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Non-Hermitian quantum photonic systems

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Abstract

The goal of this thesis is the theoretical investigation of the production of indistinguishable photons from Quantum Electrodynamics (QED)-cavities and the formulation of an experimentally feasible system that maximizes the efficiency and the degree of the indistinguishability of the emitted photons. Firstly, we do a brief review of the quantum mechanical treatment of the interaction between light and matter, along with the physics of open quantum-optical systems, and then derive the Hamiltonian of the Dissipative Jaynes-Cummings Model. Additionally, we analytically derive the expression of the degree of the indistinguishability of the produced photons. Furthermore, we theoretically investigate the production of indistinguishable photons from a qubit placed inside an optical cavity (with dissipation). More specifically, we calculate the efficiency and the degree of the indistinguishability for various values of the system's parameters, which are the qubit-cavity coupling (g), the qubit decay rate (γ), the qubit pure dephasing rate ($\bar{\gamma}$), and the cavity decay rate (κ). Also, we develop computer code and reproduce the theoretical results numerically. Furthermore, we study the addition of an empty cavity to the system, coupled to the first cavity, and investigate possible regions of the above parameters where the efficiency and the degree of indistinguishability are maximized. Lastly, we suggest a new extension of the initial system, where we add two empty cavities, one on each side of the first cavity.

1 Introduction

Historically, the first quantum "revolution" of the last century led to great technological inventions, such as the laser and semiconductors, which were based on phenomena such as the quantization of energy and the wave-particle duality of light. Respectively in the modern era, the field of Quantum Technologies has to take advantage of other purely quantum mechanical phenomena, such as the superposition of quantum states, the entanglement and the axioms of quantum measurements. More specifically, the field of Quantum Optics, that studies the properties of light and its interactions with matter, has attracted great research interest. According to [1], for the progress of Quantum Optics the following technologies need to be developed: efficient and fast photon counters, lineal and non-linear photonic circuits, and single-photon sources. The desired features of such sources are the great efficiency, the creation of pulses of light of at most one photon with a certain polarisation and spatial/temporal state, and the indistinguishability of the produced photon, in the sense that when two of them enter a beam splitter they produce an interference pattern. Indistinguishable photons have found a plethora of applications in field such as quantum information, quantum metrology and others that are presented in great detail in [2].

The most common single-photon sources are based on non-linear frequency conversion. This sources are based on a phenomenon called spontaneous parametric down-conversion [3], where a photon passes through a non-linear optical medium (such as a non linear crystal or a $\chi^{(2)}$ medium) and two lower energy photons come out. By adjusting the conditions of the system we can achieve the production of two photons in the same frequency, polarisation and spatial mode, which then we can filter and collect high indistinguishability single photons. The biggest drawback of this method, though, is the very low efficiency of the non-linear processes, which limits their scalability.

In recent years, another single-photons source that is used is based in the spontaneous decay of single solid state quantum systems, such as semiconducting quantum dots [1], [2]. More specifically, it's of great scientific interest whether or not it is possible to conduct interference experiments using photons from two different quantum emitters. An ideal quantum emitter with two energy levels coupled to quantum fluctuations of the vacuum produces fully indistinguishable photons. In contrast with the first method, the photons produced from quantum emitters don't have a phase correlation. This happens even in the case when the two emitters are pumped from the same laser and are filtered in a way to have the same polarisation and transition frequency, because the emitters are coupled to an environment that causes different phase loss in each of them. The loss of these phase correlations between the emitters leads to a loss of phase correlations between the respective fields produced, affecting the interference pattern between them. The above phenomenon is modeled in [4], where the authors derive the expression of the indistinguishability of the photons to be:

$$I = \frac{\gamma}{\gamma + \bar{\gamma}}, \quad (1)$$

where $\gamma = \frac{1}{T_1}$ is the decay rate of the quantum emitter (spontaneous decay rate) and $\frac{\bar{\gamma}}{2}$ is the pure dephasing rate of the emitter. The fraction $\frac{\bar{\gamma}}{\gamma}$ for typical solid state quantum emitters is of the order of magnitude $10^3 - 10^6$, which consequently leads to very small values of indistinguishability.

In this diploma thesis we present a theoretical study of the production of indistinguishable photons from Cavity-QEDs with the goal of finding an experimentally feasible system that maximizes the efficiency and the indistinguishability of the produced photons. Firstly, in chapter 2, we present a brief review of the quantum mechanical treatment of the interaction between light and matter, and we describe the physics of open quantum systems and derive the

Jaynes-Cummings Hamiltonian with dissipation terms. Additionally, we analytically derive an expression for the indistinguishability of the produced photons, using the Green Function formalism. Then, in Chapter 3, we theoretically study the production of indistinguishable photons from a simple system consisting of a Qubit placed inside an optical cavity with losses [5]. Moreover, in Chapter 4 we calculate analytically and numerically the efficiency and degree of indistinguishability of the system for the various parameters of the system, i.e. the coupling constant g , the spontaneous decay γ , the cavity decay κ and the pure dephasing rate $\bar{\gamma}$. Afterwards, in Chapter 5, we add an empty cavity coupled to the above system, and investigate possible regions where the efficiency and the indistinguishability are improved. Lastly, in Chapter 6, we suggest a further extension of the setup, by adding another cavity and analysing the Identical Cavity Limit.

2 Theoretical Background

The goal of Chapter 2 is a brief presentation of the theoretical tools needed for the study of the quantum systems that we will study in the next chapters. Firstly, we will derive the Jaynes-Cummings Hamiltonian, deriving each term from first principles and explaining their physical meaning. Next, we will outline some basic concepts about the time evolution of closed and open quantum systems. Furthermore, in this chapter we will derive the expression of the indistinguishability of the photons produced from the quantum systems of interest, following the methodology of the paper of Imamoğlu [6]. For the analytical calculation of the indistinguishability, and more specifically for the time correlation functions that appear, we will use the Non-Equilibrium Green Function Formalism [7]. Lastly, we will discuss about the indistinguishability of photons using some simple limiting cases to gain some physical intuition.

2.1 Derivation of the Jaynes-Cummings Hamiltonian

The Jaynes-Cummings model, as was originally described [8], consists of a two-level atom that interacts with a single normal mode of the quantized electromagnetic field with frequency ω (almost equal to the transition frequency of the atom). Unlike the semi-classical case of the light-matter interaction, where the atom is treated quantum mechanically and the Electromagnetic (E/M) field classically, in the Jaynes Cummings (JC) model both of them are treated quantum mechanically. This model quickly led to the discovery of novel phenomena, such as the Rabi oscillations, the periodic spontaneous collapse and revival of the probability [9] and others. Since then, this model has found many applications in quantum optics, quantum information and atomic physics [10].

The Jaynes-Cummings Hamiltonian that describes the system is

$$H_{total} = H_{field} + H_{atom} + H_{interaction} \quad (2)$$

where the first term describes the quantized electromagnetic field, the second describes the quantized two level atom and the third the interaction between the two. In the following sections, we will derive the form of each term from first principles. We will follow the methodology of [11]–[13].

2.1.1 Quantization of the Electromagnetic Field in a Cavity

The Electromagnetic field is defined by two quantities, the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$, which in classical electromagnetism are functions of position and time that satisfy Maxwell's Equations:

$$\nabla \cdot \mathbf{D} = \rho, \quad (3a)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3b)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (3c)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad (3d)$$

where \mathbf{J} and ρ are the current and charge density respectively. In the vacuum there are no free currents and free charges so $\mathbf{J} = \rho = 0$ and the fields are equal to $\mathbf{D} = \epsilon_0 \mathbf{E}$ and $\mathbf{H} = \frac{1}{\mu_0} \mathbf{B}$, where ϵ_0 is the electrical susceptibility of the vacuum and μ_0 is the magnetic susceptibility

of the vacuum. So, Maxwell's Equations in the vacuum are

$$\nabla \cdot \mathbf{D} = 0, \quad (4a)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (4b)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (4c)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}. \quad (4d)$$

The fields can be written with the help of a scalar potential $\phi(\mathbf{r}, t)$ and of a vector potential $\mathbf{A}(\mathbf{r}, t)$ as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (5)$$

Since these potentials aren't uniquely defined, we can choose a gauge, i.e. a transformation of the form $\phi' = \phi - \frac{\partial \chi}{\partial t}$ and $\mathbf{A}' = \mathbf{A} + \nabla\chi$. We choose the Coulomb Gauge, where $\phi = 0$ and $\nabla \cdot \mathbf{A} = 0$ and so the fields are given by

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \quad (6a)$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (6b)$$

Thus, the equations (4) simplify to the following vector equation

$$\nabla^2 \mathbf{A}(\mathbf{r}, t) = \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2}. \quad (7)$$

By using the method of separation of variables we can write the vector potential as

$$\mathbf{A}(\mathbf{r}, t) = \sum_k c_k \mathbf{u}_k(\mathbf{r}) a_k(t) + c_k^* \mathbf{u}_k^*(\mathbf{r}) a_k^*(t). \quad (8)$$

Substituting the above expression in the wave equation (7) we get

$$(\nabla^2 + \omega_k^2) \mathbf{u}(\mathbf{r}) = 0, \quad (9)$$

$$\left(\frac{\partial^2}{\partial t^2} + \omega_k^2\right) a_k(t) = 0. \quad (10)$$

and thus we find

$$\mathbf{A}(\mathbf{r}, t) = -i \sum_k \sqrt{\frac{\hbar}{2\omega_k \epsilon_0}} [\mathbf{u}_k(\mathbf{r}) a_k e^{-i\omega_k t} + \mathbf{u}_k^*(\mathbf{r}) a_k^* e^{i\omega_k t}] \quad (11)$$

Since the E/M is placed inside a cavity, u_k must satisfy the boundary conditions of the cavity, thus only certain modes of the field are allowed. The radiation modes satisfy the orthogonality conditions

$$\int d^3\mathbf{r} u_n^*(\mathbf{r}) u_m(\mathbf{r}) = \delta_{nm}. \quad (12)$$

We define the coordinates

$$q_k = \sqrt{\frac{\hbar}{2\omega_k}}(a + a^*), \quad (13a)$$

$$p_k = -i\sqrt{\frac{\omega_k\hbar}{2}}(a - a^*), \quad (13b)$$

Using the above expansion, we can write the Hamiltonian as

$$\begin{aligned} H_{field} &= \frac{1}{2} \int_V d^3r (\epsilon_0 \mathbf{E}^2 + \mu_0^{-1} \mathbf{B}^2) \\ &= \sum_k \frac{1}{2} (p_k^2(t) + \omega_k^2 q_k^2(t)) \end{aligned} \quad (14)$$

The quantization of the electromagnetic field is straightforward, we just substitute p, q with operators $\hat{p}(\hat{q})$, which satisfy the commutation relations

$$\begin{aligned} [q_i, q_j] &= [p_i, p_j] = 0 \\ [q_i, p_j] &= i\hbar\delta_{ij} \end{aligned} \quad (15)$$

and define the annihilation and creation operators

$$\begin{aligned} a_n &= \frac{1}{\sqrt{2\hbar\omega_n}}(\omega_n q_n - ip_n), \\ a_n^\dagger &= \frac{1}{\sqrt{2\hbar\omega_n}}(\omega_n q_n + ip_n), \end{aligned} \quad (16)$$

that satisfy

$$[a_m, a_n^\dagger] = \delta_{mn}, \quad (17a)$$

$$[a_m, a_n] = 0, \quad (17b)$$

$$[a_m^\dagger, a_n^\dagger] = 0. \quad (17c)$$

So the Hamiltonian of the field becomes

$$\begin{aligned} H_{field} &= \frac{1}{2} \int_V d^3r (\epsilon_0 \mathbf{E}^2 + \mu_0^{-1} \mathbf{B}^2) \\ &= \sum_k \hbar\omega_k (a_k^\dagger a_k + \frac{1}{2}) \\ &\stackrel{\text{renormalization}}{=} \sum_k \hbar\omega_k a_k^\dagger a_k. \end{aligned} \quad (18)$$

The eigenstates of the operator $a_k^\dagger a_k$ are call Fock states $|n_k\rangle$ and satisfy

$$\langle m_k | n_{k'} \rangle = \delta_{mn} \delta_{kk'}, \quad (19)$$

$$a_k^\dagger a_k |n_k\rangle = n_k |n_k\rangle, \quad (20)$$

$$a_k |n_k\rangle = \sqrt{n_k} |n_k - 1\rangle, \quad (21)$$

$$a_k^\dagger |n_k\rangle = \sqrt{n_k + 1} |n_k + 1\rangle, \quad (22)$$

$$|n_k\rangle = \frac{(a_k^\dagger)^{n_k}}{(n_k!)^{\frac{1}{2}}} |0\rangle. \quad (23)$$

2.1.2 Quantum Two Level System

Since we found the Hamiltonian of the field, next we have to find the Hamiltonian of the atom. We assume that the lower energy state (ground state) $|g\rangle$ has zero energy $E_g = 0$ and that the upper energy state (excited state) $|e\rangle$ has energy $E_e = \hbar\omega_a$. Thus the Hamiltonian is

$$H_{atom} = E_e |e\rangle\langle e| + E_g |g\rangle\langle g| = \hbar\omega_a |e\rangle\langle e| + 0 |g\rangle\langle g| = \hbar\omega_a |e\rangle\langle e|. \quad (24)$$

It is convenient to define a set of lowering(raising) operators $\sigma^- (\sigma^+)$, which in the basis $|g\rangle |e\rangle$ are written as

$$\sigma^+ = |e\rangle\langle g|, \quad \sigma^- = |g\rangle\langle e|. \quad (25)$$

From the above we conclude that

$$|e\rangle\langle e| = \sigma^+ \sigma^-, \quad |g\rangle\langle g| = \sigma^- \sigma^+. \quad (26)$$

And so, the Hamiltonian of the atom is written as

$$H_{atom} = \hbar\omega_a \sigma^+ \sigma^-. \quad (27)$$

2.1.3 Atom-E/M Field Interaction

In the previous section we found the Hamiltonian of each part of the system, without describing a mechanism where the two interact. In our analysis, we assume the electronic transitions of the atom to be coupled to the electric part of the E/M field inside the cavity. In this case, the atom-field interaction is caused by the coupling of the electric dipole moment of the atom with the electric part of the field. We assume the field is monochromatic and so it's written as

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= -i \sqrt{\frac{\hbar}{2\omega\epsilon_0}} [\mathbf{u}(\mathbf{r}) a e^{-i\omega t} + \mathbf{u}^*(\mathbf{r}) a^* e^{i\omega t}] \\ &= \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \hat{\mathbf{e}} \left[a \left\{ e^{i\mathbf{k}\cdot\mathbf{r}} \right\} + \text{c.c.} \right], \end{aligned} \quad (28)$$

where $\{\}$ symbolises a superposition of plane waves that satisfy the boundary conditions. We also assume that the wavelength of the field is much greater than the size of the atom. That approximation is called Dipole Approximation and mathematically is given by

$$r/\lambda \ll 1 \xrightarrow{k=\frac{2\pi}{\lambda}} \mathbf{k} \cdot \Delta\mathbf{r} \ll 1 \xrightarrow{\Delta\mathbf{r}=R-\mathbf{r}} e^{i\mathbf{k}\cdot\mathbf{r}} \approx e^{i\mathbf{k}\cdot\mathbf{R}}. \quad (29)$$

Thus, the electric field is written as

$$\mathbf{E}(\mathbf{r}, t) = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \hat{\mathbf{e}} [aE_0 + \text{c.c.}], \quad (30)$$

with $E_0 = \{e^{ik_n R}\}$. Using the above approximation, the interaction Hamiltonian is

$$H_{int} = -q\mathbf{r} \cdot \mathbf{E}(\mathbf{R}), \quad (31)$$

where \mathbf{R} is the vector of the center of mass of the atom and $q\mathbf{r}$ is the electric dipole moment. To proceed, we need to describe the dipole moment quantum mechanically. We follow the steps of (24) and find

$$q\mathbf{r} = \sum_{i,j} |i\rangle\langle i| q\mathbf{r} |j\rangle\langle j|, \quad (32)$$

where the sum is taken over the eigenstates of the qubit $|g\rangle, |e\rangle$. If we also assume that the atom is center-symmetric, then it won't have a permanent dipole moment and thus $q\mathbf{r}$ will have only off-diagonal elements. So we define

$$\mathbf{p} = q \langle e|\mathbf{r}|g\rangle, \quad (33a)$$

$$\mathbf{p}^* = q \langle g|\mathbf{r}|e\rangle, \quad (33b)$$

and we can write the dipole moment as

$$q\mathbf{r} = \mathbf{p} |e\rangle\langle g| + \mathbf{p}^* |g\rangle\langle e| = \mathbf{p}\sigma^+ + \mathbf{p}^*\sigma^-. \quad (34)$$

Using all of the above, we find that the interaction Hamiltonian is given by

$$H_{int} = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \hat{\mathbf{e}} \cdot (E_0 a + E_0^* a^\dagger) (\mathbf{p}\sigma^+ + \mathbf{p}^*\sigma^-). \quad (35)$$

For simplicity, we consider a real electric field ($E_0 \in \mathbb{R}$) and a real dipole moment ($p \in \mathbb{R}$) and we get

$$H_{int} = \hbar g (a + a^\dagger) (\sigma^+ + \sigma^-) = \hbar g (a\sigma^+ + a\sigma^- + a^\dagger\sigma^+ + a^\dagger\sigma^-), \quad (36)$$

where g is the coupling constant between the field and the atom and is equal to $g = pE_0 \sqrt{\frac{\omega}{\hbar\epsilon_0 V}}$. Each term of (36) correspond to a physical process. More specifically,

- $a \sigma^+ \rightarrow$ Photon Annihilation and Atom Excitation
- $a \sigma^- \rightarrow$ Photon Annihilation and Atom De-Excitation
- $a^\dagger \sigma^+ \rightarrow$ Photon Creation and Atom Excitation
- $a^\dagger \sigma^- \rightarrow$ Photon Creation and Atom De-Excitation

We observe that the 2nd and 3rd term violate the conservation of energy principle. The approximation that neglects these terms is called the Rotating Wave Approximation and its use is justified mathematically if we treat the problem in the Interaction Picture. An operator A in the Schrödinger picture can be expressed in the Interaction Picture using the formula $A_I = e^{\frac{iH_0 t}{\hbar}} A_S e^{-\frac{iH_0 t}{\hbar}}$, where H_0 is the unperturbed Hamiltonian, i.e. $H_0 = H_{field} + H_{atom}$.

Thus we find that

$$a_I = ae^{-i\omega t}, \quad a_I^\dagger = ae^{i\omega t}, \quad (37a)$$

$$\sigma_I^- = \sigma^- e^{-i\omega_a t}, \quad \sigma_I^+ = \sigma^+ e^{i\omega_a t}, \quad (37b)$$

and the interaction Hamiltonian in the Interaction Picture is written as

$$H_{int,I} = \hbar g(a\sigma^+ e^{i(\omega_a - \omega)t} + a\sigma^- e^{-i(\omega_a + \omega)t} + a^\dagger\sigma^+ e^{i(\omega_a + \omega)t} + a^\dagger\sigma^- e^{-i(\omega_a - \omega)t}). \quad (38)$$

Assuming that the system is close to resonance ($\omega \approx \omega_a$) then we observe that the term $e^{\pm i(\omega_a - \omega)t}$ goes to one and the term $e^{\pm i(\omega_a + \omega)t}$ oscillates rapidly and can thus be ignored. So, the Hamiltonian after the Rotating Wave Approximation becomes

$$H_{total} = \hbar\omega a^\dagger a + \hbar\omega_a \sigma^+ \sigma^- + \hbar g(a^\dagger \sigma^- + a\sigma^+). \quad (39)$$

2.2 Time Evolution of Quantum Systems

2.2.1 Closed Quantum Systems

According to Quantum Mechanics, the wave function ψ of a closed quantum system evolves in time according to the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle. \quad (40)$$

The solution of the Schrödinger equation can be expressed via the unitary operator $U(t, t_0)$, that transforms the initial state of the system $|\psi(t_0)\rangle$ at time t_0 to the state $|\psi(t)\rangle$ at time t .

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle. \quad (41)$$

The operator U is called time evolution operator and it satisfies the relations

$$i\hbar \frac{\partial}{\partial t} U(t, t_0) = H(t)U(t, t_0), \quad (42a)$$

$$U(t_0, t_0) = I. \quad (42b)$$

From Eq.(42a) and because the Hamiltonian is a Hermitian operator it's proven that $U(t, t_0)^\dagger U(t, t_0) = U(t, t_0)U(t, t_0)^\dagger = I$. Also, in the case of closed systems the Hamiltonian is time independent and the differential equation (41) is solved by integration $U(t, t_0) = e^{-i\frac{H(t-t_0)}{\hbar}}$. There are cases when the Hamiltonian is time dependent, e.g. controlled by external fields. In this case the time evolution operator is given by $U(t, t_0) = \text{Exp}[-\frac{i}{\hbar} \int_{t_0}^t d\tau H(\tau)]$. If the system initially is prepared in a mixed state that corresponds to a statistical ensemble then we need to introduce the density operator ρ . For $t = t_0$ we have

$$\rho(t_0) = \sum_i w_i |\psi_i(t_0)\rangle\langle\psi_i(t_0)|, \quad (43)$$

where w_i are positive numbers corresponding to the statistical weight of the state $\psi_i(t_0)$. So, at time t the density matrix will be

$$\begin{aligned} \rho(t) &= \sum_i w_i |\psi_i(t)\rangle\langle\psi_i(t)| = \sum_i w_i U(t, t_0) |\psi_i(t_0)\rangle\langle\psi_i(t_0)| U^\dagger(t, t_0) \\ &= U(t, t_0)\rho_0 U^\dagger(t, t_0). \end{aligned} \quad (44)$$

If we differentiate the last part of the equation we have

$$\begin{aligned}
\dot{\rho}(t) &= \dot{U}(t, t_0)\rho_0 U^\dagger(t, t_0) + U(t, t_0)\dot{\rho}_0 U^\dagger(t, t_0) + U(t, t_0)\rho_0 \dot{U}^\dagger(t, t_0) \\
&= \frac{1}{i\hbar}H(t)U(t, t_0)\rho_0 U^\dagger(t, t_0) + 0 + U(t, t_0)\rho_0 \frac{-1}{i\hbar}H(t)U^\dagger(t, t_0) \\
&= \frac{1}{i\hbar}(H(t)\rho(t) - \rho(t)H(t)) = \frac{1}{i\hbar}[H(t), \rho(t)].
\end{aligned} \tag{45}$$

2.2.2 Open Quantum Systems

$$S + B, H_S \otimes H_B, \rho = \rho_S \otimes \rho_B$$

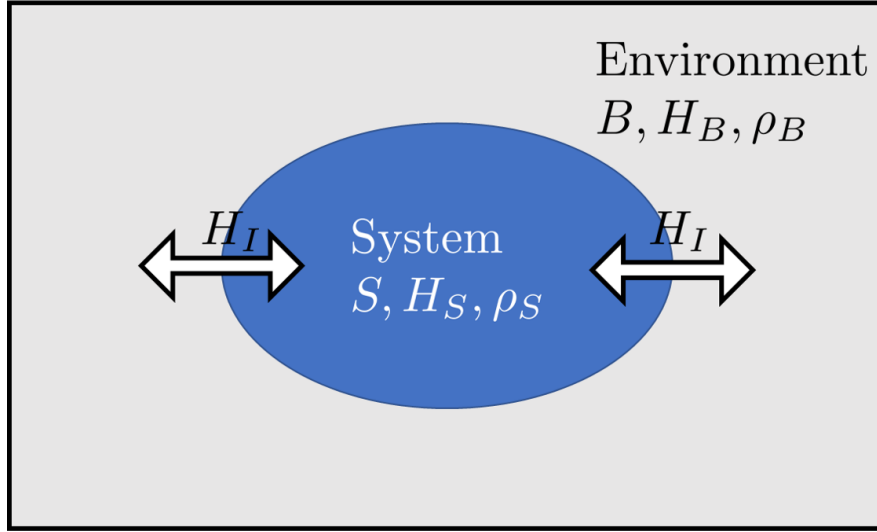


Figure 1: Illustration of an open quantum system S .

