4 our assumptions do not exclude sequences of initial distributions that have a condensate at some macroscopic point $u \in \mathbb{T}^d$.

We turn now to the proof of the relative compactness of the sequence Q^N, $N \in \mathbb{N}$, of the empirical distributions of the ZRP starting from μ^N and speeded up by N^2.

Proposition 4.1.4 *Let $\{\mu_0^N \in \mathbb{P}\mathbb{M}_N^d\}$ be a sequence of distributions associated to a macroscopic profile $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$. We suppose that either the jump rate g of the ZRP is bounded and $\mu_0^N \in \mathbb{P}_1\mathbb{M}_N^d$ or either the jump rate is g is Lipschitz, $\mu_0^N \in \mathbb{P}_4\mathbb{M}_N^d$ and*

$$K_* := \limsup_{N\uparrow+\infty} \frac{1}{N^d} H(\mu_0^N|\nu_{\rho_*}^N) < +\infty \tag{4.10}$$

for some $\rho_* \in (0, \rho_c)$. Then the sequence $Q^N \in \mathbb{P}D(\mathbb{R}_+, \mathcal{M}_+(\mathbb{T}^d))$, $N \in \mathbb{N}$, of the empirical distributions of the ZRP starting from μ_0^N and speeded up by N^2, defined in (4.4) is relatively compact in the weak topology of the space of probability measures over the Skorohod space $D(\mathbb{R}_+, \mathcal{M}_+(\mathbb{T}^d))$, and any limit point Q of the sequence $\{Q^N\}$ is concentrated on trajectories $\pi \in D(\mathbb{R}_+; \mathcal{M}_+(\mathbb{T}^d))$ such that $\pi_0 = \mu_0$.

Proof As we know by the description of the relatively compact subsets of $\mathbb{P}D(\mathbb{R}_+, \mathcal{M}_+^d)$ in order to prove that $\{Q^N\}$ is relatively compact it suffices to prove that for some countable subset $\{G_k | k \in \mathbb{N}\} \subseteq C(\mathbb{T}^d)$ such that $G_1 \equiv 1$, the sequence

$$\bar{F}_{G_k *} Q^N \in \mathbb{P}D(\mathbb{R}_+, \mathbb{R}), \quad N \in \mathbb{N}$$

is relatively compact for all $k \in \mathbb{N}$, where for all $G \in C(\mathbb{T}^d)$ we set $F_G := \langle G, \cdot \rangle : \mathcal{M}_+^d \longrightarrow \mathbb{R}$ and

$$\bar{F}_G : D(\mathbb{R}_+, \mathcal{M}_+^d) \longrightarrow D(\mathbb{R}_+, \mathbb{R})$$

is the mapping induced on the Skorohod spaces by F_G. In particular it suffices to prove that the sequence $\{\bar{F}_{G*}Q^N\}$ is relatively compact for all $G \in C^\infty(\mathbb{T}^d)$.

So let $G \in C^\infty(\mathbb{T}^d)$. In order to prove the relative compactness of $\{\bar{F}_{G*}Q^N\}_{N\in\mathbb{N}}$ it suffices to prove that: (a) for all $t \in \mathbb{R}_+$

$$\lim_{A\uparrow+\infty} \sup_{N\in\mathbb{N}} \bar{F}_{G*}Q^N \big\{ f \in D(\mathbb{R}_+, \mathbb{R}) \, \big| \, |f_t| > A \big\} = 0$$

and (b) the condition of Aldous, i.e. that for all $\varepsilon, T > 0$ we have

$$\lim_{\delta\to 0} \limsup_{N\to+\infty} \sup_{\substack{\tau\in\mathfrak{T}^T(\mathcal{F}_+^{\mathbb{R}}) \\ \theta\leq\delta}} \bar{F}_{G*}Q^N \Big\{ f \in D(\mathbb{R}_+, \mathbb{R}) \, \Big| \, \big| f_{\tau(f)} - f_{[\tau(f)+\theta]\wedge T} \big| > \varepsilon \Big\} = 0$$

where $\mathfrak{T}^T(\mathcal{F}^{\mathbb{R}}_+)$ is the set of all stopping times $\tau : D(\mathbb{R}_+, \mathbb{R}) \longrightarrow [0, T]$ with respect to the continuation $\mathcal{F}^{\mathbb{R}} = (\mathcal{F}^{0,\mathbb{R}}_{t+})_{t\geq 0}$ of the natural filtration $(\mathcal{F}^{0,\mathbb{R}}_t)_{t\geq 0}$ in $D(\mathbb{R}_+, \mathbb{R})$.
(a) Let $t \in \mathbb{R}_+$. Of course we can assume that $\|G\|_u \neq 0$ or else we have nothing to prove, and for all $N \in \mathbb{N}$ and all $A > 0$ we have that

$$\begin{aligned}
\bar{F}_{G*}Q^N\big\{f \in D(\mathbb{R}_+, \mathbb{R})\,\big|\,|f_t| > A\big\} &= Q^N\big\{\pi \in D(\mathbb{R}_+, \mathcal{M}^d_+)\,\big|\,|\langle G, \pi_t\rangle| > A\big\} \\
&\leq Q^N\big\{\pi \in D(\mathbb{R}_+, \mathcal{M}^d_+)\,\big|\,\langle \pi_t, 1\rangle > A/\|G\|_u\big\} \\
&= P^N\big\{\eta \in D(\mathbb{R}_+, \mathbb{M}^d_N)\,\big|\,|\eta_t| > N^d A/\|G\|_u\big\} \\
&= P^N\big\{\eta \in D(\mathbb{R}_+, \mathbb{M}^d_N)\,\big|\,|\eta_0| > N^d A/\|G\|_u\big\} \\
&= \mu^N\big\{\eta \in \mathbb{M}^d_N\,\big|\,|\eta| > N^d A/\|G\|_u\big\} \\
&= \mu^N\big\{\langle \pi^N, 1\rangle > A/\|G\|_u\big\}.
\end{aligned}$$

Therefore since $\mu^N \in \mathbb{P}_1\mathbb{M}^d_N$ and $\{\mu^N\}$ is associated to $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$, (a) follows by lemma 3.1.2.
(b) We prove now the Aldous condition. So let $\varepsilon, T > 0$ be fixed. As we know, given any continuous function $F : M \longrightarrow N$ between polish spaces the induced mapping $\bar{F} : D(\mathbb{R}_+, M) \longrightarrow D(\mathbb{R}_+, N)$ is $(\mathcal{F}^M_t, \mathcal{F}^N_t)$-measurable for all $t \geq 0$, where $(\mathcal{F}^X)$ is the (right) continuation of the natural filtration $(\mathcal{F}^{0,X}_t)_{t\geq 0}$ in $D(\mathbb{R}_+, X)$, $X = M, N$, which shows that

$$\mathfrak{T}^T(\mathcal{F}^N) \circ \bar{F} := \big\{\tau \circ \bar{F}\,\big|\,\tau \in \mathfrak{T}^T(\mathcal{F}^N)\big\} \subseteq \mathfrak{T}^T(\mathcal{F}^M),$$

and we obviously have that

$$\bar{F}(x)_{\tau(\bar{F}(x))} = F\big(x_{\tau\circ\bar{F}(x)}\big) \qquad \forall\, x \in D(\mathbb{R}_+, M),\ \tau \in \mathfrak{T}^T(\mathcal{F}^N).$$

In our particular case we have that $\mathfrak{T}^T(\mathcal{F}^{\mathbb{R}}) \circ \bar{F}_G \subseteq \mathfrak{T}^T(\mathcal{F}^{\mathcal{M}^d_+}_+)$ and if for each stopping time $\tau \in \mathfrak{T}^T(\mathcal{F}^{\mathbb{R}})$ we set $\tau_G := \tau \circ \bar{F}_G$ then $\langle G, \mu\rangle_{\tau(\langle G,\pi\rangle)} = \langle G, \mu_{\tau_G(\pi)}\rangle$ and so

$$\begin{aligned}
&\bar{F}_{G*}Q^N\Big\{f \in D(\mathbb{R}_+, \mathbb{R})\,\Big|\,\big|f_{\tau(f)} - f_{[\tau(f)+\theta]\wedge T}\big| > \varepsilon\Big\} \\
&\qquad = Q^N\Big\{\pi \in D(\mathbb{R}_+, \mathcal{M}^d_+)\,\Big|\,\big|\langle G, \pi_{\tau_G(\pi)}\rangle - \langle G, \pi_{[\tau_G(\pi)+\theta]\wedge T}\rangle\big| > \varepsilon\Big\}
\end{aligned}$$

for all $\theta > 0$ and all $\tau \in \mathfrak{T}^T(\mathcal{F}^{\mathbb{R}})$. It follows that for all $\delta > 0$ we have

$$\sup_{\substack{\tau\in\mathfrak{T}^T(\mathcal{F}^{\mathbb{R}})\\ \theta\leq\delta}} \bar{F}_{G*}Q^N\Big\{\big|f_\tau - f_{[\tau+\theta]\wedge T}\big| > \varepsilon\Big\} \leq \sup_{\substack{\tau\in\mathfrak{T}^T(\mathcal{F}^{\mathcal{M}^d_+})\\ \theta\leq\delta}} Q^N\Big\{\big|\langle G, \pi_\tau - \pi_{[\tau+\theta]\wedge T}\rangle\big| > \varepsilon\Big\},$$

where of course in the inequality above, f and π are the canonical cadlag process f and π on the Skorohod spaces $D(\mathbb{R}_+, \mathbb{R})$ and $D(\mathbb{R}_+, \mathcal{M}^d_+)$ respectively. With similar reasoning we get that

$$\sup_{\substack{\tau\in\mathfrak{T}^T(\mathcal{F}^{\mathcal{M}^d_+})\\ \theta\leq\delta}} Q^N\Big\{\big|\langle G, \pi_\tau - \pi_{[\tau+\theta]\wedge T}\rangle\big| > \varepsilon\Big\} \leq \sup_{\substack{\tau\in\mathfrak{T}^T(\mathcal{F}^{\mathbb{M}^d_N})\\ \theta\leq\delta}} P^N\Big\{\big|\langle G, \pi^N_\tau - \pi^N_{[\tau+\theta]\wedge T}\rangle\big| > \varepsilon\Big\}$$

for all $\delta > 0$, where here of course $\pi^N = (\pi^N_t)_{t\geq 0}$ is the empirical process.

Let now $A^{N,G}$ be the martingale associated by proposition 4.1.1 to the ZRP. By proposition 4.1.3 it follows that there exists a constant $C = C(G,p,g) \geq 0$ such that

$$\left|\langle G, \pi_t^N\rangle - \langle G, \pi_s^N\rangle - \frac{1}{2}\int_s^t \langle \Delta_\Sigma G, \sigma_r^N\rangle dr - (A_t^{N,G} - A_s^{N,G})\right| \leq \frac{C}{N^{d+1}}\int_s^t |\eta_u| du,$$

for all $0 \leq s \leq t$. It follows that

$$|\langle G, \pi_t^N - \pi_s^N\rangle| \leq \left|A_t^{N,G} - A_s^{N,G}\right| + \frac{1}{2}\int_s^t \left|\langle \Delta_\Sigma G, \sigma_r^N\rangle\right| dr + C\int_s^t \langle \pi_r^N, 1\rangle dr.$$

But we obviously have that

$$\left|\langle \Delta_\Sigma G, \sigma^N\rangle\right| \leq \|\Delta_\Sigma G\|_u \langle 1, \sigma^N\rangle \leq \|\Delta_\Sigma G\|_u \|g'\|_u \langle 1, \pi^N\rangle$$

and therefore, taking into account the conservation of the total number of particles by the dynamics of the ZRP, we can write that

$$|\langle G, \pi_t^N - \pi_s^N\rangle| \leq \left|A_t^{N,G} - A_s^{N,G}\right| + C_1 \cdot (t-s)\langle \pi_0^N, 1\rangle$$

P^N-a.s. for some constant $C_1 \geq 0$, namely $C_1 = C + \frac{1}{2}\|\Delta G\|_u \|g'\|_u$. It follows that

$$\left|\langle G, \pi_{[\tau+\theta]\wedge T}^N - \pi_\tau^N\rangle\right| \leq \left|A_{[\tau+\theta]\wedge T}^{N,G} - A_\tau^{N,G}\right| + C_1 \delta \langle \pi_0^N, 1\rangle$$

for all $\tau \in \mathfrak{T}^T(\mathcal{F}^{\mathbb{M}_N^d})$ and all $0 < \theta \leq \delta$, and therefore

$$\begin{aligned} \sup_{\substack{\tau\in\mathfrak{T}^T \\ \theta\leq\delta}} P^N\left\{\left|\langle G, \pi_{[\tau+\theta]\wedge T}^N - \pi_\tau^N\rangle\right| > \varepsilon\right\} \quad &\leq \quad \sup_{\substack{\tau\in\mathfrak{T}^T \\ \theta\leq\delta}} P^N\Big\{\left|A_{[\tau+\theta]\wedge T}^{N,G} - A_\tau^{N,G}\right| > \frac{\varepsilon}{2}\Big\} \\ &+ \quad \mu^N\Big\{C_1\delta\langle \pi_0^N, 1\rangle > \frac{\varepsilon}{2}\Big\} \end{aligned}$$

for all $\delta > 0$. So since the term $\mu^N\{C_1\delta\langle\pi_0^N, 1\rangle > \varepsilon/2\}$ converges to 0 as $\delta \to 0$ uniformly over N by (3.10), in order to prove Aldous's criterion it remains to prove that

$$\lim_{\delta\to 0}\limsup_{N\to+\infty} \sup_{\substack{\tau\in\mathfrak{T}^T(\mathcal{F}^{\mathbb{M}_N^d}) \\ \theta\leq\delta}} P^N\left\{\left|A_{[\tau+\theta]\wedge T}^{N,G} - A_\tau^{N,G}\right| > \varepsilon\right\} = 0,$$

and by the Chebyshev-Markov inequality it suffices to prove that

$$\lim_{\delta\to 0}\limsup_{N\to+\infty} \sup_{\substack{\tau\in\mathfrak{T}^T(\mathcal{F}^{\mathbb{M}_N^d}) \\ \theta\leq\delta}} \mathbb{E}^{\mu^N}\left(A_{[\tau+\theta]\wedge T}^{N,G} - A_\tau^{N,G}\right)^2 = 0. \tag{4.11}$$

To prove (4.11) in the case that g is Lipschitz we use Doob's optional stopping theorem, proposition 4.1.2 and the conservation of the total number of particles to get that

$$\begin{aligned} E_{N,\tau,\theta} \quad &:= \quad \mathbb{E}^{\mu^N}\left(A_{[\tau+\theta]\wedge T}^{N,G} - A_\tau^{N,G}\right)^2 = \mathbb{E}^{\mu^N}\left(\langle A^{N,G}\rangle_{[\tau+\theta]\wedge T} - \langle A^{N,G}\rangle_\tau\right) \\ &= \quad \frac{1}{N^{2d-2}}\mathbb{E}^{\mu^N}\int_\tau^{[\tau+\theta]\wedge T} \sum_{x,z\in\mathbb{T}_N^d} \Big[G\Big(\frac{x+z}{N}\Big) - G\Big(\frac{x}{N}\Big)\Big]^2 g\big(\eta_s(x)\big)p_N(z)ds \\ &= \quad \frac{\|\nabla G\|_u^2\|g'\|_u}{N^{2d}}\mathbb{E}^{\mu^N}\int_\tau^{[\tau+\theta]\wedge T} \sum_{x,z\in\mathbb{T}_N^d} |z|^2\eta_s(x)p(z)ds \\ &= \quad \frac{\|\nabla G\|_u^2\|g'\|_u\mathbb{V}(p)}{N^d}\mathbb{E}^{\mu^N}\int_\tau^{[\tau+\theta]\wedge T} \langle \pi_s^N, 1\rangle ds \\ &\leq \quad \frac{\|\nabla G\|_u^2\|g'\|_u\mathbb{V}(p)\theta}{N^d}\mathbb{E}^{\mu^N}\langle \pi_0^N, 1\rangle = \frac{\|\nabla G\|_u^2\|g'\|_u\mathbb{V}(p)\theta}{N^d}\int \langle \pi^N, 1\rangle d\mu^N. \end{aligned}$$

It follows that

$$\sup_{\substack{\tau\in\mathfrak{T}^T(\mathcal{F}^{\mathbb{M}^d_N})\\ \theta\le\delta}} \mathbb{E}^{\mu^N}\big(A^{N,G}_{[\tau+\theta]\wedge T}-A^{N,G}_{\tau}\big)^2 \le \frac{\|\nabla G\|_u^2\|g'\|_u\mathbb{V}(p)\delta}{N^d}\int\langle\pi^N,1\rangle d\mu^N$$

and therefore in order to prove (4.11) it suffices to prove that $\int\langle\pi^N,1\rangle d\mu^N \le O(N^d)$, i.e. that

$$\limsup_{N\to+\infty}\frac{1}{N^d}\int\langle\pi^N,1\rangle d\mu^N < +\infty. \tag{4.12}$$

Here the assumption that $\{\mu^N\}$ is associated to an initial macroscopic profile does not suffice to give (4.12), but (4.12) can be derived by the entropy assumption (4.10). By the relative entropy inequality we have that

$$\int\langle\pi^N,1\rangle d\mu^N \le \frac{1}{\theta N^d}\Big\{\log\int e^{\theta N^d\langle\pi^N,1\rangle}d\nu_{\rho_*}+H(\mu^N|\nu_{\rho_*})\Big\}$$

for all $\theta>0$ and all $N\in\mathbb{N}$. But

$$\int e^{\theta N^d\langle\pi^N,1\rangle}d\nu_{\rho_*}=\int\prod_{x\in\mathbb{T}^d_N}e^{\theta\eta(x)}d\nu_{\rho_*}=M_{\nu^1_{\rho_*}}(\theta)^{N^d}$$

and therefore

$$\int\langle\pi^N,1\rangle d\mu^N \le \frac{1}{\theta}\Big\{\Lambda_{\rho_*}(\theta)+\frac{1}{N^d}H(\mu^N|\nu_{\rho_*})\Big\}$$

for all $\theta>0$ and all $N\in\mathbb{N}$. It follows that

$$\limsup_{N\to+\infty}\int\langle\pi^N,1\rangle d\mu^N \le \frac{\Lambda_{\rho_*}(\theta)+K_*}{\theta}$$

for all $\theta>0$. But $\rho_*<\rho_c$ and thus $\nu^1_{\rho_*}$ has exponential moments, and therefore by choosing $\theta_*\in\mathcal{D}_{\Lambda_{\rho_*}}\setminus\{0\}$ in the inequality above we get that

$$\limsup_{N\to+\infty}\int\langle\pi^N,1\rangle d\mu^N < +\infty \tag{4.13}$$

This proves (4.12) and completes the proof of the relative compactness of $\{Q^N\}$.

It remains to prove that any limit point Q of the sequence $\{Q_N\}$ is concentrated on trajectories $\pi\in D(\mathbb{R}_+,\mathcal{M}_+(\mathbb{T}^d))$ such that $\pi_0=\mu_0$. So let Q be a limit point of $\{Q_N\}$. As we know, the evaluation mapping $e_t : D(\mathbb{R}_+;\mathcal{M}_+(\mathbb{T}^d))\longrightarrow\mathcal{M}_+(\mathbb{T}^d)$ given by $e_t(\pi)=\pi_t$ is continuous at each $\pi\in D(\mathbb{R}_+;\mathcal{M}_+(\mathbb{T}^d))$ that is continuous at t. In particular the evaluation $e_0 : D(\mathbb{R}_+;\mathcal{M}_+(\mathbb{T}^d))\longrightarrow\mathcal{M}_+(\mathbb{T}^d)$ is continuous and therefore for all $G\in C(\mathbb{T}^d)$ the composite mapping $I_G\circ e_0 : D(\mathbb{R}_+;\mathcal{M}_+(\mathbb{T}^d))\longrightarrow\mathbb{R}$, where $I_G:\mathcal{M}_+(\mathbb{T}^d)\longrightarrow\mathbb{R}$ is the mapping $I_G(\pi)=\langle G,\pi\rangle$, is continuous. Therefore, for all $G\in C(\mathbb{T}^d)$ and all $\varepsilon>0$ we have by the portmanteau theorem that

$$\begin{aligned} Q\big\{|\langle G,\pi_0\rangle-\langle G,\mu_0\rangle|>\varepsilon\big\} &\le \liminf_{N\to\infty}Q^N\big\{|\langle G,\pi_0\rangle-\langle G,\mu_0\rangle|>\varepsilon\big\}\\ &= \liminf_{N\to\infty}\mu^N\big\{|\langle G,\pi^N\rangle-\langle G,\mu_0\rangle|>\varepsilon\big\}=0, \end{aligned}$$

since the sequence $\{\mu^N\}$ is associated to the macroscopic profile $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$. Since this holds for all $G \in C(\mathbb{T}^d)$ and all $\varepsilon > 0$ it follows that

$$Q\big\{|\langle G, \pi_0\rangle - \langle G, \mu_0\rangle| = 0\big\} = 1$$

for all $G \in C(\mathbb{T}^d)$, and then if we choose a countable dense subset $D \subseteq C(\mathbb{T}^d)$ in the uniform norm it follows that

$$Q\{\pi_0 = \mu_0\} = Q\bigg(\bigcap_{G\in D} \{|\langle G, \pi_0\rangle - \langle G, \mu_0\rangle| = 0\}\bigg) = 1,$$

as required and the proof is complete. $\square$

4.2 The Empirical Current Process

In this section we define the empirical current process W^N and show that pair (π^N, W^N) where as always π^N is the empirical density, satisfy the continuity equation

$$\partial_t \pi^N + \mathrm{div} W^N = 0$$

at the microscopic level. We begin the considerations for a general step distribution $p \in \mathbb{P}_\infty \mathbb{Z}^d$ but in the end we mainly focus on the case that p is the n.n. elementary step distribution $p = \sum_{j=1}^d (\delta_{-e_j} + \delta_{e_j})$ where here we renormalize the nearest neighbor distribution to have a total probability equal to $2d$.

4.2.1 The Empirical Current

Definition 4.2.1 The *current along the bond* $(x, y) \in \mathbb{T}_N^d \times \mathbb{T}_N^d$ for the ZRP in the discrete torus $\mathbb{T}_N^d$ is the function $W_{x,y}^N : \mathbb{M}_N^d \longrightarrow \mathbb{R}$ given by

$$\begin{aligned} W_{x,y}^N(\eta) &= L^N(\eta, \eta^{x,y}) - L^N(\eta, \eta^{y,x}) = g(\eta_x)p(y-x) - g(\eta_y)p(x-y) \\ &= [\,g(\eta_x) - g(\eta_y)]p(y-x) \end{aligned}$$

for all $\eta \in \mathbb{M}_N^d$.

We want to obtain an expression of the action $L^N\langle G, \pi^N\rangle(\eta)$ of the generator on the function $\langle G, \pi^N\rangle$ involving the currents. For this we could use the already established formula for $L^N\langle G, \pi^N\rangle$ of the previous section. We do it however by calculating this action of the generator in a different way, to give one more example of calculations on the action of the generator. By the definition of the function $\langle G, \pi^N\rangle : \mathbb{M}_N^d \longrightarrow \mathbb{R}$ and the linearity of L^N we have that

$$L^N\langle G_s, \pi^N\rangle = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} G_s\Big(\frac{x}{N}\Big) L^N \eta(x). \tag{4.14}$$

But $\eta(x) \in B_1(\mathbb{M}_N^d)$ for all $x \in \mathbb{T}_N^d$ and therefore L^N can act on $\eta(x)$ to give

$$\begin{aligned} L^N\eta(x)(\eta) &= \sum_{y:y\neq x} \sum_{z \in \mathbb{T}_N^d} (\eta_x^{y,y+z} - \eta_x) g(\eta_y) p(z) + \sum_{z \in \mathbb{T}_N^d} (\eta_x^{x,x+z} - \eta_x) g(\eta_x) p(z) \\ &= \sum_{y:y\neq x} (\eta_x^{y,x} - \eta_x) g(\eta_y) p(x-y) + \sum_{z \in \mathbb{T}_N^d} (\eta_x^{x,x+z} - \eta_x) g(\eta_x) p(z) \\ &= \sum_{z\neq 0} (\eta_x^{x-z,x} - \eta_x) g(\eta_{x-z}) p(z) + \sum_{z\neq 0} (\eta_x^{x,x+z} - \eta_x) g(\eta_x) p(z) \\ &= \sum_{z\neq 0} [g(\eta_{x-z}) - g(\eta_x)] p(z) \end{aligned}$$

Therefore since we assume $p \in \mathbb{P}\mathbb{Z}^d$ to be symmetric we have that

$$\begin{aligned} L^N\eta(x) &= \frac{1}{2} \sum_{z\neq 0} \big[g\big(\eta(x-z)\big) - g\big(\eta(x)\big)\big] p(z) + \frac{1}{2} \sum_{z\neq 0} \big[g\big(\eta(x+z)\big) - g\big(\eta(x)\big)\big] p(z) \\ &= \frac{1}{2} \sum_{z \in \mathbb{T}_N^d} \big[g\big(\eta(x+z)\big) + g\big(\eta(x-z)\big) - 2g\big(\eta(x)\big)\big] p(z) \end{aligned}$$

In particular if p is the nearest neighbor elementary step distribution we have that

$$L^N\eta(x) = \sum_{j=1}^{d} \big[g\big(\eta(x+e_j)\big) + g\big(\eta(x-e_j)\big) - 2g\big(\eta(x)\big)\big]p(e_j)$$

The currents satisfy the identity

$$\begin{aligned} W^N_{x-z,x} - W^N_{x,x+z} &= [\,g(\eta_{x-z}) - g(\eta_x)]p(z) - [\,g(\eta_x) - g(\eta_{x+z})]p(z) \\ &= \big[g\big(\eta(x+z)\big) - g\big(\eta(x-z)\big) - 2g\big(\eta(x)\big)\big]p(z) \end{aligned}$$

for all $x, z \in \mathbb{T}^d_N$ and therefore we can write that

$$L^N\eta(x) = \frac{1}{2}\sum_{z\in\mathbb{T}^d_N} [W^N_{x-z,x} - W^N_{x,x+z}].$$

It follows by (4.14) that

$$L^N\langle G_s, \pi^N\rangle = \frac{1}{2N^d}\sum_{x\in\mathbb{T}^d_N} G_s\Big(\frac{x}{N}\Big)\sum_{z\in\mathbb{T}^d_N}[W^N_{x-z,x} - W^N_{x,x+z}].$$

By using the formula of integration by parts on the discrete torus $\mathbb{T}^d_N$:

$$\sum_{x\in\mathbb{T}^d_N}[g(x+z) - g(x)]f(x) = -\sum_{x\in\mathbb{T}^d_N} g(x)[f(x) - f(x-z)],$$

we get that

$$L^N\langle G_s, \pi^N\rangle = \frac{1}{2N^d}\sum_{x,z\in\mathbb{T}^d_N}\Big[G_s\Big(\frac{x+z}{N}\Big) - G_s\Big(\frac{x}{N}\Big)\Big]W^N_{x,x+z}.$$

Therefore the martingale $A^{N,G}$ can be rewritten as

$$\begin{aligned} A^{N,G}_t \quad &= \quad \langle G_t, \pi^N_t\rangle - \langle G_0, \pi^N_0\rangle - \int_0^t \langle \partial_s G_s, \pi^N_s\rangle ds \qquad (4.15) \\ &\quad - \frac{1}{2N^{d-2}}\int_0^t \sum_{x,z\in\mathbb{T}^d_N}\Big[G_s\Big(\frac{x+z}{N}\Big) - G_s\Big(\frac{x}{N}\Big)\Big]W^N_{x,x+z}(\eta_s)ds. \end{aligned}$$

In what follows we focus on the case where the p is the n.n. distribution renormalized so that $p(\mathbb{Z}^d) = 2d$ and define the *empirical current* $W^N : \mathbb{M}^d_N \longrightarrow \mathcal{M}(\mathbb{T}^d; \mathbb{R}^d)$ as the random vector measure given by

$$W^N = \frac{1}{N^{d-1}}\sum_{x\in\mathbb{T}^d_N}\Big[\sum_{j=1}^{d} W^N_{x,x+e_j}\cdot e_j\Big]\delta_{\frac{x}{N}} = -\frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}\nabla^N g\big(\eta(x)\big)\delta_{\frac{x}{N}}, \qquad (4.16)$$

where

$$\nabla^N g\big(\eta(x)\big) = N\sum_{j=1}^{d}\big[g\big(\eta(x+e_j)\big) - g\big(\eta(x)\big)\big]\cdot e_j,$$

is the discrete gradient of the function $g\big(\eta(x)\big)$. The empirical current is an $\mathbb{R}^d$-valued measure and acts on vector valued functions $F : \mathbb{T}^d \longrightarrow \mathbb{R}^d$ by the formula

$$\langle F, W^N\rangle := \int F \cdot dW^N = \frac{1}{N^{d-1}} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} F^j\Big(\frac{x}{N}\Big) W^N_{x,x+e_j}.$$

Proposition 4.2.1 *Let $G \in C_c^3(\mathbb{R}_+ \times \mathbb{T}^d)$ and let $A^{N,G}$ be the martingale associated by proposition 4.1.1 to the ZRP of parameters (p, g) on the discrete torus $\mathbb{T}_N^d$. Then there exists a constant $C = C(G, g, d) \geq 0$ such that*

$$\Big|\langle G_t, \pi_t^N\rangle - \langle G_0, \pi_0^N\rangle - \int_0^t \big[\langle \partial_s G_s, \pi_s^N\rangle + \langle \nabla G_s, W_s^N\rangle\big] ds - A_t^{N,G}\Big| \leq \frac{C}{N}\int_0^t \langle \pi_s^N, 1\rangle ds$$

for all $t \geq 0$, where $(\pi_t^N)_{t\geq 0}$ is the empirical ZRP and $(W_t^N)_{t\geq 0}$ its empirical current.

Proof According to the computations performed above, in the case that p is the n.n. step distribution the martingale $A_t^{N,G}$ can be written as

$$\begin{aligned} A_t^{N,G} &= \langle G_t, \pi_t^N\rangle - \langle G_0, \pi_0^N\rangle - \int_0^t \langle \partial_s G_s, \pi_s^N\rangle ds \qquad (4.17)\\ &\quad - \frac{1}{N^{d-2}} \int_0^t \sum_{j=1}^{d} \sum_{x\in\mathbb{T}_N^d} \Big[G_s\Big(\frac{x+e_j}{N}\Big) - G_s\Big(\frac{x}{N}\Big)\Big] W^N_{x,x+e_j}(\eta_s) ds. \end{aligned}$$

For any $G \in C_c^3(\mathbb{R}_+ \times \mathbb{T}^d)$, by the Taylor's expansion in (4.9) and the bound on the remainder there, for all $N \in \mathbb{N}$ and all $j = 1, \ldots, d$ we have that

$$G_s\Big(\frac{x+e_j}{N}\Big) - G_s\Big(\frac{x}{N}\Big) = \frac{1}{N}\partial_j G_s\Big(\frac{x}{N}\Big) + \frac{1}{2N^2}\partial^2_{jj} G_s\Big(\frac{x}{N}\Big) + R_s^G\Big(\frac{x}{N}, \frac{e_j}{N}\Big),$$

with the R_s^G term satisfying

$$\sup_{s\geq 0} \sup_{x\in\mathbb{T}_N^d} \sup_{j=1,\ldots,d} \Big|R_s^G\Big(\frac{x}{N}, \frac{e_j}{N}\Big)\Big| \leq \frac{\|D^3 G\|_{u,\infty}}{6N^3}.$$

Using this Taylor expansion and the empirical current we can write the second integral term $N^2 L^N\langle G_s, \pi^N\rangle(\eta_s)$ in the expression of $A^{N,G}$ as

$$\begin{aligned} N^2 L^N\langle G_s, \pi^N\rangle(\eta_s) &= \frac{1}{N^{d-2}} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} \Big[G_s\Big(\frac{x+e_j}{N}\Big) - G_s\Big(\frac{x}{N}\Big)\Big] W^N_{x,x+e_j}(\eta_s)\\ &= \langle \nabla G, W_s^N\rangle + \frac{1}{2N^d} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} \partial^2_{jj} G_s\Big(\frac{x}{N}\Big) W^N_{x,x+e_j}(\eta_s)\\ &\quad + \frac{1}{N^{d-2}} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} R_s\Big(\frac{x}{N}, \frac{e_j}{N}\Big) W^N_{x,x+e_j}(\eta_s). \qquad (4.18) \end{aligned}$$

Furthermore, we obviously have the inequality

$$\sum_{j=1}^{d} \sum_{x\in\mathbb{T}_N^d} \Big|W^N_{x,x+e_j}(\eta)\Big| \leq \|g'\|_u \sum_{j=1}^{d} \sum_{z\in\mathbb{T}_N^d} \big(|\eta| + |\tau_{e_j}\eta|\big) \leq 2d\|g'\|_u |\eta|_1$$

and therefore the third term in the sum in the right hand side of (4.18), which we set R_s^N, is bounded above by

$$
\begin{aligned}
|R_s^N| &\le \frac{1}{N^{d-2}}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\Big|R_s\Big(\frac{x}{N},\frac{e_j}{N}\Big)W_{x,x+e_j}^N(\eta_s)\Big| \\
&\le \frac{1}{N^{d-2}}\frac{\|D^3G\|_{u,\infty}}{6N^3}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\Big|W_{x,x+e_j}^N(\eta_s)\Big| \le 2d\frac{\|g'\|_u\|D^3G\|_{u,\infty}}{6N^{d+1}}|\eta_s|.
\end{aligned}
$$

Likewise, by an integration by parts, the middle term in (4.18) which we set H_s^N is

$$
H_s^N = \frac{1}{2N^d}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\Big[\partial_{jj}^2 G_s\Big(\frac{x}{N}\Big)-\partial_{jj}^2 G_s\Big(\frac{x-e_j}{N}\Big)\Big]g\big(\eta_s(x)\big)
$$

ans therefore since G is in $C_c^3(\mathbb{R}_+\times\mathbb{T}^d)$ we have that

$$
\begin{aligned}
\big|H_s^N\big| &\le \frac{1}{2N^d}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d\Big|\partial_{jj}^2 G_s\Big(\frac{x}{N}\Big)-\partial_{jj}^2 G_s\Big(\frac{x-e_j}{N}\Big)\Big|g\big(\eta_s(x)\big) \\
&\le \frac{\|D^3G\|}{2N^d}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d\Big|\frac{e_j}{N}\Big|g\big(\eta_s(x)\big) \le d\frac{\|g'\|_u\|D^3G\|}{2N^{d+1}}\langle 1,\pi_s^N\rangle.
\end{aligned}
$$

According to these calculations, up to some terms bounded absolutely by $\frac{C}{N}\int_0^t\langle\pi_s^N,\eta_s\rangle ds$, we can write the martingale $A^{N,G}$ as the process

$$
V_t^{N,G} := \langle G_t,\pi_t^N\rangle - \langle G_0,\pi_0^N\rangle - \int_0^t \big[\langle\partial_s G_s,\pi_s^N\rangle + \langle\nabla G_s, W_s^N\rangle\big]ds,\quad t\ge 0.
$$

Indeed, according to our estimates on the remainder terms $R_s^{N,G}$ and $H_s^{N,G}$ we have that

$$
|V_t^{N,G}-A_t^{N,G}| = \big|N^2L^N\langle G_s,\pi^N\rangle(\eta_s)-\langle\nabla G_s,W_s^N\rangle\big| \le |R_s^{N,G}|+|H_s^{N,G}| \le \frac{C}{N}\langle\pi_s^N,1\rangle,
$$

where $C=C(G,g,d)$ is the constant $C=\frac{d}{2}\|g'\|_u\|D^3G\|_{u,\infty}$ and the proof is complete.□

By this estimate it is obvious that

$$
\lim_{N\to\infty}P^N\big\{\sup_{0\le t\le T}|V_t^{N,G}-A_t^{N,G}|\ge\delta\big\}=0 \tag{4.19}
$$

for all $\delta,T>0$. Indeed, by the conservation of the number of particles we have that $\int_0^t\langle\pi_s^N,1\rangle ds = t\langle\pi_0^N,1\rangle$ P^N-a.s. and therefore

$$
P^N\big\{\sup_{0\le t\le T}|V_t^{N,G}-A_t^{N,G}|\ge\delta\big\}\le P^N\big\{\langle\pi_0^N,1\rangle\ge\frac{N\delta}{CT}\big\}\overset{N\to\infty}{\longrightarrow}0
$$

since μ^N is associated to an integrable initial profile $\rho_0\in L^{+,1}(\mathbb{T}^d)$. Furthermore, by using the quadratic variation of the martingale $A^{N,G}$ and Doob's inequality it is shown that the martingale $A^{N,G}$ is itself asymptotically negligible.

Proposition 4.2.2 *For any function* $G \in C_c^1(\mathbb{R}_+ \times \mathbb{T}^d)$ *the martingale*

$$A_t^{N,G} := \langle G_t, \pi_t^N \rangle - \langle G_0, \pi_0^N \rangle - \int_0^t (\partial_s + N^2 L^N)\langle G_s, \pi^N \rangle(\eta_s^N) ds, \quad t \geq 0,$$

given in proposition (4.1.1) is asymptotically negligible, that is

$$\lim_{N\to\infty} P^N \Big\{ \sup_{0 \leq t \leq T} |A_t^{N,G}| \geq \delta \Big\} = 0$$

for all $\delta, T > 0$

Proof By the Doob-Chebyshev inequality ([27], theorem II.1.7)

$$P^N \Big\{ \sup_{0\leq t\leq T} |A_t^{N,G}| \geq \delta \Big\} \leq \frac{1}{\delta^2} \mathbb{E}^N |A_T^{N,G}|^2$$

for each $\delta > 0$. But if $\langle A^{N,G} \rangle$ denotes the quadratic variation of the martingale $A^{N,G}$ then the process $\{(A_t^{N,G})^2 - \langle A_t^{N,G} \rangle\}_{t\geq 0}$ is a mean zero martingale and therefore

$$\mathbb{E}^N |A_T^{N,G}|^2 = \mathbb{E}^N \langle A_T^{N,G} \rangle.$$

Since $G \in C_c^1(\mathbb{R}_+ \times \mathbb{T}^d)$ we have that $C(G) := \sup_{(s,u)\in\mathbb{R}_+\times\mathbb{T}^d} \|\nabla G_s(u)\|_u^2 < +\infty$ and therefore for N large enough by the formula of the quadratic variation $\langle A^{N,G} \rangle$ given in proposition 4.1.2, the mean value theorem and the conservation of particles,

$$\begin{aligned}
\langle A_T^{N,G} \rangle &= \frac{1}{N^{2d-2}} \int_0^T \sum_{x,y\in\mathbb{T}_N^d} \Big[G_s\Big(\frac{y}{N}\Big) - G_s\Big(\frac{x}{N}\Big) \Big]^2 g\big(\eta_s(x)\big) p(x,y) ds \\
&\leq \frac{C(G)}{N^{2d}} \int_0^T \sum_{x,y\in\mathbb{T}_N^d} |x-y|^2 g\big(\eta_s(x)\big) p(x,y) \\
&\leq \frac{C(G)\|g'\|_u}{N^{2d}} \int_0^T \sum_{x,z\in\mathbb{T}_N^d} |z|^2 p(z) \eta_s(x) \\
&= \frac{C(G)\|g'\|_u \mathbb{V}(p)}{N^d} \int_0^T \langle 1, \pi_s^N \rangle ds \\
&\overset{P^N-a.s.}{=} \frac{C(G)\|g'\|_u \mathbb{V}(p) T}{N^d} \langle 1, \pi_0^N \rangle.
\end{aligned}$$

It follows that

$$P^N \Big\{ \sup_{0\leq t\leq T} |A_t^{N,G}| \geq \delta \Big\} \leq \frac{C(G)\|g'\|_u \mathbb{V}(p) T}{\delta^2 N^d} \int \langle 1, \pi^N \rangle d\mu_0^N,$$

where $\{\mu_0^N \in \mathbb{PM}_N^d\}$ is the family of initial distributions of the ZRP, and by (4.13) taking the limit in the inequality above, proves the claim. □

Using the asymptotic negligibility of the martingale $A^{N,G}$ and (4.19) it easily follows that

$$\lim_{N\to\infty} P^N \Big\{ \sup_{0\leq t\leq T} \Big| \langle G_t, \pi_t^N \rangle - \langle G_0, \pi_0^N \rangle - \int_0^t \big[\langle \partial_s G_s, \pi_s^N \rangle + \langle \nabla G_s, W_s^N \rangle \big] ds \Big| \geq \delta \Big\} = 0$$

for all $G \in C_c^3(\mathbb{R}_+ \times \mathbb{T}^d)$ and $\delta > 0$. In particular if G has compact support n $(0,T)$ $G \in C_c^3((0,T) \times \mathbb{T}^d)$ then

$$\lim_{N\to\infty} P^N \Big\{ \Big| \int_0^T \big[\langle \partial_s G_s, \pi_s^N \rangle + \langle \nabla G_s, W_s^N \rangle \big] ds \Big| \geq \delta \Big\} = 0 \tag{4.20}$$

Likewise, if $B^{N,G}$ denotes the process defined in (4.8), by proposition 4.1.3 we get that

$$\lim_{N\to\infty} P^N \Big\{ \sup_{0\leq t\leq T} |B_t^{N,G} - A_t^{N,G}| \geq \delta \Big\} = 0 \tag{4.21}$$

for all $\delta, T > 0$ and by the asymptotic negligibility of the martingale $A^{N,G}$ we get that

$$\lim_{N\to\infty} P^N \Big\{ \sup_{0\leq t\leq T} \Big| \langle G_t, \pi_t^N \rangle - \langle G_0, \pi_0^N \rangle - \int_0^t \big[\langle \partial_s G_s, \pi_s^N \rangle + \langle \Delta G_s, \sigma_s^N \rangle \big] ds \Big| \geq \delta \Big\} = 0$$

for all $G \in C_c^3(\mathbb{R}_+ \times \mathbb{T}^d)$ and $\delta > 0$, where here we do not have the coefficient $\frac{1}{2}$ in front of the Laplacian because we are working with n.n. step distribution of total probability equal to $2d$. We can interpret these limits as saying that at the microscopic level the triple (π^N, σ^N, W^N) satisfies the continuity equation

$$\partial_t \pi_t^N = \Delta \sigma^N = -\mathrm{div} W_t^N \quad \text{as } N \to \infty. \tag{4.22}$$

where of course here the Laplacian and the divergence are spatial operators.

Next we would like to prove the that the laws of the empirical jump rate σ^N and the empirical current W^N are relatively compact in order to obtain the analogue of the continuity equation (4.22) at the macroscopic level. Since σ^N and W^N do not describe conserved quantities, the proof of the Aldous compactness criterion on the Skorohod space regarding the oscillations is too hard to obtain. For this reason we will consider the processes σ^N and W^N as taking values in spaces with weaker topologies that on one hand allow to obtain the relative compactness of their laws but that are strong enough to yield on the other hand that the limit points of their laws are concentrated on trajectories (π, σ, W) that satisfy the continuity equation. The appropriate spaces will be L^∞-spaces of Banach space-valued curves.

4.2.2 Weak L^∞-spaces of Banach-Valued Curves

In this subsection we review some the basic facts on L^∞-spaces of vector-valued curves that will be used in defining the state spaces of the empirical current processes

$$W_s^N(\eta) = W_{\eta_s}^N, \quad \eta \in D(0,T; \mathbb{M}_N^d),\ s \geq 0$$

and the empirical jump rate process

$$\sigma_s^N(\eta) = \sigma_{\eta_s}^N, \quad \eta \in D(0,T; \mathbb{M}_N^d),\ s \geq 0.$$

We begin with some preliminaries. We denote by $\mathcal{M}^d := \mathcal{M}(\mathbb{T}^d; \mathbb{R}^d)$ the Banach space of all Borel $\mathbb{R}^d$-valued measures on the torus $\mathbb{T}^d$ equipped with the total variation norm $\|\cdot\|_{TV} : \mathcal{M}^d \longrightarrow \mathbb{R}_+$ defined by

$$\|\mu\|_{TV} = \sup_{\Pi \in \mathcal{P}_{\mathbb{T}^d}} \sum_{A\in\Pi} |\mu(A)|_2, \tag{4.23}$$

where $\mathcal{P}_{\mathbb{T}^d}$ denotes the set of all finite measurable partitions of $\mathbb{T}^d$ and $|\cdot|_2$ denotes the Euclidean norm on $\mathbb{R}^d$ and we denote by $C^d := C(\mathbb{T}^d;\mathbb{R}^d)$ the space of all continuous vector valued functions on $\mathbb{T}^d$ equipped with the uniform norm

$$\|G\|_\infty := \big\||G|_2\big\|_\infty, \qquad G \in C^d. \tag{4.24}$$

The Riesz representation theorem (theorem 7.2 in [16] for instance) is easily extended for $\mathbb{R}^d$-valued functions.

Proposition 4.2.3 *The function $I : \mathcal{M}(\mathbb{T}^d;\mathbb{R}^d) \longrightarrow C(\mathbb{T}^d;\mathbb{R}^d)^*$ defined by*

$$I(W) = \int G \cdot dW := \sum_{i=1}^{d} \int G^i dW^i,$$

where $G^i \equiv \langle G, e_i\rangle$ and $W^i \equiv \langle W, e_i\rangle$, is a linear surjective isometry, where $\mathcal{M}(\mathbb{T}^d;\mathbb{R}^d)$ is equipped with the total variation norm defined in (4.23) and $C(\mathbb{T}^d;\mathbb{R}^d)$ is equipped with the uniform norm defined in (4.24).

Proof The map I is obviously linear. It is also injective since if $I(W) = I(V)$ for some $W, V \in \mathcal{M}^d$ then for any $h \in C(\mathbb{T}^d)$ and any $i = 1, \dots, d$ we have that

$$\int h dW^i = I(W)(h \cdot e_i) = I(V)(h \cdot e_i) = \int h dV^i,$$

which implies that $W^i = V^i$ since measures are characterized by their action on continuous functions. Furthermore, it also surjective. Indeed, let $T \in (C^d)^*$. Then for each $i = 1, \dots, d$ the functional $T^i : C(\mathbb{T}^d) \longrightarrow \mathbb{R}$ defined by

$$T^i(h) = T(h \cdot e_i)$$

is a bounded linear functional with $\|T_i\| \leq \|T\|$ and by the scalar version of the Riesz representation theorem there exists $W^i \in \mathcal{M}(\mathbb{T}^d)$ such that

$$T^i(h) = \int h dW^i \qquad \text{for all } h \in C(\mathbb{T}^d).$$

But then $W := \sum_{i=1}^d W^i \cdot e_i \in \mathcal{M}(\mathbb{T}^d;\mathbb{R}^d)$ and for each $G \in C^d$ we have that

$$\int G \cdot W = \sum_{i=1}^{d} G^i dW^i = \sum_{i=1}^{d} T^i(G^i) = \sum_{i=1}^{d} T(G^i e_i) = T(G).$$

It remains to show that the map I is an isometry, i.e. that

$$\|I(W)\| := \sup_{\|G\|_\infty \leq 1} |I(W)(G)| = \|W\|_{TV}.$$

We prove first the inequality $\|I(W)\| \leq \|W\|_{TV}$. For this we show that for any bounded measurable function $F : \mathbb{T}^d \longrightarrow \mathbb{R}^d$ it holds that

$$\left| \int F \cdot dW \right| \leq \|F\|_\infty \|W\|_{TV}. \tag{4.25}$$

Indeed, let us consider first the case that of a simple function $\phi : \mathbb{T}^d \longrightarrow \mathbb{R}^d$. Then $\phi = \sum_{k=1}^m a_k \mathbb{1}_{A_k}$ for some $a_k \in \mathbb{R}^d$ and some pairwise disjoint measurable sets $A_k \subseteq \mathbb{T}^d$, $k = 1, \dots, m$ and

$$\begin{aligned} \left| \int \phi dW \right| &= \left| \sum_{k=1}^m \langle a_k, W(A_k) \rangle \right| \leq \sum_{k=1}^n |a_k|_2 \cdot |W(A_k)|_2 \leq \max_{1 \leq k \leq m} |a_k|_2 \sum_{k=1}^m |W(A_k)|_2 \\ &= \|\phi\|_\infty \sum_{k=1}^m |W(A_k)|_2 \leq \|\phi\|_\infty \|W\|_{TV}. \end{aligned}$$

Now, any real-valued bounded measurable function can be approximated uniformly by simple functions, so given any bounded measurable function $F : \mathbb{T}^d \longrightarrow \mathbb{R}^d$ we can apply this separately to each coordinate to find a sequence ϕ^n of simple functions such that $\|\phi^n - F\|_\infty \longrightarrow 0$. Then since convergence theorems for integrals hold for each coordinate, taking the limit as $n \to \infty$ in the inequality

$$\left| \int \phi^n \cdot dW \right| \leq \|\phi^n\|_\infty \|W\|_{TV}$$

we obtain (4.25). So for any $G \in C^d$ with $\|G\|_\infty \leq 1$ we have that

$$\left| \int G \cdot dW \right| \leq \|G\|_\infty \|W\|_{TV} \leq \|W\|_{TV}$$

which proves the required inequality.

We prove finally the converse inequality. By the definition of the total variation, given $W \in \mathcal{M}^d$ and $\varepsilon > 0$ there exists a partition $\Pi = \{A_1, \dots, A_m\}$ of $\mathbb{T}^d$ such that

$$\|W\|_{TV} \leq \sum_{k=0}^m |W(A_k)|_2 + \frac{\varepsilon}{2}.$$

Then if $F : \mathbb{T}^d \longrightarrow \mathbb{R}^d$ is the simple function given by

$$F = \sum_{k=1}^m \operatorname{sgn}\big(W(A_k)\big) \cdot \mathbb{1}_{A_k}$$

where for any $w \in \mathbb{R}^d$ we set

$$\operatorname{sgn}(w) := \begin{cases} \frac{w}{|w|_2} & \text{if } w \neq 0 \\ 0 & \text{if } w = 0 \end{cases}$$

we have that

$$\left| \int F \cdot dW \right| = \left| \sum_{k=1}^m \langle \operatorname{sgn}\big(W(A_k)\big), W(A_k) \rangle \right| = \sum_{k=1}^m |W(A_k)|_2 \geq \|W\|_{TV} - \frac{\varepsilon}{2}.$$

But by approximating then separately each coordinate of F by continuous functions, either by using Lusin's theorem (e.g. [16], theorem 7.10) or lemma A.3.2 in the appendix of the thesis, we can find $G \in C^d$ such that $\|G\|_\infty \leq 1$ and

$$\left| \int (F - G) \cdot dW \right| < \frac{\varepsilon}{2}$$

which shows that

$$\|W\|_{TV} \leq \left| \int G \cdot W \right| + \varepsilon \leq \|I(W)\| + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary this proves the required inequality and completes the proof.□

The Riesz representation theorem allows us to consider the w^*-topology on $\mathcal{M}^d$ via the duality $\mathcal{M}^d \cong (C^d)^*$. The w^*-topology is characterized by the requirement that a net $\{\mu_a\}_{a\in\mathcal{A}} \subseteq \mathcal{M}^d$ converges to some $\mu \in \mathcal{M}^d$ in the w^*-topology if

$$\int_{\mathbb{T}^d} \langle f, d\mu_a \rangle \stackrel{a\in\mathcal{A}}{\longrightarrow} \int_{\mathbb{T}^d} \langle f, d\mu \rangle$$

for all $f \in C^d$. The usefulness of the w^*-topology of $\mathcal{M}^d$ comes from Alaoglou's theorem according to which norm bounded subsets of $\mathcal{M}^d$ are relatively compact in the w^*-topology of $\mathcal{M}^d$. Furthermore, $(\mathcal{M}^d, w^*)$ is completely regular as topological vector space and since C^d is separable $(\mathcal{M}^d, w^*)$ is also submetrizable according to proposition A.4.1.

Since the space $\mathcal{M}^d := \mathcal{M}(\mathbb{T}^d; \mathbb{R}^d)$ of $\mathbb{R}^d$-valued measures is a Banach space it makes sense to consider various L^∞ spaces of curves in $\mathcal{M}^d$. We begin by the strong notion of considering $L^\infty(0, T; \mathcal{M})$ as a Banach space. Namely let $X = (X, \|\cdot\|_X)$ be a Banach space. We denote by $\mathcal{L}([0, T]; X)$ the set of all strongly measurable curves in X. Of course if X is separable, strong and usual measurability coincide. Then we can define for each $p \geq 1$ the function $\|\cdot\|_p : \mathcal{L}(0, T; X) \longrightarrow [0, \infty]$ by the formula

$$\|W\|_{p;X} = \big\| \|W\|_X \big\|_{L^p([0,T])}$$

where for each curve $W \in \mathcal{L}(0, T; X)$ we denote by $\|W\|_X \in \mathcal{L}^+([0, T])$ the function given by $t \mapsto \|W_t\|_X$. Then as usual we define the L^p space $L^p(0, T; X)$ as the set of a.s. equality equivalence classes of functions $W \in \mathcal{L}([0, T]; X)$ such that $\|W\|_X \in L^p([0, T])$.

Lemma 4.2.1 *If X is a separable normed space, then $L^1(0, T; X)$ is separable.*

Proof Let $f \in L^1(0, T; X)$, $\varepsilon > 0$. Fix a dense countable subset $D \subseteq X$ of X and let $\mathcal{A}$ denote the collection of all finite unions of open intervals with rational endpoints. The collection $\mathcal{A}$ is obviously countable and so the set $\mathcal{D} \subseteq L^1(0, T; X)$ consisting of all functions of the form

$$\sum_{j=1}^{n} q_j \mathbb{1}_{A_j}, \quad q_j \in D,\ A_j \in \mathcal{A},\ n \in \mathbb{N}$$

is obviously countable. We will show that it iss also dense in X. Indeed, since $f \in L^1(0, T; X)$ there exists a simple function $\phi = \sum_{k=1}^m x_k \mathbb{1}_{E_k} \in L^1(0, T; X)$ such that $\|\phi - f\|_{L^1(0,T;X)} < \varepsilon/2$. We set $M := \max_{1\leq k\leq m} \|x_k\|_X$. By proposition 1.20 in [16], for each $k = 1, \ldots, m$ there exists $A_k \in \mathcal{A}$ such that $m(E_k \triangle A_k) < \varepsilon/4mM$ and since D is dense in X, for each $k = 1, \ldots, m$ there exists $q_k \in D$ such that $\|q_k - x_k\|_X < \varepsilon/4T$. Then $\psi := \sum_{k=1}^m q_k \mathbb{1}_{A_k} \in \mathcal{D}$ and

$$\|\psi - f\|_{L^1(0,T;X)} \leq \|\psi - \phi\|_{L^1(0,T;X)} + \frac{\varepsilon}{2}.$$

But

$$\begin{aligned}
\|\psi-\phi\|_{L^1(0,T;X)} &\leq \left\|\psi-\sum_{k=1}^m x_k \mathbb{1}_{A_k}\right\|_{L^1(0,T;X)}+\left\|\sum_{k=1}^m x_k \mathbb{1}_{A_k}-\phi\right\|_{L^1(0,T;X)} \\
&\leq \sum_{k=1}^m\left(\int_0^T\|q_k-x_k\|_X \mathbb{1}_{A_k}(x)dx+\int_0^T\|x_k\|_X \mathbb{1}_{A_k\triangle E_k}(x)dx\right) \\
&\leq \frac{\varepsilon}{4T}\sum_{k=1}^m\int_0^T \mathbb{1}_{A_k}+M\sum_{k=1}^m m(E_k\triangle A_k)<\frac{\varepsilon}{2},
\end{aligned}$$

and so we have found an element of $\mathcal{D}$ ε-close to $f \in L^1(0,T;X)$. □

Proposition 4.2.4 *Let X be any Banach space and let $T>0$. There is an isometric injection*

$$i : L^\infty(0,T;X^*) \hookrightarrow L^1(0,T;X)^*$$

and the space $iL^\infty(0,T;X^*) \leq L^1(0,T;X)^*$ is a closed subspace of $L^1(0,T;X)^*$.

Proof Let $\langle\cdot,\cdot\rangle : X\times X^*$ denote the pairing $(x,x^*)\mapsto x^*(x)$ and let $W\in L^\infty(I;X^*)$. Then $W_t\in X^*$ for all $t\in I$ and therefore we can define a function $i^W : L^1(I;X)\longrightarrow \mathbb{R}$ by

$$i^W G=\int_0^T\langle G_t,W_t\rangle dt.$$

The function i^W is well defined since by the strong measurability of G and W we have that the function $\langle G,W\rangle : I\longrightarrow\mathbb{R}$ given by

$$\langle G,W\rangle_t=\langle G_t,W_t\rangle$$

is measurable, and it is real-valued since

$$|i^W G|\leq\int_0^T|\langle G_t,W_t\rangle|dt\leq\int_0^T\|G_t\|_X\|W_t\|_{X^*}dt\leq\|W\|_{\infty;X^*}\|G\|_{1;X}<\infty.$$

In particular the functional $i^W\in L^1(I;X)^*$ is bounded with operator norm

$$\|i^W\| := \sup_{\|G\|_{L^1(I;X)}=1}|i^W G|\leq\|W\|_{\infty;X^*}.$$

Consequently, the operator $T : L^\infty([0,T];X^*)\longrightarrow L^1([0,T];X)^*$ defined by

$$L^\infty([0,T];X^*)\ni W\mapsto T^W\in L^1([0,T];X)^*$$

is a bounded linear operator with operator norm $\|i\|\leq 1$.

We prove next that T is a (not necessarily surjective) isometry. For this it suffices of course to prove the converse inequality, i.e. that for all $W\in L^\infty(I;X^*)$ we have

$$\|W\|_{\infty,X^*}\leq\|i^W\|.$$

Since $L^\infty(I;X^*)\subseteq L^1(I;X^*)$ by Lebesgue's differentiation theorem for the Bochner integral there exists a measurable set $E\subseteq I$ of full measure $m(E)=m(I)$ such that

$$W_t=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}W_s ds$$

for all $t \in E$. Furthermore we note that for any $x \in X$ with $\|x\|_X = 1$, all $t \in E$ and all $\varepsilon > 0$ we have that the function $G^\varepsilon_{t,x} \in L^1(I;X)$ defined by the formula

$$G^\varepsilon_{t,x}(s) = \frac{1}{m(I \cap [t-\varepsilon, t+\varepsilon])} \mathbb{1}_{I \cap [t-\varepsilon,t+\varepsilon]}(s)x$$

satisfies $\|G^\varepsilon_{t,x}\|_{L^1([0,T];X)} = 1$ since

$$\begin{aligned} \|G^\varepsilon_x\|_{L^1([0,T];X)} &= \int_0^T \|G^\varepsilon_x(s)\|_X ds = \frac{1}{m(I \cap [t-\varepsilon, t+\varepsilon])} \int_{I\cap[t-\varepsilon,t+\varepsilon]} \|x\|_X ds \\ &= \|x\|_X = 1. \end{aligned}$$

It follows that

$$\|i^W\| = \sup_{\|G\|_{L^1(I;X)}=1} |i^W G| \geq \sup_{t\in E} \sup_{\|x\|_X=1} \sup_{\varepsilon\in(0,t\wedge(T-t))} |i^W G^\varepsilon_{t,x}|.$$

Now, for fixed $(t,x) \in E \times \{\|\cdot\|_X = 1\}$ we have for all $\varepsilon > 0$ that

$$|i^W G^\varepsilon_{t,x}| = \left|\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} \langle x, W_s\rangle ds\right| = \left|\left\langle x, \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} W_s ds\right\rangle\right| \xrightarrow{\varepsilon\to 0} |\langle x, W_t\rangle|,$$

since $\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} W_s ds \longrightarrow W_t$ strongly in X^*, and thus also in the w^*-topology of X^*. Therefore, for all $(t,x) \in E \times \{\|\cdot\|_X = 1\}$ we have that

$$\sup_{0<\varepsilon<t\wedge(T-t)} |i^W G^\varepsilon_{t,x}| \geq |\langle x, W_t\rangle|$$

and therefore

$$\|i^W\| \geq \sup_{t\in E} \sup_{\|x\|_X=1} |\langle x, W_t\rangle| = \sup_{t\in E} \|W_t\|_{X^*} \geq \|W\|_{\infty;X^*},$$

where the last inequality follows due to the fact that because $m(E) = m(I)$ we have that

$$\|W\|_{\infty;X^*} = \inf\big\{C \geq 0 \,\big|\, m\big\{t \in I : \|W_t\|_{X^*} \geq C\big\} = 0\big\} \leq \sup_{t\in E} \|W_t\|_{X^*}.$$

It remains to show that $iL^\infty(I;X^*) \leq L^1(I;X)^*$ is a closed subspace of $L^1(I;X)^*$. But this follows easily from the completeness of $L^\infty(I;X^*)$. Indeed, let $\{V^N\}_{N\in\mathbb{N}} \subseteq iL^\infty(I;X^*)$ such that $V^N \longrightarrow V \in L^1(I;X)^*$. Since $\{V^N\} \subseteq iL^\infty(I;X^*)$, for each $N \in \mathbb{N}$ there exists $W_N \in L^\infty(I;X^*)$ such that $i^{W_N} = V^N$, and since $\{V_N\}$ converges in the operator norm $\|\cdot\|$ of $L^1(I;X)^*$ it is $\|\cdot\|$-Cauchy. But then since i is an isometry it follows that $\{W_N\}$ is also $\|\cdot\|_{\infty;X^*}$-Cauchy. Indeed, given $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $\|V^N - V^M\| < \varepsilon$ for all $N, M \geq N_0$ and then

$$\|W_N - W_M\|_{\infty;X^*} = \|i^{(W_N - W_M)}\| = \|V^N - V^M\| < \varepsilon$$

for all $N, M \geq N_0$. Therefore $\{W_N\}$ is $\|\cdot\|_{X^*}$-Cauchy and by the completeness of $L^\infty(I;X^*)$ it follows that there exists $W \in L^\infty(I;X^*)$ such that $\|W_N - W\|_{\infty;X^*} \longrightarrow 0$. But then by the continuity of the linear injection i we have that

$$V = \lim_{N\to\infty} V_N = \lim_{N\to\infty} i^{W_N} = i\Big(\lim_{N\to\infty} W_N\Big) = i^W,$$

which shows that $V \in iL^\infty(I;X^*) \leq L^1(I;X)^*$ as required and completes the proof. $\square$

We will refer to the linear isometric injection of proposition 4.2.4 as *the natural injection of $L^\infty(I;X^*)$ in $L^1(I;X)^*$*. By identifying $L^\infty(I;X^*)$ with its closed image $iL^\infty(I;X^*)$ in $L^1(I;X)^*$ through the natural injection, we can consider the space $L^\infty(I;X^*)$ equipped with the restriction of the w^*-topology of $L^1(I;X)^*$. However since unless X is reflexive (in which case the natural injection of $L^\infty(I;X^*)$ in $L^1(I;X)^*$ is an isometry) the w^*-closure of $L^\infty(I;X^*)$ need not coincide with its norm closure, we can not use Alaoglou's theorem for the restriction of the w^*-topology on $L^\infty(I;X^*)$.

We give next an exact description of the space $L^1(I;X)^*$ following [10]. Let $\mathcal{L}^\infty_{w^*}(I;X^*)$ denote the space of all w^*-measurable functions $W : I \longrightarrow X^*$ for which there exists $w \in \mathcal{L}^\infty(I)$ such that

$$\|W_t\|_{X^*} \leq w(t) \quad \text{a.s.-}\forall\, t \in I.$$

Consider in $\mathcal{L}^\infty_{w^*}(I;X^*)$ the relation $\backsim$ given by $W \backsim V$ iff for all $F \in X$ we have that

$$\langle W_t, F\rangle = \langle V_t, F\rangle \quad \text{a.s.-}\forall\, t \in I.$$

We denote by $L^\infty_{w^*}(I;X^*)$ the quotient space and the equivalence class of $W \in \mathcal{L}^\infty_{w^*}(I;X^*)$ by $[W]$. The relation $\backsim$ is obviously a linear equivalence relation and the quotient space $L^\infty_{w^*}(I;X^*)$ becomes a vector space with the induced operations. Then, if for each $[W]$ we define the set $\mathcal{A}_{[W]}$ of $[W]$-admissible functions as

$$\mathcal{A}_{[W]} = \big\{w \in \mathcal{L}^\infty(I) \,\big|\, \exists\, V \in [W] \text{ such that } \|V_t\|_{X^*} \leq w(t) \quad \text{a.s.-}\forall\, t \in I\big\}$$

the function $\|\cdot\|_{L^\infty_{w^*}} : L^\infty_{w^*}(I;X^*) \longrightarrow \mathbb{R}_+$ defined by

$$\|[W]\|_{L^\infty_{w^*}} = \inf_{w \in \mathcal{A}_{[W]}} \|w\|_{L^\infty(I)}$$

is a norm that makes $L^\infty_{w^*}(I;X^*)$ a Banach space. Obviously $L^\infty(I;X^*)$ is isometrically embedded in $L^\infty_{w^*}(I;X^*)$ and

Proposition 4.2.5 *The linear operator $T : L^\infty_{w^*}(I;X^*) \longrightarrow L^1(I;X)^*$ given by*

$$T([W])(F) = \int_I \langle W_t, F_t\rangle dt$$

is an isometric isomorphism and for each $[W] \in L^\infty_{w^*}(I;X^*)$ there exists $\widetilde{W} \in [W]$ such that the function $I \ni t \mapsto \|\widetilde{W}_t\|_{X^*}$, which we denote by $\|\widetilde{W}\|_{X^*}$ is measurable and belongs in $L^\infty(I)$, and

$$\|[W]\|_{L^\infty_{w^*}} = \big\|\|\widetilde{W}\|_{X^*}\big\|_{L^\infty(I)}.$$

Proof For the proof which is based on Radon-Nikodym type theorems for Banach space-valued measures we refer to Theorems 1.5.4 and 1.5.5 in [10]. $\square$

We note that in the case that X is separable the equivalence relation $\backsim$ in $\mathcal{L}^\infty_{w^*}(I;X^*)$ is exactly the relation of almost sure equality. By Alaoglou's theorem we have the following

Proposition 4.2.6 *Let X be a normed space. Any norm bounded subset of $B \subseteq L^\infty_{w^*}(I;X^*)$ is relatively compact with respect to the w^*-topology of $L^\infty_{w^*}(I;X^*)$.*

Applying proposition A.4.1 to the separable space $L^1(0,T;X)$ we have the following.

Proposition 4.2.7 *Suppose that X is a separable normed space. Then there exists an auxiliary metric $d : L^\infty_{w^*}(I;X^*) \times L^\infty_{w^*}(I;X^*) \longrightarrow \mathbb{R}_+$ such that the restriction $d|_{B\times B}$ of d on any norm bounded subset $B \subseteq L^\infty(I;X^*)$ metrizes the restriction of the w^*-topology of $L^\infty_{w^*}(I;X^*) \cong L^1(I;X)^*$ on B.*

These results apply to the space $L^\infty(I;\mathcal{M}^d)$ of vector measure-valued curves, where $\mathcal{M}^d$ is the Banach space of all finite vector-valued measures. By the Riesz representation theorem we have that $(C^d)^* = \mathcal{M}^d$ and according to proposition 4.2.5 we have an isomorphism $T : L^\infty_{w^*}(I;\mathcal{M}^d) \longrightarrow L^1(I;C^d)^*$ given by

$$T([W])(G) = \int_0^T \int_{\mathbb{T}^d} G_t \cdot dW_t dt,$$

and we can consider the w^*-topology on $L^\infty_{w^*}(I;\mathcal{M}^d)$ according to which a net $\{[W^a]\}_{a\in\mathcal{A}} \subseteq L^\infty_{w^*}(I;\mathcal{M}^d)$ converges to $[W] \in L^\infty_{w^*}(I;\mathcal{M}^d)$ iff

$$\lim_{a\in\mathcal{A}} \int_0^T \int_{\mathbb{T}^d} G_t dW^a_t dt = \int_0^T \int_{\mathbb{T}^d} G_t dW_t dt, \quad \forall\, G \in L^1(I;C^d).$$

Now, if $W^N : \mathbb{M}^d_N \longrightarrow \mathcal{M}^d$ denotes the empirical current function defined in (4.16), we have for all $\eta \in \mathbb{M}^d_N$ that

$$\begin{aligned} \|W^N\|_{TV} &= \frac{1}{N^d}\Big\| \sum_{x\in\mathbb{T}^d_N} \nabla^N g\big(\eta(x)\big)\delta_{\frac{x}{N}} \Big\|_{TV} \\ &\le \frac{1}{N^{d-1}} \sum_{x\in\mathbb{T}^d_N}\sum_{j=1}^d \big|g\big(\eta(x+e_j)\big) - g\big(\eta(x)\big)\big| \le \frac{2d\cdot\|g'\|}{N^{d-1}} \sum_{x\in\mathbb{T}^d_N} \eta(x) \\ &= 2d\cdot\|g'\|_u N\langle 1,\pi^N\rangle. \end{aligned}$$

It is easy to see that the subspace Ω_N of the Skorohod space $D(0,T;\mathbb{M}^d_N)$ consisting of curves $\eta : [0,T] \longrightarrow \mathbb{M}^d_N$ satisfying $\langle 1,\pi^N_{\eta_t}\rangle = \langle 1,\pi^N_{\eta_0}\rangle$ is a closed, and thus Borel, subset of the Skorohod space $D(0,T;\mathbb{M}^d_N)$.

Proposition 4.2.8 *The set*

$$\Omega_N := \Big\{\eta \in D(\mathbb{R}_+;\mathbb{M}^d_N) \Big| \sum_{x\in\mathbb{T}^d_N} \eta_t(x) = \sum_{x\in\mathbb{T}^d_N} \eta_0(x),\ \forall\, t\in\mathbb{R}_+\Big\}$$

is a closed subset of the Skorohod space.