In general, an open quantum system [14], [15] consists of a quantum system S that is coupled to another, much larger and with infinite degrees of freedom, quantum system B called the environment (or bath). The total system $S + B$ is considered to be closed, so it will follow the dynamics discussed previously. In contrast with a closed system, the state of the system S will change because of its internal processes and because of its interaction with the environment. The dynamics of the system S is also called reduced, since it comes by eliminating the environment from the dynamics of the whole system. We define as \mathcal{H}_S the Hilbert space of the reduced system, as \mathcal{H}_B the Hilbert Space of the environment and as $\mathcal{H}_{tot} = \mathcal{H}_S \otimes \mathcal{H}_B$ the total. So the Hamiltonian is.

$$H(t) = H_S \otimes I_B + I_S \otimes H_B + H_I(t), \tag{46}$$

where H_S is the Hamiltonian of the system, H_B is the Hamiltonian of the environment and H_I is the interaction Hamiltonian between the system and the environment. If the state of the total system is described by the density matrix ρ , then the average value of an operator of the system $A \otimes I_B$ is

$$\langle A \rangle = \text{tr}_S\{A\rho_S\}, \tag{47}$$

where

$$\rho_S = \text{tr}_B \rho, \tag{48}$$

is the reduced density matrix given by the partial trace with respect to the degrees of freedom of the environment (tr_B). Since the total density matrix evolves as ($\rho(t) = U(t, t_0)\rho_0U^\dagger(t, t_0)$), the reduced density matrix evolves as

$$\rho_S(t) = \text{tr}_B\{U(t, t_0)\rho_0U^\dagger(t, t_0)\}. \quad (49)$$

Following the steps of (45), we find that the differential equation that describes the time evolution of ρ_S is

$$\frac{d}{dt}\rho_S(t) = \frac{1}{i\hbar} \text{tr}_B [H(t), \rho(t)]. \quad (50)$$

2.2.3 Born-Markov Equation

Although Eq.(50) fully describes the time evolution of the reduced density matrix, it isn't easily evaluated analytically or numerically. That's why, after a series of approximations as described in [11] we will derive the Lindblad Equation.

It's easier to describe the dynamics of the system in the interaction picture (we will denote the operators in the interaction picture as $\tilde{\rho}$), where the time evolution of the system is

$$\frac{d}{dt}\tilde{\rho}(t) = \frac{1}{i\hbar} [\tilde{H}_I, \tilde{\rho}], \quad (51)$$

which can be integrated and gives

$$\tilde{\rho}(t) = \tilde{\rho}_0 + \frac{1}{i\hbar} \int_0^t ds [\tilde{H}_I(s), \tilde{\rho}(s)]. \quad (52)$$

The above expression is problematic because of the integral. We substitute the integral expression (52) to (51) and we find

$$\frac{d}{dt}\tilde{\rho}(t) = \frac{1}{i\hbar} [\tilde{H}_I(t), \tilde{\rho}_0] - \frac{1}{\hbar^2} \int_0^t ds [\tilde{H}_I(t), [\tilde{H}_I(s), \tilde{\rho}(s)]]. \quad (53)$$

We are interested in the time evolution of the system, so we trace over the degrees of freedom of the environment

$$\frac{d}{dt}\tilde{\rho}_S = \frac{1}{i\hbar} \text{tr}_B\{[\tilde{H}_I(t), \tilde{\rho}_0]\} - \frac{1}{\hbar^2} \int_0^t ds \text{tr}_B [\tilde{H}_I(t), [\tilde{H}_I(s), \tilde{\rho}(s)]]. \quad (54)$$

In order to treat the quantum systems in the next chapters, we need to define the interaction Hamiltonian of the open system. Without loss of generality, we can write $H_I(t)$ of (46) as

$$H_I(t) = \sum_i S_i \otimes B_i, \quad (55)$$

with $S_i \in \mathcal{H}_S$ and $B_i \in \mathcal{H}_B$ hermitian operators.

The expression (54) can be further simplified, because the density matrix of the environment for $t = 0$ is diagonal with respect to the energy eigenbasis, since the environment is in thermal equilibrium. So

$$\text{tr}_B\{[\tilde{H}_I(t), \tilde{\rho}_0]\} = \sum_i \text{tr}_B\{[\tilde{S}_i \otimes \tilde{B}_i, \tilde{\rho}_{S,0} \otimes \tilde{\rho}_{B,0}]\} = \sum_i \tilde{S}_i \tilde{\rho}_{S,0} \text{tr}\{[\tilde{B}_i, \tilde{\rho}_{B,0}]\} = 0 \quad (56)$$

So we arrive at the equation

$$\frac{d}{dt}\tilde{\rho}_S = -\frac{1}{\hbar^2} \int_0^t ds \operatorname{tr}_B \left[\tilde{H}_I(t), \left[\tilde{H}_I(s), \tilde{\rho}(s) \right] \right]. \quad (57)$$

This time evolution equation isn't closed, since it depends on the total density matrix $\tilde{\rho}$, so we need to make two extra approximations, the Born approximation and the Markov approximation.

Born Approximation: We suppose that the coupling between the system and the environment is weak. That means that the state of the environment isn't influenced by the interaction with the system and thus stays unperturbed ($\rho_B(t) = \rho_{B,0}$). Thus $\rho(t) \approx \rho_S(t) \otimes \rho_{B,0}$ and Eq.(57) becomes

$$\frac{d}{dt}\tilde{\rho}_S = -\frac{1}{\hbar^2} \int_0^t ds \operatorname{tr}_B \left[\tilde{H}_I(t), \left[\tilde{H}_I(s), \tilde{\rho}_S(s) \otimes \tilde{\rho}_{B,0} \right] \right]. \quad (58)$$

Markov Approximation: The "future" evolution of the system $\rho_S(t + \delta t)$, doesn't depend only on the present $\rho_S(t)$ but also on previous times because of the integral with $\rho_S(s)$ in (58). In general, the previous states of the system change the state of the environment and these changes of the environment change the future states of the system (information flows from the system to the environment and returns back). In the Markov Approximation we assume that the small changes in the state of the environment because of the interaction with the system S is weak or the environment is large. In other words, the environment quickly "forgets" the information it gets from the system. So, we don't consider the case where the environment returns the information it got from the system back to the system. So, the density matrix of the system S at time t only depends on t , so we substitute $\rho_S(s) = \rho_S(t)$ and the Eq.(58) becomes

$$\frac{d}{dt}\tilde{\rho}_S = -\frac{1}{\hbar^2} \int_0^t ds \operatorname{tr}_B \left[\tilde{H}_I(t), \left[\tilde{H}_I(s), \tilde{\rho}_S(t) \otimes \tilde{\rho}_{B,0} \right] \right]. \quad (59)$$

An important note is that the time evolution does depend on the initial condition of the system S . Because of the Markov Approximation, that dependence is negligible, and by changing variables $s \rightarrow t - s$ and extending the upper limit of integration to infinity we get the Born-Markov Master Equation

$$\frac{d}{dt}\tilde{\rho}_S = -\frac{1}{\hbar^2} \int_0^\infty ds \operatorname{tr}_B \left[\tilde{H}_I(t), \left[\tilde{H}_I(t-s), \tilde{\rho}_S(t) \otimes \tilde{\rho}_{B,0} \right] \right]. \quad (60)$$

2.2.4 Lindblad Equation

In order to treat the quantum systems in the next chapters, we use an interaction Hamiltonian $H_I(t)$ of (55). Firstly, we calculate the trace inside the integral of the Born-Markov equation,

expanding the commutators

$$\begin{aligned}
\mathrm{tr}_B \left[\tilde{H}_I(t), \left[\tilde{H}_I(t'), \tilde{\rho}_S(t) \otimes \tilde{\rho}_{B,0} \right] \right] &= \sum_{i,j} \mathrm{tr}_B \left[\tilde{S}_i(t) \otimes \tilde{B}_i(t), \left[\tilde{S}_j(t') \otimes \tilde{B}_j(t'), \tilde{\rho}_S(t) \tilde{\rho}_{B,0} \right] \right] \\
&= \sum_{i,j} \mathrm{tr}_B \left[\tilde{S}_i(t) \tilde{B}_i(t), \tilde{S}_j(t') \tilde{B}_j(t') \tilde{\rho}_S(t) \tilde{\rho}_{B,0} - \tilde{\rho}_S(t) \tilde{\rho}_{B,0} \tilde{S}_j(t') \tilde{B}_j(t') \right] \\
&= \sum_{i,j} \mathrm{tr}_B \left(\tilde{S}_i(t) \tilde{B}_i(t) \tilde{S}_j(t') \tilde{B}_j(t') \tilde{\rho}_S(t) \tilde{\rho}_{B,0} - \tilde{S}_i(t) \tilde{B}_i(t) \tilde{\rho}_S(t) \tilde{\rho}_{B,0} \tilde{S}_j(t') \tilde{B}_j(t') \right. \\
&\quad \left. - \tilde{S}_j(t') \tilde{B}_j(t') \tilde{\rho}_S(t) \tilde{\rho}_{B,0} \tilde{S}_i(t) \tilde{B}_i(t) + \tilde{\rho}_S(t) \tilde{\rho}_{B,0} \tilde{S}_j(t') \tilde{B}_j(t') \tilde{S}_i(t) \tilde{B}_i(t) \right).
\end{aligned} \tag{61}$$

Then, we will use the property of the partial trace $\mathrm{tr}_B(A_S \otimes A'_B) = A_S \mathrm{tr}\{A'_B\}$, which gives

$$\begin{aligned}
\mathrm{tr}_B \left[\tilde{H}_I(t), \left[\tilde{H}_I(t'), \tilde{\rho}_S(t) \otimes \tilde{\rho}_{B,0} \right] \right] &= \sum_{i,j} \tilde{S}_i(t) \tilde{S}_j(t') \tilde{\rho}_S(t) \mathrm{tr} \left(\tilde{B}_i(t) \tilde{B}_j(t') \tilde{\rho}_{B,0} \right) \\
&\quad - \tilde{S}_i(t) \tilde{\rho}_S(t) \tilde{S}_j(t') \mathrm{tr} \left(\tilde{B}_i(t) \tilde{\rho}_{B,0} \tilde{B}_j(t') \right) \\
&\quad - \tilde{S}_j(t') \tilde{\rho}_S(t) \tilde{S}_i(t) \mathrm{tr} \left(\tilde{B}_j(t') \tilde{\rho}_{B,0} \tilde{B}_i(t) \right) \\
&\quad + \tilde{\rho}_S(t) \tilde{S}_j(t') \tilde{S}_i(t) \mathrm{tr} \left(\tilde{\rho}_{B,0} \tilde{B}_j(t') \tilde{B}_i(t) \right).
\end{aligned} \tag{62}$$

Lastly, we will use the cyclic property of the trace $\mathrm{tr}\{ABC\} = \mathrm{tr}\{BCA\} = \mathrm{tr}\{CAB\}$, which gives

$$\begin{aligned}
\mathrm{tr} \left(\tilde{B}_i(t) \tilde{B}_j(t') \tilde{\rho}_{B,0} \right) &= \mathrm{tr} \left(\tilde{B}_j(t') \tilde{\rho}_{B,0} \tilde{B}_i(t) \right) = \mathrm{tr} \left(\tilde{\rho}_{B,0} \tilde{B}_i(t) \tilde{B}_j(t') \right), \\
\mathrm{tr} \left(\tilde{B}_j(t') \tilde{B}_i(t) \tilde{\rho}_{B,0} \right) &= \mathrm{tr} \left(\tilde{B}_i(t) \tilde{\rho}_{B,0} \tilde{B}_j(t') \right) = \mathrm{tr} \left(\tilde{\rho}_{B,0} \tilde{B}_j(t') \tilde{B}_i(t) \right),
\end{aligned} \tag{63}$$

and thus we find that

$$\begin{aligned}
\mathrm{tr}_B \left[\tilde{H}_I(t), \left[\tilde{H}_I(t'), \tilde{\rho}_S(t) \otimes \tilde{\rho}_{B,0} \right] \right] &= \left(\tilde{S}_i(t) \tilde{S}_j(t') \tilde{\rho}_S(t) - \tilde{S}_j(t') \tilde{\rho}_S(t) \tilde{S}_i(t) \right) \mathrm{tr} \left(\tilde{B}_i(t) \tilde{B}_j(t') \tilde{\rho}_{B,0} \right) \\
&\quad + \left(\tilde{\rho}_S(t) \tilde{S}_j(t') \tilde{S}_i(t) - \tilde{S}_i(t) \tilde{\rho}_S(t) \tilde{S}_j(t') \right) \mathrm{tr} \left(\tilde{B}_j(t') \tilde{B}_i(t) \tilde{\rho}_{B,0} \right) \\
&= \left(\tilde{S}_i(t) \tilde{S}_j(t-s) \tilde{\rho}_S(t) - \tilde{S}_j(t-s) \tilde{\rho}_S(t) \tilde{S}_i(t) \right) \mathrm{tr} \left(\tilde{B}_i(t) \tilde{B}_j(t-s) \tilde{\rho}_{B,0} \right) \\
&\quad + \left(\tilde{\rho}_S(t) \tilde{S}_j(t-s) \tilde{S}_i(t) - \tilde{S}_i(t) \tilde{\rho}_S(t) \tilde{S}_j(t-s) \right) \mathrm{tr} \left(\tilde{B}_j(t-s) \tilde{B}_i(t) \tilde{\rho}_{B,0} \right).
\end{aligned} \tag{64}$$

Because of the Markov approximation we have that

$$\begin{aligned}
\mathrm{tr} \left(\tilde{B}_i(t) \tilde{B}_j(t-s) \tilde{\rho}_{B,0} \right) &= \mathrm{tr} \left(\tilde{B}_i(s) \tilde{B}_j(0) \tilde{\rho}_{B,0} \right), \\
\mathrm{tr} \left(\tilde{B}_j(t-s) \tilde{B}_i(t) \tilde{\rho}_{B,0} \right) &= \mathrm{tr} \left(\tilde{B}_j(0) \tilde{B}_i(s) \tilde{\rho}_{B,0} \right),
\end{aligned} \tag{65}$$

meaning that the average value (correlation function) depends only on the time difference s .

We can define the quantities

$$G_{ij} = \frac{1}{\hbar^2} \int_0^\infty ds \mathrm{tr} \left(\tilde{B}_i(s) \tilde{B}_j(0) \tilde{\rho}_{B,0} \right), \tag{66}$$

$$\bar{G}_{ij} = \frac{1}{\hbar^2} \int_0^\infty ds \mathrm{tr} \left(\tilde{B}_j(0) \tilde{B}_i(s) \tilde{\rho}_{B,0} \right). \tag{67}$$

With the help of the above expressions, we get the Lindblad equation in the Interaction Picture

$$\dot{\rho}_S = \sum_{i,j} G_{ij} \left(\tilde{S}_j \tilde{\rho}_S \tilde{S}_i - \tilde{S}_i \tilde{S}_j \tilde{\rho}_S \right) + \bar{G}_{ij} \left(\tilde{S}_i \tilde{\rho}_S \tilde{S}_j - \tilde{\rho}_S \tilde{S}_j \tilde{S}_i \right) \quad (68)$$

If we assume that G_{ij}, \bar{G}_{ij} are real, and that $\tilde{S}_i = S_i$ and $\tilde{S}_j = S_i^\dagger \delta_{ij}$, then in the Schrödinger picture the Lindblad equation becomes

$$\dot{\rho}_S = \frac{i}{\hbar} [\rho_S, H] + \sum_i \gamma_i \left(S_i \rho_S S_i^\dagger - \frac{1}{2} \{ S_i^\dagger S_i, \rho_S \} \right). \quad (69)$$

In the case when G_{ij}, \bar{G}_{ij} are complex, we have to include in the Hamiltonian an extra term to make up for the imaginary part. In this diploma thesis, we will assume this term negligible and we won't include it in our analysis.

2.2.5 Non-Hermitian-Hamiltonian Quantum Dynamics

As we proved in the previous section, the equation that fully describes the quantum system is the Lindblad Eq.(69). There we assumed that the Hamiltonian is hermitian $H = H^\dagger$ and that the dissipation is described by the Lindblad terms.

Another way of describing dissipation is by introducing non-hermitian Hamiltonians, which won't necessarily have real eigenvalues. To find the non-hermitian Hamiltonian of the Optical Quantum Systems we are interested in we begin by rewriting Eq.69 as

$$\dot{\rho}_S = \frac{i}{\hbar} \left\{ \left(H - i\hbar \sum_i \gamma_i S_i^\dagger S_i \right) \rho - \rho \left(H - i\hbar \sum_i \gamma_i S_i S_i^\dagger \right) \right\} + 2 \sum_i S_i \rho S_i^\dagger. \quad (70)$$

where $2 \sum_i S_i \rho S_i^\dagger$ is the Jump (super)operator. We then define as the non-Hermitian Hamiltonian the term inside the parenthesis

$$H' = H - i\hbar \sum_i \gamma_i S_i^\dagger S_i. \quad (71)$$

2.3 Photon Indistinguishability

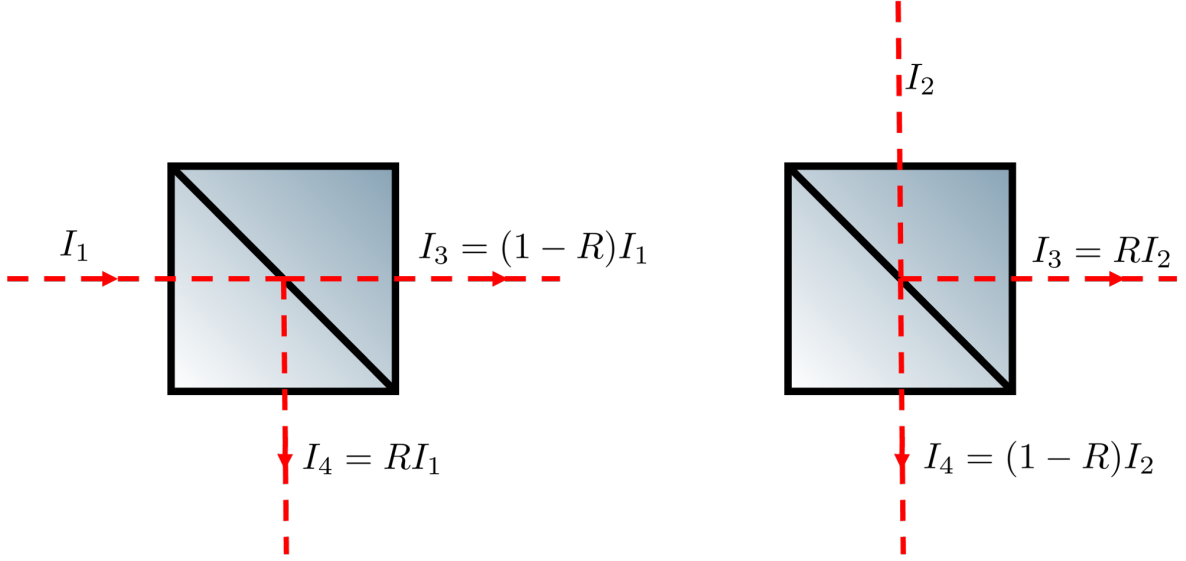


Figure 2: Illustration of a Beam-Splitter and the intensities of the input and output beams.

When a photon passes through a beam splitter, there are two possible outcomes, it either is reflected or transmitted. The probability of each outcome depends on the reflectance of the Beam-Splitter. In the classical case, if we have classical fields in input 1 and 2, \mathbf{E}_1 and \mathbf{E}_2 , and the fields exiting from output 3 and 4 are \mathbf{E}_3 and \mathbf{E}_4 , then the later will be given by

$$\begin{pmatrix} \mathbf{E}_3 \\ \mathbf{E}_4 \end{pmatrix} = \begin{pmatrix} r_{13} & t_{23} \\ t_{14} & r_{24} \end{pmatrix} \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \end{pmatrix}. \quad (72)$$

The above transformation must satisfy the energy conservation, so it must satisfy $|\mathbf{E}_1|^2 + |\mathbf{E}_2|^2 = |\mathbf{E}_3|^2 + |\mathbf{E}_4|^2$. So

$$|r_{13}|^2 + |t_{14}|^2 = |r_{24}|^2 + |t_{23}|^2 = 1, \quad (73)$$

$$r_{13}t_{23}^* + t_{14}r_{24}^* = 0. \quad (74)$$

Applying the above relations, we easily find that the BS Transformation is

$$\begin{pmatrix} \mathbf{E}_3 \\ \mathbf{E}_4 \end{pmatrix} = \begin{pmatrix} \sqrt{1-R} & -\sqrt{R}e^{-i\phi} \\ \sqrt{R}e^{i\phi} & \sqrt{1-R} \end{pmatrix} \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \end{pmatrix}. \quad (75)$$

where $R \equiv |r_{13}| = |r_{24}|$ and $T \equiv 1 - R = |t_{14}| = |t_{23}|$.

Quantum mechanically [16], the BS acts on the initial state of the system $|\psi_{in}\rangle = |1\rangle_i = a_i^\dagger |0\rangle$ as

$$\begin{aligned} a_3^\dagger &= \sqrt{1-R}a_1^\dagger - \sqrt{R}e^{-i\phi}a_2^\dagger, & a_1^\dagger &= \sqrt{1-R}a_3^\dagger + \sqrt{R}e^{i\phi}a_4^\dagger, \\ a_4^\dagger &= \sqrt{R}e^{i\phi}a_1^\dagger + \sqrt{1-R}a_2^\dagger, & a_2^\dagger &= -\sqrt{R}e^{-i\phi}a_3^\dagger + \sqrt{1-R}e^{i\phi}a_4^\dagger. \end{aligned} \quad (76)$$

But, when two indistinguishable photons interfere in a BS then an interesting phenomenon occurs, that was first experimentally observed by Chung Ki Hong, Zhe Yu Ou and Leonard Mandel in 1967 [17]. This interference is named after them (HOM interference) and has many applications in quantum optics and quantum technologies. Quantum mechanically, the state

of the two incoming photons is

$$|\psi_{in}\rangle = |1; i\rangle_1 \otimes |1; j\rangle_2 = a_{1,i}^\dagger a_{2,j}^\dagger |0\rangle. \quad (77)$$

The state of the output photons is

$$\begin{aligned} |\psi_{out}\rangle &= U_{BS} a_{1,i}^\dagger a_{2,j}^\dagger |0\rangle = \left(\sqrt{1-R} a_{3,i}^\dagger + \sqrt{R} e^{i\phi} a_{4,i}^\dagger \right) \left(-\sqrt{R} e^{-i\phi} a_{3,j}^\dagger + \sqrt{1-R} e^{i\phi} a_{4,j}^\dagger \right) |0\rangle \\ &= \left(-\sqrt{R(1-R)} e^{-i\phi} a_{3,i}^\dagger a_{3,j}^\dagger + \sqrt{R(1-R)} e^{i\phi} a_{4,i}^\dagger a_{4,j}^\dagger + (1-R) a_{3,i}^\dagger a_{4,j}^\dagger - R a_{4,i}^\dagger a_{3,j}^\dagger \right) |0\rangle. \end{aligned} \quad (78)$$

In HOM Interference we are interested in the so called Coincidence Probability p_c , which is defined as the probability that both detectors in the exit of the BS click. From the Coincidence Probability we can calculate the Indistinguishability I as

$$I = 1 - 2p_c. \quad (79)$$

2.3.1 Example of Fully Distinguishable Photons

Suppose that the properties i, j described above are the polarization of the electric field of the photons. Suppose that their polarizations are linear and orthogonal, i.e. $i = H$ and $j = V$, then Eq.(78) gives

$$\begin{aligned} |\psi_{out}\rangle &= \left(-\sqrt{R(1-R)} e^{-i\phi} a_{3,H}^\dagger a_{3,V}^\dagger + \sqrt{R(1-R)} e^{i\phi} a_{4,H}^\dagger a_{4,V}^\dagger \right. \\ &\quad \left. + (1-R) a_{3,H}^\dagger a_{4,V}^\dagger - R a_{4,H}^\dagger a_{3,V}^\dagger \right) |0\rangle \\ &= -\sqrt{R(1-R)} e^{-i\phi} |1; H\rangle_3 |1; V\rangle_3 + \sqrt{R(1-R)} e^{i\phi} |1; H\rangle_4 |1; V\rangle_4 \\ &\quad + (1-R) |1; H\rangle_3 |1; V\rangle_4 - R |1; V\rangle_3 |1; H\rangle_4. \end{aligned} \quad (80)$$

For $R = 0.5$ and $\phi = \pi$ we find

$$|\psi_{out}\rangle = \frac{1}{2} (|1; H\rangle_3 |1; V\rangle_3 - |1; H\rangle_4 |1; V\rangle_4 + |1; H\rangle_3 |1; V\rangle_4 - |1; V\rangle_3 |1; H\rangle_4). \quad (81)$$

The first term correspond to the state where the photons both exit from output 3 3 a), the second on the case when both exit from output 4 3 b), the third term on the case when both photons are transmitted 3 c) and the forth term on the case when both are reflected 3 d).