Proof Indeed, since $\mathbb{M}^d_N$ has the discrete topology, the function $m_{N,1} : \mathbb{M}^d_N \longrightarrow \mathbb{Z}_+$ given by

$$m_{N,1}(\eta) = \sum_{x\in\mathbb{T}^d_N} \eta(x)$$

is continuous. Then by corollary A.1.1 the induced function $\bar{m}_{N,1} : D(\mathbb{R}_+; \mathbb{M}_N^d) \longrightarrow D(\mathbb{R}_+; \mathbb{Z}_+)$ is continuous and since the subset $C(\mathbb{R}_+; \mathbb{Z}_+) \subseteq D(\mathbb{R}_+; \mathbb{Z}_+)$ consists exactly of the constant functions, the set

$$\Omega_N = \bar{m}_{N,1}^{-1}\big(C(\mathbb{R}_+; \mathbb{Z}_+)\big)$$

is closed as the inverse image of a closed set by a continuous map. $\square$

Therefore since the law P^N of the (diffusively rescaled) ZRP on the discrete torus $\mathbb{T}_N^d$ is concentrated on Ω, we can consider the empirical current process as a random variable

$$W^N : (\Omega, P^N|_\Omega) \longrightarrow L^\infty_{w^*}(0, T; \mathcal{M}^d)$$

since for any $\eta \in \Omega$ and any $t \in [0, T]$ we have $\|W_t^N\| \le 2dN\|g'\|_u\langle 1, \pi_{\eta_0}^N\rangle$, and thus

$$\|W^N(\eta)\|_{TV,\infty} \le 2dN\|g'\|_u\langle 1, \pi_{\eta_0}^N\rangle < +\infty$$

for all $\eta \in \Omega$. Although this definition makes sense, due to the factor N present in the last inequality one cannot obtain the tightness of the corresponding laws. To circumvent this difficulty we follow a standard strategy: We consider the current as taking values in a larger space with a weaker topology, prove the required relative compactness there, and then prove regularity results for the limiting objects that allow to conclude in the end that they are indeed vector valued measures. This is the object of the next section where we study the Kantorovich-Rubinstein measures.

4.2.3 Kantorovich-Rubinstein Vector Measures

We consider next for $M = \mathbb{R}^d$ or $\mathbb{T}^d$ the subspace

$$\mathcal{M}_0^d(M) = \big\{W \in \mathcal{M}^d(M) \,\big|\, W(\mathbb{T}^d) = 0\big\}$$

of all Borel currents on M with zero total current. In the case $M = \mathbb{R}^d$ we impose on elements of $\mathcal{M}_0^d(M)$ the additional requirement that they must have finite first moment. We note that by definition the empirical current function $W^N : \mathbb{M}_N^d \longrightarrow \mathcal{M}^d(\mathbb{T}^d)$ given by

$$W_\eta^N = -\nabla^N \sigma^N = N \sum_{j=1}^d (\tau_{\frac{e_j}{N}} \sigma^N - \sigma^N) e_j = \frac{1}{N^{d-1}} \sum_{x \in \mathbb{T}_N^d} \vec{W}_x^N \delta_{\frac{x}{N}}$$

takes in fact values in the subspace $\mathcal{M}_0^d(\mathbb{T}^d) \le \mathcal{M}^d(\mathbb{T}^d)$. We consider the subspace $\mathcal{M}_0^d(M)$ equipped with the *Kantorovich-Rubinstein norm* defined by

$$\|W\|_{KR} := \sup_{\|G\|_{\mathrm{Lip}} \le 1} \int G \cdot dW \tag{4.26}$$

where the supremum is taken over all $G \in \mathrm{Lip}(M; \mathbb{R}^d)$ with Lipschitz norm ≤ 1. As is well known, the Lipschitz norm $\|\cdot\|_{\mathrm{Lip}} : \mathrm{Lip}(\mathbb{T}^d; \mathbb{R}^d) \longrightarrow \mathbb{R}_+$ defined by

$$\|G\|_{\mathrm{Lip}} := \sup_{x \ne y} \frac{|G(x) - G(y)|}{d_M(x, y)}$$

defines a seminorm on $\mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$ with $\|G\|_{\mathrm{Lip}}=0$ iff $G\equiv c$, for some constant $c\in\mathbb{R}^d$, and therefore it induces a norm $\|\cdot\|_{\mathrm{Lip}_0}$ on the quotient space

$$\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d) := {}^{\mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)}/_{\mathbb{R}^d} \overset{\text{iso}}{=} \big\{G\in \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)\,\big|\, G(0)=0\big\}.$$

Also, since W is of zero total current the action of W on a function $G\in\mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$ through its integration depends only on the class of G modulo constants, that is if $G\equiv H+c$, $c\in\mathbb{R}^d$ then

$$\int_{\mathbb{T}^d} G\cdot dW = \int_{\mathbb{T}^d}(H+c)\cdot dW = \int_{\mathbb{T}^d} H\cdot dW + c\cdot W(\mathbb{T}^d) = \int_{\mathbb{T}^d} H\cdot dW$$

and so we can rewrite the KR-norm as

$$\|W\|_{KR} := \sup_{\substack{\|G\|_{\mathrm{Lip}}\le 1\\ G(0)=0}} \int G\cdot dW = \sup_{\|G\|_{\mathrm{Lip}_0}\le 1}\int G\cdot dW.$$

Note that the Kantorovich-Rubinstein norm on $\mathcal{M}_0^d(M)$ is exactly the norm that makes the injection

$$\mathcal{M}_0^d(M)\ni W\hookrightarrow i_W := \langle\cdot, W\rangle \in \mathrm{Lip}_0(M;\mathbb{R}^d)^*$$

defined by the pairing $\langle\cdot,\cdot\rangle : \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)\times\mathcal{M}_0^d \longrightarrow \mathbb{R}$ given by

$$\langle G, W\rangle = \int G\cdot dW$$

an isometry. However the injection $i:\mathcal{M}_0^d\hookrightarrow \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*$ thus defined does not have closed range in $\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*$ since $\mathcal{M}_0^d$ is not complete. Even worse, we have the isomorphism $\mathcal{M}_0^d(M)^* = \mathrm{Lip}_0(M;\mathbb{R}^d)$ and therefore by Goldstine's theorem, ([7], lemma 3.4) according to which every Banach space X is w^*-dense in its double dual X^{**}, it follows that the w^*-closure of $\mathcal{M}_0^d(M)$ in $\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^* = \mathcal{M}_0^d(M)^{**}$ is the whole space $\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*$.

Proposition 4.2.9 *The space $\mathcal{M}_0^d(M)$ is Lipschitz isomorphic to the product space*

$$\mathcal{M}_0^1(M)^d := \prod_{j=1}^d \mathcal{M}_0^1(M),$$

where $\mathcal{M}_0^1(M)$ is the space of all Borel charges on $M=\mathbb{R}^d$ or $\mathbb{T}^d$ with zero total charge equipped with the Kantorovich-Rubinstein norm

$$\|\mu\|_{KR} = \sup_{\substack{f\in\mathrm{Lip}_0(M;\mathbb{R})\\ \|f\|_{\mathrm{Lip}_0(M;\mathbb{R})}\le 1}} \int f d\mu,\quad \mu\in\mathcal{M}_0^1(M).$$

Proof Of course on the product space $\mathcal{M}_0^1(M)^d$ we can consider any of the p-product norms, $1\le p\le\infty$. Here we consider the ∞-product norm. We will show that the function $T:\mathcal{M}_0^d(M)\longrightarrow\mathcal{M}_0^1(M)^d$ given by the formula

$$T(W) = \big(\langle W, e_j\rangle\big)_{j=1}^d =: \big(T^j(W)\big)_{j=1}^d =: (W^j)_{j=1}^d$$

is a bi-Lispchitz isomorphism.

For each $j = 1, \dots, d$ and any $f \in \mathrm{Lip}_0(M;\mathbb{R})$ the function $F_j := f \cdot e_j$ belongs in $\mathrm{Lip}_0(M;\mathbb{R}^d)$ with $\|F_j\|_{\mathrm{Lip}_0(M;\mathbb{R}^d)} = \|f\|_{\mathrm{Lip}_0(M;\mathbb{R})}$ and

$$\int F_j \cdot dW = \int f dW^j.$$

Therefore we have that

$$\|T^j(W)\|_{KR} = \sup_{\substack{f \in \mathrm{Lip}_0(M;\mathbb{R}) \\ \|f\|_{\mathrm{Lip}_0(M;\mathbb{R})} \leq 1}} \int f dW^j \leq \sup_{\substack{F \in \mathrm{Lip}_0(M;\mathbb{R}^d) \\ \|F\|_{\mathrm{Lip}_0(M;\mathbb{R}^d)} \leq 1}} \int F \cdot dW = \|W\|_{KR}.$$

On the other hand, since for all $F \in \mathrm{Lip}_0(M;\mathbb{R}^d)$ and all $j = 1, \dots, d$ we have that $\|F^j\|_{\mathrm{Lip}_0(M;\mathbb{R})} \leq \|F\|_{\mathrm{Lip}_0(M;\mathbb{R}^d)}$, we have that

$$\begin{aligned} \|W\|_{KR} &= \sup_{\substack{F \in \mathrm{Lip}_0(M;\mathbb{R}^d) \\ \|F\|_{\mathrm{Lip}_0(M;\mathbb{R}^d)} \leq 1}} \sum_{j=1}^{d} \int F^j dW^j \leq \sum_{j=1}^{d} \sup_{\substack{F \in \mathrm{Lip}_0(M;\mathbb{R}^d) \\ \|F\|_{\mathrm{Lip}_0(M;\mathbb{R}^d)} \leq 1}} \int F^j dW^j \\ &\leq \sum_{j=1}^{d} \sup_{\substack{f \in \mathrm{Lip}_0(M;\mathbb{R}) \\ \|f\|_{\mathrm{Lip}_0(M;\mathbb{R})} \leq 1}} \int f dW^j = \sum_{j=1}^{d} \|W^j\|_{KR}. \end{aligned}$$

Therefore it follows that

$$\max_{j=1,\dots,d} \|W^j\|_{KR} \leq \|W\|_{KR} \leq d \max_{j=1,\dots,d} \|W^j\|_{KR}$$

and the proof is complete. □

Proposition 4.2.10 *For all $W \in \mathcal{M}_0^d(\mathbb{T}^d)$ the supremum*

$$\|W\|_{KR} = \sup_{\|G\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)} \leq 1} \int G \cdot dW \tag{4.27}$$

is attained.

Proof Let $W \in \mathcal{M}_0^d(\mathbb{T}^d)$ and let $\{G_k\} \subseteq \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ be a maximizing sequence for the supremum (4.27), i.e. $\sup_{k\in\mathbb{N}} \|G_k\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)} \leq 1$ and

$$\lim_{k\to\infty} \int G_k \cdot dW = \|W\|_{KR}.$$

Since we are dealing with functions modulo constants we can assume that $G_k(0) = 0$ for all $k \in \mathbb{N}$. Then for all $x \in \mathbb{T}^d$ we have that $\{G_k(x)\}_{k\in\mathbb{N}} \subseteq [-1, 1]$ and since the family $\{G_k\}_{k\in\mathbb{N}}$ is uniformly Lipschitz, it is equicontinuous and therefore by the Arzela-Ascoli theorem there exists a subsequence $\{G_{n_k}\}$ of $\{G_k\}$ and $G \in C(\mathbb{T}^d;\mathbb{R}^d)$ such that $G_{n_k} \longrightarrow G$ uniformly as $n \to \infty$. Furthermore since $\{G_k\}$ is uniformly 1-Lipschitz and $G_{n_k} \longrightarrow G$ uniformly it follows that G is 1-Lipschitz. Therefore,

$$\|W\|_{KR} = \int G \cdot dW$$

and the supremum is attained at $G = \lim_{k\to\infty} G_{n_k}$. $\square$

In what follows we will consider the torus $\mathbb{T}^d$ as the quotient space ${}^{\mathbb{R}^d}/_{\mathbb{Z}^d} \cong \mathbb{T}^d$. In this way the torus inherits a natural group structure as a quotient of Abelian groups with respect to which it is a smooth Lie manifold. The universal covering $p : \mathbb{R}^d \longrightarrow \mathbb{T}^d$ of the torus is given in this interpretation of the torus by

$$p(x) = x + \mathbb{Z}^d$$

and it is obviously a homomorphism of groups. The torus can be then equipped with the quotient group norm

$$|x|_{\mathbb{T}^d} := \min_{\widetilde{x}\in x+\mathbb{Z}^d} |\widetilde{x}|_2$$

and the corresponding metric

$$d_{\mathbb{T}^d}(x, y) := |x - y|_{\mathbb{T}^d}$$

coincides with the metric resulting from the Riemannian structure of the torus with diameter $\frac{1}{2}$. It is obvious that the universal covering is a local isometry since whenever $\widetilde{x} \in [-\frac{1}{2}, \frac{1}{2})^d \subseteq \mathbb{R}^d$ we obviously have that

$$|p(\widetilde{x})|_{\mathbb{T}^d} = \min_{z\in\widetilde{x}+\mathbb{Z}^d} |z|_2 = |\widetilde{x}|_2.$$

Proposition 4.2.11 *Let $p : \mathbb{R}^d \longrightarrow \mathbb{T}^d \cong {}^{\mathbb{R}^d}/_{\mathbb{Z}^d}$ be the universal covering of the torus, $p(x) = x + \mathbb{Z}^d$. Then the induced linear function $\widetilde{p} : \mathrm{Lip}_0(\mathbb{T}^d; \mathbb{R}^d) \longrightarrow \mathrm{Lip}_0(\mathbb{R}^d; \mathbb{R}^d)$ via the formula*

$$\widetilde{p}([F]) = [F \circ p]$$

is an isometric injection.

Proof We consider first the mapping $\bar{p} : \mathrm{Lip}(\mathbb{T}^d; \mathbb{R}^d) \longrightarrow \mathrm{Lip}(\mathbb{R}^d; \mathbb{R}^d)$ given by $\bar{p}(F) = F \circ p$. The covering p is 1-Lipschitz, that is

$$d_{\mathbb{T}^d}\big(p(x), p(y)\big) \leq |x - y|$$

for all $x, y \in \mathbb{R}^d$ and therefore $\bar{p}$ is well defined, i.e. $\bar{p}(F)$ is Lipschitz whenever F is Lipschitz. Furthermore, whenever $F, G \in \mathrm{Lip}(\mathbb{T}^d; \mathbb{R}^d)$ are such that $F - G \equiv c \in \mathbb{R}$, we obviously have that $\bar{p}(F) - \bar{p}(G) \equiv c \in \mathbb{R}$ and therefore $\bar{p}$ induces the well defined map $\widetilde{p} : \mathrm{Lip}_0(\mathbb{T}^d; \mathbb{R}^d) \longrightarrow \mathrm{Lip}_0(\mathbb{R}^d; \mathbb{R}^d)$ on the respective quotient spaces modulo constants.

We will prove next that $\widetilde{p}$ is an isometric injection. First, $\widetilde{p}$ is obviously injective, and for all $F \in \mathrm{Lip}_0(\mathbb{T}^d; \mathbb{R}^d)$ and all $x, y \in \mathbb{R}^d$ we have that

$$\begin{aligned} \|\widetilde{p}(F)\|_{\mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)} &= \sup_{\substack{x,y\in\mathbb{R}^d\\ x\neq y}} \frac{|F\circ p(x) - F\circ p(y)|}{|x-y|} \leq \sup_{\substack{x,y\in\mathbb{R}^d\\ x\neq y}} \frac{|F\circ p(x) - F\circ p(y)|}{d_{\mathbb{T}^d}\big(p(x), p(y)\big)} \\ &= \|F\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)} \end{aligned}$$

thus $\widetilde{p}$ is a contraction.

It remains to show the converse inequality. The universal covering $p : \mathbb{R}^d \longrightarrow \mathbb{T}^d$ is a local isometry, since whenever $x, y \in \mathbb{R}^d$ are such that $|x-y| \leq \frac{1}{2}$ (here we consider the circle $\mathbb{S}^1$ as having circumference length $= 1$) then $d_{\mathbb{T}^d}\big(p(x), p(y)\big) = |x-y|$. Therefore

$$\begin{aligned}
\|\widetilde{p}(F)\|_{\mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)} &= \sup_{\substack{x,y\in\mathbb{R}^d \\ x\neq y}} \frac{|\widetilde{p}(F)(x) - \widetilde{p}(F)(y)|}{|x-y|} \geq \sup_{\substack{x,y\in[0,2)^d \\ 0<|x-y|\leq\frac{1}{2}}} \frac{|\widetilde{p}(F)(x) - \widetilde{p}(F)(y)|}{|x-y|} \\
&= \sup_{\substack{x,y\in[0,2)^d \\ 0<|x-y|\leq\frac{1}{2}}} \frac{|F\big(p(x)\big) - F\big(p(y)\big)|}{d_{\mathbb{T}^d}\big(p(x),p(y)\big)} = \sup_{\substack{x,y\in\mathbb{T}^d \\ x\neq y}} \frac{|F(x)-F(y)|}{d_{\mathbb{T}^d}(x,y)} \\
&= \|F\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)},
\end{aligned}$$

as required, which completes the proof. □

The range $\widetilde{\mathrm{Lip}}_0(\mathbb{T}^d;\mathbb{R}^d) := \widetilde{p}\big(\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)\big)$ of $\widetilde{p}$ is exactly the space of all $\mathbb{Z}^d$-periodic Lipschitz functions $F \in \mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)$, a closed subspace of $\mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)$. Therefore if we consider $\widetilde{p}$ as taking values in $\widetilde{\mathrm{Lip}}_0(\mathbb{T}^d;\mathbb{R}^d)$ then it is invertible. We will show that the inverse $\widetilde{p}^{\,-1} : \widetilde{\mathrm{Lip}}_0(\mathbb{T}^d;\mathbb{R}^d) \longrightarrow \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ is induced by the Borel injection $i : \mathbb{T}^d \cong [0,1)^d \hookrightarrow \mathbb{R}^d$. For this it suffices to check that the induced operator

$$\widetilde{i} : \widetilde{\mathrm{Lip}}_0(\mathbb{T}^d;\mathbb{R}^d) \longrightarrow \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d) \tag{4.28}$$

via $\widetilde{i}(F) = F \circ i$ is well defined, i.e. that whenever $F \in \mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)$ is $\mathbb{Z}^d$-periodic then $F \circ i : \mathbb{T}^d \longrightarrow \mathbb{R}^d$ is Lipschitz, for then we would have that

$$\widetilde{p}(\widetilde{i}(F)) = F \circ p \circ i = F$$

for all $F \in \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ since $p \circ i = id_{\mathbb{T}^d}$ and

$$\widetilde{i}(\widetilde{p}(F)) = F \circ i \circ p = F$$

for all $F \in \widetilde{\mathrm{Lip}}_0(\mathbb{T}^d;\mathbb{R}^d)$ since $\widetilde{\mathrm{Lip}}_0(\mathbb{T}^d;\mathbb{R}^d)$ consists exactly of the $\mathbb{Z}^d$-periodic Lipschitz functions of $\mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)$. Now, the operator $\widetilde{i}$ in (4.28) is indeed well defined since if $F \in \widetilde{\mathrm{Lip}}_0(\mathbb{T}^d;\mathbb{R}^d)$ then F is $\mathbb{Z}^d$-periodic and therefore $F = F \circ i \circ p$. So we can write $F = \widetilde{p}(F \circ i) = \widetilde{p}(\widetilde{i}(F))$ and since $\widetilde{p}$ preserves the norms (even if we allow the domain and target spaces to be the spaces ${}^{B(\mathbb{T}^d;\mathbb{R}^d)}/_{\mathbb{R}}$ and ${}^{B(\mathbb{R}^d;\mathbb{R}^d)}/_{\mathbb{R}}$ and the Lipschitz norms to take the value $+\infty$) we have that

$$+\infty > \|F\|_{\mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)} = \|\widetilde{p}(\widetilde{i}(F))\|_{\mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)} = \|\widetilde{i}(F)\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)}$$

and therefore $\widetilde{i}(F)$ is Lipschitz whenever $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is Lipschitz and $\mathbb{Z}^d$-periodic as required.

Furthermore, by the representation of the dual of subspaces we have that

$$\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^* \cong \widetilde{\mathrm{Lip}}_0(\mathbb{T}^d;\mathbb{R}^d)^* \cong {}^{\mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)^*}/_{\widetilde{\mathrm{Lip}}_0(\mathbb{T}^d;\mathbb{R}^d)^\perp}$$

and therefore when considering functions $F \in \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ as $\mathbb{Z}^d$-periodic functions in $\mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)$ we can act on them by equivalence classes of linear functionals in

$\operatorname{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)^*$, with two such functionals being equivalent iff their difference vanishes on all $\mathbb{Z}^d$-periodic functions.

Furthermore, since $\widetilde{p}$ is an isometric injection, by the Hahn-Banach theorem its dual

$$\widetilde{p}^* : \operatorname{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)^* \longrightarrow \operatorname{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*$$

is a surjective contraction. Of course on the domain $\mathcal{M}_0^d(\mathbb{R}^d) \le \operatorname{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)^*$ the operator $\widetilde{p}^*$ restricts to the push forward operator $p_* : \mathcal{M}_0^d(\mathbb{R}^d) \longrightarrow \mathcal{M}_0^d(\mathbb{T}^d)$ and thus the push forward p_* is a surjective contraction with respect to Kantorovich-Rubinstein norms. Since $\widetilde{p}^*$ is an extension of p_*, we write $\widetilde{p}^* = p_*$.

A space that will be especially useful in the considerations regarding the continuity equation is the Banach space $C^1(\mathbb{T}^d;\mathbb{R}^d)$ equipped with the uniform C^1-norm $\|\cdot\|_{C^1}$ given by

$$\|G\|_{C^1} = \|G\|_\infty + \|DG\|_\infty := \|G\|_\infty + \big\||DG|_{Fr}\|_\infty, \quad G \in C^1(\mathbb{T}^d;\mathbb{R}^d).$$

Here, taking advantage of the interpretation of functions $G \in C^1(\mathbb{T}^d;\mathbb{R}^d)$ as $\mathbb{Z}^d$-periodic functions on Euclidean space, for $x \in \mathbb{T}^d$ we set

$$DG(x) = D(G \circ p)(\widetilde{x}) \quad \text{for any } \widetilde{x} \in \mathbb{R}^d \text{ such that } p(\widetilde{x}) = x$$

where p is the universal covering of the torus and for any matrix $A = (a_{ij}) \in \mathbb{R}^{d\times d}$ we denote by $|A|_{Fr}$ its Frobenius norm,

$$|A|_{\mathrm{Fr}} := \operatorname{tr}(A^T A)^{\frac{1}{2}} = \Big(\sum_{i,j=1}^{d} |a_{ij}|^2\Big)^{\frac{1}{2}}.$$

Of course since $G \in C^1(\mathbb{T}^d;\mathbb{R}^d)$ the function $\mathbb{R}^d \ni \widetilde{x} \mapsto |D(G\circ p)(\widetilde{x})|_{\mathrm{Fr}}$ is continuous and periodic and thus bounded. So the norm $\|\cdot\|_{C^1}$ is well defined. We note also that the Frobenius norm on $\mathbb{R}^{d\times d}$ is induced by the Euclidean inner product $\langle\cdot,\cdot\rangle_F$ on $\mathbb{R}^{d\times d}$ given by

$$\langle A, B\rangle_F = \operatorname{tr}(A^T B) = \sum_{i,j=1}^{d} a_{ij} b_{ij}$$

for all $A, B \in \mathbb{R}^{d\times d}$. Furthermore, by the Cauchy-Schwartz inequality, for any $A \in \mathbb{R}^{d\times d}$, $x \in \mathbb{R}^d$,

$$|Ax|_2^2 = \sum_{i=1}^{d}\Big|\sum_{j=1}^{d} a_{ij}x_j\Big|^2 \le \sum_{i=1}^{d}\Big(\sum_{j=1}^{d} a_{ij}^2\Big)\Big(\sum_{j=1}^{d} x_j^2\Big) = |A|_{\mathrm{Fr}}^2 |x|_2^2. \tag{4.29}$$

The following lemma is an adaptation of a similar lemma for real valued functions in compact subsets of $\mathbb{R}^d$ found in [6].

Lemma 4.2.2 *There exists a family* $\{S_\varepsilon\}_{\varepsilon>0}$ *of linear operators*

$$S_\varepsilon : C(\mathbb{T}^d;\mathbb{R}^d) \longrightarrow C^\infty(\mathbb{T}^d;\mathbb{R}^d)$$

and constants $C = C(\rho, d), C_\varepsilon = C_\varepsilon(\rho, d) \ge 0$, $\varepsilon > 0$, *such that* $S_\varepsilon F \overset{\varepsilon\to 0}{\longrightarrow} F$ *uniformly for all* $F \in C(\mathbb{T}^d;\mathbb{R}^d)$ *and*

(a) $\|S_\varepsilon F\|_{\mathrm{Lip}} \le C_\varepsilon \|F\|_u$, *for all* $F \in C(\mathbb{T}^d;\mathbb{R}^d)$ *and all* $\varepsilon > 0$,

(b) $\|S_\varepsilon F\|_{\mathrm{Lip}} \le \|F\|_{\mathrm{Lip}}$ *for all* $F \in \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$ *and all* $\varepsilon > 0$,

(c) $\|S_\varepsilon F - F\|_u \le \varepsilon C \|F\|_{\mathrm{Lip}}$, *for all* $F \in \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$ *and all* $\varepsilon > 0$.

(d) $\|S_\varepsilon F\|_{C^1} \le \|F\|_{C^1}$ *for all* $F \in C^1(\mathbb{T}^d;\mathbb{R}^d)$ *and all* $\varepsilon > 0$.

Proof We define a smooth approximation $(\rho_\varepsilon)_{\varepsilon\in(0,1)} \subseteq C^\infty(\mathbb{T}^d)$ of the identity via convolution through the usual molifiers. Let $\rho : \mathbb{R}^d \longrightarrow [0,1]$ be a radially symmetric C^∞ function such that $\rho(0) = 0$, $\mathrm{supp}\rho \subseteq D(0,\frac{1}{2})$ and $\int_{\mathbb{R}^d} \rho = 1$. For all $\varepsilon \in (0,1)$ we define $\rho_\varepsilon : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ by the formula

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\Big(\frac{x}{\varepsilon}\Big).$$

Then $\mathrm{supp}\rho_\varepsilon \subseteq D(0,\frac{\varepsilon}{2})$, for all $\varepsilon \in (0,1)$. We define $\widetilde{S}_\varepsilon : C(\mathbb{R}^d;\mathbb{R}^d) \longrightarrow C^\infty(\mathbb{R}^d;\mathbb{R}^d)$ by

$$\widetilde{S}_\varepsilon F = F * \rho_\varepsilon,$$

that is $S_\varepsilon F$ is given by the formula

$$S_\varepsilon F(x) = \int_{\mathbb{R}^d} F(x-y)\rho_\varepsilon(y)dy = \int_{D(0,\frac{\varepsilon}{2})} F(x-y)\rho_\varepsilon(y)dy = \int_{D(x,\frac{\varepsilon}{2})} F(y)\rho_\varepsilon(x-y)dy.$$

Of course by standard results in convolutions $\widetilde{S}_\varepsilon$ takes values in the space of smooth functions. Indeed, for all $x \in \mathbb{R}^d$ and $h > 0$ we have that

$$\begin{aligned}\frac{F*\rho_\varepsilon(x+he_j) - F*\rho_\varepsilon(x)}{h} &= \int_{\mathbb{R}^d} \frac{F(x+he_j-y) - F(x-y)}{h}\rho_\varepsilon(y)dy \\ &= \int_{\mathbb{R}^d} F(y)\frac{\rho_\varepsilon(x+he_j-y) - \rho_\varepsilon(x-y)}{h}dy \\ &= \int_{K_x} F(y)\frac{\rho_\varepsilon(x+he_j-y) - \rho_\varepsilon(x-y)}{h}dy,\end{aligned}$$

for some compact ball K_x such that $D(x,\frac{\varepsilon}{2}) \cup D(x+he_j,\frac{\varepsilon}{2}) \subseteq K_x$ for all $h \in (0,1)$. Now,

$$\rho_\varepsilon(x+he_j-y) - \rho_\varepsilon(x-y) = \tau_{he_j}(\tau_x\rho_\varepsilon)(-y) - (\tau_x\rho_\varepsilon)(-y)$$

and since $\tau_x\rho_\varepsilon \in C_c^2(\mathbb{R}^d)$ we have that

$$\frac{\tau_{he_j}\tau_x\rho_\varepsilon - \tau_x\rho_\varepsilon}{h} \overset{h\to 0}{\longrightarrow} \partial_j\tau_x\rho_\varepsilon = \tau_x\partial_j\rho_\varepsilon \quad \text{uniformly.}$$

It follows that the limit

$$\begin{aligned}\partial_j(F*\rho_\varepsilon)(x) &= \lim_{h\to 0}\frac{F*\rho_\varepsilon(x+he_j) - F*\rho_\varepsilon(x)}{h} = \int_{K_x} F(y)\tau_x\partial_j\rho_\varepsilon(-y)dy \\ &= \int_{D(x,\frac{\varepsilon}{2})} F(y)\partial_j\rho_\varepsilon(x-y)dy\end{aligned}$$

exists for all $x \in \mathbb{R}^d$. In the same way one proves by induction that the partial derivative

$$\partial^a(F*\rho_\varepsilon)(x) = \int_{D(x,\frac{\varepsilon}{2})} F(y)\partial^a\rho_\varepsilon(x-y)dy$$

exists for all $x \in \mathbb{R}^d$ and all multi-indices $a \in \mathbb{Z}_+^d$, and thus $\widetilde{S}_\varepsilon F$ is C^∞.

Next, it is easy to see that $\widetilde{S}_\varepsilon$ maps $\mathbb{Z}^d$-periodic functions to $\mathbb{Z}^d$-periodic functions. Indeed, if $F \in C(\mathbb{T}^d;\mathbb{R}^d)$ is $\mathbb{Z}^d$-periodic and $k \in \mathbb{Z}^d$, then by the definition of $S_\varepsilon F \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ we have that for all $x \in \mathbb{R}^d$,

$$S_\varepsilon F(x+k) = \int_{\mathbb{R}^d} F(x+k-y)\rho_\varepsilon(y)dy = \int_{\mathbb{R}^d} F(x-y)\rho_\varepsilon(y)dy = S_\varepsilon F(x),$$

and thus $S_\varepsilon F$ is $\mathbb{Z}^d$-periodic. Therefore the restriction

$$S_\varepsilon := \widetilde{S}_\varepsilon|_{C(\mathbb{T}^d;\mathbb{R}^d)} : C(\mathbb{T}^d;\mathbb{R}^d) \longrightarrow C^\infty(\mathbb{T}^d;\mathbb{R}^d)$$

gives a well defined function.

We check next that $S_\varepsilon F \longrightarrow F$ uniformly. So let $F \in C(\mathbb{T}^d;\mathbb{R}^d)$. For all $x \in \mathbb{T}^d$, $\varepsilon > 0$, we have that

$$\begin{aligned} S_\varepsilon F(x) - F(x) &= \int \big(F(x-y) - F(x)\big)\rho_\varepsilon(y)dy \\ &= \int \big(\tau_{-\varepsilon y}F(x) - F(x)\big)\rho(y)dy \end{aligned}$$

and therefore

$$\|S_\varepsilon F - F\|_u \leq \int \|\tau_{-\varepsilon y}F - F\|_u \rho(y)dy.$$

Let $f_\varepsilon : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ denote the function $f_\varepsilon(y) = \|\tau_{-\varepsilon y}F - F\|_u$. Obviously the family $\{f_\varepsilon\}_{\varepsilon\in(0,1)}$ is uniformly bounded by $2\|F\|_u$. Furthermore, $f_\varepsilon \longrightarrow 0$ pointwise. Indeed, for fixed $y \in \mathbb{R}^d \setminus \{0\}$, we have that

$$f_\varepsilon(y) = \sup_{x\in[-\frac{1}{2},\frac{1}{2}]^d} |F(x-\varepsilon y) - F(x)|.$$

But since F is continuous, it is uniformly continuous on, say $[-1,1]^d$, and therefore for each $\gamma > 0$ there exists $\delta_\gamma > 0$ such that

$$x, z \in [-1,1]^d, \quad |z - x| < \delta_\gamma \quad \Longrightarrow \quad |F(z) - F(x)| < \gamma.$$

So if we choose $\varepsilon_{\gamma,y} > 0$ such that $\varepsilon_{\gamma,y} < \delta_\gamma/\|y\|$, then for all $x \in [-\frac{1}{2},\frac{1}{2})^d$ and all $0 < \varepsilon < \varepsilon_{\gamma,y}$ we have that $|F(x-\varepsilon y) - F(x)| < \gamma$, and so for all $\varepsilon \in (0,\varepsilon_{\gamma,y})$ we have that $f_\varepsilon(y) < \gamma$. Since $\gamma > 0$ was arbitrary, $\lim_{\varepsilon\to 0} f_\varepsilon(y) = 0$, as required.

It is easy to see that in the case that in addition $F \in \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$, one has the estimate

$$|\tau_{-\varepsilon y}F(x) - F(x)| \leq \varepsilon\|F\|_{\mathrm{Lip}}|y|$$

for all $x \in \mathbb{R}^d$. Therefore, $\|\tau_{-\varepsilon y}F - F\|_u \leq \varepsilon\|F\|_{\mathrm{Lip}}|y|$ for all $y \in \mathbb{R}^d$, and

$$\|S_\varepsilon F - F\|_u \leq \varepsilon\|F\|_{\mathrm{Lip}} \int_{\mathbb{R}^d} |y|\rho(y)dy,$$

which proves that the operators S_ε, $\varepsilon > 0$, satisfy (c) with constant $C = \int_{\mathbb{R}^d} |\cdot|\rho \geq 0$.
(b) Let $F \in \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$. Then for all $x, y \in \mathbb{R}^d$ and all $\varepsilon \in (0,1)$ we have that

$$\begin{aligned}|S_\varepsilon F(y) - S_\varepsilon F(x)| &= \left| \int \big(F(y-z) - F(x-z)\big)\rho_\varepsilon(z)dz \right| \\ &\leq \int |F(y-z) - F(x-z)|\rho_\varepsilon(z)dz \leq \|F\|_{\mathrm{Lip}}|y-x|,\end{aligned}$$

which implies that $\|S_\varepsilon F\|_{\mathrm{Lip}} \leq \|F\|_{\mathrm{Lip}}$ for all $\varepsilon \in (0,1)$, as required.
(a) Let $F \in C(\mathbb{T}^d;\mathbb{R}^d)$. Then for all $x, y \in \mathbb{R}^d$ and all $\varepsilon \in (0,1)$ we have that

$$S_\varepsilon F(y) - S_\varepsilon F(x) = \int F(z)\big(\rho_\varepsilon(y-z) - \rho_\varepsilon(x-z)\big)dz.$$

Now, by proposition 4.2.11 we have that

$$\|S_\varepsilon F\|_{\mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)} = \sup_{\substack{x,y\in[-1,1]^d \\ |x-y|\leq\frac{1}{2}}} \frac{S_\varepsilon F(y) - S_\varepsilon F(x)}{|x-y|},$$

and for all $x, y \in [-1,1]^d$ we have that

$$\begin{aligned}S_\varepsilon F(y) - S_\varepsilon F(x) &\leq \|F\|_u \int_{D(x,\frac{\varepsilon}{2})\cup D(y,\frac{\varepsilon}{2})} |\rho_\varepsilon(y-z) - \rho_\varepsilon(x-z)|dz \\ &\leq \|\rho_\varepsilon\|_{\mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^d)}\|F\|_u|x-y|m_{\mathbb{R}^d}\left(D\Big(x,\frac{\varepsilon}{2}\Big) \cup D\Big(y,\frac{\varepsilon}{2}\Big)\right) \\ &\leq \|\rho_\varepsilon\|_{\mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^d)}m_{\mathbb{R}^d}\left(D\Big([-1,1]^d,\frac{1}{2}\Big)\right)\|F\|_u|x-y| \\ &\leq 3^d\|\rho_\varepsilon\|_{\mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^d)}\|F\|_u|x-y|,\end{aligned}$$

which proves that (a) holds with constant $C_\varepsilon := 3^d\|\rho_\varepsilon\|_{\mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^d)} \geq 0$.
(d). Let $F \in C^1(\mathbb{T}^d;\mathbb{R}^d)$. Then of course $\|S_\varepsilon F\|_u \leq \|F\|_u$ for all $\varepsilon > 0$ and since $F \in C^1(\mathbb{T}^d;\mathbb{R}^d)$ we have that $D(S_\varepsilon F) = S_\varepsilon DF$ for all $\varepsilon \in (0,1)$, and therefore

$$\|S_\varepsilon F\|_{C^1} = \|S_\varepsilon F\|_u + \|D(S_\varepsilon F)\|_u = \|S_\varepsilon F\|_u + \|S_\varepsilon(DF)\|_u \leq \|F\|_u + \|DF\|_u = \|F\|_{C^1}$$

for all $F \in C^1(\mathbb{T}^d;\mathbb{R}^d)$ and all $\varepsilon \in (0,1)$, as required. □

Let $\widehat{\mathcal{M}}_0^d(M)$ denote the completion of $\mathcal{M}_0^d(M)$ with respect to the Kantorovich-Rubinstein norm or equivalently the norm-closure of $\mathcal{M}_0^d(M)$ in $\mathrm{Lip}_0(M;\mathbb{R}^d)^*$. Elements of $\widehat{\mathcal{M}}_0^d(\mathbb{T}^d)$ can be characterized by the following continuity property. Recall by functional analysis that for any Banach space X we have $(X^*, w^*)^* = X \leq X^{**}$. In our case, due to the isomorphism $\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d) = \widehat{\mathcal{M}}_0^d(\mathbb{T}^d)^*$ we have that

$$\widehat{\mathcal{M}}_0^d(\mathbb{T}^d) = \big(\widehat{\mathcal{M}}_0^d(\mathbb{T}^d)^*, w^*\big)^* \subseteq \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*$$

Proposition 4.2.12 *Let $W \in \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*$. Then $W \in \widehat{\mathcal{M}}_0^d(\mathbb{T}^d)$ iff for all sequences $\{F_n\}_{n\in\mathbb{N}} \subseteq \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$ the following implication holds:*

$$\|F_n - c\|_u \longrightarrow 0, \quad c \in \mathbb{R}^d, \quad \sup_{n\in\mathbb{N}} \|F_n\|_{\mathrm{Lip}} < +\infty \quad \Longrightarrow \quad \lim_{n\to\infty} \langle W, F_n\rangle = 0. \quad (4.30)$$

Proof We have to prove that W satisfies (4.30). So let $\{F_n\} \subseteq C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ be such that $\lim_{n\to\infty} \|F_n - c\|_u = 0$ for some $c \in \mathbb{R}^d$ and $C := \sup_{n\in\mathbb{N}} \|F_n\|_{\mathrm{Lip}} < +\infty$. By definition, for each $\varepsilon > 0$ there exists W^ε such that $\|W - W^\varepsilon\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*} < \varepsilon$. Then for all $n \in \mathbb{N}$, $\varepsilon > 0$, we have that

$$|\langle T_W, F_n\rangle| \leq |\langle W^\varepsilon, F_n\rangle| + |\langle W - W^\varepsilon, F_n\rangle| \leq |\langle W^\varepsilon, F_n\rangle| + C\varepsilon.$$

But since $W^\varepsilon \in \mathcal{M}_0^d(\mathbb{T}^d)$, we have that

$$\lim_{n\to\infty} \langle W^\varepsilon, F_n\rangle = \langle W^\varepsilon, c\rangle = 0,$$

and therefore

$$\limsup_{n\to\infty} |\langle W, F_n\rangle| \leq C\varepsilon,$$

which since $\varepsilon > 0$ was arbitrary proves that $\langle W, F_n\rangle \longrightarrow 0$.

We prove now the converse. Let $S_\varepsilon : C(\mathbb{T}^d;\mathbb{R}^d) \longrightarrow C^\infty(\mathbb{T}^d;\mathbb{R}^d)$, $\varepsilon \in (0,1)$, be the family of operators of lemma 4.2.12. We set $W^\varepsilon := W \circ S_\varepsilon$ for all $\varepsilon \in (0,1)$. Then by property (a) of the operators S_ε we have that for all $F \in C(\mathbb{T}^d;\mathbb{R}^d)$

$$\langle W^\varepsilon, F\rangle = \langle W, S_\varepsilon F\rangle \leq \|W\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*} \|S_\varepsilon F\|_{\mathrm{Lip}} \leq C_\varepsilon \|W\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*} \|F\|_u.$$

Therefore, for all $\varepsilon \in (0,1)$, W_ε defines a bounded linear function on $C(\mathbb{T}^d;\mathbb{R}^d)$ and can thus be identified with a measure in $\mathcal{M}(\mathbb{T}^d;\mathbb{R}^d)$ which we continue to denote by W_ε, with $\|W_\varepsilon\|_{TV} \leq C_\varepsilon \|W\|_{\mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)^*}$. Furthermore, since by definition the constants $c \in \mathbb{R}^d$ are invariant by the action of approximation operators S_ε, i.e. $S_\varepsilon c = c$ for all $c \in \mathbb{R}^d$, $\varepsilon \in (0,1)$, we have that

$$\langle W_\varepsilon, c\rangle = \langle W, c\rangle = 0$$

for all $c \in \mathbb{R}^d$ and thus $W_\varepsilon \in \mathcal{M}_0^d(\mathbb{T}^d)$. Furthermore, by property (b) of the operators S_ε it easily follows that $\|W_\varepsilon\|_{KR} \leq \|W\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*}$ for all $\varepsilon \in (0,1)$, since for all $F \in \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$ we have that

$$\langle W_\varepsilon, F\rangle = \langle W, S_\varepsilon F\rangle \leq \|W\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*} \|S_\varepsilon F\|_{\mathrm{Lip}} \leq \|W\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*} \|F\|_{\mathrm{Lip}}.$$

Now, by definition, for each $\varepsilon \in (0,1)$ there exists $F_\varepsilon \in \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$ with $\|F_\varepsilon\|_{\mathrm{Lip}} \leq 1$ such that

$$\|W_\varepsilon - W\|_{\mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)^*} \leq \langle W_\varepsilon - W, F_\varepsilon\rangle + \varepsilon = \langle W, S_\varepsilon F_\varepsilon - F_\varepsilon\rangle + \varepsilon. \tag{4.31}$$

But by properties (b) and (c) of the operators S_ε we have that

$$\sup_{\varepsilon\in(0,1)} \|S_\varepsilon F_\varepsilon - F_\varepsilon\|_{\mathrm{Lip}} \leq 2 \quad \text{and} \quad \|S_\varepsilon F_\varepsilon - F_\varepsilon\|_u \leq C\varepsilon \|F_\varepsilon\|_{\mathrm{Lip}} \leq C\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} 0,$$

which, since W is assumed to satisfy implication (4.30), proves according to (4.31) that $\|W_\varepsilon - W\|_{\mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)^*} \longrightarrow 0$ and thus $W \in \widehat{\mathcal{M}}_0^d(\mathbb{T}^d)$ as required. □

According to the following proposition, every $W \in \widehat{\mathcal{M}}_0^d(\mathbb{T}^d)$, is determined uniquely by its action on C^1-functions.

Corollary 4.2.1 *The linear operator* $T : \widehat{\mathcal{M}}_0^d(\mathbb{T}^d) \longrightarrow \big(C^1(\mathbb{T}^d;\mathbb{R}^d), \|\cdot\|_{C^1}\big)^*$ *defined through restriction by* $T_W := W|_{C^1(\mathbb{T}^d;\mathbb{R}^d)}$ *is injective contraction.*

Proof We suppose that $W|_{C^1(\mathbb{T}^d;\mathbb{R}^d)} \equiv 0$ and we will prove that $\langle W, F\rangle = 0$ for all $F \in \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$. So let $F \in \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$. We consider then the sequence $\{F_n := S_{\frac{1}{n}} F\}_{n\in\mathbb{N}} \subseteq C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ given by lemma 4.2.2. Then by lemma 4.2.2 (b) the sequence $\{F_n - F\} \subseteq \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$ satisfies

$$\sup_{n\in\mathbb{N}} \|F_n - F\|_{\mathrm{Lip}} \leq 2\|F\|_{\mathrm{Lip}}$$

and by the same lemma $F_n - F \longrightarrow 0$ uniformly as $n \to \infty$. Therefore, by implication (4.30), we have that

$$\lim_{n\to\infty} \langle W, F_n - F\rangle = 0.$$

Since by assumption $\langle W, F_n\rangle = 0$ for all $n \in \mathbb{N}$ this proves that $\langle W, F\rangle = 0$ as required.

We prove next that T is indeed a contraction. But this easy, since for all $(F, W) \in C^1(\mathbb{T}^d;\mathbb{R}^d) \times \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*$ we have that

$$\langle F, W\rangle \leq \|F\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)} \|W\|_{KR} \leq \|F\|_{C^1} \|W\|_{KR}$$

and therefore

$$\|T_W\|^*_{C^1} := \sup_{\|F\|_{C^1(\mathbb{T}^d;\mathbb{R}^d)} \leq 1} \langle F, W\rangle \leq \|W\|_{KR}$$

for all $W \in \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$. So T is a bounded operator of norm $\|T\| \leq 1$. □

Proposition 4.2.13 *Let* $L \in C^1(\mathbb{T}^d;\mathbb{R}^d)^*$. *The following are equivalent:*
(a) *L is of the form $L = T_W$ for some $W \in \widehat{\mathcal{M}}_0^d(\mathbb{T}^d)$*
(b) *For every sequence $\{F_n\}_{n\in\mathbb{N}} \subseteq C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ the following implication holds:*

$$\|F_n - c\|_u \longrightarrow 0, \quad c \in \mathbb{R}^d, \quad \sup_{n\in\mathbb{N}} \|DF_n\|_u < +\infty \quad \Longrightarrow \quad \lim_{n\to\infty} \langle L, F_n\rangle = 0. \tag{4.32}$$

(c) *For every $c \in \mathbb{R}^d$ and all $\varepsilon, C > 0$ there exists $\delta > 0$ such that*

$$F \in C^\infty(\mathbb{T}^d;\mathbb{R}^d),\ \|F - c\|_u < \delta,\ \|DF\|_u \leq C \quad \Longrightarrow \quad |\langle L, F\rangle| < \varepsilon.$$

Proof First, since for any function $F \in C^1(\mathbb{T}^d;\mathbb{R}^d)$ we have that $\|F\|_{\mathrm{Lip}} \leq \|DF\|_u$, it is obvious by proposition 4.2.12 that (a) implies (b).
(b)$\Longrightarrow$(c) We suppose that L satisfies (b) not (c) to derive a contradiction. Indeed, if L does not satisfy (c), then there exists $c \in \mathbb{R}^d$ and $\varepsilon, C > 0$ such that for all $\delta > 0$ there exists $F_\delta \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ such that

$$\|F_\delta - c\|_u < \delta,\ \|DF_\delta\|_u \leq C \ \text{ and } \ |\langle L, F_\delta\rangle| \geq \varepsilon.$$

Then if for all $n \in \mathbb{N}$ we pick $G_n := F_{\frac{1}{n}}$ we have that

$$\|G_n - c\|_u \overset{n\to\infty}{\longrightarrow} 0,\ \sup_{n\in\mathbb{N}} \|DG_n\|_u \leq C \ \text{ and } \ |\langle L, G_n\rangle| \geq \varepsilon,$$

comes to contradiction with implication (4.32) since L is assumed to satisfy (b).
(c)$\Longrightarrow$(b) Let $\{F_n\}_{n\in\mathbb{N}} \subseteq C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ be a sequence such that $\|F_n - c\|_u \longrightarrow 0$ for some constant $c \in \mathbb{R}^d$ and such that $C := \sup_{n\in\mathbb{N}} \|DF_n\|_u < \infty$. We have to prove that $\langle L, F_n\rangle \longrightarrow 0$ as $n \to \infty$. So let $\varepsilon > 0$. Since L satisfies (c), there exists $\delta > 0$ such that

$$F \in C^\infty(\mathbb{T}^d;\mathbb{R}^d),\ \|F - c\|_u < \delta,\ \|DF\|_u \leq C \quad \Longrightarrow \quad |\langle L, F\rangle| < \varepsilon.$$

But then if we choose $n_0 \in \mathbb{N}$ such that $\|F_n - c\|_u < \delta$ for all $n \geq n_0$, we then have that

$$n \geq n_0 \quad \Longrightarrow \quad |\langle L, F_n\rangle| < \varepsilon,$$

which proves that L satisfies (b).
(b)$\wedge$(c)$\Longrightarrow$(a) Let $L \in C^1(\mathbb{T}^d;\mathbb{R}^d)^*$ be a distribution satisfying (b) and (c). We will prove that there exists $W \in \widehat{\mathcal{M}}_0^d(\mathbb{T}^d)$ such that $L = T_W$. We show first that L can be extended to a linear operator $\bar{L}$ on the domain $\mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$. Indeed, let $F \in \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$ and let $\{F_n := S_{\frac{1}{n}} F\}_{n\in\mathbb{N}} \subseteq C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ be the sequence given by lemma 4.2.2. Then $F_n - F \longrightarrow 0$ uniformly and

$$\mathrm{Lip}_{\{F_n\}} := \sup_{n\in\mathbb{N}} \|F_n\|_{\mathrm{Lip}} \leq \|F\|_{\mathrm{Lip}} < +\infty.$$

We claim that the sequence $\{\langle L, F_n\rangle\}$ is Cauchy. Indeed, since L satisfies (c), given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$G \in C^\infty(\mathbb{T}^d;\mathbb{R}^d),\ \|G\|_u < \delta,\ \|DG\|_u \leq 2\mathrm{Lip}_{\{F_n\}} \quad \Longrightarrow \quad |\langle L, G\rangle| < \varepsilon.$$

But now, since $F_n \longrightarrow F$ uniformly there exists $n_0 \in \mathbb{N}$ such that

$$n, m \geq m_0 \quad \Longrightarrow \quad \|F_n - F_m\|_u < \delta.$$

Therefore, since $\|D(F_n - F_m)\|_u \leq \|F_n\|_{\mathrm{Lip}} + \|F_m\|_{\mathrm{Lip}} \leq 2\mathrm{Lip}_{\{F_n\}}$ for all $n, m \in \mathbb{N}$, we have that

$$|\langle L, F_n\rangle - \langle L, F_m\rangle| = |\langle L, F_n - F_m\rangle| < \varepsilon$$

for all $n, m \geq n_0$, which proves that $\{\langle L, F_n\rangle\}$ is Cauchy. Therefore we can define an operator $\bar{L}$ on $\mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$ by defining

$$\langle \bar{L}, F\rangle = \lim_{n\to\infty} \langle L, S_{\frac{1}{n}} F\rangle$$

for all $F \in \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$. Of course it remains to be proved that $\bar{L}$ extends L.

First, it is easy to see that the definition of $\bar{L}$ does not depend on the particular choice of the sequence approximating sequence $\{F_n\} \subseteq C^\infty(\mathbb{T}^d;\mathbb{R}^d)$, among those that satisfy

$$\|F_n - F\|_u \longrightarrow 0 \quad \text{and} \quad \mathrm{Lip}_{\{F_n\}} := \sup_{n\in\mathbb{N}} \|F_n\|_{\mathrm{Lip}} < \infty. \tag{4.33}$$

Indeed, let $\{\widetilde{F}_n\} \subseteq C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ another sequence such that $\|\widetilde{F}_n - F\|_u \longrightarrow 0$ and $\mathrm{Lip}_{\{\widetilde{F}_n\}} < \infty$. Then $\{F_n - \widetilde{F}_n\} \subseteq C^\infty(\mathbb{T}^d;\mathbb{R}^d)$, $\|F_n - \widetilde{F}_n\|_u \longrightarrow 0$ and

$$\sup_{n\in\mathbb{N}} \|F_n - \widetilde{F}_n\|_{\mathrm{Lip}} \leq \mathrm{Lip}_{\{F_n\}} + \mathrm{Lip}_{\{\widetilde{F}_n\}} < +\infty,$$

which since L satisfies (b) implies that

$$\lim_{n\to\infty}\langle L, F_n - \widetilde{F}_n\rangle = 0,$$

and so the definition of $\bar{L}$ does not depend on the approximating sequence $\{F_n\}$.

In turn, this implies that the formula

$$\langle \bar{L}, F\rangle = \lim_{n\to\infty}\langle L, F_n\rangle, \quad F \in \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d),$$

where the limit is taken along any sequence $\{F_n\}_{n\in\mathbb{N}} \subseteq C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ satisfying (4.33), defines an extension of $L \in C^1(\mathbb{T}^d;\mathbb{R}^d)^*$ on $\mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)$. Indeed, if $F \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ then the constant sequence defined by $F_n = F$ for all $n \in \mathbb{N}$ satisfies (4.33) and therefore

$$\langle \bar{L}, F\rangle = \lim_{n\to\infty}\langle L, F_n\rangle = \langle L, F\rangle.$$

Therefore $\bar{L}|_{C^\infty(\mathbb{T}^d;\mathbb{R}^d)} \equiv L|_{C^\infty(\mathbb{T}^d;\mathbb{R}^d)}$ and so, since $C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ is dense in $C^1(\mathbb{T}^d;\mathbb{R}^d)$ in the C^1-uniform norm, if we show that the functional $\bar{L}|_{C^1(\mathbb{T}^d;\mathbb{R}^d)} : C^1(\mathbb{T}^d;\mathbb{R}^d) \longrightarrow \mathbb{R}$ is bounded with respect to the C^1-uniform norm of $C^1(\mathbb{T}^d;\mathbb{R}^d)$ it will follow that $\bar{L}|_{C^1(\mathbb{T}^d;\mathbb{R}^d)} \equiv L|_{C^1(\mathbb{T}^d;\mathbb{R}^d)}$ as required. But indeed, since $L \in C^1(\mathbb{T}^d;\mathbb{R}^d)^*$, for all $F \in C^1(\mathbb{T}^d;\mathbb{R}^d)$ we have that

$$\langle \bar{L}, F\rangle = \lim_{n\to\infty}\langle L, S_{\frac{1}{n}}F\rangle \le \|L\| \limsup_{n\to\infty}\|S_{\frac{1}{n}}F\|_{C^1} \le \|L\|\|F\|_{C^1}$$

and therefore $\bar{L}$ is bounded, as required.

Then it is immediate to check that the extension $\bar{L}$ is linear. Indeed, if we continue to denote $F_n := S_{\frac{1}{n}}F$, $n \in \mathbb{N}$, for all $F \in C(\mathbb{T}^d;\mathbb{R}^d)$, then for all $F, G \in \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ and all $a, b \in \mathbb{R}$ we have that

$$\langle \bar{L}, aF + bG\rangle = \lim_{n\to\infty}\langle \bar{L}, (aF+bG)_n\rangle = \lim_{n\to\infty}\langle \bar{L}, aF_n + bG_n\rangle = a\langle \bar{L}, F\rangle + b\langle \bar{L}, G\rangle.$$

Furthermore, it is easy to see that any distribution $L \in C^1(\mathbb{T}^d;\mathbb{R}^d)^*$ satisfying one of the equivalent conditions (b) and (c) vanishes on all constant functions $c \in \mathbb{R}^d$. Indeed, let $c \in \mathbb{R}^d$. Since L satisfies, say (c), for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F \in C^\infty(\mathbb{T}^d;\mathbb{R}^d),\ \|F - c\|_u < \delta,\ \|F\|_{\mathrm{Lip}} \le 1 \quad \Longrightarrow \quad |\langle L, F\rangle| < \varepsilon$$

and in particular $|\langle L, c\rangle| < \varepsilon$ which since $\varepsilon > 0$ was arbitrary proves that $\langle L, c\rangle = 0$. Consequently, the linear function $\bar{L} : \mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d) \longrightarrow \mathbb{R}$ passes to a well defined bounded linear function $W := {}^{\bar{L}}/_{\mathbb{R}^d} : \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d) \longrightarrow \mathbb{R}$. Let us check that W is indeed bounded. First, for all $F \in C^1(\mathbb{T}^d;\mathbb{R}^d)$ we have that

$$\langle W, F\rangle = \langle L, F - F(0)\rangle \le \|L\|\|F - F(0)\|_{C^1} \le 2\|L\|\|F\|_{\mathrm{Lip}}.$$

Let now $F \in \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ and let $\{F_n\} := \{S_{\frac{1}{n}}F\} \subseteq C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ be the approximating sequence given by lemma 4.2.2. Then in particular $\sup_n \|F_n\|_{\mathrm{Lip}} \le \|F\|_{\mathrm{Lip}}$ and so

$$\langle W, F_n\rangle = \langle L, F_n - F_n(0)\rangle \le \|L\|\|F_n - F_n(0)\|_{C^1} \le 2\|L\|\|F_n\|_{\mathrm{Lip}} \le 2\|L\|\|F\|_{\mathrm{Lip}}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ it follows that $\langle W, F\rangle \leq 2\|L\|\|F\|_{\rm Lip}$, which since $F \in {\rm Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ was arbitrary, proves that W is indeed bounded, with norm $\|W\|_{KR} = \|W\|_{{\rm Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*} \leq 2\|L\|$.

Let us prove next that the extended operator $W = \bar{L}$ satisfies implication (4.30) for all sequences $\{F_n\} \subseteq {\rm Lip}(\mathbb{T}^d;\mathbb{R}^d)$. So let $\{F_n\} \subseteq {\rm Lip}(\mathbb{T}^d;\mathbb{R}^d)$ be such that $\|F_n - c\|_u \longrightarrow 0$ for some $c \in \mathbb{R}^d$ and $C_F := \sup_{n\in\mathbb{N}} \|F_n\|_{\rm Lip} < \infty$. By the definition of $\bar{L}$ for each $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ such that

$$|\langle \bar{L}, F_n\rangle - \langle L, S_{\varepsilon_n} F_n\rangle| < \frac{1}{n}, \tag{4.34}$$

and of course we can assume that the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ has been chosen so that $\varepsilon_n \downarrow 0$ as $n \to \infty$. Then, since the $\{F_n\} \subseteq {\rm Lip}(\mathbb{T}^d;\mathbb{R}^d)$, by properties (b) and (c) of the operators S_ε, $\varepsilon \in (0, 1)$, of lemma 4.2.2 we have that $\|D(S_{\varepsilon_n} F_n)\|_u \leq \|S_{\varepsilon_n} F_n\|_{\rm Lip} \leq \|F_n\|_{\rm Lip} \leq C_F$ for all $n \in \mathbb{N}$ and

$$\|S_{\varepsilon_n} F_n - F_n\|_u \leq C\varepsilon_n \|F_n\|_{\rm Lip} \leq CC_F\varepsilon_n \overset{n\to\infty}{\longrightarrow} 0,$$

and therefore since $\{S_{\varepsilon_n} F_n\} \subseteq C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ and L satisfies the continuity property (b) of this proposition we get that

$$\lim_{n\to\infty} \langle L, S_{\varepsilon_n} F_n\rangle = 0,$$

which in turn implies by (4.34) that $\lim_{n\to\infty}\langle \bar{L}, F_n\rangle = 0$, as required.

So far we have proved that given a distribution $L \in C^1(\mathbb{T}^d;\mathbb{R}^d)^*$ satisfying one of the equivalent conditions (b) and (c) there exists unique $W = W_L \in {\rm Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*$ such that $T_W := W|_{C^1(\mathbb{T}^d;\mathbb{R}^d)} = L$. In addition, we have also shown that this $W \in {\rm Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ satisfies implication (A.1) for all $F \in {\rm Lip}(\mathbb{T}^d;\mathbb{R}^d)$, which according to proposition 4.2.12 proves that $W \in \widehat{\mathcal{M}}^d_0(\mathbb{T}^d)$, and the proof is complete. □

According to this proposition, if we denote by $\widehat{\mathcal{M}}^d_{0,C^1}(\mathbb{T}^d)$ the vector subspace of $C^1(\mathbb{T}^d;\mathbb{R}^d)^*$ consisting of all $L \in C^1(\mathbb{T}^d;\mathbb{R}^d)^*$ satisfying implication (4.32), then the extension operator $^- : \widehat{\mathcal{M}}^d_{0,C^1}(\mathbb{T}^d) \longrightarrow \widehat{\mathcal{M}}^d_0(\mathbb{T}^d)$ given by

$$\begin{aligned} \bar{L} &:= \text{the unique } W \in \widehat{\mathcal{M}}^d_0(\mathbb{T}^d) \text{ such that } W|_{C^1(\mathbb{T}^d;\mathbb{R}^d)} = L \\ &= \text{pw-}\lim_{\varepsilon\to 0} L \circ S_\varepsilon \end{aligned}$$

is well defined.

Corollary 4.2.2 *The extension operator* $^- : \widehat{\mathcal{M}}^d_{0,C^1}(\mathbb{T}^d) \longrightarrow \widehat{\mathcal{M}}^d_0(\mathbb{T}^d)$ *is a bi-Lipschitz linear isomorphism.*

Proposition 4.2.14 *Let* $\nabla : L^1(\mathbb{T}^d) \longrightarrow {\rm Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*$ *be defined by*

$$\langle \nabla\sigma, F\rangle = -\int_{\mathbb{T}^d} \sigma(x){\rm div}F(x)dx.$$

Then ∇ *is a bounded linear operator and* $\nabla L^1(\mathbb{T}^d) \leq \widehat{\mathcal{M}}^d_0(\mathbb{T}^d)$.

Proof Obviously for all $\sigma \in L^1(\mathbb{T}^d)$ and all $F \in \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ we have that

$$\langle \nabla\sigma, F\rangle \leq \|\mathrm{div}F\|_{L^\infty(\mathbb{T}^d)}\|\sigma\|_{L^1(\mathbb{T}^d)} \leq 2\|\sigma\|_{L^1(\mathbb{T}^d)}\|F\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)},$$

and therefore ∇ is well defined and bounded with $\|\nabla\| \leq 2$.

We prove now that $\nabla L^1(\mathbb{T}^d) \subseteq \widehat{\mathcal{M}}_0^d(\mathbb{T}^d)$. First we note that if $\sigma \in C^\infty(\mathbb{T}^d)$ then for all $F \in \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ we can perform integration by parts to get

$$\langle \nabla\sigma, F\rangle = -\int \sigma(x)\mathrm{div}F(x)dx = \int_{\mathbb{T}^d} \nabla\sigma(x)\cdot F(x)dx,$$

and therefore

$$\nabla\sigma = \nabla\sigma\, d_{m_{\mathbb{T}^d}} \in \mathcal{M}_0^d(\mathbb{T}^d) \leq \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*.$$

Let now $\sigma \in L^1(\mathbb{T}^d)$. Since $C^\infty(\mathbb{T}^d)$ is dense in $L^1(\mathbb{T}^d)$ there exists a sequence $\{\sigma_n\} \subseteq C^\infty(\mathbb{T}^d)$ such that $\|\sigma - \sigma_n\|_{L^1(\mathbb{T}^d)} \longrightarrow 0$ as $n \to \infty$, and therefore since ∇ is bounded with $\|\nabla\| \leq 2$, we have that

$$\|\nabla\sigma - \nabla\sigma_n\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*} \leq 2\|\sigma - \sigma_n\|_{L^1(\mathbb{T}^d)} \longrightarrow 0,$$

which, since $\nabla C^\infty(\mathbb{T}^d) \subseteq \mathcal{M}_0^d(\mathbb{T}^d)$, proves that $\sigma \in \widehat{\mathcal{M}}_0^d(\mathbb{T}^d)$. □

Proposition 4.2.15 *The gradient operator*

$$\nabla : L^1(\mathbb{T}^d) \leq \mathcal{M}(\mathbb{T}^d) \cong C(\mathbb{T}^d)^* \longrightarrow \widehat{\mathcal{M}}_0^d(\mathbb{T}^d) \leq C^1(\mathbb{T}^d;\mathbb{R}^d)^*$$

is w^*-continuous.

Proof Indeed, let $\{\sigma_\alpha\}_{\alpha\in\mathcal{A}} \subseteq L^1(\mathbb{T}^d)$ be a net such w^*-$\lim_a \sigma_\alpha = \sigma \in L^1(\mathbb{T}^d)$, i.e. such that

$$\int_{\mathbb{T}^d} f\sigma_\alpha = \int_{\mathbb{T}^d} f\sigma, \quad \forall\, f \in C(\mathbb{T}^d).$$

Then, since $\mathrm{div}F \in C(\mathbb{T}^d)$ for all $F \in C^1(\mathbb{T}^d;\mathbb{R}^d)$, we have that

$$\langle \nabla\sigma_a, F\rangle = -\int \sigma_\alpha \mathrm{div}F \longrightarrow -\int \sigma \mathrm{div}F = \langle \nabla\sigma, F\rangle$$

for all $F \in C^1(\mathbb{T}^d;\mathbb{R}^d)$, and therefore $\nabla\sigma_\alpha \longrightarrow \nabla\sigma$ in the w^*-topology of $\widehat{\mathcal{M}}_0^d(\mathbb{T}^d)$ when considered as a subspace of $C^1(\mathbb{T}^d;\mathbb{R}^d)^*$. □

4.2.4 The State Space of the Empirical Current Process

As we have already seen in the end of section 4.2.2, the (diffusively rescaled) empirical current process can be regarded as the process

$$W^N : \Omega_N \longrightarrow L^\infty_{w^*}(0,T;\mathcal{M}^d)$$

given by the formula

$$W^N(\eta,t) = W^N_{\eta_t},$$

where Ω_N is the closed subset

$$\Omega_N := \{\eta \in D(\mathbb{R}_+ \times \mathbb{M}_N^d) \big| \langle 1, \pi_{\eta t}^N \rangle = \langle 1, \pi_{\eta_0}^N \rangle, \ \forall\, t \geq 0\} \tag{4.35}$$

and

$$\|W\|_{TV,\infty} \leq 2dN\langle 1, \pi_{\eta_0}^N \rangle.$$

As we have seen the extra factor N above poses difficulties in the proof of the relative compactness of the law $W_*^N P^N \in \mathbb{P}L_{w^*}^\infty(0,T;\mathcal{M}^d)$.

However, as we have noted the empirical current function $W^N : \mathbb{M}_N^d \longrightarrow \mathcal{M}^d$ takes in fact values in the space $\mathcal{M}_0^d(\mathbb{T}^d)$ of vector-valued measures with zero total measure, where $\mathcal{M}_0^d(\mathbb{T}^d)$ is considered equipped with the Kantorovich-Rubinstein norm. By an integration by parts and the Cauchy Schwartz inequality we see that for any $G \in \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$,

$$\begin{aligned}\int G \cdot dW^N &= \frac{1}{N^{d-1}} \sum_{j=1}^d \sum_{x\in\mathbb{T}_N^d} \Big[G^j\Big(\frac{x}{N}\Big) - G^j\Big(\frac{x-e_j}{N}\Big)\Big] g\big(\eta(x)\big) \\ &\leq \frac{1}{N^{d-1}} \sum_{x\in\mathbb{T}_N^d} g\big(\eta(x)\big)\sqrt{d}\Big|G\Big(\frac{x}{N}\Big) - G\Big(\frac{x-e_j}{N}\Big)\Big|_2 \\ &\leq \|g'\|_u \sqrt{d}\|G\|_{\mathrm{Lip}(\mathbb{T}^d;\mathbb{R}^d)}\langle 1, \pi^N \rangle.\end{aligned}$$

Therefore

$$\|W^N\|_{KR} = \sup_{\substack{G\in\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d) \\ \|G\|_{\mathrm{Lip}}\leq 1}} \int G \cdot dW_\eta^N \leq \sqrt{d}\|g'\|_u\langle 1, \pi_\eta^N \rangle. \tag{4.36}$$

Then, if Ω_N is the Borel subset of $D(\mathbb{R}_+;\mathbb{M}_N^d)$ defined in (4.35) we have that

$$\|W_\eta^N\|_{L^\infty(0,T;\mathcal{M}_0^d)} \leq \sqrt{d}\|g'\|_u\langle 1, \pi_{\eta_0}^N \rangle$$

for all $\eta \in \Omega_N$. Consequently we can regard the empirical current process as the mapping

$$(\Omega_N, P_N) \xrightarrow{W^N} L^\infty(0,T;\mathcal{M}_0^d) \leq L_{w^*}^\infty(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*) \cong L^1(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))^*.$$

In the same way, in the case that the jump rate g is bounded we get the estimate

$$\|W_\eta^N\|_{L^\infty(0,T;\mathcal{M}_0^d)} \leq \sqrt{d}\|g\|_u \tag{4.37}$$

and so in this case the empirical current process takes values in the ball of radius $\sqrt{d}\|g\|_u$ in $L_{w^*}^\infty(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)*)$. Of course in order for this to be meaningful we also have to ensure that it is a Borel random variable with respect to the Borel σ-algebra of the Skorohod space and the Borel σ-algebra of the w^*-topology of $L_{w^*}^\infty\big(I;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*\big)$ so that the laws $W_*^N P^N$ are well-defined. This follows by the next proposition

Proposition 4.2.16 *The empirical current mapping*

$$W_N : \Omega_N \longrightarrow L^1(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))^*$$

is continuous with respect to the Skorohod topology on Ω_N and the w^*-topology on $L^1(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))^*$ and thus Borel measurable with respect to the corresponding σ-algebras

Proof We have to prove that for every $F \in L^1(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))$ the function $\langle F, W^N\rangle_{[0,T]} : D(0,T;\mathbb{M}_N^d) \longrightarrow \mathbb{R}$ given by

$$\langle F, W^N\rangle_{[0,T]}(\eta) = \int_0^T \langle F_t, W^N_{\eta_t}\rangle dt$$

is continuous. So let $F \in L^1(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))$. By definition F is strongly measurable and the function $t \mapsto \|F_t\|_{\mathrm{Lip}_0}$ is in $L^\infty(0,T)$. Let $\{\eta^k\}_{k\in\mathbb{N}} \subseteq D(0,T;\mathbb{M}_N^d)$ be a sequence of cadlag paths such that $\eta^k \longrightarrow \eta \in D(0,T;\mathbb{M}_N^d)$ in the Skorohod topology and we will prove that $\lim_{k\to\infty}\langle F, W^N\rangle_{[0,T]}(\eta^k) = \langle F, W^N\rangle_{[0,T]}(\eta)$. First, as we have seen in the proof of proposition 4.2.8 the function $m_{N,1} : \Omega_N \longrightarrow C(0,T;\mathbb{Z}_+)$ given by

$$m_{N,1}(\eta)(t) = \sum_{x\in\mathbb{T}_N^d} \eta_t(x)$$

is continuous and therefore $m_{N,1}(\eta^k) \longrightarrow m_{N,1}(\eta)$ uniformly on $[0,T]$ as $k \to \infty$. Consequently there exists $k_0 \in \mathbb{N}$ such that

$$k \geq k_0 \quad \Longrightarrow \quad \sum_{x\in\mathbb{T}_N^d} \eta_t^k(x) = \sum_{x\in\mathbb{T}_N^d} \eta_0(x), \quad \forall\, t \in [0,T].$$

Next, since the function $W^N : \mathbb{M}_N^d \longrightarrow \mathcal{M}_0^d(\mathbb{T}^d) \leq \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*$ is obviously continuous the induced function

$$\bar{W}^N : D(0,T;\mathbb{M}_N^d) \longrightarrow D\big(0,T;\mathcal{M}_0^d(\mathbb{T}^d)\big)$$

on the Skorohod spaces is continuous. Therefore $W^N_{\eta^k} \longrightarrow W^N_\eta$ in the Skorohod topology and consequently $W_{\eta_t^k} \longrightarrow W^N_{\eta_t^k}$ in $\mathcal{M}_0^d(\mathbb{T}^d)$ for all continuity points of $t \in [0,T]$ of η. But η has at most countable discontinuity points and so

$$\|W_{\eta_t^k} \longrightarrow W^N_{\eta_t^k}\|_{KR} \longrightarrow 0$$

for almost all $t \in [0,T]$. Consequently, since $F_t \in \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ for all $t \in [0,T]$, we have that

$$\langle F_t, W^N_{\eta_t^k}\rangle \longrightarrow \langle F_t, W^N_{\eta_t}\rangle, \quad \text{a.s.-}\forall\, t \in [0,T].$$

Furthermore, for all $k \geq k_0$ by the bound (4.36),

$$\begin{aligned} |\langle F_t, W^N_{\eta_t^k}\rangle| &\leq \|F_t\|_{\mathrm{Lip}_0}\|W^N_{\eta_t^k}\|_{KR} \leq \sqrt{d}\|g'\|_u\langle 1, \pi^N_{\eta_t^k}\rangle\|F_t\|_{\mathrm{Lip}_0} \\ &= \sqrt{d}\|g'\|_u\langle 1, \pi^N_{\eta_0}\rangle\|F_t\|_{\mathrm{Lip}_0} \in L^1(0,T) \end{aligned}$$

and so by the dominated convergence theorem it follows that

$$\lim_{k\to\infty}\langle F, W^N\rangle_{[0,T]}(\eta^k) = \lim_{k\to\infty}\int_0^T \langle F_t, W^N_{\eta_t^k}\rangle dt = \int_0^T \langle F_t, W^N_{\eta_t}\rangle dt = \langle F, W^N\rangle_{[0,T]}(\eta)$$

as required for the w^*-continuity of the empirical current process. □

Due to the fact that $\mathrm{cl}_{(\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*,w^*)}\mathcal{M}_0^d = \mathrm{Lip}_0(M;\mathbb{R}^d)^*$, the injection

$$L^\infty_{w^*}(0,T;\widehat{\mathcal{M}}_0^d(M)) \hookrightarrow L^\infty_{w^*}\big(0,T;\mathrm{Lip}_0(M;\mathbb{R}^d)^*\big)$$

does not have w^*-closed range and that is why we consider the empirical current as taking values on the whole space $L^1(0,T;\mathrm{Lip}_0(M;\mathbb{R}^d))^*$, since any limiting argument with respect to the w^*-topology would anyway take us out of the smaller space $L^\infty_{w^*}(0,T;\widehat{\mathcal{M}}^d_0(M))$.