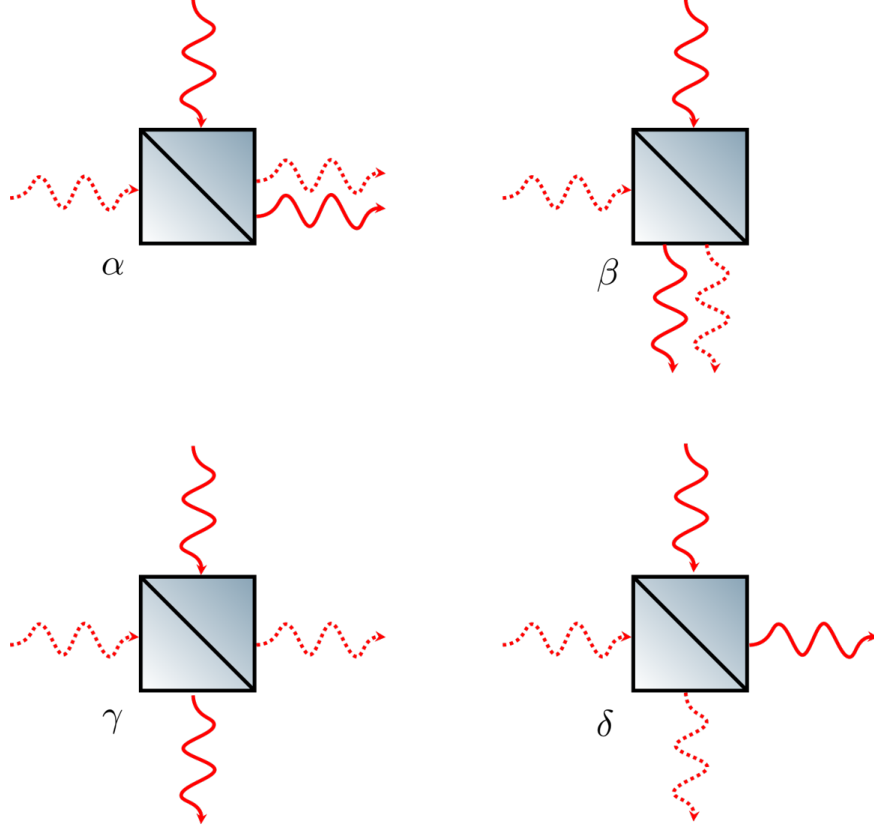


Figure 3: Example of Fully Distinguishable Photons.

It's straightforward to calculate the Coincidence Probability and the Indistinguishability

$$p_c = \left| c_{|1;H\rangle_3|1;V\rangle_4} \right|^2 + \left| c_{|1;V\rangle_3|1;H\rangle_4} \right|^2 = \left| \frac{1}{2} \right|^2 + \left| -\frac{1}{2} \right|^2 = \frac{1}{2}, \quad (82)$$

$$I = 1 - 2p_c = 1 - 2 \cdot \frac{1}{2} = 0. \quad (83)$$

2.3.2 Example of Fully Indistinguishable Photons

This time we assume the photons have identical polarization, e.g. $i = j = H$. Then (78) becomes

$$\begin{aligned} |\psi_{out}\rangle &= \left(-\sqrt{R(1-R)}e^{-i\phi}a_{3,H}^\dagger a_{3,H}^\dagger + \sqrt{R(1-R)}e^{i\phi}a_{4,H}^\dagger a_{4,H}^\dagger \right. \\ &\quad \left. + (1-R)a_{3,H}^\dagger a_{4,H}^\dagger - Ra_{4,H}^\dagger a_{3,H}^\dagger \right) |0\rangle \\ &= -\sqrt{2R(1-R)}e^{-i\phi} |2; H\rangle_3 + \sqrt{2R(1-R)}e^{i\phi} |2; H\rangle_4 \\ &\quad + (1-R) |1; H\rangle_3 |1; H\rangle_4 - R |1; H\rangle_3 |1; H\rangle_4. \end{aligned} \quad (84)$$

The first term corresponds to the case when both photons exit from output 3 and the second term to the case when both exit from output 4, and the last two on the case when one photon exits from each output.

For $R = 0.5$ and $\phi = \pi$ we find

$$\begin{aligned} |\psi_{out}\rangle &= \frac{1}{2} (|2; H\rangle_3 - |2; H\rangle_4 + |1; H\rangle_3 |1; H\rangle_4 - |1; H\rangle_3 |1; H\rangle_4) \\ &= \frac{1}{2} (|2; H\rangle_3 - |2; H\rangle_4). \end{aligned} \quad (85)$$

The first term corresponds to the case when both photons exit from output 3 4 a) and the second on the case when both exit from output 4 4 b). We observe that there is no case where both detectors click, so the Coincidence Probability is $p_c = 0$ and the Indistinguishability is $I = 1$.

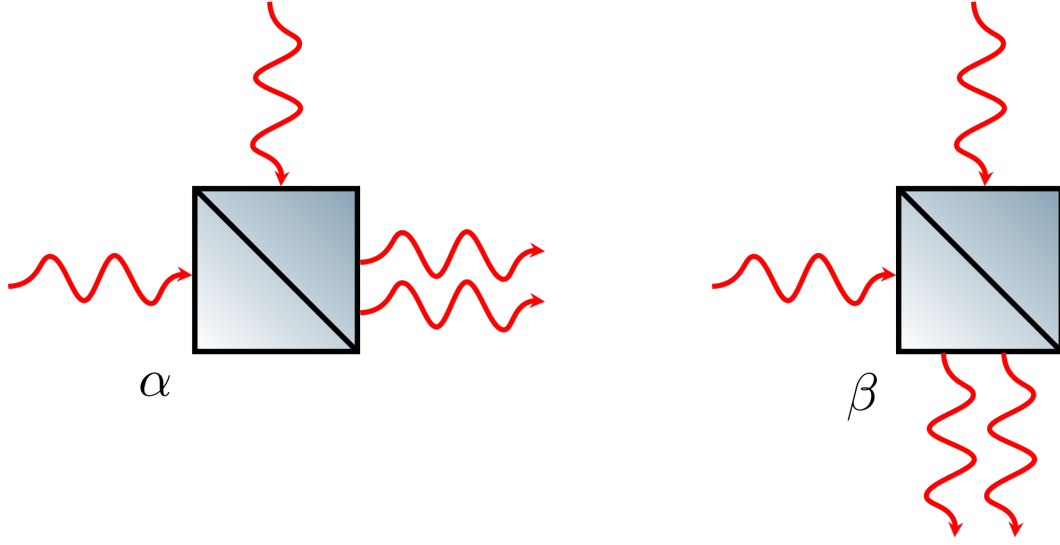


Figure 4: Example of Fully Indistinguishable Photons.

2.3.3 Derivation of Indistinguishability

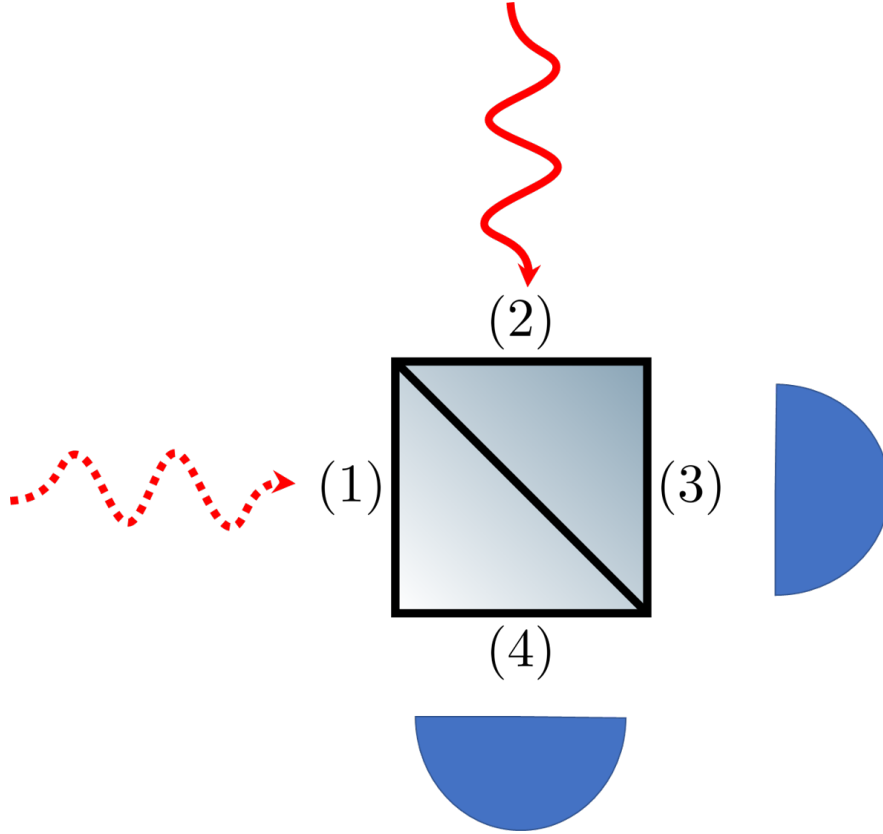


Figure 5: Illustration of the general case of the indistinguishability between two photons.

In contrast with the above examples, in our system it's not possible to identify all the properties of the produced photons that make them distinguishable, as illustrated in Fig(5). That's why we need to use the correlation functions of first and second order. The correlation function of first order is defined as

$$G^{(1)}(t_1, t_2; \mathbf{r}_1, \mathbf{r}_2) = \text{tr}\{\rho E^-(t_1, \mathbf{r}_1)E^+(t_2, \mathbf{r}_2)\}. \quad (86)$$

The correlation function of second order is defined as

$$G^{(2)}(t_1, t_2, t_3, t_4; \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = \text{tr}\{\rho E^-(t_1, \mathbf{r}_1)E^-(t_2, \mathbf{r}_2)E^+(t_3, \mathbf{r}_3)E^+(t_4, \mathbf{r}_4)\}. \quad (87)$$

Suppose now that two photons created from two different systems and interfered in a BS and were detected by two different detectors that measure the time they reached them. The result of this experiment is quantified using the normalised second order correlation function $g^{(2)}(\tau)$ that is defined as

$$g^{(2)}(\tau) = \frac{G^{(2)}(t_1 = t_3 = t, t_2 = t_4 = t + \tau; \mathbf{r}_1 = \mathbf{r}_3, \mathbf{r}_2 = \mathbf{r}_4)}{G^{(1)}(t_1 = t, \mathbf{r}_3)G^{(1)}(t_2 = t + \tau, \mathbf{r}_4)}. \quad (88)$$

According to the analysis of [6], the coincidence probability is

$$p_c = \frac{\int dt \int d\tau G_{34}^{(2)}(t, \tau)}{\int dt G_3^{(1)}(t) \int d\tau G_4^{(1)}(t + \tau)}. \quad (89)$$

Firstly we calculate $G_{34}^{(2)}(t, \tau)$

$$\begin{aligned} G_{34}^{(2)}(t, \tau) &= \left\langle a_3^\dagger(t) a_4^\dagger(t + \tau) a_3(t) a_4(t + \tau) \right\rangle \\ &= \frac{1}{4} \left\langle \left(a_1^\dagger(t) - e^{i\phi} a_2^\dagger(t) \right) \left(e^{-i\phi} a_1^\dagger(t + \tau) + a_2^\dagger(t + \tau) \right) \right. \\ &\quad \left. \left(a_1(t) - e^{-i\phi} a_2(t) \right) \left(e^{i\phi} a_1(t + \tau) + a_2(t + \tau) \right) \right\rangle. \end{aligned} \quad (90)$$

If we expand the product we have terms of the form

$$\begin{aligned} a_i^\dagger(t) a_i^\dagger(t + \tau) a_i(t) a_i(t + \tau), & \quad a_i^\dagger(t) a_i^\dagger(t + \tau) a_i(t) a_j(t + \tau), & \quad a_i^\dagger(t) a_i^\dagger(t + \tau) a_j(t) a_i(t + \tau), \\ a_i^\dagger(t) a_j^\dagger(t + \tau) a_i(t) a_i(t + \tau), & \quad a_j^\dagger(t) a_i^\dagger(t + \tau) a_i(t) a_i(t + \tau), & \quad a_i^\dagger(t) a_i^\dagger(t + \tau) a_j(t) a_j(t + \tau), \\ a_i^\dagger(t) a_j^\dagger(t + \tau) a_i(t) a_j(t + \tau), & \quad a_i^\dagger(t) a_j^\dagger(t + \tau) a_j(t) a_i(t + \tau), & \end{aligned} \quad (91)$$

where $i = 1, 2$ and $j \neq i$. From this terms only remain the last two, because the rest either create or destroy two photons in one of the inputs of the BS. So we get

$$\begin{aligned} G_{34}^{(2)}(t, \tau) &= \frac{1}{4} \left(\left\langle a_1^\dagger(t) a_2^\dagger(t + \tau) a_1(t) a_2(t + \tau) \right\rangle + \left\langle a_2^\dagger(t) a_1^\dagger(t + \tau) a_1(t) a_2(t + \tau) \right\rangle \right. \\ &\quad \left. - \left\langle a_1^\dagger(t) a_2^\dagger(t + \tau) a_2(t) a_1(t + \tau) \right\rangle - \left\langle a_2^\dagger(t) a_1^\dagger(t + \tau) a_1(t) a_2(t + \tau) \right\rangle \right). \end{aligned}$$

The systems we are interested in, produce only one photon per excitation, so we need a way to define the indistinguishability of one photon. The way we measure it is by exciting the system twice, delaying the first produced photon by making it take a longer path and then interfering the two photons in a BS, as illustrated in Fig.(6).

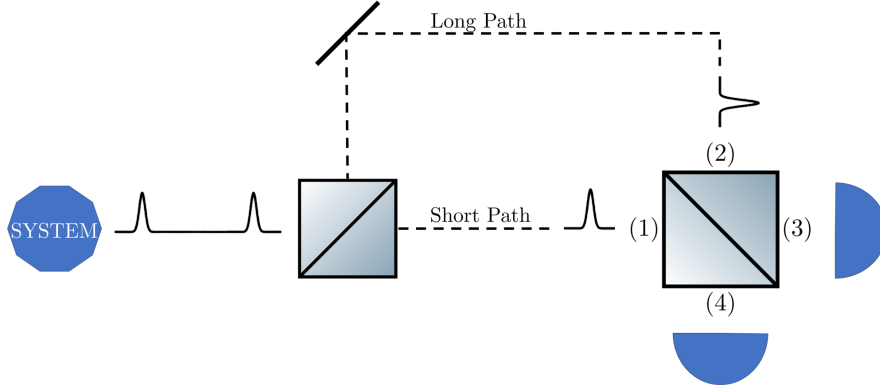


Figure 6: Illustration of the experimental setup to measure the indistinguishability of one photon produced by our system.

To calculate the indistinguishability of the photon of our system, we then assume that there is no correlation between the incoming photons, since they are produced in different systems, so we can eliminate the indices,

$$\begin{aligned} G_{34}^{(2)}(t, \tau) &= \frac{1}{2} \left(\left\langle a^\dagger(t) a^\dagger(t + \tau) a(t) a(t + \tau) \right\rangle - \left\langle a^\dagger(t) a^\dagger(t + \tau) a(t) a(t + \tau) \right\rangle \right) \\ &= \frac{1}{2} \left(\left\langle a^\dagger(t) a(t) \right\rangle \left\langle a^\dagger(t + \tau) a(t + \tau) \right\rangle - \left| \left\langle a^\dagger(t + \tau) a(t) \right\rangle \right|^2 \right). \end{aligned} \quad (92)$$

So the expression of the indistinguishability is

$$I = \frac{\int_0^\infty dt \int_0^\infty d\tau |\langle a^\dagger(t+\tau)a(t) \rangle|^2}{\int_0^\infty dt \int_0^\infty d\tau \langle a^\dagger(t)a(t) \rangle \langle a^\dagger(t+\tau)a(t+\tau) \rangle}. \quad (93)$$

2.3.4 Calculation of the Time Correlation Functions

As we see in Eq.(93), for the calculation of the indistinguishability it's necessary to calculate the two time correlation functions $A(t)B(t+\tau)$. That's why we will use the Non Equilibrium Green's Functions Formalism described in [7]. We will define lesser, greater and retarded Green function

$$G^<(x, t; x', t') = \langle \psi^\dagger(x', t') \psi(x, t) \rangle, \quad (94a)$$

$$G^>(x, t; x', t') = \langle \psi(x, t) \psi^\dagger(x', t') \rangle, \quad (94b)$$

$$G^R(x, t; x', t') = \Theta(t - t') \langle \{ \psi(x, t), \psi^\dagger(x', t') \} \rangle. \quad (94c)$$

where $\psi(x, t) = (\psi_1(x, t), \psi_2(x, t), \dots)$ is an row vector of operators. The retarded Green function is different than zero for $t \geq t'$ and is used for the response of the system due to an external cause. The lesser and greater Green functions are used to find measurable quantities of the system, such as the number of particles. Another important property of the lesser Green function is

$$\rho(t) = G^<(t, t). \quad (95)$$

Additionally, we have to define the Self-Energies of the system, to describe the markovian dissipation terms of the system. The lesser and greater Self-Energies are analogously defined as

$$\Sigma^<(t, t + \tau) = \langle H_{int}(t + \tau) G^<(t, t + \tau) H_{int}(t) \rangle, \quad (96)$$

$$\Sigma^R(t, t + \tau) = \langle H_{int}(t + \tau) G^R(t, t + \tau) H_{int}(t) \rangle. \quad (97)$$

Because of our Hamiltonian, the retarded Green function depends only on one variable, so we can express it in terms of frequency (energy) as

$$G^R(\omega) = -i \int d\omega e^{i\omega t} G^R(\tau). \quad (98)$$

Using the definitions, we can derive the Dyson-Green equation, which is of the form

$$(H + \Sigma^R(\omega)) |\psi\rangle = \hbar\omega |\psi\rangle, \quad (99)$$

$$(\hbar\omega - H - \Sigma^R(\omega)) G^R(\omega) = \mathbb{1}. \quad (100)$$

The time evolution of the Non-Equilibrium Green functions is given by the Kadanoff-Baym equations. It's proven that for markovian self-energies we have

$$G^<(t, t + \tau) = G^R(\tau) G^<(t, t) = G^R(\tau) \rho(t). \quad (101)$$

We are only interested in $\langle a^\dagger(t + \tau)a(t) \rangle$, which is

$$\begin{aligned}
\langle a^\dagger(t + \tau)a(t) \rangle &= \langle g, 1 | G^<(t, t + \tau) | g, 1 \rangle = \langle g, 1 | G^R(\tau) \rho(t) | g, 1 \rangle \\
&= (0 \quad 1) \begin{pmatrix} G_{ee}^R(\tau) & G_{ec}^R(\tau) \\ G_{ce}^R(\tau) & G_{cc}^R(\tau) \end{pmatrix} \begin{pmatrix} \rho_{ee}(t) & \rho_{ec}(t) \\ \rho_{ce}(t) & \rho_{cc}(t) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= G_{cc}^R(\tau) \rho_{cc}(t) + G_{ce}^R(\tau) \rho_{ec}(t).
\end{aligned} \tag{102}$$

So we conclude that for the calculation of the two time correlation function we need to solve both the time evolution equation for the density matrix (Lindblad) and the time evolution equation for the retarded Green function (Kadanoff-Baym). The later is derived by taking the Fourier transform of Eq.(99)

$$\boxed{i \frac{d}{d\tau} G^R(\tau) = i\delta(\tau)\mathbf{1} + (H - i\Sigma^R(0)) G^R(\tau)}. \tag{103}$$

3 Methods: The One Cavity System

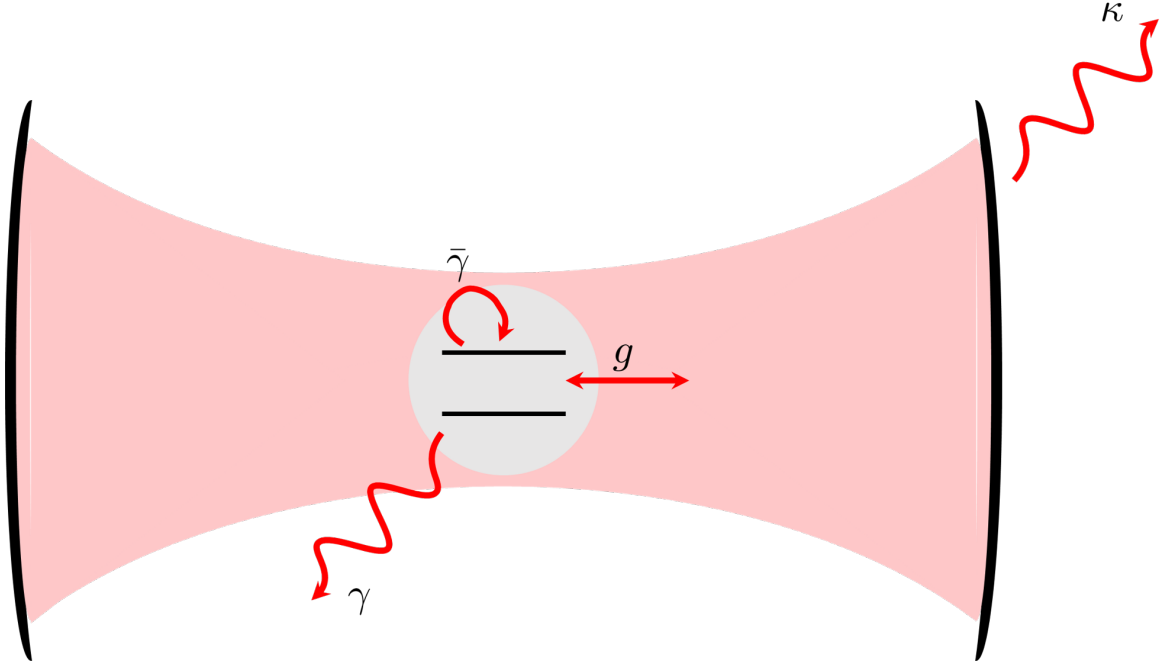


Figure 7: Illustration of the One Cavity System.

The goal of Chapter 3 is the investigation of the quantum system of Fig.(7), following the methodology of [5]. The quantum system consists of one optical cavity and an atom (gray area) inside it. The atom is modeled as a two-level system (two black lines) and is coupled to one normal mode of the E/M field of the cavity (pink area), with coupling constant g . Additionally, the system has dissipation terms, since photons leave the system to the environment (curly red lines). More specifically, the photon leaving the atom is due to spontaneous emission (rate γ) and the one leaving the cavity is due to the cavity decay rate κ . Finally, there is another dissipative non-radiative term that corresponds to the pure dephasing rate of the atom ($\bar{\gamma}$) and which is illustrated as a line that starts and ends to atom.

3.1 Optical Master Equation of the single Cavity System

As we can see from Fig.(7), the system consists of a two level atom coupled to a normal mode of the EM field inside the cavity, so the Hamiltonian that describes the system is

$$H = \hbar\omega_{atom}\sigma^+\sigma^- + \hbar\omega_{cav}a^\dagger a + \hbar g(\sigma^+ a + \sigma^- a^\dagger). \quad (104)$$

We assume that the atom initially is in the excited state and that the cavity has no photons, so the state of the system is $|\psi_0\rangle = |e, 0\rangle$. Also, we assume that the temperature is low, such as that thermal photons are negligible. It's straightforward to show that $[H, \sigma^+\sigma^- + a^\dagger a] = 0$, so the total number of energy quanta is conserved. So, the basis that we will use to describe the system is $\left\{ |e, 0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |g, 1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. In this basis the Hamiltonian is written as

$$\begin{aligned} H &= \hbar\omega_{atom} |e, 0\rangle\langle e, 0| + \hbar\omega_{cav} |g, 1\rangle\langle g, 1| + \hbar g(|e, 0\rangle\langle g, 1| + |g, 1\rangle\langle e, 0|), \\ H &= \hbar\Delta |g, 1\rangle\langle g, 1| + \hbar g(|e, 0\rangle\langle g, 1| + |g, 1\rangle\langle e, 0|), \end{aligned} \quad (105)$$

or in matrix form

$$H = \hbar \begin{pmatrix} 0 & g \\ g & \Delta \end{pmatrix}, \quad (106)$$

where $\Delta = \omega_{cav} - \omega_{atom}$ is the detuning of the atom and the field, which is 0. Then, we define the density matrix elements

$$\rho(t) = \begin{pmatrix} \rho_{ee}(t) = \langle \sigma^+(t)\sigma^-(t) \rangle & \rho_{ec}(t) = \langle \sigma^+(t)a(t) \rangle \\ \rho_{ce}(t) = \langle a^\dagger(t)\sigma^-(t) \rangle & \rho_{cc}(t) = \langle a^\dagger(t)a(t) \rangle \end{pmatrix}. \quad (107)$$

The time evolution is given by the Lindblad equation

$$\dot{\rho} = i[\rho, H] + \sum_i \mathcal{L}_i[\rho], \quad (108)$$

$$\dot{\rho} = \frac{i}{\hbar}[\rho, H] + \sum_i \gamma_i \left(L_i \rho L_i^\dagger - \frac{1}{2} \{ L_i^\dagger L_i, \rho \} \right). \quad (109)$$

In our analysis the dissipation terms are the spontaneous decay of the atom γ , the cavity decay rate κ and the pure dephasing of the atom $\bar{\gamma}$. They are described by the Lindbladian of the operators $L = \sigma^-$, $L = a$ and $L = \sigma^+\sigma^-$ respectively.

We calculate the Lindblad terms in our basis

$$\mathcal{L}_{QE}[\rho] = -\gamma \begin{pmatrix} \rho_{ee} & \rho_{ec}/2 \\ \rho_{ce}/2 & 0 \end{pmatrix}, \quad (110)$$

$$\mathcal{L}_{cav}[\rho] = -\kappa \begin{pmatrix} 0 & \rho_{ec}/2 \\ \rho_{ce}/2 & \rho_{cc} \end{pmatrix}, \quad (111)$$

$$\mathcal{L}_{deph}[\rho] = -\bar{\gamma} \begin{pmatrix} 0 & \rho_{ec}/2 \\ \rho_{ce}/2 & 0 \end{pmatrix}, \quad (112)$$

and also the commutator

$$[\rho, H] = \begin{pmatrix} g(\rho_{ec} - \rho_{ce}) & g(\rho_{ee} - \rho_{cc}) + \Delta\rho_{ec} \\ g(\rho_{cc} - \rho_{ee}) - \Delta\rho_{ce} & g(\rho_{ce} - \rho_{ec}) \end{pmatrix}. \quad (113)$$

and we derive the system of time evolution equations

$$\begin{aligned} \dot{\rho}_{ee} &= ig(\rho_{ec} - \rho_{ce}) - \gamma\rho_{ee}, \\ \dot{\rho}_{ec} &= ig(\rho_{ee} - \rho_{cc}) - \frac{\gamma + \kappa + \bar{\gamma}}{2}\rho_{ec}, \\ \dot{\rho}_{ce} &= -ig(\rho_{ee} - \rho_{cc}) - \frac{\gamma + \kappa + \bar{\gamma}}{2}\rho_{ce}, \\ \dot{\rho}_{cc} &= -ig(\rho_{ec} - \rho_{ce}) - \kappa\rho_{cc}. \end{aligned} \quad (114)$$

The above system can't be solved analytically for random values of our parameters $\{g, \gamma, \kappa, \bar{\gamma}\}$.

3.2 Calculation of the Time Correlation Functions

In our system the Self-Energies are

$$\Sigma_{QE}^R(t, t + \tau) = \frac{1}{2}\delta(\tau)\Theta(\tau)\gamma\sigma^+\sigma^-, \quad \Sigma_{QE}^<(t, t + \tau) = 0, \quad (115)$$

$$\Sigma_{deph}^R(t, t + \tau) = \frac{1}{2}\delta(\tau)\Theta(\tau)\bar{\gamma}\sigma^+\sigma^-, \quad \Sigma_{deph}^<(t, t + \tau) = \delta(\tau)\bar{\gamma}\sigma^+\sigma^-G^<(t, t + \tau)\sigma^+\sigma^-, \quad (116)$$

$$\Sigma_{cav}^R(t, t + \tau) = \frac{1}{2}\delta(\tau)\kappa\Theta(\tau)a^\dagger a, \quad \Sigma_{cav}^<(t, t + \tau) = 0, \quad (117)$$

So, the total lesser(retarder) self-energy, i.e. $\Sigma_{tot}^{<,(R)}(t, t + \tau) = \Sigma_{QE}^{<,(R)}(t, t + \tau) + \Sigma_{deph}^{<,(R)}(t, t + \tau) + \Sigma_{cav}^{<,(R)}(t, t + \tau)$ in matrix form in our basis is

$$\Sigma_{tot}^R(t, t + \tau) = \delta(\tau)\Theta(\tau) \begin{pmatrix} \frac{(\gamma+\bar{\gamma})}{2} & 0 \\ 0 & \frac{\kappa}{2} \end{pmatrix}, \quad (118)$$

$$\Sigma_{tot}^<(t, t + \tau) = \delta(\tau) \begin{pmatrix} \bar{\gamma}G_{ee}^<(t, t + \tau) & 0 \\ 0 & 0 \end{pmatrix}. \quad (119)$$

Because of our Hamiltonian, the retarded Green function depends only on one variable, so we can express it in terms of frequency (energy) as

$$G^R(\omega) = -i \int d\omega e^{i\omega t} G^R(\tau). \quad (120)$$

Using the definitions, we can derive the Dyson-Green equation, which is of the form

$$(H + \Sigma^R(\omega)) |\psi\rangle = \hbar\omega |\psi\rangle, \quad (121)$$

$$(\hbar\omega - H - \Sigma^R(\omega)) G^R(\omega) = \mathbb{1}. \quad (122)$$

In matrix form, the Retarded Green function in the frequency domain is written as

$$G^R(\omega) = \begin{pmatrix} \omega + \frac{i}{2}(\gamma + \bar{\gamma}) & g \\ g & \omega + \frac{i}{2}\kappa \end{pmatrix}^{-1}. \quad (123)$$

The time evolution of the Non-Equilibrium Green functions are the Kadanoff-Baym. It's proven that for markovian self-energies we have

$$G^<(t, t + \tau) = G^R(\tau)G^<(t, t) = G^R(\tau)\rho(t). \quad (124)$$

We are only interested in $\langle a^\dagger(t + \tau)a(t) \rangle$, which is

$$\begin{aligned} \langle a^\dagger(t + \tau)a(t) \rangle &= \langle g, 1 | G^<(t, t + \tau) | g, 1 \rangle = \langle g, 1 | G^R(\tau)\rho(t) | g, 1 \rangle \\ &= (0 \quad 1) \begin{pmatrix} G_{ee}^R(\tau) & G_{ec}^R(\tau) \\ G_{ce}^R(\tau) & G_{cc}^R(\tau) \end{pmatrix} \begin{pmatrix} \rho_{ee}(t) & \rho_{ec}(t) \\ \rho_{ce}(t) & \rho_{cc}(t) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= G_{cc}^R(\tau)\rho_{cc}(t) + G_{ce}^R(\tau)\rho_{ec}(t). \end{aligned} \quad (125)$$

4 Results for the One Cavity System

The goal of chapter 4 is the investigation of the indistinguishability of the produced photons from the one cavity system we described in chapter 3 alongside with its efficiency. Firstly, we will work on the limit of coherent coupling between the atom and the field and solve analytically the Lindblad equation and the Kadanoff-Baym, in order to find an analytical expression for the efficiency and the indistinguishability. Then, we will study the incoherent coupling regime, where the dissipative terms dominate and so we can adiabatically eliminate the coherences of the density matrix. Furthermore, we study this region in the good cavity limit and the bad cavity limit and derive analytical expressions for the indistinguishability and the efficiency. Finally, we numerically calculate the indistinguishability and the efficiency in the other regions and summarize our results and compare them to our analytical predictions.

4.1 Coherent coupling regime

The coherent coupling regime corresponds to the region where $2g > \gamma + \kappa + \bar{\gamma}$.

4.1.1 Time Evolution of the Density Matrix

We start from the time evolution equations for the density matrix (114) in matrix form

$$\begin{pmatrix} \dot{\rho}_{ee}(t) \\ \dot{\rho}_{ec}(t) \\ \dot{\rho}_{ce}(t) \\ \dot{\rho}_{cc}(t) \end{pmatrix} = \begin{pmatrix} -\gamma & ig & -ig & 0 \\ ig & -\frac{\gamma+\bar{\gamma}+\kappa}{2} & 0 & -ig \\ -ig & 0 & -\frac{\gamma+\bar{\gamma}+\kappa}{2} & ig \\ 0 & -ig & ig & -\kappa \end{pmatrix} \begin{pmatrix} \rho_{ee}(t) \\ \rho_{ec}(t) \\ \rho_{ce}(t) \\ \rho_{cc}(t) \end{pmatrix}, \quad (126)$$

We assume that the initial state of the system is $\rho_0 = |e, 0\rangle\langle e, 0|$, so $\begin{pmatrix} \rho_{ee}(0) & \rho_{ec}(0) \\ \rho_{ce}(0) & \rho_{cc}(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We know that the term $\bar{\gamma}$ is much greater than the other dissipation terms (γ, κ), so in the coherent regime where $2g \gg \gamma + \kappa + \bar{\gamma}$ we can ignore these terms when they appear individually. We define the parameters $\phi = \frac{\gamma+\bar{\gamma}+\kappa}{2}$ and $\Omega = 64g^2 - (\gamma + \bar{\gamma} + \kappa)^2$ and we find that

$$\begin{pmatrix} \rho_{ee}(t) & \rho_{ec}(t) \\ \rho_{ce}(t) & \rho_{cc}(t) \end{pmatrix} = e^{-\phi t} \begin{pmatrix} \frac{(\gamma+\bar{\gamma}+\kappa)}{2\sqrt{\Omega}} \sin \frac{\sqrt{\Omega}t}{4} + \frac{1}{2} \cos \frac{\sqrt{\Omega}t}{4} & -\frac{1}{2i} \sin \frac{\sqrt{\Omega}t}{4} \\ \frac{1}{2i} \sin \frac{\sqrt{\Omega}t}{4} & -\frac{(\gamma+\bar{\gamma}+\kappa)}{2\sqrt{\Omega}} \sin \frac{\sqrt{\Omega}t}{4} - \frac{1}{2} \cos \frac{\sqrt{\Omega}t}{4} \end{pmatrix} + e^{-(\gamma+\kappa)t} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \quad (127)$$

In the coherent regime $2g \gg \gamma + \bar{\gamma} + \kappa$ so we can simplify the expression

$$\frac{\gamma + \gamma + \bar{\gamma}}{2g} \approx 0, \quad (128a)$$

$$\Omega = 64g^2 \left(1 - \left(\frac{\gamma + \bar{\gamma} + \kappa}{2g} \right)^2 \right) \approx 64g^2, \quad (128b)$$

$$\frac{\gamma + \gamma + \bar{\gamma}}{\sqrt{\Omega}} \approx 0, \quad (128c)$$

$$\begin{aligned} \cos \frac{\sqrt{\Omega}t}{4} &\approx \cos 2gt = \cos^2(gt) - \sin^2(gt) \\ &= 2 \cos^2(gt) - 1 = 1 - 2 \sin^2(gt). \end{aligned} \quad (128d)$$

and thus we find that

$$\begin{pmatrix} \rho_{ee}(t) & \rho_{ec}(t) \\ \rho_{ce}(t) & \rho_{cc}(t) \end{pmatrix} = e^{-\phi t} \begin{pmatrix} \cos^2(gt) & -\frac{1}{2i} \sin(2gt) \\ \frac{1}{2i} \sin(2gt) & \sin^2(gt) \end{pmatrix} + \left(e^{-\frac{\gamma+\kappa}{2}t} - e^{-\phi t} \right) \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \quad (129)$$

4.1.2 Calculation of the Green Function

We start from the Eq.(103), which in matrix form gives

$$i \begin{pmatrix} \dot{G}_{ee}^R(\tau) & \dot{G}_{ec}^R(\tau) \\ \dot{G}_{ce}^R(\tau) & \dot{G}_{cc}^R(\tau) \end{pmatrix} = i \begin{pmatrix} \delta(\tau) & 0 \\ 0 & \delta(\tau) \end{pmatrix} + \begin{pmatrix} -i\frac{\gamma+\bar{\gamma}}{2} & g \\ g & -i\frac{\kappa}{2} \end{pmatrix} \begin{pmatrix} G_{ee}^R(\tau) & G_{ec}^R(\tau) \\ G_{ce}^R(\tau) & G_{cc}^R(\tau) \end{pmatrix}, \quad (130)$$

We assume $G^R(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and we define the parameters $\phi = \frac{\gamma+\bar{\gamma}+\kappa}{2}$ and $\Omega = [4g + (\kappa - \gamma - \bar{\gamma})][4g - (\kappa - \gamma - \bar{\gamma})]$.

Then we solve the above system and apply the inverse Laplace Transform and we find

$$\begin{pmatrix} G_{ee}^R(\tau) & G_{ec}^R(\tau) \\ G_{ce}^R(\tau) & G_{cc}^R(\tau) \end{pmatrix} = e^{-\phi\tau} \begin{pmatrix} \frac{\kappa-\gamma-\bar{\gamma}}{\sqrt{\Omega}} \sin \frac{\sqrt{\Omega}\tau}{4} + \cos \frac{\sqrt{\Omega}\tau}{4} & -\frac{4ig}{\sqrt{\Omega}} \sin \frac{\sqrt{\Omega}\tau}{4} \\ -\frac{4ig}{\sqrt{\Omega}} \sin \frac{\sqrt{\Omega}\tau}{4} & -\frac{\kappa-\gamma-\bar{\gamma}}{\sqrt{\Omega}} \sin \frac{\sqrt{\Omega}\tau}{4} + \cos \frac{\sqrt{\Omega}\tau}{4} \end{pmatrix}. \quad (131)$$

In the coherent regime $2g \gg \gamma + \bar{\gamma} + \kappa$ so we can make the approximations

$$\frac{\kappa \pm (\gamma + \bar{\gamma})}{4g} \approx 0, \quad (132a)$$

$$\Omega = 16g^2 \left(1 + \frac{\kappa - \gamma - \bar{\gamma}}{4g} \right) \left(1 - \frac{\kappa - \gamma - \bar{\gamma}}{4g} \right) \approx 16g^2, \quad (132b)$$

$$\frac{\pm(\kappa - \gamma - \bar{\gamma})}{\sqrt{\Omega}} \approx 0. \quad (132c)$$

and thus we find the Green function to be

$$\begin{pmatrix} G_{ee}^R(\tau) & G_{ec}^R(\tau) \\ G_{ce}^R(\tau) & G_{cc}^R(\tau) \end{pmatrix} = e^{-\frac{\gamma+\bar{\gamma}+\kappa}{4}\tau} \begin{pmatrix} \cos g\tau & -i \sin g\tau \\ -i \sin g\tau & \cos g\tau \end{pmatrix}. \quad (133)$$

4.1.3 Calculation of the efficiency and the indistinguishability

The efficiency is given by

$$\begin{aligned} \beta &= \kappa \int_0^\infty dt \rho_{cc}(t) \\ &= \kappa \int_0^\infty dt e^{-\frac{\gamma+\kappa+\bar{\gamma}}{2}t} \left(\sin^2(gt) - \frac{1}{2} \right) + \frac{1}{2} e^{-\frac{\gamma+\kappa}{2}t} \\ &= \kappa \frac{\bar{\gamma}(\gamma + \kappa + \bar{\gamma}) + 16g^2}{(\gamma + \kappa) ((\gamma + \kappa + \bar{\gamma})^2 + 16g^2)} = \kappa \frac{2g \left(\bar{\gamma} \frac{\gamma+\kappa+\bar{\gamma}}{2g} + 8g \right)}{4g^2(\gamma + \kappa) \left(\left(\frac{\gamma+\kappa+\bar{\gamma}}{2g} \right)^2 + 4 \right)} \\ &\approx \kappa \frac{16g^2}{16g^2(\gamma + \kappa)} = \frac{\kappa}{\gamma + \kappa}. \end{aligned} \quad (134)$$

The two-time correlation function $\langle a^\dagger(t + \tau)a(t) \rangle$ is given by (125), which gives

$$\begin{aligned} \left| \langle a^\dagger(t + \tau)a(t) \rangle \right|^2 &= |G_{cc}^R(\tau)\rho_{cc}(t) + G_{ce}^R(\tau)\rho_{ce}(t)|^2 \\ &= \left| \cos g\tau \left(e^{-\frac{\gamma+\kappa+\bar{\gamma}}{2}t} (\sin^2(gt) - \frac{1}{2}) + \frac{e^{-\frac{\gamma+\kappa}{2}t}}{2} \right) - \frac{\sin(g\tau)}{2} e^{-\frac{\gamma+\kappa+\bar{\gamma}}{2}t} \sin 2gt \right|^2 e^{-(\gamma+\bar{\gamma}+\kappa)\tau}. \end{aligned} \quad (135)$$

Since $2g \gg \gamma + \bar{\gamma} + \kappa$, we can make the approximation

$$\sin^n(gt), \cos^n(gt) \rightarrow \begin{cases} 0, & n = 1, 3, 5 \dots \\ \frac{1}{2}, & n = 0, 2 \end{cases}, \quad (136)$$

which gives

$$\left\langle \left| \langle a^\dagger(t + \tau)a(t) \rangle \right|^2 \right\rangle = \left(\frac{e^{-(\gamma+\kappa+\bar{\gamma})t}}{8} + \frac{e^{-(\gamma+\kappa)t}}{8} \right) e^{-(\gamma+\kappa+\bar{\gamma})\tau}. \quad (137)$$

So the integral of the numerator of the indistinguishability is

$$\begin{aligned} \int_0^\infty \int_0^\infty dt d\tau \left\langle \left| \langle a^\dagger(t + \tau)a(t) \rangle \right|^2 \right\rangle &= \int_0^\infty dt \frac{1}{8} \left(e^{-(\gamma+\kappa+\bar{\gamma})t} + e^{-(\gamma+\kappa)t} \right) \int_0^\infty d\tau e^{-(\gamma+\kappa+\bar{\gamma})\tau} \\ &= \frac{\gamma + \kappa + \frac{\bar{\gamma}}{2}}{2(\gamma + \kappa)(\gamma + \kappa + \bar{\gamma})^2}. \end{aligned} \quad (138)$$

Lastly, the denominator is

$$\begin{aligned} \int_0^\infty \int_0^\infty dt d\tau \langle a^\dagger(t)a(t) \rangle \langle a^\dagger(t + \tau)a(t + \tau) \rangle &= \int_0^\infty dt \langle a^\dagger(t)a(t) \rangle \int_t^\infty d\tau \langle a^\dagger(\tau)a(\tau) \rangle \\ &\approx \frac{1}{2} \left| \int_0^\infty dt \langle a^\dagger(t)a(t) \rangle \right|^2 = \frac{1}{\sqrt{2}(\kappa + \bar{\gamma})}. \end{aligned} \quad (139)$$

So, the indistinguishability is

$$I = \frac{\gamma + \kappa + \frac{\bar{\gamma}}{2}}{2(\gamma + \kappa)(\gamma + \kappa + \bar{\gamma})^2} \left(\frac{1}{\sqrt{2}(\gamma + \kappa)} \right)^{-2} = \frac{(\gamma + \kappa)(\gamma + \kappa + \frac{\bar{\gamma}}{2})}{(\gamma + \kappa + \bar{\gamma})}. \quad (140)$$

4.2 The incoherent coupling regime

The next interesting limit we will investigate is the incoherent coupling regime between atom and cavity, which corresponds to $2g < \gamma + \kappa + \bar{\gamma}$.

4.2.1 Adiabatic Elimination

From Eq.(114), we observe that the time evolution of the off-diagonal density matrix elements is given by

$$\dot{\rho}_{ec} = -\frac{\gamma + \kappa + \bar{\gamma}}{2} \rho_{ec} + ig(\rho_{ee} - \rho_{cc}), \quad (141a)$$

$$\dot{\rho}_{ce} = -\frac{\gamma + \kappa + \bar{\gamma}}{2} \rho_{ce} - ig(\rho_{ee} - \rho_{cc}). \quad (141b)$$

In the incoherent regime ($2g \ll \gamma + \kappa + \bar{\gamma}$), the first term dominates, so the coherences are exponentially decaying as $(\rho_{ec} \propto e^{-\frac{\gamma+\kappa+\bar{\gamma}}{2}t})$, so in great times ($t \gg \frac{1}{\gamma+\kappa+\bar{\gamma}}$) the derivative will be $\dot{\rho}_{ec} = 0$

$$\rho_{ec}(t) = \frac{2ig(\rho_{ee}(t) - \rho_{cc}(t))}{\gamma + \kappa + \bar{\gamma}}, \quad (142a)$$

$$\rho_{ce}(t) = -\frac{2ig(\rho_{ee}(t) - \rho_{cc}(t))}{\gamma + \kappa + \bar{\gamma}}. \quad (142b)$$

We substitute the expressions of the coherences to the equations of the populations and get

$$\dot{\rho}_{ee} = -(\gamma + R)\rho_{ee} + R\rho_{cc}, \quad (143a)$$

$$\dot{\rho}_{cc} = -(\kappa + R)\rho_{cc} + R\rho_{ee}. \quad (143b)$$

where we define the atom-cavity exchange rate as

$$R = \frac{4g^2}{\gamma + \kappa + \bar{\gamma}}. \quad (144)$$

If the system is initially in the state $\rho_0 = |e, 0\rangle\langle e, 0|$, then the equations become

$$\begin{pmatrix} \dot{\rho}_{ee}(t) \\ \dot{\rho}_{cc}(t) \end{pmatrix} = \begin{pmatrix} -(\gamma + R) & R \\ R & -(\kappa + R) \end{pmatrix} \begin{pmatrix} \rho_{ee}(t) \\ \rho_{cc}(t) \end{pmatrix} \rightarrow \begin{pmatrix} s + \gamma + R & -R \\ -R & s + \kappa + R \end{pmatrix} \begin{pmatrix} \tilde{\rho}_{ee}(s) \\ \tilde{\rho}_{cc}(s) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (145)$$

$$\begin{aligned} \rho_{ee}(t) &= \frac{e^{-\frac{t}{2}(2R+\gamma+\kappa)} (A \cosh(\frac{At}{2}) + (\kappa - \gamma) \sinh(\frac{At}{2}))}{A}, & \rho_{cc}(t) &= \frac{e^{-\frac{t}{2}(2R+\gamma+\kappa)} R \sinh(\frac{At}{2})}{A}, \\ \rho_{ec}(t) &= \frac{2ige^{-\frac{t}{2}(2R+\gamma+\kappa)}}{\gamma + \kappa + \bar{\gamma}} \left[\left(1 - \frac{R}{A}\right) \cosh\left(\frac{At}{2}\right) + \frac{\kappa - \gamma - R}{A} \sinh\left(\frac{At}{2}\right) \right], \end{aligned} \quad (146)$$

where $A = \sqrt{4R^2 + (\gamma - \kappa)^2}$. It's straightforward to find the efficiency

$$\beta = \kappa \int_0^\infty \rho_{cc}(t) = \kappa \int_0^\infty \frac{e^{-\frac{t}{2}(2R+\gamma+\kappa)} R \sinh(\frac{At}{2})}{A} = \frac{\kappa R}{\kappa R + \gamma(\kappa + R)}. \quad (147)$$