This definition of the empirical current process will allows us to prove the relative compactness of the law $\{W^N_* P^N\}_{N\in\mathbb{N}}$ of the empirical current. However, since the space $\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ is not separable, the space $L^\infty_{w^*}(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$ is not sub-metrizable and the Prokhorov-Le Cam theorem (A.4.1 in the appendix) will give us only a convergent sub-net of the sequence $\{W^N_* P^N\}$. In order to get a convergent sub-sequence we will also prove the relative compactness of the empirical current in a slightly weaker topology, in the space $L^\infty_{w^*}(0,T;C^1(\mathbb{T}^d;\mathbb{R}^d)^*)$.

It is easy to see with the same reasoning that the empirical current also gives rise to a well defined map

$$W^N : (\Omega_N, P^N) \longrightarrow L^\infty_{w^*}(0,T;C^1(\mathbb{T}^d;\mathbb{R}^d)^*) \cong L^1(0,T;C^1(\mathbb{T}^d;\mathbb{R}^d))^*. \qquad (4.38)$$

Let us recall the mean value theorem for vector valued functions.

Lemma 4.2.3 *Let $G \in C^1(\mathbb{R}^d;\mathbb{R}^d)$. Then*

$$G(y) - G(x) = \Big(\int_0^1 DG((1-t)x + ty))dt\Big)(y-x)$$

for all $x, y \in \mathbb{R}^d$ and in particular

$$|G(y) - G(x)|_2 \le \sup_{z\in[x,y]} |DG(z)|_{\mathrm{Fr}} |y-x|_2.$$

Proof Let $x, y \in \mathbb{R}^d$ and let G^j, $j = 1, \ldots, d$ denote the coordinate functions of G. For each $j = 1, \ldots, d$ we define the function $g_j : [0,1] \longrightarrow \mathbb{R}$ by the formula

$$g_j(t) = G^j((1-t)x + ty).$$

Then by the fundamental theorem of calculus we have

$$G^j(y) - G^j(x) = g_j(1) - g_j(0) = \int_0^1 g_j'(t)dt = \int_0^t \langle \nabla G^j((1-t)x + ty), y - x\rangle dt$$

and since for any $p \in \mathbb{R}^d$

$$DG(p)(y-x) = \sum_{j=1}^d \langle \nabla G^j(p), y - x\rangle \cdot e_j$$

it follows that

$$G(y) - G(x) = \int_0^1 DG\big((1-t)x + ty\big)(y-x)dt = \Big(\int_0^1 DG\big((1-t)x + ty\big)dt\Big)(y-x).$$

Furthermore, since G is C^1, the function $\mathbb{R}^d \ni z \mapsto DG(z) \in \mathbb{R}^{d\times d}$ is continuous and therefore by the compactness of the segment $[x,y]$ we have that

$$\sup_{z\in[x,y]} |DG(z)|_{\mathrm{Fr}} < +\infty$$

and therefore

$$\left| \int_0^1 DG\big((1-t)x+ty\big)dt \right|_{\mathrm{Fr}} \leq \sup_{z\in[x,y]} |DG(z)|_{\mathrm{Fr}} < +\infty.$$

Consequently, since by (4.29) the Frobenius norm bounded the ℓ_2-operator norm we have that

$$|G(y)-G(x)|_2 \leq \sup_{z\in[x,y]} |DG(z)|_{\mathrm{Fr}}|x-y|_2$$

and the proof is complete. ☐

Corollary 4.2.3 *Let $G \in C^1(\mathbb{T}^d;\mathbb{R}^d)$ and let $p : \mathbb{R}^d \longrightarrow \mathbb{T}^d$ denote the universal covering of the torus. Then for all $x, y \in \mathbb{T}^d \cong \mathbb{R}^d/_{\mathbb{Z}^d}$,*

$$|G(y)-G(x)|_2 \leq \big\| |D(G\circ p)|_{\mathrm{Fr}} \big\|_{C(\mathbb{R}^d)} d_{\mathbb{T}^d}(x,y) < +\infty.$$

Proof Recall the considerations on the torus before proposition 4.2.11 and let $p : \mathbb{R}^d \longrightarrow \mathbb{T}^d$, $p(\widetilde{x}) = \widetilde{x} + \mathbb{Z}^d$, denote the universal covering of the torus. Since p is smooth, the function $G\circ p$ belongs in $C^1(\mathbb{R}^d;\mathbb{R}^d)$ and since p is local isometry, given $x, y \in \mathbb{T}^d$ we can choose $\widetilde{x}, \widetilde{y} \in \mathbb{R}^d$ such that $p(\widetilde{x}) = x$, $p(\widetilde{y}) = y$ and

$$|\widetilde{x}-\widetilde{y}|_2 = \min_{z\in\widetilde{y}+\mathbb{Z}^d} |\widetilde{x}-z|_2 = d_{\mathbb{T}^d}(x,y).$$

Then by applying the mean value theorem to the function $G\circ p$ we have

$$|G(y)-G(x)|_2 = |G\circ p(\widetilde{y}) - G\circ p(\widetilde{x})|_2 \leq \big\| |D(G\circ p)|_{\mathrm{Fr}} \big\|_{C(\mathbb{R}^d)} |\widetilde{x}-\widetilde{y}|_2 = d(x,y)_{\mathbb{T}^d}$$

as required. ☐

Returning to the definition of the empirical current process, by an integration by parts, the Cauchy-Schwartz inequality and the mean value theorem we have for any $G \in C^1(\mathbb{T}^d;\mathbb{R}^d)$ that ,

$$\begin{aligned}
\int G\cdot dW^N &= \frac{1}{N^{d-1}} \sum_{j=1}^d \sum_{x\in\mathbb{T}_N^d} \Big[G^j\Big(\frac{x}{N}\Big) - G^j\Big(\frac{x-e_j}{N}\Big)\Big] g\big(\eta(x)\big) \\
&\leq \frac{1}{N^{d-1}} \sum_{x\in\mathbb{T}_N^d} g\big(\eta(x)\big)\sqrt{d}\Big| G\Big(\frac{x}{N}\Big) - G\Big(\frac{x-e_j}{N}\Big)\Big|_2 \\
&\leq \|g'\|_u \sqrt{d} \big\| |D(G\circ p)|_{\mathrm{Fr}} \big\|_{C(\mathbb{R}^d)} \langle 1, \pi^N \rangle \\
&\leq \sqrt{d}\|g'\|_u \langle 1, \pi^N\rangle \|G\|_{C^1(\mathbb{T}^d;\mathbb{R}^d)}
\end{aligned}$$

and therefore

$$\|W^N\|_{C^1(\mathbb{T}^d;\mathbb{R}^d)^*} \leq \sqrt{d}\|g'\|_u \langle 1, \pi^N\rangle. \tag{4.39}$$

Finally it is obvious by proposition 4.2.16 that the empirical current process also as considered in (4.38) is continuous with respect to the Skorohod and w^*-topologies and thus Borel measurable.

4.3 The Continuity Equation on the Torus

In this section we consider the continuity equation

$$\partial_t \mu + \mathrm{div}_x W = 0 \quad \text{in } [0,T] \times \mathbb{T}^d$$

on the torus in the sense of distributions. Here $\mu = (\mu_t)_{0<t<T} \subseteq L^\infty_{w^*}(0,T;\mathcal{M}_+)$ is a curve of finite non-negative measures describing the density and $W = (\mu_t)_{0<t<T}$ is either an element of the space $L^\infty(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$ or of $L^1_{w^*}(0,T;C^1(\mathbb{T}^d;\mathbb{R}^d)^*)$ describing the density flux. As we will see even with these more general interpretation of the current the continuity equation makes sense and by an adaptation of lemma 8.1.2 in [2] its solutions $\mu : \mathbb{R}_+ \longrightarrow \mathcal{M}_+(\mathbb{T}^d)$ are weakly-continuous in time.

Lemma 4.3.1 *Let X, Y be normed spaces. Then*

$$(X \times_1 Y)^* \overset{\text{iso}}{=} X^* \times_\infty Y^*.$$

Proof Let $1 \leq p \leq q \leq \infty$ be conjugate exponents and let $\ell \in (X \times^p Y)^*$ be a linear functional. Then the functionals $\ell_X : X \longrightarrow \mathbb{R}$ and $\ell_Y : Y \longrightarrow \mathbb{R}$ defined by

$$\ell_X(x) = \ell(x,0), \qquad \ell_Y(y) = \ell(0,y)$$

are obviously linear. They are also bounded with $\|\ell_X\| \vee \|\ell_Y\| \leq \|\ell\|$ since for all $x \in X$ we have that

$$|\ell_X(x)| = |\ell(x,0)| \leq \|\ell\| \|(x,0)\|_p = \|\ell\| \|x\|_X$$

and likewise we see that $\ell_Y \in Y^*$ with $\|\ell_Y\| \leq \|\ell\|$. Note that we can also express ℓ_X and ℓ_Y as $\ell_Z = \ell \circ i_Z$, $Z = X, Y$, where $i_Z : Z \hookrightarrow X \times_p Y$ are the natural injections defined by $i_X(x) = (x,0)$ and $i_Y(y) = (y,0)$.

Now, the function $T : (X \times_p Y)^* \longrightarrow X^* \times_q Y^*$ defined by

$$T(\ell) = (\ell \circ i_X, \ell \circ i_Y)$$

is obviously a linear bijection. We will show that it is an isometry. We recall that by duality in $\ell^2_p := (\mathbb{R}^2, |\cdot|_p)$ we have that

$$\sup_{(x,y) \in \mathbb{R}^d \setminus \{0\}} \frac{ax + by}{\|(x,y)\|_p} = \|(a,b)\|_q.$$

Therefore, since for all $(x,y) \in X \times_p Y \neq \{0\}$ we have that

$$\frac{\ell(x,y)}{\|(x,y)\|_p} = \frac{\ell_X(x) + \ell_Y(y)}{\big\|(\|x\|, \|y\|)\big\|_p} \leq \frac{\|\ell_X\|\|x\| + \|\ell_Y\|\|y\|}{\big\|(\|x\|, \|y\|)\big\|_p}$$

we see that

$$\|\ell\| = \sup_{(x,y) \neq 0} \frac{\ell(x,y)}{\|(x,y)\|_p} \leq \sup_{(x,y) \neq 0} \frac{\|\ell_X\|\|x\| + \|\ell_Y\|\|y\|}{\big\|(\|x\|, \|y\|)\big\|_p} = \big\|(\|\ell_X\|, \|\ell_Y\|)\big\|_q = \|T\ell\|,$$

or in a more explicit notation

$$\|\ell\|_{(X \times_p Y)^*} \leq \|T\ell\|_{X^* \times_q Y^*}.$$

On the other hand we obviously have that

$$\|T\ell\|_{X^*\times_q Y^*} = \big\|(\|\ell_X\|, \|\ell_Y\|)\big\|_q = \Big(\|\ell_X\|^q + \|\ell_Y\|^q\Big)^{\frac{1}{q}} \leq 2^{\frac{1}{q}}\|\ell\|_{(X\times_p Y)^*}$$

which together with the previous inequality gives

$$\|\ell\|_{(X\times_p Y)^*} \leq \|T\ell\|_{X^*\times_q Y^*} \leq 2^{\frac{1}{q}}\|\ell\|_{(X\times_p Y)^*}. \tag{4.40}$$

In particular, with the obvious modifications in the case $(p,q) = (1,\infty)$ this proves that

$$\|T\ell\|_{X^*\times_\infty Y^*} = \|\ell\|_{(X\times_1 Y)^*}$$

and completes the proof. $\square$

Definition 4.3.1 We say that *a density-current curve*

$$(\mu, W) \in L^\infty\big(I; \mathcal{M}_+(\mathbb{T}^d)\big) \times L^\infty_{w^*}\big(I; \mathrm{Lip}_0(\mathbb{T}^d; \mathbb{R}^d)^*\big)$$

satisfies the continuity equation

$$\partial_t \mu_t + \mathrm{div} W_t = 0, \tag{4.41}$$

in $I^o \times \mathbb{T}^d$ iff for all smooth $\mathbb{Z}^d$-periodic test functions $G \in C^\infty(I^o \times \mathbb{R}^d)$ of compact support in I^o we have that

$$\int_0^T \bigg(\int_{\mathbb{R}^d} \partial_t G_t d\widetilde{\mu}_t + \langle \nabla G_t, \widetilde{W}_t\rangle\bigg) dt = 0 \tag{4.42}$$

for any representative

$$(\widetilde{\mu}, \widetilde{W}) \in L^\infty\big(I; \mathcal{M}_+(\mathbb{R}^d)\big) \times L^\infty_{w^*}\big(I; \mathrm{Lip}_0(\mathbb{R}^d, \mathbb{R}^d)^*\big)$$

of (μ, W), i.e. for any such pair $(\widetilde{\mu}, \widetilde{W})$ for which $p_*(\mu, W) := (p_*\widetilde{\mu}, p_*\widetilde{W}) = (\mu, W)$.

In this case we write $\langle \nabla G_t, W_t\rangle$ instead of $\int_M \nabla G_t \cdot dW_t$ to emphasize that $W_t \in \mathrm{Lip}_0(\mathbb{T}^d; \mathbb{R}^d)^*$ is not necessarily a measure. A few remarks are in order. First, the requirement in (4.41) is not empty due to the surjectivity of the maps $p_* : L^\infty(I; \mathcal{M}_+(\mathbb{R}^d)) \longrightarrow L^\infty(I; \mathcal{M}_+(\mathbb{T}^d))$ and $p_* : L^\infty_{w^*}(I; \mathrm{Lip}_0(\mathbb{R}^d, \mathbb{R}^d)^*) \longrightarrow L^\infty_{w^*}(I; \mathrm{Lip}_0(\mathbb{T}^d, \mathbb{R}^d)^*)$. Secondly, in order for (4.41) to make sense we have to make sure that for each $\mathbb{Z}^d$-periodic function $G \in C^\infty(I^o \times \mathbb{R}^d)$ of compact support in I^o the curve $I \ni t \mapsto \nabla_x G_t$ belongs in $L^1(I; \mathrm{Lip}_0(\mathbb{R}^d; \mathbb{R}^d))$. But this is obvious since due to the fact that G is $\mathbb{Z}^d$-periodic in space and of compact support in I^o we have that

$$\|\nabla_x G_t\|_{\mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)} \leq \|D^2_x G_t\|_u \leq \sup_{t\in I}\|D^2_x G_t\|_u < +\infty.$$

To prove the weak continuity of the solutions $\mu : \mathbb{R}_+ \longrightarrow \mathcal{M}_+(\mathbb{T}^d)$ of the continuity equation we need a description of the dual of$C^2_0(M)$, $M = \mathbb{R}^d$ or $\mathbb{T}^d$, which is defined as the closure of $C^2_c(M)$ in $C^2(M)$ with respect to the usual C^2-uniform norm $\|\cdot\|_{C^2}$ defined by

$$\|f\|_{C^2} = \|f\|_u + \|\nabla f\|_u + \|D^2 f\|_u, \quad f \in C^2(M).$$

Here we consider the space $\mathbb{R}^{d\times d}$ of $d\times d$-matrices equipped with the Frobenius norm $\|\cdot\|_{\mathrm{Fr}}$. Since the Frobenius norm is Euclidean, by considering $\mathbb{R}^{d\times d}$ equipped with the Frobenius norm we have by the Riesz representation theorem for vector-space valued measures that $C_0(M;\mathbb{R}^{d\times d})\cong\mathcal{M}(M;\mathbb{R}^{d\times d})$.

To describe the linear functionals on $C_0^2(M)$ we consider first the natural injection

$$i : C_0^2(M)\longrightarrow X := C_0(M)\times_1 C_0(M;\mathbb{R}^d)\times_1 C_0(M;\mathbb{R}^{d\times d})$$

defined by the formula $i(f)=(f,\nabla f,D^2f)$. This is an isometric injection with closed range and by the Hahn-Banach theorem the dual mapping $i^*:X^*\longrightarrow C_0^2(M)^*$ is a surjection with $\|i^*\|=\|i\|=1$. It follows that the quotient mapping

$$\widetilde{i^*} : {}^{X^*}/_{\ker i^*}\longrightarrow C_0^2(M)^*$$

is an isometry. Of course by the Riesz representation theorem and lemma 4.3.1 we have that $X^*:=\mathcal{M}\times\mathcal{M}^d\times\mathcal{M}^{d\times d}$ and since $\ker(i^*)=R(i)^\perp=iC_0^2(M)^\perp$ we get an isometry

$$T : {}^{\mathcal{M}\times\mathcal{M}^d\times\mathcal{M}^{d\times d}}/_{iC_0^2(M)^\perp}\longrightarrow C_0^2(M)^*.$$

Consequently, any linear function $\ell\in C_0^2(M)^*$ can be represented in the form

$$\ell(f)=T_{[\mu,W,Q]}(f)=\int fd\mu+\int\nabla f\cdot dW+\int\langle D^2f,dQ\rangle_{\mathrm{Fr}}$$

for some $(\mu,W,Q)\in\mathcal{M}\times\mathcal{M}^d\times\mathcal{M}^{d\times d}$, and two such triples (μ,W,Q) and (ν,V,R) in $\mathcal{M}\times\mathcal{M}^d\times\mathcal{M}^{d\times d}$ give rise to the same functional $\ell\in C_0^2(M)^*$ iff $(\mu-\nu,W-V,Q-R)\in iC_0^2(M)$.

Lemma 4.3.2 *Let $M=\mathbb{T}^d$ or $\mathbb{R}^d$ and consider $C_0(M)^*=\mathcal{M}$ as a subspace of $C_0^2(M)^*$ through the submetric injection $i^*:C_0(M)^*\longrightarrow C_0^2(M)^*$ where $i:C_0^2(M)\longrightarrow C_0(M)$ is the inclusion injection. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence in $C_0(M)$ such that*

$$\|\mu_n-\ell\|_{C_0^2(M)^*}\longrightarrow 0$$

for some $\ell\in C_0^2(M)^*$. If $\{\mu_n\}\subseteq\mathcal{M}$ is TV-norm bounded, and in addition tight in the case $M=\mathbb{R}^d$, then $\ell\in\mathcal{M}$.

Proof Since $\mu_n\longrightarrow\ell$ in $C_0^2(M)^*$, for any $f\in C_0^2(M)$ we have that

$$|\mu_n(f)-\ell(f)|\leq\|\mu_n-\ell\|_{C_0^2(M)^*}\|f\|_{C^2}\longrightarrow 0$$

as $n\to\infty$. Now, ℓ is of the form $\ell=T_{(\nu,V,R)}$ for some $(\nu,V,R)\in\mathcal{M}\times_\infty\mathcal{M}^d\times\mathcal{M}^{d\times d}$ and therefore by the limit above we have that

$$\lim_{n\to\infty}\int fd\mu_n=\int fd\nu+\int\langle\nabla f,dV\rangle+\int\langle D^2f,dR\rangle_{\mathrm{Fr}},\quad\forall\, f\in C_0^2(M).\quad(4.43)$$

On the other hand, $\{\mu_n\}\subseteq C_0(\mathbb{T}^d)^*\equiv\mathcal{M}$ is norm bounded and tight and therefore it is relatively compact in the weak topology of $\mathcal{M}$. So there exists a subsequence $\{\mu_{k_n}\}$ of $\{\mu_n\}$ and $\mu\in\mathcal{M}$ such that

$$\lim_{n\to\infty}\int fd\mu_{k_n}=\int fd\mu,\quad\forall\, f\in BC(M).$$

Then for all $f \in C_0^2(\mathbb{T}^d)$ we have that

$$\int f d\nu + \int \nabla f \cdot dV + \int \langle D^2 f, dR\rangle_{\mathrm{Fr}} = \lim_{n\to\infty}\int f d\mu_{k_n} = \int f d\mu$$

and therefore by (4.43) we have that

$$\lim_{n\to\infty}\int f d\mu_n = \int f d\mu, \quad \forall\, f \in C_0^2(M). \tag{4.44}$$

We shall show now that $\mu_n \longrightarrow \mu$ weakly. Since $\{\mu_n\}$ is supposed to be bounded, it is contained in some metrizable for the weak topology subspace $B_{\mathcal{M}}(0, M) \subseteq \mathcal{M}$, $M > 0$, and therefore it suffices to prove that any subsequence of $\{\mu_n\}$ has a further subsequence which converges weakly to μ. So let $\{\mu_{k_n}\}$ be a subsequence of $\{\mu_n\}$. Since $\{\mu_n\}$ is weakly relatively compact there exists a further subsequence $\{\mu_{m_{k_n}}\}$ such that $\mu_{m_{k_n}} \longrightarrow \mu_0$ weakly for some $\mu_0 \in \mathcal{M}$. But then by (4.44) we have that

$$\int f d\mu_0 = \lim_{n\to\infty}\int f d\mu_{m_{k_n}} = \int f d\mu, \quad \forall\, f \in C_0^2(M),$$

which shows that $\mu = \mu_0$ and completes the proof. □

Proposition 4.3.1 *Let $(\mu, W) \in L^\infty\big(I;\mathcal{M}_+(\mathbb{T}^d)\big)\times L^\infty_{w^*}\big(I;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*\big)$ be a density-current curve satisfying the continuity equation. Then there exists a weakly continuous curve $\bar\mu$ in the class of μ in $L^\infty\big(I;\mathcal{M}_+(\mathbb{T}^d)\big)$, and for this continuous representative we have that*

$$\int_{\mathbb{T}^d} G_t d\bar\mu_t - \int_{\mathbb{T}^d} G_s d\bar\mu_s = \int_s^t \left(\int_{\mathbb{T}^d}\partial_r G_r d\bar\mu_r + \langle \nabla G_r, W_r\rangle\right) dr,$$

for all $G \in C^\infty(I\times\mathbb{T}^d)$ and all $0 \le s \le t \le T$, in the sense that for any representative $\widetilde W \in L^\infty_{w^*}\big(I;\mathrm{Lip}_0(\mathbb{R}^d,\mathbb{R}^d)^*\big)$ of W it holds that

$$\int_{\mathbb{R}^d} G_t d\widetilde{\bar\mu}_t - \int_{\mathbb{R}^d} G_s d\widetilde{\bar\mu}_s = \int_s^t \left(\int_{\mathbb{R}^d}\partial_r G_r d\widetilde{\bar\mu}_r + \langle \nabla G_r, \widetilde W_r\rangle\right) dr, \tag{4.45}$$

where $(\widetilde{\bar\mu}_t)_{t\in I} := (i_*\bar\mu_t)_{t\in I}$ and $i : \mathbb{T}^d \hookrightarrow \mathbb{R}^d$ is the Borel injection.

Proof We fix an arbitrary $\mathbb{Z}^d$-periodic function $\zeta \in C^\infty(\mathbb{R}^d)$ and let $\varphi_\zeta : (0,T) \longrightarrow \mathbb{R}$ denote the function defined a.s. by

$$\varphi_\zeta(t) = \int_{\mathbb{R}^d} \zeta\circ i d\mu_t,$$

where of course $i : \mathbb{T}^d \cong [0,1)^d \hookrightarrow \mathbb{R}^d$ is the Borel injection. Then due to our assumption that $\mu \in L^\infty\big(I;\mathcal{M}_+(\mathbb{T}^d)\big)$ we have that $\varphi_\zeta \in L^\infty(I)$ since

$$|\varphi_\zeta(t)| \le \|\zeta\|_u \mu_t(\mathbb{T}^d) \le \|\zeta\|_u\|\mu\|_{\infty;TV} < +\infty,$$

for almost all $t \in I$.

Let now $G \in C_c^\infty(I^o\times\mathbb{R}^d)$ be any function of the form $G(t,x) = f(t)\zeta(x)$ for some function $f \in C_c^\infty(I^o)$ and some $\mathbb{Z}^d$-periodic function $\zeta \in C_c^\infty(\mathbb{R}^d)$. Then since the pair $(\mu, W) \in L^\infty(I^o; \mathcal{M}_+(\mathbb{T}^d)) \times L^\infty_{w^*}(I; \mathrm{Lip}_0(\mathbb{T}^d; \mathbb{R}^d))$ satisfies the continuity equation, we have by (4.42) that

$$\int_0^T f'(t)\varphi_\zeta(t)dt = \int_0^T f'(t)\int_M \zeta d\widetilde{\mu}_t dt = -\int_0^T f(t)\langle \nabla\zeta, \widetilde{W}_t\rangle dt$$

for any $(\widetilde{\mu}, \widetilde{W}) \in L^\infty(I^o; \mathcal{M}_+(\mathbb{R}^d)) \times L^\infty_{w^*}(I; \mathrm{Lip}_0(\mathbb{R}^d; \mathbb{R}^d))$ such that $p_*(\widetilde{\mu}, \widetilde{W}) = (\mu, W)$. Therefore, since the equality above holds for all $f \in C_c^\infty(I^o)$ we see that the measurable function $\psi_\zeta : I \longrightarrow \mathbb{R}$ defined a.s. by

$$\psi_\zeta(t) = \langle \nabla\zeta, \widetilde{W}_t\rangle$$

is the weak derivative of the function φ_ζ. But since $\widetilde{W} \in L^\infty_{w^*}(I; \mathrm{Lip}_0(\mathbb{R}^d; \mathbb{R}^d)^*)$ the function ψ_ζ is in $L^\infty(I)$ since for almost all $t \in T$ we have that

$$|\psi_\zeta(t)| \leq \|\nabla\zeta\|_{\mathrm{Lip}_0}\|\widetilde{W}_t\|_{KR} \leq \|\nabla\zeta\|_{\mathrm{Lip}_0}\|\widetilde{W}\|_{\infty;KR} < +\infty.$$

Therefore $\varphi_\zeta \in W^{1,\infty}(I^o)$ with distributional derivative ψ_ζ. Consequently, the equivalence class ϕ_ζ contains a Lipschitz representative $\bar{\varphi}_\zeta$ with Lipschitz constant

$$\|\bar{\varphi}_\zeta\|_{\mathrm{Lip}} \leq \|\psi_\zeta\|_{L^\infty(I)} \leq \|\nabla\zeta\|_{\mathrm{Lip}_0}\|\widetilde{W}\|_{\infty;KR}.$$

Let now $\mathcal{Z}$ be a countable subset of $\widetilde{C}^\infty(\mathbb{T}^d)$ that is dense in $\widetilde{C}^2(\mathbb{T}^d)$ in the usual C^2-norm $\|\cdot\|_{C^2}$ given by

$$\|\zeta\|_{C^2} = \|\zeta\|_u + \|\nabla\zeta\|_u + \|D^2\zeta\|_u$$

for $\zeta \in \widetilde{C}^2(\mathbb{T}^d)$. Then obviously $\mathcal{Z}$ is also dense in $\widetilde{C}(\mathbb{T}^d)$ with the uniform norm $\|\cdot\|_u$ and we set

$$I_{\mathcal{Z}} := \bigcap_{\zeta\in\mathcal{Z}} \big\{t \in I \,\big|\, \varphi_\zeta(t) = \bar{\varphi}_\zeta(t)\big\}.$$

Then $I_{\mathcal{Z}}$ is of full Lebesgue measure in I. We denote by $\hat{\mu} : I_{\mathcal{Z}} \longrightarrow \mathcal{M}_+$ the restriction of $\mu \in L^\infty(I; \mathcal{M}_+(\mathbb{T}^d))$ on $I_{\mathcal{Z}}$. Then since $\mathcal{M}_+(\mathbb{T}^d) \leq C(\mathbb{T}^d)^*$ and $C(\mathbb{T}^d)^*$ is naturally injected in $C^2(\mathbb{T}^d)^*$ through restriction of domains, that is through the mapping

$$C(\mathbb{T}^d)^* \ni \ell \mapsto \ell|_{C^2(\mathbb{T}^d)} \in C^2(\mathbb{T}^d)^*,$$

we can regard $\hat{\mu}$ as a function $\hat{\mu} : I_{\mathcal{Z}} \longrightarrow C^2(\mathbb{T}^d)^*$. As such the function $\hat{\mu}$ is Lipschitz, with Lipschitz constant $\leq \|\widetilde{W}\|_{\infty;KR}$. Indeed, for all $s, t \in I_{\mathcal{Z}}$ and all $\zeta \in \mathcal{Z}$ we have that

$$\begin{aligned}|\hat{\mu}_t(\zeta) - \hat{\mu}_s(\zeta)| &= |\bar{\phi}_\zeta(t) - \bar{\phi}_\zeta(s)| \leq \|\hat{\varphi}_\zeta\|_{\mathrm{Lip}}|t-s| \leq \|\widetilde{W}\|_{\infty;KR}\|\nabla\zeta\|_{\mathrm{Lip}_0}|t-s| \\ &\leq \|\widetilde{W}\|_{\infty;KR}\|\zeta\|_{C^2}|t-s|,\end{aligned}$$

which since $\mathcal{Z}$ is dense in $C^2(\mathbb{T}^d)$ in the C^2-norm $\|\cdot\|_{C^2}$ shows that

$$\|\hat{\mu}_t - \hat{\mu}_s\|_{C^2(\mathbb{T}^d)^*} = \sup_{\zeta\in\mathcal{Z}} \frac{|\hat{\mu}_t(\zeta) - \hat{\mu}_s(\zeta)|}{\|\zeta\|_{C^2}} \leq \|\widetilde{W}\|_{\infty;KR}|t-s|.$$

Therefore $\hat{\mu} : I_{\mathcal{Z}} \longrightarrow C^2(\mathbb{T}^d)^*$ has a Lipschitz extension $\bar{\mu} : I \longrightarrow C^2(\mathbb{T}^d)^*$ with the same Lipschitz constant $\|\widetilde{W}\|_{\infty;KR}$.

Now, since μ belongs in $L^\infty(I;\mathcal{M}_+(\mathbb{T}^d))$ by hypothesis, we can assume that $I_{\mathcal{Z}}$ has been chosen so that

$$\|\mu_t\|_{TV} = \mu_t(\mathbb{T}^d) \leq \|\mu\|_{\infty;TV} < +\infty$$

for all $t \in I_{\mathcal{Z}}$. Therefore, since $\{\mu_t\}_{t\in I_{\mathcal{Z}}}$ is in addition tight since $\{\mu_t\}_{t\in I_{\mathcal{Z}}} \subseteq \mathcal{M}_+(\mathbb{T}^d)$ it follows by lemma 4.3.2 that the Lipschitz extension $\bar{\mu} : I \longrightarrow C^2(\mathbb{T}^d)^*$ takes values in $\mathcal{M}_+(\mathbb{T}^d)$ and is weakly continuous.

We prove finally (4.45). So let $G \in C^\infty(I \times \mathbb{R}^d)$ be a $\mathbb{Z}^d$-periodic function, let $0 \leq s < t \leq T$ and let $\widetilde{W}$ be any curve in $L^\infty_{w^*}(I;\mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)^*)$ such that $p_*\widetilde{W} = W$. Let $f_\varepsilon \in C_c^\infty\big((s,t);[0,1]\big)$, $\varepsilon > 0$, be such that $f_\varepsilon \longrightarrow \mathbb{1}_{(s,t)}$ pointwise in I and such that

$$\lim_{\varepsilon\to 0}\int_I f'_\varepsilon(r)h(r)dr = h(s) - h(t), \quad \forall\, h \in C([0,T]). \tag{4.46}$$

Then since the pair (μ, W) satisfies the continuity equation we have for all $\varepsilon > 0$ that

$$\begin{aligned} 0 &= \int_0^T \left(\int_{\mathbb{R}^d} \partial_r[f_\varepsilon(r)G_r]d\widetilde{\bar{\mu}}_r + \langle \nabla_x[f_\varepsilon(r)G_r], \widetilde{W}_r\rangle \right) dr \\ &= \int_0^T \left(\int_{\mathbb{R}^d} [f'_\varepsilon(r)G_r + f_\varepsilon(r)\partial_r G_r]d\widetilde{\bar{\mu}}_r + \langle f_\varepsilon(r)\nabla_x G_r, \widetilde{W}_r\rangle \right) dr \\ &= \int_0^T f'_\varepsilon(r)\int_{\mathbb{R}^d} G_r d\widetilde{\bar{\mu}}_r + \int_0^T f_\varepsilon(r)\bigg(\int_{\mathbb{R}^d} \partial_r G_r d\widetilde{\bar{\mu}}_r + \langle \nabla_x G_r, \widetilde{W}_r\rangle \bigg) dr. \end{aligned}$$

Now, since the function $G \in C^\infty(I \times \mathbb{R}^d)$ is $\mathbb{Z}^d$-periodic, the function $\widetilde{G} : I \times \mathbb{T}^d \longrightarrow \mathbb{R}$ given $I \times \mathbb{T}^d \ni (r,x) \mapsto G\big(r, i(x)\big)$ is C^∞. In particular the curve $I \ni r \mapsto \widetilde{G}_r \in C(\mathbb{T}^d)$ is continuous with respect to the uniform norm in $C(\mathbb{T}^d)$ and therefore due to the weak continuity of $\bar{\mu}$, the function

$$I \ni r \mapsto \int G_r d\widetilde{\bar{\mu}} = \int \widetilde{G}_r d\bar{\mu}_r$$

is continuous. Therefore taking the limit $\varepsilon \to 0$ and using (4.46) in the first term of the sum above and the bounded convergence theorem in the second term, we get that

$$\int_{\mathbb{R}^d} G_t d\widetilde{\bar{\mu}}_t - \int_{\mathbb{R}^d} G_s d\widetilde{\bar{\mu}}_s = \int_s^t \left(\int_{\mathbb{R}^d} \partial_r G_r d\widetilde{\bar{\mu}}_r + \langle \nabla_x G_r, \widetilde{W}_r\rangle \right) dr$$

for the arbitrary representative $\widetilde{W}$ of W in $L^\infty_{w^*}(I;\mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)^*)$, as required. □

All the considerations in this section also remain valid if we consider the empirical current as taking values in the space $L^1(0,T;C^1(\mathbb{T}^d;\mathbb{R}^d))^*$ and there is no need to repeat the arguments.

4.4 Relative Compactness of the Empirical Current

In this section we prove the relative compactness of the empirical current and the empirical jump-rate processes and prove that the limit points of their laws are concentrated on solutions of the continuity equation. In particular, with any additional work it will follow by the general considerations of section 4.2.3 that the law of the empirical current is supported by trajectories $W : [0,T] \longrightarrow \widehat{\mathcal{M}}_0^d(\mathbb{T}^d)$ of Kantorovich-Rubinstein measures.

4.4.1 The Empirical Current

According to the results in appendix A.4 on completely regular Hausdorff topological spaces and in particular the Prokhorov-Le Cam theorem A.4.1, in order to prove the relative compactness of the sequence

$$W_*^N P^N \in \mathbb{P}L_{w^*}^\infty\big(I; \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*\big), \quad N \in \mathbb{N},$$

it suffices to check that it consists of Radon measures and that is uniformly tight. It is easy to see that the sequence $\{W_*^N P^N\}_{N\in\mathbb{N}}$ consists of Radon measures. Indeed, all Borel probability measures on a polish space are Radon and so since $P^N \in \mathbb{P}\Omega_N$ and Ω_N is a polish space with the restriction of the Skorohod metric the measures P^N are Radon. But from proposition 4.2.16 the empirical current function

$$W^N : \Omega_N \longrightarrow L_{w^*}^\infty(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$$

is a continuous function with respect to the restriction of the Skorohod topology on Ω_N and w^* topology on $L_{w^*}^\infty(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$ and so according to proposition A.4.3 the laws $W_*^N P^N$ are Radon on the Borel σ-algebra of the w^*-topology, i.e.

$$W_*^N P^N \in \mathbb{P}_R\big(L_{w^*}^\infty(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*), w^*\big).$$

Proposition 4.4.1 *The sequence*

$$Q^N := W_*^N P^N \in \mathbb{P}L_{w^*}^\infty(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*), \quad N \in \mathbb{N},$$

of the distributions of the empirical current process is relatively compact in the weak topology of $\mathbb{P}L_{w^*}^\infty(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$ *that is induced by the* w^**-topology.*

Proof Since by the Banach-Alaoglou theorem norm bounded subsets of a dual space X^* are relatively compact in the w^*-topology, it suffices to prove that

$$\lim_{A\uparrow\infty} \sup_{N\in\mathbb{N}} Q^N\big\{B_{L_{w^*}^\infty(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)}(0,A)^c\big\} = 0.$$

But for each $N \in \mathbb{N}$, we have by the bound (4.36) that

$$\begin{aligned} Q^N\big\{B_{L_{w^*}^\infty(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)}(0,A)^c\big\} &= P^N\big\{\|W^N\|_{L_{w^*}^\infty(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)} > A\big\} \\ &\le P^N\big\{\sqrt{d}\|g'\|_u\langle 1,\pi_0^N\rangle > A\big\} \\ &= \mu_0^N\Big\{\langle 1,\pi^N\rangle > \frac{A}{\sqrt{d}\|g'\|_u}\Big\}, \end{aligned}$$

where μ_0^N is the initial distribution of the law P^N of the diffusively rescaled ZRP. Therefore the required limit follows from (3.10). $\square$

We set now

$$DL^\infty_{w^*}(0,T;\mathcal{M}_+\times \mathrm{Lip}_0^*) := D(0,T;\mathcal{M}_+(\mathbb{T}^d))\times L^\infty_{w^*}(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$$

and consider the empirical density-current pair process

$$(\pi^N,W^N):(\Omega_N,P^N)\longrightarrow DL^\infty_{w^*}(0,T)$$

and the rest of this section is devoted to proving that any limit point R of the distributions

$$R^N := (\pi^N,W^N)_*P^N\in \mathbb{P}DL^\infty_{w^*}(0,T\mathcal{M}_+,\mathrm{Lip}_0^*)$$

of the sequence of the empirical density-current processes is concentrated on solutions of the continuity equation. We recall (4.20) according to which for all $G\in C_c^3((0,T)\times\mathbb{T}^d)$ and all $\delta>0$ we have that

$$\lim_{N\to\infty}\mathbb{P}^N\bigg\{\bigg|\int_0^T\big[\langle\partial_sG_s,\pi_s^N\rangle+\langle\nabla G_s,W_s^N\rangle\big]ds\bigg|>\delta\bigg\}=0.$$

If we denote by

$$\pi : DL^\infty_{w^*}(0,T;\mathcal{M}_+\times \mathrm{Lip}_0^*)\longrightarrow D(I;\mathcal{M}_+(\mathbb{T}^d)),$$

$$W : DL^\infty_{w^*}(0,T;\mathcal{M}_+\times \mathrm{Lip}_0^*)\longrightarrow L^\infty_{w^*}(I;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$$

the natural projections then the above limit can be rewritten as

$$\lim_{N\to\infty}R^N\bigg\{\bigg|\int_0^T\big[\langle\partial_tG_t,\pi_t\rangle+\langle\nabla G_t,W_t\rangle\big]dt\bigg|>\delta\bigg\}=0.$$

We claim that for all $G\in C_c^\infty(I^o\times\mathbb{T}^d)$ the function $f^G : DL^\infty_{w^*}(0,T;\mathcal{M}_+\times\mathrm{Lip}_0^*)\longrightarrow\mathbb{R}$ given by

$$f^G(\pi,W)=\int_0^T\big[\langle\partial_tG_t,\pi_t\rangle+\langle\nabla G_t,W_t\rangle\big]dt$$

is continuous. Indeed, we write $f^G(\pi,W)=f^{1,G}(\pi)+f^{2,G}(W)$ where

$$f^{1,G}(\pi)=\int_0^T\langle\partial_tG_t,\pi_t\rangle dt,\qquad f^{2,G}(W)=\int_0^T\langle\nabla_xG_t,W_t\rangle dt. \tag{4.47}$$

We begin first with the function $f_t^{1,G}$. We note that for any function $G\in C^1([0,T]\times\mathbb{T}^d)$ the induced function $I_G(\cdot,\cdot)\langle G,\cdot\rangle_{(\cdot)}:\mathbb{R}_+\times\mathcal{M}_+(\mathbb{T}^d)\longrightarrow\mathbb{R}$ defined by

$$I_G(t,\mu)\equiv\langle G,\pi\rangle_t=\langle G_t,\mu\rangle=\int_{\mathbb{T}^d}G_td\mu$$

satisfies properties (A.1) and (A.2) of proposition A.1.6. Indeed, any compact subset $\mathcal{K}$ of $\mathcal{M}_+(\mathbb{T}^d)$ is contained in some compact subset of the form

$$\mathcal{M}_+^L := \{\mu\in\mathcal{M}_+(\mathbb{T}^d)|\langle 1,\mu\rangle\le L\}$$

for some $L = L(K) \in \mathbb{R}_+$ and therefore

$$\begin{aligned}\sup_{\mu\in\mathcal{K}}\left|I_G(t+h,\mu)-I_G(t,\mu)\right| &\leq \sup_{\mu\in\mathcal{M}_+^L}\left|I_G(t+h,\mu)-I_G(t,\mu)\right| \\ &\leq L\sup_{x\in\mathbb{T}^d}\left|G(t+h,x)-G(t,x)\right| \stackrel{h\to 0}{\longrightarrow} 0\end{aligned}$$

which proves condition (A.1). For the second condition we consider the space $\mathcal{M}_+(\mathbb{T}^d)$ equipped with the Dudley metric $d_{\mathcal{D}}$ defined via the norm in (1.55). Of course we can do this since as we have seen this norm metrizes the weak topology on $\mathcal{M}_+(\mathbb{T}^d)$ and by proposition A.1.2 the Skorohod topology on $D(0,T;M)$ depends only the topology of M and not on the particular metric defining the topology. Then, for any $t \in [0,T]$, $\mu,\nu \in \mathcal{M}_+(\mathbb{T}^d)$ we have that

$$\left|I_G(t,\mu)-I_G(t,\nu)\right| = \left|\int_{\mathbb{T}^d} G_t d(\mu-\nu)\right| \leq \|G_t\|_{BL} d_{\mathcal{D}}(\mu,\nu) \leq \|G_t\|_{C^1} d_{\mathcal{D}}(\mu,\nu),$$

where we recall that $\|\cdot\|_{BL}$ is the bounded-Lipschitz norm defined in 1.56 and $\|\cdot\|_{C^1}$ is the uniform C^1-norm on $C^1(\mathbb{T}^d)$ given by $\|G\|_{C^1} := \|G\|_u + \|\nabla G\|_u$. Consequently, if we set

$$C_T^1(G) := \sup_{0\leq t\leq T}\|G_t\|_{C^1}$$

then we have that

$$\sup_{0\leq t\leq T}\left|I_G(t,\mu)-I_G(t,\nu)\right| \leq C_T^1(G) d_{\mathcal{D}}(\mu,\nu) \stackrel{d_{\mathcal{D}}(\mu,\nu)\to 0}{\longrightarrow} 0,$$

which proves the second condition (A.2).

Consequently, the induced function

$$D(0,T;\mathcal{M}_+) \ni (\pi_t)_{t\in\mathbb{R}_+} \mapsto \left(\int G_t d\pi_t\right)_{t\in\mathbb{R}_+} \in D(0,T;\mathbb{R})$$

is continuous. In our case, since $G \in C_c^3((0,T)\times\mathbb{T}^d)$ we have that $\partial_t G \in C_c^2((0,T)\times\mathbb{T}^d)$ and so the function

$$D(0,T;\mathcal{M}_+) \ni (\pi_t)_{t\in\mathbb{R}_+} \stackrel{I_{\partial G}}{\mapsto} \left(\int \partial_t G d\pi_t\right)_{t\in\mathbb{R}_+} \in D(0,T;\mathbb{R})$$

is continuous. Furthermore, by proposition A.1.7 the function

$$D(0,T,\mathbb{R}) \ni (x_t)_{t\in\mathbb{R}_+} \stackrel{I}{\mapsto} \left(\int_0^t x_s ds\right)_{t\in\mathbb{R}_+} \in C(0,T;\mathbb{R})$$

is continuous, and so finally since the evaluation mappings $e_t : C(0,T;\mathbb{R}) \longrightarrow \mathbb{R}$, defined by $e_t(x) = x(t)$ for all $x \in C(0,T;\mathbb{R})$ and all $t \in \mathbb{R}_+$ are continuous we see that $f_t^{1,G}$ is continuous since $f_t^{1,G} = e_t \circ I \circ I_{\partial G}$.

We prove next the continuity of $f^{2,G}$. We note that given any function $G \in C_c^1((0,T)\times\mathbb{T}^d;\mathbb{R}^d)$ the formula $I \ni t \mapsto G_t \in {}^{C^1(\mathbb{T}^d;\mathbb{R}^d)}/_{\mathbb{R}} \leq \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ defines an element G of $L^1(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))$ with

$$\|G\|_{L^1(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))} = \int_0^T \|G_t\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)} dt \leq \int_0^T \|D_x G_t\|_{C(\mathbb{T}^d;\mathbb{R}^{d\times d})} dt < +\infty.$$

In particular for any $G \in C_c^2((0,T)\times\mathbb{T}^d)$ the function $(\nabla_x G) : [0,T] \longrightarrow \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ given by

$$t \mapsto \nabla_x G_t \in {}^{C^1(\mathbb{T}^d;\mathbb{R}^d)}/_{\mathbb{R}} \leq \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$$

defines an element $\nabla_x G$ of $L^1(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))$, and then $f^{2,G} \equiv \langle \nabla_x G, \cdot\rangle_{[0,T]}$, where of course here $\langle\cdot,\cdot\rangle_{[0,T]}$ denotes the duality on the product $L^1(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)) \times L^\infty_{w^*}(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$. But by the definition of the w^*-topology a net $\{W^\alpha\}_{\alpha\in\mathcal{A}} \subseteq L^\infty_{w^*}(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$ converges to $W \in L^\infty_{w^*}(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$ iff

$$\lim_{\alpha\in\mathcal{A}} \langle G, W^\alpha\rangle_{[0,T]} = \langle G, W\rangle_{[0,T]}$$

for all $G \in L^1(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))$ and so in particular given $G \in C^\infty((0,T)\times\mathbb{T}^d)$, for any converging net $\{W^\alpha\}_{\alpha\in\mathcal{A}}$ with limit W we have that

$$f^{2,G}(W^\alpha) = \langle \nabla_x G, W^\alpha\rangle_{[0,T]} \longrightarrow \langle \nabla_x G, W\rangle_{[0,T]} = f^{2,G}(W)$$

which proves the continuity of $f^{2,G}$, and thus also of f^G.

Now, since f^G is continuous the set $\{|f^G| > \delta\}$ is open and so by the portmanteau theorem it follows that for any limit point R of the sequence $\{R^N\}$ along a subnet $(R^\alpha)_{\alpha\in\mathcal{A}}$ of R^N we have that

$$\begin{aligned} R\{|f^G| > \delta\} &= R\bigg\{\bigg|\int_0^T \big[\langle \partial_t G_t, \pi_t\rangle + \langle \nabla G_t, W_t\rangle\big]dt\bigg| > \delta\bigg\} \\ &\leq \liminf_{\alpha\in\mathcal{A}} R^\alpha\bigg\{\bigg|\int_0^T \big[\langle \partial_t G_t, \pi_t\rangle + \langle \nabla G_t, W_t\rangle\big]dt\bigg| > \delta\bigg\} = 0 \end{aligned}$$

for all $\delta > 0$ and all $G \in C_c^3((0,T)\times\mathbb{T}^d)$. Since this holds for all $\delta > 0$ it follows that

$$R\bigg\{\int_0^T \big[\langle \partial_t G_t, \pi_t\rangle + \langle \nabla G_t, W_t\rangle\big]dt = 0\bigg\} = 1, \quad \forall\, G \in C_c^3((0,T)\times\mathbb{T}^d).$$

Therefore if we can find a countable family $\mathcal{G} \subseteq C_c^3((0,T\times\mathbb{T}^d)$ such that

$$\bigcap_{G\in C_c^3((0,T)\times\mathbb{T}^d)} \{f^G = 0\} = \bigcap_{G\in\mathcal{G}} \{f^G = 0\} \tag{4.48}$$

it will follow that

$$R\bigg(\bigcap_{G\in C_c^3((0,T)\times\mathbb{T}^d)} \bigg\{\int_0^T \big[\langle \partial_t G_t, \pi_t\rangle + \langle \nabla G_t, W_t\rangle\big]dt = 0\bigg\}\bigg) = 1,$$

i.e. that R is concentrated on solutions of the continuity equation. To this end, let $\mathcal{G} \subseteq C_c^\infty((0,T)\times\mathbb{T}^d)$ be a countable set of smooth functions dense in $C_c^2((0,T)\times\mathbb{T}^d)$ (and thus also in $C_c^3((0,T)\times\mathbb{T}^d)$) with respect to the usual C^2-uniform norm of $C_c^2((0,T)\times\mathbb{T}^d)$, given by

$$\|G\|_{C_c^2((0,T)\times\mathbb{T}^d)} := \|G\|_u + \|\nabla G\|_u + \|D^2 G\|_u$$

where differentiation is with respect both to the time and space variables, the uniform norms in the right hand side are taken in the spaces $C((0,T)\times\mathbb{T}^d)$, $C((0,T)\times$

$\mathbb{T}^d;\mathbb{R}^{d+1})$ and $C((0,T)\times\mathbb{T}^d;\mathbb{R}^{(d+1)\times(d+1)})$, respectively, and $\mathbb{R}^{(d+1)\times(d+1)}$ is considered equipped with the Frobenius norm. Note that if we show that for any sequence $\{G^k\}\subseteq C_c^\infty((0,T)\times\mathbb{T}^d)$ such that $G^k\longrightarrow G\in C_c^2((0,T)\times\mathbb{T}^d)$ with respect to the C^2-uniform norm we have that $f^{G^k}\longrightarrow f^G$ pointwise in $DL^\infty_{w^*}(0,T;\mathcal{M}_+\times\mathrm{Lip}_0^*)$ then we will have that (4.48) holds. Indeed, if this is true, and $(\pi,W)\in\bigcup_{G\in\mathcal{G}}\{f^G=0\}$ then given any $G\in C_c^2((0,T)\times\mathbb{T}^d)$ there exists a sequence $\{G^k\}$ in $\mathcal{G}$ such that $\|G-G^k\|_{C_c^2((0,T)\times\mathbb{T}^d)}\longrightarrow 0$ and thus

$$f^G(\pi,W)=\lim_{k\to\infty}f^{G^k}(\pi,W)=0,$$

which shows that (4.48) holds.

So let $\{G^k\}_{k\in\mathbb{N}}\subseteq C_c^\infty((0,T)\times\mathbb{T}^d)$ be a sequence such that $\|G^k-G\|_{C_c^2((0,T)\times\mathbb{T}^d)}\longrightarrow 0$. Of course it suffices to prove that the sequences $\{f^{1,G^k}\}$ and $\{f^{2,G^k}(W)\}$ converge pointwise on the spaces $D(0,T;\mathcal{M}_+(\mathbb{T}^d))$ and $L^\infty_{w^*}(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$ to the functions $f^{1,G}$ and $f^{2,G}$, respectively, where for any $G\in C_c^2((0,T\times\mathbb{T}^d)$ the functions $f^{1,G},f^{2,G}$ are defined as in (4.47). For the sequence $\{f^{1,G^k}\}$ we begin by noting that if $G\in C_c^2((0,T)\times\mathbb{T}^d)$ and $\partial G\in C^1((0,T)\times\mathbb{T}^d)$ is the function given by $(t,x)\mapsto\partial_tG(t,x)$ we have that for all $t\in I$,

$$\|\partial_tG_t^k-\partial_tG_t\|_{C(\mathbb{T}^d)}\le\|\partial G^k-\partial G\|_{C((0,T)\times\mathbb{T}^d)}\le\|G^k-G\|_{C_c^2((0,T)\times\mathbb{T}^d)}\longrightarrow 0$$

and therefore, given $\pi\in D(0,T;\mathcal{M}_+(\mathbb{T}^d))$, for all $t\in I$ we have that

$$\lim_{k\to\infty}\int_{\mathbb{T}^d}\partial_tG^kd\pi_t=\int_{\mathbb{T}^d}\partial_tGd\pi_t.$$

For a given function $G\in C((0,T)\times\mathbb{T}^d)$ now, the function $\langle G,\pi\rangle$ given by $t\mapsto\int_{\mathbb{T}^d}G_td\pi_t$ is in $L^\infty(0,T)$, since $D(I;\mathcal{M}_+(\mathbb{T}^d))\subseteq L^\infty(0,T;\mathcal{M}_+(\mathbb{T}^d))$ and

$$\langle G,\pi\rangle_t=\int G_td\pi_t\le\|G_t\|_{C(\mathbb{T}^d)}\pi_t(\mathbb{T}^d)\le\|G\|_{C(I^o\times\mathbb{T}^d)}\|\pi\|_{L^\infty(0,T;\mathcal{M}_+(\mathbb{T}^d))}$$

for all $t\in I$. In our case, since $G^k\longrightarrow G$ in the C^2-uniform norm, there exists a constant $C\ge 0$ such that

$$\sup_{k\in\mathbb{N}}\|\partial G^k\|_{C_c((0,T)\times\mathbb{T}^d)}\le\sup_{k\in\mathbb{N}}\|G^k\|_{C_c^2((0,T)\times\mathbb{T}^d)}\le C<+\infty,$$

and therefore for the sequence $\{\langle\partial G^k,\pi\rangle\}\subseteq D(0,T;\mathbb{R})\subseteq L^\infty(0,T)$ we have that

$$\int\partial G_t^kd\pi_t\le C\|\pi\|_{L^\infty(0,T;\mathcal{M}_+(\mathbb{T}^d))}<+\infty$$

for all $t\in I$. Therefore since $\{\langle\partial G^k,\pi\rangle\}$ is uniformly bounded and converges pointwise to $\langle\partial G,\pi\rangle$ we have by the bounded convergence theorem that

$$\lim_{k\to\infty}\int_0^T\langle\partial G^k,\pi\rangle_tdt=\int_0^T\langle\partial G,\pi\rangle_tdt,\tag{4.49}$$

which proves that $f^{1,G^k}\longrightarrow f^{1,G}$ pointwise on $D(0,T;\mathcal{M}_+(\mathbb{T}^d))$.

Next, for the second term $f^{2,G}$ of the function f^G, as we have already noted, for any $G\in C_c^2((0,T)\times\mathbb{T}^d)$ the function $(\nabla_xG):I\longrightarrow\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ given by

$$t\mapsto\nabla_xG_t\in{}^{C^1(\mathbb{T}^d;\mathbb{R}^d)}/_{\mathbb{R}}\le\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$$

defines an element of $L^1(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))$ and for these such elements that are defined by the C^2-converging sequence $\{G^k\}\subseteq C_c^\infty((0,T)\times\mathbb{T}^d)$ we have that

$$\begin{aligned}\|\nabla_x G^k-\nabla_x G\|_{L^1(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))} &= \int_I \|\nabla_x(G_t^k-G_t)\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)}dt\\ &\leq \int_I \|D_x^2(G_t^k-G_t)\|_{C(\mathbb{T}^d;\mathbb{R}^{d\times d})}dt\\ &\leq T\cdot\|G^k-G\|_{C_c^2((0,T)\times\mathbb{T}^d)}\overset{k\to\infty}{\longrightarrow}0\end{aligned}$$

Therefore, for any $W\in L_{w^*}^\infty(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)\cong L^1(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))^*$ we have that

$$\int_0^T\langle\nabla_x G_t^k,W_t\rangle dt=W(\nabla_x G^k)\overset{k\to\infty}{\longrightarrow}W(\nabla_x G)=\int_0^T\langle\nabla_x G_t,W_t\rangle dt. \tag{4.50}$$

which proves that the sequence of functions $\{f^{2,G^k}\}_{k\in\mathbb{N}}$, converges pointwise to $f^{2,G}$ in $L_{w^*}^\infty(0,T;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$. So $\{f^{G^k}\}$ converges pointwise to f^G on $DL_{w^*}^\infty(0,T;\mathcal{M}_+\times\mathrm{Lip}_0^*)$ whenever $G_k\longrightarrow G$ in the C^2-uniform norm, as required for the proof of (4.48).

We note that exactly the same result can proved if in considers the empirical current process as taking values in the space $L_{w^*}^\infty(0,T;C^1(\mathbb{T}^d;\mathbb{R}^d)^*)$. Furthermore, in this case due to the separability of $C^1(\mathbb{T}^d;\mathbb{R}^d)$, according to lemma 4.2.1 proposition A.4.1 the space $L_{w^*}^\infty(0,T;C^1(\mathbb{T}^d;\mathbb{R}^d)^*)$ is submetrizable and thus by the Prokhorov-Le Cam theorem, in this case the sequence $\{W_*^NP^N\}\subseteq\mathbb{P}L_{w^*}^\infty(0,T;C^1(\mathbb{T}^d;\mathbb{R}^d)^*)$ is in addition sequentially relatively compact.

4.4.2 The Empirical Jump Rate

We continue to denote by P^N the sequence of distributions of the ZRPs starting from a sequence of initial measures μ^N associated to a macroscopic profile $\mu_0\in\mathcal{M}_+(\mathbb{T}^d)$, speeded up by N^2. As a shorthand we will occasionally write $I:=[0,T]$, $I^o:=(0,T)$. Also *for simplicity in this section we will assume that jump rate function g is bounded.* As we have seen (with the convention that $p(\mathbb{Z}^d)=2d$) we have that

$$\lim_{N\to\infty}\mathbb{P}^N\left\{\left|\langle G_t,\pi_t^N\rangle-\langle G_0,\pi_0^N\rangle-\int_0^t\langle\partial_sG_s,\pi_s^N\rangle+\langle\Delta G_s,\sigma_s^N\rangle ds\right|\right\}=0 \tag{4.51}$$

for all $G\in C^3(I\times\mathbb{T}^d)$ and all $t\in\mathbb{R}_+$. We consider the space

$$DL_{w^*}^\infty(I;\mathcal{M}_+(\mathbb{T}^d)):=D\big(I;\mathcal{M}_+(\mathbb{T}^d)\big)\times L_{w^*}^\infty\big(I;\mathcal{M}(\mathbb{T}^d)\big)$$

and set

$$R^{N,g}:=(\pi^N,\sigma^N)_*P^N\in\mathbb{P}DL_{w^*}^\infty(I;\mathcal{M}_+(\mathbb{T}^d)). \tag{4.52}$$

Of course if we denote by $id_{DL_{w^*}^\infty(I;\mathcal{M}_+(\mathbb{T}^d))}=(\pi,\sigma)$ the natural projections then (4.51) can be rewritten as

$$\lim_{N\to\infty}R^{N,g}\left\{\left|\langle G_t,\pi_t\rangle-\langle G_0,\pi_0\rangle-\int_0^t\langle\partial_sG_s,\pi_s\rangle+\langle\Delta G_s,\sigma_s\rangle ds\right|\right\}=0 \tag{4.53}$$

We denote the marginals of $R^{N,g}$ on $\mathbb{P}D\big(I;\mathcal{M}_+(\mathbb{T}^d)\big)$ and $\mathbb{P}L^\infty_{w^*}\big(I;\mathcal{M}_+(\mathbb{T}^d)\big)$ by P^N and $Q^{N,g}$ respectively. Since for the function $\sigma^N:\mathbb{M}^d_N\longrightarrow\mathcal{M}_+(\mathbb{T}^d)$ we have that

$$\|\sigma^N\|_{TV}=\sigma^N(\mathbb{T}^d)=\frac{1}{N^d}\sum_{x\in\mathbb{T}^d}g\big(\eta(x)\big)\le\|g\|_u$$

we also have that the process $\sigma^N:D(I;\mathbb{M}^d_N)\longrightarrow L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$ takes values in the norm-bounded ball $B_{L^\infty_{w^*}(I;\mathcal{M}(\mathbb{T}^d))}(0,\|g\|_u)$, that is

$$\{Q^{N,g}\}_{N\in\mathbb{N}}\subseteq\mathbb{P}B_{L^\infty_{w^*}(I;\mathcal{M}(\mathbb{T}^d))}(0,\|g\|_u).$$

Lemma 4.4.1 *The subset*

$$L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d)):=\big\{\nu\in L^\infty(I;\mathcal{M}(\mathbb{T}^d))\,\big|\,\nu_t\in\mathcal{M}_+(\mathbb{T}^d)\ \ a.s.\text{-}\forall\,t\in I\big\}$$

is a w^*-closed subset of $L^\infty_{w^*}(I;\mathcal{M}(\mathbb{T}^d))$.

Proof Let $C_+(\mathbb{T}^d)$ denote the set of all non-negative continuous functions on the torus. Then as we know

$$\mathcal{M}_+(\mathbb{T}^d)=\big\{\nu\in\mathcal{M}(\mathbb{T}^d)\,|\,\langle f,\nu\rangle\ge0\ \ \forall\,f\in C_+(\mathbb{T}^d)\big\}.$$

We claim that

$$L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))\overset{(*)}{=}\left\{\nu\in L^\infty(I;\mathcal{M}(\mathbb{T}^d))\,\Big|\,\int_I\langle f_t,\nu_t\rangle dt\ge0\ \ \forall f\in L^1(I;C_+(\mathbb{T}^d))\right\}.$$

Indeed, one inclusion us obvious, so let ν belonging in the set in the right hand side and we will prove that it belongs in $L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$. Let $f\in C_+(\mathbb{T}^d)$. Then by the Lebesgue differentiation theorem we have that for almost all $t\in I$,

$$0\le\lim_{\varepsilon\to0}\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\langle f,\nu_s\rangle ds=\langle f,\nu_t\rangle,$$

and since $C(\mathbb{T}^d)$ is separable it follows that $\nu\in L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$. Let now $\{\nu^\alpha\}_{\alpha\in\mathcal{A}}\subseteq L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$ be a net such that

$$\nu^\alpha\longrightarrow\nu\in L^\infty_{w^*}(I;\mathcal{M}(\mathbb{T}^d))$$

in the w^*-topology and we have to show that $\nu_t\in\mathcal{M}_+(\mathbb{T}^d)$ for almost all $t\in I$. Since $\{\nu^\alpha\}_{\alpha\in\mathcal{A}}\subseteq L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$, for all $f\in L^1(I;C_+(\mathbb{T}^d))$ we have that

$$\int_I\int_{\mathbb{T}^d}f_t d\nu_t=\lim_{\alpha\in\mathcal{A}}\int_I\int_{\mathbb{T}^d}f_t d\nu^\alpha_t\ge0$$

which according to equality $(*)$ proves the claim. □

Lemma 4.4.2 *Let F be a closed subset of the polish space X. Then $\mathbb{P}F$ is a closed subspace of $\mathbb{P}X$.*

Proof Let $\{\mu_n\} \subseteq \mathbb{P}F$ be a sequence converging to $\mu \in \mathbb{P}X$ weakly. Then

$$\text{supp}\mu \subseteq \text{K-}\liminf_{n\to\infty} \text{supp}\mu_n,$$

that is for all $x \in \text{supp}\mu$ there exists a sequence $x_n \in \text{supp}\mu_n$, $n \in \mathbb{N}$ such that $x_n \longrightarrow x$, and since $\{\mu_n\} \subseteq \mathbb{P}F$ we have that $\{x_n\} \subseteq F$. Since F is closed it follows that $x \in F$, which since $x \in \text{supp}\mu$ was arbitrary shows that $\text{supp}\mu \subseteq F$, and so $\mu \in \mathbb{P}F$. $\square$

Proposition 4.4.2 *The sequence $\{R^{N,g}\}_{N\in\mathbb{N}} \subseteq \mathbb{P}DL^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$ is relatively compact, i.e. there exists $R^g \in \mathbb{P}DL^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$ and a subsequence $\{R^{k_N,g}\}$ of $\{R^{N,g}\}$ such that $R^{k_N,g} \longrightarrow R^g$ weakly.*

Proof We already know that the sequence $\{P^N\}$ of the first marginals of $\{R^{N,g}\}$ is relatively compact and so we only have to prove that the sequence $\{Q^{N,g}\}$ of the second marginals is relatively compact. Since the sequence $\{Q^{N,g}\}$ is supported by the compact metrizable ball $B_{L^\infty_{w^*}(I;\mathcal{M}(\mathbb{T}^d))}$ with respect to the w^*-topology of $L^\infty_{w^*}(I;\mathcal{M}(\mathbb{T}^d))$ we have by the Banach Alaoglou theorem that there exists $Q^g \in \mathbb{P}L^\infty_{w^*}(I;\mathcal{M}(\mathbb{T}^d))$ and a subsequence $\{Q^{k_N,g}\}$ of $\{Q^{N,g}\}$ such that $Q^{k_N,g} \longrightarrow Q^g$ weakly. But since $\{Q^{N,g}\}_{N\in\mathbb{N}} \subseteq \mathbb{P}L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$ and $L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$ is a w^*-closed subset of $L^\infty_{w^*}(I;\mathcal{M}(\mathbb{T}^d))$ it follows by lemmas 4.4.1 and 4.4.2 that $Q^g \in \mathbb{P}L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$ as required. $\square$

Definition 4.4.1 We say that *a density-diffusion rate pair*

$$(\pi,\sigma) \in L^\infty_{w^*}\big(I;\mathcal{M}_+(\mathbb{T}^d)\big) \times L^\infty\big(\mathcal{M}_+(\mathbb{T}^d)\big)$$

satisfies the heat equation

$$\partial_t \pi_t = \Delta_x \sigma, \tag{4.54}$$

in $I^o \times \mathbb{T}^d$ iff for all smooth $\mathbb{Z}^d$-periodic test functions $G \in C^\infty(I^o \times \mathbb{R}^d)$ of compact support in I^o we have that

$$\int_0^T \bigg(\int_{\mathbb{R}^d} \partial_t G_t d\widetilde{\pi}_t + \int_{\mathbb{R}^d} \Delta_x G_t d\widetilde{\pi}^g_t \bigg) dt = 0 \tag{4.55}$$

for any representative

$$(\widetilde{\pi},\widetilde{\sigma}) \in L^\infty\big(I;\mathcal{M}_+(\mathbb{R}^d)\big)^2$$

of (π,σ), i.e. for any such pair $(\widetilde{\pi},\widetilde{\sigma})$ for which $p_*(\widetilde{\pi},\widetilde{\sigma}) := (p_*\widetilde{\pi}, p_*\widetilde{\sigma}) = (\pi,\sigma)$.

We consider the gradient operator $\nabla : \mathcal{M}_+(\mathbb{R}^d) \longrightarrow \text{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)^*$ defined by

$$\nabla\mu(F) = \langle F, \nabla\mu \rangle = -\int_{\mathbb{R}^d} \text{div}F d\mu,$$

for all $(\mu, F) \in \mathcal{M}_+(\mathbb{R}^d) \times \text{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)$. Let us check that this operator is well defined. First, we check that given $\mu \in \mathcal{M}_+(\mathbb{R}^d)$ the functional $\nabla\mu$ is bounded. So let $F \in \text{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)$. Then

$$\|\text{div}F\|_{L^\infty(\mathbb{R}^d)} \le \sum_{j=1}^d \|\partial_j F_j\|_{L^\infty(\mathbb{R}^d)} \le \sum_{j=1}^d \|F_j\|_{\text{Lip}_0(\mathbb{R}^d;\mathbb{R})} \le 2\|F\|_{\text{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)}$$

and so for each $\mu \in \mathcal{M}_+(\mathbb{R}^d)$, $\nabla\mu$ is bounded with

$$\|\nabla\mu\|_{\mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)^*} \leq 2\|\mu\|_{TV}.$$

We consider also the gradient operator $\nabla : \mathcal{M}_+(\mathbb{T}^d) \longrightarrow \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*$ defined by

$$\nabla\mu = p_*(\nabla\widetilde{\mu})$$

where $\widetilde{\mu} \in \mathcal{M}(\mathbb{R}^d)$ is any measure such that $p_*\widetilde{\mu} = \mu$. Obviously, this definition does not depend on the choice of $\widetilde{\mu}$, since if $\widetilde{\mu}, \widetilde{\widetilde{\mu}} \in \mathcal{M}(\mathbb{R}^d)$ are two measures such that $p_*\widetilde{\mu} = p_*\widetilde{\widetilde{\mu}}$ then

$$\int G d\widetilde{\mu} = \int G d\widetilde{\widetilde{\mu}}$$

for all bounded $\mathbb{Z}^d$-periodic functions, and therefore for all $F \in \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ we have that

$$\langle F, p_*(\nabla\widetilde{\mu})\rangle = \langle F\circ p, \nabla\widetilde{\mu}\rangle = -\int_{\mathbb{R}^d} \mathrm{div}(F\circ p)d\widetilde{\mu} = -\int_{\mathbb{R}^d} \mathrm{div}(F\circ p)d\widetilde{\widetilde{\mu}} = \langle F, p_*(\nabla\widetilde{\widetilde{\mu}})\rangle.$$

In particular the action of the gradient operator on measures on the torus can be given by

$$\langle F, \nabla\mu\rangle = \langle F\circ p, \nabla(i_*\mu)\rangle = -\int_{\mathbb{R}^d} \mathrm{div}(F\circ p)di_*\mu$$

for all $(\mu, F) \in \mathcal{M}_+(\mathbb{T}^d)\times\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$, where $i : \mathbb{T}^d \hookrightarrow \mathbb{R}^d$ is the Borel injection.