4.2.2 The bad cavity limit

In the bad cavity limit the cavity decay rate is greater than the other dissipation terms ($\kappa \gg \gamma + \bar{\gamma}$). That's why we can adiabatically eliminate both the coherences and the cavity population, as such

$$\left. \begin{aligned} \dot{\rho}_{cc} &= -\kappa\rho_{cc} + ig(\rho_{ce} - \rho_{ec}) = 0 \\ \dot{\rho}_{ec} &= -\frac{\gamma + \kappa + \bar{\gamma}}{2}\rho_{ec} + ig(\rho_{ee} - \rho_{cc}) = 0 \end{aligned} \right\} \begin{aligned} \rho_{cc} &= \frac{ig(\rho_{ce} - \rho_{ec})}{\kappa} \\ \rho_{ec} &= \frac{2ig(\rho_{ee} - \rho_{cc})}{\gamma + \kappa + \bar{\gamma}} \end{aligned} \right\} \begin{aligned} \rho_{cc}(t) &= \frac{R}{\kappa + R}\rho_{ee}(t), \\ \rho_{ec}(t) &= \frac{i\kappa}{2g}\rho_{cc}(t) = \frac{i\kappa R}{2g(\kappa + R)}\rho_{ee}(t). \end{aligned} \quad (148)$$

So, from the reduced population rate equations in the incoherent regime (143) we get

$$\begin{aligned}\dot{\rho}_{ee} &= -(\gamma + R)\rho_{ee} + \frac{R^2}{\kappa + R}\rho_{ee} \\ \rho_{ee}(t) &= e^{-(\gamma+R-\frac{R^2}{\kappa+R})t} \approx e^{-(\gamma+R)t}.\end{aligned}\quad (149)$$

So, the density matrix is

$$\begin{pmatrix} \rho_{ee}(t) & \rho_{ec}(t) \\ \rho_{ce}(t) & \rho_{cc}(t) \end{pmatrix} = e^{-(\gamma+R)t} \begin{pmatrix} 1 & \frac{i\kappa R}{2g(\kappa+R)} \\ -\frac{i\kappa R}{2g(\kappa+R)} & \frac{R}{\kappa+R} \end{pmatrix}.\quad (150)$$

It's straightforward to calculate the efficiency

$$\beta = \kappa \int_0^\infty dt \frac{R}{\kappa + R} e^{-(\gamma+R)t} = \frac{\kappa R}{(\kappa + R)(\gamma + R)} \approx \frac{\kappa R}{\kappa R + \gamma(\kappa + R)}.\quad (151)$$

On the other hand, we can apply the adiabatic elimination on the Retarded Green Function. For times $\tau \gg \frac{1}{\kappa}$, we adiabatically eliminate the term G_{ce}^R , substituting to (130) $\frac{\partial G_{ce}^R}{\partial \tau} = 0$ which gives

$$G_{ce}^R(\tau) = -2i\frac{g}{\kappa}G_{ee}^R(\tau).\quad (152)$$

We thus have to solve a reduced system of ODE's

$$\begin{aligned}i\dot{G}_{cc}^R &= gG_{ec}^R - i\frac{\kappa}{2}G_{cc}^R = i\frac{2g^2}{\kappa}G_{cc}^R - i\frac{\kappa}{2}G_{cc}^R, \\ i\dot{G}_{ee}^R &= gG_{ce}^R - i\frac{\gamma + \bar{\gamma}}{2}G_{ee}^R = -i\frac{2g^2}{\kappa}G_{ee}^R - i\frac{\gamma + \bar{\gamma}}{2}G_{ee}^R.\end{aligned}\quad (153)$$

We notice that

$$\frac{2g^2}{\kappa} = \frac{1}{2} \frac{4g^2}{\gamma + \kappa + \bar{\gamma}} \frac{\gamma + \kappa + \bar{\gamma}}{\kappa} \approx \frac{1}{2} \frac{4g^2}{\gamma + \kappa + \bar{\gamma}} = \frac{R}{2}.\quad (154)$$

which further simplifies the system

$$\left. \begin{aligned}i\dot{G}_{cc}^R &= -\frac{\kappa}{2}G_{cc}^R \\ i\dot{G}_{ee}^R &= -\frac{\gamma + \bar{\gamma} + R}{2}G_{ee}^R\end{aligned} \right\} \begin{aligned}G_{cc}^R &= e^{-\frac{\kappa}{2}\tau}, \\ G_{ee}^R &= e^{-\frac{\gamma + \bar{\gamma} + R}{2}\tau}.\end{aligned}$$

In matrix form, the solution is

$$\begin{pmatrix} G_{ee}^R(\tau) & G_{ec}^R(\tau) \\ G_{ce}^R(\tau) & G_{cc}^R(\tau) \end{pmatrix} = \begin{pmatrix} e^{-\frac{\gamma + \bar{\gamma} + R}{2}\tau} & 2i\frac{g}{\kappa}e^{-\frac{\gamma + \bar{\gamma} + R}{2}\tau} \\ -2i\frac{g}{\kappa}e^{-\frac{\gamma + \bar{\gamma} + R}{2}\tau} & e^{-\frac{\kappa}{2}\tau} \end{pmatrix}.\quad (155)$$

We now can calculate the indistinguishability, by first calculating the two-time correlation function

$$\begin{aligned}
\langle a^\dagger(t+\tau)a(t) \rangle &= G_{cc}^R(\tau)\rho_{cc}(t) + G_{ce}^R(\tau)\rho_{ce}(t) \\
&= e^{-\frac{\kappa}{2}\tau} \frac{R}{R+\kappa} e^{-(\gamma+R)t} + \frac{2g}{i\kappa} e^{-\frac{\gamma+\bar{\gamma}+R}{2}\tau} \frac{i\kappa R}{2g(\kappa+R)} e^{-(\gamma+R)t} \\
&\approx \frac{R}{\kappa} e^{-\frac{\gamma+\bar{\gamma}+R}{2}\tau} e^{-(\gamma+R)t}.
\end{aligned} \tag{156}$$

The respective integrals are

$$\int_0^\infty \int_0^\infty dt d\tau \left| \langle a^\dagger(t+\tau)a(t) \rangle \right|^2 = \frac{1}{R+\kappa}, \tag{157a}$$

$$\int_0^\infty \int_0^\infty dt d\tau \langle a^\dagger(t)a(t) \rangle \langle a^\dagger(t+\tau)a(t+\tau) \rangle = \frac{1}{2(\gamma+\kappa)(\gamma+\bar{\gamma}+R)}. \tag{157b}$$

We substitute the above expressions to the indistinguishability (93) and get

$$I = \frac{\gamma+R}{\gamma+R+\bar{\gamma}}. \tag{158}$$

4.2.3 The good cavity limit

In the incoherent coupling region and the good cavity limit the atom dissipation terms dominate, i.e. the spontaneous decay and the pure dephasing. In that limit we have $2g \ll \gamma + \kappa + \bar{\gamma}$ and $\gamma + \bar{\gamma} \gg \kappa$. As in the case of the bad cavity limit, we can adiabatically eliminate both the coherences and the atom population (since in this case the atom dissipation dominates)

$$\left. \begin{aligned}
\dot{\rho}_{ee} &= -\gamma\rho_{ee} + ig(\rho_{ec} - \rho_{ce}) = 0 \\
\dot{\rho}_{ec} &= -\frac{\gamma+\kappa+\bar{\gamma}}{2}\rho_{ec} + ig(\rho_{ee} - \rho_{cc}) = 0
\end{aligned} \right\} \begin{aligned}
\rho_{ee} &= \frac{ig(\rho_{ec} - \rho_{ce})}{\gamma} \\
\rho_{ec} &= \frac{2ig(\rho_{ee} - \rho_{cc})}{\gamma + \kappa + \bar{\gamma}}
\end{aligned} \tag{159}$$

$$\begin{aligned}
\rho_{ee}(t) &= \frac{R}{\gamma+R}\rho_{cc}(t) \\
\rho_{ec}(t) &= -\frac{i\gamma}{2g}\rho_{ee}(t) = \frac{-i\gamma R}{2g(\gamma+R)}\rho_{cc}(t)
\end{aligned}$$

and get the cavity population rate equation is

$$\begin{aligned}
\dot{\rho}_{cc} &= -(\kappa+R)\rho_{cc} + \frac{R^2}{\gamma+R}\rho_{cc} \\
\rho_{cc} &= e^{-(\kappa+R-\frac{R^2}{\gamma+R})t} \approx e^{-(\kappa+R)t}
\end{aligned} \tag{160}$$

We can find the efficiency

$$\beta = \kappa \int_0^\infty dt e^{-(\kappa+R)t} = \frac{\kappa}{\kappa+R} \approx \frac{\kappa R}{\kappa R + \gamma(\kappa+R)} \tag{161}$$

Then, we apply the adiabatic elimination in the Retarded Green Function. For times $\tau \gg \frac{1}{\gamma+\bar{\gamma}}$ we adiabatically eliminate G_{ce}^R , substitute in (130) $\frac{\partial G_{ce}^R}{\partial \tau} = 0$ and find

$$G_{ec}^R(\tau) = -2i \frac{g}{\gamma+\bar{\gamma}} G_{cc}^R(\tau). \tag{162}$$

So, the reduced time evolution of G_{cc}^R is

$$i\dot{G}_{cc}^R = gG_{ec}^R - i\frac{\kappa}{2}G_{cc}^R = -i\frac{2g^2}{\gamma + \bar{\gamma}}G_{cc}^R - i\frac{\kappa}{2}G_{cc}^R. \quad (163)$$

We notice that

$$\frac{2g^2}{\gamma + \bar{\gamma}} = \frac{1}{2}\frac{4g^2}{\gamma + \bar{\gamma} + \kappa}\frac{\gamma + \bar{\gamma} + \kappa}{\gamma + \bar{\gamma}} \approx \frac{R}{2}. \quad (164)$$

which further simplifies the rate equations

$$i\dot{G}_{cc}^R = -\left(\frac{R}{2} + \frac{\kappa}{2}\right)G_{cc}^R \rightarrow G_{cc}^R = e^{-\frac{\kappa+R}{2}\tau}, \quad (165a)$$

$$G_{ce}^R = 2i\frac{g}{\gamma + \bar{\gamma}}G_{cc}^R \rightarrow G_{ce}^R = i\frac{2g}{\gamma + \bar{\gamma}}e^{-\frac{\kappa+R}{2}\tau}. \quad (165b)$$

To find the indistinguishability, we calculate the two-time correlation function

$$\begin{aligned} \langle a^\dagger(t + \tau)a(t) \rangle &= G_{cc}^R(\tau)\rho_{cc}(t) + G_{ce}^R(\tau)\rho_{ce}(t) \\ &= e^{-\frac{\kappa+R}{2}\tau}\rho_{cc}(t) + i\frac{2g}{\gamma + \bar{\gamma}}e^{-\frac{\kappa+R}{2}\tau}\rho_{ce}(t) \approx e^{-\frac{\kappa+R}{2}\tau}\rho_{cc}(t). \end{aligned} \quad (166)$$

So, the indistinguishability is

$$I = \frac{\int_0^\infty dt \rho_{cc}^2(t) \int_0^\infty d\tau e^{-(\kappa+R)\tau}}{\frac{1}{2}|\int_0^\infty dt \rho_{cc}(t)|^2}. \quad (167)$$

We substitute the expression of ρ_{cc} from (146) and find

$$\int_0^\infty dt \rho_{cc}(t) = \frac{R}{\kappa R + \gamma R + \kappa \gamma}, \quad (168a)$$

$$\int_0^\infty dt \rho_{cc}(t)^2 = \frac{R^2}{2(2R + \gamma + \kappa)(\kappa R + \gamma R + \kappa \gamma)}, \quad (168b)$$

$$\int_0^\infty d\tau e^{-(\kappa+R)\tau} = \frac{1}{\kappa + R}. \quad (168c)$$

So, the indistinguishability is

$$I = \frac{\frac{R^2}{2(2R + \gamma + \kappa)(\kappa R + \gamma R + \kappa \gamma)} \frac{1}{\kappa + R}}{\frac{1}{2} \frac{R^2}{(\kappa R + \gamma R + \kappa \gamma)^2}} = \frac{\kappa R + \gamma R + \kappa \gamma}{(2R + \gamma + \kappa)(\kappa + R)} = \frac{\gamma + \frac{\kappa R}{\kappa + R}}{2R + \gamma + \kappa}. \quad (169)$$

4.3 Results for the One cavity System

Graph 8 shows the efficiency of the one cavity system as a function of the cavity decay rate κ and the atom-cavity coupling g , for fixed values of the spontaneous decay rate $\gamma = 1$ and the pure dephasing $\bar{\gamma} = 10^4$. We distinguish a large area where the efficiency is close to one, which corresponds to the limit where the coupling and the cavity decay are both large ($\kappa, g \gg \bar{\gamma}$). This is the ideal region because of its efficiency, but the problem is that it is difficult to achieve such high values of g and κ . Close to that region, there is a "band" of values where the efficiency isn't one but takes sufficiently big values 20 – 80%. That "band" corresponds to the region where $R = \gamma$ and $\kappa > \gamma$ and to the region where $\kappa \approx \gamma$ and $g > 500\gamma$

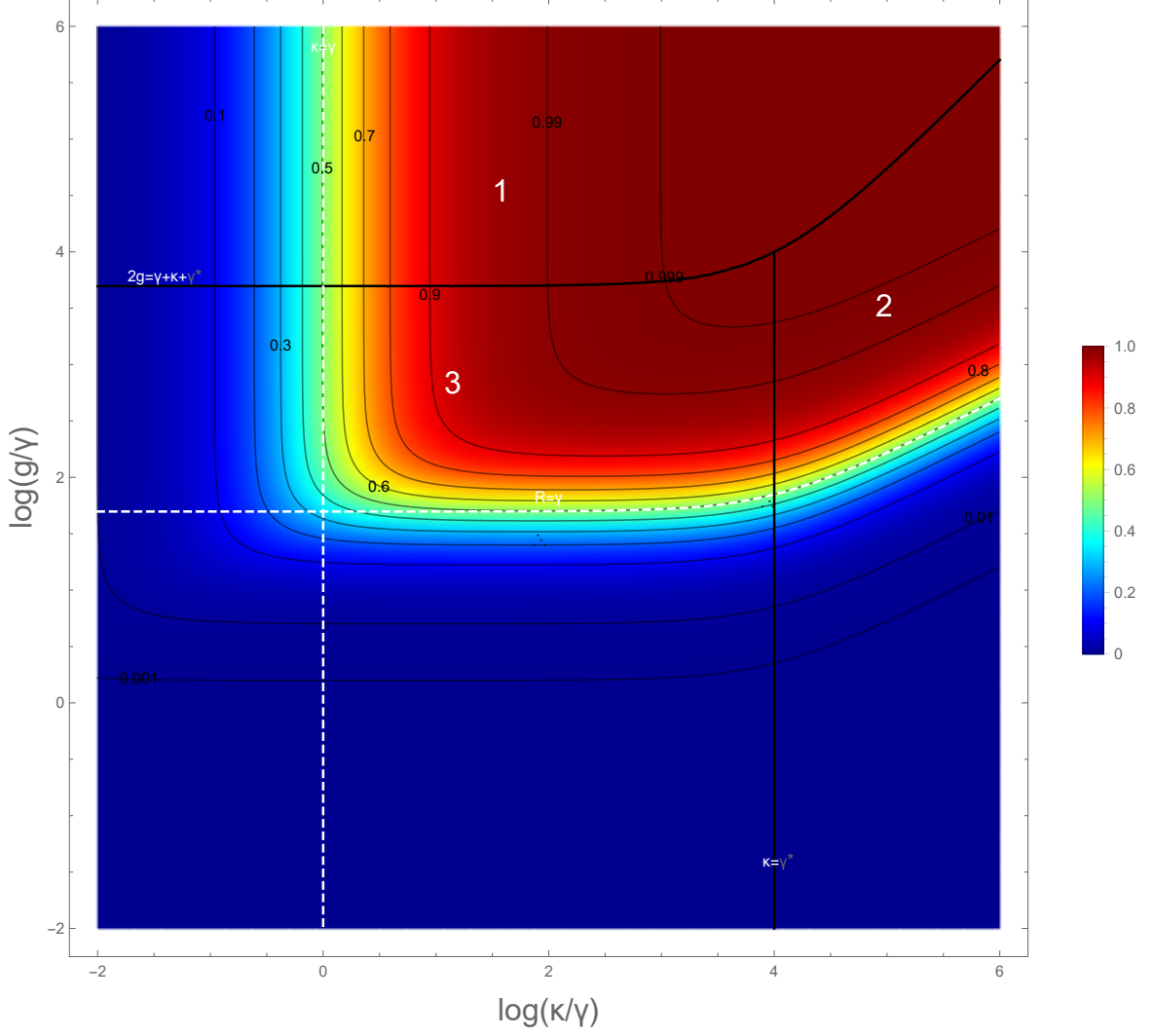


Figure 8: Graph of the system's efficiency as a function of the cavity decay κ and the atom-cavity coupling g . Also, we divide the parameter space into the regions of interest we described in the main text.

Fig.(9) shows the indistinguishability of the photons as a function of the cavity decay rate κ and the atom-cavity coupling g , for fixed values of the spontaneous decay rate $\gamma = 1$ and the pure dephasing $\bar{\gamma} = 10^4$. We distinguish 2 regions where the indistinguishability is close to one. The first one corresponds to the limit where the coupling and the cavity decay are both large ($\kappa, g \gg \bar{\gamma}$), the same as the region in Fig.(8) which experimentally isn't reachable. The second region corresponds to the region where $\kappa < \gamma$ and both κ and g satisfy the inequality $R < \gamma$. That second region is experimentally feasible and is interesting, even though the efficiency is very low.

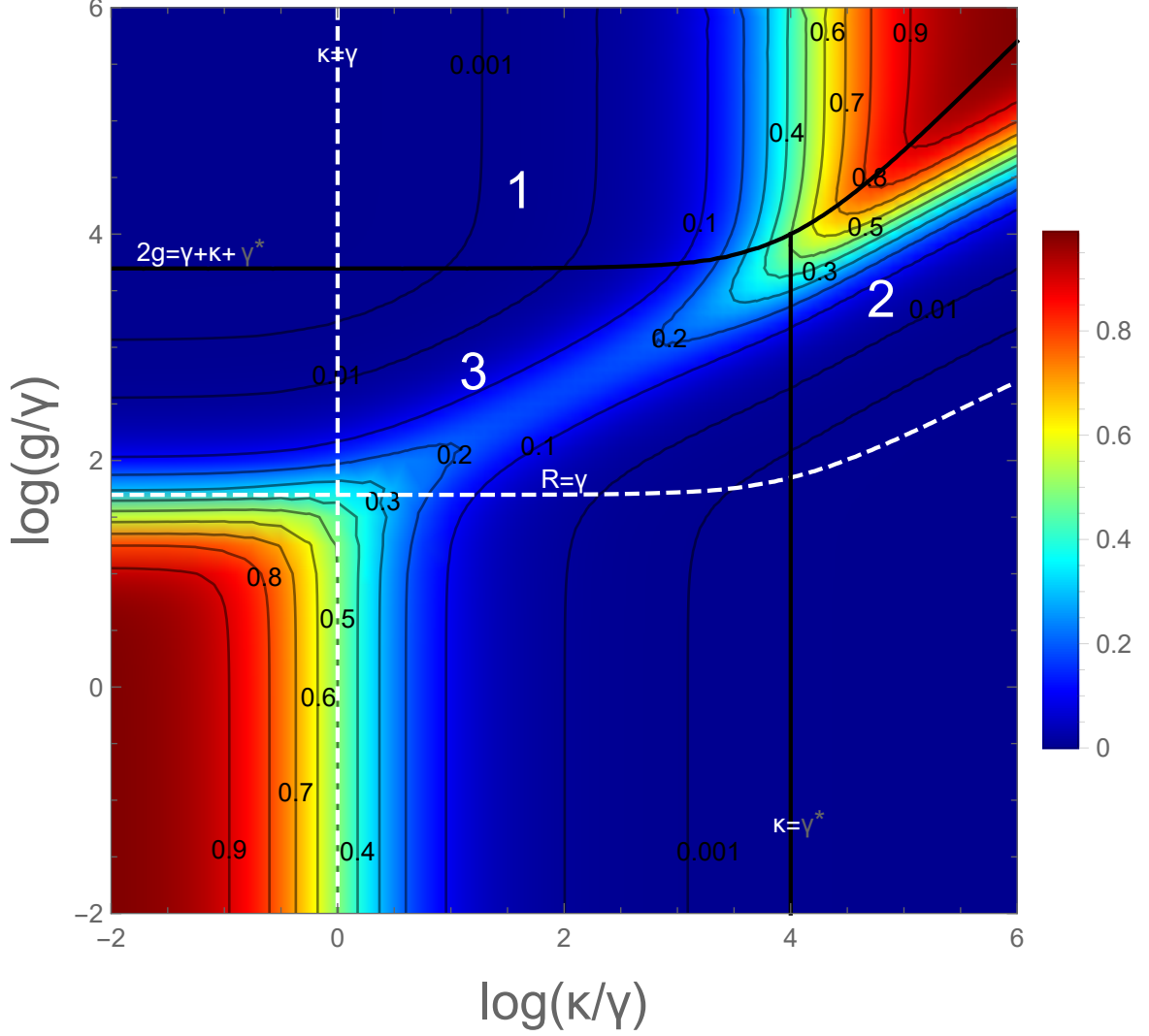


Figure 9: Graph of the produced photons' indistinguishability as a function of the cavity decay κ and the atom-cavity coupling g .

According to our analysis, we can split these graphs into 3 regions of interest, the coherent coupling regime (region 1), the incoherent coupling and good cavity regime (region 2), and the incoherent coupling and bad cavity regime (region 3). We will summarize the strategy we need to follow to maximize the efficiency and the indistinguishability of each region and we will compare them with our theoretical predictions.

In the coherent coupling regime between the atom and the cavity, the time evolution of the system is oscillatory, it exhibits Rabi Oscillations, with decreasing amplitude due to the dissipative terms of the system. In the limit ($2g \gg \gamma + \kappa + \bar{\gamma}$), the indistinguishability is given by $I_{cc} = \frac{(\gamma+\kappa)(\gamma+\kappa+\bar{\gamma}/2)}{(\gamma+\kappa+\bar{\gamma})^2}$. We notice that the indistinguishability is independent of the atom-cavity coupling constant, which means that as long as the condition $g \gg \gamma + \kappa + \bar{\gamma}$ is satisfied, we don't need to change it to improve the indistinguishability.

As we see from Fig.(10), in the coherent coupling regime (where g is sufficiently big, green line), to increase the indistinguishability we need to increase the cavity decay rate κ . But, when κ becomes comparable with g , the indistinguishability reaches its maximum value and then quickly decreases, since we aren't in the coherent coupling regime. The green line shows the numerical value of the indistinguishability when $g = 10^5 \gamma \gg \bar{\gamma}$, and we see the maximum

occurring when $\kappa \approx 10^5\gamma$, while the blue line corresponds to the case when the cavity decay is comparable to the dephasing, $\kappa = \bar{\gamma}$, and it shows a similar behaviour but with the maximum been at $\kappa \approx 10^4$. Our analytical solution matches the numerical results in the appropriate regime and starts diverging when $g \approx \bar{\gamma} + \kappa$, so we can use it only in the limit $2g \gg \gamma + \bar{\gamma} + \kappa$.