Note that by definition the gradient operator commutes with the push forward operators $p_* : \mathcal{M}(\mathbb{R}^d) \longrightarrow \mathcal{M}(\mathbb{T}^d)$ and $p_* : \mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)^* \longrightarrow \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*$ induced by the universal covering $p : \mathbb{R}^d \longrightarrow \mathbb{T}^d$ of the torus, i.e. that

$$\nabla(p_*\mu) = p_*(\nabla\mu) \tag{4.56}$$

for all $\mu \in \mathcal{M}(\mathbb{R}^d)$.

We will also consider the discrete gradient operators

$$\nabla^N : \mathcal{M}_+(M) \longrightarrow \mathcal{M}_0^d(M) \leq \mathrm{Lip}_0(M;\mathbb{R}^d)^*, \quad N \in \mathbb{N},$$

where $M = \mathbb{R}^d$ or $\mathbb{T}^d$, given by

$$\nabla^N\mu = N\sum_{j=1}^d \left(\mu - \tau_{\frac{e_j}{N}*}\mu\right)\cdot e_j.$$

Then, if for all $F \in \mathrm{Lip}_0(M;\mathbb{R}^d)$ we denote by

$$\mathrm{div}^N F(x) := N\sum_{j=1}^d \Big(F^j\Big(x + \frac{e_j}{N}\Big) - F^j(x)\Big)$$

the discrete divergence of F, we have that

$$\nabla^N\mu(F) = -\int \mathrm{div}^N F d\mu.$$

We note also that the discrete divergences define a family of bounded linear operators $\mathrm{div}^N : \mathrm{Lip}_0(M;\mathbb{R}^d) \longrightarrow C(M)$, $N \in \mathbb{N}$, with uniformly bounded operator norm by 2, that is

$$\sup_{N\in\mathbb{N}} \|\mathrm{div}^N F\|_u \leq 2\|F\|_{\mathrm{Lip}_0(M;\mathbb{R}^d)}, \quad \forall F \in \mathrm{Lip}_0(M;\mathbb{R}^d).$$

Also we note that since the covering map $p : \mathbb{R}^d \longrightarrow \mathbb{T}^d$ is homomorphism, for all $\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$ we have that

$$\begin{aligned}
\mathrm{div}^N(F \circ p) &= N\sum_{j=1}^{d}\Big(F^j\Big(p\big(x+\frac{e_j}{N}\big)\Big) - F^j\big(p(x)\big)\Big) \\
&= N\sum_{j=1}^{d}\Big(F^j\Big(p(x)+p\big(\frac{e_j}{N}\big)\Big) - F^j\big(p(x)\big)\Big) = [\mathrm{div}^N F]\circ p.
\end{aligned}$$

It is obvious that the discrete gradient operators are w^*-continuous in the sense that

$$\mu = w^*\text{-}\lim_{k\to\infty}\mu_k \quad \Longrightarrow \quad \nabla^N\mu = w^*\text{-}\lim_{k\to\infty}\nabla^N\mu_k.$$

Proposition 4.4.3 (a) *The operators* $\nabla^N : L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d)) \longrightarrow L^\infty_{w^*}(I;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$, $N \in \mathbb{N}$, *induced by the discrete gradient operators is* w^**-continuous, i.e. whenever*

$$\int_0^T \langle f_t, \mu_t\rangle dt = \lim_{\alpha\to\infty}\int_0^T \langle f_t, \mu_t^\alpha\rangle dt, \quad \forall f \in L^1(I;C(\mathbb{T}^d)) \tag{4.57}$$

it follows that

$$\int_0^T \langle F_t, \nabla^N\mu_t\rangle dt = \lim_{\alpha\to\infty}\int_0^T \langle F_t, \nabla^N\mu_t^\alpha\rangle dt, \quad \forall\, F \in L^1(I;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))$$

(b) *The restriction of the gradient operators on* $L^\infty_{w^*}(I;\mathcal{M}_{+,ac}(\mathbb{T}^d))$ *converges pointwise to the gradient operator with respect to the* w^**-topology of* $L^\infty_{w^*}(I;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$.
(c) *The gradient operator* $\nabla : L^\infty_{w^*}(I;\mathcal{M}_{+,ac}(\mathbb{T}^d)) \longrightarrow L^\infty_{w^*}(I;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$ *is* $(\mathcal{B},\mathfrak{Ba})$*-measurable, where by* $\mathcal{B}$ *and* $\mathfrak{Ba}$ *we denote the Borel and Baire* σ*-algebras induced by the* w^**-topology on the domain and the target space, respectively.*

Proof (a) Let $F \in L^1(I;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))$. Then for almost all $t \in I$ we have that $\langle F_t, \nabla^N\mu_t\rangle = \langle \mathrm{div}^N F_t, \mu_t\rangle$. Obviously the curve $I \ni t \mapsto \mathrm{div}^N F_t \in C(\mathbb{T}^d)$ belongs in $L^1(I;C(\mathbb{T}^d))$ since

$$\int_0^T \|\mathrm{div}^N F_t\|_u dt \leq 2\int_0^T \|F_t\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)} dr < +\infty.$$

Therefore, given a net $(\mu^\alpha) \subseteq L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$ satisfying (4.57), we have that

$$\begin{aligned}
\int_0^T \langle F_t, \nabla^N\mu_t\rangle dt &= \int_0^T \langle \mathrm{div}^N F_t, \mu_t\rangle dt = \lim_{\alpha\to\infty}\int_0^T \langle \mathrm{div}^N F_t, \mu_t^\alpha\rangle dt \\
&= \lim_{\alpha\to\infty}\int_0^T \langle F_t, \nabla^N\mu_t^\alpha\rangle dt
\end{aligned}$$

for all $F \in L^1(I;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))$ as required.
(b) Let $\mu \in L^\infty_{w^*}(I;\mathcal{M}_{+,ac}(\mathbb{T}^d))$. We have to prove that $\nabla^N\mu \longrightarrow \nabla\mu$ as $N\to\infty$ in the w^*-topology of $L^\infty_{w^*}(I;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*)$. So let $F \in L^1(I;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))$ be arbitrary and we have to prove that

$$\lim_{N\to\infty}\langle F,\nabla^N\mu\rangle_I = \langle F,\nabla\mu\rangle_I. \tag{4.58}$$

We have that

$$\langle F,\nabla^N\mu\rangle_I = \int_0^T\int \mathrm{div}^N F_t d\mu_t dt.$$

Let $t\in I$. Since $F_t \in \mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)$, we have that $\mathrm{div}^N F_t \longrightarrow \mathrm{div}F_t$ pointwise almost surely with respect to Lebesgue measure in $\mathbb{T}^d$. Since $\mu_t \ll m_{\mathbb{T}^d}$ it follows by the bounded convergence theorem that

$$\lim_{N\to\infty}\int \mathrm{div}^N F_t d\mu_t = \int \mathrm{div}F_t d\mu_t =: h(t)$$

for all $t\in I$. But since $(\mu,F)\in L^\infty_{w^*}(I;\mathcal{M}_{+,ac}(\mathbb{T}^d))\times L^1(I;\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d))$ we have that

$$\sup_{N\in\mathbb{N}}\int \mathrm{div}^N F_t d\mu_t \le 2\|F_t\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)}\|\mu\|_{\infty;TV}$$

for all $t\in I$. Therefore the sequence $\{h^N\}$ of functions defined by $h^N(t) = \langle \mathrm{div}^N F_t,\mu\rangle$ for $t\in I$, is dominated by the function $\bar{h}\in L^1(I)$ given by $\bar{h}(t) = 2\|\mu\|_{\infty;TV}\|F_t\|_{\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)}$ and so since $h^N \longrightarrow h$ pointwise, it follows by the dominated convergence theorem that (4.58) holds. (c) is a consequence of (b). ☐

Proposition 4.4.4 *A density-diffusion rate pair* $(\pi,\sigma)\in DL^\infty_{w^*}\big(I;\mathcal{M}_+(\mathbb{T}^d)\times\mathcal{M}_+(\mathbb{T}^d)\big)$ *satisfies the diffusion equation iff the pair* $(\pi,\nabla\sigma)\in DL^\infty_{w^*}\big(I;\mathcal{M}_+(\mathbb{T}^d)\times\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*\big)$ *satisfies the continuity equation.*

Proof Let $i:\mathbb{T}^d\cong[0,1)^d\longrightarrow\mathbb{R}^d$ denote the Borel injection. We assume first that the pair (π,σ) satisfies the diffusion equation. Then

$$\int_0^T\int \partial_t G_t d\widetilde{\pi}_t dt = -\int_0^T\int \Delta_x G d\widetilde{\sigma}_t dt = \int_0^T\langle\nabla_x G,\nabla\widetilde{\sigma}_t\rangle dt \tag{4.59}$$

for all $G\in C^\infty((0,T)\times\mathbb{R}^d)$ that are $\mathbb{Z}^d$-periodic and of bounded support in time, and all $(\widetilde{\pi},\widetilde{\sigma})\in DL(0,T;\mathcal{M}_+(\mathbb{R}^d)\times\mathcal{M}_+(\mathbb{R}^d))$ such that $p_*(\widetilde{\pi},\widetilde{\sigma}) = (\pi,\sigma)$. Let now $(\widetilde{\pi},\widetilde{W})\in DL^\infty_{w^*}\big(0,T;\mathcal{M}_+(\mathbb{R}^d)\times\mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)\big)$ be such that $p_*(\widetilde{\pi},\widetilde{W}) = (\pi,\nabla\sigma)$. If we show that

$$\int_0^T\langle\nabla_x G,\widetilde{W}_t\rangle dt = \int_0^T\langle\nabla_x G,\nabla\widetilde{\sigma}_t\rangle dt \tag{4.60}$$

for all $\mathbb{Z}^d$-periodic and of bounded support in time functions $G\in C^\infty((0,T)\times\mathbb{R}^d)$, it will follow that the pair $(\pi,\nabla\sigma)$ satisfies the continuity equation

$$\partial_t\pi = -\mathrm{div}(\nabla\sigma)$$

on $(0,T)\times\mathbb{T}^d$ in the sense of distributions. But (4.60) indeed holds, since by (4.56) it follows that for all $t\in I$,

$$p_*\widetilde{W}_t=\nabla\sigma_t=\nabla(p_*\widetilde{\sigma}_t)=p_*(\nabla\widetilde{\sigma}_t),$$

which obviously implies (4.60).

Suppose conversely that the pair $(\pi,\nabla\sigma)$ satisfies the continuity equation. Then

$$\int_0^T\int\partial_tG_td\widetilde{\pi}_tdt=\int_0^T\langle\nabla_xG,\widetilde{W}_t\rangle dt \tag{4.61}$$

for all $G\in C^\infty((0,T)\times\mathbb{R}^d)$ that are $\mathbb{Z}^d$-periodic and of bounded support in time, and all $(\widetilde{\pi},\widetilde{W})\in DL(I;\mathcal{M}_+(\mathbb{R}^d)\times\mathrm{Lip}_0(\mathbb{R}^d;\mathbb{R}^d)^*)$ such that $p_*(\widetilde{\pi},\widetilde{W})=(\pi,\nabla\sigma)$. Let now $\widetilde{\sigma}\in L^\infty_{w^*}(I;\mathcal{M}(\mathbb{R}^d))$ be such that $p_*\widetilde{\sigma}=\sigma$. Then the functional $\widetilde{W}:=\nabla\widetilde{\sigma}\in\mathrm{Lip}_0(\mathbb{T}^d;\mathbb{R}^d)^*$ satisfies $p_*\widetilde{W}=p_*(\nabla\widetilde{\sigma})=\nabla(p_*\widetilde{\sigma})=\nabla\sigma$ and therefore by (4.61) we have that

$$\int_0^T\int\partial_tG_td\widetilde{\pi}_tdt=\int_0^T\langle\nabla_xG,\nabla\widetilde{\sigma}_t\rangle dt=-\int_0^T\langle\Delta_xG,\widetilde{\sigma}_t\rangle dt$$

for all $\mathbb{Z}^d$-periodic and of bounded support in time functions $G\in C^\infty((0,T)\times\mathbb{R}^d)$, and thus the pair (π,σ) satisfies the diffusion equation, as required. □

Corollary 4.4.1 *Let $(\pi,\sigma)\in L^\infty\big(I;\mathcal{M}_+(\mathbb{T}^d)\big)^2$ be a density-diffusion rate pair satisfying the diffusion equation. Then there exists a weakly continuous representative $\bar{\pi}$ in the class of π in $L^\infty(I;\mathcal{M}_+(\mathbb{T}^d))$ modulo a.s. equality, and for this continuous representative we have that*

$$\int_{\mathbb{T}^d}G_td\bar{\pi}_t-\int_{\mathbb{T}^d}G_sd\bar{\pi}_s=\int_s^t\left(\int_{\mathbb{T}^d}\partial_rG_rd\bar{\pi}_r+\langle\Delta_xG_r,\sigma_r\rangle\right)dr,$$

for all $G\in C^\infty(I\times\mathbb{T}^d)$ and all $0\le s\le t\le T$, in the sense that for any representative $\widetilde{\sigma}\in L^\infty_{w^*}\big(I;\mathcal{M}_+(\mathbb{R}^d)\big)$ of σ it holds that

$$\int_{\mathbb{R}^d}G_t\circ id\bar{\pi}_t-\int_{\mathbb{R}^d}G_s\circ id\bar{\pi}_s=\int_s^t\left(\int_{\mathbb{R}^d}(\partial_rG_r)\circ id\bar{\pi}_r+\langle\Delta_xG_r,\widetilde{\sigma}_r\rangle\right)dr, \tag{4.62}$$

for all $\mathbb{Z}^d$-periodic in space functions $G\in C^\infty(I\times\mathbb{R}^d)$ and all $0\le s\le t\le T$, where $i:\mathbb{T}^d\hookrightarrow\mathbb{R}^d$ is the Borel injection.

Proof By the previous proposition the density-current pair $(\pi,-\nabla\sigma)$ satisfies the continuity equation and so by proposition 4.3.1 there exists a continuous representative $\pi\in C(I;\mathcal{M}_+(\mathbb{T}^d))$ in the class of π and for this continuous representative we have that for any $\widetilde{W}\in L^\infty_{w^*}\big(I;\mathrm{Lip}_0(\mathbb{R}^d,\mathbb{R}^d)^*\big)$ such that $p_*\widetilde{W}=-\nabla\sigma$ it holds that

$$\int_{\mathbb{R}^d}G_t\circ id\bar{\pi}_t-\int_{\mathbb{R}^d}G_s\circ id\bar{\pi}_s=\int_s^t\left(\int_{\mathbb{R}^d}(\partial_rG_r)\circ id\bar{\pi}_r+\langle\nabla G_r,\widetilde{W}_r\rangle\right)dr,$$

for all $\mathbb{Z}^d$-periodic in space functions $G\in C^\infty(I\times\mathbb{R}^d)$ and all $0\le s\le t\le T$. But as we have seen in the proof of the previous proposition whenever $\widetilde{\sigma}\in L^\infty(I;\mathcal{M}(\mathbb{R}^d))$ is such that $p_*\widetilde{\sigma} = \sigma$ then we have $p_*\nabla\widetilde{\sigma} = \nabla\sigma$ and therefore

$$\begin{aligned}\int_{\mathbb{R}^d} G_t \circ id\bar{\pi}_t - \int_{\mathbb{R}^d} G_s \circ id\bar{\pi}_s &= \int_s^t \left(\int_{\mathbb{R}^d} (\partial_r G_r) \circ id\bar{\pi}_r - \langle \nabla G_r, \nabla\widetilde{\sigma}_r \rangle \right) dr \\ &= \int_s^t \left(\int_{\mathbb{R}^d} (\partial_r G_r) \circ id\bar{\pi}_r + \langle \Delta_x G_r, \widetilde{\sigma}_r \rangle \right) dr\end{aligned}$$

for all $\mathbb{Z}^d$-periodic in space functions $G \in C^\infty(I \times \mathbb{R}^d)$ and all $0 \leq s \leq t \leq T$. □

Proposition 4.4.5 *Let R^g be any limit point of the sequence $\{R^{N,g}\}_{N\in\mathbb{N}}$. Then the second marginal Q^g of R^g is concentrated on paths $(\sigma_t)_{0\leq t\leq T} \in L^\infty(I; \mathcal{M}_+(\mathbb{T}^d))$ consisting of measures absolutely continuous with respect to the Lebesgue measure with density uniformly bounded by $\|g\|_u$, i.e.*

$$Q^g\left\{\sigma \in L^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d)) \,\Big|\, \sigma_t \ll m_{\mathbb{T}^d},\ \left\|\frac{d\pi^g_t}{dm_{\mathbb{T}^d}}\right\|_{L^\infty(\mathbb{T}^d)} \leq \|g\|_u,\ \ a.s.\text{-}\forall\, t \in I\right\} = 1$$

and R^g is concentrated on pairs $(\pi, \sigma) \in DL^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d) \times L^\infty(\mathbb{T}^d))$ satisfying the weak diffusion equation, i.e. it holds that

$$R^g\left\{(\pi, \sigma) \in DL^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d) \times L^\infty(\mathbb{T}^d)) \,\Big|\, \partial_t\pi = \Delta_x\sigma \ \text{on}\ I^o \times \mathbb{T}^d\right\} = 1.$$

Proof Let $G \in C(\mathbb{T}^d)$. For all $N \in \mathbb{N}$ we have that

$$|\langle G, \sigma^N\rangle| \leq \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \left|G\left(\frac{x}{N}\right)\right| g(\eta(x)) \leq \frac{\|g\|_u}{N^d} \sum_{x\in\mathbb{T}^d_N} \left|G\left(\frac{x}{N}\right)\right|.$$

Now, since G is continuous we have that

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \left|G\left(\frac{x}{N}\right)\right| = \int_{\mathbb{T}^d} |G(x)| dx$$

and so given $\varepsilon > 0$ there exists $N_0 = N_0(G, \varepsilon) \in \mathbb{N}$ such that

$$N \geq N_0 \quad \Longrightarrow \quad \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \left|G\left(\frac{x}{N}\right)\right| \leq \int_{\mathbb{T}^d} |G| + \frac{\varepsilon}{\|g\|_u}$$

It follows that

$$Q^{N,g}\left\{\sigma \in L^\infty(I; \mathcal{M}_+(\mathbb{T}^d)) \,\Big|\, \operatorname*{ess\,sup}_{0\leq t\leq T} |\langle G, \sigma_t\rangle| \leq \|g\|_u \int_{\mathbb{T}^d} |G| + \varepsilon\right\}$$

for all $N \geq N_0$. We claim now that the function

$$L^\infty(I; \mathcal{M}_+(\mathbb{T}^d)) \ni \sigma = (\sigma_t)_{0\leq t\leq T} \overset{h^G}{\mapsto} \|\langle G, \sigma\rangle\|_{L^\infty(I)} = \operatorname*{ess\,sup}_{0\leq t\leq T} |\langle G, \sigma_t\rangle| \in \mathbb{R}_+$$

is lower semicontinuous. We consider first the operator $I_G : L^\infty(I; \mathcal{M}_+(\mathbb{T}^d)) \longrightarrow L^\infty(I)$ given by

$$I_G(\mu)(t) = \langle G, \mu_t\rangle$$

and we will show that I_G is w^*-continuous. Indeed, if $\{\mu^\alpha\}_{\alpha\in\mathcal{A}} \subseteq L^\infty(I;\mathcal{M}_+(\mathbb{T}^d))$ is a net converging to $\mu \in L^\infty(I;\mathcal{M}_+(\mathbb{T}^d))$ in the w^*-topology, i.e.

$$\lim_\alpha \int_0^T \int F_t d\mu_t^\alpha dt = \int_0^T \int F_t d\mu_t dt, \qquad \forall\, F \in L^1(I;C(\mathbb{T}^d)),$$

then for all $f \in L^1(I)$ we have that

$$\begin{aligned}\lim_\alpha \int_0^T f(t) I_G(\mu^\alpha)(t)dt &= \lim_\alpha \int_0^T \int f(t)G(x)d\mu_t^\alpha(x)dt = \int_0^T \int f(t)G(x)d\mu_t(x)dt \\ &= \int_0^T f(t)I_G(\mu)(t)dt,\end{aligned}$$

since whenever $G \in C(\mathbb{T}^d)$ and $f \in L^1(I)$ the function given by $F(t,x) = f(t)G(x)$ for $(t,x) \in I \times \mathbb{T}^d$ is in $L^1(I;C(\mathbb{T}^d))$. Therefore the function I_G is w^*-continuous. It follows then that the function h^G is lower semicontinuous as it is the composition of the w^*-continuous function I_G and the w^*-lower semicontinuous function $\|\cdot\|_{L^\infty(I)} : L^\infty(I) \longrightarrow \mathbb{R}$, i.e. $h_G = \|I_G\|_{L^\infty(I)}$.

By the lower semicontinuity of h_G now, it follows that the set

$$E_G^\varepsilon := \left\{ \sigma \in L^\infty(I;\mathcal{M}_+(\mathbb{T}^d)) \,\middle|\, \operatorname*{ess\,sup}_{0\le t\le T} |\langle G, \sigma_t\rangle| \le \|g\|_u \int_{\mathbb{T}^d} |G| + \varepsilon \right\}$$

is closed for all $G \in C(\mathbb{T}^d)$, $\varepsilon > 0$. Therefore, by the portmanteau theorem it follows that

$$Q^g(E_G^\varepsilon) \ge \limsup_{N\to\infty} Q^{N,g}(E_G^\varepsilon) = 1$$

for all $G \in C(\mathbb{T}^d)$, $\varepsilon > 0$. Now, obviously the set

$$E_G := \bigcap_{n\in\mathbb{N}} E_G^{\frac{1}{n}} = \left\{ \sigma \in L^\infty(I;\mathcal{M}_+(\mathbb{T}^d)) \,\middle|\, \operatorname*{ess\,sup}_{0\le t\le T} |\langle G, \sigma_t\rangle| \le \|g\|_u \int_{\mathbb{T}^d} |G| \right\},$$

is of full Q^g-measure, i.e. $Q^g(E_G) = 1$, for all $G \in C(\mathbb{T}^d)$.

Let now $D \subseteq C(\mathbb{T}^d)$ be a countable subset dense in $C(\mathbb{T}^d)$. We claim that

$$\bigcap_{G\in D} E_G = \bigcap_{G\in C(\mathbb{T}^d)} E_G.$$

In order to prove this it suffices to show that

$$\sup_{G\in C(\mathbb{T}^d)} \left(\|\langle G,\sigma\rangle\|_{L^\infty(I)} - \|g\|_u \int_{\mathbb{T}^d} |G| \right) = \sup_{G\in D} \left(\|\langle G,\sigma\rangle\|_{L^\infty(I)} - \|g\|_u \int_{\mathbb{T}^d} |G| \right)$$

Q^g-a.s. for all $\sigma \in L^\infty(I;\mathcal{M}_+(\mathbb{T}^d))$. Let $c_1 \in C(\mathbb{T}^d)$ denote the constant function $c_1 \equiv 1$. Then $Q^g(E_{c_1}) = 1$ and we will show that the equality above holds for all $\sigma \in E_{c_1}$. Indeed, let $\varepsilon > 0$ and $\sigma \in E_{c_1}$. Since D is dense in $C(\mathbb{T}^d)$ for each $G \in C(\mathbb{T}^d)$ there exists $G^\varepsilon \in D$ such that $\|G - G^\varepsilon\|_u \le \varepsilon/(\|g\|_u + 1)$. Then obviously

$$\int_{\mathbb{T}^d} |G - G^\varepsilon| < \frac{\varepsilon}{\|g\|_u + 1}$$

and since $\sigma \in E_{c_1}$ we have that $\|\langle \sigma, 1\rangle\|_{L^\infty(I)} \le \|g\|_u$ and therefore

$$\operatorname*{ess\,sup}_{0\le t\le T} \langle |G - G^\varepsilon|, \sigma_t\rangle < \varepsilon.$$

It follows that for all $G \in C(\mathbb{T}^d)$ we have that

$$\begin{aligned} \|\langle G, \sigma\rangle\|_{L^\infty(I)} - \|g\|_u \int_{\mathbb{T}^d} |G| &\le 2\varepsilon + \|\langle G^\varepsilon, \sigma\rangle\|_{L^\infty(I)} - \|g\|_u \int_{\mathbb{T}^d} |G^\varepsilon| \\ &\le 2\varepsilon + \sup_{G\in D}\left(\|\langle G, \sigma\rangle\|_{L^\infty(I)} - \|g\|_u \int_{\mathbb{T}^d} |G| \right). \end{aligned}$$

Taking the supremum over all $G \in C(\mathbb{T}^d)$ and then letting ε tend to zero yields the required equality.

It follows that the set

$$\begin{aligned} E &:= \bigcap_{G\in C(\mathbb{T}^d)} E_G \\ &= \left\{ \sigma \in L^\infty(I; \mathcal{M}_+(\mathbb{T}^d)) \,\middle|\, \|\langle G, \sigma\rangle\|_{L^\infty(I)} \le \|g\|_u \int_{\mathbb{T}^d} |G|, \;\; \forall\, G \in C(\mathbb{T}^d) \right\} \end{aligned}$$

is of full Q^g-measure (where we always work inside E_{c_1}).

For each $G \in D$, $\sigma \in E$ let $I_{G,\sigma} \subseteq I$ be a set of full measure satisfying

$$\sup_{t\in I_{G,\sigma}} \left(|\langle G, \sigma_t\rangle| - \|g\|_u \int_{\mathbb{T}^d} |G| \right) \le 0.$$

Then the set $I_\sigma := \bigcap_{G\in D} I_{G,\sigma}$ is of full measure in I and

$$\sup_{G\in C(\mathbb{T}^d)} \sup_{t\in I_\sigma} \left(|\langle G, \sigma_t\rangle| - \|g\|_u \int_{\mathbb{T}^d} |G| \right) \le 0 \tag{4.63}$$

for all $\sigma \in L^\infty(I; \mathcal{M}_+(\mathbb{T}^d))$. Indeed, let $t \in I_\sigma$ and let $G \in C(\mathbb{T}^d)$. Then for all $G \in D$ we have that $|\langle G, \sigma_t\rangle| \le \|g\|_u \int_{\mathbb{T}^d} |G|$. Therefore, if given $\varepsilon > 0$ we choose $G^\varepsilon \in D$ such that $\|G - G^\varepsilon\|_u \le \varepsilon/(\|g\|_u + 1)$ then we have that

$$|\langle G, \sigma_t\rangle| - \|g\|_u \int_{\mathbb{T}^d} |G| \le 2\varepsilon + |\langle G^\varepsilon, \sigma_t\rangle| - \|g\|_u \int_{\mathbb{T}^d} |G^\varepsilon| \le 2\varepsilon,$$

which since $\varepsilon > 0$ was arbitrary, proves (4.63).

Since for measures $\mu \in \mathcal{M}_+(\mathbb{T}^d)$ it holds that

$$\mu \in \mathcal{M}_{+,ac}(\mathbb{T}^d), \; \left\| \frac{d\mu}{dm_{\mathbb{T}^d}} \right\|_{L^\infty(\mathbb{T}^d)} \le C \quad \Longleftrightarrow \quad |\langle G, \mu\rangle| \le C \int_{\mathbb{T}^d} |G|, \quad \forall\, G \in C(\mathbb{T}^d)$$

the first claim is proven.

For the second claim, we consider for each $G \in C_c^2(I^o \times \mathbb{T}^d)$ the function $f^G : DL^\infty_{w^*} \longrightarrow \mathbb{R}$ given by

$$f^G(\pi, \sigma) = \int_0^T \langle \partial_t G_t, \pi_t\rangle + \langle \Delta_x G_t, \sigma_t\rangle dt.$$

In the previous section we have shown that the function $f_1^G : D(I;\mathcal{M}_+(\mathbb{T}^d))$ given by

$$f_1^G(\pi) = \int_0^T \langle \partial_t G_t, \pi_t \rangle dt$$

is continuous and so if we show that the function $f_2^G : L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$ given by

$$f_2^G(\sigma) = \int_0^T \langle \Delta_x G_t, \sigma_t \rangle dt$$

is continuous it will follow that f^G is continuous. So we prove the continuity of $f^{2,G}$. We note that given any function $G \in C_c^2(I^o \times \mathbb{T}^d)$ the formula

$$I \ni t \mapsto \Delta_x G_t \in C(\mathbb{T}^d)$$

defines an element $\Delta_x G$ of $L^1(I;C(\mathbb{T}^d))$ with

$$\|\Delta_x G\|_{L^1(I;C(\mathbb{T}^d))} = \int_I \|\Delta_x G_t\|_{C(\mathbb{T}^d)} dt \leq T \|\Delta_x G\|_{C_c^2(I^o \times \mathbb{T}^d)} < +\infty,$$

and then $f^{2,G} \equiv \langle \Delta_x G, \cdot \rangle_I$, where $\langle \cdot, \cdot \rangle_I$ is the duality on $L^1(I;C(\mathbb{T}^d)) \times L^\infty(I;\mathcal{M}_+(\mathbb{T}^d))$. By the definition of the w^*-topology on $L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$ a net $\{\pi^{g,\alpha}\}_{\alpha \in \mathcal{A}}$ converges to $\sigma \in L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$ iff

$$\lim_{\alpha \in \mathcal{A}} \langle G, \pi^{g,\alpha} \rangle_I = \langle G, \sigma \rangle_I, \quad \forall\, G \in L^1(I;C(\mathbb{T}^d))$$

and so in particular given $G \in C_c^2(I^o \times \mathbb{T}^d)$, for any converging net $\{\pi^{g,\alpha}\}_{\alpha \in \mathcal{A}}$ with limit σ we have that

$$f^{2,G}(\pi^{g,\alpha}) = \langle \Delta_x G, \pi^{g,\alpha} \rangle_I \longrightarrow \langle \nabla_x G, \sigma \rangle_I = f^{2,G}(\sigma)$$

which proves the continuity of $f^{2,G}$, and thus also of f^G.

Now, since f^G is continuous the set $\{|f^G| > \delta\}$ is open and so by the portmanteau theorem it follows that for any limit point R^g of the sequence $\{R^{N,g}\}$ along a subsequence $(R^{k_N,g})_{N \in \mathbb{N}}$ of $\{R^{N,g}\}$ we have that

$$\begin{aligned} R^g\big\{|f^G| > \delta\big\} &= R^g\bigg\{\bigg|\int_0^T \big[\langle \partial_t G_t, \pi_t \rangle + \langle \Delta_x G_t, \sigma_t \rangle\big] dt\bigg| > \delta\bigg\} \\ &\leq \liminf_{N \to \infty} R^{k_N,g}\bigg\{\bigg|\int_0^T \big[\langle \partial_t G_t, \pi_t \rangle + \langle \Delta_x G_t, \sigma_t \rangle\big] dt\bigg| > \delta\bigg\} = 0 \end{aligned}$$

for all $\delta > 0$ and all $G \in C_c^3(I^o \times \mathbb{T}^d)$. Since this holds for all $\delta > 0$ it follows that

$$R^g\bigg\{\int_0^T \big[\langle \partial_t G_t, \pi_t \rangle + \langle \Delta_x G_t, \sigma_t \rangle\big] dt = 0\bigg\} = 1, \quad \forall\, G \in C_c^3(I^o \times \mathbb{T}^d).$$

Therefore if we can find a countable family $\mathcal{G} \subseteq C_c^3(I^o \times \mathbb{T}^d)$ such that

$$\bigcap_{G \in C_c^3(I^o \times \mathbb{T}^d)} \{f^G = 0\} = \bigcap_{G \in \mathcal{G}} \{f^G = 0\} \tag{4.64}$$

it will follow that

$$R\Big(\bigcap_{G\in C_c^3(I^o\times\mathbb{T}^d)}\Big\{\int_0^T\big[\langle\partial_t G_t,\pi_t\rangle+\langle\Delta_x G_t,\sigma_t\rangle\big]dt=0\Big\}\Big)=1,$$

i.e. that R^g is concentrated on solutions of the diffusion equation. To this end, let $\mathcal{G}\subseteq C_c^\infty(I^o\times\mathbb{T}^d)$ be a countable set of smooth functions dense in $C_c^2(I^o\times\mathbb{T}^d)$ (and thus also in $C_c^3(I^o\times\mathbb{T}^d)$) with respect to the usual C^2-uniform norm of $C_c^2(I^o\times\mathbb{T}^d)$, given by

$$\|G\|_{C_c^2(I^o\times\mathbb{T}^d)}:=\|G\|_u+\|\nabla G\|_u+\|D^2G\|_u$$

where differentiation is with respect both to the time and space variables, the uniform norms in the right hand side are taken in the spaces $C(I^o\times\mathbb{T}^d)$, $C(I^o\times\mathbb{T}^d;\mathbb{R}^{d+1})$ and $C(I^o\times\mathbb{T}^d;\mathbb{R}^{(d+1)\times(d+1)})$, respectively, and $\mathbb{R}^{(d+1)\times(d+1)}$ is considered equipped with the Frobenius norm. Note that if we show that for any sequence $\{G^k\}\subseteq C_c^\infty(I^o\times\mathbb{T}^d)$ such that $G^k\longrightarrow G\in C_c^2(I^o\times\mathbb{T}^d)$ with respect to the C^2-uniform norm we have that $f^{G^k}\longrightarrow f^G$ pointwise in $DL_{w^*}^\infty(I,\mathbb{T}^d)$ then we will have that (4.64) holds. Indeed, if this is true, and $(\pi,\sigma)\in\bigcup_{G\in\mathcal{G}}\{f^G=0\}$ then given any $G\in C_c^2(I^o\times\mathbb{T}^d)$ there exists a sequence $\{G^k\}$ in $\mathcal{G}$ such that $\|G-G^k\|_{C_c^2(I^o\times\mathbb{T}^d)}\longrightarrow 0$ and thus

$$f^G(\pi,\sigma)=\lim_{k\to\infty}f^{G^k}(\pi,\sigma)=0,$$

which shows that (4.64) holds.

So let $\{G^k\}_{k\in\mathbb{N}}\subseteq C_c^\infty(I^o\times\mathbb{T}^d)$ be a sequence such that $\|G^k-G\|_{C_c^2(I^o\times\mathbb{T}^d)}\longrightarrow 0$. The pointwise convergence of $\{f_1^{G^k}\}$ to f_1^G on the space $D(I;\mathcal{M}_+(\mathbb{T}^d))$ has been proved in the previous section and so we have to prove that $\{f_2^{G^k}\}$ converges pointwise on $L^\infty(I;\mathcal{M}_+(\mathbb{T}^d))$ to the function f_2^G. As we have already noted, for any $G\in C_c^2(I^o\times\mathbb{T}^d)$ the function $\Delta_xG:I\longrightarrow C(\mathbb{T}^d)$ given by $t\mapsto\Delta_xG_t\in C(\mathbb{T}^d)$ defines an element of $L^1(I;C(\mathbb{T}^d))$ and for these such elements that are defined by the C^2-converging sequence $\{G^k\}\subseteq C_c^\infty(I^o\times\mathbb{T}^d)$ we have that

$$\begin{aligned}\|\Delta_xG^k-\Delta_xG\|_{L^1(I;C(\mathbb{T}^d))} &= \int_I\|\Delta_x(G_t^k-G_t)\|_u dt\\ &\le \int_I\|D_x^2(G_t^k-G_t)\|_{C(\mathbb{T}^d;\mathbb{R}^{d\times d})}dt\\ &\le T\cdot\|G^k-G\|_{C_c^2(I^o\times\mathbb{T}^d)}\overset{k\to\infty}{\longrightarrow}0\end{aligned}$$

Therefore, for any $\sigma\in L_{w^*}^\infty(I;\mathcal{M}_+(\mathbb{T}^d))\cong L^1(I;C(\mathbb{T}^d))^*$ we have that

$$f_2^{G_k}(\sigma)=\langle\Delta_xG^k,\sigma\rangle_I\overset{k\to\infty}{\longrightarrow}\langle\Delta_xG,\sigma\rangle_I=f_2^G(\sigma). \tag{4.65}$$

which proves that the sequence of functions $\{f^{2,G^k}\}_{k\in\mathbb{N}}$, converges pointwise to $f^{2,G}$ in $L_{w^*}^\infty(I;\mathcal{M}_+(\mathbb{T}^d))$. Therefore $\{f^{G^k}\}$ converges pointwise to f^G on $DL_{w^*}^\infty(I,\mathbb{T}^d)$ whenever $G_k\longrightarrow G$ in the C^2-uniform norm, as required for the proof of (4.64).

Combining this proposition with corollary 4.4.1 we can rephrase this result as follows.

Proposition 4.4.6 *Let $\{\eta^N = (id_{D(\mathbb{R}_+,\mathbb{M}_N^d)}, P^N)\}_{N\in\mathbb{N}}$ be the sequence of nearest neighbor ZR processes with bounded jump rate g on the discrete toruses $\mathbb{T}_N^d$ starting from a sequence $\{\mu_0^N \in \mathbb{P}_1\mathbb{M}_N^d\}$ of initial distributions associated to the macroscopic profile $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$. Then the sequence of distributions of the diffusively scaled empirical density-diffusion rate pair processes associated to $\{\eta^N\}_{N\in\mathbb{N}}$ given by*

$$R^{N,g} := \Big[(\pi^N_{tN^2}, \sigma^N_{tN^2})_{t\geq 0}\Big]_* P^N \in \mathbb{P}DL^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d) \times \mathcal{M}_+(\mathbb{T}^d))$$

is relatively compact and any limit point R^g of the sequence is concentrated on pairs $(\pi, \sigma) \in DL^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d) \times \mathcal{M}_+(\mathbb{T}^d))$ such that:

(a) $(\pi, \sigma) \in CL^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d) \times L^\infty(\mathbb{T}^d))$*, where* $L^\infty(\mathbb{T}^d) \leq L^1(\mathbb{T}^d) \leq \mathcal{M}_+(\mathbb{T}^d)$.

(b) *the density-diffusion rate pair (π, σ) satisfies the weak diffusion equation with initial condition $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$, i.e. (π, σ) satisfies*

$$\begin{cases} \partial_t \pi_t = \Delta_x \sigma \\ \pi_0 = \mu_0 \end{cases} \qquad \textit{in } I \times \mathbb{T}^d$$

in the sense of distributions.

In the next proposition we regard $\widehat{\mathcal{M}}_0(\mathbb{T}^d)$ as a subspace of $C^1(\mathbb{T}^d; \mathbb{R}^d)^*$.

Proposition 4.4.7 *Let $\{R^{N,g}\}_{N\in\mathbb{N}} \subseteq \mathbb{P}DL^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d)^2)$ denote the sequence of the diffusively rescaled distributions of the empirical density-diffusion rate pair process*

$$(\pi^N, \sigma^N) : D(I; \mathbb{M}_N^d) \longrightarrow DL^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d) \times \mathcal{M}_+(\mathbb{T}^d))$$

and let $\{R^N\}$ denote the sequence of the diffusively rescaled distributions of the empirical density-current pair process

$$(\pi^N, W^N) : D(I; \mathbb{M}_N^d) \longrightarrow DL^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d) \times \widehat{\mathcal{M}}_0^d(\mathbb{T}^d)).$$

We denote by

$$\mathcal{R}^g \subseteq \mathbb{P}CL^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d) \times L^\infty(\mathbb{T}^d)), \quad \mathcal{R} \subseteq \mathbb{P}CL^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d) \times C^1(\mathbb{T}^d; \mathbb{R}^d)^*)$$

the sets of subsequential limits of the sequences $\{R^{N,g}\}$ and $\{R^N\}$, respectively. Then

$$[id \times \nabla]_* \mathcal{R}^g = \mathcal{R}$$

where $\nabla : L^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d)) \longrightarrow L^\infty_{w^*}(I; C^1(\mathbb{T}^d; \mathbb{R}^d)^*)$ is the operator induced by the gradient operator $\nabla : \mathcal{M}_+(\mathbb{T}^d) \longrightarrow C^1(\mathbb{T}^d; \mathbb{R}^d)^*$ and $id = id_{C(I;\mathcal{M}_+(\mathbb{T}^d))}$ is the identity function. In particular $\mathcal{R} \subseteq \mathbb{P}CL^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d) \times \widehat{\mathcal{M}}_0^d(\mathbb{T}^d))$.

Proof By definition we have that $W^N = \nabla^N \sigma^N$ for all $N \in \mathbb{N}$. We consider first a limit point R^g of $\{R^{N,g}\}$ and we will prove that the distribution $[id \times \nabla]_* R^g$ is a limit point of the sequence $\{R^N\}$. There exists a subsequence of $\{R^{N,g}\}$ which we continue to denote by $\{R^{N,g}\}$ such that $R^{N,g} \longrightarrow R^g$ in the weak topology of probability measures.

As we have seen the gradient operator $\nabla : \mathcal{M}_+(\mathbb{T}^d) \longrightarrow C^1(\mathbb{T}^d;\mathbb{R}^d)^*$ is w^*-continuous. Likewise the induced operator

$$\nabla : L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d)) \longrightarrow L^\infty_{w^*}(I;C^1(\mathbb{T}^d;\mathbb{R}^d)^*)$$

is w^*-continuous. Indeed, let $\{\mu_\alpha\}_{\alpha\in\mathcal{A}} \subseteq L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$ be a net converging in the w^*-topology to $\mu \in L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$ and let $G \in L^1(I;C^1(\mathbb{T}^d;\mathbb{R}^d))$. Then the function $\mathrm{div}G$ obviously belongs in $L^1(I;C(\mathbb{T}^d))$ and therefore since $w^*\text{-}\lim_\alpha \mu^\alpha = \mu$ we have that

$$\langle G,\nabla\mu_\alpha\rangle_I = -\int_0^T \langle \mathrm{div}G_t,\mu^\alpha_t\rangle dt \longrightarrow -\int_0^T \langle \mathrm{div}G_t,\mu_t\rangle dt = \langle G,\nabla\mu\rangle_I.$$

It follows that

$$\lim_{N\to\infty}(id\times\nabla)_* R^{N,g} = (id\times\nabla)_* R^g.$$

We claim that the corresponding subsequence $\{R^N\}$ converges to $(id\times\nabla)_*R^g$, and for this it suffices to prove that any subsequence $\{R^{N_k}\}$ of $\{R^N\}$ has a further subsequence that converges weakly to $(id\times\nabla)_*R^g$. For this we consider the sequence of probability measures

$$\widetilde{R}^N := (\pi^N, W^N, \nabla\sigma^N)_*\mathbb{P}^N \in \mathbb{P}(M_1\times M_2\times M_3)$$

where

$$M_1 := D(I;\mathcal{M}_+(\mathbb{T}^d)),\ \ M_2 = M_3 := L^\infty_{w^*}(I;C^1(\mathbb{T}^d;\mathbb{R}^d)^*).$$

Then denoting by $x^i : M_1\times M_2\times M_3 \longrightarrow X_i$ the natural projections we have that

$$(x^1,x^2)_*\widetilde{R}^N = R^N \quad \text{and} \quad (x^1,x^3)_*\widetilde{R}^N = (id\times\nabla)_*R^{N,g}.$$

The corresponding subsequence $\{\widetilde{R}^{N_k}\}$ is obviously relatively compact and therefore there exists $\widetilde{R}\in\mathbb{P}(M_1\times M_2\times M_3)$ and a further subsequence $\{\widetilde{R}^{N_{k_m}}\}$ converging to $\widetilde{R}$. Then

$$R^{N_{k_m}} = (x^1,x^2)_*\widetilde{R}^{N_{k_m}} \longrightarrow (x^1,x^2)_*\widetilde{R} =: R$$

and

$$(id\times\nabla)_*R^{N_{k_m},g} = (x^1,x^3)_*\widetilde{R}^{N_{k_m}} \longrightarrow (x^1,x^3)_*\widetilde{R} = (id\times\nabla)_*R^g.$$

We claim that $(x^2,x^3)_*\widetilde{R}\in\mathbb{P}(M_2\times M_3)$ is the identity plan, i.e. that it is concentrated in the diagonal of $M_2\times M_3$, which since

$$(x^1,x^3)_*\widetilde{R} = \big[(x^2,x^3)_*\widetilde{R}\big]\circ\big[(x^1,x^2)_*\widetilde{R}\big]$$

will prove the claim.

So let $G\in L^1(I;C^1(\mathbb{T}^d;\mathbb{R}^d))$. Then, for all $\varepsilon>0$ we have that

$$\begin{aligned}
\widetilde{R}\big\{|\langle G,x^2\rangle_I - \langle G,x^3\rangle_I| > \varepsilon\big\} &\le \liminf_{m\to\infty}\widetilde{R}^{N_{k_m}}\big\{|\langle G,x^2\rangle_I - \langle G,x^3\rangle_I|>\varepsilon\big\}\\
&= \liminf_{m\to\infty}\mathbb{P}^{N_{k_m}}\big\{|\langle G,W^{N_{k_m}}\rangle_I - \langle G,\nabla\sigma^{N_{k_m}}\rangle_I|>\varepsilon\big\}\\
&= \liminf_{m\to\infty}\mathbb{P}^{N_{k_m}}\big\{|\langle \mathrm{div}G - \mathrm{div}^{N_{k_m}}G,\sigma^{N_{k_m}}\rangle_I|>\varepsilon\big\}\\
&\le \varlimsup_{m\to\infty}\mathbb{P}^{N_{k_m}}\Big\{\|\mathrm{div}G-\mathrm{div}^{N_{k_m}}G\|_{L^1(I;C^1)} > \frac{\varepsilon}{\|g\|_u}\Big\}.
\end{aligned}$$

Therefore we see that if we prove that

$$G \in L^1(I; C^2(\mathbb{T}^d;\mathbb{R}^d)) \quad \Longrightarrow \quad \lim_{N\to\infty} \int_0^T \|\mathrm{div} G_t - \mathrm{div}^N G_t\|_u dt = 0, \tag{4.66}$$

where we regard $C^2(\mathbb{T}^d;\mathbb{R}^d)$ as a subspace of $(C^1(\mathbb{T}^d;\mathbb{R}^d), \|\cdot\|_{C^1})$, it will follow that

$$\widetilde{R}\big\{|\langle G, x^2\rangle_I - \langle G, x^3\rangle_I| > \varepsilon\big\} = 0$$

for all $G \in L^1(I; C^2(\mathbb{T}^d;\mathbb{R}^d))$ and all $\varepsilon > 0$, which will then imply that

$$\widetilde{R}\big\{\langle G, x^2\rangle = \langle G, x^3\rangle\big\} = 1$$

for all $G \in L^1(I; C^2(\mathbb{T}^d;\mathbb{R}^d))$. But $L^1(I; C^2(\mathbb{T}^d;\mathbb{R}^d))$ is dense in $L^1(I; C^1(\mathbb{T}^d;\mathbb{R}^d))$ and therefore by choosing a countable subset $D \subseteq L^1(I; C^2(\mathbb{T}^d;\mathbb{R}^d))$ that is dense in $L^1(I; C^1(\mathbb{T}^d;\mathbb{R}^d))$, this will in turn imply that

$$1 = \widetilde{R}\bigg(\bigcap_{G\in D} \big\{\langle G, x^2\rangle = \langle G, x^3\rangle\big\}\bigg) = \widetilde{R}\big\{x^2 = x^3\big\},$$

which proves that $(x^2, x^3)_* \widetilde{R}$ is concentrated in the diagonal of $M_2 \times M_3$, as required.

Finally, (4.66) follows easily by the bounded convergence theorem, since as we know whenever $G \in C^2(\mathbb{T}^d;\mathbb{R}^d)$ then $\|\mathrm{div} G - \mathrm{div}^N G\|_u \longrightarrow 0$, and therefore for all $G \in L^1(C^2(\mathbb{T}^d;\mathbb{R}^d))$ the bounded function $h^N(t) := \|\mathrm{div} G_t - \mathrm{div}^N G_t\|_u$, $0 \leq t \leq T$, with $\sup_{N\in\mathbb{N}} \|h^N\|_u \leq 2\|G\|_{C^1(\mathbb{T}^d;\mathbb{R}^d)}$, converges pointwise to zero. $\square$

4.5 Regularity Results for the Limiting Empirical Diffusion Rate and Current

4.5.1 An Energy Estimate

In this section we adapt to the case at hand the regularity estimates of section 5.7 in [25] to conclude that in fact the trajectories of the empirical current consist of vector measures absolutely continuous with respect to the Lebesgue measure. We fix $T > 0$, set $I := [0, T]$ and recall that we have denoted by $Q^{N,g}$ and Q^N the laws of the empirical diffusion-rate process $(\sigma_t^N)_{0\leq t\leq T}$ and the empirical current process $(W_t^N)_{0\leq t\leq T}$, starting from a sequence $\{\mu_0^N\}_{N\in\mathbb{N}}$ of initial distributions satisfying the $O(N^d)$-entropy assumption, on the spaces $L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$ and $L^\infty_{w^*}(I;\mathcal{M}_0^d(\mathbb{T}^d))$ respectively. In section 4.4 we have shown that the sequences

$$\{Q^{N,g}\}_{N\in\mathbb{N}} \subseteq \mathbb{P}L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$$

and

$$\{Q^N\}_{N\in\mathbb{N}} \subseteq \mathbb{P}L^\infty_{w^*}(I;\widehat{\mathcal{M}}_0^d(\mathbb{T}^d)) \leq \mathbb{P}\big(L^1(I;C^1(\mathbb{T}^d;\mathbb{R}^d))^*, w^*\big),$$

are relatively compact. Furthermore, as we have shown in section 4.4, denoting by $\mathcal{Q}^g$ and $\mathcal{Q}$ the set of all subsequential limits of the sequences $\{Q^{N,g}\}$ and $\{Q^N\}$ respectively, we have that

$$\mathcal{Q}^g \subseteq \mathbb{P}L^\infty_{w^*}(I;L^\infty(\mathbb{T}^d)) \quad \text{and} \quad \mathcal{Q} \subseteq \mathbb{P}L^\infty_{w^*}(I;\widehat{\mathcal{M}}_0^d(\mathbb{T}^d)).$$

Our main goal in this section is to prove regularity results for the trajectories on which elements of $\mathcal{Q}^g$ and $\mathcal{Q}$ are concentrated. Namely, we aim to prove that

$$\mathcal{Q}^g \subseteq \mathbb{P}L^\infty_{w^*}(I;H^1(\mathbb{T}^d)), \tag{4.67}$$

where as usual $H^1(\mathbb{T}^d) = W^{1,2}(\mathbb{T}^d)$ denotes the Hilbert-Sobolev space of functions possessing first order weak derivatives. Then since as we have shown, we have

$$\mathcal{Q} = \nabla\mathcal{Q}^g,$$

is will follow by (4.67) that

$$\mathcal{Q} \subseteq \mathbb{P}L^\infty_{w^*}(I;\mathcal{M}_0^d(\mathbb{T}^d)), \tag{4.68}$$

i.e. that all elements of $\mathcal{Q}$ are concentrated on trajectories whose points are in fact vector-valued measures, and not just elements of the Kantorovich-Rubinstein completion $\widehat{\mathcal{M}}_0^d(\mathbb{T}^d)$ in the Lipschitz norm of $\mathcal{M}_0^d(\mathbb{T}^d)$.

Our first regularity result is based on the following lemma which is in essence contained in lemma in [25]. Before stating this lemma let us fix $j = 1,\dots,d$ and define for each $N \in \mathbb{N}$, $\varepsilon > 0$ and each function $H \in C(\mathbb{T}^d)$ the function $V^N(\varepsilon,H) \equiv V^{N,j}(\varepsilon,H) : \mathbb{M}_N^d \longrightarrow \mathbb{R}$ by the formula

$$\begin{aligned} V^N(\varepsilon,H) \quad = \quad & \frac{1}{N^{d-1}}\sum_{x\in\mathbb{T}_N^d} H\Big(\frac{x}{N}\Big)\frac{g\big(\eta(x)\big) - g\big(\eta(x+[N\varepsilon]e_j)\big)}{[N\varepsilon]} \\ & -\frac{2}{N^d}\sum_{x\in\mathbb{T}_N^d} H\Big(\frac{x}{N}\Big)^2\frac{1}{[N\varepsilon]}\sum_{k=0}^{[N\varepsilon]} g\big(\eta(x+ke_j)\big). \end{aligned}$$

Then for each $H \in C^1(I \times \mathbb{T}^d)$ and $t \in I$ we set

$$V_t^N(\varepsilon, H)(\eta) = V^N(\varepsilon, H_t)(\eta_t), \qquad \eta \in D(\mathbb{R}_+, \mathbb{M}_N^d),$$

for the induced process $V^N(\varepsilon, H) : I \times D(I; \mathbb{M}_N^d) \longrightarrow \mathbb{R}$.

Lemma 4.5.1 *Let $\{H^i\}_{i=1}^m \subseteq C^1(I \times \mathbb{T}^d)$, $m \in \mathbb{N}$, be a finite sequence of functions and let $\{\mu_0^N \in \mathbb{P}_1\mathbb{M}_N^d\}_{N\in\mathbb{N}}$ be a sequence of initial distributions satisfying the $O(N^d)$-entropy assumption for some finite constant $C_0 > 0$. Then for all $\varepsilon > 0$ we have that*

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_0^N}\left\{ \max_{1\le i\le m} \int_0^T V_t^N(\varepsilon, H)dt \right\} \le C_0.$$

Proof The proof follows that of lemma 5.7.3 in [25]. □

Corollary 4.5.1 *Let $\{H^i\}_{i=1}^m \subseteq C^1(I \times \mathbb{T}^d)$, $m \in \mathbb{N}$, be a finite sequence of functions and let $Q^g \in \mathcal{Q}^g \subseteq \mathbb{P}L_{w^*}^\infty(I; L^\infty(\mathbb{T}^d))$. Then*

$$\int \left\{ \max_{1\le i\le m} \int_0^T \int_{\mathbb{T}^d} \big[\partial_j H_t^i(x) - H_t^i(x)^2\big]\sigma_t(x)dxdt \right\} dQ^g(\sigma) \le C_0.$$

Proof We begin by noting that for each $H \in C(\mathbb{T}^d)$ and $\varepsilon > 0$ by a simple summation by parts we can write $V^N(\varepsilon, H)$ as

$$\begin{aligned}
V^N(\varepsilon, H) &= \frac{1}{N^{d-1}} \sum_{x\in\mathbb{T}_N^d} \frac{H(\frac{x}{N}) - H\big(\frac{x-[N\varepsilon]e_j}{N}\big)}{[N\varepsilon]} g\big(\eta(x)\big) \\
&\quad - \frac{2}{N^d} \sum_{x\in\mathbb{T}_N^d} \frac{1}{[N\varepsilon]} \sum_{k=0}^{[N\varepsilon]} H\Big(\frac{x - ke_j}{N}\Big)^2 g\big(\eta(x)\big) \\
&= \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \frac{N}{[N\varepsilon]} \Big[H\Big(\frac{x}{N}\Big) - \tau_{-\frac{[N\varepsilon]}{N}e_j} H\Big(\frac{x}{N}\Big)\Big] g\big(\eta(x)\big) \\
&\quad - \frac{2}{N^d} \sum_{x\in\mathbb{T}_N^d} \frac{1}{[N\varepsilon]} \sum_{k=0}^{[N\varepsilon]} \tau_{-\frac{k}{N}e_j} H\Big(\frac{x}{N}\Big)^2 g\big(\eta(x)\big) \\
&= \Big\langle \frac{N}{[N\varepsilon]} \Big[H - \tau_{-\frac{[N\varepsilon]}{N}e_j} H\Big] - \frac{2}{[N\varepsilon]} \sum_{k=0}^{[N\varepsilon]} \tau_{-\frac{k}{N}e_j} H^2, \sigma^N \Big\rangle.
\end{aligned}$$

Therefore if for each $H \in C(\mathbb{T}^d)$ we denote by $\upsilon_\varepsilon^N(H) \in C(\mathbb{T}^d)$ the function

$$\upsilon_\varepsilon^N(H) := \frac{N}{[N\varepsilon]} \Big[H - \tau_{-\frac{[N\varepsilon]}{N}e_j} H\Big] - \frac{2}{[N\varepsilon]} \sum_{k=0}^{[N\varepsilon]} \tau_{-\frac{k}{N}e_j} H^2$$

then we can write $V^N(\varepsilon, H) = \langle \upsilon_\varepsilon^N(H), \sigma^N \rangle$, and

$$\mathbb{E}^{\mu_0^N}\left\{ \max_{1\le i\le m} \int_0^T V_t^N(\varepsilon, H)dt \right\} = \int \left\{ \max_{1\le i\le m} \int_0^T \int_{\mathbb{T}^d} \upsilon_\varepsilon^N(H_t^i) d\pi_t^g dt \right\} dQ^{N,g}(\sigma).$$

We claim next that

$$v_\varepsilon^N(H) \longrightarrow \frac{H-\tau_{-\varepsilon e_j}H}{\varepsilon} - \frac{2}{\varepsilon}\int_0^\varepsilon H(\cdot - te_j)^2 dt =: v_\varepsilon(H) \tag{4.69}$$

uniformly as $N \to \infty$. The fact that

$$\frac{N}{[N\varepsilon]}\Big[H - \tau_{-\frac{[N\varepsilon]}{N}e_j}H\Big] \longrightarrow \frac{H-\tau_{-\varepsilon e_j}H}{\varepsilon}$$

uniformly as $N \to \infty$ is obvious and so we have to prove that

$$\frac{1}{[N\varepsilon]}\sum_{k=0}^{[N\varepsilon]} \tau_{-\frac{k}{N}e_j}H^2 \longrightarrow \frac{1}{\varepsilon}\int_0^\varepsilon H(\cdot - te_j)^2 dt$$

uniformly as $N \to \infty$. For each $u \in \mathbb{T}^d$ we have

$$\begin{aligned}
\frac{1}{[N\varepsilon]}\sum_{k=0}^{[N\varepsilon]} \tau_{-\frac{k}{N}e_j}H(u)^2 &= \frac{N}{[N\varepsilon]}\sum_{k=0}^{[N\varepsilon]} H\Big(u-\frac{k}{N}e_j\Big)^2\frac{1}{N} \\
&= \frac{N}{[N\varepsilon]}\sum_{k=0}^{[N\varepsilon]} \int_{[\frac{k}{N},\frac{k+1}{N})} H\Big(u-\frac{k}{N}e_j\Big)^2 dt \\
&= \frac{N}{[N\varepsilon]}\sum_{k=0}^{[N\varepsilon]} \int_{[\frac{k}{N},\frac{k+1}{N})} H\Big(u-\frac{[Nt]}{N}e_j\Big)^2 dt \\
&= \frac{N}{[N\varepsilon]}\int_0^{\frac{[N\varepsilon]+1}{N}} H\Big(u-\frac{[Nt]}{N}e_j\Big)^2 dt.
\end{aligned}$$

So for all $u \in \mathbb{T}^d$ we have that

$$\begin{aligned}
\Delta_\varepsilon^N(H)(u) &:= \frac{1}{[N\varepsilon]}\sum_{k=0}^{[N\varepsilon]} \tau_{-\frac{k}{N}e_j}H(u)^2 - \frac{1}{\varepsilon}\int_0^\varepsilon H(u-te_j)^2 dt \\
&= \frac{1}{\varepsilon}\int_0^\varepsilon \Big[H\Big(u-\frac{[Nt]}{N}e_j\Big)^2 - H(u-te_j)^2\Big] dt \\
&\quad + \Big(\frac{N}{[N\varepsilon]} - \frac{1}{\varepsilon}\Big)\int_0^\varepsilon H\Big(u-\frac{[Nt]}{N}e_j\Big)^2 dt \\
&\quad + \frac{[N\varepsilon]}{N}\int_\varepsilon^{\frac{[N\varepsilon]+1}{N}} H\Big(u-\frac{[Nt]}{N}e_j\Big)^2 dt
\end{aligned}$$

and therefore

$$\begin{aligned}
\big|\Delta_\varepsilon^N(H)(u)\big| &\leq \frac{2\|H\|_u}{\varepsilon}\int_0^\varepsilon \Big|H\Big(u-\frac{[Nt]}{N}e_j\Big) - H(u-te_j)\Big| dt + \varepsilon\|H\|_u^2\Big(\frac{N}{[N\varepsilon]} - \frac{1}{\varepsilon}\Big) \\
&\quad + \frac{[N\varepsilon]}{\varepsilon}\|H\|_u^2\Big(\frac{[N\varepsilon]+1}{N} - \varepsilon\Big).
\end{aligned}$$

Consequently, in order to prove that $\|\Delta_\varepsilon^N(H)\|_u \longrightarrow 0$ as required it suffices to show that

$$\sup_{u\in\mathbb{T}^d}\int_0^\varepsilon \Big|H\Big(u-\frac{[Nt]}{N}e_j\Big) - H(u-te_j)\Big| dt \overset{N\to\infty}{\longrightarrow} 0.$$

But this limit is also obvious since every function $H \in C(\mathbb{T}^d)$ is a uniformly continuous function. To summarize, we have proved that for all $H \in C(\mathbb{T}^d)$ and all $\varepsilon > 0$,

$$\lim_{N\to\infty} \left\| \upsilon_\varepsilon^N(H) - \upsilon_\varepsilon(H) \right\|_u = 0. \tag{4.70}$$

Furthermore, for all $H \in C(\mathbb{T}^d)$ and all large enough N we have that

$$\|\upsilon_\varepsilon(H)\|_u \vee \|\upsilon_\varepsilon^N(H)\|_u \leq \frac{4}{\varepsilon}\|H\|_u + 2\|H\|_u^2 < +\infty. \tag{4.71}$$

Next, for each function $H \in C(I\times\mathbb{T}^d)$ we denote by $\upsilon_\varepsilon^N(H) \in C(I\times\mathbb{T}^d)$ the function given by

$$\upsilon_\varepsilon^N(H)(t,u) \equiv \upsilon_\varepsilon^N(H)_t(u) \equiv \upsilon_\varepsilon^N(H_t)(u), \qquad (t,u) \in I \times \mathbb{T}^d.$$

With this notation, it follows by (4.70) and (4.71) that for all functions $H \in C(I \times \mathbb{T}^d)$,

$$\lim_{N\to\infty} \left\| \upsilon_\varepsilon^N(H)_t - \upsilon_\varepsilon(H)_t \right\|_u = 0, \quad \forall\, t \in I,$$

and

$$\sup_{t\in I} \left\{ \|\upsilon_\varepsilon(H)_t\|_u \vee \|\upsilon_\varepsilon^N(H)_t\|_u \right\} \leq \frac{4}{\varepsilon}\|H\|_{C(I\times\mathbb{T}^d)} + 2\|H\|_{C(I\times\mathbb{T}^d)}^2 < +\infty,$$

and therefore for all $H \in C(I \times \mathbb{T}^d)$ we have by the bounded convergence theorem that

$$\|\upsilon_\varepsilon^N(H) - \upsilon_\varepsilon(H)\|_{L^1(I;C(\mathbb{T}^d))} = \int_0^T \|\upsilon_\varepsilon^N(H)_t - \upsilon_\varepsilon(H)_t\|_u dt \overset{N\to\infty}{\longrightarrow} 0. \tag{4.72}$$

Recall that for each function $F \in L^1(I;C(\mathbb{T}^d))$ we denote by $\langle F, \cdot\rangle_I$ the functional $\langle F, \cdot\rangle_I : L^\infty_{w^*}(I;\mathcal{M}(\mathbb{T}^d)) \longrightarrow \mathbb{R}$ given by

$$\langle F, \pi\rangle_I = \int_0^T \langle F_t, \pi_t\rangle dt, \qquad \pi \in L^\infty_{w^*}(I;\mathcal{M}(\mathbb{T}^d)).$$

Let now $Q^g \in \mathcal{Q}^g(\{\mu_0^N\})$ and consider a subsequence of $\{Q^{N,g}\}_{N\in\mathbb{N}}$, which we continue to denote by $\{Q^{N,g}\}$, converging weakly to Q^g. Then, using the elementary inequality

$$\max_{1\leq i\leq m} a_i - \max_{1\leq i\leq m} b_i \leq \max_{1\leq i\leq m}(a_i - b_i)$$

which holds for all finite sequences $\{a_i\}_{i=1}^m$, $\{b_i\}_{i=1}^m$ of real numbers, we write

$$\begin{aligned}\int \max_{1\leq i\leq m} \langle \upsilon_\varepsilon(H^i), \sigma\rangle_I dQ^{N,g}(\sigma) \quad &\leq \quad \int \max_{1\leq i\leq m} \langle \upsilon_\varepsilon^N(H^i), \sigma\rangle_I dQ^{N,g}(\sigma) \\ &\quad + \int \max_{1\leq i\leq m} \langle \upsilon_\varepsilon(H^i) - \upsilon_\varepsilon^N(H^i), \sigma\rangle_I dQ^{N,g}(\sigma).\end{aligned}$$

The function

$$\sigma \mapsto \max_{1\leq i\leq m} \langle \upsilon_\varepsilon(H^i), \sigma\rangle_I$$

is continuous in the w^*-topology of $L^\infty_{w^*}(I;\mathcal{M}(\mathbb{T}^d))$ as a maximum of a finite number of continuous functionals and therefore since $Q^{N,g}$ converges weakly to Q^g we have that

$$\lim_{N\to\infty} \int \max_{1\leq i\leq m} \langle \upsilon_\varepsilon(H^i), \sigma\rangle_I dQ^{N,g}(\sigma) = \int \max_{1\leq i\leq m} \langle \upsilon_\varepsilon(H^i), \sigma\rangle_I dQ^g(\sigma)$$

and on the other hand, by lemma 4.5.1 we have that

$$\limsup_{N\to\infty}\int \max_{1\le i\le m}\langle \upsilon_\varepsilon^N(H^i),\sigma\rangle_I dQ^{N,g}(\sigma)\le C_0.$$

It follows that

$$\int \max_{1\le i\le m}\langle \upsilon_\varepsilon(H^i),\sigma\rangle_I dQ^{g}(\sigma)\le C_0+\overline{\lim_{N\to\infty}}\int \max_{1\le i\le m}\langle \upsilon_\varepsilon(H^i)-\upsilon_\varepsilon^N(H^i),\sigma\rangle_I dQ^{N,g}(\sigma).$$

We prove next that

$$\lim_{N\to\infty}\left|\int \max_{1\le i\le m}\langle \upsilon_\varepsilon(H^i)-\upsilon_\varepsilon^N(H^i),\sigma\rangle_I dQ^{N,g}(\sigma)\right|=0, \tag{4.73}$$

in order to conclude that

$$\int \max_{1\le i\le m}\langle \upsilon_\varepsilon(H^i),\sigma\rangle_I dQ^{g}(\sigma)\le C_0. \tag{4.74}$$

So we proceed with the proof of (4.73). The quantity that we want to prove that tends to zero is bounded above by

$$\int \max_{1\le i\le m}\big|\langle \upsilon_\varepsilon(H^i)-\upsilon_\varepsilon^N(H^i),\sigma\rangle_I\big| dQ^{N,g}(\sigma) \tag{4.75}$$

and for all $i=1,\dots,m$ we have that for $Q^{N,g}$-a.s. all $\sigma\in L^\infty_{w^*}(I;\mathcal{M}(\mathbb{T}^d))$,

$$\begin{aligned}\big|\langle \upsilon_\varepsilon(H^i)-\upsilon_\varepsilon^N(H^i),\sigma\rangle_I\big| &\le \int_0^T \big|\langle \upsilon_\varepsilon(H^i)_t-\upsilon_\varepsilon^N(H^i)_t,\sigma_t\rangle\big|dt\\ &\le \|g\|_u\int_0^T \|\upsilon_\varepsilon(H^i)_t-\upsilon_\varepsilon^N(H^i)_t\|_u dt.\end{aligned}$$

It follows that the quantity in (4.75) is bounded above by

$$\|g\|_u\max_{1\le i\le m}\int_0^T \|\upsilon_\varepsilon(H^i)_t-\upsilon_\varepsilon^N(H^i)_t\|_u dt$$

which obviously tends to zero as $N\to\infty$ by (4.72).

We study next the behavior of the quantity in the left hand side of (4.74) as $\varepsilon\to 0$ for functions $H\in C^1(I\times\mathbb{T}^d)$. We note first that for each function $H\in C^1(\mathbb{T}^d)$ we have by the differentiability of H and Lebesgue's differentiation theorem that

$$\upsilon_\varepsilon(H)\overset{\varepsilon\to 0}{\longrightarrow}\partial_j H-2H^2=:\upsilon(H),$$

pointwise in $\mathbb{T}^d$, since the set $(0,\varepsilon)$ shrinks well to zero. Consequently, using the notation $\upsilon(H)_t(u)=\upsilon(H)(t,u)=\upsilon(H_t)(u)$ for functions $H\in C^1(I\times\mathbb{T}^d)$, we have that

$$\upsilon_\varepsilon(H)_t\longrightarrow \upsilon(H)_t$$

pointwise in $\mathbb{T}^d$ for all $t\in I$, for all $H\in C^1(I\times\mathbb{T}^d)$. Furthermore, for $H\in C^1(I\times\mathbb{T}^d)$ we have for fixed $t\in I$ that

$$\sup_{\varepsilon>0}\|\upsilon_\varepsilon(H)_t\|_u\le \|H_t\|_{\mathrm{Lip}}+2\|H_t\|_u^2$$

and therefore by the bounded convergence theorem we have that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{T}^d}\upsilon_\varepsilon(H)_t d\sigma_t = \int_{\mathbb{T}^d}\upsilon(H)_t d\sigma_t$$

for all $t \in I$. Continuing we have for Q^g-a.s. all $\sigma \in L^\infty_{w^*}(I;\mathcal{M}(\mathbb{T}^d))$ that

$$\begin{aligned}\sup_{\varepsilon>0}\sup_{0\le t\le T}\left|\int_{\mathbb{T}^d}\upsilon_\varepsilon(H)_t d\sigma_t\right| &\le \|g\|_u \sup_{\varepsilon>0}\sup_{0\le t\le T}\|\upsilon_\varepsilon(H)_t\|_u \\ &\le \sup_{0\le t\le T}\left\{\|H_t\|_{\mathrm{Lip}} + 2\|H_t\|_u^2\right\} < +\infty,\end{aligned}$$

since $H \in C^1(I\times\mathbb{T}^d)$, and therefore by the bounded convergence theorem again, we have that

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{T}^d}\upsilon_\varepsilon(H)_t d\sigma_t dt = \int_0^T\int_{\mathbb{T}^d}\upsilon(H)_t d\sigma_t dt$$

for all $H \in C^1(I\times\mathbb{T}^d)$. Finally, for Q^g-a.s. all $\sigma \in L^\infty_{w^*}(I;\mathcal{M}(\mathbb{T}^d))$ we have that

$$\sup_{\varepsilon>0}\left|\max_{1\le i\le m}\int_0^T\int_{\mathbb{T}^d}\upsilon_\varepsilon(H^i)_t d\sigma_t dt\right| \le \|g\|_u \max_{1\le i\le m}\sup_{\varepsilon>0}\int_0^T \|\upsilon_\varepsilon(H^i)_t\|_u dt < +\infty$$

and so applying the bounded convergence theorem one last time we get that

$$\int \max_{1\le i\le m}\langle \upsilon(H^i),\sigma\rangle_I dQ^g(\sigma) = \lim_{\varepsilon\to 0}\int \max_{1\le i\le m}\langle \upsilon_\varepsilon(H^i),\sigma\rangle_I dQ^g(\sigma) \le C_0,$$

which completes the proof. □

Corollary 4.5.2 *If $\{\mu_0^N \in \mathbb{P}_1\mathbb{M}_N^d\}$ is a sequence of initial distributions satisfying the $O(N^d)$-entropy assumption for some constant $C_0 > 0$, then for all $Q^g \in \mathcal{Q}^g(\{\mu_0^N\})$ we have that*

$$\int\left\{\sup_{H\in C^1(I\times\mathbb{T}^d)}\int_0^T\int_{\mathbb{T}^d}\left[\partial_j H_t(x) - H_t(x)^2\right]\sigma_t(x)dxdt\right\}dQ^g(\sigma) \le C_0.$$

Proof Let $\{H_i\}_{i\in\mathbb{N}} \subseteq C^1(I\times\mathbb{T}^d)$ be a sequence dense in $C^1(I\times\mathbb{T}^d)$ in the usual C^1 uniform norm $\|\cdot\|_{C^1}$. Then for all $\sigma \in L^\infty_{w^*}(I;L^\infty(\mathbb{T}^d))$ we obviously have that

$$\begin{aligned}&\sup_{H\in C^1(I\times\mathbb{T}^d)}\int_0^T\int_{\mathbb{T}^d}\left[\partial_j H_t(x) - H_t(x)^2\right]\sigma_t(x)dxdt \\ &\qquad = \lim_{m\to\infty}\max_{1\le i\le m}\int_0^T\int_{\mathbb{T}^d}\left[\partial_j H_t^i(x) - H_t^i(x)^2\right]\sigma_t(x)dxdt.\end{aligned}$$

So the claim follows by the monotone convergence theorem and the previous corollary. □

Proposition 4.5.1 *Let $\{\mu_0^N \in \mathbb{P}_1\mathbb{M}_N^d\}$ be a sequence of initial distributions satisfying the $O(N^d)$-entropy assumption for some constant $C_0 > 0$ and let $Q^g \in \mathcal{Q}^g(\{\mu_0^N\})$. Then Q^g is concentrated on paths $\sigma \in L^\infty_{w^*}(I;L^\infty(\mathbb{T}^d))$ with the property that there exist $L^2(I\times\mathbb{T}^d)$ functions denoted by $\partial_j\sigma$, $j = 1,\dots,d$, such that*

$$\int_0^T\int_{\mathbb{T}^d}\partial_j H_t(x)\sigma(t,x)dxdt = -\int_0^T\int_{\mathbb{T}^d}H_t(x)\partial_j\sigma(t,x)dxdt \tag{4.76}$$

and

$$\int_0^T \int_{\mathbb{T}^d} \frac{|\nabla \sigma(t,x)|_2^2}{\sigma(t,x)} dxdt < +\infty. \tag{4.77}$$

Proof Fix $j = 1, \dots, d$. By corollary 4.5.2 for Q^g-a.s. all paths $\sigma \in L^\infty(I; L^\infty(\mathbb{T}^d))$ we have that

$$B_\sigma := \sup_{H \in C^1(I \times \mathbb{T}^d)} \int_0^T \left(\int_{\mathbb{T}^d} \left[\partial_j H(t,x) - 2H(t,x)^2\right] \sigma(t,x) dx \right) dt < +\infty. \tag{4.78}$$

Fix such a path $\sigma \in L^\infty(I; L^\infty(\mathbb{T}^d))$ and consider on $C^1(I \times \mathbb{T}^d)$ the inner product $\langle \cdot, \cdot \rangle_\sigma$ defined by

$$\langle H, G \rangle_\sigma = \int_0^T \int_{\mathbb{T}^d} H(t,x) G(t,x) \sigma(t,x) dxdt,$$

and denote by L^2_σ the Hilbert space resulting from the completion of $C^1(I \times \mathbb{T}^d)$ with respect to the inner product $\langle \cdot, \cdot \rangle_\sigma$.

Let now $\ell \equiv \ell_j : C^1(I \times \mathbb{T}^d) \longrightarrow \mathbb{R}$ denote the linear function given by the formula

$$\ell(H) = \int_0^T \int_{\mathbb{T}^d} \partial_j H(t,x) \sigma(t,x) dxdt.$$

It follows from estimate (4.78) that

$$a\ell(H) - 2a^2 \|H\|^2_{L^2_\sigma} \leq B_\sigma$$

for all $a \in \mathbb{R}$ and all $H \in C^1(I \times \mathbb{T}^d)$. The maximum over all $a \in \mathbb{R}$ of the quantity in the left hand side of the inequality above is achieved at $a = \ell(H)/\|2H\|^2_{L^2_\sigma}$, and therefore

$$\frac{\ell(H)^2}{8\|H\|^2_{L^2_\sigma}} = \frac{\ell(H)^2}{4\|H\|^2_{L^2_\sigma}} - 2\frac{\ell(H)^2}{16\|H\|^4_{L^2_\sigma}} \|H\|^2_{L^2_\sigma} \leq B_\sigma$$

for all $H \in C^1(I \times \mathbb{T}^d)$. It follows that

$$|\ell(H)| \leq 2\sqrt{2B_\sigma} \|H\|_{L^2_\sigma}$$

for all $H \in C^1(I \times \mathbb{T}^d)$ and thus ℓ can be extended to a bounded linear function $\ell : L^2_\sigma \longrightarrow \mathbb{R}$ with norm $\|\ell\| \leq 2\sqrt{2B_\sigma}$.

By the Riesz representation theorem now, there exists an L^2_σ function, which we denote by $\partial_j(\log \sigma)$, such that

$$\ell(H) = -\langle H, \partial_j(\log \sigma) \rangle_\sigma = \int_0^T \int_{\mathbb{T}^d} H(t,x) \partial_j(\log \sigma)(t,x) \sigma(t,x) dxdt \tag{4.79}$$

for all $H \in C^1(I \times \mathbb{T}^d)$. Of course, since $\partial_j(\log \sigma) \in L^2_\sigma$ we have that

$$\|\partial_j(\log \sigma)\|^2_{L^2_\sigma} = \int_0^T \int_{\mathbb{T}^d} \left[\partial_j(\log \sigma)(t,x)\right]^2 \sigma(t,x) dxdt < +\infty. \tag{4.80}$$

Therefore, since as we now from proposition 4.4.5 for Q^g-a.s. all $\sigma \in L^\infty_{w^*}(I; L^\infty(\mathbb{T}^d))$ we have that

$$\big\| \|\sigma_t\|_{L^\infty(\mathbb{T}^d)} \big\|_{L^\infty(I)} \leq \|g\|_u,$$

the function $\partial_j \sigma := \sigma \cdot \partial_j(\log \sigma)$ satisfies

$$\int_{I\times\mathbb{T}^d} \big(\partial_j\sigma\big)^2 = \int_0^T \int_{\mathbb{T}^d} \big[\partial_j(\log\sigma)(t,x)\big]^2 \sigma(t,x)^2 dxdt \le \|g\|_u \|\partial_j(\log\sigma)\|_{L^2_\sigma}^2 < +\infty,$$

and thus is in $L^2(I\times\mathbb{T}^d)$, and by (4.79) $\partial_j\sigma$ satisfies property (4.76) and is thus the required L^2weak derivative of σ. Finally, by the identity $\partial_j = \sigma\cdot\partial_j(\log\sigma)$ we have that

$$\frac{|\nabla_x\sigma(t,x)|_2^2}{\sigma(t,x)} = \sigma(t,x)|\nabla(\log\sigma)(t,x)|_2^2$$

and therefore (4.77) follows from (4.80). □

Proposition 4.5.2 *Let $\{\mu_0^N \in \mathbb{P}_1\mathbb{M}_N^d\}$ be a sequence of initial distributions satisfying the $O(N^d)$-entropy assumption for some constant $C_0 > 0$ and let $Q^g \in \mathcal{Q}^g(\{\mu_0^N\})$. Then Q^g is concentrated on paths $\sigma \in L^\infty_{w^*}(I; H^1(\mathbb{T}^d))$, i.e.*

$$Q^g\big(\sigma|\sigma_t \in H^1(\mathbb{T}^d)) \text{ a.s. for all } t\in[0,T]\big) = 1. \tag{4.81}$$

Proof By the previous proposition we know that Q^g is concentrated on paths $\sigma \in L^\infty_{w^*}(I; L^\infty(\mathbb{T}^d))$ such that there exist functions $\partial_j\sigma \in L^2(I\times\mathbb{T}^d)$, $j=1,\dots,d$, satisfying (4.76) for all $H\in C^1(I\times\mathbb{T}^d)$. We fix such a path $\sigma\in L^\infty_{w^*}(I;L^\infty(\mathbb{T}^d))$ and we will show that $\sigma_t \in H^1(\mathbb{T}^d)$ for almost all $t\in[0,T]$. For each $t\in[0,T]$ and $\varepsilon>0$ we consider a sequence of smooth functions $\{f_{t,\varepsilon}^N\}_{N\in\mathbb{N}}$ defined on $[0,T]$ such that $f_{t,\varepsilon}^N \le \mathbb{1}_{[t-\varepsilon,t+\varepsilon]}$ for all $N\in\mathbb{N}$ and $f_{t,\varepsilon}^N \longrightarrow \mathbb{1}_{(t-\varepsilon,t+\varepsilon)}$ pointwise as $N\to\infty$. Then for all functions $H\in C^1(I\times\mathbb{T}^d)$ we have by (4.76) that

$$\int_0^T\int_{\mathbb{T}^d} f_{t,\varepsilon}^N(s)\partial_j H_s(x)\sigma(s,x)dxds = -\int_0^T\int_{\mathbb{T}^d} f_{t,\varepsilon}^N(s) H_s(x)\partial_j\sigma(s,x)dxds.$$

Then taking the limit as $N\to\infty$ in both sides of the inequality above, we get that

$$\int_{t-\varepsilon}^{t+\varepsilon}\int_{\mathbb{T}^d} \partial_j H_s(x)\sigma(s,x)dxds = -\int_{t-\varepsilon}^{t+\varepsilon}\int_{\mathbb{T}^d} H_s(x)\partial_j\sigma(s,x)dxds.$$

Then taking the limit as $\varepsilon\to 0$ in both sides of the equality above, it follows by Lebesgue's differentiation theorem that for each $H\in C^1(I\times\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} \partial_j H_t(x)\sigma(t,x)dx = -\int_{\mathbb{T}^d} H_t(x)\partial_j\sigma(t,x)dx \tag{4.82}$$

for all $t\in E_H$, for some measurable set $E_H\subseteq I$ of full measure $m(E_H)=T$. Taking then a sequence $\{H^i\}_{i\in\mathbb{N}}\subseteq C^1(I\times\mathbb{T}^d)$ dense in $C^1(I\times\mathbb{T}^d)$ in the C^1-uniform norm $\|\cdot\|_{C^1}$, we have that the set $E:=\bigcap_{i\in\mathbb{N}} E_{H^i}$ is of full measure $m(E)=T$, and for each $t\in E$ we have that (4.82) holds for all $H\in C^1(I\times\mathbb{T}^d)$. In particular, since $C^1(\mathbb{T}^d)$ can be considered as a subspace of $C^1(I\times\mathbb{T}^d)$ it follows that

$$\int_{\mathbb{T}^d} \partial_j H(x)\sigma(t,x)dx = -\int_{\mathbb{T}^d} H(x)\partial_j\sigma(t,x)dx, \qquad \forall\,(t,H)\in E\times C^1(\mathbb{T}^d).$$

Consequently, σ_t is weakly differentiable for almost all $t \in I$, with weak j-th partial derivative $\partial_j \sigma_t$. Finally, since $\partial_j \sigma \in L^2(I \times \mathbb{T}^d)$ we have that

$$\int_0^T \|\partial_j \sigma_t\|^2_{L^2(\mathbb{T}^d)} dt = \|\partial_j \sigma\|^2_{L^2(I \times \mathbb{T}^d)} < +\infty,$$

and therefore $\|\partial_j \sigma_t\|_{L^2(\mathbb{T}^d)} < +\infty$ for almost all $t \in I$. Consequently, $\sigma_t \in H^1(\mathbb{T}^d)$ for almost all $t \in I$, as required. □

Of course, by the discussion in the beginning of this section we get that (4.68) holds.

4.5.2 An Upper Bound On the Limiting Empirical Jump Rate

In this section we continue restricting our attention to the case that the jump rate g is bounded, and we will investigate the relation between the limiting laws of the families of the distributions of the processes $\{\sigma^N\}_{N \in \mathbb{N}}$, $\{\sigma^{N,\ell}\}_{(N,\ell) \in \mathbb{N}^2}$ and $\{\sigma^{N,\ell,\Phi}\}_{(N,\ell) \in \mathbb{N}^2}$. These are defined in (3.3), (3.6) and (3.4) respectively.

Lemma 4.5.2 *Let $\{x_n\}$ be a sequence in a metric space X and let $\mathrm{Lim}\{x_n\}$ denote the set of all subsequential limit points of $\{x_n\}$. Then*

$$\mathrm{Lim}\{x_n\} = \bigcap_{m=1}^{\infty} \overline{\{x_n | n \geq m\}},$$

and in particular $\mathrm{Lim}\{x_n\}$ is a closed set.

Proposition 4.5.3 *We suppose that g is a bounded jump rate function and that the sequence $\{\mu_0^N \in \mathbb{P}_1 \mathbb{M}_N^d\}$ of initial distributions is associated to a macroscopic profile $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$. We set*

$$Q^N := \left[\left(\sigma_t^N\right)_{t \in I}\right]_* P^N \in \mathbb{P} L^\infty(I; \mathcal{M}_+(\mathbb{T}^d))$$
$$Q^{N,\ell} := \left[\left(\sigma_t^{N,\ell}\right)_{t \in I}\right]_* P^N \in \mathbb{P} L^\infty(I; \mathcal{M}_+(\mathbb{T}^d))$$

for all $N \in \mathbb{N}$. We also set $\mathcal{Q}^\infty$ denote the set of all subsequential limits of $\{Q^N\}$ in $\mathbb{P} L^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d))$ and for each $\ell \in \mathbb{N}$ we denote by $\mathcal{Q}^{\infty,\ell}$ the set of subsequential limit points of the sequence $\{Q^{N,\ell}\}_{N \in \mathbb{N}}$ in $\mathbb{P} L^\infty(I; \mathcal{M}_+(\mathbb{T}^d))$. Then if $\mathcal{Q}^{\infty,\infty}$ denotes the set of limit points along subsequences of all sequences $\{Q^{\infty,\ell}\}_{\ell \in \mathbb{N}}$ such that $Q^{\infty,\ell} \in \mathcal{Q}^{\infty,\ell}$ for all $\ell \in \mathbb{N}$, we have $\mathcal{Q}^\infty = \mathcal{Q}^{\infty,\infty}$

Proof We note first that

$$\sup_{(N,\ell) \in \mathbb{N} \times \mathbb{N}} \sup_{\eta \in \mathbb{M}_N^d} \|\sigma_\eta^{N,\ell}\|_{TV} \leq \|g\|_u < +\infty,$$

which shows that the family $\{Q^{N,\ell}\}_{(N,\ell) \in \mathbb{N} \times \mathbb{N}}$ is contained in the compact subspace $\mathbb{P} B_{L^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d))}(0, \|g\|_u)$. In particular

$$\emptyset \neq \mathcal{Q}^{\infty,\ell} \subseteq \mathbb{P} B_{L^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d))}(0, \|g\|_u),$$

and therefore any sequence $\{Q^{\infty,\ell}\}_{\ell\in\mathbb{N}}$ such that $Q^{\infty,\ell}\in\mathcal{Q}^{\infty,\ell}$ for all $\ell\in\mathbb{N}$ is contained in the compact subspace $\mathbb{P}B_{L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))}(0,\|g\|_u)$ and therefore has a convergent subsequence. In particular,

$$\emptyset\neq\mathcal{Q}^{\infty,\infty}\subseteq\mathbb{P}B_{L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))}(0,\|g\|_u).$$

We prove next that $\mathcal{Q}^\infty=\mathcal{Q}^{\infty,\infty}$. To this aim we consider the measures

$$\widetilde{Q}^{N,\ell}:=\big[(\sigma^N_t,\sigma^{N,\ell}_t)_{t\geq 0}\big]_*P^N\in\mathbb{P}L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d)\times\mathcal{M}_+(\mathbb{T}^d)).$$

Then denoting by $x^i:L^\infty_{w^\infty}(I;\mathcal{M}_+(\mathbb{T}^d))\times L^\infty_{w^\infty}(I;\mathcal{M}_+(\mathbb{T}^d))\longrightarrow L^\infty_{w^\infty}(I;\mathcal{M}_+(\mathbb{T}^d))$, $i=1,2$, the natural projections, we have that

$$x^1_*\widetilde{Q}^{N,\ell}=Q^N\quad\text{and}\quad x^2_*\widetilde{Q}^{N,\ell}=Q^{N,\ell}$$

and the family $\{\widetilde{Q}^{N,\ell}\}_{(N,\ell)\in\mathbb{N}\times\mathbb{N}}$ is relatively compact.

We prove first that $\mathcal{Q}^\infty\subseteq\mathcal{Q}^{\infty,\infty}$. So let $Q\in\mathcal{Q}^\infty$ and let $\{Q^{k_N}\}_{N\in\mathbb{N}}$ be a subsequence of $\{Q^N\}_{N\in\mathbb{N}}$ converging to Q. Then, for each $\ell\in\mathbb{N}$, the sequence $\{\widetilde{Q}^{k_N,\ell}\}_{N\in\mathbb{N}}$ is relatively compact and as such it has a subsequential limit point $\widetilde{Q}^{\infty,\ell}$, along some subsequence $\{\widetilde{Q}^{k_{m^\ell_N},\ell}\}_{N\in\mathbb{N}}$ of $\{\widetilde{Q}^{k_N,\ell}\}_{N\in\mathbb{N}}$. Then since the projections x^1,x^2 are continuous we obviously have that

$$x^1_*\widetilde{Q}^{\infty,\ell}=x^1_*\Big(\lim_{N\to\infty}\widetilde{Q}^{k_{m^\ell_N},\ell}\Big)=\lim_{N\to\infty}x^1_*\Big(\widetilde{Q}^{k_{m^\ell_N},\ell}\Big)=\lim_{N\to\infty}Q^{k_{m^\ell_N}}=Q$$

and

$$\lim_{N\to\infty}Q^{k_{m^\ell_N},\ell}=\lim_{N\to\infty}x^2_*\widetilde{Q}^{k_{m^\ell_N},\ell}=x^2_*\Big(\lim_{N\to\infty}\widetilde{Q}^{k_{m^\ell_N},\ell}\Big)=x^*_2\widetilde{Q}^{\infty,\ell}$$

for all $\ell\in\mathbb{N}$.

We claim that

$$\lim_{\ell\to\infty}\widetilde{Q}^{\infty,\ell}=(id,id)_*Q,\qquad id=id_{L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))}.\tag{4.83}$$

Obviously this implies that

$$Q^{\infty,\ell}=x^2_*\widetilde{Q}^{\infty,\ell}\longrightarrow x^2_*\big[(id,id)_*Q\big]=Q$$

and proves the inclusion $\mathcal{Q}^\infty\subseteq\mathcal{Q}^{\infty,\infty}$. Since the sequence $\{Q^{\infty,\ell}\}_{\ell\in\mathbb{N}}$ is obviously relatively compact in order to prove (4.83) it suffices to prove that any subsequential limit point of $\{\widetilde{Q}^{\infty,\ell}\}$ is equal to $(id,id)_*Q$. So let $\{\widetilde{Q}^{\infty,n_\ell}\}_{\ell\in\mathbb{N}}$ be a subsequence of $\{\widetilde{Q}^{\infty,\ell}\}$ converging weakly to some probability measure $\widetilde{Q}^{\infty,\infty}\in\mathbb{P}\big(L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))^2\big)$. Since $x^1_*\widetilde{Q}^{\infty,\ell}=Q$ for all $\ell\in\mathbb{N}$ in order to prove that $\widetilde{Q}^{\infty,\infty}=(id,id)_*Q$ it suffices to prove that $\widetilde{Q}^{\infty,\infty}$ is consentrated on the diagonal of $L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))\times L^\infty_{w^*}(I;\mathcal{M}_+(\mathbb{T}^d))$. But this is true indeed. By the portmanteau theorem and proposition 3.1.3 we have for all $f\in L^1(I;C(\mathbb{T}^d))$ and all $\varepsilon>0$ that

$$\begin{aligned}\widetilde{Q}^{\infty,\infty}\big\{|\langle f,x^1\rangle_I-\langle f,x^2\rangle_I|>\varepsilon\big\}&\leq\liminf_{\ell\to\infty}\widetilde{Q}^{\infty,n_\ell}\big\{|\langle f,x^1\rangle_I-\langle f,x^2\rangle_I|>\varepsilon\big\}\\&\leq\liminf_{\ell\to\infty}\liminf_{N\to\infty}\widetilde{Q}^{k_{m^{n_\ell}_N},n_\ell}\big\{|\langle f,x^1\rangle_I-\langle f,x^2\rangle_I|>\varepsilon\big\}\\&=\lim_{\ell\to\infty}\lim_{N\to\infty}P^{k_{m^{n_\ell}_N}}\big\{|\langle f,\sigma^{k_{m^{n_\ell}_N}}-\sigma^{k_{m^{n_\ell}_N},n_\ell}\rangle_I|>\varepsilon\big\}\\&\leq\varlimsup_{\ell\to\infty}\varlimsup_{N\to\infty}P^N\big\{|\langle f,\sigma^N-\sigma^{N,n_\ell}\rangle_I|>\varepsilon\big\}\\&\leq\varlimsup_{\ell\to\infty}\varlimsup_{N\to\infty}P^N\big\{|\langle f,\sigma^N-\sigma^{N,\ell}\rangle_I|>\varepsilon\big\}=0.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary it follows that

$$\widetilde{Q}^{\infty,\infty}\big\{|\langle f, x^1\rangle_I - \langle f, x^2\rangle_I| > \varepsilon\big\} = 0, \qquad \forall\, f \in L^1(I; C(\mathbb{T}^d)),$$

which since $L^1(I; C(\mathbb{T}^d))$ is separable implies that

$$\widetilde{Q}^{\infty,\infty}\big\{x^1 = x^2\big\} = \widetilde{Q}^{\infty,\infty}\Bigg(\bigcap_{f\in L^1(I;C(\mathbb{T}^d))}\big\{|\langle f, x^1\rangle_I - \langle f, x^2\rangle_I| = 0\big\}\Bigg) = 0,$$

and therefore $\widetilde{Q}^{\infty,\infty}$ is concentrated on the diagonal and (4.83) holds.

We prove next the inclusion $\mathcal{Q}^{\infty,\infty} \subseteq \mathcal{Q}^{\infty}$. So let $Q^{\infty,\infty} \in \mathcal{Q}^{\infty,\infty}$. There exists then a subsequence $\{Q^{\infty,n_\ell}\}_{\ell\in\mathbb{N}}$ of $\{Q^{\infty,\ell}\}$ converging to $Q^{\infty,\infty}$, and for each $\ell \in \mathbb{N}$ there exists a subsequence $\{Q^{k^\ell_N,n_\ell}\}_{N\in\mathbb{N}}$ of $\{Q^{N,n_\ell}\}$ converging to Q^{∞,n_ℓ}. We consider the transport plans $\widetilde{Q}^{k^\ell_N,n_\ell} \in \Pi(Q^{k^\ell_N}, Q^{k^\ell_N,n_\ell})$, $(N,\ell) \in \mathbb{N}$. The family of these transport plans is relatively compact and therefore for each $\ell \in \mathbb{N}$ we can choose a convergent subsequence $\{\widetilde{Q}^{k^\ell_{m^\ell_N},n_\ell}\}$ of $\{\widetilde{Q}^{k^\ell_N,n_\ell}\}$ and set

$$\widetilde{Q}^{\infty,n_\ell} := \lim_{N\to\infty} \widetilde{Q}^{k^\ell_{m^\ell_N},n_\ell}.$$

Then obviously

$$x^2_*\widetilde{Q}^{\infty,n_\ell} = \lim_{N\to\infty} x^2_*\widetilde{Q}^{k^\ell_{m^\ell_N},n_\ell} = \lim_{N\to\infty} Q^{k^\ell_{m^\ell_N},n_\ell} = Q^{\infty,n_\ell},$$

and the sequence $\{\widetilde{Q}^{\infty,n_\ell}\}_{\ell\in\mathbb{N}}$ is relatively compact and therefore has a convergent subsequence $\{\widetilde{Q}^{\infty,n_{i_\ell}}\}_{\ell\in\mathbb{N}}$ converging to some $\widetilde{Q}^{\infty,\infty} \in \mathbb{P}\big[L^\infty_{w^*}\big(I; \mathcal{M}_+(\mathbb{T}^d)\big)^2\big]$ with

$$x^2_*\widetilde{Q}^{\infty,\infty} = \lim_{\ell\to\infty} x^2_*\widetilde{Q}^{\infty,n_{i_\ell}} = \lim_{\ell\to\infty} Q^{\infty,n_{i_\ell}} = Q^{\infty,\infty}.$$

Therefore, in order to prove that $Q^{\infty,\infty} \in \mathcal{Q}^{\infty}$ it suffices to prove that

$$x^1_*\widetilde{Q}^{\infty,\infty} \in \mathcal{Q}^{\infty} \tag{4.84}$$

and that $\widetilde{Q}^{\infty,\infty}$ is concentrated on the diagonal of $L^\infty_{w^*}\big(I; \mathcal{M}_+(\mathbb{T}^d)\big)^2$.

We prove first that $x^1_*\widetilde{Q}^{\infty,\infty} \in \mathcal{Q}^{\infty}$. Of course since x^1_* is continuous we have that

$$x^1_*\widetilde{Q}^{\infty,n_{i_\ell}} = \lim_{N\to\infty} x^1_*\widetilde{Q}^{k^{i_\ell}_{m^{i_\ell}_N},n_{i_\ell}} = \lim_{N\to\infty} Q^{k^{i_\ell}_{m^{i_\ell}_N}} \in \mathcal{Q}^{\infty}$$

for all $\ell \in \mathbb{N}$. Likewise, since x^1_* is continuous and $\widetilde{Q}^{\infty,n_{i_\ell}}$ converges to $\widetilde{Q}^{\infty,\infty}$ we have that

$$x^1_*\widetilde{Q}^{\infty,\infty} = \lim_{\ell\to\infty} x^1_*\widetilde{Q}^{\infty,n_{i_\ell}} \in \overline{\mathcal{Q}^{\infty}}. \tag{4.85}$$

But by lemma 4.5.2 the set

$$\mathcal{Q}^{\infty} = \mathrm{Lim}\{Q^N\} = \bigcap_{m=1}^{\infty} \overline{\{Q^N | N \geq m\}}$$

is closed and therefore (4.84) follows from (4.85).

We prove finally that $\widetilde{Q}^{\infty,\infty}$ is concentrated on the diagonal. So let $f \in L^1(I; C(\mathbb{T}^d))$ and $\varepsilon > 0$ be arbitrary. Then by the portmanteau theorem we have that

$$\begin{aligned}
\widetilde{Q}^{\infty,\infty}\big\{|\langle f, x^1\rangle_I - \langle f, x^2\rangle_I| > \varepsilon\big\} &\le \liminf_{\ell\to\infty} \widetilde{Q}^{\infty,n_{i_\ell}}\big\{|\langle f, x^1\rangle_I - \langle f, x^2\rangle_I| > \varepsilon\big\} \\
&\le \liminf_{\ell\to\infty}\liminf_{N\to\infty} \widetilde{Q}^{k^{i_\ell}_{m^{i_\ell}_N}, n_{i_\ell}}\big\{|\langle f, x^1\rangle_I - \langle f, x^2\rangle_I| > \varepsilon\big\} \\
&= \underline{\lim}_{\ell\to\infty}\,\underline{\lim}_{N\to\infty}\, P^{k^{i_\ell}_{m^{i_\ell}_N}}\big\{|\langle f, \sigma^{k^{i_\ell}_{m^{i_\ell}_N}} - \sigma^{k^{i_\ell}_{m^{i_\ell}_N}, n_{i_\ell}}\rangle_I| > \varepsilon\big\} \\
&\le \overline{\lim}_{\ell\to\infty}\,\overline{\lim}_{N\to\infty}\, P^N\big\{|\langle f, \sigma^N - \sigma^{N,n_\ell}\rangle_I| > \varepsilon\big\} \\
&\le \overline{\lim}_{\ell\to\infty}\,\overline{\lim}_{N\to\infty}\, P^N\big\{|\langle f, \sigma^N - \sigma^{N,\ell}\rangle_I| > \varepsilon\big\} = 0.
\end{aligned}$$

Since $f \in L^1(I; C(\mathbb{T}^d))$ and $\varepsilon > 0$ were arbitrary this proves that $\widetilde{Q}^{\infty}\{x^1 = x^2\} = 1$, i.e. that $\widetilde{Q}^{\infty,\infty}$ is concentrated in the diagonal and completes the proof. □

Following these arguments and the version of the One-Block estimate proved in section 3.1 with respect to functions in $L^1(0,T; C(\mathbb{T}^d))$ one can also prove the following

Proposition 4.5.4 *We suppose that g is a bounded jump rate function and that the sequence $\{\mu_0^N \in \mathbb{P}_1\mathbb{M}_N^d\}$ of initial distributions is associated to a macroscopic profile $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$. We set*

$$Q^N := \big[(\sigma_t^N)_{t\in I}\big]_* P^N \in \mathbb{P}L^\infty(I; \mathcal{M}_+(\mathbb{T}^d))$$

$$Q^{N,\ell,\Phi} := \big[(\sigma_t^{N,\ell,\Phi})_{t\in I}\big]_* P^N \in \mathbb{P}L^\infty(I; \mathcal{M}_+(\mathbb{T}^d))$$

for all $N \in \mathbb{N}$. We also set $\mathcal{Q}^\infty$ denote the set of all subsequential limits of $\{Q^N\}$ in $\mathbb{P}L^\infty_{w^*}(I; \mathcal{M}_+(\mathbb{T}^d))$ and for each $\ell \in \mathbb{N}$ we denote by $\mathcal{Q}^{\infty,\ell,\Phi}$ the set of subsequential limit points of the sequence $\{Q^{N,\ell,\Phi}\}_{N\in\mathbb{N}}$ in $\mathbb{P}L^\infty(I; \mathcal{M}_+(\mathbb{T}^d))$. Then if $\mathcal{Q}^{\infty,\infty,\Phi}$ denotes the set of limit points along subsequences of all sequences $\{Q^{\infty,\ell,\Phi}\}_{\ell\in\mathbb{N}}$ such that $Q^{\infty,\ell,\Phi} \in \mathcal{Q}^{\infty,\ell,\Phi}$ for all $\ell \in \mathbb{N}$, we have $\mathcal{Q}^\infty = \mathcal{Q}^{\infty,\infty,\Phi}$

It follows that

$$Q^g\big(\sigma \,\big|\, \|\sigma_t\|_\infty \le \varphi_c \text{ a.s. for all } t \in [0,T]\big) = 1$$

and with this the proof of proposition 4.0.2 is complete.

4.6 A Conjecture

In the general case, in order to describe the condensation phenomenon of ZRPs in the hydrodynamic limit, the solutions of the non-linear diffusion equation should be allowed to be measure-valued.

Let us for the moment concentrate on the case of the Evans model for $0 \leq b \leq 2$. In this case the mean jump rate $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is C^1, and an appropriate interpretation of the non-linear heat equation $\partial_t \mu = \Delta\Phi(\mu)$ that allows measure-valued solutions μ can be given via the theory of Wasserstein gradient flows, at least in dimension $d = 1$. In the one-dimensional case, a promising way to model the non-linear diffusion equation so that it allows measure-valued solutions, is as the gradient flow with respect to the L^2-Wasserstein metric of the functional $H_E : \mathcal{M}_+(\mathbb{T}) \longrightarrow (-\infty, +\infty]$ defined by

$$H_E(\mu) = \int E(\mu^{ac}) dm_{\mathbb{T}^d} + E'(\infty)\mu^s(\mathbb{T}), \tag{4.86}$$

where $\mu = \mu^{ac} m_{\mathbb{T}} + \mu^s$ is the Radon-Nikodym decomposition of μ with respect to the Lebesgue measure on the torus $\mathbb{T}$, $E : [0, \infty) \longrightarrow \mathbb{R}$ is the convex internal energy given by

$$E(\rho) = -\Phi(\rho) - \rho \int_\rho^{+\infty} \frac{\Phi'(r)}{r} dr$$

and

$$E'(\infty) := \lim_{t \to +\infty} \frac{E(t)}{t}.$$

We consider the one-dimensional case, since in the condensing case we have

$$\lim_{\rho \to \infty} \Phi(\rho) = \varphi_c < +\infty, \tag{4.87}$$

which implies that for $d \geq 2$ the Mac-Cann conditions

$$\rho\Phi'(\rho) - \Big(1 - \frac{1}{d}\Big)\Phi(\rho) \geq 0, \quad \rho \geq 0$$

fail to hold, and thus the functional H_E fails to be λ-displacement convex, at least for $\lambda \geq 0$. However, in dimension $d = 1$ it follows by the characterizations of displacement convexity of functionals on the Wasserstein space of probability measures given by Villani ([32], theorem 17.15 and remarks 17.18 and 17.20) that the functional H_E is displacement convex. So by the work of Shin-ichi Ohta ([22], theorem 5.11) on Gradient flows on Wasserstein spaces over compact Alexandrov spaces, it induces a well defined gradient flow:

Given any initial distribution μ_0 there exists a gradient curve $t \mapsto \mu_t$ of H_E starting from μ_0, and given two gradient curves (μ_t), (ν_t) starting from μ_0, ν_0 respectively we have

$$W_2(\mu_t, \nu_t) \leq W_2(\mu_0, \nu_0).$$

Even in the case $d = 1$, the condition

$$\lim_{\rho \to \infty} \rho^{\frac{1}{d}} \Big(E'(\infty) - \frac{E(\rho)}{\rho}\Big) = +\infty$$

which forces the gradient curves to be absolutely continuous with respect to Lebesgue measure, fails to hold. In fact in dimension $d = 1$,

$$\rho^{\frac{1}{d}}\left(E'(\infty) - \frac{E(\rho)}{\rho}\right) = -E(\rho) \overset{\rho\to\infty}{\longrightarrow} \varphi_c < +\infty.$$

In the article [17] Fornaro, Lisini, Savare and Toscani in 2010 study the gradient flow of the functional H_E defined in (4.86) in the case of the real line, under the assumption (4.87), in the presence of a C^2 driving potential such that

$$\inf_{x\in\mathbb{R}} V''(x) > 0 \quad \text{and} \quad \liminf_{|x|\to\infty} \frac{V(x)}{|x|^2} \geq 0.$$

There, they study the problem

$$\partial_t \mu - \operatorname{div}(\nabla\Phi(\mu) + V'\mu) = 0 \tag{4.88}$$

as the gradient flow of the energy functional

$$\mathcal{E}(\mu) = H_E(\mu) + \int_{\mathbb{R}} V(x) d\mu(x). \tag{4.89}$$

Definition 4.6.1 Let $\mathcal{M}_{+,\rho_c}(M)$, $M = \mathbb{R}$ or $\mathbb{T}$ denote the set of all finite measures of finite quadratic moment whose absolutely continuous part has a continuous representative $\rho \in C(M; [0, \rho_c])$ such that

$$\mu^{\perp}(\{\rho < \rho_c\}) = 0, \quad \mathcal{L}^1\big(M \setminus \{\rho < \rho_c\}\big) = 0 \quad \text{and} \quad \mu^{ac} = \rho dx.$$

In the article [17], $\rho_c = +\infty$. There they define $J : \mathcal{M}_+(\mathbb{T}^d) \longrightarrow \mathbb{R}_+$ by

$$J(\mu) = \begin{cases} \int \left|\frac{\nabla(\Phi(\rho))}{\rho} + V'\right|^2 \rho dx + \int |V'|^2 d\mu^{\perp} & \text{if } \mu = \rho + \mu^{\perp} \in \mathcal{M}_{+,\rho_c},\ \Phi(\rho) \in W^{1,1}_{\text{loc}} \\ J(\mu) = +\infty & \text{otherwise} \end{cases}$$

and prove the following characterization of the gradient flow of the functional $\mathcal{E}$ as a solution of problem (4.88) in the sense of distributions:

Proposition 4.6.1 *A curve $\mu \in C([0,\infty); \mathcal{M}_{+,\rho_c}(\mathbb{R}))$ is a gradient flow of the functional $\mathcal{E}$ defined in (4.89) if*

$$\text{(a)} \quad \mu_t = \rho_t + \mu_t^{\perp} \in \mathcal{M}_{+,\rho_c}(\mathbb{R}) \quad a.s.\ \forall\, t \geq 0,$$

$$\text{(b)} \quad \int_0^T J(\mu_t) dt < +\infty \quad \forall\, T > 0,$$

and

$$\text{(c)}\ \partial_t \mu - \operatorname{div}(\nabla\Phi(\rho) + V'\mu) = 0$$

holds in the sense of distributions, i.e. for all $G \in C_c^\infty((0,\infty) \times \mathbb{R})$

$$\int_0^\infty \int_{\mathbb{R}} \Big[\partial_t G - (\partial_x G) V'\Big] d\mu_t dt + \int_0^\infty \int_{\mathbb{R}} (\partial^2_{xx} G) \Phi(\rho_t) dx dt = 0.$$

We believe that this result can be extended to the case of the one-dimensional torus $\mathbb{T}$, without the presence of the potential V, i.e. when $V \equiv 0$. Then, this result together with the validity of the continuity equation and the regularity estimate (4.2) would imply that in order to close the equation and complete the proof of the hydrodynamic limit it suffices to prove that

$$\text{(a) } \mu_t \in \mathcal{M}_{+,\rho_c}(\mathbb{T}^d) \text{ a.s. for all } t \in [0,T] \quad \text{and} \quad \text{(b) } \sigma = \Phi(\mu^{ac}). \tag{4.90}$$

Indeed, then by(4.90) and the regularity estimate (4.2) property (b) of Savare's result follows and by (4.90b) and the validity of the continuity equation the property (c) of Savare's result, i.e. that $\partial_t \mu = \Delta\Phi(\mu^{ac})$ in the sense of distributions, follows. Note also that (in dimension 1) if one has (4.90b), then by Morrey's inequality and the regularity estimate (4.2) it follows that $\rho = \mu^{ac}$ is continuous, and thus in order to prove (4.90a) one needs to show that $\mathrm{supp}\mu^{\perp} \subseteq \{\mu^{ac} = \rho_c\}$. By the uniqueness of solutions we would then get that the gradient flow of the functional H_E is the hydrodynamic limit of the ZRP. We conjecture that this is true.

If the above program succeeds, further adaptations of the results in [17], allowing for instance ρ_c to be finite while Φ is still assumed C^1, would give the hydrodynamic limit of Evans' ZRP for values $b \in (2,3]$. Furthermore, allowing the function Φ to be non-differentiable at $\rho_c < +\infty$ would give the hydrodynamic limit for all values of $b \geq 0$.

Appendix A

Appendix

A.1 The Skorohod Space

Let $I := \mathbb{R}_+$ or $[0, T]$ for some $T > 0$ and let (M, d) be a polish space. A path $x : I \longrightarrow M$ is called cadlag if it right continuous and has left hand limits. Any such path can have at most countable jump discontinuities. We denote by $D(I; M)$ the set of all cadlag paths $x : I \longrightarrow M$. We let $\Lambda(I)$ denote the set of all strictly increasing (and thus continuous) functions $\lambda : I \longrightarrow I$ and set $\Lambda_L(I)$ the set of all Lipschitz functions $\lambda \in \Lambda(I)$ satisfying

$$\gamma(\lambda) := \operatorname*{ess\,sup}_{t \geq 0} |\log \lambda'(t)| = \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| < +\infty.$$

Since $|\ln x| = \ln x \vee \ln(1/x)$, $x > 0$, by definition any function $\lambda \in \Lambda_L(I)$ is bi-Lipschitz and

$$\gamma(\lambda) = \gamma(\lambda^{-1}).$$

Note also that for any $\lambda_1, \lambda_2 \in \Lambda_L(I)$,

$$\gamma(\lambda_1 \circ \lambda_2) \leq \gamma(\lambda_1) + \gamma(\lambda_2).$$

Since the set $\Lambda_L(I)$ is a group with respect to the composition operation, the above relations show that γ is a group norm on $\Lambda_L(I)$. Convergence in γ-norm implies uniform converge in compact subsets of the real line, i.e.

$$\lim_{n\to\infty} \gamma(\lambda_n) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \sup_{0 \leq t \leq T} |\lambda_n(t) - t| = 0 \quad \text{for all } T > 0.$$

In the case that $I := [0, T]$ we define the Skorohod metric d_S on $D(0, T; M)$ by the formula

$$d_S(x, y) = \max\left\{ \inf_{\lambda \in \Lambda_L([0,T])} \gamma(\lambda), \sup_{0 \leq t \leq T} d(x_t, y_{\lambda(t)}) \right\}$$

In the case that $I = \mathbb{R}_+$ the convergence in the space $D(\mathbb{R}_+; M)$ is loosely speaking the convergence in the Skorohod metric for compact subsets of $\mathbb{R}_+$. One metric characterizing this topology is defined (see [14], section 3.5) as follows. For $x, y \in D(\mathbb{R}_+; M)$,

$\lambda \in \Lambda_L(\mathbb{R}_+)$, $T > 0$, one sets

$$\rho(x, y, \lambda, T) := \sup_{t \geq 0} \bar{d}\big(x(t \wedge T), y(\lambda(t) \wedge T)\big),$$

where $\bar{d} := d \wedge 1$ and defines the metric d_S by

$$d_S(x, y) = \inf_{\lambda \in \Lambda_L(\mathbb{R}_+)} \max \left\{ \gamma(\lambda), \int_0^\infty e^{-s} \rho(x, y, \lambda, s) ds \right\}.$$

Proposition A.1.1 *Let $\{x_n\}_{n \in \mathbb{N}} \subseteq D(\mathbb{R}_+; M)$ and $x \in D(\mathbb{R}_+; M)$. The following are equivalent.*
(a) *$d_S(x_n, x) \longrightarrow 0$ as $n \to \infty$.*
(b) *There exists $\{\lambda_n\} \subseteq \Lambda_L(\mathbb{R}_+)$ such that*

$$\lim_{n\to\infty} \gamma(\lambda_n) = 0 \quad \textit{and} \quad \lim_{n\to\infty} \sup_{0 \leq t \leq T} d\big(x_n(t), x((\lambda_n(t)))\big) = 0 \quad \textit{for all } T > 0.$$

For each $T > 0$ there exists $\{\lambda_n\}_{n \in \mathbb{N}}$ (possibly depending on $T > 0$) such that

$$\lim_{n\to\infty} \sup_{0 \leq t \leq T} |\lambda_n(t) - t| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \sup_{0 \leq t \leq T} d\big(x_n(t), x((\lambda_n(t)))\big) = 0.$$

Proof See proposition 3.5.3 in [14]. □

As noted in [14] the right hand side conditions in (b) in the proposition above can be replaced by

$$\lim_{n\to\infty} \sup_{0 \leq t \leq T} d\big(x_n\big(\lambda_n(t)\big), x(t)\big) = 0.$$

Note that since the uniform limit of continuous functions is again a continuous function this implies that the subset $C(\mathbb{R}_+; M)$ is a closed subspace of $D(\mathbb{R}_+; M)$ in the Skorohod topology.

Proposition A.1.2 *Let (M, d) be a metric space. The topology defined on $D(\mathbb{R}_+; M)$ by the Skorohod metric d_S depends only on the topology of M and not on the particular choice of the metric d defining the topology of M.*

Proof This is a particular case of theorem 1.3 in [24]. □

Proposition A.1.3 *If M is separable then $D(\mathbb{R}_+; M)$ is separable. If (M, d) is complete then $\big(D(\mathbb{R}_+; M), d_s\big)$ is complete.*

Proof See proposition 3.5.6 in [14]. □

This proposition allows one to describe the compact subsets of $D(\mathbb{R}_+; M)$ via the general result on complete metric spaces according to which the compact subsets are exactly the complete and totally bounded subsets. However this result is to general to be useful. The main tool for a useful compactness criterion is a modified modulus of continuity which allows for the generalization of the Arzela-Ascoli theorem on the Skorohod space $D(\mathbb{R}_+; M)$, which in turns permits to specialize Prokhorov's theorem on the space $\mathbb{P}D(\mathbb{R}_+;M)$. Recall that given a path $x:\mathbb{R}_+\longrightarrow M$ the usual modulus of continuity of x on $A\subseteq\mathbb{R}_+$ is defined by

$$w_x(A) := \sup_{s,t\in A} d(x_t,x_s).$$

Definition A.1.1 Let $\Pi(\delta,T)$, $\delta,T>0$, denote the set of all partitions

$$\Delta := \{0=t_0<t_1<\cdots<t_n\},\quad n\in\mathbb{N},$$

such that

$$t_n\geq T \quad\text{and}\quad \min_{1\leq i\leq n}|t_i-t_{i-1}|>\delta.$$

For each partition $\Delta\in\Pi(\delta,T)$ we set

$$w_x^{(\Delta)} := \max_{1\leq i\leq n} w_x[t_{i-1},t_i)$$

and define the *modified modulus of continuity* $w'_x:(0,\infty)^2\longrightarrow\mathbb{R}_+$ *of a path* $x:I\longrightarrow M$ by the formula

$$w'_x(\delta,T) := \inf_{\Delta\in\Pi(\delta,T)} w_x^{(\Delta)}.$$

Given $x\in D(\mathbb{R}_+;M)$, the function w'_x is obviously increasing with respec to the variables δ,T and for a path $x:\mathbb{R}_+\longrightarrow M$ we have

$$x\in D(\mathbb{R}_+;M)\quad\text{iff}\quad \lim_{\delta\to 0} w'_x(\delta,T)=0\quad\text{for all } T>0.$$

Furthermore, w'_x is right continuous with respect to the vriable δ and the function

$$D(\mathbb{R}_+;M)\ni x\mapsto w'_x(\delta,T+) := \lim_{\varepsilon\downarrow 0} w'_x(\delta,T+\varepsilon)$$

is upper-semicontinuous for each $\delta,T>0$. By the upper-semicontinuity of this function and the equality

$$w'_x(\delta,T) = \lim_{n\to\infty} w'_x\big(\delta,(T-\frac{1}{n})+\big)$$

which holds for all $\delta,T>0$, the Borel measurability of the function

$$D(\mathbb{R}_+;M)\ni x\mapsto w'_x(\delta,T)$$

follows for each fixed $\delta,T>0$. Using the modified modulus of continuity the Arzela-Ascoli theorem is generalized on $D(\mathbb{R}_+;M)$ as follows.

Proposition A.1.4 *If M is complete, then a set $K\subseteq D(\mathbb{R}_+;M)$ is relatively compact iff for each $T>0$,*
(a) *The set $K([0,T]) := \{x_t|x\in K,\ t\in[0,T]\}$ is relatively compact in M, and*
(b)$\lim_{\delta\to 0}\sup_{x\in K} w'_x(\delta,T)=0$.

Proof See theorem 3.6.3 in [14] and the remark following it. □

Through this characterization of the relatively compact subsets of $D(\mathbb{R}_+;M)$ Prokhorov's theorem takes the following form on $D(\mathbb{R}_+;M)$.

Theorem A.1.1 *If M is polish then a set $\mathcal{K} \subseteq \mathbb{P}D(\mathbb{R}_+; M)$ is relatively compact iff:*
(a) *For each $t \in \mathbb{R}_+$ and each $\varepsilon > 0$ there exists a relatively compact subset $\Gamma_{t,\varepsilon} \subseteq M$ such that*

$$\sup_{P \in \mathcal{K}} P\big\{x \in D(\mathbb{R}_+; M) \big| x_t \notin \Gamma_{t,\varepsilon}\big\} \leq \varepsilon.$$

(b) *For each $\varepsilon, T > 0$,*

$$\lim_{\delta \to 0} \sup_{P \in \mathcal{K}} P\big\{x \in D(\mathbb{R}_+; M) \big| w'_x(\delta, T) > \varepsilon\big\} = 0.$$

Proof See theorem 3.7.2 in [14]. □

In the case that the set $\mathcal{K}$ above is the image of a sequence, $\mathcal{K} = \{P^N\}_{N \in \mathbb{N}}$, then the conditions (a) and (b) above can be replaced by the conditions:
(a') For each $t \in \mathbb{R}_+$ and each $\varepsilon > 0$ there exists a relatively compact subset $\Gamma_{t,\varepsilon} \subseteq M$ such that

$$\limsup_{N \to \infty} P^N\big\{x \in D(\mathbb{R}_+; M) \big| x_t \notin \Gamma_{t,\varepsilon}\big\} \leq \varepsilon,$$

and (b') for each $\varepsilon, T > 0$,

$$\lim_{\delta \to 0} \limsup_{N \to \infty} P^N\big\{x \in D(\mathbb{R}_+; M) \big| w'_x(\delta, T) > \varepsilon\big\} = 0.$$

Of course the hardest of the two conditions to check is condition (b) which relies on the behavior of the paths on intervals $[0, T] \subseteq \mathbb{R}_+$ and just at a fixed time point as condition (a). However, in [1] Aldous gave a very useful criterion that ensures the validity of the requirement (b') for the tightness of the laws $\{P_N\} \subseteq \mathbb{P}D(\mathbb{R}_+; M)$. In essence, the validity of Aldous' criterion to be stated below, ensures that one can make a good choice of partitions in the involved in the modified modulus of continuity w'.

Theorem A.1.2 *Let $(F_t)_{t \geq 0}$ denote the minimal right-continuous filtration containing the natural filtration (i.e. the one generated by the coordinate projections $D(\mathbb{R}_+; M) \ni X \mapsto x_t \in M$, $t \in \mathbb{R}_+$) of $D(\mathbb{R}_+; M)$ and let $\mathcal{T}^T \equiv \mathcal{T}^T(\mathcal{F}_t)$ denote the set of all $(\mathcal{F}_t)$-stopping times bounded by $T > 0$. If for all $\varepsilon, T > 0$ the sequence $\{P_N\}_{N \in \mathbb{N}} \subseteq \mathbb{P}D(\mathbb{R}_+; M)$ satisfies*

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \sup_{\substack{\tau \in \mathcal{T}^T \\ \theta \leq \delta}} P_N\big\{x \in D(\mathbb{R}_+; M) \,\big|\, d(x_\tau, x_{(\tau+\theta) \wedge T}) > \varepsilon\big\} = 0,$$

then the condition (b) of theorem A.1.1 is satisfied.

Proof See proposition 4.1.6 in [25] for a proof. □

We will apply theorems A.1.1 and A.1.2 in the case that $M = \mathcal{M}_+ := \mathcal{M}_+(\mathbb{T}^d)$ is the metric space of all non-negative finite Borel measures on the torus equipped with the metric δ defined in 1.53 and for the sequence of the laws of the empirical density process of the ZRP. The next result reduces the problem of establishing the relative compactness of the laws of a sequence of $\mathcal{M}_+$-valued processes to the case of real valued processes, by projecting $\mathcal{M}_+$-valued processes to real-valued ones via functions of the form $I_G(\mu) = \langle G, \mu \rangle := \int G d\mu$, where $G \in C(\mathbb{T}^d)$.

Proposition A.1.5 *Let $\{G_k\}_{k\in\mathbb{N}}$ be a dense subset of $C(\mathbb{T}^d)$ with $G_1 \equiv 1$. Then a sequence $\{P_N\}_{N\in\mathbb{N}} \subseteq \mathbb{P}D(\mathbb{R}_+;\mathcal{M}_+)$ is relatively compact iff for each $k \in \mathbb{N}$ the sequence $\{I_{G_k*}P_N\}_{N\in\mathbb{N}}$ is relatively compact, where $I_{G_k*} : \mathbb{P}D(\mathbb{R}_+;\mathcal{M}_+) \longrightarrow \mathbb{P}D(\mathbb{R}_+;\mathbb{R})$ denotes the push-forward of measures induced by the function*

$$D(\mathbb{R}_+;\mathcal{M}_+) \ni \mu \mapsto \left(\int G_k d\mu_t \right)_{t\geq 0} \in D(\mathbb{R}_+;\mathbb{R}).$$

Proof See proposition 4.1.7 in [25] for a proof. □

We close this section by reviewing ways of producing continuous mappings between Skorohod spaces that will be useful in the main text.

Lemma A.1.1 *Let $x : [0,T] \longrightarrow M$ be a cadlag path in a metric space M. Then the image $x([0,T])$ of x is relatively compact in M.*

Proof Let $\{y_n\}_{n\in\mathbb{N}} \subseteq x([0,T])$ be any sequence. Then $y_n = x(t_n)$ for some sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$. Since $[0,T]$ is compact there exists then a subsequence $\{t_{k_n}\}_{n\in\mathbb{N}}$ of $\{t_n\}$ converging to some point $t \in [0,T]$. If $t \in [0,T]$ is a continuity point of x then $y_{k_n} = x(t_{k_n}) \longrightarrow x(t)$ and $\{y_n\}$ has a converging subsequence. On other hand if t is discontinuity point of x then either there exists a further subsequence $\{t_{m_{k_n}}\}$ of $\{t_{k_n}\}$ that converges to t from the right or either there exists a further subsequence $\{t_{m_{k_n}}\}$ of $\{t_{k_n}\}$ that converges to t from the left. In the first case, by the right continuity of x we have that $y_{k_{m_n}} \longrightarrow x(t)$ while in the second case by the existence of left hand limits we have that $y_{k_{m_n}} \longrightarrow x(t-)$. In either case there is again a convergent subsequence of $\{y_n\}$, we shows that any subsequence of $x([0,T])$ has a converging subsequence and thus $x([0,T])$ is relatively compact. □

Proposition A.1.6 *Let M, N be metric spaces and let $G : \mathbb{R}_+ \times M \longrightarrow N$ be continuous function that is continuous in time uniformly over compact subsets of space, i.e. for each $t \in [0,T]$ and any compact $K \subseteq M$*

$$\lim_{h\to 0} \sup_{x\in K} d\big(G_{t+h}(x), G_t(x)\big) = 0 \tag{A.1}$$

and uniformly continuous in space uniformly over compact subsets of time, i.e.

$$\lim_{d(x,y)\to 0} \sup_{0\leq t\leq T} d\big(G_t(x), G_t(y)\big) = 0 \tag{A.2}$$

for each $T > 0$. Then the induced function $\bar{G} : D(0,T;M) \longrightarrow D(0,T;N)$ on the Skorohod spaces given by

$$\bar{G}(x)(t) = G\big(t, x(t)\big)$$

is continuous.

Proof Let $\{x_n\}_{n\in\mathbb{N}} \subseteq D(0,T;M)$, $x \in D(0,T;M)$ such that $d_S(x_n,x) \longrightarrow 0$. We have to show that $d_S\big(\bar{G}(x_n), \bar{G}(x)\big) \longrightarrow 0$. Since $d_S(x_n,x) \longrightarrow 0$ there exists $\lambda \in \Lambda_L([0,T])$ such that $\gamma(\lambda_n) \longrightarrow 0$ and

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} d\big(x_n(t), x\big(\lambda_n(t)\big)\big) = 0. \tag{A.3}$$

We will show that for this sequence $\{\lambda_n\}$ we have

$$\lim_{n\to\infty} \sup_{0\le t\le T} d\big(\bar{G}(x_n)(t), \bar{G}(x)\big(\lambda_n(t)\big)\big) = 0,$$

which proves that $\bar{G}(x_n) \longrightarrow \bar{G}(x)$ in the Skorohod topology of $D(0,T;N)$. By ht previous lemma there exists a compact set $K \subseteq M$ such that $x([0,T]) \subseteq K$. Then for each $0 \le t \le T$ we have that

$$\begin{aligned} d\big(\bar{G}(x_n)(t), \bar{G}(x)\big(\lambda_n(t)\big)\big) &\le d\big[G\big(t, x_n(t)\big), G\big(t, x(\lambda_n(t))\big)\big] \\ &\quad + d\big[G\big(t, x(\lambda_n(t))\big), G(\lambda_n(t), x(\lambda_n(t)))\big] \\ &\le \sup_{0\le s\le T} d\big[G\big(s, x_n(t)\big), G\big(s, x(\lambda_n(t))\big)\big] \\ &\quad + \sup_{x\in K} d\big[G(t,x), G\big(\lambda_n(t), x\big)\big] \end{aligned}$$

and therefore

$$\begin{aligned} \sup_{0\le t\le T} d\big(\bar{G}(x_n)(t), \bar{G}(x)\big(\lambda_n(t)\big)\big) &\le \sup_{0\le s,t\le T} d\big[G\big(s, x_n(t)\big), G\big(s, x(\lambda_n(t))\big)\big] \\ &\quad + \sup_{0\le t\le T}\sup_{x\in K} d\big[G(t,x), G\big(\lambda_n(t), x\big)\big]. \end{aligned} \tag{A.4}$$

Let now $\varepsilon > 0$. By (A.1) for each $t \in [0,T]$ there exists $\delta_t > 0$ such that

$$|h| < \delta_t \quad \Longrightarrow \quad \sup_{x\in K} d\big(G(t+h, x), G(t,x)\big) < \frac{\varepsilon}{4}. \tag{A.5}$$

Let $\delta_0 > 0$ be the Lebesgue number of the open cover $\{(t-\delta_t, t+\delta_t)\}_{t\in[0,T]}$ of the compact set $[0,T]$ and choose $n_1 \in \mathbb{N}$ such that

$$n \ge n_1 \quad \Longrightarrow \quad \sup_{0\le t\le T} |\lambda_n(t) - t| < \frac{\delta_0}{2}.$$

Then for each $t \in [0,T]$ the set $A_t := \{t\} \cup \{\lambda_n(t) | n \ge n_1\}$ has diameter less than δ_0 and since δ_0 is the Lebesgue number of the cover $\{(t-\delta_t, t+\delta_t)\}_{t\in[0,T]}$ there exists for each $t \in [0,T]$ an $s_t \in [0,T]$ such that $A_t \subseteq (s_t - \delta_{s_t}, s_t + \delta_{s_t})$. Then obviously $|s_t - t| < \delta_{s_t}$ and $|\lambda_n(t) - s_t| < \delta_{s_t}$ for all $n \ge n_1$, and therefore by (A.5) we have that

$$\sup_{x\in K} d\big[G(t,x), G\big(\lambda_n(t), x\big)\big] \le \sup_{x\in K} d\big[G(t,x), G(s_t, x)\big] + \sup_{x\in K} d\big[G(s_t,x), G(\lambda_n(t), x)\big] < \frac{\varepsilon}{2}$$

for all $n \ge n_1$. Since this holds for all $t \in [0,T]$ we have shown that

$$n \ge n_1 \quad \Longrightarrow \quad \sup_{0\le t\le T}\sup_{x\in K} d\big[G(t,x), G\big(\lambda_n(t), x\big)\big] < \frac{\varepsilon}{2},$$

which deals with the second term in the right hand side of (A.4). Next, by assumption (A.2) on G we can pick $\delta > 0$ such that

$$x, y \in M,\ d(x,y) < \delta \quad \Longrightarrow \quad \sup_{0\le s\le T} d\big[G(s,x), G(s,y)\big] < \frac{\varepsilon}{2} \tag{A.6}$$

and by (A.3) there exists $n_2 \in \mathbb{N}$ such that

$$n \geq n_2 \quad \Longrightarrow \quad \sup_{0 \leq t \leq T} d\big[x_n(t), x\big(\lambda_n(t)\big)\big] < \delta. \tag{A.7}$$

Then by (A.7) and (A.6) it follows that for all $n \geq n_2$,

$$\sup_{0 \leq s,t \leq T} d\big[G\big(s, x_n(t)\big), G\big(s, x\big(\lambda_n(t)\big)\big)\big] < \frac{\varepsilon}{2},$$

which deals with first summand in the right hand side of (A.4) and so for all $n \geq n_1 \vee n_2$ we have that

$$\sup_{0 \leq t \leq T} d\big[\bar{G}(x_n)(t), \bar{G}(x)\big(\lambda_n(t)\big)\big] < \varepsilon,$$

which completes the proof. □

Corollary A.1.1 *Let $f : M \longrightarrow N$ be a continuous function between metric spaces. Then the induced function $\bar{f} : D(\mathbb{R}_+; M) \longrightarrow D(\mathbb{R}_+; N)$ given by*

$$f(x)(t) = f\big(x(t)\big)$$

is continuous.

Proposition A.1.7 *The function $I : D(\mathbb{R}_+; \mathbb{R}) \longrightarrow C(\mathbb{R}_+; \mathbb{R}) \subseteq D(\mathbb{R}_+; \mathbb{R})$ given by the formula*

$$I(x)(t) = \int_0^t x(s)ds$$

is continuous.

A.2 Martingales of Markov Jump Processes

Given any countable discrete pointed Polish space $(M, o, |\cdot|)$ and any measurable function $F : \mathbb{R}_+ \times M \longrightarrow \mathbb{R}$ we will denote by F_t the function $F(t, \cdot) : M \longrightarrow \mathbb{R}$, $t \geq 0$. If F is differentiable with respect to the time variable we denote by $\partial F : \mathbb{R}_+ \times M \longrightarrow \mathbb{R}$ its partial derivative with respect to time,

$$\partial F(t, x) := \partial_t F(t, x) = \partial_t F_t(x) = (\partial F)_t(x),$$

and for any measurable space M we denote by $B_r C_{\mathbb{R}_+}(M)$ the space of all measurable functions $F : \mathbb{R}_+ \times M \longrightarrow \mathbb{R}$ such that

$$\sup_{(t,x)\in\mathbb{R}_+\times M} \frac{|F(t,x)|}{(1+|x|^{|r|})^{\mathrm{sgn} r}} := \|F\|_r < +\infty,$$

and by $B_r C^1_{\mathbb{R}_+}(M)$ the space of all functions $F \in B_r C(M)$ for which the curve $\mathbb{R}_+ \ni t \mapsto F_t \in B_r(M)$ is C^1 in $B_r(M)$, that is for which

$$\partial F \in B_r C_{\mathbb{R}_+}(M), \qquad \lim_{h\downarrow 0} \left\| \frac{F_{t+h} - F_t - h\partial_t F_t}{h} \right\|_{u,r} = 0.$$

Of course $B_0 C_{\mathbb{R}_+}(M) = BC(\mathbb{R}_+ \times M)$ obviously $B_r(M) \subseteq B_r C^1_{\mathbb{R}_+}(M)$ for all $r \in \mathbb{R}$ in the sense that if $x : \mathbb{R}_+ \times M \longrightarrow M$ is the projection on the space coordinate then $B_r C(M) \circ x := \{F(x) | F \in B_r C(M)\} \subseteq B_r C^1_{\mathbb{R}_+}(M)$.

The following proposition gives a sufficient condition for functions $F \in B_r C_{\mathbb{R}_+}(M)$, such that $\partial F \in B_r C_{\mathbb{R}_+}(M)$ exists, to belong in $B_r C^1_{\mathbb{R}_+}(M)$.

Proposition A.2.1 *Let $F \in B_r C_{\mathbb{R}_+}(M)$ such that $\partial F \in B_r C_{\mathbb{R}_+}(M)$ exists. If ∂F is Holder continuous with respect to time in $B_r(M)$ with exponent $a \in (0, 1]$, that is if*

$$\sup_{s,t\in\mathbb{R}_+ : t\neq s} \frac{\|\partial_t F_t - \partial_s F_s\|_{u,r}}{|t-s|^a} < +\infty, \tag{A.8}$$

then $F \in B_r C^1_{\mathbb{R}_+}(M)$.

Proof Indeed, then there exists a constant $C \geq 0$ such that

$$|\partial_t F_t(x) - \partial_s F_s(x)| \leq C(1 + |x|^{|r|})^{\mathrm{sgn} r} |t-s|^a$$

for all $t, s \in \mathbb{R}$, $x \in M$, and therefore

$$\left| \frac{F_{t+h} - F_t - h\partial F_t}{h(1+|\cdot|^{|r|})^{\mathrm{sgn} r}} \right| = \left| \frac{1}{h} \int_t^{t+h} \frac{\partial F_s - \partial F_t}{(1+|\cdot|^{|r|})^{\mathrm{sgn} r}} ds \right| \leq \frac{C}{h} \int_t^{t+h} |s-t|^a ds = \frac{C}{1+a} h^a.$$

It follows that

$$\left\| \frac{F_{t+h} - F_t - h\partial_t F_t}{h} \right\|_{u,r} \leq \frac{C}{1+a} h^a \overset{h\downarrow 0}{\longrightarrow} 0$$

as required. □

Obviously if $F \in B_r C_{\mathbb{R}_+}(M)$ is pointwise C^2 with respect to the time variable with $\partial^2 F \in B_r C_{\mathbb{R}_+}(M)$ then by the mean value theorem we get that (A.8) holds with $a = 1$. This gives a stronger but more simple sufficient condition instead of (A.8).

Proposition A.2.2 *Let (X, P) be a Markov jump process on the pointed polish state space $(M, o, |\cdot|)$ with partially bounded Lipschitz continuous jump rate function λ, transition probability p, such that for all states $x \in M$ the set of all states $y \in M$ accessible from x according to p are contained in the $|x|$-spheres of the metric $|\cdot|$, i.e. such that*

$$M_x := \{y \in M \mid \exists\, m \geq 0 : p^m(x, y) > 0\} \subseteq \{|\cdot| = |x|\}. \tag{A.9}$$

for all $x \in M$. Let $L : B_{r-1}(M) \longrightarrow B_r(M)$, $r \geq 1$, be the generator of (X, P) in $B_r(M)$ with respect to uniform convergence on bounded subsets and let $\mu \in \mathbb{P}_r M$ be any initial measure. Then, for all $F \in B_{r-1}C^1_{\mathbb{R}_+}(M)$ the real process

$$M_t^{X,F} := F_t(X_t) - F_0(X_0) - \int_0^t (\partial_s + L)F_s(X_s)ds, \quad t \geq 0,$$

defined on the filtered probability space $\big(D(\mathbb{R}_+, M), (\mathcal{F}_t)_{t\geq 0}, P^\mu\big)$ is a mean-zero martingale, where $(\mathcal{F}_t)$ is the minimal right continuous filtration to which X is adapted and the P^μ is the distribution of (X, P) starting from μ. Furthermore, if in addition the jump rate function λ is bounded then $M^{X,F}$ is a martingale for all $F \in B_r C^1_{\mathbb{R}_+}(M)$.

Proof Let $F \in B_{r-1}C^1_{\mathbb{R}_+}(M)$. We note first that $M_t^{X,F} \in L^1(P^\mu)$ for all $t \geq 0$. Indeed, by hypothesis (A.9) we have that

$$p|\cdot|^r(x) = \int |y|^r dp_x(y) = \int_{M_x} |y|^r dp_x(y) = |x|^r$$

for all $r \geq 0$ and therefore by proposition 1.1.4 the semigroup $P_t : B_r(M) \longrightarrow B_r(M)$, $t \geq 0$ is defined, and since $|X_t| = x$ P^x-a.s. for all $x \in M$ according to (A.9), it satisfies

$$P_t|\cdot|^r(x) = \mathbb{E}^x|X_t|^r = |x|^r, \qquad x \in M,\ t \geq 0, \tag{A.10}$$

for all $r \geq 0$ and $X_{t*}P^\mu = \mu P_t \in \mathbb{P}_r M$ for all $t \geq 0$. Therefore, given $r \geq 1$, since $\mathbb{P}_r M \subseteq \mathbb{P}_{r-1}M$, we have that

$$\begin{aligned} \mathbb{E}^\mu|F_t(X_t)| &\leq \|F\|_{r-1}\mathbb{E}^\mu(1 + |X_t|^{r-1}) = \|F\|_{r-1}\int (1 + |x|^{r-1})d\mu P_t(x) \\ &\leq \|F\|_{r-1}\int (1 + |x|^{r-1})d\mu < +\infty \end{aligned}$$

and the term $F_t(X_t) - F_0(X_0)$ is P^μ-integrable. For the other term we have first of all that $\partial F \in B_{r-1}C_{\mathbb{R}_+}(M)$ and therefore for all $t \geq 0$ we have by Tonelli's theorem that

$$\begin{aligned} \mathbb{E}^\mu\left|\int_0^t \partial_s F_s(X_s)ds\right| &\leq \|\partial F\|_{r-1}\mathbb{E}^\mu\int_0^t (1 + |X_s|^{r-1})ds \\ &= \|\partial F\|_{r-1}\int_0^t \mathbb{E}^\mu(1 + |X_s|^{r-1})ds \\ &= \|\partial F\|_{r-1}\int_0^t \int (1 + |x|^{r-1})d\mu P_s(x)ds \\ &= \|\partial F\|_{r-1}\int_0^t \int (1 + P_s|\cdot|^{r-1})d\mu ds \\ &\leq \|\partial F\|_{r-1}t\Big(1 + \int |x|^{r-1}d\mu(x)\Big) < +\infty, \end{aligned}$$

since $\mu \in \mathbb{P}_r M \subseteq \mathbb{P}_{r-1} M$. Finally, for the term $\int_0^t LF_s(X_s)ds$, $t \geq 0$, we note as in the proof of proposition 1.1.6 it follows that $L : B_{r-1}(M) \longrightarrow B_r(M)$ is a bounded operator with $\|L\| \leq 4|\lambda'|$ where $|\lambda'|$ is the Lipschitz constant of the jump rate function λ. Therefore for all $0 \leq s \leq t$ we have that

$$\begin{aligned} |LF_s(x)| &\leq 4|\lambda'|\|F_s\|_{u,r-1}(1+|x|^r) \leq 4|\lambda'|\Big(\sup_{0\leq s\leq t}\|F_s\|_{u,r-1}\Big)(1+|x|^r) \\ &\leq 4|\lambda'|\|F\|_{r-1}(1+|x|^r) \end{aligned}$$

and so by Tonelli's theorem we have that

$$\mathbb{E}^\mu\left|\int_0^t LF_s(X_s)ds\right| \leq 4|\lambda'|\|F\|_{r-1}t\Big(1+\int |x|^r d\mu(x)\Big) < +\infty,$$

which proves that $M_t^{X,F} \in L^1(P^\mu)$ for all $t \geq 0$.