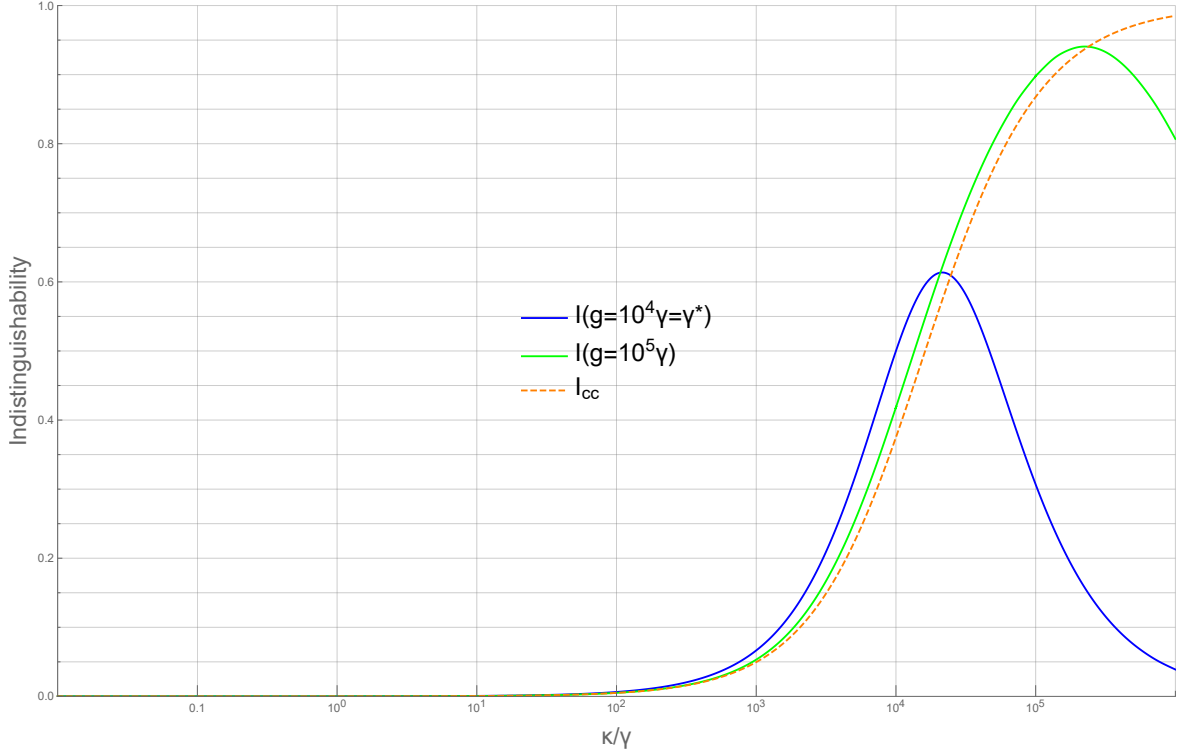


Figure 10: Coherent coupling regime: Comparison of the analytical (dashed line) and computational expression (blue for $g = 10^4\gamma$ and green for $g = 10^5\gamma$ solid lines) of the indistinguishability. Note that the analytical expression is independent of g , but is only valid in the strong coupling limit.

The analytical expression for the efficiency in the coherent coupling regime is $\beta = \frac{\kappa}{\kappa + \gamma}$. From the above expression it's easy to conclude that in order to increase the efficiency we need to increase the cavity decay rate κ without it surpassing the coupling g . This behaviour matches the numerical results, as we can see in Fig.(11).

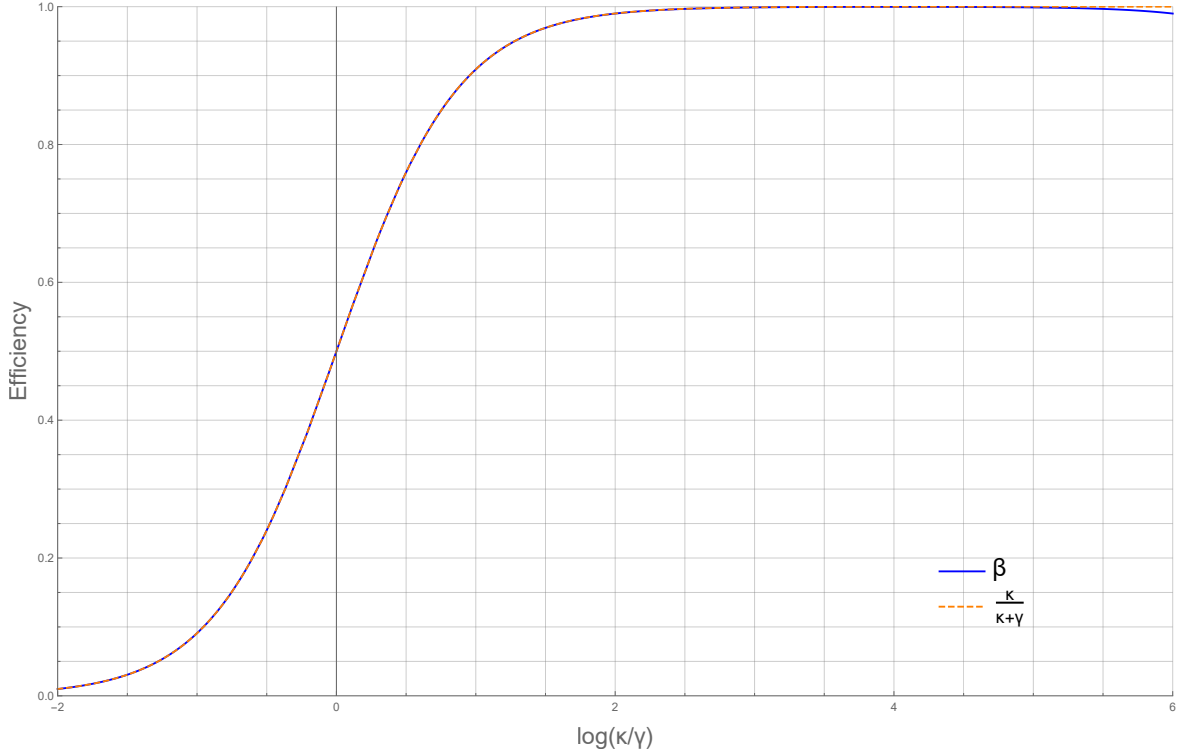


Figure 11: Coherent coupling regime: Comparison of the analytical (dashed line) and computational (solid line) expression of the efficiency, for $g = 10^5\gamma$.

In the incoherent coupling regime, the time evolution of the system is governed by the terms of the incoherent population transfer from the atom to the cavity, described by an effective transfer rate $R = \frac{4g^2}{\gamma + \kappa + \bar{\gamma}}$. That region can be split in two sub-regions, the good cavity limit ($\kappa < \gamma, g, \bar{\gamma}$) and the bad cavity limit ($\kappa > \gamma, g, \bar{\gamma}$). In the bad cavity limit, the off-diagonal terms of the density matrix and the Retarded Green function can be adiabatically eliminated, giving rise to an effective system described by an effective atom with decay rate $\gamma + R$. The analytical expression of the indistinguishability in the bad cavity limit ($2g \ll \gamma + \kappa + \bar{\gamma}$ and $\kappa \gg \gamma + \bar{\gamma}$) is given by $I_{bc} = \frac{\gamma + R}{\gamma + \bar{\gamma} + R}$ (see green line in Fig.(12)). Since $\gamma, \bar{\gamma}$ are constant, in order to increase the indistinguishability we have to increase R , which can be achieved by increasing the coupling constant such that $2g \gg \bar{\gamma}$. Since we are in the incoherent coupling regime, we simultaneously have to increase the cavity decay κ , which is difficult to achieve experimentally. On the other hand, in the good cavity region the cavity keeps photons for time greater or equal to the pure dephasing time of the atom $1/\bar{\gamma}$. In the good cavity limit ($2g \ll \gamma + \kappa + \bar{\gamma}$ and $\kappa \ll \gamma + \bar{\gamma}$) the expression of the indistinguishability is $I_{gc} = \frac{\gamma + \frac{\kappa R}{\kappa + R}}{\gamma + \kappa + 2R}$ (see red line in Fig.(12)). To increase the indistinguishability either $\kappa < \gamma$ or $R < \gamma$. The first condition is due to the fact that the cavity doesn't immediately takes the photon, but after time $1/\gamma$, in which the population ρ_{cc} takes its maximum value. Then, after the initial filling of the cavity, there still exists an incoherent exchange between the atom and the cavity with rate R . The more frequent the exchange the greater the effect of the dephasing so the smaller the indistinguishability, explaining the condition $R < \gamma$.

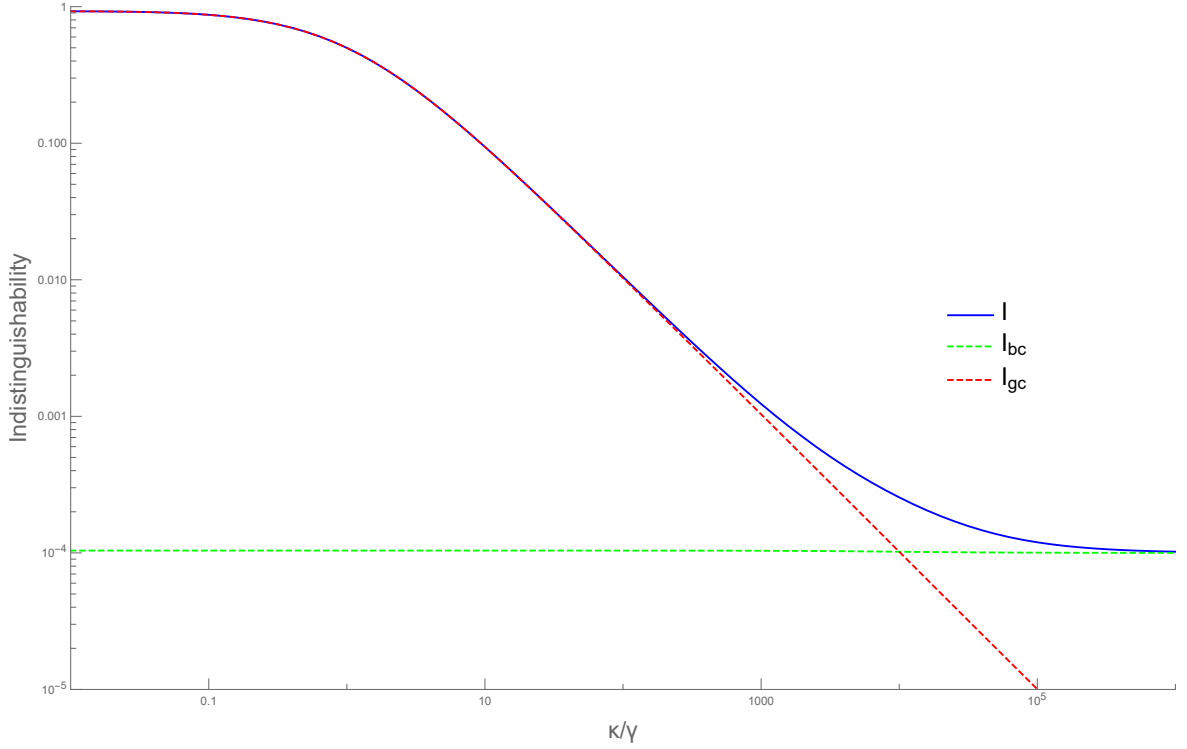


Figure 12: incoherent coupling regime: Comparison of the analytical expressions in the bad and good cavity limit (green and red lines) with the computational expression (blue line) of the indistinguishability, for $g = 10\gamma$. Note that the analytical expressions are only valid in the bad/good limit.

The analytical expression of the efficiency in the incoherent coupling regime is $\beta = \frac{\kappa R}{\kappa R + \gamma(\kappa + R)}$. We observe that in order to achieve great efficiency we need $R > \gamma$ and $\kappa > R$. As we see from the Fig.(13), for small values of κ , where $R > \kappa$, the cavity doesn't emit many photons because of the small decay rate and so the efficiency is $\beta = \frac{\kappa}{\kappa + \gamma}$. On the other hand, when κ is big, and $\kappa > R$, then the efficiency is given by $\beta = \frac{R}{R + \gamma}$.

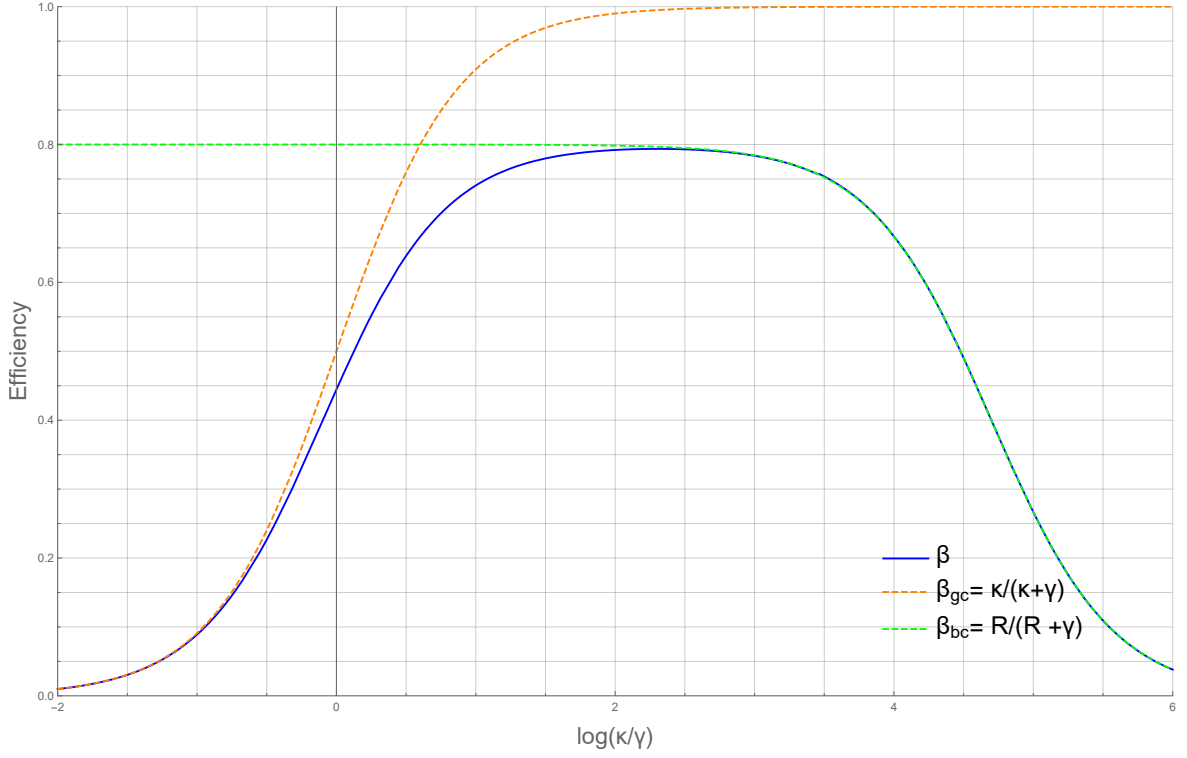


Figure 13: Incoherent coupling regime: Comparison of the analytical expressions in the bad and good cavity limit (green and orange lines) with the computational expression (blue line) of the indistinguishability, for $g = 100\gamma$. Note that the analytical expressions are only valid in the bad/good limit.

Finally, the analytical expression of the efficiency in the incoherent cavity regime is $\beta = \frac{\kappa R}{\kappa R + \gamma(\kappa + R)}$, which also matches the expression in the coherent coupling regime as seen in 14. Indeed we have

$$\beta_{th} = \frac{\kappa R}{\kappa R + \gamma(\kappa + R)} = \begin{cases} \beta_{cc} & g \gg \gamma + \kappa + \bar{\gamma} \rightarrow R \gg \kappa \\ \beta_{gc} & g \ll \gamma + \kappa + \bar{\gamma} \rightarrow R \gg \kappa \\ \beta_{bc} & g \ll \gamma + \kappa + \bar{\gamma}, R \ll \kappa \end{cases} \quad (170)$$

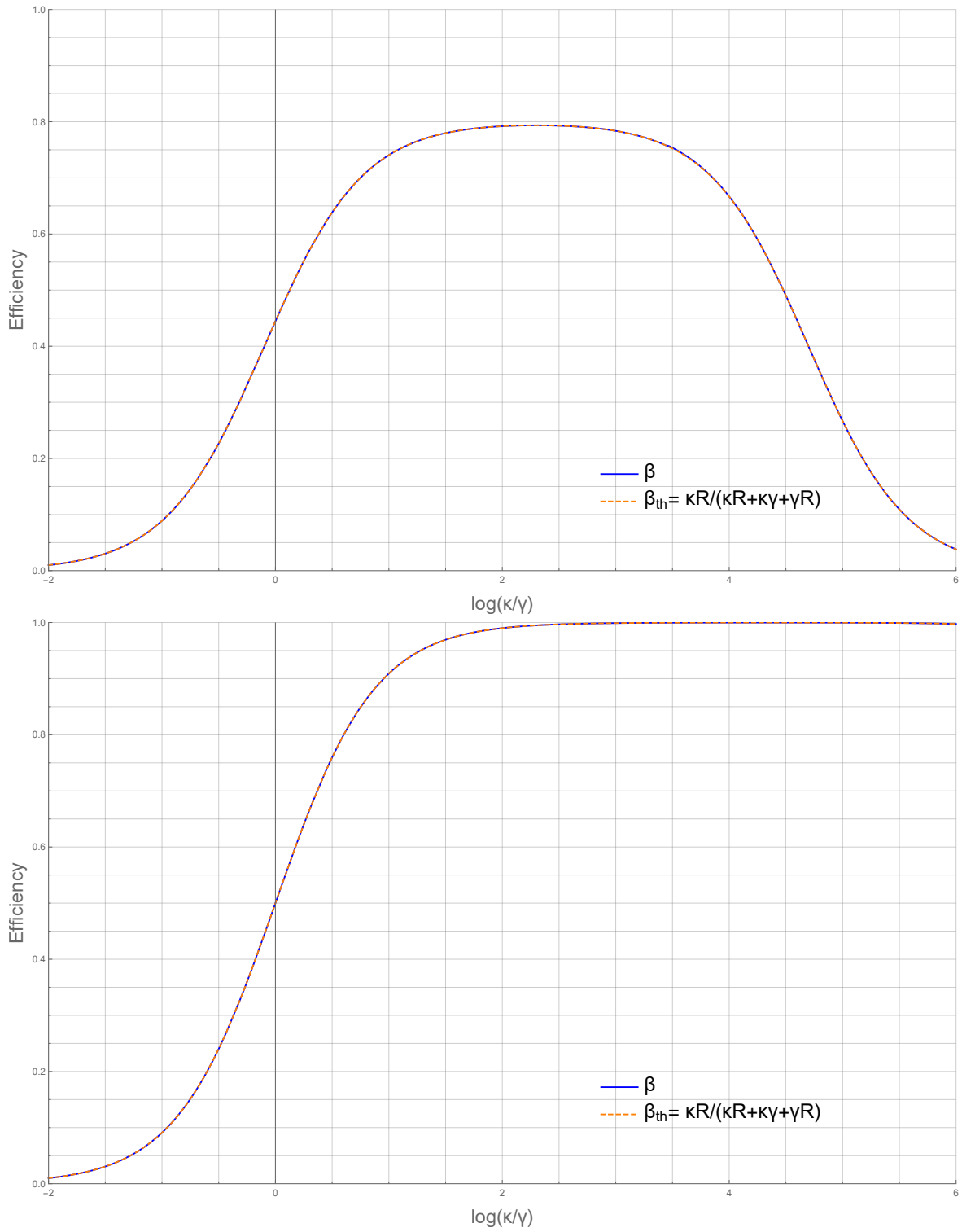


Figure 14: Comparison of the analytical expressions (dashed lines) of the efficiency in the incoherent regime (first graph) and coherent regime (second graph) with the computational expression (solid lines), for $g = 100\gamma$ and $g = 10^5\gamma$ respectively.

5 Analysis of the Two Cavity System

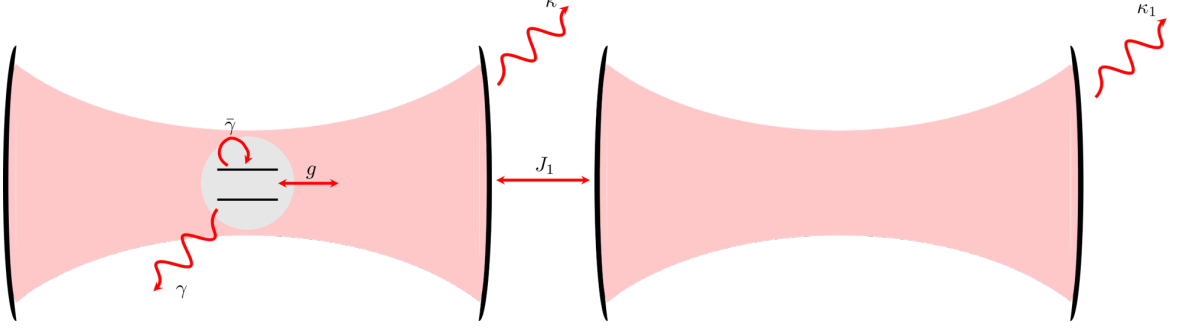


Figure 15: Illustration of the two cavity system.

The goal of Chapter 5 is the study of the quantum system of two cavities Fig.(15), following the methodology of the papers [18] and [19]. This quantum system is an extension of the system we studied in Chapters 2 and 3, since it consists of an optical cavity (cavity c), that includes an atom with two energy levels, and another empty optical cavity (cavity a). These two cavities are coupled to each other, with coupling constant J_1 , giving rise to additional degrees of freedom that determine the indistinguishability of the produced photons. The characteristics of the cavity with the atom are the same as in Chapter 2. The second cavity has only one dissipative term, the cavity decay rate κ_1 .

Also, we will analytically study the region where $g, J_1 \ll \bar{\gamma}$ and using the adiabatic elimination we will simplify the system of ODE's we have to solve. Furthermore, we will apply a transformation to reduce the system of two cavities to a system of one effective cavity, and then calculate the efficiency and the indistinguishability of the photons produced from Cavity a .