To prove now that $(M_t^{X,F})_{t\geq 0}$ is a $(\mathcal{F}_t, P^\mu)$-martingale it remains to show that

$$\mathbb{E}^\mu(M_t^{X,F}|\mathcal{F}_s) = M_s^{X,F}, \quad P^\mu\text{-a.s.}, \quad \forall\, 0 \leq s \leq t < +\infty,$$

or equivalently that

$$\mathbb{E}^\mu\big(F_t(X_t)|\mathcal{F}_s\big) - F_s(X_s) = \mathbb{E}^\mu\left(\int_s^t (\partial_u + L)F_u(X_u)du \,\Big|\, \mathcal{F}_s\right) \tag{A.11}$$

P^μ-a.s. for all $0 \leq s \leq t$. So let $0 \leq s < t$. Since (X, P^μ) is a Markov process with Markov semigroup $P_t : B_r(M) \longrightarrow B_r(M)$ the term in left hand side in (A.11) is equal to

$$\mathbb{E}^\mu\big(F_t(X_t)|X_s\big) - F_s(X_s) = P_{t-s}F_t(X_s) - F_s(X_s) = (P_{t-s}F_t - F_s)(X_s) \tag{A.12}$$

P^μ-a.s.. On the other hand, as we have seen, the function

$$\mathbb{R}_+ \times D(\mathbb{R}_+, M) \ni (t,\omega) \mapsto (\partial_t + L)F_t\big(X_t(\omega)\big)$$

is in $L^1(P^\mu \otimes \mathbb{1}_{[0,T]}(s)ds)$ and therefore by the conditional Fubini theorem the term in the right hand side in (A.11) is equal to

$$\int_s^t \mathbb{E}^\mu\big((\partial_u + L)F_u(X_u)|\mathcal{F}_s\big)du = \int_s^t P_{u-s}\big[(\partial_u + L)F_u\big](X_s)du. \tag{A.13}$$

Therefore if we show the pointwise Leibniz formula

$$\partial_t(P_tF_t) = LP_tF_t + P_t(\partial_t F_t), \quad t \geq 0, \tag{A.14}$$

and the pointwise formula

$$LP_tF_t = P_tLF_t, \quad t \geq 0, \tag{A.15}$$

on M it will follow that

$$\begin{aligned} P_{t-s}F_t - F_s &= \int_0^{t-s} \partial_u(P_uF_{u+s})du = \int_0^{t-s} LP_uF_{u+s}du + \int_0^{t-s} P_u(\partial_u F_{u+s})du \\ &= \int_s^t P_{u-s}LF_u du + \int_s^t P_{u-s}(\partial_u F_u)du = \int_s^t P_{u-s}\big[(\partial_u + L)F_u\big]du, \end{aligned}$$

which according to (A.12) and (A.13) when applied at the point X_s, gives us (A.11), that is

$$\begin{aligned}\mathbb{E}^\mu\big(F_t(X_t)|\mathcal{F}_s\big) - F_s(X_s) &= (P_{t-s}F_t - F_s)(X_s) = \int_s^t P_{u-s}\big[(\partial_u + L)F_u\big](X_s)du \\ &= \int_s^t \mathbb{E}^\mu\big((\partial_u + L)F_u(X_u)|\mathcal{F}_s\big)du.\end{aligned}$$

So it remains to prove (A.14) and (A.15). We prove first (A.14). So let $t \geq 0$ and $h > 0$. Then

$$\frac{1}{h}\big(P_{t+h}F_{t+h} - P_tF_t\big) = \frac{1}{h}\big(P_{t+h}F_t - P_tF_t\big) + P_{t+h}\Big(\frac{1}{h}(F_{t+h} - F_t)\Big)$$

Now, since (P_t) satisfies (A.10) for all $r \in \mathbb{R}$ and $F_t \in B_{r-1}(M)$ we have that $P_tF_t \in B_{r-1}C(M)$ and since $L : B_{r-1}(M) \longrightarrow B_r(M)$ is contained in the generator of (P_t) in $B_r(M)$ with respect to uniform convergence on bounded subsets, we have that

$$\lim_{h\downarrow 0}\frac{1}{h}\big(P_{t+h}F_t - P_tF_t\big) = LP_tF_t$$

uniformly on bounded subsets of M. Therefore in order to prove (A.14) it suffices to prove that

$$\lim_{h\downarrow 0} P_{t+h}\Big(\frac{1}{h}(F_{t+h} - F_t)\Big) = P_t(\partial_t F_t). \tag{A.16}$$

Now, since $F \in B_{r-1}C^1_{\mathbb{R}_+}(M)$ we have that

$$\lim_{h\downarrow 0}\frac{F_{t+h} - F_t}{h} = \partial_t F_t \quad \text{in} \;\; B_{r-1}(M)$$

and therefore in order to prove (A.16) it suffices to prove that for any family of functions $\{G_h\}_{h\geq 0} \subseteq B_{r-1}(M)$ we have that

$$\lim_{h\downarrow 0}\|G_h - G_0\|_{u,r-1} = 0 \quad \Longrightarrow \quad P_{t+h}G_h \stackrel{h\downarrow 0}{\longrightarrow} P_tG_0 \;\text{ pointwise in } M. \tag{A.17}$$

As we will see, in fact the convergence in the right hand side of (A.17) is uniform on bounded subsets of M.

In order to prove implication (A.17) we note first that $P_tf \stackrel{t\downarrow 0}{\longrightarrow} f$ uniformly on bounded subsets of M for all $f \in B_{r-1}(M)$. Indeed, since $L : B_{r-1}(M) \longrightarrow B_r(M)$ is the generator of (P_t) with respect to uniform convergence on bounded subsets, for all $f \in B_{r-1}(M)$, given $R > 0$, there exists $\delta_1 = \delta_{f,R} > 0$ such that

$$0 < t < \delta_1 \quad \Longrightarrow \quad \sup_{|x|\leq R}\left|\frac{1}{t}(P_tf - f)(x)\right| \leq \sup_{|x|\leq R}|Lf(x)| + 1.$$

But then for all $0 < t < \delta_1$ we have that

$$\sup_{|x|\leq R}\big|P_tf(x) - f(x)\big| = t\sup_{|x|\leq R}\left|\frac{1}{t}(P_tf - f)(x)\right| \leq t\Big(\sup_{|x|\leq R}|Lf(x)| + 1\Big) \stackrel{t\downarrow 0}{\longrightarrow} 0,$$

as claimed .

But, given any family of functions $\{G_h\}_{h\geq 0}\subseteq B_{r-1}(M)$ by the triangle inequality

$$|P_{t+h}G_h - P_tG_0| \leq |P_{t+h}(G_h-G_0)| + |(P_{t+h}-P_t)G_0|, \tag{A.18}$$

and by the previous paragraph

$$\lim_{h\downarrow 0} P_{t+h}G_0 = \lim_{h\downarrow 0} P_h\big(P_tG_0\big) = P_tG_0$$

uniformly on bounded subsets of M. Therefore the second term in the right hand side of (A.18) converges to 0 uniformly on bounded subsets of M as $h\downarrow 0$, and so in order to prove implication (A.17) it remains to prove that $P_{t+h}(G_h-G_0)$ converges to 0 uniformly on bounded subsets of M as $h\downarrow 0$. But, indeed, since $G_h\longrightarrow G_0$ in $B_{r-1}(M)$ and therefore given $\varepsilon>0$ there exists $h_0>0$ such that

$$0<h<h_0 \quad\Longrightarrow\quad |G_h(x)-G_0(x)|\leq \varepsilon(1+|x|^{r-1}),\ \ \forall x\in M,$$

and thus for all $0<h<h_0$ and all $x\in M$ we have that

$$\begin{aligned}\big|P_{t+h}(G_h-G_0)(x)\big| &\leq \int |G_h(y)-G_0(y)|dP^x_{t+h}(y)\leq \varepsilon\int(1+|y|^{r-1})dP^x_{t+h}(y)\\ &\leq \varepsilon(1+|x|^{r-1}).\end{aligned}$$

Therefore for all $0<h<h_0$ we have that $\|P_{t+h}(G_h-G_0)\|_{u,r-1}\leq\varepsilon$ which proves that $P_{t+h}(G_h-G_0)\longrightarrow 0$ as $h\downarrow 0$ in $B_{r-1}(M)$, and in particular uniformly in bounded subsets of M. This proves (A.17) and consequently (A.16) and (A.14).

It remains thus to prove (A.15). On one hand, since $P_tF_t\in B_{r-1}(M)$ and $L: B_{r-1}(M)\longrightarrow B_r(M)$ is contained in the generator of (P_t) with respect to uniform convergence on bounded subsets of M we have that for all $t\geq 0$,

$$LP_tF_t = \lim_{h\downarrow 0}\frac{P_{t+h}F_t - P_tF_t}{t} = \lim_{h\downarrow 0} P_t\bigg(\frac{P_hF_t-F_t}{h}\bigg) \tag{A.19}$$

uniformly on bounded subsets of M.

On the other hand, since $F_t\in B_{r-1}(M)$, we have that

$$\lim_{h\downarrow 0}\frac{P_hF_t-F_t}{h} = LF_t$$

uniformly on bounded subsets of M. We define $\widetilde{F}:\mathbb{R}_+\times M\longrightarrow\mathbb{R}$ to be the function given by

$$\widetilde{F}(t,x) = \frac{e^{-|x|}}{1+|x|}F(t,x).$$

Then

$$|\widetilde{F}(t,x)|\leq \frac{e^{-|x|}}{1+|x|}\|F\|_{r-1}(1+|x|^{r-1})\leq 2e^{-|x|}(1+|x|^{|r-2|})^{\mathrm{sgn(r-2)}},$$

and therefore the function $e^{|\cdot|}\widetilde{F}_t(\cdot)$ belongs in $B_{r-2}(M)$ for all $t\geq 0$, that is $\widetilde{F}_t\in e^{-|\cdot|}B_{r-2}(M)$. But as we know, $e^{-|\cdot|}B_{r-2}(M)$ is contained in the domain of the generator $L: D_{L_r}\subseteq B_r(M)\longrightarrow B_r(M)$ of (P_t) in $B_r(M)$, and therefore we have that

$$\lim_{h\downarrow 0}\frac{P_h\widetilde{F}_t-\widetilde{F}_t}{h} = L\widetilde{F}_t \quad\text{in } B_r(M)$$

for all $t \geq 0$. As we have seen, if $\{G_h\}_{h\geq 0} \subseteq B_r(M)$ then

$$\lim_{h\downarrow 0} \|G_h - G_0\|_{u,r} = 0 \quad \Longrightarrow \quad \lim_{h\downarrow 0} \|P_t G_h - P_t G_0\|_{u,r} = 0$$

for all $t \geq 0$, and therefore

$$\lim_{h\downarrow 0} P_t\bigg(\frac{P_h\widetilde{F}_t - \widetilde{F}_t}{h}\bigg) = P_t L\widetilde{F}_t \quad \text{in } B_r(M).$$

for all $t \geq 0$. But then by hypothesis (A.9) we have on one hand that

$$P_h\widetilde{F}_t(x) = \mathbb{E}^x\widetilde{F}_t(X_h) = \mathbb{E}^x\bigg(\frac{e^{-|X_h|}}{1+|X_h|}F_t(X_h)\bigg) = \frac{e^{-|x|}}{1+|x|}\mathbb{E}^x F_t(X_h) = \frac{e^{-|x|}}{1+|x|}P_h F_t(x)$$

and therefore

$$P_t\bigg(\frac{P_h\widetilde{F}_t - \widetilde{F}_t}{h}\bigg) = P_t\bigg(\frac{e^{-|\cdot|}}{1+|\cdot|}\,\frac{P_hF_t - F_t}{h}\bigg) = \frac{e^{-|\cdot|}}{1+|\cdot|}P_t\bigg(\frac{P_hF_t - F_t}{h}\bigg)$$

and on the other hand, by hypothesis (A.9) again, we have that

$$L\widetilde{F}_t(x) = \sum_{y\in M_x}\big(\widetilde{F}_t(y) - \widetilde{F}_t(x)\big)\lambda(x)p(x,y) = \frac{e^{-|x|}}{1+|x|}LF_t(x)$$

for all $x \in M$, $t \geq 0$, and thus $P_tL\widetilde{F}_t = \frac{e^{-|\cdot|}}{1+|\cdot|}P_tLF_t$ for all $t \geq 0$. It follows that

$$\lim_{h\downarrow 0}\frac{e^{-|\cdot|}}{1+|\cdot|}P_t\bigg(\frac{P_hF_t - F_t}{h}\bigg) = \frac{e^{-|\cdot|}}{1+|\cdot|}P_tLF_t$$

in $B_r(M)$, which proves that

$$\lim_{h\downarrow 0} P_t\bigg(\frac{P_hF_t - F_t}{h}\bigg) = P_tLF_t$$

uniformly on bounded subsets of M. This together with (A.19) proves (A.15) and completes the proof in the case of unbounded jump rate function λ. The case of bounded jump rate function λ follows similarly. □

Proposition A.2.3 *Let (X, P) be a Markov jump process on the pointed polish state space $(M, o, |\cdot|)$ with partially bounded Lipschitz continuous jump rate function λ and transition probability p such that (A.9) holds for all $x \in M$ and let $\mu \in \mathbb{P}_{2r}M$, $r \geq 1$, ($\mu \in \mathbb{P}_{2r-1}M$ if λ is bounded), be any initial measure. Then, for all $F \in B_{r-1}C^1_{\mathbb{R}_+}(M)$ the P^μ-martingale $M^F \equiv M^{X,F}$ of proposition A.2.2 is square integrable and its quadratic variation $\langle M^F\rangle$ is given by*

$$\langle M^F\rangle_t = \int_0^t \Big\{L(F_s^2)(X_s) - 2F_s(X_s)LF_s(X_s)\Big\}ds, \quad t \geq 0, \tag{A.20}$$

where $L : B_{r-1}(M) \longrightarrow B_r(M)$ is the generator of (X, P) in $B_r(M)$ with respect to uniform convergence on bounded subsets.

Proof We prove first that the P^μ-martingale M^F is square integrable, and of course for this it suffices to prove that the term $F_t(X_t)$ and the integral term $\int_0^t(\partial_s+L)F_s(X_s)ds$ are square integrable for all $t\geq 0$. For the term $F_t(X_t)$ we have that

$$\begin{aligned}\mathbb{E}^\mu F_t^2(X_t) &\leq \|F_t\|^2_{u,r-1}\int(1+|\cdot|^{r-1})^2 d\mu P_t \leq 2\|F_t\|^2_{u,r-1}\int(1+|\cdot|^{2r-2})d\mu P_t\\ &\leq 2\|F_t\|^2_{u,r-1}\int(1+|\cdot|^{2r-2})d\mu<+\infty\end{aligned}$$

since $\mu\in\mathbb{P}_{2r}M\subseteq\mathbb{P}_{2r-2}M$, and therefore $F_t(X_t)\in L^2(P^\mu)$.

For the integral term we note first that since F is assumed to belong in $B_{r-1}C^1_{\mathbb{R}_+}(M)$ we have that $\partial F\in B_{r-1}C_{\mathbb{R}_+}(M)$ and therefore

$$\begin{aligned}\mathbb{E}^\mu\left(\int_0^t\partial_sF_s(X_s)ds\right)^2 &\leq \mathbb{E}^\mu\int_0^t\big(\partial_sF_s(X_s)\big)^2ds\leq 2\|\partial F\|^2_{r-1}\mathbb{E}^\mu\int_0^t(1+|X_s|^{2r-2})ds\\ &= 2\|\partial F\|^2_{r-1}\iint_0^t(1+|X_s|^{2r-2})dsdP^\mu\\ &= 2\|\partial F\|^2_{r-1}\int_0^t\int(1+|X_s|^{2r-2})dP^\mu ds\\ &\leq 2t\|\partial F\|^2_{r-1}\int(1+|\cdot|^{2r-2})d\mu<+\infty.\end{aligned}$$

Finally, for the term $\int_0^t LF_s(X_s)ds$, $t\geq 0$, since $L:B_{r-1}(M)\longrightarrow B_r(M)$ is a bounded operator with $\|L\|\leq 4|\lambda'|$ we have that $\|LF_s\|_{u,r}\leq 4|\lambda'|\|F_s\|_{u,r-1}$ which implies that

$$|LF_s(x)|^2\leq 32|\lambda'|^2\|F\|^2_{u,r-1}(1+|x|)^{2r}$$

for all $x\in M$, $s\geq 0$ and $F\in B_{r-1}C_{\mathbb{R}_+}(M)$ and therefore

$$\begin{aligned}\mathbb{E}^\mu\left(\int_0^t LF_s(X_s)ds\right)^2 &\leq \mathbb{E}^\mu\int_0^t LF_s(X_s)^2ds\leq 32|\lambda'|^2\|F\|^2_{r-1}\mathbb{E}^\mu\int_0^t(1+|X_s|^{2r})ds\\ &\leq 32t|\lambda'|^2\|F\|^2_{r-1}\int(1+|x|^{2r})d\mu(x)<+\infty.\end{aligned}$$

This proves the square integrability of the martingale M^F.

Let $(V_t^F)_{t\geq 0}$ denote the real process in the right hand side of (A.20). Obviously since $\mu\in\mathbb{P}_{2r}M$ it is easy to see that the process V^F is integrable and to prove that the quadratic variation of M^F is V^F it suffices to prove that the real process $N^F:=(M^F)^2-V^F$ is a martingale, i.e. that

$$\mathbb{E}^\mu\big[(M_t^F)^2-(M_s^F)^2\big|\mathcal{F}_s\big]=\mathbb{E}^\mu\big[V_t^F-V_s^F\big|\mathcal{F}_s\big] \tag{A.21}$$

for all $0\leq s<t$. Now since M^F is a square integrable martingale we have that

$$\mathbb{E}^\mu\big[(M_t^F)^2-(M_s^F)^2|\mathcal{F}_s\big]=\mathbb{E}\big[(M_t^F-M_s^F)^2|\mathcal{F}_s\big]$$

for all $0 \le s < t$. For simplicity in the notation we set $\widetilde{L} := \partial + L$ the sum of the generator L of X and the time derivative operator ∂ and for all $0 \le s < t$ we have that

$$\begin{aligned}\mathbb{E}^\mu\big[(M_t^F)^2 - (M_s^F)^2|\mathcal{F}_s\big] &= \mathbb{E}^\mu\bigg\{F_t^2(X_t) - F_s^2(X_s) + \Big(\int_s^t \widetilde{L}F_u(X_u)du\Big)^2 \\ &\quad -2F_t(X_t)\int_s^t \widetilde{L}F_u(X_u)du\Big|\mathcal{F}_s\bigg\} \\ &\quad -2F_s(X_s)\mathbb{E}^\mu\bigg\{F_t(X_t) - F_s(X_s) - \int_s^t \widetilde{L}F_u(X_u)du\Big|\mathcal{F}_s\bigg\}.\end{aligned}$$

Now since M^F is a martingale the second term in the sum above is equal to

$$-2F_s(X_s)\mathbb{E}^\mu(M_t^F - M_s^F|\mathcal{F}_s) = 0,$$

and therefore

$$\begin{aligned}\mathbb{E}^\mu\big[(M_t^F)^2 - (M_s^F)^2|\mathcal{F}_s\big] &= \mathbb{E}^\mu\bigg\{F_t^2(X_t) - F_s^2(X_s) + \Big(\int_s^t \widetilde{L}F_r(X_u)du\Big)^2\Big|\mathcal{F}_s\bigg\} \\ &\quad -2\mathbb{E}^\mu\bigg\{F_t(X_t)\int_s^t \widetilde{L}F_u(X_u)dr\Big|F_s\bigg\} \qquad \text{(A.22)}\end{aligned}$$

for all $0 \le s < t$. We compute next the term

$$Y_{s,t} := \mathbb{E}^\mu\bigg\{F_t(X_t)\int_s^t \widetilde{L}F_u(X_u)du\Big|F_s\bigg\}. \qquad \text{(A.23)}$$

For this term we note first that

$$\begin{aligned}\int_s^t \mathbb{E}^\mu|F_t(X_t)\widetilde{L}F_u(X_r)|dP^\mu du &\le C\int_s^t\int(1+|X_t|^{r-1})(1+|X_u|^r)dP^\mu du \\ &= C\int_s^t\int(1+|x|^r)\int(1+|y|^{r-1})dP_{t-u}^x(y)d\mu P_u(x)du \\ &\le C\int_s^t\int(1+|x|^r)(1+|x|^{r-1})d\mu P_u(x)du \\ &\le 2C\int_s^t(1+|x|^{2r-1})d\mu P_u(x)du \\ &\le 2C(t-s)\int(1+|x|^{2r-1})d\mu(x) < +\infty\end{aligned}$$

for some constant $C \ge 0$ and therefore by the conditional Fubini theorem we have that

$$\begin{aligned}Y_{s,t} &:= \mathbb{E}^\mu\bigg\{F_t(X_t)\int_s^t \widetilde{L}F_r(X_r)dr\Big|\mathcal{F}_s\bigg\} = \int_s^t \mathbb{E}^\mu\{F_t(X_t)\widetilde{L}F_u(X_u)|\mathcal{F}_s\}du \\ &= \int_s^t \mathbb{E}^\mu\{F_t(X_t)\widetilde{L}F_u(X_u)|\mathcal{F}_u|\mathcal{F}_s\}du = \int_s^t \mathbb{E}^\mu\big\{\widetilde{L}F_u(X_u)\mathbb{E}^\mu[F_t(X_t)|\mathcal{F}_u]\big|\mathcal{F}_s\big\}du \\ &= \mathbb{E}^\mu\bigg\{\int_s^t \mathbb{E}^\mu\{F_t(X_t)|\mathcal{F}_u\}\widetilde{L}F_u(X_u)du\Big|\mathcal{F}_s\bigg\} \qquad \text{(A.24)}\end{aligned}$$

Again since M^F is a martingale we have that

$$\mathbb{E}^\mu[F_t(X_t)|\mathcal{F}_u] = F_u(X_u) + \mathbb{E}^\mu\bigg\{\int_u^t \widetilde{L}F_v(X_v)dv\Big|\mathcal{F}_u\bigg\}$$

and therefore continuing from (A.24) we have that

$$\begin{aligned} Y_{s,t} &= \mathbb{E}^{\mu}\Big\{\int_s^t F_u(X_u)\widetilde{L}F_u(X_u)du\Big|\mathcal{F}_s\Big\} \\ &\quad +\mathbb{E}^{\mu}\Big\{\int_s^t \mathbb{E}^{\mu}\Big\{\int_u^t \widetilde{L}F_v(X_v)dv\big|\mathcal{F}_u\Big\}\widetilde{L}F_u(X_u)du\Big|\mathcal{F}_s\Big\}. \end{aligned}$$

We keep the first term in the above sum and estimate the second term which we denote by $Z_{s,t}$ by using the conditional Fubini theorem:

$$\begin{aligned} Z_{s,t} : &= \mathbb{E}^{\mu}\Big\{\int_s^t \mathbb{E}^{\mu}\Big\{\int_u^t \widetilde{L}F_v(X_v)dv\big|\mathcal{F}_u\Big\}\widetilde{L}F_u(X_u)du\Big|\mathcal{F}_s\Big\} \\ &= \mathbb{E}^{\mu}\Big\{\int_s^t \mathbb{E}^{\mu}\Big\{\widetilde{L}F_u(X_u)\int_u^t \widetilde{L}F_v(X_v)dv\big|\mathcal{F}_u\Big\}du\Big|\mathcal{F}_s\Big\} \\ &= \mathbb{E}^{\mu}\Big\{\mathbb{E}^{\mu}\Big\{\int_s^t \widetilde{L}F_u(X_u)\int_u^t \widetilde{L}F_v(X_v)dvdu\big|\mathcal{F}_u\Big\}\Big|\mathcal{F}_s\Big\} \\ &= \mathbb{E}^{\mu}\Big\{\int_s^t\int_u^t \widetilde{L}F_u(X_u)\widetilde{L}F_v(X_v)dvdu\Big|\mathcal{F}_s\Big\}. \end{aligned}$$

But by symmetry, for any bounded function $h : [s,t]\times[s,t] \longrightarrow \mathbb{R}$ of the form $h(u,y) = g(u)g(v)$ for some function $g \in B([s,t])$ we have that

$$\int_s^t\int_u^t g(u)g(v)dvdu = \frac{1}{2}\Big(\int g(u)du\Big)^2$$

and therefore

$$Z_{s,t} = \frac{1}{2}\mathbb{E}^{\mu}\Big\{\Big(\int_s^t \widetilde{L}F_u(X_u)du\Big)^2\Big|\mathcal{F}_s\Big\}.$$

It follows that

$$Y_{s,t} = \mathbb{E}^{\mu}\Big\{\int_s^t F_u(X_u)\widetilde{L}F_u(X_u)du\Big|\mathcal{F}_s\Big\} + \frac{1}{2}\mathbb{E}^{\mu}\Big\{\Big(\int_s^t \widetilde{L}F_u(X_u)du\Big)^2\Big|\mathcal{F}_s\Big\}$$

and therefore by (A.22) we have that

$$\begin{aligned} \mathbb{E}^{\mu}\big[(M_t^F)^2 - (M_s^F)^2|\mathcal{F}_s\big] &= \mathbb{E}^{\mu}\Big\{F_t^2(X_t) - F_s^2(X_s) + \Big(\int_s^t \widetilde{L}F_r(X_u)du\Big)^2\Big|\mathcal{F}_s\Big\} - 2Y_{s,t} \\ &= \mathbb{E}^{\mu}\big\{F_t^2(X_t) - F_s^2(X_s)\big|\mathcal{F}_s\big\} \\ &\quad -2\mathbb{E}^{\mu}\Big\{\int_s^t F_u(X_u)\widetilde{L}F_u(X_u)du\Big|\mathcal{F}_s\Big\} \qquad \text{(A.25)} \end{aligned}$$

We show next that if $F \in B_{r-1}C_{\mathbb{R}_+}(M)$ then $F^2 \in B_{2r-2}C_{\mathbb{R}_+}(M)$ with $\partial F^2 = 2F\cdot\partial F$. Indeed,

$$\begin{aligned} \left|\frac{F_{t+h}^2 - F_t^2 - 2hF_t\partial F_t}{h(1+|\cdot|^{2r-2})}\right| &= \left|\frac{F_{t+h}^2 - F_tF_{t+h} + F_tF_{t+h} - F_t^2 - 2hF_t\partial F_t}{h(1+|\cdot|^{2r-2})}\right| \\ &\le \left|\frac{F_{t+h}(F_{t+h}-F_t) - hF_t\partial_tF_t}{h(1+|\cdot|^{2r-2})}\right| + \left|F_t\frac{F_{t+h}-F_t-h\partial_tF_t}{h(1+|\cdot|^{2r-2})}\right| \end{aligned}$$

and for the second term in this sum we have that

$$\begin{aligned}\left|F_t\frac{F_{t+h}-F_t-h\partial_t F_t}{h(1+|\cdot|^{2r-2})}\right| &\leq \|F\|_{u,r-1}\left|\frac{(F_{t+h}-F_t-h\partial_t F_t)(1+|\cdot|^{r-1})}{h(1+|\cdot|^{2r-2})}\right|\\ &\leq 2\|F\|_{u,r-1}\left|\frac{(F_{t+h}-F_t-h\partial_t F_t)}{h(1+|\cdot|^{r-1})}\right|\\ &\leq 2\|F\|_{u,r-1}\left\|\frac{(F_{t+h}-F_t-h\partial_t F_t)}{h}\right\|_{u,r-1}\end{aligned}$$

while for the first term we have that

$$\left|\frac{F_{t+h}(F_{t+h}-F_t)-hF_t\partial_t F_t}{h(1+|\cdot|^{2r-2})}\right| \leq \left|\frac{(F_{t+h}-F_t)^2}{h(1+|\cdot|^{2r-2})}\right| + 2\|F\|_{u,r-1}\left\|\frac{F_{t+h}-F_t-h\partial_t F_t}{h}\right\|_{u,r-1}$$

and

$$\left|\frac{(F_{t+h}-F_t)^2}{h(1+|\cdot|^{2r-2})}\right| \leq 2|h|\left|\frac{F_{t+h}-F_t}{h(1+|\cdot|^{r-1})}\right|^2 \leq 2|h|\left\|\frac{F_{t+h}-F_t}{h}\right\|^2_{u,r-1}.$$

But since $F \in B_{r-1}C_{RR_+}(M)$ we have that $\frac{1}{h}(F_{t+h}-F_t) \longrightarrow \partial_t F_t$ in $B_{r-1}(M)$ and in particular the $\|\cdot\|_{u,r-1}$-norm of $\frac{1}{h}(F_{t+h}-F_t)$ is bounded above by some constant $C \geq 0$ uniformly for all $x \in M$ and $h > 0$, and therefore

$$\left\|\frac{F^2_{t+h}-F^2_t-2hF_t\partial F_t}{h}\right\|_{u,2r-2} \leq 2C^2|h| + 4\|F\|_{u,r-1}\left\|\frac{F_{t+h}-F_t-h\partial_t F_t}{h}\right\|_{u,r-1} \overset{h\to+\infty}{\longrightarrow} 0$$

which proves that $F^2 \in B_{2r-2}C_{\mathbb{R}_+}(M)$ with $\partial F^2 = 2F\cdot\partial F$.

Therefore, since $\mu \in \mathbb{P}_{2r}M$, it follows by proposition A.2.2 that the process $M^{G,X}$, $G = F^2$, is an integrable martingale and so for all $0 \leq s < t$ we have that

$$\mathbb{E}^\mu\big\{F^2_t(X_t) - F^2_s(X_s)\big|\mathcal{F}_s\big\} = \mathbb{E}^\mu\Big\{\int_s^t \widetilde{L}(F^2_u)(X_u)du\Big|\mathcal{F}_s\Big\}$$

and so it follows by (A.25) that

$$\mathbb{E}^\mu\big[(M^F_t)^2 - (M^F_s)^2|\mathcal{F}_s\big] = \mathbb{E}^\mu\Big\{\int_s^t \widetilde{L}(F^2_u)(X_u) - 2F_u(X_u)\widetilde{L}F_u(X_u)du\Big|\mathcal{F}_s\Big\}$$

which proves (A.21) as required to complete the proof since we obviously have that

$$\widetilde{L}(F^2_u) - 2F_u\widetilde{L}F_u = \partial_u F^2_u + L(F^2_u) - 2F_u\partial_u F_u - 2F_u LF_u = L(F^2_u) - 2F_u LF_u. \qquad \square$$

The following simple Leibniz-type formula for the generator of a Markov jump process (X, P) allows to express the quadratic variation of the martingale $M^F = M^{F,X}$ in an equivalent useful form.

Proposition A.2.4 *Let X be a Markov jump process in M with partially bounded jump rate function λ and skeleton chain p such that $p|\cdot|^{2r} \leq |\cdot|^{2r}$, $r \geq 0$, and let L be the generator of X. Then*

$$L(f^2)(x) = 2f(x)Lf(x) + \sum_{y\in M}[f(y)-f(x)]^2\lambda(x)p(x,y)$$

for all $f \in B_r(M)$.

Proof Indeed, for any $x, y \in M$ and $f \in B_r(M)$ we have that

$$\begin{aligned} L(f^2)(x) &= \lambda(x) \sum_{y\in M} [f(y)^2 - f(x)^2] p(x,y) \\ &= \lambda(x) \sum_{y\in M} \big[\big(f(y)-f(x)\big)^2 + 2f(x)\big(f(y)-f(x)\big)\big] p(x,y) \\ &= \lambda(x) \sum_{y\in M} \big[f(y)-f(x)\big]^2 p(x,y) + 2f(x)Lf(x) \end{aligned}$$ □

According to this proposition the quadratic variation given in proposition A.2.3 for the martingale $M^F \equiv M^{F,X}$ can be written equivalently as

$$\langle M^F\rangle_t = \int_0^t \sum_{y\in M} [F_s(y) - F_s(X_s)]^2 \lambda(X_s) p(X_s, y) ds.$$

Proposition A.2.5 *Let $M = (M_t)_{t\geq 0} \subseteq \mathcal{L}^2(P)$ be a right continuous $(\mathcal{F}_t)_{t\geq 0}$-martingale with quadratic variation $\langle M\rangle$. Then for any bounded $(\mathcal{F}_t)_{t\geq 0}$-stopping times $\sigma \leq \tau \leq T$, $T > 0$,*

$$\mathbb{E}\big[M_\tau^2 - M_\sigma^2 \big| \mathcal{F}_\sigma\big] = \mathbb{E}\big[(M_\tau - M_\sigma)^2 \big| \mathcal{F}_\sigma\big] = \mathbb{E}\big[\langle M\rangle_\tau - \langle M\rangle_\sigma \big| \mathcal{F}_\sigma\big]$$

and so in particular

$$\mathbb{E}(M_\tau^2 - M_\sigma^2) = \mathbb{E}(M_\tau - M_\sigma)^2 = \mathbb{E}\big(\langle M\rangle_\tau - \langle M\rangle_\sigma\big).$$

Proof Since M is a right continuous martingale and σ, τ are bounded stopping times Doob's optimal stopping theorem ([27], theorem II.3.2) applies and so $\mathbb{E}(M_\tau|\mathcal{F}_\sigma) = M_\sigma$. Consequently

$$\begin{aligned} \mathbb{E}\big[(M_\tau - M_\sigma)^2|\mathcal{F}_\sigma\big] &= \mathbb{E}(M_\tau^2|\mathcal{F}_\sigma) - 2M_\sigma \mathbb{E}(M_\tau|\mathcal{F}_\sigma) + M_\sigma^2 \\ &= \mathbb{E}(M_\tau^2|\mathcal{F}_\sigma) - M_\sigma^2 = \mathbb{E}(M_\tau^2 - M_\sigma^2|\mathcal{F}_s), \end{aligned}$$

which proves the first claimed equality.

For the other inequality, since $\langle M\rangle$ is the quadratic variation of M, the process $N_t := M_t^2 - \langle M\rangle_t$, $t \geq 0$ is also a right continuous $(\mathcal{F}_t)$-martingale and so by Doob's optimal stopping theorem again we have that $\mathbb{E}(N_\tau - N_\sigma|\mathcal{F}_s) = 0$ which implies that

$$\mathbb{E}(M_\tau^2 - M_\sigma^2|\mathcal{F}_\sigma) = \mathbb{E}(\langle M\rangle_\tau - \langle M\rangle_\sigma|\mathcal{F}_\sigma)$$

as required. □

A.3 Relative Entropy and Entropy Production

A.3.1 Relative Entropy Functionals

Definition A.3.1 Let $\mu \in \mathbb{P}M$. The *relative entropy function with respect to* μ is the functional $H(\cdot|\mu) : \mathbb{P}M \longrightarrow [0,\infty]$ given by

$$H(\nu|\mu) = \begin{cases} \int f \log f d\mu, & \nu \ll \mu,\ f = \frac{d\nu}{d\mu}, \\ +\infty, & \nu \not\ll \mu. \end{cases}$$

We note first that the function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}$ given by $g(x) = x \log x$ is strictly convex in $\mathbb{R}_+$ and thus by Jensen's inequality, we have for $f = \frac{d\nu}{d\mu}$ that

$$0 = 1 \cdot 0 = \int f d\mu \cdot \log \int f d\mu = g\left(\int f d\mu\right) \leq \int g(f) d\mu = H(\mu|\nu),$$

for all $\mu, \nu \in \mathbb{P}M$, with equality iff the r.v. (f, μ) is a.s. constant, say $f = c$ μ-$\sigma.\beta.$. But since $f = \frac{d\nu}{d\mu}$ and μ, ν are probability measures, then we necessarily have that

$$c = \int f d\mu = \int 1 d\nu = 1,$$

and thus $H(\mu|\nu) = 0$ iff $\frac{d\nu}{d\mu} = 1$, that is iff $\mu = \nu$.

Therefore $H : \mathbb{P}M \times \mathbb{P}M \longrightarrow [0,\infty]$ takes real positive values and $H(\mu|\nu) = 0$ iff $\mu = \nu$. Nevertheless, neither H, nor its symmetrization $H^S(\mu,\nu) := \frac{1}{2}[H(\nu|\mu) + H(\mu|\nu)]$ defines a metric in $\mathbb{P}M$. Now obviously for any $\mu, \nu \in \mathbb{P}M$ with $\nu \ll \mu$ we have that

$$H(\nu|\mu) = \int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} d\mu = \int \log \frac{d\nu}{d\mu} d\nu.$$

and if $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is the strictly convex continuous function

$$h(t) = t \log t - t + 1,$$

where $h(0) = 1$, then for all $\mu, \nu \in \mathbb{P}M$ such that $\nu \ll \mu$ we have that

$$H(\nu|\mu) = \int g\Big(\frac{d\nu}{d\mu}\Big) d\mu = \int h\Big(\frac{d\nu}{d\mu}\Big) d\mu,$$

which shows that $(\nu|\mu) \geq 0$ with equality iff $\nu = \mu$.

Also, by the strict convexity of h, it follows that the relative entropy with respect to μ is strictly linearly convex, i.e. for all $\nu_1, \nu_2 \in \mathbb{P}M$ and all $a, b \geq 0$ such that $a + b = 1$ it holds that

$$\begin{aligned} H(a\nu_1 + b\nu_2|\mu) &= \int h\left(a\frac{d\nu_1}{d\mu} + b\frac{d\nu_2}{d\mu}\right) d\mu \\ &\leq a \int h\left(\frac{d\nu_1}{d\mu}\right) d\mu + b \int h\left(\frac{d\nu_2}{d\mu}\right) d\mu = aH(\nu_1|\mu) + bH(\nu_2|\mu), \end{aligned}$$

with strict inequality whenever $\nu_1 \neq \nu_2$ and $a > 0$.

Another useful property of relative entropy is the following formula for changing the reference measure μ. In particular, let $V : M \longrightarrow (-\infty, \infty]$ be a Borel functional with sup-polynomial growth of order $p \geq 0$, i.e. such that $|V(x)| \leq C\big(1 + d^p(x, x_0)\big)$ for all $x \in M$ for some constant $C \geq 0$, and such that $d\widetilde{\mu} := e^{-V} d\mu$ is a probability measure. Then,

$$H(\nu|\widetilde{\mu}) = H(\nu|\mu) + \int V d\nu,$$

for all $\nu \in \mathbb{P}_p M$. Indeed, we obviously have that $\nu \ll \widetilde{\mu}$ iff $\nu \ll \mu$ and in this case the required equality is trivially satisfied as $\infty = \infty$, while if $\nu \ll \mu$, then

$$\frac{d\nu}{d\widetilde{\mu}} = \frac{d\nu}{d\mu}\frac{d\mu}{d\widetilde{\mu}} = e^V \frac{d\nu}{d\mu},$$

and thus

$$\begin{aligned} H(\nu|\widetilde{\mu}) &= \int h\left(\frac{d\nu}{d\widetilde{\mu}}\right) d\widetilde{\mu} = \int \frac{d\nu}{d\mu} \cdot \big[V + \log \frac{d\nu}{d\mu}\big] e^V d\widetilde{\mu} = \int \frac{d\nu}{d\mu} \cdot \big[V + \log \frac{d\nu}{d\mu}\big] d\mu \\ &= \int V d\nu + H(\nu|\mu). \end{aligned}$$

Particularly useful is the following variational characterization of the relative entropy.

Proposition A.3.1 *For all $\mu, \nu \in \mathbb{P}M$ we have that*

$$H(\nu|\mu) = \sup_{f \in BC(M)} \left\{ \int f d\nu - \log \int e^f d\mu \right\}. \tag{A.26}$$

This characterization of relative entropy will follow from the more general variational characterization the generalized relative h-entropy functionals, one of which is relative entropy itself with $h(t) = t \log t - t + 1$ for $t \geq 0$ and $h(t) = +\infty$ for $t < 0$.

Definition A.3.2 Let $h : \mathbb{R} \longrightarrow [0, \infty]$ be a lower semicontinuous convex functional with non-trivial domain $\mathcal{D}_h \supset [0, \varepsilon)$, $\varepsilon > 0$. The functional $\mathcal{H} \equiv \mathcal{H}_h : \mathbb{P}M \times \mathbb{P}M \longrightarrow [0, \infty]$ given by the formula

$$\mathcal{H}(\nu|\mu) = \begin{cases} \int_M h\big(\frac{d\nu}{d\mu}\big) d\mu, & \text{if} \nu \ll \mu, \\ +\infty, & \text{otherwise.} \end{cases}$$

is called the *relative h-entropy functional.*

Obviously, as in the case of relative entropy, due to the convexity of h we have by Jensen's inequality that

$$0 \leq h(1) = h\left(\int \frac{d\nu}{d\mu} d\mu\right) \leq \int h\left(\frac{d\nu}{d\mu}\right) d\mu = \mathcal{H}(\nu|\mu)$$

for all $\nu \ll \mu$. Therefore $\mathcal{H}(\nu|\mu) \geq h(1)$ for all $\mu, \nu \in \mathbb{P}M$ with equality iff $\nu = \mu$. Furthermore $\mathcal{H}_h$ is obviously linearly convex.

Note that since ν is a positive measure we have that $\frac{d\nu}{d\mu} \geq 0$ whenever $\nu \ll \mu$ and thus the valued of h in $(-\infty, 0)$ do not play a part in the definition of $\mathcal{H}_h$. In what follows for any function $h : \mathbb{R} \;\; \mathbb{R}_+ \longrightarrow [0,\infty]$ we will denote by $\widetilde{h} : \mathbb{R} \longrightarrow [0,\infty]$ the function given by the formula

$$\widetilde{h}(t) = \begin{cases} h(t), & t \geq 0 \\ +\infty, & t < 0 \end{cases},$$

and of course for any convex function $h : \mathbb{R}$ or $\mathbb{R}_+ \longrightarrow [0,\infty]$ we have that $\mathcal{H}_h = \mathcal{H}_{\widetilde{h}}$. The variational characterization of the generalized relative h-entropy functionals requires the notion of the right semi-Legendre transform.

Definition A.3.3 Let $h : \mathbb{R} \longrightarrow [0,\infty]$ be a function. The *right semi-Legendre transform h^+ of h* is the function $h^+ : \mathbb{R} \longrightarrow [-h(0),\infty]$ given by the formula

$$h^+(s) = \sup_{t\geq 0}\{ts - h(t)\}$$

We note first that indeed $h^+ \geq -h(0)$, since for all $s \in \mathbb{R}$ we have that

$$h^+(s) \equiv \sup_{t\geq 0}\{st - h(t)\} \geq s \cdot 0 - h(0) = -h(0).$$

Furthermore, h is always lower semicontinuous as a supremum of affine functionals and obviously

$$h^+ = \widetilde{h}^*,$$

where $\widetilde{h}^*$ is the Legendre transform of $\widetilde{h}$ in $\mathbb{R}$, that is

$$\widetilde{h}^*(s) = \sup_{t\in\mathbb{R}}\{ts - \widetilde{h}(t)\}.$$

Now, $\widetilde{h}$ is lower semicontinuous whenever h is, and thus if h is lower semicontinuous then by the duality $f^{**} = f$ for lower semicontinuous convex functions f we have that

$$h^{+*} = \widetilde{h}.$$

According to the variational characterization of relative h-entropy functionals to be proved, for any lower semicontinuous and convex function $h :\longrightarrow [0,\infty]$ with non-trivial proper domain $\mathcal{D}_h \subseteq [0,\infty)$ such that $\mathcal{D}_{h^+} = \mathbb{R}$ we have that

$$\mathcal{H}_h(\nu|\mu) = \sup_{f\in BC(M)}\left\{\int f d\nu - \int h^+(f)d\mu\right\}. \tag{A.27}$$

The variational characterization (A.26) of relative entropy follows from the variational (A.27) of generalized entropy functionals. Indeed, relative entropy is given as we have seen by the function $h : \mathbb{R} \longrightarrow [0,+\infty)$ defined by $h(t) = t\log t - t + 1$ in $\mathcal{D}_h = [0,\infty)$ and the function

$$\mathbb{R}_+ \ni t \overset{g_s}{\mapsto} g_s(t) := st - h(t) = (s+1)t - t\log t - 1$$

has obviously a global maximum at $t = e^s$, and so

$$h^+(s) = s \cdot e^s - h(e^s) = se^s - e^s \log e^s + e^s - 1 = e^s - 1.$$

Therefore for any bounded and continuous function $f : M \longrightarrow \mathbb{R}$ we have that

$$\int f d\nu - \int h^+(f) d\mu = \int f d\nu - \int e^f d\mu + 1.$$

Consequently, since $h(t) = t\log t - t + 1$ satisfies condition $\mathcal{D}_{h^+} = \mathbb{R}$, if the variational characterization of generalized functionals had been proved we would have that

$$\begin{aligned} H(\nu|\mu) &= \sup_{f\in BC(M)} \left\{ \int f d\nu - \int h^+(f) d\mu \right\} \\ &= \sup_{f\in BC(M)} \sup_{c\in\mathbb{R}} \left\{ \int (f+c) d\nu - \int h^+(f+c) d\mu \right\} \\ &= \sup_{f\in BC(M)} \sup_{c\in\mathbb{R}} \left\{ c + \int f d\nu - e^c \int e^f d\mu + 1 \right\}. \end{aligned} \tag{A.28}$$

But by elementary calculus we see that the mapping

$$c \mapsto c + \int f d\nu - e^c \int e^f d\mu + 1$$

has global maximum $\int f d\nu - \log \int e^f d\mu$ at the point $c = -\log \int e^f d\mu$, which according to (A.28) proves the variational characterization of relative entropy.

An obvious and useful consequence of the variational characterization (A.26) is the lower semicontinuity of the relative entropy $H(\cdot|\mu)$ as a supremum of affine functionals with respect to the weak topology in $\mathbb{P}M$.

Another useful application of the variational characterization of relative entropy is that it allows us to estimate integrals with respect to ν through integrals with respect to the reference measure μ ant the relative entropy $H(\nu|\mu)$. Indeed, by the variational characterization the so called entropy inequality

$$\int f d\nu \leq \inf_{a>0} \frac{1}{a} \left\{ \log \int e^{af} d\mu + H(\nu|\mu) \right\}$$

follows easily for all $f \in B(M)$. For indicator functions $f = \mathbb{1}_A$, $A \in \mathcal{B}_M$ the entropy inequality takes a simpler form.

Proposition A.3.2 *Let $A \in \mathcal{B}_M$ be a Borel subset of M. Then for all $\nu, \mu \in \mathbb{P}M$ we have that*

$$\nu(A) \leq \begin{cases} \frac{\log 2 + H(\nu|\mu)}{\log(1+\frac{1}{\mu(A)})}, & \mu(A) > 0, \\ \lim\limits_{a\uparrow+\infty} \frac{1}{a} H(\nu|\mu), & \mu(A) = 0 \end{cases}.$$

Proof For any $a > 0$, we have that $e^{a\mathbb{1}_A} = e^a\mathbb{1}_A + \mathbb{1}_{A^C}$ and thus $\int e^{a\mathbb{1}_A} d\mu = 1 + (e^a - 1)\mu(A)$. So by the entropy inequality we have for all $a > 0$ that

$$\nu(A) \leq \frac{1}{a} \log\left[1 + (e^a - 1)\mu(A)\right] + \frac{1}{a} H(\nu|\mu).$$

Taking the limit as $a \to +\infty$ if $\mu(A) = 0$ and choosing $a > 0$ so that $(e^a - 1)\mu(A) = 1$ if $\mu(A) \neq 0$ we get the required inequality. □

For the proof now of the variational characterization of generalized entropy functionals we need some formula for the right semi-Legendre transform. Such a formula is given in the following lemma. We recall first that any lower semicontinuous function $h:\mathbb{R}\longrightarrow(-\infty,\infty]$ is continuous in $\mathcal{D}_h^o$, one-sidedly continuous in each endpoint of $\mathcal{D}_h$ that is contained in $\mathcal{D}_h$ divergent to $+\infty$ in each endpoint of $\mathcal{D}_h$ not contained in $\mathcal{D}_h$. Furthermore, the one-sided derivatives h'_- and h'_+ of h exist in $\mathcal{D}_h^o$, are left and right continuous respectively, and for all $r,s,t\in\mathcal{D}_h^o$ such that $r<s<t$ satisfy

$$h_+(r)\leq h'_-(s)\leq h'_+(s)\leq h'_-(t),$$

and if $a,b\in[-\infty,\infty]$ is the left and right endpoint of $\mathcal{D}_h$ respectively, then $h'_+(a)$ ($h'_-(b)$ resp.) is defined in $[-\infty,\infty)$ ($(-\infty,\infty]$ resp.) if $a\in\mathcal{D}_h$ ($b\in\mathcal{D}_h$ resp.).

In our case now, in the right semi-Legendre transform, we are essentially working with the functional $\widetilde{h}$ for which $\widetilde{h}|_{(-\infty,0)}\equiv+\infty$ and since the values of h in $(-\infty,0)$ do not play a role in the definition of $\mathcal{H}_h$ we restrict our attention to functions h such that $h=\widetilde{h}$ and set

$$\begin{array}{ll} a:=\inf\mathcal{D}_h=0, & b:=\sup\mathcal{D}_h\in(0,\infty] \\ a':=\inf_{t>a}h'_+(t)\in[-\infty,\infty), & b':=\sup_{t<b}h'_-(t)\in(-\infty,+\infty]. \end{array} \tag{A.29}$$

Of course, by the one-sided continuity of h'_- and h'_+ we have that whenever some of the endpoints a,b belongs in $\mathcal{D}_h$ then $a'=h'_+(a)=h'_+(0)$ and $b':=h'_-(b)$.

Lemma A.3.1 *Let $h:\mathbb{R}\longrightarrow[0,\infty]$ be a lower semi-continuous function with non-trivial proper domain $\mathcal{D}_h\subseteq[0,\infty)$, let a,b,a',b' be the numbers given in (A.29), and let $(h'_+)^{-1}:(a',b')\longrightarrow[0,\infty]$ be the generalized inverse of the right continuous and increasing function $h'_+:(0,b)\longrightarrow(a',b')$, given by the formula*

$$(h'_+)^{-1}(s):=\inf\big\{t\in(0,b)\big|s\leq h'_+(t)\big\}.$$

Then $(h'_+)^{-1}(a',b'):=\{(h'_+)^{-1}(t)|t\in(a',b')\}\subseteq\mathcal{D}_h^o\equiv(0,b)$ and the right semi-Legendre transform h^+ of h os given by the formula

$$h^+(s)=\begin{cases}-h(0), & \text{if } s\leq a'.\\ s\cdot(h'_+)^{-1}(s)-h\big((h'_+)^{-1}(s)\big), & \text{if } a'<s<b'\\ bs-h(b), & \text{if } b'\leq s, b\in\mathcal{D}_h,\\ \lim\limits_{t\uparrow+\infty}\big(ts-h(t)\big), & \text{if } b'\leq s, b=+\infty.\end{cases} \tag{A.30}$$

Furthermore, in the case that $b'<+\infty$ and $b=+\infty$ we have that $h^+(s)=+\infty$ for all $s>b'$ while for $s=b'$ both cases, $h^+(b')<+\infty$ and $h^+(b')=+\infty$, are possible.

Proof First we note that if $a'=-\infty$ then the first branch of (A.30) does not exist. So suppose that $a'>-\infty$ and let $s\leq a'$. Then

$$s\leq a'=\inf_{t>0}h'_+(t)=\inf_{t>0}\frac{h(t)-h(0)}{t},$$

and thus $ts-h(t)\leq-h(0)$ for all $t\geq0$, which gives us that

$$h^+(s)=\sup_{t\geq0}\{ts-h(t)\}\leq-h(0).$$

But as we have seen we always have $h^+ \geq -h(0)$ and so $h^+(s) = -h(0)$ which proves the formula in the first branch of (A.30).

Suppose on the other hand that $b' < +\infty$ and let $s \geq 0$. Then necessarily either $b \in \mathcal{D}_h$ or either $b = +\infty$. Indeed, if $b' < +\infty$ and $b \notin \mathcal{D}_h$ then

$$\sup_{0<s<t<b} \frac{h(t) - h(s)}{t - s} \leq b' < +\infty,$$

and thus

$$h(t) - h(s) \leq b'(t - s) \tag{A.31}$$

for all $0 < s < t < b$. But as we have already noted, since h is lower semicontinuous and $b \notin \mathcal{D}_h$ we have that $\lim_{t\uparrow b} h(t) = +\infty$ and so by taking the limit in (A.31) as $t \uparrow b$ for some $s \in (0, b)$ we get that

$$+\infty = \lim_{t\to b} \big(h(t) - h(s)\big) \leq \lim_{t\to b} b'(t - s) = b'(b - s)$$

and thus $b' = +\infty$.

So, in the case that $b' \leq s < +\infty$, either $b \in \mathcal{D}_h$ or either $b = +\infty$. If $b \in \mathcal{D}_h$ then $h'_-(b) = b' \leq s < +\infty$ and thus

$$\sup_{0<t<b} \frac{h(b) - h(t)}{b - t} = b' \leq s < \infty.$$

Therefore for all $s \geq b'$ we have that $h(b) - h(t) \leq s(b - t)$ for all $0 < t \leq b$, which shows that

$$st - h(t) \leq sb - h(b), \quad \forall 0 < t \leq b,$$

and thus

$$h^+(s) = \sup_{t\geq 0}\{st - h(t)\} = \sup_{t\in\mathcal{D}_h}\{st - h(t)\} \leq s \cdot b - h(b).$$

On the other hand we obviously have that $sb - h(b) \leq h^+(s)$, which proves the formula in the third branch of (A.30).

Suppose now that $b' \leq s < +\infty$ and $b = +\infty$. Then

$$\sup_{0<r<t<\infty} \frac{h(t) - h(r)}{t - r} \leq b' \leq s$$

and therefore

$$r < t \quad \Longrightarrow \quad sr - h(r) \leq st - h(t),$$

i.e. for all $s \geq b'$ the function $t \mapsto st - h(t)$ is increasing and so

$$h^+(s) \equiv \sup_{t\geq 0}\{st - h(t)\} = \lim_{t\to\infty} (st - h(t)),$$

which proves the formula in the fourth branch of (A.30).

It remains to check the case $a' < s < b'$. First, we note that if some $s_{t_0} \in \mathbb{R}$ is a supporting point of h at $t_0 \in \mathcal{D}_h^o$ then $h(t) \geq s_{t_0}(t - t_0) + h(t_0)$ for all $t \in \mathbb{R}$ and thus

$$h^+(s_{t_0}) = \sup_{t\geq 0}\{t \cdot s_{t_0} - h(t)\} \leq t_0 \cdot s_{t_0} - h(t_0) \leq h^+(s_{t_0}). \tag{A.32}$$

Indeed, let $s \in (a', b')$. By the definition of a', b' there exist $\theta, m \in \mathcal{D}_h^o = (0, b)$ such that

$$a' \leq h'_+(\theta) < s < h'_-(m) \leq b'. \tag{A.33}$$

We will show that s is a supporting point of h at some $t_s \in [\theta, m] \subseteq (0, b)$. Indeed, let

$$t_s := (h'_+)(s) \equiv \inf\{t \in \mathcal{D}_h | s \leq h'_+(t)\}. \tag{A.34}$$

By the definition of m in (A.33) we have that $m \in E_s := \{t \in \mathcal{D}_h^o | s \leq h'_+(t)\}$. Therefore the set E_s is non-empty and since h'_+ is increasing we have that E_s is an interval with right endpoint $+\infty$ and left endpoint the number $t_s \in [0, b)$. But by (A.33) $\theta \notin E_s$, and therefore $t_s \in [\theta, m] \subseteq \mathcal{D}_h^o$. Also, by the right continuity of h'_+ we have that

$$h'_+(t_s) = \lim_{t \downarrow t_s} h'_+(t) \geq s \tag{A.35}$$

i.e. that $t_s \in E_s$. Now, according to(A.35)on order for s to be a supporting point of h at t_s it remains to prove that $h'_-(t_s) \leq s$. But this is indeed true, since for all $t < t_s$ we have that $h'_-(t) \leq h'_+(t) < s$ and thus by the left continuity of h'_- we get that

$$h'_-(t_s) = \lim_{t \uparrow t_s} h'_-(t) \leq s.$$

Now since $s \in (a', b')$ is a supporting point of h at $t_s := (h'_+)^{-1}(s)$ it follows by (A.32) that for all $s \in (a', b')$:

$$h^+(s) = s \cdot t_s - h(t_s) = s \cdot (h'_+)^{-1}(s) - h\big((h'_+)^{-1}(s)\big),$$

as required for the second branch of (A.30).

We prove now the last claim of the lemma. So suppose that $b' < +\infty$, $b = +\infty$ and let $s > b'$. We note that

$$\lim_{t \uparrow +\infty} \frac{h(t)}{t} \leq \sup_{t \geq 0} h'_+(t) = b'.$$

Indeed, we obviously have that $b := \sup_{t \geq 0} h'_-(t) = \sup_{t \geq 0} h'_+(t)$ and by the convexity of h in $(0, b) = (0, \infty)$ the function

$$(1, \infty) \ni t \mapsto \frac{h(t) - h(1)}{t - 1}$$

is increasing and bounded above by b', and therefore there exists the limit

$$\lim_{t \uparrow \infty} \frac{h(t)}{t} = \lim_{t \uparrow +\infty} \frac{h(t) - h(1)}{t - 1} \leq b'.$$

So, since $\delta := \frac{1}{2}(s - b') > 0$, there exists $M > 0$ such that

$$t \geq M \quad \Longrightarrow \quad h(t) \leq b't + \delta t$$

and then for all $t > M$ we have that

$$st - h(t) \geq (s - b' - \delta)t = \frac{t}{2}(s - b') \stackrel{t \uparrow \infty}{\longrightarrow} +\infty,$$

as required.

Finally to show that in the case that $b' < +\infty$ and $b = +\infty$ we can either have $h^+(b') < +\infty$ or $h^+(b') = +\infty$ we consider the convex functions $h_i : \mathbb{R} \longrightarrow (-\infty, \infty]$ given by the formulas

$$h_1(t) = \log \frac{1}{t} \quad \text{and} \quad h_2(t) = \frac{1}{t}$$

on their common proper domain $\mathcal{D}_{h_1} = \mathcal{D}_{h_2} = (0, \infty)$. Then for these two functions we have that

$$b_i' := \sup_{t>0} h_i{}'_-(t) = 0, \quad i = 1, 2,$$

while as we can easily check $h_1^+(0) = +\infty$ and $h_2^+(0) = 0$. □

For the proof of the variational characterization of generalized relative h-entropy functionals we will need the following lemma, which essentially generalizes Lusin's theorem so that the approximation of a given measurable and bounded function f by bounded and continuous functions can be achieved by the same approximating sequence in any finite number of L^1 spaces of regular measures.

Lemma A.3.2 *For any $f \in B(M)$ and any $\mu, \nu \in \mathbb{P}M$ there exists a sequence $\{f_n\} \subseteq BC(M)$ such that*

$$\begin{gathered} f_n \overset{n\uparrow+\infty}{\longrightarrow} f \quad \text{in } L^1(\mu) \text{ and in } L^1(\nu), \\ f_n \overset{n\uparrow+\infty}{\longrightarrow} f \quad \mu - \sigma.\beta. \quad \text{and} \quad \nu - \sigma.\beta., \\ \inf_{x\in M} f(x) + \tfrac{1}{n} \leq f_n \leq \sup_{x\in M} f(x) - \tfrac{1}{n}, \quad \forall\, n \in \mathbb{N}. \end{gathered} \tag{A.36}$$

Proof Obviously, since any $f \in B(M)$ can be approximated uniformly by simple functions it suffices to consider the case in which f is the indicator function $\mathbb{1}_E$ of some Borel set $E \subseteq M$. So let $E \subseteq M$ be a Borel set and let $\varepsilon > 0$. We want to find $g \in BC(M)$ such that

$$\int |g - \mathbb{1}_E| d\mu \vee \int |g - \mathbb{1}_E| d\nu < \varepsilon.$$

Since μ, ν are finite Borel measures in polish space, they are regular and thus there exist compact sets $K_\mu, K_\nu \subseteq E$ open sets $A_\mu, A_\nu \supseteq E$ such that

$$K_\rho \subseteq E \subseteq A_\rho, \quad \rho(A_\rho \setminus K_\rho) < \varepsilon, \quad \rho = \mu, \nu.$$

But then, if we set $K := K_\mu \cup K_\nu$, $A := A_\mu \cap A_\nu$, the set $K$is compact, is open and

$$\rho(A \setminus K) \leq \rho(A_\rho \setminus K) \leq \rho(A_\rho \setminus K_\rho) < \varepsilon, \quad \rho = \mu, \nu.$$

By choosing then a continuous function $g \in BC(M)$ such that $\mathbb{1}_K \leq g \leq \mathbb{1}_A$ we obviously have that $|g - \mathbb{1}_E| \leq \mathbb{1}_{A\setminus K}$ and therefore

$$\int |g - \mathbb{1}_E| d\rho \leq \rho(A \setminus K) < \varepsilon, \quad \rho = \mu, \nu,$$

as required. This proves that for any $f \in B(M)$ there exists $\{g_n\} \subseteq BC(M)$ such that $g_n \longrightarrow f$ in $L^1(\mu)$ in $L^1(\nu)$. By passing if necessary to a subsequence $\{g_n\}$ which we continue to denote by $\{g_n\}$ we can assume in addition that $\{g_n\}$ converges pointwise to f μ-$\sigma.\beta.$ and ν-$\sigma.\beta.$. But then if we define

$$f_n := (-\inf_{x\in M} f(x) + \frac{1}{n}) \vee \big[g_n \wedge (\sup_{x\in M} f(x) - \frac{1}{n})\big], \quad \forall n \in \mathbb{N},$$

we obviously have that $f_n \longrightarrow f$ pointwise μ-$\sigma.\beta.$ and ν-$\sigma.\beta.$, and since $\|f_n\|_u \le \|f\|_u$ for all $n \in \mathbb{N}$ it follows by the bounded convergence theorem that $f_n \longrightarrow f$ in $L^1(\mu)$ and in $L^1(\nu)$, and therefore $\{f_n\}$ satisfies (A.36). □

Proposition A.3.3 *Let $h : \mathbb{R} \longrightarrow [0,\infty]$ be a lower semicontinuous functional with non-trivial proper domain $\mathcal{D}_h \subseteq [0,\infty)$. Then*

$$\mathcal{H}_h(\nu|\mu) = \sup_{f\in B(M)} \left\{ \int f d\nu - \int h^+(f) d\mu \right\} =: \mathcal{H}'_h(\nu|\mu). \tag{A.37}$$

Furthermore, $\mathcal{D}_{h^+} = \mathbb{R}$ iff $b' = +\infty$ or $b \in \mathcal{D}_h$, and in this case for all $\mu,\nu \in \mathbb{P}M$ we have that

$$\mathcal{H}_h(\nu|\mu) = \sup_{f\in BC(M)} \left\{ \int f d\nu - \int h^+(f) d\mu \right\} =: \mathcal{H}''_h(\nu|\mu). \tag{A.38}$$

In any case (A.38) holds for all $\mu,\nu \in \mathbb{P}M$ such that $\nu \ll \mu$.

Finally, if $\mathcal{D}_{h^+} \subsetneqq \mathbb{R}$ then $b := \sup \mathcal{D}_h = +\infty$,

$$b' := \sup_{t<b} h'_-(t) = \sup \mathcal{D}_{h^+} = \lim_{t\uparrow+\infty} \frac{h(t)}{t} = \sup_{t>0} \frac{h(t)}{t} < +\infty, \tag{A.39}$$

and if the reference measure $\mu \in \mathbb{P}M$ has support $\mathrm{supp}\mu = M$*, then for all $\nu \in \mathbb{P}M$ we have that*

$$\underline{\mathcal{H}}_h(\nu|\mu) := \mathcal{H}_h(\nu_{ac|\mu}|\mu) + b'\nu_{s|\mu}(M) = \sup_{f\in BC(M)} \left\{ \int f d\nu - \int h^+(f) d\mu \right\}, \tag{A.40}$$

where $\nu = \nu_{ac|\mu} + \nu_{s|\mu}$ is the Radon-Nykodim decomposition of ν with respect to μ with absolutely continuous part $\nu_{ac|\mu} \ll \mu$ and singular part $\nu_{s|\mu} \perp \mu$.

Proof We prove first that (A.37). We note first that by the definition of h^+ we have that $ts \le h(t) + h^+(s)$ for any $t \ge 0$, $s \in \mathbb{R}$, and therefore if $\mu,\nu \in \mathbb{P}$ and $\nu \ll \mu$, then for any $f \in B(M)$ we have that

$$f \cdot \frac{d\nu}{d\mu} \le h\left(\frac{d\nu}{d\mu}\right) + h^+(f),$$

which shows that

$$\int f d\nu - \int h^+(f) d\mu \le \mathcal{H}_h(\nu|\mu),$$

for all $f \in B(M)$. So by taking the supremum over all $f \in B(M)$ it follows that

$$\mathcal{H}''_h(\nu|\mu) \le \mathcal{H}'_h(\nu|\mu) := \sup_{f\in B(M)} \left\{ \int f d\nu - \int h^+(f) d\mu \right\} \le \mathcal{H}_h(\nu|\mu)$$

for all $\mu, \nu \in \mathbb{P}M$, and therefore in order to prove (A.37) it remains to show that $\mathcal{H}_h(\nu|\mu) \leq \mathcal{H}'_h(\nu|\mu)$.

First, if $\nu \not\ll \mu$, there exists a Borel set $E \subseteq M$ such that $\nu(E) > 0$ and $\mu(E) = 0$, and by considering the sequence $f_n = n\mathbb{1}_E$, $n \in \mathbb{N}$ we have that for all $n \in \mathbb{N}$,

$$\mathcal{H}'_h(\nu|\mu) \geq \int f_n d\nu - \int h^*(f_n) d\mu = n\nu(E) - h^+(0) \overset{n\to+\infty}{\longrightarrow} +\infty,$$

and thus the required inequality holds as $\infty = \infty$ in this case.

We note next that for any constant $c \in \mathbb{R}$ we have $\mathcal{H}_{h+c} = c + \mathcal{H}_h$, and since

$$(h+c)^+(s) = \sup_{t\geq 0}\{ts - h(t) - c\} = -c + h^+(s)$$

for all $s \in \mathbb{R}$ it follows that

$$\mathcal{H}'_{h+c} = c + \mathcal{H}'_h, \quad \mathcal{H}''_{h+c} = c + \mathcal{H}''_h.$$

Therefore in order to prove the variational characterization we can, modulo some constant $c \in \mathbb{R}$, assume in addition that

$$-h^+(0) = \inf_{t\geq 0} h(t) = 0. \tag{A.41}$$

Then since $0 \in BC(M)$ by the additional assumption that $h^+(0) = 0$ it follows that for all $\nu, \mu \in \mathbb{P}M$ we have that

$$\mathcal{H}'_h(\nu|\mu) \geq \mathcal{H}''_h(\nu|\mu) \geq \int 0 d\nu - \int h^+(0) d\mu = 0. \tag{A.42}$$

Let now $\nu \ll \mu$ and suppose that

$$\mu\left\{x \in M \,\middle|\, \frac{d\nu}{d\mu}(x) \notin \overline{\mathcal{D}_h}\right\} = \mu\left\{\frac{d\nu}{d\mu} > b\right\} > 0. \tag{A.43}$$

Then obviously

$$\mu\left\{h\left(\frac{d\nu}{d\mu}\right) = +\infty\right\} > 0,$$

and thus $\mathcal{H}_h(\nu|\mu) = +\infty$. Therefore in order to prove the claim in the case that $\nu, \mu \in \mathbb{P}M$ satisfy (A.43) we must show that $\mathcal{H}'_h(\nu|\mu) = +\infty$. Let $\{f_N\}_{N=1}^{\infty} \subseteq B(M)$ be the sequence given by $f_N := N\mathbb{1}_{\{\frac{d\nu}{d\mu} > b\}}$, $N \in \mathbb{N}$. According to our convention that $h^+(0) = 0$ modulo some constant $c \in \mathbb{R}$ we have that

$$h^+(f_N) = h^+(N)\mathbb{1}_{\{\frac{d\nu}{d\mu} > b\}} \tag{A.44}$$

for all $N \in \mathbb{N}$. Then by the definition of $\mathcal{H}'_h$ and (A.44) we have that

$$\begin{aligned}\mathcal{H}'_h(\nu|\mu) &\geq \int f_N d\nu - \int h^+(f_N) d\mu = \int \left[f_N \frac{d\nu}{d\mu} - h^+(f_N)\right] d\mu \\ &= \int_{\{\frac{d\nu}{d\mu} > b\}} \left[\frac{d\nu}{d\mu} N - h^+(N)\right] d\mu\end{aligned}$$

So in order to prove the claim it suffices to show that given (A.43), which of course can hold only if $b < +\infty$, we have that

$$a_N := \int_{\{\frac{d\nu}{d\mu} > b\}} \left[\frac{d\nu}{d\mu} N - h^+(N) \right] d\mu \overset{N\to\infty}{\longrightarrow} +\infty \tag{A.45}$$

For the proof of (A.45) we separate cases on whether $b' < +\infty$ or $b' = +\infty$.
$b' < +\infty$: Then by the lower semicontinuity of h we have that $b \in \mathcal{D}_h$. Consequently for all $N \in \mathbb{N}$ such that $N > b'$ we have by lemma A.3.1 that $h^+(N) = bN - h(b)$ and therefore for all $N > b'$ we have that

$$a_N = \int_{\{\frac{d\nu}{d\mu} > b\}} \left[\left(\frac{d\nu}{d\mu} - b \right) N + h(b) \right] d\mu.$$

Now since $\mu\{\frac{d\nu}{d\mu} > b\} > 0$ there exists $\varepsilon_0 > 0$ such that $\mu\{\frac{d\nu}{d\mu} \geq b + \varepsilon_0\} > 0$, and thus

$$a_N \geq \int_{\{\frac{d\nu}{d\mu} \geq b+\varepsilon_0\}} \left[\left(\frac{d\nu}{d\mu} - b \right) N + h(b) \right] d\mu \geq \varepsilon_0 N + h(b) \overset{N\to\infty}{\longrightarrow} +\infty.$$

$b' = \infty$: In this case, by lemma A.3.1 we have that

$$h^+(N) = N \cdot (h'_+)^{-1}(N) - h\big((h'_+)^{-1}(N)\big)$$

and so in this case, since h is non-negative and since $(h'_+)^{-1}(a', b') \subseteq (0, b)$ according to A.3.1, we have that

$$\begin{aligned} a_N &\geq \int_{\{\frac{d\nu}{d\mu} \geq b+\varepsilon_0\}} \left[\left(\frac{d\nu}{d\mu} - (h'_+)^{-1}(N) \right) N + h\big((h'_+)^{-1}(N)\big) \right] d\mu \\ &\geq \varepsilon_0 N + h\big((h'_+)^{-1}(N)\big) \geq \varepsilon_0 N \longrightarrow +\infty. \end{aligned}$$

Another case in which the relative h-entropy $\mathcal{H}_h(\nu|\mu)$ is infinite obviously occurs when

$$\text{(a)} \ \ \mu\left\{ \frac{d\nu}{d\mu} = 0 \right\} > 0 \ \text{ and } \ 0 \notin \mathcal{D}_h \qquad \text{or} \qquad \text{(b)} \ \ \mu\left\{ \frac{d\nu}{d\mu} = b \right\} > 0 \ \text{ and } \ b \notin \mathcal{D}_h$$

and as we will show in this case we have that $\mathcal{H}'_h(\nu|\mu) = +\infty$. Indeed, if (a) holds we consider the sequence $\{f_N\}_{N=1}^{\infty} \subseteq B(M)$ given by $f := -N\mathbb{1}_{\{\frac{d\nu}{d\mu}=0\}}$, for which $h^+(f_N) = h^+(-N)\mathbb{1}\{\frac{d\nu}{d\mu} = 0\}$, and then

$$\begin{aligned} \mathcal{H}'_h(\nu|\mu) &\geq \int \left[f_N \frac{d\nu}{d\mu} - h^+(f_N) \right] d\mu = \int_{\{\frac{d\nu}{d\mu}=0\}} \left[N \frac{d\nu}{d\mu} - h^+(N) \right] d\mu \\ &= -h^+(-N)\mu\{\frac{d\nu}{d\mu} = 0\} \end{aligned} \tag{A.46}$$

But since $0 \notin \mathcal{D}_h$, by the lower semicontinuity of h we have that $a' = -\infty$ and therfore by lemma A.3.1 we have that

$$h^+(-N) = -N \cdot (h'_+)^{-1}(-N) - h\big((h'_+)^{-1}(-N)\big).$$

Therefore, continuing from (A.46) and taking in account that $(h'_+)^{-1}(-N) > 0$ since by lemma A.3.1 we have that $(h'_+)^{-1}(a', b') \subseteq (0, b)$, we get that

$$\frac{\mathcal{H}'_h(\nu|\mu)}{\mu\{\frac{d\nu}{d\mu} = 0\}} \geq N \cdot (h'_+)^{-1}(-N) + h\big((h'_+)^{-1}(-N)\big) \geq h\big((h'_+)^{-1}(-N)\big),$$

and so in order to prove the claim in the case that (a) holds it suffices to prove that $(h'_+)^{-1}(s) \longrightarrow 0$ as $s \downarrow a' = -\infty$, since due to the fact that $0 \notin \mathcal{D}_h$ we have by the lower semicontinuity of h that $\lim_{t\downarrow 0} h(t) = a' = -\infty$. So let $\varepsilon > 0$. Then $h'_+(\varepsilon) > -\infty$ and for all $s < h'_+(\varepsilon)$ we have that $\varepsilon \in \{t \in \mathcal{D}_h | s \leq h'_+(t)\}$, and thus for all $s < h'_+(\varepsilon)$ we obviously have that

$$0 < (h'_+)^{-1}(s) = \inf\{t \in \mathcal{D}_h | s \leq h'_+(t)\} \leq \varepsilon.$$

We consider next the case that (b) holds. Similarly to (a) we consider the sequence $\{f_N\}_{N\in\mathbb{N}} \subseteq B(M)$ given by the formula $f_N = N\mathbb{1}_{\{\frac{d\nu}{d\mu}=b\}}$. In this case, $h^+(f_N) = h^+(N)\mathbb{1}_{\{\frac{d\nu}{d\mu}=b\}}$ and

$$\begin{aligned}
\mathcal{H}'_h(\nu|\mu) &\geq \int \left[f_N \frac{d\nu}{d\mu} - h^+(f_N)\right] d\mu = \int_{\{\frac{d\nu}{d\mu}=b\}} \left[N\frac{d\nu}{d\mu} - h^+(N)\right] d\mu \\
&= [bN - h^+(N)]\mu\{\frac{d\nu}{d\mu} = b\} \\
&= \left[\big(b - (h'_+)^{-1}(N)\big)N + h\big((h'_+)^{-1}(N)\big)\right] \mu\{\frac{d\nu}{d\mu} = b\} \\
&\geq h\big((h'_+)^{-1}(N)\big)\mu\{\frac{d\nu}{d\mu} = b\}.
\end{aligned}$$

But $(h'_+)^{-1}(s) \uparrow b$ as $s \uparrow b' = +\infty$. Indeed, let $\varepsilon > 0$. Then $h'_+(b-\varepsilon) < +\infty$ and for all $s > h'_+(b-\varepsilon)$ we have that $b - \varepsilon \notin \{t \in \mathcal{D}_h | s \leq h'_+(t)\}$ and therefore

$$b > (h'_+)^{-1}(s) > b - \varepsilon,$$

which proves that $\lim_{s\uparrow+\infty}(h'_+)^{-1}(s) = b$. Here, again by the lower semicontinuity of h and the fact that $b \notin \mathcal{D}_h$ we have that $\lim_{t\uparrow b} h(t) = +\infty$. This proves the claim in the case that (b) holds.

So it remains to show inequality $\mathcal{H}''_h(\nu|\mu) \leq \mathcal{H}_h(\nu|\mu)$ in the case that $\nu \ll \mu$

$$\mu\left\{\frac{d\nu}{d\mu} \notin \overline{\mathcal{D}_h}\right\} = 0, \tag{A.47}$$

$$\mu\left\{\frac{d\nu}{d\mu} = 0\right\} = 0 \text{ or } 0 \in \mathcal{D}_h \qquad \text{and} \qquad \mu\left\{\frac{d\nu}{d\mu} = b\right\} = 0 \text{ or } b \in \mathcal{D}_h \tag{A.48}$$

Due to (A.47), the function $f : M \longrightarrow [a', b'] \subseteq [-\infty, +\infty]$ given by

$$f = \begin{cases} a', & \text{in } \{\frac{d\nu}{d\mu} = 0\} \\ h'_+\big(\frac{d\nu}{d\mu}\big), & \text{in } \{\frac{d\nu}{d\mu} \in \mathcal{D}^o_h\} \\ b', & \text{in } \{\frac{d\nu}{d\mu} = b\} \end{cases}, \tag{A.49}$$

is well defined on M μ-a.s.. In the set $\{\frac{d\nu}{d\mu}\in\mathcal{D}_h^o\}$, f is a supporting point of h at $\frac{d\nu}{d\mu}$ and therefore $h^+(f)=f\cdot\frac{d\nu}{d\mu}-h\big(\frac{d\nu}{d\mu}\big)$ and takes μ-$\sigma.\beta.$ real values. So by (A.48) the only case in which f may not take a.s. real values is when

$$\text{(a)}\quad 0\in\mathcal{D}_h\ \text{ and }\ a'=-\infty\qquad\text{or}\qquad\text{(b)}\quad b\in\mathcal{D}_h\ \text{ and }\ b'=+\infty \tag{A.50}$$

Consequently, if (A.50) does not hold, then $\mu\{|f|<\infty\}=1$, and the composition $h^+\circ f$ is well defined and given by the formula

$$h^+(f)=\begin{cases}h^+(a'), & \text{in }\ \{\frac{d\nu}{d\mu}=0\}\\ f\cdot\frac{d\nu}{d\mu}-h\big(\frac{d\nu}{d\mu}\big), & \text{in }\ \{\frac{d\nu}{d\mu}\in\mathcal{D}_h^o\}\\ h^+(b'), & \text{in }\ \{\frac{d\nu}{d\mu}=b\}\end{cases}.$$

But if it does not hold that $0\in\mathcal{D}_h$ and $a'=-\infty$, then either $0\notin\mathcal{D}_h$ in which case the first branch of (A.49) corresponds to a set of μ-measure 0 by (A.48), or either $a'\in\mathbb{R}$, in which case by the formula for right semi-Legendre transform of lemma A.3.1 we obviously have that in the set $\{\frac{d\nu}{d\mu}=0\}$,

$$h^+(f)=h^+(a')=-h(0)=f\cdot 0-h(0)=f\cdot\frac{d\nu}{d\mu}-h\big(\frac{d\nu}{d\mu}\big).$$

Similarly, if it does not hold that $b\in\mathcal{D}_h$ and $b'=+\infty$, then either $b\notin\mathcal{D}_h$ in which case the third branch of (A.49) corresponds to a set of μ-measure 0 by (A.48), or either $b'\in\mathbb{R}$, in which case by the formula for right semi-Legendre transform we have that in the set $\{\frac{d\nu}{d\mu}=b\}$,

$$h^+(f)=h^+(b')=b'b-h(b)=f\cdot\frac{d\nu}{d\mu}-h\Big(\frac{d\nu}{d\mu}\Big).$$

Therefore if (A.50) does not hold then in any case we have by (A.47) and (A.48) that $f,h^+(f)$ take real values, the composition $h^+(f)=h^+\circ f$ is well defined and we have that

$$h_+(f)=f\cdot\frac{d\nu}{d\mu}-h\left(\frac{d\nu}{d\mu}\right),\quad \mu-a.s., \tag{A.51}$$

and therefore

$$\mathcal{H}_h(\nu|\mu)=\int h\left(\frac{d\nu}{d\mu}\right)d\mu=\int\left[f\cdot\frac{d\nu}{d\mu}-h_+(f)\right]d\mu. \tag{A.52}$$

So, if f is bounded, as is for instance the case when $-\infty<a'<b'<+\infty$, then

$$\mathcal{H}_h(\nu|\mu)\le\mathcal{H}_h'(\nu|\mu).$$

In the case that f is not necessarily bounded and (A.50) does not hold it suffices to approximate f by an appropriate sequence $\{f_N\}_{N=1}^\infty\subseteq B(M)$ such that

$$\int\left[f_N\frac{d\nu}{d\mu}-h^+(f_N)\right]d\mu\overset{N\to+\infty}{\longrightarrow}\int\left[f\cdot\frac{d\nu}{d\mu}-h^+(f)\right]d\mu=\mathcal{H}_h(\nu|\mu).$$

So we set

$$f_N := f\mathbb{1}_{\{|f|\le N\}}, \quad N\in\mathbb{N}.$$

Then according to our convention that $h^+(0)=0$ we have that

$$h^+(f_N) = h^+(f)\mathbb{1}_{\{|f|\le N\}}$$

and since (A.50) does not hold we have by (A.51) that

$$\int\left[f_N\frac{d\nu}{d\mu}-h^+(f_N)\right]d\mu = \int_{\{|f|\le N\}}\left[f\frac{d\nu}{d\mu}-h^+(f)\right]d\mu = \int_{\{|f|\le N\}} h\left(\frac{d\nu}{d\mu}\right)d\mu.$$

But $\mu\{|f|<+\infty\}=1$ since (A.50) does not hold and thus since $h\ge 0$ we have by the monotone convergence theorem that

$$\lim_{n\to+\infty}\int\left[f\cdot\frac{d\nu}{d\mu}-h^+(f)\right]d\mu = \int_{\{|f|<+\infty\}}\left[f\cdot\frac{d\nu}{d\mu}-h^+(f)\right]d\mu = \mathcal{H}_h(\nu|\mu),$$

as required.

So it remains to investigate what happens when (A.50) holds. We consider first the case that both (A.50) and (A.50). We define then the sequence $\{f_N\}_{N\in\mathbb{N}}\subseteq B(M)$ by the formula

$$f_N = \begin{cases} h'_+\left(\frac{1}{N}\right), & \text{if } \{0\le \frac{d\nu}{d\mu}\le\frac{1}{N}\} \\ h'_+\big(\frac{d\nu}{d\mu}\big), & \text{if } \{\frac{1}{N}\le\frac{d\nu}{d\mu}\le b-\frac{1}{N}\} \\ h'_+\left(b-\frac{1}{N}\right), & \text{if } \{b-\frac{1}{N}\le\frac{d\nu}{d\mu}\le b\} \end{cases}.$$

and then for all $N\in\mathbb{N}$ we have that

$$f_N\cdot\frac{d\nu}{d\mu}-h^+(f_N) = \begin{cases} \big(\frac{d\nu}{d\mu}-\frac{1}{N}\big)h'_+\left(\frac{1}{N}\right)+h\left(\frac{1}{N}\right), & 0\le\frac{d\nu}{d\mu}\le\frac{1}{N} \\ h\big(\frac{d\nu}{d\mu}\big), & \frac{1}{N}\le\frac{d\nu}{d\mu}\le b-\frac{1}{N} \\ \big(\frac{d\nu}{d\mu}-b+\frac{1}{N}\big)h'_+\left(b-\frac{1}{N}\right)+h\left(b-\frac{1}{N}\right), & b-\frac{1}{N}\le\frac{d\nu}{d\mu}\le b \end{cases}.$$

By the formula above it follows that the sequence $\{g_N\}:=\{f_N\cdot\frac{d\nu}{d\mu}-h^+(f_N)\}_{N=1}^{\infty}$ converges uniformly to $h(\frac{d\nu}{d\mu})$ in $\{\frac{d\nu}{d\mu}\in\overline{\mathcal{D}_h}\}$. Indeed, we note first that $\{g_N\}$ is eventually identically equal to $h(\frac{d\nu}{d\mu})$ in any subset of $\{\frac{d\nu}{d\mu}\in\overline{\mathcal{D}_h}\}$ of the form $\{\frac{d\nu}{d\mu}\in[a_0,b_0]\}$, where $[a_0,b_0]\subseteq(0,b)$. Also since $-a'=b'=+\infty$ it follows that

$$\exists\,\delta_0\in(0,b/2)\quad\text{such that}\quad h'_+|_{[0,\delta_0]}<0<h'_+|_{[b-\delta_0,b]}. \tag{A.53}$$

But then for all $\in\mathbb{N}$ such that $\frac{1}{N}\le\delta_0$ we have that $h'_+(\frac{1}{N})<0$, and so since obviously

$$-\frac{1}{N}\le\left(\frac{d\nu}{d\mu}-\frac{1}{N}\right)\Big|_{\{\frac{d\nu}{d\mu}\le\frac{1}{N}\}}\le 0$$

it follows that

$$\begin{aligned} h\big(\frac{1}{N}\big) &\le g_N|_{\{\frac{d\nu}{d\mu}\le\frac{1}{N}\}} = \left(\frac{d\nu}{d\mu}-\frac{1}{N}\right)\Big|_{\{\frac{d\nu}{d\mu}\le\frac{1}{N}\}}h'_+\big(\frac{1}{N}\big)+h\big(\frac{1}{N}\big) \\ &\le -\frac{1}{N}h'_+\big(\frac{1}{N}\big)+h\big(\frac{1}{N}\big). \end{aligned} \tag{A.54}$$

But $\lim_{t\to 0} th'_+(t) = 0$. Indeed, since h is convex, it is Lipschitz continuous, and thus absolutely continuous, in any interval of the form $[\varepsilon, b-\varepsilon]$, $0 < \varepsilon < b/2$. So, for all $0 < \varepsilon < \delta_0$ we have that $h(\delta_0) = h(\varepsilon) + \int_\varepsilon^{\delta_0} h'_+(t)dt$ and thus

$$\lim_{\varepsilon\to 0} \int_\varepsilon^{\delta_0} h'_+(t)ds = h(\delta_0) - h(0) \in (-\infty, 0).$$

But h is negative in the set $(0, \delta_0]$ and therefore the family of functions $\{\mathbb{1}_{(\varepsilon,\delta_0]} h\}_{0<\varepsilon<\delta_0}$ is a family of negative functions decreasing to $\mathbb{1}_{(0,\delta_0]} h$ as $\varepsilon \downarrow 0$, and by the monotone convergence theorem it follows that

$$\int_0^{\delta_0} h'_+(t)dt = h(\delta_0) - h(0) \in (-\infty, 0).$$

But then, if it were not true that $\lim_{t\downarrow 0} th_+(t) = 0$, since h'_+ is negative in $(0, \delta_0]$ there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists $t_\delta \in (0, \delta)$ such that $t_\delta h'_+(t_\delta) \le -\varepsilon$, and so there exists a sequence $\{t_n\}_{n=0}^\infty \subseteq (0, \delta_0]$ such that $t_0 = \delta_0$, $0 < t_n \le \frac{1}{2} t_{n-1}$ and $t_n h'_+(t_n) \le -\varepsilon$ for all $n \in \mathbb{N}$, which leads to a contradiction since then

$$\begin{aligned} -\infty &< \int_0^{\delta_0} h'_+(t)dt \le \int_0^{\delta_0} \sum_{n=1}^\infty h'_+(t_{n-1}) \mathbb{1}_{[t_n,t_{n-1})}(t)dt \\ &\le -\varepsilon \sum_{n=1}^\infty \frac{1}{t_{n-1}}(t_{n-1} - t_n) = -\varepsilon \sum_{n=1}^\infty \left(1 - \frac{t_n}{t_{n-1}}\right) \le -\varepsilon \sum_{n=1}^\infty \frac{1}{2} = -\infty. \end{aligned}$$

So we indeed have that $\lim_{t\downarrow 0} th'_+(t) = 0$ and so by (A.54) we have that for all $N \ge \frac{1}{\delta_0}$,

$$\begin{aligned} \left\| g_N|_{\{\frac{d\nu}{d\mu} \le \frac{1}{N}\}} - h\Big(\frac{1}{N}\Big) \right\|_u &= \sup_{x:\, 0 \le \frac{d\nu}{d\mu}(x) \le \frac{1}{N}} \left(g_N(x) - h\Big(\frac{1}{N}\Big)\right) \\ &\le -\frac{1}{N} h'_+\Big(\frac{1}{N}\Big) \overset{n\to\infty}{\longrightarrow} 0. \end{aligned} \tag{A.55}$$

Similarly, for the right endpoint b, for all $N \in \mathbb{N}$ we have that

$$0 \le \left(\frac{d\nu}{d\mu} - b + \frac{1}{N}\right)\Big|_{\{b - \frac{1}{N} \le \frac{d\nu}{d\mu} \le b\}} \le \frac{1}{N}$$

and therefore for all $N \ge \frac{1}{\delta_0}$, in which case $h'_+\big(b - \frac{1}{N}\big) > 0$, we have in the set $\{b - \frac{1}{N} \le \frac{d\nu}{d\mu} \le b\}$ that

$$h\big(b - \frac{1}{N}\big) \le g_N \le \frac{1}{N} h'_+\big(b - \frac{1}{N}\big) + h\big(b - \frac{1}{N}\big)$$

and thus

$$\left\| g_N|_{\{b - \frac{1}{N} \le \frac{d\nu}{d\mu} \le b\}} - h\Big(b - \frac{1}{N}\Big) \right\|_u \le \frac{1}{N} h'_+\big(b - \frac{1}{N}\big). \tag{A.56}$$

But as before, we also have here that

$$\lim_{N\to+\infty} \frac{1}{N} h'_+\big(b - \frac{1}{N}\big) = 0.$$

Indeed, first it is obvious that to prove that limit in (A.56) we can equivalently prove that $\lim_{t\uparrow b}(b-t)h'_+(t)=0$. The calculation of this limit is similar to the one for the left endpoint of $\mathcal{D}_h$. In particular, as before it follows by the monotone convergence theorem that

$$\int_{b-\delta_0}^{b} h'_+(t)dt = \lim_{\varepsilon\downarrow 0}\int_{b-\delta_0}^{b-\varepsilon} h'_+(t)dt = h(b)-h(b-\delta_0)\in(0,\infty).$$

So, if it is not true that this limit exists and is equal to 0, then since $h>0$ in $[b-\delta_0,b)$, it follows that there exists a sequence $\{t_n\}_{n=0}^{\infty}\subseteq[b-\delta_0,b)$ such that

$$t_0=b-\delta_0,\qquad (b-t_n)\leq\frac{1}{2}(b-t_{n-1}),\quad (b-t_n)h'_+(t_n)\geq\varepsilon,\quad \forall n\in\mathbb{N}.$$

But this leads to a contradiction, since obviously for all $n\in\mathbb{N}$ we have that $(b-t_n)\leq \frac{1}{2}(b-t_{n-1})$ iff $b-t_n\leq t_n-t_{n-1}$, and thus

$$\begin{aligned}+\infty \quad &> \quad \int_{b-\delta_0}^{b} h'_+(t)dt \geq \int_{b-\delta_0}^{b}\sum_{n=1}^{\infty} h'_+(t_{n-1})\mathbb{1}_{[t_{n-1},t_n)}(t)dt\\ &\geq \quad \varepsilon\sum_{n=1}^{\infty}\frac{t_n-t_{n-1}}{b-t_{n-1}}=\varepsilon\sum_{n=1}^{\infty}1=+\infty.\end{aligned}$$

Therefore $\lim_{t\uparrow b}(b-t)h'_+(t)=0$ and so by (A.56) it follows that

$$\left\|g_N|_{\{b-\frac{1}{N}\leq\frac{d\nu}{d\mu}\leq b\}}-h\Big(b-\frac{1}{N}\Big)\right\|_u=\sup_{\{b-\frac{1}{N}\leq\frac{d\nu}{d\mu}\leq b\}}\left|g_N-h\big(b-\frac{1}{N}\big)\right|\longrightarrow 0. \quad (A.57)$$

By (A.55) and (A.57) we easily get the uniform convergence $g_n\longrightarrow h$ in the set $\{0\leq\frac{d\nu}{d\mu}\leq b\}$. Indeed, let $\varepsilon>0$. Since h is continuous in $[0,b]$ we can assume that the number $\delta_0>0$ in (A.53) has been chosen so that

$$s,t\in(0,\delta_0)\quad\Longrightarrow\quad |h(t)-h(s)|<\frac{\varepsilon}{2} \quad (A.58)$$

and

$$s,t\in(b-\delta_0,b)\quad\Longrightarrow\quad |h(t)-h(s)|<\frac{\varepsilon}{2}. \quad (A.59)$$

Next, by (A.55) and (A.57) there exists $N_0\in\mathbb{N}$, $_0\geq\frac{1}{\delta_0}$, such that

$$N\geq N_0\quad\Longrightarrow\quad\left\|g_N|_{\{\frac{d\nu}{d\mu}\leq\frac{1}{N}\}}-h\Big(\frac{1}{N}\Big)\right\|_u<\frac{\varepsilon}{2}. \quad (A.60)$$

and

$$N\geq N_0\quad\Longrightarrow\quad\left\|g_N|_{\{b-\frac{1}{N}\leq\frac{d\nu}{d\mu}\leq b\}}-h\Big(b-\frac{1}{N}\Big)\right\|_u<\frac{\varepsilon}{2} \quad (A.61)$$

But then for all $N\geq N_0$ we have that

$$g_N|_{[\frac{1}{N},b-\frac{1}{N}]}\equiv h\left(\frac{d\nu}{d\mu}\right)\Big|_{[\frac{1}{N},b-\frac{1}{N}]}, \quad (A.62)$$

while for all $x \in \{0 \leq \frac{d\nu}{d\mu} \leq \frac{1}{N}\}$ we have by (A.58) and (A.60) that

$$\left|g_N(x) - h\left(\frac{d\nu}{d\mu}(x)\right)\right| \leq \left|g_N(x) - h\left(\frac{1}{N}\right)\right| + \left|h\left(\frac{1}{N}\right) - h\left(\frac{d\nu}{d\mu}(x)\right)\right| < \varepsilon$$

and it similarly follows by (A.59) and (A.61) that

$$\left|g_N - h\left(\frac{d\nu}{d\mu}\right)\right| < \varepsilon$$

in the set $\{b - \frac{1}{N} \leq \frac{d\nu}{d\mu} \leq b\}$. Consequently, by the two inequalities above and (A.62) it follows that

$$\sup_{\{0 \leq \frac{d\nu}{d\mu} \leq b\}} \left|g_N - h\left(\frac{d\nu}{d\mu}\right)\right| < \varepsilon$$

for all $N \geq N_0$ which proves the uniform convergence $g_N \longrightarrow h$ in $\{0 \leq \frac{d\nu}{d\nu}\}$.

But now, since $\{g_N\}$ converges uniformly to h on the set $\{0 \leq \frac{d\nu}{d\mu} \leq b\}$, which set by hypothesis supports the measure μ, it follows that

$$\int g_N d\mu = \int \left[f_N \cdot \frac{d\nu}{d\mu} - h^+(f_N)\right] d\mu \longrightarrow \int h\left(\frac{d\nu}{d\mu}\right) d\mu = \mathcal{H}_h(\nu|\mu),$$

which proves the required inequality in this case.

We investigate finally the case in which (A.50) is satisfied but (A.50) is not, since the other case that's left, i.e. the one in which (A.50) is satisfied but not (A.50) is proved similarly.

So let's suppose (A.50) holds and that (A.50) doesn't. We define then the sequence of functions $\{\widetilde{f}_N\}$ by the formula

$$\widetilde{f}_N = \begin{cases} h'_+\left(\frac{d\nu}{d\mu}\right), & \{0 \leq \frac{d\nu}{d\mu} \leq b - \frac{1}{N}\} \\ h'_+\left(b - \frac{1}{N}\right), & \{b - \frac{1}{N} \leq \frac{d\nu}{d\mu} \leq b\} \end{cases}.$$

By the proof in the case that (A.47) and (A.48) are satisfied but (A.50) is not, we know that $\widetilde{f}_N$ takes μ-a.s. finite values and the composition

$$h^+(\widetilde{f}_N) = \begin{cases} h'_+\left(\frac{d\nu}{d\mu}\right) \cdot \frac{d\nu}{d\mu} - h\left(\frac{d\nu}{d\mu}\right), & \{0 \leq \frac{d\nu}{d\mu} \leq b - \frac{1}{N}\} \\ \left(b - \frac{1}{N}\right)h'_+(b - \frac{1}{N}) - h\left(b - \frac{1}{N}\right), & \{b - \frac{1}{N} \leq \frac{d\nu}{d\mu} \leq b\} \end{cases}$$

is well defined. So,

$$\begin{aligned} \widetilde{g}_N &:= \widetilde{f}_N \frac{d\nu}{d\mu} - h^+(\widetilde{f}_N) \\ &= \begin{cases} h\left(\frac{d\nu}{d\mu}\right), & \{0 \leq \frac{d\nu}{d\mu} \leq b - \frac{1}{N}\} \\ \left(\frac{d\nu}{d\mu} - b + \frac{1}{N}\right)h'_+(b - \frac{1}{N}) + h\left(b - \frac{1}{N}\right), & \{b - \frac{1}{N} \leq \frac{d\nu}{d\mu} \leq b\} \end{cases}, \end{aligned}$$

and since (A.50) holds it follows by (A.61) that $\{\widetilde{g}_N\}$ converges uniformly to $h\left(\frac{d\nu}{d\mu}\right)$ in the set $\{0 \leq \frac{d\nu}{d\mu} \leq b\}$ as $N \to \infty$, and therefore

$$\lim_{N \to +\infty} \int \left[\widetilde{f}_N \frac{d\nu}{d\mu} - h^+(\widetilde{f}_N)\right] d\mu = \mathcal{H}_h(\mu|\nu).$$

Now, if $a' \neq -\infty$, then the $\widetilde{f}_N$'s are bounded and the above limit proves the claim in this case. Suppose, on the other hand that $a' = -\infty$. Since $h \geq 0$ and $b' = +\infty$ it follows that $h(b) > 0$. By the continuity of h in b we choose first $\delta > 0$ such that

$$b - \delta < t \leq b \quad \Longrightarrow \quad |h(b) - h(t)| < \frac{h(b)}{2}.$$

Next, given $\varepsilon > 0$ we choose $N_\varepsilon \in \mathbb{N}$, $N_\varepsilon > \frac{1}{\delta}$ such that

$$\left\| \left(\widetilde{g}_{N_\varepsilon} - h\Big(\frac{d\nu}{d\mu}\Big) \right) \Big|_{\{0 \leq \frac{d\nu}{d\mu} \leq b\}} \right\|_u < \frac{h(b)}{2}$$

and

$$\int \widetilde{g}_{N_\varepsilon} d\mu \geq [\mathcal{H}_h(\nu|\mu) - \varepsilon] \wedge \frac{1}{\varepsilon}. \tag{A.63}$$

In the set $\{0 \leq \frac{d\nu}{d\mu} \leq b - \frac{1}{N_\varepsilon}\}$ we obviously have that $\widetilde{g}_{N_\varepsilon} = h\big(\frac{d\nu}{d\mu}\big) \geq 0$ while if $[\delta, b-\delta] \subseteq [\frac{1}{N_\varepsilon}, b - \frac{1}{N_\varepsilon}]$, for all $x \in \{b - \frac{1}{N_\varepsilon} \leq \frac{d\nu}{d\mu} \leq b\}$ we have that

$$\begin{aligned} \widetilde{g}_{N_\varepsilon}(x) &= h(b) + \widetilde{g}_{N_\varepsilon}(x) - h\Big(\frac{d\nu}{d\mu}(x)\Big) - h(b) + h\Big(\frac{d\nu}{d\mu}(x)\Big) \\ &> h(b) - \frac{h(b)}{2} - \frac{h(b)}{2} = 0, \end{aligned}$$

and therefore $\widetilde{g}_{N_\varepsilon}$ is non-negative in the set $\{0 \leq \frac{d\nu}{d\mu} \leq b\}$.

So, if we set $f_N := \widetilde{f}_{N_\varepsilon} \mathbb{1}_{\{\widetilde{f}_{N_\varepsilon} > -N\}} \in B(M)$ for all $N \in \mathbb{N}$ we have by our additional assumption $h^+(0) = 0$ that

$$h^+(f_N) = h^+(\widetilde{f}_{N_\varepsilon}) \mathbb{1}_{\{\widetilde{f}_{N_\varepsilon} > -N\}}$$

and therefore

$$g_N := f_N \frac{d\nu}{d\mu} - h^+(f_N) = \left(\widetilde{f}_{N_\varepsilon} \frac{d\nu}{d\mu} - h^+(\widetilde{f}_{N_\varepsilon}) \right) \mathbb{1}_{\{\widetilde{f}_{N_\varepsilon} > -N\}} = \widetilde{g}_{N_\varepsilon} \mathbb{1}_{\{\widetilde{f}_{N_\varepsilon} > -N\}} \geq 0.$$

for all $N \in \mathbb{N}$. Now since (A.50a) does not hold, in the case $a' = -\infty$ that we are investigating, we have by (A.48) that necessarily $\mu\{\frac{d\nu}{d\mu} = 0\} = 0$ and thus $\mu\{\widetilde{f}_{N_\varepsilon} > -\infty\} = 1$. Therefore $\{g_N\}$ is pointwise increasing to $\widetilde{g}_{N_\varepsilon}$ μ-a.s. and therefore by the monotone convergence theorem we have that

$$\lim_{N \to +\infty} \int g_N d\mu \longrightarrow \int \widetilde{g}_{N_\varepsilon} d\mu.$$

Therefore there exists $N'_\varepsilon \in \mathbb{N}$ such that

$$\int g_{N'_\varepsilon} d\mu \geq \left(\int \widetilde{g}_{N_\varepsilon} d\mu - \varepsilon \right).$$

But then for the function $f := f_{N'_\varepsilon} \in B(M)$ we have by (A.63) that

$$\int \left[f \frac{d\nu}{d\mu} - h^+(f) \right] d\mu = \int g_{N'_\varepsilon} d\mu \geq [\mathcal{H}_h(\nu|\mu) - 2\varepsilon] \wedge \Big(\frac{1}{\varepsilon} - \varepsilon\Big),$$

which since $\varepsilon > 0$ was arbitrary proves the variational characterization $\mathcal{H}_h \equiv \mathcal{H}'_h$ of the generalized relative entropy functionals for bounded functions.

We prove next that in the case that $\mathcal{D}_{h^+} = \mathbb{R}$, equality (A.38) holds, i.e. that we can restrict the supremum in (A.37) on bounded and continuous functions $f \in BC(M)$. First, by the formula of the right semi-Legendre transform

$$\mathcal{D}_{h^+} = \begin{cases} \mathbb{R}, & \text{if } b \in \mathcal{D}_h,\ b' < \infty \\ (-\infty, b'], & \text{if } b = +\infty, b' < +\infty,\ \lim_{t\uparrow+\infty}(tb' - h(t)) < +\infty \\ (-\infty, b'), & \text{otherwise} \end{cases}$$

and it is obvious that $\mathcal{D}_{h^+} = \mathbb{R}$ iff $b' = +\infty$ or $b \in \mathcal{D}_h$. To prove the claim it suffices to prove that given distributions $\mu, \nu \in \mathbb{P}M$, a function $f \in B(M)$ and $\varepsilon > 0$ there exists $\widetilde{f} \in BC(M)$ such that

$$\int \widetilde{f} d\nu - \int h^+(\widetilde{f}) d\mu \geq \int f d\nu - \int h^+(f) d\mu - \varepsilon. \tag{A.64}$$

So let $\mu, \nu \in \mathbb{P}M$ and $f \in B(M)$ and let $\{f_n\}$ be a sequence satisfying (A.36) of the double Lusin lemma. Then by the definition of $\{f_n\}$ we have that

$$\int f_n d\nu \longrightarrow \int f d\nu$$

and so if we show that

$$\lim_{n\uparrow+\infty} \int h^+(f_n) d\mu \longrightarrow \int h^+(f) d\mu \tag{A.65}$$

then by choosing $\widetilde{f} = f_{n_0}$ for some n_0 large enough, (A.64) is satisfied. But indeed, h^+ is continuous as a convex function with proper domain $\mathcal{D}_{h^+} = \mathbb{R}$, and so obviously $h^+(f_n) \longrightarrow h^+(f)$ pointwise μ-a.s. But h^+ is bounded on compact intervals as a continuous functions and so since $|f_n(x)| \leq \|f\|_u$ for all $n \in \mathbb{N}$ and all $x \in M$ we have that

$$\|h^+(f_n)\|_u \leq \sup_{-\|f\|_u \leq t \leq \|f\|_u} |h^+(t)| < +\infty$$

for all $n \in \mathbb{N}$. Therefore (A.65) follows by the bounded convergence theorem. This proves (A.38) in the case that $\mathcal{D}_{h^+} = +\infty$.

We prove next that for measures $\nu, \mu \in \mathbb{P}M$ such that $\nu \ll \mu$, (A.38) continues to hold even when $\mathcal{D}_{h^+} \neq \mathbb{R}$. We note first that since $(-\infty, b') \subseteq \mathcal{D}_{h^+}$, for all $f \in B(M)$ the function $h^+(f)$ is bounded below and so the integral $\int h^+(f) d\mu$ is defined for all $f \in B(M)$ and

$$h^+(f) \notin L^1(\mu) \quad \Longleftrightarrow \quad \int h^+(f) d\mu = +\infty.$$

Consequently, functions $f \in B(M)$ such that $h^+(f) \notin L^1(\mu)$ do not contribute in the definition of $\mathcal{H}'_h$ and $\mathcal{H}''_h$, since they give $\int f d\nu - \int h^+(f) d\mu = -\infty$ and obviously for functions $f \in B(M)$ such that $h^+(f) \in L^1(\mu)$ we have that

$$\mu\{f \leq b'\} \geq \mu\{f \in \mathcal{D}_{h^+}\} = \mu\{h^+(f) < +\infty\} = 1. \tag{A.66}$$

We separate cases on whether $\mathcal{D}_{h^+} = (-\infty, b']$ or $(-\infty, b')$, $b' < +\infty$.
$\mathcal{D}_{h^+} = (-\infty, b']$: Since $b' \in \mathcal{D}_{h^+}$ we have that $b' < +\infty$ and $h^+(b') < +\infty$, and by (A.66) we have that

$$-\|f\|_u \leq f \leq b', \quad \mu-\sigma.\beta..$$

Let $\{f_n\} \subseteq BC(M)$ be a sequence satisfying (A.36) of the double Lusin lemma for the function f and the measures $\mu, \nu \in \mathbb{P}M$. We set $\widetilde{f}_n := f_n \wedge \|f\|_{L^\infty(\mu)}$. By the definition of the L^∞-norm we have that $\mu\{|f| \geq \|f\|_{L^\infty(\mu)}\} = 0$ and since $\nu \ll \mu$ it follows that $\nu\{|f| \geq \|f\|_{L^\infty(\mu)}\} = 0$, and thus

$$\|f\|_{L^\infty(\nu)} = \inf\left\{C \geq 0 \,\middle|\, \nu\{|f| \geq C\} = 0\right\} \leq \|f\|_{L^\infty(\mu)}. \tag{A.67}$$

Then,

$$-\|f\|_u \leq \widetilde{f}_n(x) \leq \|f\|_{L^\infty(\mu)} \leq b'$$

for all $x \in M$. Also, by the definition of $\{f_n\}$ we have that $f_n \longrightarrow f$ ν-a.s. and therefore

$$f_n \wedge \|f\|_{L^\infty(\nu)} \longrightarrow f \wedge \|f\|_{L^\infty(\nu)} = f, \quad \nu\text{-}\sigma.\beta..$$

It follows by the bounded convergence theorem that

$$\lim_{n\to+\infty} \int f_n \wedge \|f\|_{L^\infty(\nu)} d\nu = \int f d\nu.$$

But by (A.67) we have that

$$\int f_n \wedge \|f\|_{L^\infty(\nu)} d\nu \leq \int \widetilde{f}_n d\nu \leq \int f_n d\nu$$

for all $n \in \mathbb{N}$ from where it follows that

$$\lim_{n\to+\infty} \int \widetilde{f}_n d\nu = \int f d\nu.$$

On the other hand,h^+ is continuous and bounded on $[-\|f\|_u, b']$ and therefore since $\widetilde{f}_n \longrightarrow f$ μ-a.s. and

$$\sup_{n\in\mathbb{N}} \|h^+(\widetilde{f}_n)\|_u \leq \sup_{-\|f\|_u \leq t \leq b'} |h^+(t)| < +\infty$$

it follows by the bounded convergence theorem that $\int h^+(\widetilde{f}_n) d\mu \longrightarrow \int h^+(f) d\mu$.
$\mathcal{D}_{h^+} = (-\infty, b') \neq \mathbb{R}$: By (A.66) we have that $-\|f\|_u \leq f < b'$ μ-$\sigma.\beta..$ Since $b' \notin \mathcal{D}_{h^+}$ we have that $\lim_{t\uparrow b'} h^+(t) = +\infty$. Therefore since h is convex there exists $\delta_0 > 0$ such that h^+ positive and increasing in $[b'-\delta_0, b')$. So we set $f_\delta := f \wedge (b'-\delta)$ for all $\delta \in (0, \delta_0)$. Since $\nu \ll \mu$ we obviously have that $\|f\|_{L^\infty(\nu)} \leq \|f\|_{L^\infty(\mu)} \leq b'$ and therefore $f_\delta \uparrow f$ in $L^\infty(\nu)$ and in $L^\infty(\mu)$ as $\delta \downarrow 0$. In particular $\int f_\delta d\nu \longrightarrow \int f d\nu$ as $\delta \downarrow 0$. Furthermore, for all $\delta \in (0, \delta_0)$ we have that

$$\int h^+(f_\delta) d\mu = \int_{\{f < b'-\delta_0\}} h^+(f) d\mu + \int_{\{b'-\delta_0 \leq f < b'\}} h^+(f_\delta) d\mu$$

since obviously $f = f_\delta$ in the set $\{f < b' - \delta_0\}$ for all $\delta \in (0, \delta_0)$. But in the set $[b' - \delta_0, b')$ the function h^+ is positive and increasing and so since $\{f_\delta\}_{0<\delta<\delta_0}$ increases to f μ-a.s. as δ decreases to 0, it follows that in the set $\{b' - \delta_0 \leq f < b'\}$, the sequence $\{h^+(f_\delta)\}_{0<\delta<\delta_0}$ is increasing to $h^+(f)$ μ-a.s. as δ decreases to 0. So by the monotone convergence theorem we have that

$$\begin{aligned}
\lim_{\delta \downarrow 0} \int h^+(f_\delta) d\mu &= \int_{\{f<b'-\delta_0\}} h^+(f) d\mu + \lim_{\delta \downarrow 0} \int_{\{b'-\delta_0 \leq f < b'\}} h^+(f_\delta) d\mu \\
&= \int_{\{f<b'-\delta_0\}} h^+(f) d\mu + \int_{\{b'-\delta_0 \leq f < b'\}} h^+(f) d\mu \\
&= \int h^+(f) d\mu.
\end{aligned}$$

Therefore, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int f_\delta d\nu - \int h^+(f_\delta) d\mu \geq \int f d\mu - \int h^+(f) d\mu - \varepsilon$$

Byt he double Lusin lemma now, there exists a sequence $\{f_n\}_{n=1}^\infty \subseteq BC(M)$ such that

$$-\|f_\delta\|_u \leq f_n \leq \sup_{x \in M} f_\delta(x) \leq b' - \delta$$

and such that $f_n \longrightarrow f_\delta$ in $L^1(\mu)$ and $L^1(\nu)$ and μ-a.s. and ν-a.s.. Thus, since h^+ is continuous and bounded in $[-\|f_\delta\|_u, b' - \delta]$ it follows by the bounded convergence theorem that

$$\lim_{n \to +\infty} \left\{ \int f_n d\nu - \int h^+(f_n) d\mu \right\} = \int f_\delta d\nu - \int h^+(f_\delta) d\mu.$$

Consequently, given $\varepsilon > 0$ we can find a continuous function $\widetilde{f} \in BC(M)$ such that

$$\int \widetilde{f} d\nu - \int h^+(\widetilde{f}) d\mu \geq \int f d\nu - \int h^+(f) d\mu - \varepsilon,$$

which completes the proof of (A.38) in the case that $\nu \ll \mu$.

We suppose finally that $\mathcal{D}_{h^+} \neq \mathbb{R}$. Then obviously $b = +\infty$ and (A.39) holds. Let $\mu \in \mathbb{P}M$ be such that $\text{supp}\mu = M$. It remains to prove that for all $\nu \in \mathbb{P}M$ (A.40) is satisfied. Here we consider separate cases in whether $\mathcal{D}_{h^+} = (-\infty, b']$ or $(-\infty, b')$.
$\mathcal{D}_{h^+} = (-\infty, b']$: As we have seen, function $f \in BC(M)$ such that $f \notin L^1(\mu)$ do not contribute to the supremum in the definition of $\mathcal{H}_h''$ and so according to (A.66) we can restrict the supremum in (A.40) to functions $f \in BC(M)$ such that $f \leq b' < +\infty$ μ $\sigma.\beta.$. But for any $f \in BC(M)$ such that $f \leq b' < +\infty$ μ a.s. we have by the continuity of f that $f \leq b'$ everywhere on $\text{supp}\mu = M$ and therefore we can write that

$$\mathcal{H}_h''(\nu|\mu) = \sup_{f \in BC(M),\, f \leqq b'} \left\{ \int f d\nu - \int h^+(f) d\mu \right\}.$$

Therefore, if $\nu = \nu_{ac} + \nu_s$ is the Radon-Nikodym decomposition of ν with respect to μ with $\nu_{ac} \ll \mu$ and $\nu_s \perp \mu$ then

$$\begin{aligned}
\mathcal{H}_h''(\nu|\mu) &= \sup_{f \in BC(M),\, f \leqq b'} \left\{ \int f d\nu_{ac} + \int f d\nu_s - \int h^+(f) d\mu \right\} \\
&\overset{(*)}{\leq} \sup_{f \in BC(M),\, f \leqq b'} \left\{ \int f d\nu_{ac} - \int h^+(f) d\mu \right\} + \sup_{f \in BC(M),\, f \leqq b'} \int f d\nu_s \\
&= \mathcal{H}_h''(\nu_{ac}|\mu) + b'\nu_s(M).
\end{aligned}$$

But as we have seen, equality (A.38) holds for measures $\nu, \nu \in \mathcal{M}^+(M)$ such that $\nu \ll \mu$ and therefore $\mathcal{H}''_h(\nu_{ac}|\mu) = \mathcal{H}_h(\nu_{ac}|\mu)$. It follows that

$$\mathcal{H}''_h(\nu|\mu) \le \mathcal{H}_h(\nu_{ac}|\mu) + b'\nu_s(M) = \underline{\mathcal{H}}_h(\nu|\mu), \tag{A.68}$$

which prove one of the inequalities of (A.40), with $(*)$ being the only inequality that appears.

Let now $M_s \subseteq M$ be a Borel set such that $\nu_s(M \setminus M_s) = 0$ and $\mu(M_s) = 0$. For all $f \in BC(M)$ such that $f \le b'$ in M we consider the function

$$\widetilde{f} := b'\mathbb{1}_{M_s} + f\mathbb{1}_{M_s^C} = (b' - f)\mathbb{1}_{M_s} + f \in B(M).$$

Then obviously $f = \widetilde{f}$ μ-a.s. and consequently also $f = \widetilde{f}$ ν_{ac}-a.s., and therefore

$$\begin{aligned} \int \widetilde{f} d\nu - \int h^+(\widetilde{f}) d\mu &= \int f d\nu_{ac} + \int \widetilde{f} d\nu_s - \int h^+(f) d\mu \\ &= \int f d\nu_{ac} - \int h^+(f) d\mu + b'\nu_s(M). \end{aligned}$$

So, if given $\varepsilon > 0$ we pick $f_\varepsilon \in BC(M)$ so that $f_\varepsilon \le b'$ and

$$\mathcal{H}_h(\nu_{ac}|\mu) = \mathcal{H}''_h(\nu_{ac}|\mu) \le \int f_\varepsilon d\nu_{ac} - \int h^+(f_\varepsilon) + \frac{\varepsilon}{2}$$

then

$$\begin{aligned} \int \widetilde{f}_\varepsilon d\nu - \int h^+(\widetilde{f}_\varepsilon) d\mu &= \int f_\varepsilon d\nu_{ac} - \int h^+(f_\varepsilon) d\mu + b'\nu_s(M) \\ &\ge \mathcal{H}_h(\nu_{ac}|\mu) + b'\nu_s(M) - \frac{\varepsilon}{2}. \end{aligned}$$

Therefore if we prove that there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq BC(M)$ such that

$$\int f_n d\nu - \int h^+(f_n) d\mu \longrightarrow \int \widetilde{f}_\varepsilon d\nu - \int h^+(\widetilde{f}_\varepsilon) d\mu, \tag{A.69}$$

then by choosing $n_0 \in \mathbb{N}$ large enough so that

$$\int f_{n_0} d\nu - \int h^+(f_{n_0}) d\mu \ge \int \widetilde{f}_\varepsilon d\nu - \int h^+(\widetilde{f}_\varepsilon) d\mu - \frac{\varepsilon}{2}$$

we get that

$$\mathcal{H}''_h(\nu|\mu) \ge \int f_{n_0} d\nu - \int h^+(f_{n_0}) d\mu \ge \mathcal{H}_h(\nu_{ac}|\mu) + b'\nu_s(M) - \varepsilon = \underline{\mathcal{H}}_h(\nu|\mu) - \varepsilon,$$

which since $\varepsilon > 0$ was arbitrary proves and the other inequality of (A.40). To complete the proof of (A.40) in the case that $\mathcal{D}_{h^+} = (-\infty, b'] \subseteq \mathbb{R}$ it remains to prove that there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq BC(M)$ such that (A.69) holds. By the double Lusin lemma for $\mathbb{1}_{M_s}$ there exists a sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq BC(M)$ such that $g_n \longrightarrow \mathbb{1}_{M_s}$ μ-a.s. and ν-a.s. and sequences $\{K_n\}_{n\in\mathbb{N}}$ and $\{A_n\}_{n\in\mathbb{N}}$ of compact and open subset of M, respectively, such that $K_n \subseteq M_s \subseteq A_n$, $\mathbb{1}_{K_n} \le g_n \le \mathbb{1}_{A_n}$ and

$$\int |g_n - \mathbb{1}_{M_s}| d\mu \vee \int |g_n - \mathbb{1}_{M_s}| d\nu \le \mu(A_n \setminus K_n) \vee \nu(A_n \setminus K_n) \le \frac{1}{n}$$

for every $n \in \mathbb{N}$. We define the sequence $\{f_n\} \subseteq BC(M)$ by the formula

$$f_n := (b' - f_\varepsilon)g_n + f_\varepsilon$$

for all $n \in \mathbb{N}$. Then obviously $f_n \longrightarrow f_\varepsilon$ μ-a.s. and ν-a.s. as $n \to +\infty$ and

$$-\|f_\varepsilon\|_u \leq f_\varepsilon \leq f_n \leq b'.$$

Therefore by the bounded convergence theorem we have that $\int f_n d\nu \longrightarrow \int f_\varepsilon d\nu$, and since

$$\sup_{n\in\mathbb{N}} \|h^+(f_n)\|_u \leq \sup_{-\|f_\varepsilon\|_u \leq t \leq b'} |h^+(t)| < +\infty$$

we have by the bounded convergence theorem again that

$$\int h^+(f_n)d\mu \longrightarrow \int h^+(f)d\mu.$$

This proves (A.69) and completes the proof of (A.40) in the case $\mathcal{D}_{h^+} = (-\infty, b']$.

$\mathcal{D}_{h^+} = (-\infty, b') \neq \mathbb{R}$: Firstly, in this case also the supremum in the definition of $\mathcal{H}''_h$ does not change of it is restricted to functions $f \in BC(M)$ such that $h^+(f) \in L^1(\mu)$. But since in this case $b' \notin \mathcal{D}_{h^+}$, we have that $f < b'$ μ-a.s. for every $f \in BC(M)$ such that $h^+(f) \in L^1(\mu)$. So inequality (A.68) is also true in this case. So let $\varepsilon > 0$. We want to find $f_\varepsilon \in BC(M)$ such that

$$\int f_\varepsilon d\nu - \int h^+(f_\varepsilon)d\mu \geq \underline{\mathcal{H}}_h(\nu|\mu) - \varepsilon. \tag{A.70}$$

We write again $\nu = \nu_{ac} + \nu_s$ for the Radon-Nikodym decomposition of ν with respect to μ and pick initially $f \in BC(M)$ such that

$$\int f d\nu_{ac} - \int h^+(f)d\mu \geq \mathcal{H}_h(\nu_{ac}|\mu) - \frac{\varepsilon}{4}.$$

In particular $h^+(f) \in L^1(\mu)$ and therefore $f < b'$ μ-a.s.. Since $h^+(b') = +\infty$ and h^+ is convex, h^+ is increasing in some interval $(b' - \delta_0, b')$, $\delta_0 > 0$. Since $f < b'$ μ-a.s. the sequence $f_n := f \wedge (b' - \frac{1}{n}) \in BC(M)$, $n \in \mathbb{N}$, increases μ-a.s., and therefore also ν_{ac}-a.s.. So $\int f_n d\nu_{ac} \longrightarrow \int f d\nu_{ac}$ and if n_0 is such that $\frac{1}{n_0} < \delta_0$ then the sequence $\{h^+(f_n)\}_{n \geq n_0}$ is also increasing and therefore $\int h^+(f_n)d\mu \longrightarrow \int h^+(f)d\mu$ by the monotone convergence theorem. So there exists $n_1 \geq n_0$ such that

$$\int f_n d\nu_{ac} - \int h^+(f_n)d\mu \geq \mathcal{H}_h(\nu_{ac}|\mu) - \frac{\varepsilon}{3}$$

for all $n \geq n_1$. We pick next a Borel subset $M_s \subseteq M$ such that $\nu_s(M \setminus M_s) = 0$ and for every $n \in \mathbb{N}$ we set

$$\widetilde{f}_n := \Big(b' - \frac{1}{n}\Big)\mathbb{1}_{M_s} + f_n \mathbb{1}_{M_s^c} = \Big(b' - \frac{1}{n} - f_n\Big)\mathbb{1}_{M_s} + f_n \in B(M).$$

Then obviously $f_n \leq \widetilde{f}_n \leq b' - \frac{1}{n}$ and since $\nu_s \perp \mu$ we have that $\widetilde{f}_n = f_n$ μ-a.s. for all $n \in \mathbb{N}$ and therefore for all $n \geq n_1$ we have that

$$\begin{aligned} \int \widetilde{f}_n d\nu - \int h^+(\widetilde{f}_n) d\mu &= \int \widetilde{f}_n d\nu_{ac} + \int \widetilde{f}_n d\nu_s - \int h^+(\widetilde{f}_n) d\mu \\ &= \int f_n d\nu_{ac} - \int h^+(f_n) d\mu + \Big(b' - \frac{1}{n}\Big)\nu_s(M_s). \\ &\geq \mathcal{H}_h(\nu_{ac}|\mu) + \Big(b' - \frac{1}{n}\Big)\nu_s(M_s) - \frac{\varepsilon}{3}. \end{aligned}$$

By choosing next $n_2 \geq n_1$ such that $\nu_s(M_s)/n_2 \leq \varepsilon/6$ we have that

$$\int \widetilde{f}_{n_2} d\nu - \int h^+(\widetilde{f}_{n_2}) d\mu \geq \mathcal{H}_h(\nu_{ac}|\mu) + b'\nu_s(M_s) - \frac{\varepsilon}{2}.$$

But since

$$-\|f\|_u \leq f_{n_2} \leq \widetilde{f}_{n_2} \leq b' - \frac{1}{n_2}$$

and $b' - \frac{1}{n_2} \in \mathcal{D}_{h^+}$ we have moved away from the point b' h^+ and exactly as in the case where $b' \in \mathcal{D}_{h^+}$ we can approximate $\widetilde{f}_{n_2}$ μ and ν-a.s. and in $L^1(\nu)$ and $L^1(\mu)$ by some continuous function $f_\varepsilon \in BC(M)$ such that

$$\int f_\varepsilon d\nu - \int h^+(f_\varepsilon) d\mu \geq \int \widetilde{f}_{n_2} d\nu - \int h^+(\widetilde{f}_{n_2}) d\mu - \frac{\varepsilon}{2}.$$

So f_ε is the required function in(A.70), which completes the proof. □

The full solution of the variational problem

$$\sup_{f \in BC(M)} \Big\{ \int f d\nu - \int h^+(f) d\mu \Big\}, \quad \nu, \mu \in \mathbb{P}M$$

is given by the relative h-entropy functional $H_h : \mathbb{P}M \times \mathbb{P}M \longrightarrow [0, +\infty]$ given by the formula

$$H_h(\nu|\mu) = \int h\Big(\frac{d\nu_{ac|\mu}}{d\mu}\Big) d\mu + \Big[\lim_{t \uparrow +\infty} \frac{h(t)}{t}\Big] \nu_{s|\mu}(\text{supp}\mu) + \infty \cdot \nu_s(\text{supp}\mu^c), \quad \text{(A.71)}$$

where of course $\nu = \nu_{ac|\mu} + \nu_{s|\mu}$ is the Radon-Nikodym decomposition of ν with respect to μ with $\nu_{ac|\mu} \ll \mu$ and $\nu_{s|\mu} \perp \mu$ and we make the usual convention $0 \cdot (+\infty) = 0$ of integration theory.

Proposition A.3.4 *Let $h : \mathbb{R} \longrightarrow [0, \infty]$ be a lower semicontinuous and convex functional with non-trivial proper domain $\mathcal{D}_h \subseteq [0, \infty)$. Then*

$$H_h(\nu|\mu) = \sup_{f \in BC(M)} \Big\{ \int f d\nu - \int h^+(f) d\mu \Big\},$$

where H_h is the relative h-entropy functional defined in (A.71).

Proof It is easy to see that

$$\sup_{t>0} \frac{h(t)}{t} = \begin{cases} +\infty, & \text{if } \mathcal{D}_{h^+} = \mathbb{R}, \\ b' := \sup \mathcal{D}_{h^+}, & \text{if } \mathcal{D}_{h^+} \neq \mathbb{R} \end{cases}$$

and therefore by the previous proposition it follows that if supp$\mu = M$ then the functional $H_h(\cdot|\mu) : \mathbb{P}M \longrightarrow [0, \infty]$ satisfies

$$H_h(\nu|\mu) = \int h\Big(\frac{d\nu_{ac|\mu}}{d\mu}\Big) d\mu + \Big[\lim_{t\uparrow+\infty} \frac{h(t)}{t}\Big] \nu_{s|\mu}(M)$$

for all $\nu \in \mathbb{P}M$, where of course $\nu = \nu_{ac|\mu} + \nu_{s|\mu}$ is the Radon-Nikodym decomposition of ν with respect to μ.

So it remains to prove that if $b' < +\infty$, supp$\mu \neq M$ and $\nu \not\ll \mu$ with $\nu(\text{supp}\mu^c) > 0$, then

$$\mathcal{H}_h''(\nu|\mu) := \sup_{f\in BC(M)} \Big\{ \int f d\nu - \int h^+(f) d\mu \Big\} = +\infty.$$

Since ν is a finite Borel measure in a metric space and $\nu_\mu := \nu(\text{supp}\mu^c) > 0$ there exists a closed set $F \subseteq \text{supp}\mu^c$ such that $\nu(F) \geq \nu_\mu/2 > 0$. But then the sets F and suppμ are closed and disjoint subsets of the metric space M and therefore for every $N \in \mathbb{N}$ there exists a continuous function $f_N : M \longrightarrow [0, N]$ such that $f|_{\text{supp}\mu} \equiv 0$ and $f|_F \equiv N$. So, by making again the additional assumption $h^+(0) = 0$, we have that for every $N \in \mathbb{N}$

$$\mathcal{H}_h''(\nu|\mu) \geq \int f_N d\nu - \int h^+(f_N) d\mu = \int f_N d\nu_s \geq N\nu_s(F).$$

But $\nu_s(F) > 0$ and therefore by taking the limit as $N \uparrow \infty$ it follows that $\mathcal{H}_h''(\nu|\mu) = \infty$.□

Proposition A.3.5 (Pinsker's Inequality) *For every $\mu, \nu \in \mathbb{P}M$ the relative entropy $\mathcal{H} : \mathbb{P}M \longrightarrow [0, +\infty]$ satisfies the inequality*

$$\|\nu - \mu\|_{TV}^2 \leq 2\mathcal{H}(\nu|\mu),$$

where $\|\cdot\|_{TV}$ is the total variation norm $\|\nu - \mu\|_{TV} = |\nu - \mu|(M)$.

Proof The mapping

$$L^1(\mu) \ni f \mapsto f d\mu \in \mathcal{M}_{TV}(M)$$

is an isometric embedding for all $\mu \in \mathcal{M}_+(M)$, and therefore if $\nu \ll \mu$ we have that

$$\|\nu - \mu\|_{TV} = \Big\| \Big(\frac{d\nu}{d\mu} - 1\Big) d\mu \Big\|_{TV} = \int \Big| \frac{d\nu}{d\mu} - 1 \Big| d\mu.$$

For the function $h(t) = t \log t - t + 1$ we have that $h(1) = 0$, $h'(t) = \log t$ and $h''(t) = \frac{1}{t}$. Therefore by the integral representation of the remainder in the first order Taylor's theorem for h,

$$h(t) = \int_1^t \frac{(t-s)}{s} ds = (t-1)^2 \int_0^1 \frac{1-s}{1+(t-1)s} ds.$$

Therefore by setting $f := \frac{d\nu}{d\mu}$ we have by the Cauchy-Schwartz inequality that

$$\begin{aligned}
\|\nu-\mu\|_{TV}^2 &= \left(\int |f-1|d\mu\right)^2 = 4\left(\int_0^1\int |f-1|(1-s)d\mu ds\right)^2 \\
&= 4\left(\int_0^1\int\sqrt{\frac{1-s}{1+(f-1)s}}\,|f-1|\,\sqrt{(1-s)(1+(f-1)s)}d\mu ds\right)^2 \\
&\leq 4\int_0^1\int\frac{|f-1|^2(1-s)}{1+(f-1)s}d\mu ds\int_0^1\int(1-s)\big(1+(f-1)s\big)d\mu ds \\
&= 4\int h(f)d\mu\int_0^1(1-s)ds = 2\mathcal{H}(\nu|\mu),
\end{aligned}$$

and the inequality is proved. □

Another useful property of relative entropy is the following super-additivity property with respect to the marginal distributions in product spaces.

Proposition A.3.6 *Let $M := M_1 \times M_2$ be the Cartesian product of the polish spaces M_i, $i = 1,2$, let $\mu_i \in \mathbb{P}M_i$, $i = 1,2$ and let $\pi^i : M \longrightarrow M_i$ be the natural projections. Then for every $\nu \in \mathbb{P}M$ we have that*

$$\mathcal{H}(\nu|\mu_1 \otimes \mu_2) \geq \mathcal{H}(\pi^1_*\nu|\mu_1) + \mathcal{H}(\pi^2_*\nu|\mu_2),$$

with equality in the case that ν is also a product measure.

Proof Set $\mu := \mu_1 \otimes \mu_2$. If $\mathcal{H}(\nu|\mu) = +\infty$ we have nothing to prove, so we suppose that $\mathcal{H}(\nu|\mu) < +\infty$, in which case in particular we have that $\nu \ll \mu$. Then, necessarily $\nu_1 \ll \mu_1$, since if $\mu_1(A_1) = 0$ for some measurable set $A_1 \subseteq M_1$, then $\mu(A_1 \times M_2) = 0$ and therefore $\nu_1(A_1) = \nu(A_1 \times M_2) = 0$. Similarly we get that $\nu_2 \ll \mu_2$. Furthermore, as we will show, if $\{\nu_x\}_{x\in M_1}$ is the disintegration of ν with respect to its first marginal ν_1, i.e. if $\{\nu_x\}_{x\in M_1}$ is the unique ν_1-a.s. defined family of measures satisfying

$$\int_{M_1\times M_2} f(x,y)d\nu(x,y) = \iint f(x,y)d\nu_x(y)d\nu_1(x), \quad \forall\, f \in B(M), \tag{A.72}$$

then

$$\nu_1\{x \in M_1 \,|\, \nu_x(A_2) = 0\} = 1, \quad \forall\, A_2 \in \mathcal{N}_{\mu_2}, \tag{A.73}$$

where $\mathcal{N}_{\mu_2}$ is the set of all Borel sets of zero μ_2-measure. Indeed, if some Borel set $A_2 \subseteq M_2$ we have that $\mu_2(A_2) = 0$, then $\nu_2(A_2) = 0$ since $\nu_2 \ll \mu_2$, and therefore

$$0 = \nu_2(A_2) = \int \nu_x(A_2)d\nu_1(x),$$

which proves (A.73). Since M_2 is a polish space it is logical to expect that we can pass the "for every" in (A.73) inside the measure ν_1 as an intersection, in order to get

$$\nu_1\Big(\bigcap_{A_2\in\mathcal{N}_{\mu_2}}\{x \in M_1 \,|\, \nu_x(A_2) = 0\}\Big) = 1,$$

which implies that $\nu_x \ll \mu_2$ ν_1-a.s. for all $x \in M_1$. As we shall see next, this is indeed true, at last for $x \in M_1$ such that $\frac{d\nu_1}{d\mu_1}(x) \neq 0$. We define a function $g : M \longrightarrow \mathbb{R}_+$ by the formula

$$g(x,y) = \begin{cases} \frac{\frac{d\nu}{d\mu}(x,y)}{\frac{d\nu_1}{d\mu_1}(x)}, & \frac{d\nu_1}{d\mu_1}(x) \neq 0 \\ 0, & \frac{d\nu_1}{d\mu_1}(x) = 0. \end{cases}$$

for every $(x,y) \in M$. Then obviously $g\frac{d\nu_1}{d\mu_1} = \frac{d\nu}{d\mu}\mathbb{1}_{\{\frac{d\nu_1}{d\mu_1}\neq 0\}}$ and for every $f \in B(M)$ we have that

$$\begin{aligned} \iint f(x,y)g(x,y)d\mu_2(y)d\nu_1(x) &= \iint f(x,y)g(x,y)\frac{d\nu_1}{d\mu_1}(x)d\mu_2(y)d\mu_1(x) \\ &= \iint f(x,y)\frac{d\nu}{d\mu}\mathbb{1}_{(0,\infty)}\Big(\frac{d\nu_1}{d\mu_1}(x)\Big)d\mu_2(y)d\mu_1(x) \\ &= \int f(x,y)\frac{d\nu}{d\mu}\mathbb{1}_{(0,\infty)}\Big(\frac{d\nu_1}{d\mu_1}(x)\Big)d\mu(x,y) \\ &= \int f(x,y)\mathbb{1}_{(0,\infty)}\Big(\frac{d\nu_1}{d\mu_1}(x)\Big)d\nu(x,y). \end{aligned}$$

Consequently, by the uniqueness of $\{\nu_x\}$, whenever (A.72) holds, it follows that ν_1-almost for every $x \in M_1$ for which $\frac{d\nu_1}{d\mu_1}(x) \neq 0$, we have that

$$d\nu_x = g(x,\cdot)d\mu_2.$$

By the definition of g and the above equality it follows that

$$\frac{d\nu}{d\mu}(x,y) = \frac{d\nu_1}{d\mu_1}(x)\frac{d\nu_x}{d\mu_2}(y)$$

μ-almost for every $(x,y) \in M$.

Therefore, by the definition of relative entropy and the above expression of the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ we have that

$$\begin{aligned} \mathcal{H}(\nu|\mu) &= \int \log\frac{d\nu}{d\mu}d\nu = \int \log\frac{d\nu_1}{d\mu_1}(x)d\nu(x,y) + \int \log\frac{d\nu_x}{d\mu_2}(y)d\nu(x,y) \\ &= \int \log\frac{d\nu_1}{d\mu_1}(x)d\nu_1(x) + \iint \log\frac{d\nu_x}{d\mu_2}(y)d\nu_x(y)d\nu_1(x) \\ &= \mathcal{H}(\nu_1|\mu_1) + \int \mathcal{H}(\nu_x|\mu_2)d\nu_1(x) \\ &\geq \mathcal{H}(\nu_1|\mu_1) + \mathcal{H}(\nu_2|\mu_2), \end{aligned}$$

where the last inequality follows from the linear convexity of $\mathcal{H}$,

$$\mathcal{H}\left(\int \nu_x d\nu_1(x)\middle|\mu_2\right) \leq \int \mathcal{H}(\nu_x|\mu_2)d\nu_1(x),$$

for all $\{\nu_x\}_{x\in M_1} \subseteq \mathbb{P}M_2$, $\nu_1 \in \mathbb{P}M_1$, $\mu_2 \in \mathbb{P}M_2$. The case of equality is easily verified. □

A.3.2 Entropy Production and the Dirichlet Form

In this section we review some basic results on entropy production and the Dirichlet form in the context of Markov jump processes, as found in [25]

Proposition A.3.7 *Let $(P_t)_{t\geq 0}$ be a Markov semigroup on the Polish space M admitting an invariant measure $\pi \in \mathbb{P}M$. Then for any initial distribution $\mu \in \mathbb{P}M$ the relative entropy of the push forward μP_t, $t \geq 0$, with respect to the invariant measure π does not increase in time, i.e.*

$$\mathcal{H}(\mu P_t|\pi) \leq \mathcal{H}(\mu|\pi), \quad \forall\, t \geq 0. \tag{A.74}$$

Consequently, if $\mathcal{H}(\mu|\pi) < \infty$ then $\mu P_t \ll \pi$ for all $t \geq 0$.

If in addition M is countable and $(P_t)_{t\geq 0}$ is irreducible then the equality

$$\mathcal{H}(\mu P_t|\pi) = \mathcal{H}(\mu|\pi) < +\infty$$

holds for some $t > 0$ iff $\mu = \pi$.

Proof If $\mathcal{H}(\mu|\pi) = \infty$ there is nothing to prove, so we assume that $\mathcal{H}(\mu|\pi) < +\infty$ and fix $t \geq 0$. Then $\mu \ll \pi$ and for any bounded function $f \in B(M)$ we have that

$$\begin{aligned}\int f d\mu P_t &= \iint f dP_t^x d\mu(x) = \iint \frac{d\mu}{d\pi}(x) f(y) dP_t^x(y) d\pi(x) \\ &= \int \frac{d\mu}{d\pi}(x) f(y) d\pi[\delta \otimes P_t](x,y).\end{aligned} \tag{A.75}$$

Since π is (P_t)-invariant the distribution $\pi_{0,t} := \pi[\delta \otimes P_t] \in \mathbb{P}(M \times M)$, $t \geq 0$, has left and right marginals equal to π and by the disintegration theorem we can write

$$\int f d\pi_{0,t} = \int f(x,y) dQ_t^y(x) d\pi(y), \quad \forall\, f \in B(M \times M),$$

for a unique π-a.s. defined Markov kernel $Q_t \in \mathcal{L}(\pi; \mathbb{P}M)$. Obviously, the Markov operator induced by Q_t on $L^2(\pi)$, i.e. the operator $Q_t : L^2(\pi) \longrightarrow L^2(\pi)$ given by

$$Q_t f(y) = \int f dQ_t^y, \quad y \in M$$

is the adjoint P_t^* of the induced Markov operator $P_t : L^2(\pi) \longrightarrow L^2(\pi)$, since

$$\begin{aligned}\langle f, Q_t g\rangle_\pi &= \int f(y) Q_t g(y) d\pi(y) = \iint g(x) f(y) dQ_t^y(x) d\pi(y) \\ &= \iint g(x) f(y) dP_t^x(y) d\pi(x) = \int g(x) P_t f(x) d\pi(y) = \langle P_t f, g\rangle_\pi.\end{aligned}$$

Now, according to (A.75) and the definition of $Q_t = P_t^*$ we have that

$$\int f d\mu P_t = \int \left(Q_t \frac{d\mu}{d\pi}\right)(y) f(y) d\pi(y)$$

for all $f \in B(M)$, which implies that μP_t is absolutely continuous with respect π with Radon-Nikodym derivative given by

$$\frac{d(\mu P_t)}{d\pi} = P_t^* \frac{d\mu}{d\pi}. \tag{A.76}$$

Since the function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}$ given by $h(u) = u \log u$ is convex we have by Jensen's inequality that

$$h\left(P_t^* \frac{d\mu}{d\pi}\right) = h\left(\int \frac{d\mu}{d\pi} dP_t^*\right) \leq \int h\left(\frac{d\mu}{d\pi}\right) dP_t^* = P_t^* \left[h\left(\frac{d\mu}{d\pi}\right)\right]$$

and so we have that

$$\mathcal{H}(\mu P_t|\pi) = \int h\left(P_t^* \frac{d\mu}{d\pi}\right) d\pi \leq \int P_t^* \left[h\left(\frac{d\mu}{d\pi}\right)\right] d\pi = \int h\left(\frac{d\mu}{d\pi}\right) d\pi P_t^*. \tag{A.77}$$

But for all $f \in B(M)$ we have that

$$\int f d\pi P_t^* = \int P_t^* f d\pi = \int f P_t 1 d\pi = \int f d\pi$$

and therefore $\pi P_t^* = \pi$, which by (A.77) gives the required inequality $\mathcal{H}(\mu P_t|\pi) \leq \mathcal{H}(\mu|\pi)$.