5.1 Time Evolution of the System

As we see from Fig.(15), the system consists of two optical cavities-QED with one them including an atom. The Hamiltonian of the system is

$$H = \hbar\omega_{atom}\sigma^+\sigma^- + \hbar\omega_{cav}a^\dagger a + \hbar\omega_{cav_1}a_1^\dagger a_1 + \hbar g(\sigma^+ a + a^\dagger \sigma^-) + \hbar J_1(a^\dagger a_1 + a_1^\dagger a) \quad (171)$$

where a^\dagger is the creation operator of the Cavity c and a_1^\dagger the creation operator of Cavity a . We also define the coupling constant between the 2 cavities J_1 . We assume the atom is initially in the excited state and that the cavities are empty. Then, we need to find the basis of the system, since its general form is $(\{|g\rangle, |e\rangle\} \otimes |n\rangle_{cav} \otimes |m\rangle_{cav_1}$ with $n, m = 0, 1, \dots$). It's easy to show that $[H, \sigma^+\sigma^- + a^\dagger a + a_1^\dagger a_1] = 0$, meaning that the total number of energy quanta is

conserved. So, the basis we will use is $\left\{ |e, 0, 0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |g, 1, 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |g, 0, 1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

and we write the Hamiltonian in that basis

$$\begin{aligned} H &= \hbar\omega_{atom} |e, 0, 0\rangle\langle e, 0, 0| + \hbar\omega_{cav} |g, 1, 0\rangle\langle g, 1, 0| + \hbar |g, 0, 1\rangle\langle g, 0, 1| \\ &+ \hbar g(|e, 0, 0\rangle\langle g, 1, 0| + |g, 1, 0\rangle\langle e, 0, 0|) + \hbar J_1(|e, 0, 0\rangle\langle g, 0, 1| + |g, 0, 1\rangle\langle e, 0, 0|) \rightarrow \\ H &= \hbar\Delta |g, 1, 0\rangle\langle g, 1, 0| + \hbar\Delta_1 |g, 0, 1\rangle\langle g, 0, 1| \\ &+ \hbar g(|e, 0, 0\rangle\langle g, 1, 0| + |g, 1, 0\rangle\langle e, 0, 0|) + \hbar J_1(|e, 0, 0\rangle\langle g, 0, 1| + |g, 0, 1\rangle\langle e, 0, 0|). \end{aligned} \quad (172)$$

or in matrix form

$$H = \hbar \begin{pmatrix} 0 & g & 0 \\ g & \Delta & J_1 \\ 0 & J_1 & \Delta_1 \end{pmatrix}. \quad (173)$$

Then, we define the density matrix

$$\rho(t) = \begin{pmatrix} \rho_{ee}(t) = \langle \sigma^+(t)\sigma^-(t) \rangle & \rho_{ec}(t) = \langle \sigma^+(t)a(t) \rangle & \rho_{ea}(t) = \langle \sigma^+(t)a_1(t) \rangle \\ \rho_{ce}(t) = \langle a^\dagger(t)\sigma^-(t) \rangle & \rho_{cc}(t) = \langle a^\dagger(t)a(t) \rangle & \rho_{ca}(t) = \langle a^\dagger(t)a_1(t) \rangle \\ \rho_{ae}(t) = \langle a_1^\dagger(t)\sigma^-(t) \rangle & \rho_{ac}(t) = \langle a_1^\dagger(t)a(t) \rangle & \rho_{aa}(t) = \langle a_1^\dagger(t)a_1(t) \rangle \end{pmatrix}. \quad (174)$$

The time evolution of the density matrix will be given by the Lindblad equation

$$\dot{\rho} = i[\rho, H] + \sum_i \mathcal{L}_i[\rho], \quad (175)$$

$$\dot{\rho} = \frac{i}{\hbar}[\rho, H] + \sum_i \gamma_i \left(L_i \rho L_i^\dagger - \frac{1}{2} \{ L_i^\dagger L_i, \rho \} \right). \quad (176)$$

In our system, the dissipation terms are due to the spontaneous emission (γ), the pure dephasing ($\bar{\gamma}$), the cavity c decay rate (κ) and the cavity a decay rate (κ_1). These terms correspond to $L = \sigma^-$, $L = \sigma^+\sigma^-$, $L = a$ and $L = a_1$ respectively, and we find

$$\mathcal{L}_{QE}[\rho] = -\gamma \begin{pmatrix} \rho_{ee} & \rho_{ec}/2 & \rho_{ea}/2 \\ \rho_{ce}/2 & 0 & 0 \\ \rho_{ae}/2 & 0 & 0 \end{pmatrix}, \quad (177)$$

$$\mathcal{L}_{cav}[\rho] = -\kappa \begin{pmatrix} 0 & \rho_{ec}/2 & 0 \\ \rho_{ce}/2 & \rho_{cc} & \rho_{ca}/2 \\ 0 & \rho_{ac}/2 & 0 \end{pmatrix}, \quad (178)$$

$$\mathcal{L}_{cav_1}[\rho] = -\kappa_1 \begin{pmatrix} 0 & 0 & \rho_{ea}/2 \\ 0 & 0 & \rho_{ca}/2 \\ \rho_{ae}/2 & \rho_{ac}/2 & \rho_{aa} \end{pmatrix}, \quad (179)$$

$$\mathcal{L}_{deph}[\rho] = -\bar{\gamma} \begin{pmatrix} 0 & \rho_{ec}/2 & \rho_{ea}/2 \\ \rho_{ce}/2 & 0 & 0 \\ \rho_{ae}/2 & 0 & 0 \end{pmatrix}, \quad (180)$$

We also calculate the commutator of the density matrix with the Hamiltonian

$$[\rho, H] = \begin{pmatrix} g(\rho_{ec} - \rho_{ce}) & g(\rho_{ee} - \rho_{cc}) + J_1 \rho_{ea} & J_1 \rho_{ec} - g \rho_{ca} \\ g(\rho_{cc} - \rho_{ee}) - J_1 \rho_{ae} & g(\rho_{ce} - \rho_{ec}) + J_1(\rho_{ca} - \rho_{ac}) & J_1(\rho_{cc} - \rho_{aa}) - g \rho_{ea} \\ g \rho_{ac} - J_1 \rho_{ce} & J_1(\rho_{aa} - \rho_{cc}) + g \rho_{ae} & J_1(\rho_{ac} - \rho_{ca}) \end{pmatrix}. \quad (181)$$

So, we have derived the following system of equations

$$\begin{aligned}
\dot{\rho}_{ee} &= ig(\rho_{ec} - \rho_{ce}) - \gamma\rho_{ee}, \\
\dot{\rho}_{ec} &= ig(\rho_{ee} - \rho_{cc}) + iJ_1\rho_{ea} - \frac{\gamma + \kappa + \bar{\gamma}}{2}\rho_{ec}, \\
\dot{\rho}_{ea} &= iJ_1\rho_{ec} - ig\rho_{ca} - \frac{\gamma + \kappa_1 + \bar{\gamma}}{2}\rho_{ea}, \\
\dot{\rho}_{ce} &= -ig(\rho_{ee} - \rho_{cc}) - iJ_1\rho_{ae} - \frac{\gamma + \kappa + \bar{\gamma}}{2}\rho_{ce}, \\
\dot{\rho}_{cc} &= -ig(\rho_{ec} - \rho_{ce}) + iJ_1(\rho_{ca} - \rho_{ac}) - \kappa\rho_{cc}, \\
\dot{\rho}_{ca} &= iJ_1(\rho_{cc} - \rho_{aa}) - ig\rho_{ea} - \frac{\kappa + \kappa_1}{2}\rho_{ca}, \\
\dot{\rho}_{ae} &= -iJ_1\rho_{ce} + ig\rho_{ac} - \frac{\gamma + \kappa_1 + \bar{\gamma}}{2}\rho_{ae}, \\
\dot{\rho}_{ac} &= -iJ_1(\rho_{cc} - \rho_{aa}) + ig\rho_{ae} - \frac{\kappa + \kappa_1}{2}\rho_{ac}, \\
\dot{\rho}_{aa} &= iJ_1(\rho_{ac} - \rho_{ca}) - \kappa_1\rho_{aa}.
\end{aligned} \tag{182}$$

5.2 Calculation of the Two-Time Correlation Functions

The Green functions of our system are

$$G^<(t, t + \tau) = \begin{pmatrix} \langle \sigma^+(t + \tau)\sigma^-(t) \rangle & \langle \sigma^+(t + \tau)a(t) \rangle & \langle \sigma^+(t + \tau)a_1(t) \rangle \\ \langle a^\dagger(t + \tau)\sigma^-(t) \rangle & \langle a^\dagger(t + \tau)a(t) \rangle & \langle a^\dagger(t + \tau)a_1(t) \rangle \\ \langle a_1^\dagger(t + \tau)\sigma^-(t) \rangle & \langle a_1^\dagger(t + \tau)a(t) \rangle & \langle a_1^\dagger(t + \tau)a_1(t) \rangle \end{pmatrix}, \tag{183a}$$

$$G^>(t, t + \tau) = \begin{pmatrix} \langle \sigma^-(t)\sigma^+(t + \tau) \rangle & \langle a(t)\sigma^+(t + \tau) \rangle & \langle a_1(t)\sigma^+(t + \tau) \rangle \\ \langle \sigma^-(t)a^\dagger(t + \tau) \rangle & \langle a(t)a^\dagger(t + \tau) \rangle & \langle a_1(t)a^\dagger(t + \tau) \rangle \\ \langle \sigma^-(t)a_1^\dagger(t + \tau) \rangle & \langle a(t)a_1^\dagger(t + \tau) \rangle & \langle a_1(t)a_1^\dagger(t + \tau) \rangle \end{pmatrix}, \tag{183b}$$

$$G^R(t, t + \tau) = \Theta(\tau) (G^<(t, t + \tau) + G^>(t, t + \tau)). \tag{183c}$$

Also, the total Retarded Self Energy of the system is

$$\Sigma_{tot}^R(t, t + \tau) = \delta(\tau) \begin{pmatrix} \frac{\gamma + \bar{\gamma}}{2} & 0 & 0 \\ 0 & \frac{\kappa}{2} & 0 \\ 0 & 0 & \frac{\kappa_1}{2} \end{pmatrix}. \tag{184}$$

So, in matrix form, the Retarded Green function in the frequency domain is

$$G^R(\omega) = \begin{pmatrix} \omega + \frac{i}{2}(\gamma + \bar{\gamma}) & g & 0 \\ g & \omega + \frac{i}{2}\kappa & J_1 \\ 0 & J_1 & \omega + \frac{i}{2}\kappa_1 \end{pmatrix}^{-1}. \tag{185}$$

and it's time evolution is given by (103)

$$i \begin{pmatrix} \dot{G}_{ee}^R(\tau) & \dot{G}_{ec}^R(\tau) & \dot{G}_{ea}^R(\tau) \\ \dot{G}_{ce}^R(\tau) & \dot{G}_{cc}^R(\tau) & \dot{G}_{ca}^R(\tau) \\ \dot{G}_{ae}^R(\tau) & \dot{G}_{ac}^R(\tau) & \dot{G}_{aa}^R(\tau) \end{pmatrix} = i\delta(\tau)\mathbb{1} + \begin{pmatrix} -i\frac{\gamma + \bar{\gamma}}{2} & g & 0 \\ g & -i\frac{\kappa}{2} & J_1 \\ 0 & J_1 & i\frac{\kappa_1}{2} \end{pmatrix} \begin{pmatrix} G_{ee}^R(\tau) & G_{ec}^R(\tau) & G_{ea}^R(\tau) \\ G_{ce}^R(\tau) & G_{cc}^R(\tau) & G_{ca}^R(\tau) \\ G_{ae}^R(\tau) & G_{ac}^R(\tau) & G_{aa}^R(\tau) \end{pmatrix}. \tag{186}$$

We will only study the efficiency and the indistinguishability of the photons from cavity a, which are given by

$$\beta = \kappa_1 \int_0^\infty dt \rho_{aa}(t), \quad (187a)$$

$$I = \frac{\int_0^\infty \int_0^\infty dt d\tau \left| \langle a_1^\dagger(t+\tau) a_1(t) \rangle \right|^2}{\int_0^\infty \int_0^\infty dt d\tau \langle a_1^\dagger(t+\tau) a_1(t+\tau) \rangle \langle a_1^\dagger(t) a_1(t) \rangle}. \quad (187b)$$

5.3 Adiabatic elimination

In our system, the dissipation of the pure dephasing is greater than the other dissipation terms, So, we can adiabatically eliminate the coherence term between the atom and cavity a, ρ_{ea}

$$\dot{\rho}_{ea} = iJ_1\rho_{ec} - ig\rho_{ca} - \frac{\gamma + \kappa_1 + \bar{\gamma}}{2}\rho_{ea} = 0 \quad (188)$$

$$\rho_{ea} = \frac{2i(J_1\rho_{ec} - g\rho_{ca})}{\gamma + \kappa_1 + \bar{\gamma}}. \quad (189)$$

The other coherence terms (atom-cavity c, cavity c-cavity a) become

$$\dot{\rho}_{ec} = ig(\rho_{ee} - \rho_{cc}) - \frac{\gamma + \kappa + \bar{\gamma}}{2}\rho_{ec} - \frac{2J_1(J_1\rho_{ec} - g\rho_{ca})}{\gamma + \kappa_1 + \bar{\gamma}} \quad (190)$$

$$\dot{\rho}_{ca} = iJ_1(\rho_{cc} - \rho_{aa}) - \frac{\kappa + \kappa_1}{2}\rho_{ca} + \frac{2g(J_1\rho_{ec} - g\rho_{ca})}{\gamma + \kappa_1 + \bar{\gamma}}. \quad (191)$$

Then, we define the exchange constant between atom and cavity c, $\tilde{R} = \frac{4g^2}{\gamma + \kappa_1 + \bar{\gamma}}$. So, the coherence term between the two cavities becomes

$$\begin{aligned} \dot{\rho}_{ca} &= -\frac{\kappa + \kappa_1 + \frac{4g^2}{\gamma + \kappa_1 + \bar{\gamma}}}{2}\rho_{ca} + iJ_1(\rho_{cc} - \rho_{aa}) + \frac{2gJ_1}{\gamma + \kappa_1 + \bar{\gamma}}\rho_{ec} \\ &\approx -\frac{\kappa + \kappa_1 + \tilde{R}}{2}\rho_{ca} + iJ_1(\rho_{cc} - \rho_{aa}), \end{aligned} \quad (192)$$

where we used the fact that $J_1\rho_{ec} \ll g\rho_{ca}$ for times $t \gg \frac{1}{R}$. Since we are interested in the photon of cavity a, for times $t > \frac{1}{R}$ we can adiabatically eliminate the coherence term between the two cavities, so that $\dot{\rho}_{ca} = 0$ and

$$\rho_{ca} = \frac{2iJ_1(\rho_{cc} - \rho_{aa})}{\kappa + \kappa_1 + \tilde{R}}. \quad (193)$$

Then, we substitute the above terms in the time evolution equation of ρ_{ec}

$$\begin{aligned} \dot{\rho}_{ec} &= ig(\rho_{ee} - \rho_{cc}) - \frac{\gamma + \kappa + \bar{\gamma} + \frac{4J_1^2}{\gamma + \kappa_1 + \bar{\gamma}}}{2}\rho_{ec} + \frac{4iJ_1^2g}{(\gamma + \kappa_1 + \bar{\gamma})(\kappa + \kappa_1 + \tilde{R})}(\rho_{cc} - \rho_{aa}) \\ &\approx ig(\rho_{ee} - \rho_{cc}) - \frac{\gamma + \kappa + \bar{\gamma}}{2}\rho_{ec}, \end{aligned} \quad (194)$$

where we ignored the terms of order $\frac{1}{\bar{\gamma}}$. We can also adiabatically eliminate the coherence term between the atom and the cavity c for $t > \frac{1}{\bar{\gamma}}$

$$\rho_{ec} = \frac{2ig(\rho_{ee} - \rho_{cc})}{\gamma + \kappa + \bar{\gamma}}. \quad (195)$$

Substituting the above, we derived the population rate equations

$$\begin{aligned} \dot{\rho}_{ee} &= -\gamma\rho_{ee} - \frac{4g^2}{\gamma + \bar{\gamma} + \kappa}(\rho_{ee} - \rho_{cc}), \\ \dot{\rho}_{cc} &= -\kappa\rho_{cc} - \frac{4g^2}{\gamma + \bar{\gamma} + \kappa}(\rho_{cc} - \rho_{ee}) - \frac{4J_1^2}{\kappa + \kappa_1 + \bar{R}}(\rho_{cc} - \rho_{aa}), \\ \dot{\rho}_{aa} &= -\kappa_1\rho_{aa} - \frac{4J_1^2}{\kappa + \kappa_1 + \bar{R}}(\rho_{aa} - \rho_{cc}). \end{aligned} \quad (196)$$

If we introduce another exchange constant from cavity c to cavity a, $R_1 = \frac{4J_1^2}{\kappa + \kappa_1 + \bar{R}}$, the rate equations in matrix form become

$$\begin{pmatrix} \dot{\rho}_{ee} \\ \dot{\rho}_{cc} \\ \dot{\rho}_{aa} \end{pmatrix} = \begin{pmatrix} -\gamma - R & R & 0 \\ R & -\kappa - R - R_1 & R_1 \\ 0 & R_1 & -\kappa_1 - R_1 \end{pmatrix} \begin{pmatrix} \rho_{ee} \\ \rho_{cc} \\ \rho_{aa} \end{pmatrix}. \quad (197)$$

5.4 Calculation of the Efficiency

To solve the above system, we find the characteristic polynomial of the matrix

$$(s + \gamma + R) [(s + \kappa + R + R_1)(s + \kappa_1 + R_1) - R_1^2] - R^2(s + \kappa_1 + R_1) = 0. \quad (198)$$

The polynomial is of third order, so it has 3 roots, s_1, s_2, s_3 . So, the cavity a population is $\rho_{aa} = A_1e^{-s_1t} + A_2e^{-s_2t} + A_3e^{-s_3t}$, where the A_i 's satisfy the initial condition of the system

$$\begin{aligned} \rho_{aa}(0) &= 0, \\ \dot{\rho}_{aa}(0) &= -(\kappa_1 + R_1)\rho_{aa}(0) + R_1\rho_{cc}(0) = 0, \\ \dot{\rho}_{cc}(0) &= R\rho_{ee}(0) - (\kappa + R + R_1)\rho_{cc}(0) + R_1\rho_{aa}(0) = R, \\ \ddot{\rho}_{aa}(0) &= -\kappa_1\dot{\rho}_{aa}(0) - R_1(\dot{\rho}_{aa}(0) - \dot{\rho}_{cc}(0)) = RR_1. \end{aligned} \quad (199)$$

We apply the above conditions and we find

$$\beta_a = \kappa_1 \int_0^\infty dt \rho_{aa}(t) = \kappa_1 \frac{-RR_1}{s_1 s_2 s_3}. \quad (200)$$

The term $s_1 s_2 s_3$ is the constant term of the polynomial so its known. Substituting the above into the efficiency we find

$$\beta_a = \frac{\kappa_1 R_1}{\kappa(\kappa_1 + R_1) + \kappa_1 R_1}. \quad (201)$$

This analytical expression closely matches the numerical results, as seen in Fig.(16).

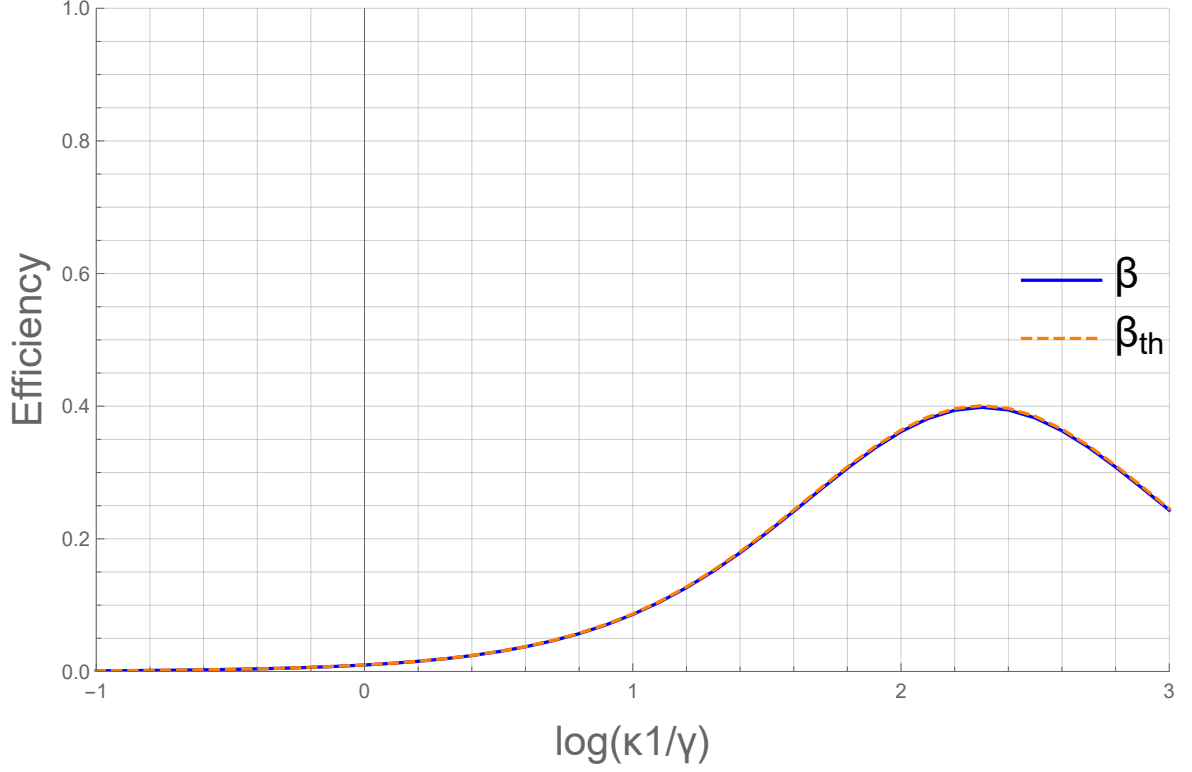


Figure 16: Comparison of the theoretical efficiency for $g = 500\gamma$, $\kappa = 100\gamma$, $J_1 = 100\gamma$ and $\bar{\gamma} = 10^4\gamma$.

5.4.1 The effective emitter model

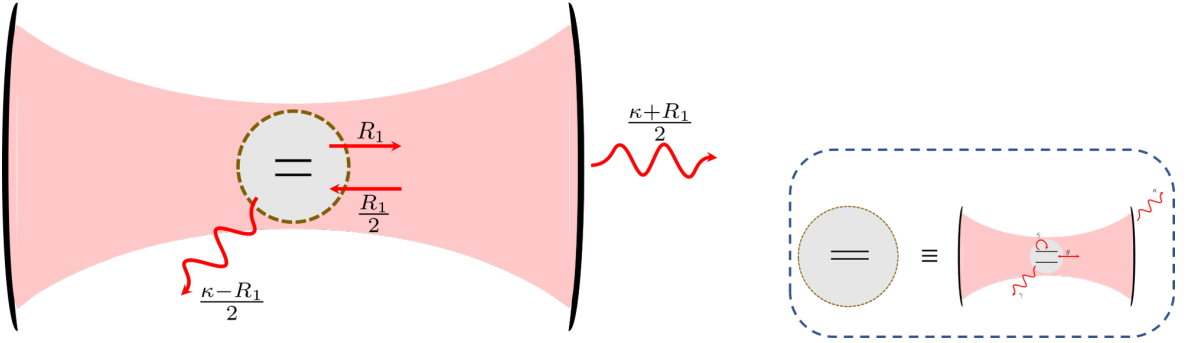


Figure 17: Illustration of the equivalent system.