We suppose next that M is countable and (P_t) is irreducible and investigate the case of equality. Obviously (A.74) holds as a finite equality for $t > 0$ iff

$$h\left(P_t^* \frac{d\mu}{d\pi}\right) = P_t^* \left[h\left(\frac{d\mu}{d\pi}\right)\right], \quad \pi - \text{a.s.}.$$

But since h is strictly convex, this holds iff

$$\frac{d\mu}{d\pi} \equiv \text{const.}, \quad (P_t^*)^y\text{-a.s.}, \qquad \pi\text{-a.s. } \forall y.$$

But if (P_t) is irreducible this implies that $\mu = \pi$. Indeed, given $x, y \in \{u \in M | \pi(u) > 0\}$ we have that there exist constants $c_x, c_y \in \mathbb{R}$ such that

$$\frac{d\mu}{d\pi} \equiv c_x, \quad (P_t^*)^x\text{-a.s.} \qquad \text{and} \qquad \frac{d\mu}{d\pi} \equiv c_y, \quad (P_t^*)^y\text{-a.s.}. \tag{A.78}$$

Since P_t^* is the adjoint of P_t in $L^2(\pi)$ for all $x, y \in M$ we have that,

$$\pi(y) P_t^*(y, x) = \langle \mathbb{1}_{\{y\}}, P_t^* \mathbb{1}_{\{x\}} \rangle_\pi = \langle P_t \mathbb{1}_{\{y\}}, \mathbb{1}_{\{x\}} \rangle_\pi = \pi(x) P_t(x, y)$$

and since (P_t) is irreducible we have that $P_t(z, x) \wedge P_t(z, y) > 0$ for all $t \geq 0$ and all $x, y, z \in M$. So if we choose $z \in M$ such that $\pi(z)$ we have that $P_t^*(x, z) \wedge P_t^*(y, z) > 0$ and therefore by (A.78) we get

$$\frac{d\mu}{d\pi}(x) = c_x = \frac{d\mu}{d\pi}(z) = c_y = \frac{d\mu}{d\pi}(y).$$

Therefore $\frac{d\mu}{d\pi}$ is π-a.s. constant, and then necessarily $\frac{d\mu}{d\pi} = 1$ π-a.s., which gives $\mu = \pi$ as required, and completes the proof. □

In what follows we restrict our attention to Markov jump processes on a countable state space M. So let $(P_t)_{t\geq 0}$ be the Markov semigroup on M corresponding to the Markov jump process defined by a transition kernel $P : M \longrightarrow \mathbb{P}M$ be and a jump rate function $\lambda : M \longrightarrow \mathbb{R}_+$ bounded on each communication class of the kernel p. In this way, if $\{M_i\}_{i=1}^m$, $m \in \mathbb{N} \cup \{\infty\}$, is the partition of M into the communication classes of p, then λ satisfies

$$\bar{\lambda}_i := \sup_{x \in M_i} \lambda(x) < +\infty, \quad \forall\, i = 1, \dots, m.$$

The generator $L : \mathcal{L}(M) \longrightarrow \mathcal{L}(M)$ of (P_t), given by

$$Lf(x) = \sum_{y \in M} [f(y) - f(x)]\lambda(x)p(x, y)$$

defines bounded linear operators $L_i : B(M_i) \longrightarrow B(M_i)$ for all $i = 1, \dots, m$ with norms $\|L_i\| \leq 2\bar{\lambda}_i$, since for all $f \in B(M_i)$, $x \in M_i$ we have that

$$|L_i f(x)| \leq \sum_{y \in M_i} |f(y) - f(x)|\lambda(x)p(x, y) \leq 2\bar{\lambda}_i \|f\|_u,$$

and obviously for all $f \in B(M)$ we have that

$$Lf = \sum_{i=1}^{m} L_i(f|_{M_i})\mathbb{1}_{M_i}.$$

Furthermore, if π is an invariant measure then $d\pi_i := \mathbb{1}_{M_i} d\pi$ is an invariant measure for all $i = 1, \dots, m$ and the operators L_i induce bounded operators $L_i : L^2(\pi_i) \longrightarrow L^2(\pi_i)$. In this context by saying the adjoint of L in $L^2(\pi)$ we mean the operator $L^* : \mathcal{L}(M) \longrightarrow \mathcal{L}(M)$ given by

$$L^* f = \sum_{i=1}^{m} L_i^*(f|_{M_i})\mathbb{1}_{M_i}$$

for all functions $f : M \longrightarrow \mathbb{R}$. As we saw in (A.76), given a Markov semigroup (P_t) the time evolution $f_t := \frac{d\mu P_t}{d\pi}$, $t \geq 0$, of the density $f := \frac{d\mu}{d\pi}$ of μ with respect to a (P_t)-invariant measure π, is given by $f_t = P_t^* f$, $t \geq 0$, where P_t^* is the adjoint operator of the Markov operator $P_t : L^2(\pi) \longrightarrow L^2(\pi)$. Therefore, if we recall that since π is (P_t)-invariant the adjoint L^* of L is the generator of the adjoint semigroup $(P_t^*)_{t\geq 0}$, we see that $(f_t)_{t\geq 0}$ is a solution of the initial value problem

$$\begin{cases} f_0 = \frac{d\mu}{d\pi} \\ \partial_t f_t = L^* f_t \end{cases}$$

This observation allows us to deduce a simple estimate on the time derivative of the entropy.

Proposition A.3.8 *Let $(P_t)_{t\geq 0}$ be the Markov jump semigroup corresponding to some skeleton kernel $p : M \longrightarrow \mathbb{P}M$ and some jump rate function $\lambda : M \longrightarrow \mathbb{R}_+$ bounded on the communication classes M_i, $i = 1, \dots, m$, $m \in \mathbb{N} \cup \{\infty\}$, of (P_t). Let $\bar{\lambda} : M \longrightarrow \mathbb{R}_+$ the function given by $\bar{\lambda} = \sum_{i=1}^{m} \bar{\lambda}_i \mathbb{1}_{M_i}$ and suppose that (P_t) possesses an invariant* *measure* $\pi \in \mathbb{P}M$ *such that* $\bar{\lambda} \in L^1(\pi)$*. Then for every distribution* $\mu \in \mathbb{P}M$ *such that* $\bar{\lambda} \in L^1(\mu)$ *and* $\mathcal{H}(\mu|\pi) \vee \mathcal{H}_{\bar{\lambda}}(\mu|\pi) < +\infty$*, where* $\mathcal{H}_{\bar{\lambda}}(\mu|\pi)$ *is the weighted entropy*

$$\mathcal{H}_{\bar{\lambda}}(\mu|\pi) := \int \bar{\lambda}\frac{d\mu}{d\pi}\log\frac{d\mu}{d\pi}d\pi$$

we have that

$$\mathcal{H}(\mu P_{t+s}|\pi) - \mathcal{H}(\mu P_t|\pi) = \int_t^{t+s}\langle f_r, L\log f_r\rangle_\pi dr \leq 2\int_t^{t+s}\langle\sqrt{f_r}, L\sqrt{f_r}\rangle_\pi dr,$$

where $f_t := \frac{d\mu P_t}{d\pi}$*,* $t \geq 0$*, is the density of* μP_t *with respect to* π*. Moreover, for all* $t \geq 0$ *we have that*

$$\langle\sqrt{f_t}, L\sqrt{f_t}\rangle_\pi = -\frac{1}{2}\sum_{x,y\in M}\left[\sqrt{f_t(y)} - \sqrt{f_t(x)}\right]^2 L(x,y)\pi(x). \qquad \text{(A.79)}$$

Proof By the definition of the relative entropy and the remark prior to the statement of the proposition we have that

$$\mathcal{H}(\mu P_{t+s}|\pi) - \mathcal{H}(\mu P_t|\pi) = \iint_t^{t+s}\partial_s[f_r\log f_r]drd\pi = \iint_t^{t+s}(1+\log f_r)L^* f_r drd\pi$$

Let as before $M = \bigsqcup_{i=1}^m M_i$, $m \in \mathbb{N}\cup\{\infty\}$, be the decomposition of M in communication classes. For all $y \in M_i$ we have that

$$\begin{aligned}|L^* f_r(y)| &\leq \sum_{x\in M}|f_r(x) - f_r(y)|L^*(y,x) \leq \lambda(y)f_r(y) + \sum_{x\in M:x\neq y} f_r(x)L^*(y,x)\\ &\leq \bar{\lambda}_i f_r(y) + \lambda(y)\sum_{x\in M} f_r(x)p^*(y,x)\end{aligned}$$

where p^* is the adjoint of p in $L^2(\pi)$ and so

$$\begin{aligned}|L^* f_r(y)| &\leq \sum_{i=1}^m \bar{\lambda}_i\Big(f_r(y) + \mathbb{1}_{\{\lambda\neq 0\}}(y)\sum_{x\in M_i} f_r(x)p^*(y,x)\Big)\mathbb{1}_{M_i}(y)\\ &= \bar{\lambda}(y)f_r(y) + \lambda(y)\sum_{x\in M} f_r(x)p^*(y,x).\end{aligned}$$

Therefore for all $s \geq 0$ we have that

$$|L^* f_r| \leq \bar{\lambda}f_r + \lambda p^* f_r$$

Note that since M_i are the communication classes of (P_t) we have that $P_s^x(M_i) = \mathbb{1}_{M_i}(x)$ for all $s > 0$, $x \in M$ and $i = 1, \ldots, m$. So since $\bar{\lambda} \in L^1(\mu)$ we have that

$$\begin{aligned}\int\bar{\lambda}f_r d\pi &= \int\bar{\lambda}d\mu P_s = \iint\bar{\lambda}(y)dP_t^x(y)d\mu(x) = \int\sum_{i=1}^m\bar{\lambda}_i P_t^x(M_i)d\mu(x)\\ &= \int\sum_{i=1}^m\bar{\lambda}_i\mathbb{1}_{M_i}(x)d\mu(x) = \int\bar{\lambda}d\mu < +\infty.\end{aligned}$$

Therefore the non-negative function $\bar{\lambda} f_r$ is in $L^1(\pi)$ and likewise

$$\begin{aligned}
\int \lambda p^* f_r d\pi &= \sum_{y \in M} \lambda(y) \sum_{x \in M} f_r(x) p^*(y,x) \pi(y) = \sum_{y \in M} \sum_{x \in M} f_r(x) \lambda(x) p(x,y) \pi(x) \\
&= \sum_{x \in M} \lambda(x) f_r(x) \pi(x) \sum_{y \in M} p(x,y) \leq \sum_{x \in M} \lambda(x) f_r(x) \pi(x) \\
&\leq \int \bar{\lambda} f_r d\pi = \int \bar{\lambda} d\mu < +\infty
\end{aligned}$$

for all $s \geq 0$. This proves that $|L^* f_r|$ is in $L^1(\pi)$-integrable function with

$$\int |L^* f_r| d\pi \leq 2 \int \bar{\lambda} d\mu.$$

Therefore, by Tonelli's theorem we have that

$$\iint_t^{t+s} |L^* f_r| dr d\pi \leq \int_t^{t+s} 2 \int \bar{\lambda} d\mu dr = 2s \int \bar{\lambda} d\mu < +\infty$$

which allows us to apply Fubini's theorem to obtain

$$\iint_t^{t+s} L^* f_r dr d\pi = \int_t^{t+s} \int L^* f_r d\pi dr = 0$$

since $L^* f_r \in L^1(\pi)$, L^* is the adjoint of the generator L and π is an invariant distribution. We have proved thus so far that

$$\mathcal{H}(\mu P_{t+s}|\pi) - \mathcal{H}(\mu P_t|\pi) = \iint_t^{t+s} (L^* f_r) \log f_r dr d\pi. \tag{A.80}$$

But

$$\begin{aligned}
(L^* f_r) \log f_r(y) &= \log f_r(y) \sum_{x \in M} L^*(y,x) f_r(x) \\
&= \sum_{x \in M, x \neq y} L^*(y,x) f_r(x) \log f_r(y) - \lambda(y) f_r(y) \log f_r(y) \\
&= \lambda(y) p^* f_r(y) \log f_r(y) - \lambda(y) f_r(y) \log f_r(y)
\end{aligned} \tag{A.81}$$

Now, if we denote by $\widetilde{h}$ the function $\widetilde{h}(u) := h(u) - u + 1 = u \log u - u + 1$, $u \geq 0$, then we have that

$$\bar{\lambda} \widetilde{h}(f_0) = \bar{\lambda} h(f_0) - \bar{\lambda} f_0 + \bar{\lambda}.$$

Since by our assumptions $\bar{\lambda} \in L^1(\mu) \cap L^1(\pi)$ and

$$-\frac{1}{e} \int \bar{\lambda} d\pi \leq \int \bar{\lambda} h(f_0) d\pi = \mathcal{H}_{\bar{\lambda}}(\mu|\pi) < +\infty,$$

we have that $\bar{\lambda} \widetilde{h}(f_0) = \bar{\lambda} h(f_0) - \bar{\lambda} f_0 + \bar{\lambda} \in L^1(\pi)$ and therefore

$$\widetilde{\mathcal{H}}_{\bar{\lambda}}(\mu|\pi) := \int \bar{\lambda} \widetilde{h}(f_0) d\pi < +\infty.$$

We shall show that the weighted entropy $\widetilde{\mathcal{H}}_{\bar{\lambda}}(\mu P_r|\pi)$ does not increase with respect to time $r \geq 0$. Indeed, since $\widetilde{h}$ is convex and non-negative, if we set $d\pi_i := \mathbb{1}_{M_i} d\pi$, we have by Jensen's inequality and the monotone convergence theorem if necessary that

$$\begin{aligned}
\widetilde{\mathcal{H}}_{\bar{\lambda}}(\mu P_r|\pi) &= \int \bar{\lambda}\widetilde{h}(P_r^* f_0) d\pi \leq \int \bar{\lambda} P_r^*[\widetilde{h}(f_0)] d\pi = \sum_{i=1}^m \bar{\lambda}_i \int_{M_i} P_r^*[\widetilde{h}(f_0)] d\pi \\
&= \sum_{i=1}^m \bar{\lambda}_i \int P_r^*[\widetilde{h}(f_0)] d\pi_i = \sum_{i=1}^m \bar{\lambda}_i \int \widetilde{h}(f_0) d\pi_i P_r^* \\
&= \sum_{i=1}^m \bar{\lambda}_i \int \widetilde{h}(f_0) d\pi_i = \int \bar{\lambda}\widetilde{h}(f_0) d\pi = \mathcal{H}_{\bar{\lambda}}(\mu|\pi).
\end{aligned}$$

Therefore for all $r \geq 0$ we have that

$$0 \leq \int \bar{\lambda}\widetilde{h}(f_r) d\pi = \widetilde{\mathcal{H}}_{\bar{\lambda}}(\mu P_r|\pi) \leq \widetilde{\mathcal{H}}_{\bar{\lambda}}(\mu|\pi) < +\infty$$

which since for all $r \geq 0$ we have $\bar{\lambda} h(f_r) = \bar{\lambda}\widetilde{h}(f_r) + \bar{\lambda} f_r - \bar{\lambda}$ and $\bar{\lambda} f_r - \bar{\lambda} \in L^1(\pi)$, gives us that $\bar{\lambda} h(f_r) \in L^1(\pi)$ for all $r \geq 0$. This proves that the second term in the right hand side of (A.81) is in $L^1(\pi)$ with

$$\|\lambda h(f_r)\|_{L^1(\pi)} \leq \mathcal{H}_{\bar{\lambda}}(\mu|\pi) + \int \bar{\lambda} d(\mu + \pi).$$

Consequently, by Tonelli's theorem we have that

$$\begin{aligned}
\iint_t^{t+s} |\lambda f_r \log f_r| dr d\pi &= \int_t^{t+s} \int |\lambda h(f_r)| d\pi dr \\
&\leq s\left[\mathcal{H}_{\bar{\lambda}}(\mu|\pi) + \int \bar{\lambda} d(\mu + \pi)\right] < +\infty. \qquad \text{(A.82)}
\end{aligned}$$

For the other term we note that by the inequality

$$uv \leq e^v + u \log u - u, \quad \forall\, u \geq 0,\ v \in \mathbb{R},$$

we have that

$$\lambda \cdot (p^* f_r) \cdot \log f_r \leq \lambda f_r + \lambda (p^* f_r) \log(p^* f_r) - \lambda p^* f_r.$$

Now, obviously $\lambda f_r - \lambda p^* f_r$ is in $L^1(\pi)$ and

$$-\infty < -\frac{1}{e} \int \lambda d\pi \leq \int \lambda (p^* f_r) \log(p^* f_r) d\pi \leq \int p^*[h(f_r)] \lambda d\pi \overset{*}{=} \int \lambda h(f_r) d\pi < +\infty,$$

where equality $(*)$ follows from the fact that π is (P_t)-invariant iff the measure $d\pi_\lambda := \lambda d\pi$ is invariant for the skeleton kernel p. Therefore the function $\lambda p^* f_r \log f_r$ is bounded above by some $L^1(\pi)$-function and consequently its positive part is in $L^1(\pi)$, with

$$\int \big(\lambda p^* f_r \log f_r\big)^+ d\pi \leq \int \int \lambda |h(f_r)| d\pi \leq \mathcal{H}_{\bar{\lambda}}(\mu|\pi) + \int \bar{\lambda} d(\mu + \pi) < +\infty$$

for all $r \geq 0$. Therefore the positive part of

$$[t, t+s] \times M \ni (r, y) \mapsto \lambda(y) p^* f_r(y) \log f_r(y)$$

is in $L^1(\mathbb{1}_{[t,t+s]}(r)dr \otimes \pi)$. Together with (A.81) and the fact that the function

$$[t, t+s] \times M \ni (r, y) \mapsto \lambda(y) f_r(y) \log f_r(y)$$

is in $L^1(\mathbb{1}_{[t,t+s]}(r)dr \otimes \pi)$ by (A.82), this allows us to apply Fubini's theorem and interchange the order of integration in (A.80) in order to obtain that

$$\mathcal{H}(\mu P_{t+s}|\pi) - \mathcal{H}(\mu P_t|\pi) = \int_t^{t+s} \langle L^* f_r, \log f_r \rangle_\pi dr = \int_t^{t+s} \langle f_r, L \log f_r \rangle_\pi dr,$$

as required, where the last equality holds due to the fact that L^* is the adjoint of L in $L^2(\pi)$. To complete the proof of the first claim of this proposition it remains to prove that for all $r \geq 0$ we have that

$$f_r \cdot L \log f_r \leq 2\sqrt{f_r} \cdot L\sqrt{f_r},$$

and indeed, by the elementary inequality

$$a[\log b - \log a] \leq 2\sqrt{a}[\sqrt{b} - \sqrt{a}], \quad \forall\, a, b \geq 0,$$

we have that

$$\begin{aligned} f_r(x) L \log f_r(x) &= f_r(x) \sum_{y \in M} [\log f_r(y) - \log f_r(x)] \lambda(x) p(x, y) \\ &\leq 2\sqrt{f_r(x)} \sum_{y \in M} [\sqrt{f_r(y)} - \sqrt{f_r(x)}] \lambda(x) p(x, y) \\ &= 2\sqrt{f_r(x)} \cdot L\sqrt{f_r}(x). \end{aligned}$$

It remains to prove that (A.79). Since L^* is the adjoint of L in $L^2(\pi)$ we have that

$$\begin{aligned} 2\langle \sqrt{f_r}, L\sqrt{f_r} \rangle_\pi &= \langle \sqrt{f_r}, L\sqrt{f_r} \rangle_\pi + \langle \sqrt{f_r}, L^*\sqrt{f_r} \rangle_\pi \\ &= \sum_{x,y \in M} \sqrt{f_r(x)} \left[\sqrt{f_r(y)} - \sqrt{f_r(x)} \right] L(x, y) \pi(x) \\ &\quad + \sum_{x,y \in M} \sqrt{f_r(y)} \left[\sqrt{f_r(x)} - \sqrt{f_r(y)} \right] L^*(y, x) \pi(y) \\ &= \sum_{x,y \in M} \sqrt{f_r(x)} \left[\sqrt{f_r(y)} - \sqrt{f_r(x)} \right] L(x, y) \pi(x) \\ &\quad + \sum_{x,y \in M} \sqrt{f_r(y)} \left[\sqrt{f_r(x)} - \sqrt{f_r(y)} \right] L(x, y) \pi(x) \\ &= -\sum_{x,y \in M} \left[\sqrt{f_r(x)} - \sqrt{f_r(y)} \right]^2 L(x, y) \pi(x), \end{aligned}$$

as required. □

In what follows we restrict for simplicity we restrict attention to Markov jump process with uniformly bounded jump rates. Then the generator L of any Markov jump semigroup $P_t : M \longrightarrow \mathbb{P}M$, $t \in \mathbb{R}_+$, defines a bounded operator $L : L^2(\pi) \longrightarrow L^2(\pi)$ for any invariant measure $\pi \in \mathbb{P}M$ of (P_t), the Dirichlet form of L is a bounded operator and the estimate proved for the entropy production can be rephrased in terms of the Dirichlet form.

Definition A.3.4 Let $(P_t)_{t\in\mathbb{R}_+}$ be a Markov jump transition semigroup with bounded jump rate function $\lambda : M \longrightarrow \mathbb{R}_+$ on the countable state space M with generator L and let $\pi \in \mathbb{P}M$ be an invariant measure of (P_t). Then *the Dirichlet form associated to* L *is* the operator $\mathfrak{D} : L^2(\pi) \longrightarrow \mathbb{R}$ given by the formula

$$\mathfrak{D}(f) = -\langle f, Lf\rangle_{L^2(\pi)} = -\sum_{x\in M} f(x)Lf(x).$$

Of course the Dirichlet form $\mathfrak{D}$ is well defined since the generator L is assumed a bounded operator, and thus

$$|\mathfrak{D}(f)| \leq \|f\|_{L^2(\pi)}\|Lf\|_{L^2(\pi)} \leq \|L\| \int f^2 d\pi < +\infty.$$

Proposition A.3.9 *The Dirichlet form* $\mathfrak{D} : L^2(\pi) \longrightarrow \mathbb{R}$ *associated to the generator* $L : L^2(\pi) \longrightarrow L^2(\pi)$ *is positive and given by the formula*

$$\mathfrak{D}(f) = \frac{1}{2} \sum_{x,y\in M} \big[f(y) - f(x)\big]^2 \pi(x)L(x,y) \geq 0. \tag{A.83}$$

Proof Let $f \in L^2(\pi)$. Then since $\pi(x)L(x,y) = \pi(y)L^*(y,x)$ for all $x, y \in M$ we have that

$$\begin{aligned} 2\langle f, Lf\rangle &= \langle f, Lf\rangle_{L^2(\pi)} + \langle L^* f, f\rangle_{L^2(\pi)} \\ &= \sum_{x,y\in M} f(x)\left[f(y) - f(x)\right] L(x,y)\pi(x) \\ &\qquad + \sum_{x,y\in M} f(y)\left[f(x) - f(y)\right] L^*(y,x)\pi(y) \\ &= -\sum_{x,y\in M} \left[f(x) - f(y)\right]^2 \pi(x)L(x,y) \leq 0 \end{aligned}$$

as required. □

In terms of the Dirichlet form the upper bounded on the entropy production of proposition A.3.8 states that for every initial distribution $\mu \in \mathbb{P}M$ of finite relative entropy with respect to the invariant measure π we have that

$$\mathcal{H}(\mu P_{t+h}|\pi) - \mathcal{H}(\mu P_t|\pi) \leq -2\int_t^{t+h} \mathfrak{D}(\sqrt{f_s})ds, \tag{A.84}$$

where $f_s := \frac{d\mu P_s}{d\pi}$, $s \in \mathbb{R}_+$. The following proposition describes some basic properties of the Dirichlet form.

Proposition A.3.10 *The Dirichlet form* $\mathfrak{D} : L^2(\pi) \longrightarrow \mathbb{R}_+$ *is linearly convex function and satisfies the following properties:*
(a) $\mathfrak{D}(f) = 0$ *iff* f *is constant on the positively recurrent communication classes of* M.
(b) *For any* 1*-Lipschitz function* $F : \mathbb{R} \longrightarrow \mathbb{R}$ *we have that*

$$\mathfrak{D}(F \circ f) \leq \mathfrak{D}(f).$$

Proof To prove that $\mathfrak{D}$ is convex let $\{f_i\}_{i=1}^{\infty} \subseteq L^2(\pi)$ and let $(p_i)_{i\in\mathbb{N}}$ be any probability measure on $\mathbb{N}$. By Jensen's inequality, for each $x \in \mathbb{R}^{\mathbb{N}}$ we have that

$$\Big(\sum_{j=1}^{\infty} p_j x_j\Big)^2 \leq \sum_{j=1}^{\infty} p_j x_j^2$$

and therefore

$$\begin{aligned}
\mathfrak{D}\Big(\sum_{i=1}^{\infty} p_i f_i\Big) &= \sum_{x,y\in M}\Big[\sum_{j=1}^{\infty} p_j f_j(x) - \sum_{j=1}^{\infty} p_j f_j(y)\Big]^2 \pi(x)L(x,y) \\
&= \sum_{x,y\in M}\Big[\sum_{j=1}^{\infty} p_j \big[f_j(x) - f_j(y)\big]\Big]^2 \pi(x)L(x,y) \\
&\leq \sum_{x,y\in M}\sum_{j=1}^{\infty} p_j \big[f_j(x) - f_j(y)\big]^2 \pi(x)L(x,y) \\
&= \sum_{j=1}^{\infty} p_j \mathfrak{D}(f_j).
\end{aligned}$$

(a) We suppose first that f is constant on the positively recurrent classes of L, taking the value c_x on the communication class $\mathcal{C}_x$ of $x \in \{\pi \neq 0\}$. Then by the explicit formula (A.83) of the Dirichlet form we have that

$$\begin{aligned}
\mathfrak{D}(f) &= \frac{1}{2}\sum_{x:\pi(x)\neq 0}\ \sum_{y:L(x,y)\neq 0} \big[f(y) - f(x)\big]^2 \pi(x)L(x,y) \\
&= \frac{1}{2}\sum_{x:\pi(x)\neq 0}\ \sum_{y:L(x,y)\neq 0} \big[c_x - c_x\big]^2 \pi(x)L(x,y) = 0.
\end{aligned}$$

Conversely suppose that $\mathfrak{D}(f) = 0$, let $\mathcal{C} \subseteq M$ be a positively recurrent class and let $x \in \mathcal{C}$. Then $\pi(x) > 0$ and $L(x,y) > 0$ for all $y \in \mathcal{C}$ and it is obvious by the explicit formula (A.83) of the Dirichlet form that $f|_{\mathcal{C}} \equiv f(x)$ is constant on $\mathcal{C}$. Finally (b) also follows obviously by the explicit formula (A.83) of the Dirichlet form. □

In the case that the invariant measure $\pi \in \mathbb{P}M$ is reversible, i.e. when it satisfies the detailed balance equations, there exists a variational formula for the Dirichlet form $\mathfrak{D}(f)$ of non-negative functions $f \in L^2(\pi)$.

Proposition A.3.11 *For every positive function $f \in L^2(\pi)$,*

$$\mathfrak{D}(f) = \sup_h\Big(-\big\langle \frac{f^2}{h}, Lh\big\rangle_{L^2(\pi)}\Big) = -\inf_h \sum_{x\in M}\frac{f^2(x)}{h(x)} Lh(x)\pi(x),$$

where the supremum is taken over all bounded positive functions bounded below by a strictly positive constant.

Proof We prove first that the supremum in the right hand side is bounded above by the Dirichlet form $\mathfrak{D}(f)$. So let $h : M \longrightarrow \mathbb{R}_+$ be a bounded function such that

$\inf_{x\in M} h(x) \geq c$ for some constant $c > 0$. We consider the function $\psi := \frac{h}{f}\mathbb{1}_{\{f>0\}} > 0$. Then since the invariant measure π is assumed reversible, the transition probabilities P_t, $t \geq 0$, of the generator L are self-adjoint in $L^2(\pi)$ and therefore

$$\begin{aligned}\Big\langle \frac{f^2}{h}, P_t h\Big\rangle_{L^2(\pi)} &= \Big\langle \frac{f}{\psi}, P_t h\Big\rangle_{L^2(\pi)} = \Big\langle \frac{f}{\psi}, P_t(f\psi)\Big\rangle_{L^2(\pi)} \\ &= \frac{1}{2}\Big(\Big\langle \frac{f}{\psi}, P_t(f\psi)\Big\rangle_{L^2(\pi)} + \Big\langle P_t\frac{f}{\psi}, f\psi\Big\rangle_{L^2(\pi)}\Big) \\ &= \frac{1}{2}\sum_{x\in M}\frac{f(x)}{\psi(x)}P_t(f\psi)(x)\pi(x) + \frac{1}{2}\sum_{x\in M}P_t\frac{f}{\psi}(x)f(x)\psi(x)\pi(x) \\ &= \frac{1}{2}\sum_{x,y\in M}\frac{f(x)}{\psi(x)}f(y)\psi(y)P_t(x,y)\pi(x) \\ &\quad + \frac{1}{2}\sum_{x,y\in M}\frac{f(y)}{\psi(y)}P_t(x,y)f(x)\psi(x)\pi(x) \\ &= \frac{1}{2}\sum_{x,y\in M}\Big(\frac{\psi(y)}{\psi(x)} + \frac{\psi(x)}{\psi(y)}\Big)f(x)f(y)P_t(x,y)\pi(x).\end{aligned}$$

But for all $a > 0$ we have that $a + \frac{1}{a} \geq 2$ and therefore

$$-\Big\langle \frac{f^2}{h}, P_t h\Big\rangle_{L^2(\pi)} \leq -\sum_{x,y\in M} f(x)f(y)P_t(x,y)\pi(x) = -\langle f, P_t f\rangle_{L^2(\pi)}$$

By adding $\|f\|_{L^2(\pi)} = \langle \frac{f^2}{h}, h\rangle_{L^2(\pi)}$ to both sides of this inequality and dividing by $t > 0$ we get

$$\langle \frac{f^2}{h}, \frac{h - P_t h}{t}\rangle_{L^2(\pi)} \leq \langle f, \frac{f - P_t f}{t}\rangle_{L^2(\pi)}.$$

Since we assume the jump rate to be bounded $\frac{P_t h - h}{t}$ converges uniformly to Lh and $\frac{P_t f - f}{t}$ converges to Lf in $L^2(\pi)$ and therefore by taking the limit as $t \to 0$ in the inequality above we get that

$$-\langle \frac{f^2}{h}, Lh\rangle_{L^2(\pi)} \leq -\langle f, Lf\rangle_{L^2(\pi)} = \mathfrak{D}(f).$$

Since h was arbitrary this proves the required inequality.

For the converse inequality we note that in the case that f is admissible in the supremum, i.e. when it is bounded and bounded below by a positive constant, we can take $h = f$ and therefore

$$\mathfrak{D}(f) = -\langle f, Lf\rangle_{L^2(\pi)} = -\Big\langle \frac{f^2}{f}, Lf\Big\rangle_{L^2(\pi)} \leq \sup_h\Big(-\Big\langle \frac{f^2}{h}, Lh\Big\rangle_{L^2(\pi)}\Big).$$

For the case of general non-negative $f \in L^2(\pi)$ we approximate f by a sequence of admissible functions. So let $f \in L^2_+(\pi)$. We set $f_n := \frac{1}{n} + f \wedge n$ for all $n \in \mathbb{N}$. Then, using again the reversibility of π, we can write

$$\begin{aligned}-\Big\langle \frac{f^2}{f_n}, Lf_n\Big\rangle_{L^2(\pi)} &= -\frac{1}{2}\sum_{x,y\in M}\Big(\frac{f^2(x)}{f_n(x)}f_n(y) + \frac{f^2(y)}{f_n(y)}f_n(x)\Big)L(x,y)\pi(x) \\ &= \frac{1}{2}\sum_{x,y\in M}\Big(\frac{f(y)^2}{f_n(y)} - \frac{f(x)^2}{f_n(x)}\Big)\big(f_n(y) - f_n(x)\big)\pi(x)L(x,y).\end{aligned}$$

Let $F_n : M \times M \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$, denote the function

$$F_n(x,y) = \Big(\frac{f(y)^2}{f_n(y)} - \frac{f(x)^2}{f_n(x)}\Big)\big(f_n(y) - f_n(x)\big).$$

We note that $F_n(x,x) = 0$ and

$$F_n(x,y) \longrightarrow \big[f(y) - f(x)\big]^2 =: F(x,y)$$

for all $x, y \in M$. Then if we consider the functions LF_n given by

$$LF_n(x) = \sum_{y:y\neq x} F_n(x,y)L(x,y) = \sum_{y\neq x} \Big(\frac{f(y)^2}{f_n(y)} - \frac{f(x)^2}{f_n(x)}\Big)\big(f_n(y) - f_n(x)\big)L(x,y)$$

it follows by Fatou's lemma that $LF \leq \liminf_{n\to\infty} LF_n$. Then we can write

$$-\big\langle \frac{f^2}{f_n}, Lf_n \big\rangle_{L^2(\pi)} = \frac{1}{2}\sum_{x\in M} LF_n(x)\pi(x)$$

and by the explicit formula (A.83) for the Dirichlet form and Fatou's lemma again,

$$\begin{aligned}
\mathcal{D}(f) &= \frac{1}{2}\sum_{x\in M} LF(x)\pi(x) \leq \liminf_{n\to\infty} \frac{1}{2}\sum_{x\in M} LF_n(x)\pi(x) \\
&= \liminf_{n\to\infty}\Big(-\big\langle \frac{f^2}{f_n}, Lf_n\big\rangle_{L^2(\pi)}\Big) \leq \sup_{n\in\mathbb{N}}\Big(-\big\langle \frac{f^2}{f_n}, Lf_n\big\rangle_{L^2(\pi)}\Big) \\
&\leq \sup_h \Big(-\big\langle \frac{f^2}{h}, Lh\big\rangle_{L^2(\pi)}\Big),
\end{aligned}$$

as required. $\square$

It often convenient to consider the functional $D : L^1_{+,1}(\pi) \longrightarrow \mathbb{R}_+$ defined on the space

$$L^1_{+,1}(\pi) := \Big\{ f \in L^1(\pi) \,\Big|\, f \geq 0, \ \int f d\pi = 1\Big\}$$

of all L^1-densities with respect to π by the formula

$$D(f) = \mathfrak{D}(\sqrt{f}).$$

It is a simple consequence of the proposition just proved that the functional D is convex and lower semi-continuous.

A.4 Prokhorov's Theorem in Completely Regular Submetrizable Hausdorff Topological Spaces

In this chapter we present the extension of the basic results (e.g. the portmanteau and Prokhorov theorems) on the weak topology of probability measures on polish spaces to probability measures in completely regular and submetrizable spaces. As it turns out Prokhorov's theorem is still valid in this more general case: uniform tightness implies relative compactness. If in addition the space submetrizable it also implies sequential relative compactness. The results of topological measure theory of this section are taken from [29] and [9]. All topological spaces considered in this section will always be Hausdorff topological spaces. Recall that a topological space M is completely regular if for every closed subset $F \subseteq M$ and every $x \in M$ there exists a continuous function $f : M \longrightarrow [0,1]$ such that

$$f|_F \equiv 0 \quad \text{and} \quad f(x) = 1.$$

Definition A.4.1 A topological space (M, τ) is called *submetrizable* if there exists a continuous (in the product topology) metric $d : M \times M \longrightarrow \mathbb{R}_+$.

It is easy to see that if M is submetrizable then any τ-continuous metric d metrizes the restriction of the topology τ on every compact subspace K of M. Indeed, let $(x_\alpha)_{\alpha \in \mathcal{A}}$ be a net in K d-converging to some $x \in K$. Then $d(x_\alpha, x) \longrightarrow 0$ and since K is τ-compact, there exists a subnet $(x_{\alpha_\beta})_\beta$ of $(x_\alpha)_{\alpha \in \mathcal{A}}$ τ-converging to some $y \in K$. But then by the continuity of d we have that $d(x_{\alpha_\beta}, y) \longrightarrow 0$, which implies that $x = y$ and therefore (x_α) τ-converges to x as claimed.

A particular category of completely regular spaces on which we will apply the Prokhorov-Le Cam theorem is the category of the duals of Banach spaces equipped with the w^*-topology. This is possible since every Hausdorff (T_1 in fact) topological group is a completely regular topological space ([21], section III.21, theorem 5). In addition the w^*-topology on X^* for separable Banach spaces X is also submetrizable:

Proposition A.4.1 *Suppose that X is separable Banach space and let $\psi : \mathbb{R}_+ \longrightarrow [0,1]$ be the function $\psi(x) = \frac{x}{x+1}$. Then the function $w : X^* \times X^* \longrightarrow \mathbb{R}_+$ given by*

$$w(x^*, y^*) = \sum_{k=1}^{d} \frac{1}{2^k} \psi\big(|\langle x_k, x^* - y^* \rangle|\big),$$

where $\{x_k\} \subseteq X$ is dense sequence in X is a translation invariant metric.

Furthermore, the topology induced on X^* by the metric w is weaker that the w^*-topology and metrizes the restriction of the w^*-topology on norm bounded subsets of X^*. In particular X^* is submetrizable.

Proof It is obvious that w is a metric and that if $x_a^* \longrightarrow X^*$ in the weak topology of X^*, then $x_a^* \longrightarrow x^*$ in the metric w and the topology of the metric w is weaker than the w^*-topology.

So we have to prove that w metrizes the w^*-topology on norm-bounded subsets of

X^*. So let $B \subseteq X^*$ be a a norm bounded subset and let $\{x_a^*\}_{a\in\mathcal{A}} \subseteq B$, $x^* \in X^*$ such that $w(x_a^*, x^*) \longrightarrow 0$. We will prove that given $x \in X$ we have that

$$\langle x, x_a^* \rangle \longrightarrow \langle x, x^* \rangle.$$

Since B is bounded there exists $C < +\infty$ such that

$$\|x^*\|_{X^*} \vee \sup_{a\in\mathcal{A}} \|x_a^*\|_{X^*} \leq C.$$

Let now $\varepsilon > 0$. We choose $k_0 \in \mathbb{N}$ such that $\|x - x_{k_0}\|_X \leq \frac{\varepsilon}{3C}$, and then

$$\big|\langle x, x^* \rangle - \langle x_{k_0}, x^* \rangle\big| \vee \sup_{a\in\mathcal{A}} \big|\langle x, x_a^* \rangle - \langle x_{k_0}, x_a^* \rangle\big| \leq C\|x - x_{k_0}\|_X \leq \frac{\varepsilon}{3}.$$

Next, since $w(x_a^*, x^*) \longrightarrow 0$ we can choose $a_0 \in \mathbb{N}$ such that

$$\big|\langle x_{k_0}, x_a^* \rangle - \langle x_{k_0}, x^* \rangle\big| \leq \frac{\varepsilon}{3} \tag{A.85}$$

for all $a \geq a_0$, and then, for all $a \geq a_0$ we have that

$$\begin{aligned} \big|\langle x, x_a^* \rangle - \langle x, x^* \rangle\big| &\leq \big|\langle x - x_{k_0}, x_a^* \rangle\big| + \big|\langle x_{k_0}, x_a^* - x^*)\rangle\big| + \big|\langle x_{k_0} - x), x^* \rangle\big| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which shows that $\langle x, x_a^* \rangle \longrightarrow \langle x, x^* \rangle$ as required and completes the proof. □

We will use the following terminology.

Definition A.4.2 Let μ be a Borel probability measure on a topological space M.
(a) A set $A \in \mathcal{B}_M$ is a called μ-regular if

$$\mu(A) = \inf_{U \supseteq A} \mu(U)$$

and μ is called regular if every Borel set is μ-regular. More generally, given any subfamily $\mathcal{F} \subseteq \mathcal{B}_M$ we say that μ is regular in $\mathcal{F}$ if every set F is μ-regular.
(b) A set $A \in \mathcal{B}_M$ is called μ-Radon if

$$\mu(A) = \sup_{K \Subset A} \mu(K),$$

where the supremum is taken over all compact subsets of A, and μ is called Radon if every Borel set is μ-Radon.
(c) The probability measure μ is called weakly Radon if it is regular and all open subsets of M are μ-Radon.
(d) The topological space M is called (weakly) Radon if all Borel probability measures on M are (weakly) Radon.

Obviously any Radon measure is regular. Furthermore, as usual a Borel probability measure μ on the topological space M is called tight if

$$\mu(M) = \sup_{K \Subset M} \mu(K).$$

We will denote by $\mathbb{P}_R M$ and $\mathbb{P}_t M$ the spaces of all of all Radon and tight probability measures, respectively. Obviously, $\mathbb{P}_R M \subseteq \mathbb{P}_t M$

Proposition A.4.2 *Let M be a topological space. Then $\mathbb{P}_t M = \mathbb{P}_R M$ iff every compact subspace of M is a Radon space.*

Proof We suppose first that every compact subspace of M is Radon and prove that $\mathbb{P}_t M \subseteq \mathbb{P}_R M$. So let $\mu \in \mathbb{P}_t M$, $B \in \mathcal{B}_M$ be any Borel subset of M and let $\varepsilon \in (0,1)$. Then since μ is tight, there exists a compact subset K of M such $\mu(M \setminus K) < \frac{\varepsilon}{2}$. By assumption, the subspace K is a Radon space, and therefore the probability measure $\bar{\mu}(\cdot) := \frac{1}{\mu(K)}\mu(\cdot \cap K) \in \mathbb{P}K$ is Radon. Therefore there exists a compact subset F of $K \cap B$ such that $\bar{\mu}([K \cap B] \setminus F) < \frac{\varepsilon}{2\mu(K)}$ and for which

$$\mu(B \setminus F) \leq \mu([K \cap B] \setminus F) + \mu(M \setminus K) < \varepsilon.$$

It is easy to see that any compact subset of the space K is compact subset of M, which since $\mu(B \setminus F) < \varepsilon$ and $\varepsilon > 0$ was arbitrary, proves that μ is Radon.

Conversely, suppose that $\mathbb{P}_t M = \mathbb{P}_R M$, let $K \subseteq M$ be compact and let $\mu \in \mathbb{P}K$. The measure $\bar{\mu}(\cdot) := \mu(K \cap \cdot) \in \mathbb{P}M$ is obviously tight and therefore by assumption it is Radon. Let now $B \in \mathcal{B}_K$ and $\varepsilon > 0$. Since μ is Radon, there exists a compact subset of M such that $F \subseteq B \subseteq K$ and $\mu(B \setminus F) < \varepsilon$. Then F is also compact in K, and therefore $\bar{\mu}(B \setminus F) = \mu(B \setminus F) < \varepsilon$. □

Corollary A.4.1 *For any submetrizable space (M, τ) it holds that $\mathbb{P}_t M = \mathbb{P}_R M$.*

Proof Indeed, since M is submetrizable there exists a continuous metric d on M, which as we have seen metrizes the restriction of τ on every compact subset $K \subseteq M$. Consequently, every compact subspace of M is metrizable, thus polish and thus Radon. □

It will be also useful to note that continuous images of Radon measures are Radon measures.

Proposition A.4.3 *Let $f : M \longrightarrow N$ be a continuous function between topological spaces and let $\mu \in \mathbb{P}_R M$ be a Radon measure. Then the push-forward measure $f_*\mu \in \mathbb{P}Y$ is Radon.*

Proof Indeed, let $B \in \mathcal{B}(Y)$ be a Borel subset of Y. Then $f^{-1}(B)$ is a Borel subset of X and therefore, given $\varepsilon > 0$, there exists a compact subset $K \subseteq f^{-1}(B) \subseteq X$ such that

$$\mu\big(f^{-1}(B) \setminus K\big) < \varepsilon.$$

Then $f(K) \subseteq f\big(f^{-1}(B)\big) \subseteq B$ and since f is continuous the set $f(K)$ is compact in Y and

$$f_*\mu\big(B \setminus f(K)\big) = \mu\big(f^{-1}(B \setminus f(K))\big) = \mu\big(f^{-1}(B) \setminus f^{-1}(f(K))\big) \leq \mu\big(f^{-1}(B) \setminus K\big) < \varepsilon,$$

which since $B \in \mathcal{B}(Y)$ and $\varepsilon > 0$ were arbitrary proves that $f_*\mu \in \mathbb{P}Y$ is Radon. □

Proposition A.4.4 *Let M be a completely regular topological space and let $\mu, \nu \in \mathbb{P}M$ be weakly Radon measures, such that*

$$\int f d\mu = \int f d\nu, \qquad \forall\, f \in BC(M). \tag{A.86}$$

Then $\mu = \nu$.

Proof Since μ, ν are Borel measures, it suffices to prove that $\mu(A) = \nu(A)$ for every open set A. But since μ, ν are weakly Radon, for every open $A \subseteq M$ we have that $\mu(A) = \sup_{K \Subset A} \mu(K)$, and likewise for ν, which shows that in order to prove that $\mu = \nu$ it suffices to prove that $\mu(K) = \nu(K)$ for compact subset K of M.

So let $K \subseteq M$ be compact. Since M is completely regular, for every $x \in M \setminus K$ there exists a function $f_x : M \longrightarrow [0, 1]$ such that $f_x(x) = 1$ and $f|_K \equiv 0$. We denote by $F(M)$ the set of all finite subsets of M, define an upwards directed set $\mathcal{A}$ given by

$$\mathcal{A} := \big\{\alpha \in F(M) | \alpha \cap K = \emptyset\big\}$$

with order the set inclusion, and define the non-decreasing net $(f_\alpha)_{\alpha \in \mathcal{A}} \subseteq C(M; [0, 1]) \subseteq BC(M)$ by

$$f_\alpha = \max_{x \in \alpha} f_x.$$

Obviously $f_\alpha|_K \equiv 0$ for every $\alpha \in \mathcal{A}$ and $f_\alpha(x) = 1$ for all $x \in \alpha$. Consequently, $f_\alpha \xrightarrow{\alpha} 1 - \mathbb{1}_K$ pointwise, since given $x \in M \setminus K$, for every $\alpha \geq \{x\} \in \mathcal{A}$ we have that

$$1 \geq f_\alpha(x) \geq f_x(x) = 1$$

and for every $x \in K$ we have that $f_a(x) = 0$ for all $\alpha \in \mathcal{A}$. Furthermore, this net is obviously increasing. In other words $\mathbb{1}_\alpha \leq f_\alpha \leq 1 - \mathbb{1}_K$ for all $\alpha \in \mathcal{A}$ and $\mathbb{1}_\alpha \longrightarrow 1 - \mathbb{1}_K$ pointwise, and

$$\int f_\alpha d\rho \leq \rho(M \setminus K) \tag{A.87}$$

for $\rho = \mu, \nu$ and all $\alpha \in \mathcal{A}$.

On the other hand, given $\varepsilon > 0$, for each $x \in M \setminus K$ we have that $f_x(x) = 1 > 1 - \varepsilon$ and therefore

$$M \setminus K \subseteq \bigcup_{x \in M \setminus K} \{f_x > 1 - \varepsilon\}.$$

Then, for any compact set $F \subseteq M \setminus K$, the family $\mathcal{U}_\varepsilon := (\{f_x > 1 - \varepsilon\})_{x \in M \setminus K}$ is an open covering of F, and so there exist $n = n(F, \mathcal{U}_\varepsilon) \in \mathbb{N}$ and $x_1, \ldots, x_n \in M \setminus K$ such that

$$F \subseteq \bigcup_{k=1}^{n} \{f_{x_k} > 1 - \varepsilon\}.$$

Then, for $\rho = \mu, \nu$, we have that for all $\alpha \geq \alpha_\varepsilon := \{x_1, \ldots, x_n\} \in \mathcal{A}$ that

$$\rho(F) \leq \rho\bigg(\bigcup_{k=1}^{n} \{f_{x_k} > 1 - \varepsilon\}\bigg) \leq \rho\big(\{f_\alpha > 1 - \varepsilon\}\big) \leq \frac{1}{1 - \varepsilon} \int f_\alpha d\rho.$$

Therefore, since $\varepsilon > 0$ is arbitrary, for fixed $F \Subset M \setminus K$ we have that

$$\rho(F) \leq \liminf_{\alpha \to \infty} \int f_\alpha d\rho$$

and since$M \setminus K$ is open and μ, ν are weakly Radon taking the supremum over all $F \Subset M \setminus K$, we get that

$$\rho(M \setminus K) \leq \liminf_{\alpha \to \infty} \int f_\alpha d\rho.$$

Together with (A.87) this proves that

$$\rho(M \setminus K) = \int f_\alpha d\rho, \qquad \text{for } \rho = \mu, \nu,$$

which by assumption (A.86) implies that $\mu(M \setminus K) = \nu(M \setminus K)$, and thus $\mu(K) = \nu(K)$ as required. $\square$

Lemma A.4.1 *Let (M, τ) be a completely regular topological space and let $f \in B(M)$ be a bounded function. Then f is lower semicontinuous iff*

$$f = \sup_{h \in BC(M),\, h \le f} h. \tag{A.88}$$

Proof We note first that we can make that additional assumption that $f \geq 0$. Indeed, if $m := \inf_{x \in M} f(x)$ and the claim holds for non-negative functions, that

$$f = m + (f - m) = m + \sup_{h \in BC(M),\, h \le f - m} h = \sup_{h \in BC(M),\, h \le f} h.$$

So in the rest of the proof we assume in addition that $f \geq 0$.

Obviously, we only have to prove that

$$f \le \sup_{h \in BC(M),\, h \le f} h$$

So in order to prove the claim it suffices to prove that for arbitrary all $x \in M$ we have that

$$f(x) \le \sup_{h \in BC(M): h \le f} h(x).$$

Since we assume f to be ≥ 0 we obviously have that

$$\sup_{h \in BC(M),\, h \le f} h \ge 0,$$

and therefore if $f(x) = 0$ we have nothing to prove. So we fix $x \in M$ such that $f(x) > 0$ and let $\varepsilon > 0 \in (0, f(x)/2)$ be arbitrary. Since f is lower semicontinuous, there exists an open neighborhood V_x of x such that

$$f(V_x) \subseteq (f(x) - \varepsilon, +\infty),$$

and since M is completely regular, there exists a continuous function $h_x : M \longrightarrow [0, f(x) - \varepsilon]$ such that $h_x(x) = f(x) - \varepsilon$ and $h_x|_{V_x^c} \equiv 0$. Then, $h_x \in BC(M)$ and $0 \le h_x \le \big[f(x) - \varepsilon\big]\mathbb{1}_{V_x} \le f$, where the last inequality follows from the choice of the neighborhood V_x. But then

$$f(x) = \varepsilon + h_x(x) \le \varepsilon + \sup_{h \in BC(M): h \le f} h(x).$$

So letting ε tend to zero we get that

$$f(x) \le \sup_{h \in BC(M): h \le f} h(x)$$

and since $x \inf^{-1}(0, \infty)$ was arbitrary, this proves the claim. The converse is obvious and does not require the complete regularity of M. $\square$

Definition A.4.3 A Borel probability measure μ in a topological space (M, τ) is called τ*-smooth* if for any upwards directed family $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of open sets we have that

$$\mu\Big(\bigcup_{\alpha \in \mathcal{A}} U_\alpha\Big) = \sup_{\alpha \in \mathcal{A}} \mu(U_\alpha).$$

It is easy to see that any weakly Radon measure on a topological space (M, τ) is τ-smooth. Indeed, let $\{U_\alpha\}_{\alpha \in \mathcal{A}} \subseteq \tau$ be an upwards directed family of open sets. We obviously have that

$$\mu\Big(\bigcup_{\alpha \in \mathcal{A}} U_\alpha\Big) \geq \sup_{\alpha \in \mathcal{A}} \mu(U_\alpha).$$

For the converse inequality, let $\varepsilon > 0$ be arbitrary. Then $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$ is open and since μ is weakly Radon there exists a compact set $K \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ such that

$$\mu\Big(\Big(\bigcup_{\alpha \in \mathcal{A}} U_\alpha\Big) \setminus K\Big) < \varepsilon,$$

and therefore

$$\mu\Big(\bigcup_{\alpha \in \mathcal{A}} U_\alpha\Big) \leq \mu\Big(\Big(\bigcup_{\alpha \in \mathcal{A}} U_\alpha\Big) \bigcap K\Big) + \varepsilon = \mu(K) + \varepsilon.$$

Now, the family $\{U_\alpha\}$ covers the compact set K, and therefore there exists $\alpha_1, \ldots, \alpha_n \in \mathcal{A}$ such that $K \subseteq \bigcup_{k=1}^n U_{\alpha_k}$. But since $\{U_\alpha\}$ is upwards directed, there exist $\alpha_0 \in \mathcal{A}$ such that $\bigcup_{k=1}^n U_{\alpha_k} \subseteq U_{\alpha_0}$, which shows that

$$\mu\Big(\bigcup_{\alpha \in \mathcal{A}} U_\alpha\Big) = \mu(K) + \varepsilon \leq \mu(U_{\alpha_0}) + \varepsilon \leq \sup_{\alpha \in \mathcal{A}} \mu(U_\alpha) + \varepsilon,$$

and proves the claim.

Lemma A.4.2 *Let (M, τ) be a topological space and let $\mu \in \mathbb{P}M$ be a τ-smooth measure. Then, if $f := \sup_{u \in \mathcal{U}} u$, where $\mathcal{U}$ is any upwards directed uniformly bounded family $\mathcal{U}$ of lower semicontinuous functions $u : M \longrightarrow \mathbb{R}$, we have that*

$$\int f d\mu = \sup_{u \in \mathcal{U}} \int u d\mu.$$

Proof We note first that we can assume in addition that $0 \leq f(x) < 1$ for all $x \in M$. Indeed, suppose this is true and let

$$b := \inf_{x \in M} f(x) \leq \sup_{x \in M} f(x) =: B.$$

Then for any $b' < b$, we have $f - b' > 0$ and the function $\bar{f} := \frac{f - b'}{B - b' + 1}$ satisfies

$$0 < \bar{f}(x) < 1, \qquad \forall\, x \in M$$

and $\bar{f} = \sup_{\bar{u} \in \bar{\mathcal{U}}} \bar{u}$ where $\bar{\mathcal{U}} = \{\frac{u - b'}{B - b' + 1} | u \in \mathcal{U}\}$. Then,

$$\int f d\mu = b' + (B - b' + 1) \int \bar{f} d\mu = b + (B - b' + 1) \sup_{\bar{u} \in \mathcal{U}} \int \bar{u} d\mu = \sup_{u \in \mathcal{U}} \int u d\mu.$$

So in what follows we assume that $f(M) \subseteq (0,1)$ and let $\varepsilon > 0$ be arbitrary. We have to prove that

$$\int f d\mu \leq \varepsilon + \sup_{u \in \mathcal{U}} \int u d\mu.$$

For each $n \in \mathbb{N}$ we have

$$\begin{aligned}
\int f d\mu &\leq \frac{1}{n}\mu\Big\{0 < f \leq \frac{1}{n}\Big\} + \frac{2}{n}\mu\Big\{\frac{1}{n} < f \leq \frac{2}{n}\Big\} + \cdots + \mu\Big\{\frac{n-1}{n} < f \leq 1\Big\} \\
&= \frac{1}{n}\sum_{k=0}^{n-1}\mu\Big\{f > \frac{k}{n}\Big\} = \frac{1}{n} + \frac{1}{n}\sum_{k=1}^{n-1}\mu\Big\{f > \frac{k}{n}\Big\}.
\end{aligned}$$

We fix $n > 2/\varepsilon$. Since $f = \sup_{u \in \mathcal{U}} u$, we have that $\{f > \frac{k}{n}\} = \bigcup_{u \in \mathcal{U}}\{u > \frac{k}{n}\}$ for each $k = 1, \ldots, n-1$. But since each $u \in \mathcal{U}$ is lower semicontinuous, for each $k = 1, \ldots, n-1$ the set $U_u^k := \{u > \frac{k}{n}\}$ is open, and the family $U^k := \{U_u^k\}_{u \in \mathcal{U}}$ is an upwards directed family of open sets for each fixed $k = 1, \ldots, n-1$. Therefore since μ is τ-smooth we have that

$$\mu\Big\{f > \frac{k}{n}\Big\} = \sup_{u \in \mathcal{U}} \mu\Big\{u > \frac{k}{n}\Big\}$$

for all $k = 1, \ldots, n-1$, and so for each $k = 1, \ldots, n-1$ we can choose $u_k \in \mathcal{U}$ such that

$$\mu\Big\{u_k > \frac{k}{n}\Big\} > \mu\Big\{f > \frac{k}{n}\Big\} - \frac{\varepsilon}{2}.$$

Then, since $\mathcal{U}$ is upwards directed, there exists $u_0 \in \mathcal{U}$ such that $u_0 \geq u_1 \vee \ldots \vee u_{n-1}$, and

$$\begin{aligned}
\int f d\mu &\leq \frac{\varepsilon}{2} + \frac{1}{n}\sum_{k=1}^{n-1}\mu\Big\{f > \frac{k}{n}\Big\} \leq \frac{\varepsilon}{2} + \frac{n-1}{n}\frac{\varepsilon}{2} + \frac{1}{n}\sum_{k=1}^{n-1}\mu\Big\{u_k > \frac{k}{n}\Big\} \\
&\leq \varepsilon + \frac{1}{n}\sum_{k=1}^{n-1}\mu\Big\{u_0 > \frac{k}{n}\Big\} = \varepsilon + \frac{1}{n}\sum_{k=1}^{n-1}k\mu\Big\{\frac{k}{n} < u_0 \leq \frac{k+1}{n}\Big\} \\
&\leq \varepsilon + \int u_0 d\mu \leq \varepsilon + \sup_{u \in \mathcal{U}} \int u d\mu.
\end{aligned}$$

Proposition A.4.5 (The portmanteau theorem) *Let (M, τ) be a completely regular topological space, let $(\mu_\alpha)_{\alpha \in \mathcal{A}}$ be a net in $\mathbb{P}M$, and let $\mu \in \mathbb{P}M$ be a τ-smooth measure. Then the following are equivalent:*
(a) *$\mu_\alpha \longrightarrow \mu \in \mathbb{P}M$ weakly.*
(b) *For every closed set $F \subseteq M$, $\limsup_\alpha \mu_\alpha(F) \leq \mu(F)$.*
(c) *For every open set $U \subseteq M$, $\liminf_\alpha \mu_\alpha(U) \geq \mu(U)$.*
(d) *For every μ-continuous set $A \subseteq M$, i.e. for every* Borel *set $A \subseteq M$ such that $\mu(\partial A) = 0$, it holds that*

$$\lim_\alpha \mu_\alpha(A) = \mu(A).$$

(b$'$) *For every bounded upper semicontinuous function $f : M \longrightarrow [-\infty, \infty)$,*

$$\limsup_\alpha \int f d\mu_\alpha \leq \int f d\mu.$$

(c′) *For every bounded lower semicontinuous function* $f : M \longrightarrow (-\infty, \infty]$,

$$\liminf_\alpha \int f d\mu_\alpha \geq \int f d\mu.$$

(d′) *For evert bounded μ-a.s. continuous function,* $\lim_\alpha \int f d\mu_\alpha = \int f d\mu$.

Proof We note first that since a Borel set $A \subseteq X$ is closed, open and μ-continuous iff $\mathbb{1}_A$ is lower semicontinuous, upper semicontinuous and μ-a.s. continuous respectively, it follows that (x′) implies (x), for $x = b, c, d$. Furthermore, (b) is obviously equivalent to (c), and (b′) is equivalent to (c′). Finally it is obvious that (d′) implies (a), and therefore it suffices to prove that (a)$\Longrightarrow$(c,′), (b)$\wedge$(c)$\Longrightarrow$(d), and that (d)$\Longrightarrow$(d′).
$(a) \Longrightarrow (c')$ Let $f : X \longrightarrow (-\infty, \infty]$ lower semicontinuous and bounded. By lemmas A.4.1 and A.4.2 we have that

$$\int f d\mu = \sup \left\{ \int h d\mu \,\Big|\, h \in BC(M),\ h \leq f \right\}.$$

which as we can easily see implies that $\liminf_\alpha \int f d\mu_\alpha \geq \int f d\mu$.
$(b) \wedge (c) \Longrightarrow (d)$ We note first that a Borel set $A \subseteq X$ is an μ-continuous set iff $\mu(A^o) = \mu(A) = \mu(\overline{A})$. So if A is an μ-continuous set, by (b) and (c) we have that

$$\begin{aligned} \mu(A^o) &\leq \liminf \mu_n(A^o) \leq \liminf \mu_n(A) \\ &\leq \limsup \mu_n(A) \leq \limsup \mu_n(\overline{A}) \leq \mu(\overline{A}), \end{aligned}$$

which according to the initial remark proves (d).
$(d) \Longrightarrow (d')$ Let $f : X \longrightarrow \mathbb{R}$ be a bounded, μ-a.s. continuous function and let $\varepsilon > 0$. Let $M_0 \in \mathcal{B}_M$ be a full measure set, $\mu(M_0) = 1$, of continuity points of f and let $a, b \in \mathbb{R}$ such that $a < f(x) < b$ for all $x \in M$. For each $r \in (a, b)$, we set $F_r := \{x \in X \mid f(x) = r\}$. The family $\{F_r\}_{r\in(a,b)}$ is a partition of M, and thus for every finite subset I of (a, b) we have that

$$\sum_{r\in I} \mu(F_r) = \mu\Big(\bigcup_{r\in I} F_r\Big) \leq 1.$$

Consequently $\sum_{r\in(a,b)} \mu(F_r) \leq 1 < +\infty$, and thus the set of all $r \in (a, b)$ for which $\mu(F_r) > 0$, is at most countable. There exists the a partition $a = a_0 < a_1 < \cdots < a_n = b$ of the interval (a, b), such that $a_i - a_{i-1} < \varepsilon$, $i = 1, \ldots, n$ and $\mu(F_{a_i}) = 0$, $i = 0, \ldots, n$. For each $i = 1, \ldots, n$, we set $E_i := f^{-1}\big([a_{i-1}, a_i)\big)$ and define the simple functions

$$\phi = \sum_{i=1}^n a_{i-1}\mathbb{1}_{E_i}, \qquad \psi = \sum_{i=1}^n a_i \mathbb{1}_{E_i}.$$

Obviously, $\phi \leq f \leq \psi$ and $\psi - \phi \leq \varepsilon$. Also, for all $i = 1, \ldots, n$ we have that $\partial E_i \subseteq F_{a_{i-1}} \cup F_{a_i} \cup (M \setminus M_0)$, and thus the E_i's are μ-continuous sets. By (d) it follows that $\lim \int \phi d\mu_n = \int \phi d\mu$ $\lim \int \psi d\mu_n = \int \psi d\mu$. Consequently,

$$\begin{aligned} \int f d\mu - \varepsilon &\leq \int \phi d\mu \leq \liminf_n \int f d\mu_n \leq \limsup_n \int f d\mu_n \\ &\leq \int \psi d\mu_n \leq \int f d\mu + \varepsilon, \end{aligned}$$

and since $\varepsilon > 0$ was arbitrary, the claim follows. □

Corollary A.4.2 *Let M be a completely regular topological space and let $(\mu_\alpha)_{\alpha\in\mathcal{A}} \subseteq \mathbb{P}_t M$ be a uniformly tight net converging to some $\mu \in \mathbb{P}M$. Then $\mu \in \mathbb{P}_t M$ is tight.*

Proof Indeed, let $\varepsilon > 0$. Since $(\mu_\alpha)_{\alpha\in\mathcal{A}}$ is uniformly tight, there exists a compact set $K \subseteq M$ such that

$$\sup_{\alpha\in\mathcal{A}} \mu_\alpha(M \setminus K) < \varepsilon.$$

But then since K is closed, we have by the portmanteau theorem that

$$\mu(M \setminus K) \leq \liminf_{\alpha} \mu_\alpha(M \setminus K) \leq \sup_{\alpha\in\mathcal{A}} \mu_\alpha(M \setminus K) < \varepsilon,$$

and thus μ is tight. □

Theorem A.4.1 (Prokhorov-Le Cam) *Let M be a completely regular topological space. Then any uniformly tight family $\mathcal{K} \subseteq \mathbb{P}_R M$ of probability measures is relatively compact in $\mathbb{P}_R M$ in the weak topology. If M is in addition submetrizable then any uniformly tight family $\mathcal{K} \subseteq \mathbb{P}_t M$ is also sequentially relatively compact in $\mathbb{P}_t M$ in the weak topology.*

Proof For the proof of the first assertion see [11], chapter 3 theorem 59, while for the second assertion we refer to [8] and [28]. □

In the case that M is completely regular and submetrizable we do not need to assume the family $\mathcal{K}$ to consist of Radon measures due to corollary A.4.1.

Bibliography

[1] David Aldous. Stopping times and tightness. *Annals of Probability*, 6(2):335–340, 1978.

[2] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savare. *Gradient Flows in Metric Spaces and in the Space of Probability Measures.* Lectures in Mathematics. Birkhauser, 2000.

[3] Inés Armendáriz, Stefan Grosskinsky, and Michail Loulakis. Zero-range condensation at criticality. *Stochastic Processes and their Applications*, 123:34663496, 2009.

[4] Inés Armendáriz and Michail Loulakis. Thermodynamic limit for the invariant measures in supercritical zero range processes. *Probability Theory and Related Fields*, 2008.

[5] G. Bouchitte and M. Valadier. Integral representation of convex functionals on a space of measures. *Journal of Functional Analysis*, 80(4):398–420, 1988.

[6] G. Bouchitti, T. Champion, and C. Jimenez. Completion of the Space of Measures in the Kantorovich Norm. *Trends in the Calculus of Variations*, pages 127–139, 2004.

[7] Haïm Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Universitext. Springer, 1983.

[8] Le Cam. Convergence in distribution of stochastic processes. *Univ. California Publ. Statist.*, 2:207–236, 1956.

[9] Charles Castaing, Paul Raynaud de Fitte, and Michel Valadier. *Young Measures on Topological Spaces, With Applications in Control Theory and Probability Theory*, volume 571 of *Mathematics and Its Applications.* Springer, 2004.

[10] Pilar Cembranos and Jose Mendoza. *Banach Spaces of Vector Valued Functions*, volume 1676 of *Lecture Notes in Mathematics.* Springer, 1997.

[11] Claude Dellacherie and Paul-André Meyer. *Probabilities and Potential*, volume 29 of *Mathematic Studies.* North Holland, 1978.

[12] Amir Dembo and Ofer Zeitouni. *Large Deviations Techinques and Applications.* Applications of Mathematics Stichastic Modelling and Applid Probability. Springer, second edition edition, 1997.

[13] J.-M. Drouffe, C. Godréche, and F. Camia. A simple stochastic model for the dynamics of condensation. *Journal of Physics A-Mathematical and General*, 31(1), 1998.

[14] Stewart N. Ethier and Thomas G. Kurtz. *Markov Processes: Characterization and Convergence*. Jon Wiley and Sons, 1986.

[15] M.R. Evans and Tom Hanney. Nonequilibrium Statistical Mechanics of the Zero-Range Process and Related Models. *Brazilian Journal of Physics*, 30(1):196–240, 2000.

[16] Gerald B. Folland. *Real Analysis, Modern Techniques and Their Applications*. Jon Wiley and Sons, 1999.

[17] S. Fornaro, S. Lisini, G. Savare, and G. Toscani. Measure valued solutions of sub-linear diffusion equations with a drift term. *Discrete and Continuous Dynamical Systems*, pages 1675–1708, 2012.

[18] Stefan Grosskinsky and Paul Chleboun. Condensation in stochastic particle systems with stationary product measures. *Journal of Statistical Physics*, 154:432–465, 2014.

[19] Stefan Grosskinsky, Gunter M. Schütz, and Herbert Spohn. Condensation in the zero range process: stationary and dynamical properties. *Journal of Statistical Physics*, 113:389–410, November 2003.

[20] M.Z. Guo, G.C. Papanikolaou, and S.R.S Varadhan. Nonlinear diffusion limit for a system with nearest neighbor interactions. *Communications in Mathematical Physics*, 118:31–59, 1988.

[21] Taqdir Husain. *Introduction to Topological Groups*. W.B. Saunders Company, 1966.

[22] Shin ichi OHTA. Gradient flown in wasserstein spaces over compact alexandrov spaces. *Amer. J. Math.*, 131:475–516, 2009.

[23] A.V. Ivanov. Quasilinear degenerate and nonuniformly elliptic and parabolic equations of second order. *Proceedings of the Steklov Institute of Mathematics*, 160:3–285, 1982.

[24] Adam Jakubowski. On the skorohod topology. *Annales de l'Institut Henri Poincaré, Section B*, 22(3):263–285, 1986.

[25] Claude Kipnis and Claudio Landim. *Scaling Limits of Interacting Particle Systems*. Springer, Germany, 1999.

[26] Olga Aleksandrovna Ladyzhenskaiya, Vsevolod Alekseevich Solonnikov, and Nina N. Ural'tseva. *Linear and Quasi-linear Equations of Parabolic Type*, volume 23 of *Translations of Mathematical Monographs*. American Mathematical Society, 2 edition, 1968.

[27] Daniel Revuz and Marc Yor. *Continuous Martingales and Brownian Motion*, volume 293 of *A Series of Comprehensive Studies in Mathematics.* Springer, 3rd edition, 1999.

[28] O.G. Smolyanov and S.V. Fomin. Measures on linear topological spaces. *Russian Mathematical Surveys*, 31(4):207–236, 1976.

[29] Flemming Topsoe. *Topology and Measure*, volume 133. Springer, 1970.

[30] Juan Luis Vazquez. *The Porous Medium Equation, Mathematical Theory.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2007.

[31] Cédric Villani. *Topics in Optimal Transportation*, volume 58 of *Graduate Studies in Mathematics.* American Mathematical Society, 2003.

[32] Cedric Villani. *Optimal transport, old and new.* Springer, 2006.

[33] Horng-Tzer Yau. Relative entropy and hydrodynamics of Ginzburg-Landau models. *Letters in Mathematical Physics*, 22:63–80, May 1991.