Since the system of equations is difficult to solve analytically, we will apply matrix transformation which will leave cavity a unchanged. In matrix form the transformation is

$$T = \begin{pmatrix} \frac{2R}{\kappa + R_1 - \sqrt{(\kappa + R_1)^2 + 4R_1^2}} & \frac{2R}{\kappa + R_1 + \sqrt{(\kappa + R_1)^2 + 4R_1^2}} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2R \gg \kappa + R_1). \quad (202)$$

The populations $\{\rho_{ee}, \rho_{cc}, \rho_{aa}\}$ are transformed into $\{\rho_{dd}, \rho_{ss}, \rho_{aa}\}$ and the coefficient matrix R to $T^{-1}RT$. The term ρ_{ss} is the sum of the populations of the atom and the cavity and ρ_{dd}

the difference. So the new system is

$$\begin{pmatrix} \dot{\rho}_{dd} \\ \dot{\rho}_{ss} \\ \dot{\rho}_{aa} \end{pmatrix} = \begin{pmatrix} -\frac{(\kappa+R_1)+2R+\sqrt{(\kappa+R_1)^2+4R^2}}{2} & 0 & -\frac{RR_1}{\sqrt{(\kappa+R_1)^2+4R^2}} \\ 0 & -\frac{(\kappa+R_1)+2R-\sqrt{(\kappa+R_1)^2+4R^2}}{2} & \frac{RR_1}{\sqrt{(\kappa+R_1)^2+4R^2}} \\ -\frac{2RR_1}{\sqrt{(\kappa+R_1)^2+4R^2}-(\kappa+R_1)} & \frac{2RR_1}{\sqrt{(\kappa+R_1)^2+4R^2}+(\kappa+R_1)} & -\kappa_1 - R_1 \end{pmatrix} \begin{pmatrix} \rho_{dd} \\ \rho_{ss} \\ \rho_{aa} \end{pmatrix}, \quad (203)$$

where we ignored γ since it's much smaller than the other terms. In the limit $2R \gg \kappa + R_1$ we get

$$\begin{pmatrix} \dot{\rho}_{dd} \\ \dot{\rho}_{ss} \\ \dot{\rho}_{aa} \end{pmatrix} = \begin{pmatrix} -(2R + \frac{\kappa+R_1}{2}) & 0 & -\frac{R_1}{2} \\ 0 & -\frac{\kappa+R_1}{2} & \frac{R_1}{2} \\ -R_1 & R_1 & -(\kappa_1 + R_1) \end{pmatrix} \begin{pmatrix} \rho_{dd} \\ \rho_{ss} \\ \rho_{aa} \end{pmatrix}. \quad (204)$$

The term $2R + \frac{\kappa+R_1}{2} \gg \kappa_1 + R_1$, so the term ρ_{dd} decays much faster than the population of cavity a, and thus can be neglected. So the system is simplified further to

$$\begin{pmatrix} \dot{\rho}_{ss} \\ \dot{\rho}_{aa} \end{pmatrix} = \begin{pmatrix} -\frac{\kappa+R_1}{2} & \frac{R_1}{2} \\ R_1 & -(\kappa_1 + R_1) \end{pmatrix} \begin{pmatrix} \rho_{ss} \\ \rho_{aa} \end{pmatrix}. \quad (205)$$

The physical meaning of the above system is the following. Starting from three separate systems, the atom and the two cavities, we have managed to combine the atom and the cavity c into one system, which then interacts with cavity a. The coupling between the two systems is asymmetric, since the cavity a pumps the system s with rate R_1 and the system s pumps the cavity with $\frac{R_1}{2}$. Solving the above system gives

$$\rho_{aa}(t) \propto e^{-\frac{\kappa(\kappa_1+R_1)+\kappa_1 R_1}{\kappa+2\kappa_1+3R_1} t}. \quad (206)$$

5.5 Calculation of Indistinguishability

Since the photons we study come from cavity a , the expression of the indistinguishability is given by

$$I_a = \frac{\int_0^\infty \int_0^\infty dt d\tau \left| \langle a_1^\dagger(t+\tau) a_1(t) \rangle \right|^2}{\int_0^\infty dt d\tau \langle a_1^\dagger(t+\tau) a_1(t+\tau) \rangle \langle a_1^\dagger(t) a_1(t) \rangle}. \quad (207)$$

This time, the two time correlation function relates to the Retarded Green function through

$$\begin{aligned} \langle a_1^\dagger(t+\tau) a_1(t) \rangle &= \langle g, 0, 1 | G^R(\tau) \rho(t) | g, 0, 1 \rangle \\ &= G_{aa}^R(\tau) \rho_{aa}(t) + G_{ac}^R(\tau) \rho_{ca}(t) + G_{ae}^R(\tau) \rho_{ea}(t). \end{aligned} \quad (208)$$

The Eq.(186), in the limit $\gamma + \bar{\gamma} - \kappa \gg g$ and $R + \kappa - \kappa_1 \gg J_1$ has the solution

$$G_{ae}^R(\tau) \propto e^{-\frac{\gamma+\bar{\gamma}-R}{2}\tau}, \quad (209a)$$

$$G_{ac}^R(\tau) \propto e^{-\frac{\kappa+R-R_1}{2}\tau}, \quad (209b)$$

$$G_{aa}^R(\tau) \propto e^{-\frac{\kappa_1+R_1}{2}\tau}. \quad (209c)$$

Thus, the dominating term is $G_{aa}^R(\tau)$ and the two-time correlation function becomes

$$\langle a_1^\dagger(t+\tau)a_1(t) \rangle \approx G_{aa}^R(\tau)\rho_{aa}(t). \quad (210)$$

If we substitute the above expression to the indistinguishability we get

$$I_a \approx \frac{\int_0^\infty dt \rho_{aa}(t)^2 \int_0^\infty d\tau |G_{aa}^R(\tau)|^2}{\frac{1}{2} \left| \int_0^\infty dt \rho_{aa}(t) \right|^2}. \quad (211)$$

We calculate the integrals

$$\int_0^\infty dt \rho_{aa}^2(t) = \frac{1}{2} \frac{\kappa + 2\kappa_1 + 3R_1}{\kappa\kappa_1 + R_1(\kappa + \kappa_1)}, \quad (212a)$$

$$\left| \int_0^\infty dt \rho_{aa}(t) \right|^2 = \left(\frac{\kappa + 2\kappa_1 + 3R_1}{\kappa\kappa_1 + R_1(\kappa + \kappa_1)} \right)^2, \quad (212b)$$

$$\int_0^\infty d\tau |G_{aa}^R(\tau)|^2 = \frac{1}{\kappa_1 + R_1}. \quad (212c)$$

and we find

$$\begin{aligned} I &= \frac{\frac{1}{2} \frac{\kappa + 2\kappa_1 + 3R_1}{\kappa\kappa_1 + R_1(\kappa + \kappa_1)} * \frac{1}{\kappa_1 + R_1}}{\frac{1}{2} \left(\frac{\kappa + 2\kappa_1 + 3R_1}{\kappa\kappa_1 + R_1(\kappa + \kappa_1)} \right)^2} = \frac{\kappa\kappa_1 + R_1(\kappa + \kappa_1)}{(\kappa_1 + R_1)(\kappa + 2\kappa_1 + 3R_1)} = \frac{\kappa + \frac{\kappa_1 R_1}{\kappa_1 + R_1}}{\kappa + 2\kappa_1 + 3R_1} \\ &= \frac{\kappa + (\kappa_1 || R_1)}{\kappa + 2\kappa_1 + 3R_1}. \end{aligned} \quad (213)$$

As we see from Fig.(18), our analytical expression matches the numerical results. We also notice that for small values of κ_1 , our analytical prediction involving the effective emitter model underestimates the true value of the indistinguishability. That happens because in that regime the terms $G_{ac}^R(\tau)\rho_{ca}(t) + G_{ae}^R(\tau)\rho_{ea}(t)$ are comparable to $G_{aa}^R(\tau)\rho_{aa}(t)$.

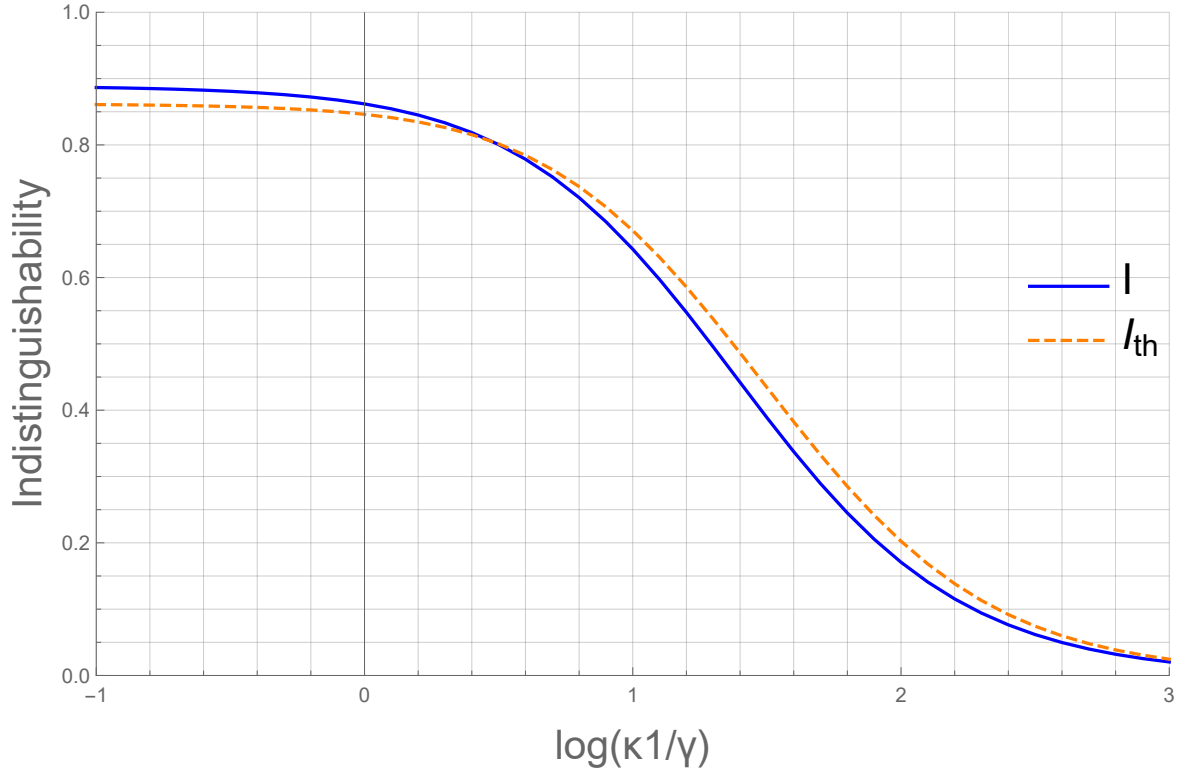


Figure 18: Comparison of the analytical expression (dashed line) of the indistinguishability with the computational expression (solid line) for $g = 500\gamma$, $\kappa = 100\gamma$, $J_1 = 100\gamma$ and $\bar{\gamma} = 10^4\gamma$.

5.6 Results

We will summarize the resulting efficiency and indistinguishability for specified values for cavity c , i.e. $g = 500\gamma$, $\kappa = 100\gamma$ and $\bar{\gamma} = 10^4\gamma$. In the case of one cavity, the efficiency is $\beta = 0.98$ and the indistinguishability is $I = 0.18$. Our goal is to achieve a higher efficiency and indistinguishability using cavity a . For instance, we can change the coupling constant J_1 by changing the distance between the two cavities. Similarly to the one cavity case, we define the population transfer rate from cavity c to cavity a , $R_1 = \frac{4J_1^2}{R + \kappa + \kappa_1}$. As we see from Fig.(19), the efficiency is maximized in the region where $\kappa_1, J_1 \gg \gamma$. There is also a "band" region around it where the efficiency isn't 1 but sufficient, similar to the one in the one cavity system (see Fig.(8)). Although the efficiency is smaller than in the case of the one cavity system, it's large enough to continue studying the system.

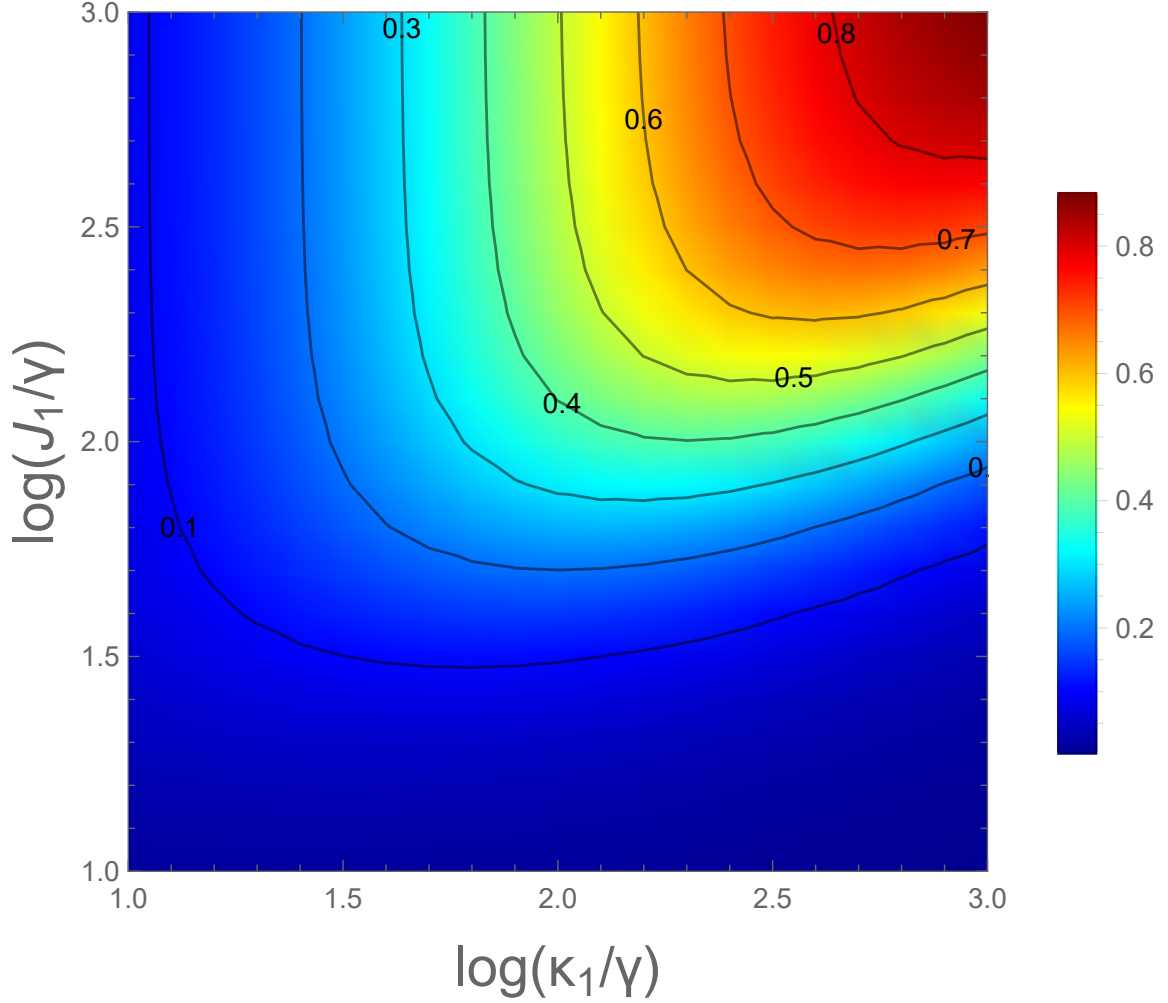


Figure 19: Graph of the efficiency of cavity a .

Then, we present the graph of the indistinguishability as a function of κ_1, J_1 in Fig.(20), for the same values of the cavity c as previously. The indistinguishability is maximized in the region where $\kappa_1, J_1 \ll \gamma$. We also distinguish a "band" region where the indistinguishability takes moderate values. Thus, the indistinguishability shows a similar behaviour as in the case of the one cavity system, in the region of the incoherent atom-cavity coupling and in the limit of the good cavity (see Fig.(9)). But, in contrast with the one cavity system, where to achieve high values of indistinguishability the values of g, κ must be 5 to 6 orders of magnitude smaller than $\bar{\gamma}$, in the two cavity system they have to be 2 to 3 orders of magnitude smaller. As a result, we have managed to increase the efficiency in the regions where the indistinguishability is large by adding a cavity.

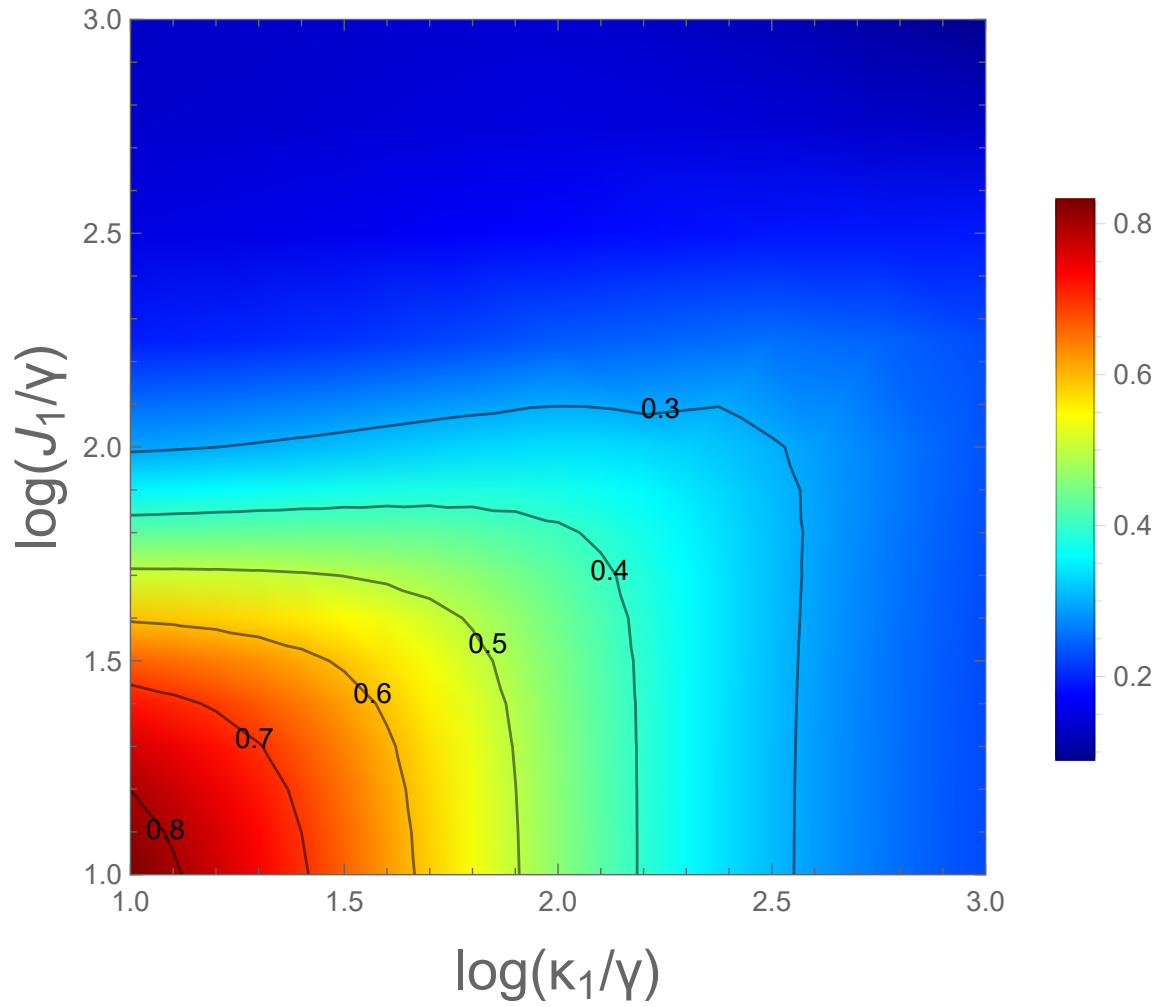


Figure 20: Graph of the Indistinguishability of the photons from Cavity a .

6 Analysis of the Three Cavity System

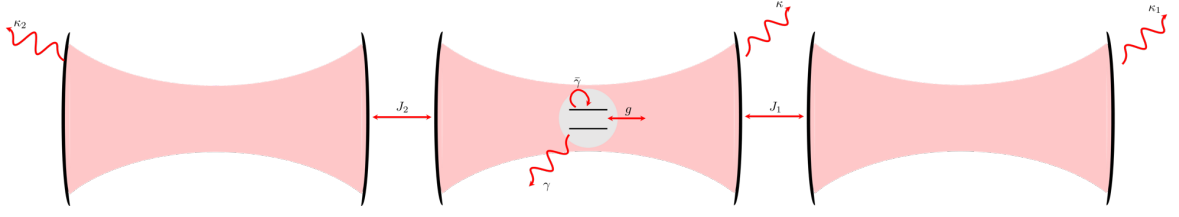


Figure 21: Illustration of the three cavity system.

The goal of Chapter 6 is the study of the quantum system illustrated in Fig.(21) following the methodology of the previous chapters. This quantum system is an extension of the system of Chapter 2. The quantum system consists of an optical cavity c that included an atom, and other two empty cavities a and b on both sides of c . Each of these cavities is coupled to cavity c with coupling constants J_1 and J_2 and have decay rates κ_1 and κ_2 respectively, giving us more parameters to achieve higher indistinguishability and efficiency. The parameters of cavity c are the same as in previous chapters. The above system is previously studied for different phenomena in [20], [21].

Additionally, we will analytically study the region where the coupling constants $g, J_1, J_2 \ll \gamma^*$, and using the adiabatic elimination method we will simplify the system of ODE's we need to solve. Also, we will present the methodology for the calculation of the efficiency and the indistinguishability of the photons produced from all three cavities. Finally, we will present the interesting limit of the same cavities, where the parameters of the empty cavities are the same, a case when the photon produced from either cavity has the same indistinguishability and thus we can collect the photon from either cavity.

6.1 Three Cavity System

As we see from Fig.(21), our system consists of three optical cavities QED, with one of them including an atom modeled as a two-level system (cavity c). Both of the empty cavities (cavities a and b) are coupled to cavity c with coupling constants J_1 and J_2 respectively. So, the Hamiltonian describing the system is

$$H = \hbar\omega_{atom}\sigma^+\sigma^- + \hbar\omega_{cav}a^\dagger a + \hbar\omega_{cav_1}a_1^\dagger a_1 + \hbar\omega_{cav_2}a_2^\dagger a_2 + \hbar g(\sigma^+ a + a^\dagger \sigma^-) + \hbar J_1(a^\dagger a_1 + a_1^\dagger a) + \hbar J_2(a^\dagger a_2 + a_2^\dagger a), \quad (214)$$

where a^\dagger is the creation operator of cavity c , a_1^\dagger is the creation operator of cavity a and a_2^\dagger is the creation operator of cavity b . The basis of the system is

$$(\{|g\rangle \cdot |e\rangle\} \otimes |n\rangle_{cav} \otimes |m\rangle_{cav_1} \otimes |l\rangle_{cav_2}, n, m, l = 0, 1, \dots).$$

Since the Hamiltonian commutes with $\sigma^+\sigma^- + a^\dagger a + a_1^\dagger a_1 + a_2^\dagger a_2$, the total number of energy quanta is conserved, so the basis becomes $|e, 0, 0, 0\rangle, |g, 1, 0, 0\rangle, |g, 0, 1, 0\rangle, |g, 0, 0, 1\rangle$.

and the Hamiltonian is written as

$$H = \hbar\omega_{atom} |e, 0, 0, 0\rangle\langle e, 0, 0, 0| + \hbar\omega_{cav} |g, 1, 0, 0\rangle\langle g, 1, 0, 0| + \hbar\omega_{cav_1} |g, 0, 1, 0\rangle\langle g, 0, 1, 0| + \hbar\omega_{cav_2} |g, 0, 0, 1\rangle\langle g, 0, 0, 1| + \hbar g(|e, 0, 0, 0\rangle\langle g, 1, 0, 0| + |g, 1, 0, 0\rangle\langle e, 0, 0, 0|) + \hbar J_1(|e, 0, 0, 0\rangle\langle g, 0, 1, 0| + |g, 0, 1, 0\rangle\langle e, 0, 0, 0|) + \hbar J_2(|e, 0, 0, 0\rangle\langle g, 0, 0, 1| + |g, 0, 0, 1\rangle\langle e, 0, 0, 0|), \quad (215)$$

or in matrix form

$$H = \hbar \begin{pmatrix} 0 & g & 0 & 0 \\ g & \Delta & J_1 & J_2 \\ 0 & J_1 & \Delta_1 & 0 \\ 0 & J_2 & 0 & \Delta_2 \end{pmatrix}. \quad (216)$$

Then, we calculate the dissipative terms of the system

$$\mathcal{L}_{QE}[\rho] = -\gamma \begin{pmatrix} \rho_{ee} & \rho_{ec}/2 & \rho_{ea}/2 & \rho_{eb}/2 \\ \rho_{ce}/2 & 0 & 0 & 0 \\ \rho_{ae}/2 & 0 & 0 & 0 \\ \rho_{ba}/2 & 0 & 0 & 0 \end{pmatrix}, \quad (217)$$

$$\mathcal{L}_{cav} = -\kappa \begin{pmatrix} 0 & \rho_{ec}/2 & 0 & 0 \\ \rho_{ce}/2 & \rho_{cc} & \rho_{ca}/2 & \rho_{cb}/2 \\ 0 & \rho_{ac}/2 & 0 & 0 \\ 0 & \rho_{bc}/2 & 0 & 0 \end{pmatrix}, \quad (218)$$

$$\mathcal{L}_{cav_1} = -\kappa_1 \begin{pmatrix} 0 & 0 & \rho_{ea}/2 & 0 \\ 0 & 0 & \rho_{ca}/2 & 0 \\ \rho_{ae}/2 & \rho_{ac}/2 & \rho_{aa} & \rho_{ab}/2 \\ 0 & 0 & \rho_{ba}/2 & 0 \end{pmatrix}, \quad (219)$$

$$\mathcal{L}_{cav_2} = -\kappa_2 \begin{pmatrix} 0 & 0 & 0 & \rho_{eb}/2 \\ 0 & 0 & 0 & \rho_{cb}/2 \\ 0 & 0 & 0 & \rho_{ab}/2 \\ \rho_{be}/2 & \rho_{bc}/2 & \rho_{ba}/2 & \rho_{bb} \end{pmatrix}, \quad (220)$$

$$\mathcal{L}_{def} = -\gamma^* \begin{pmatrix} 0 & \rho_{ec}/2 & \rho_{ea}/2 & \rho_{eb}/2 \\ \rho_{ce}/2 & 0 & 0 & 0 \\ \rho_{ae}/2 & 0 & 0 & 0 \\ \rho_{be}/2 & 0 & 0 & 0 \end{pmatrix}. \quad (221)$$

and the commutator of the density matrix with the Hamiltonian

$$[\rho, H] = \begin{pmatrix} g(\rho_{ec} - \rho_{ce}) & g(\rho_{ee} - \rho_{cc}) + J_1\rho_{ea} + J_2\rho_{eb} & J_1\rho_{ec} - g\rho_{ca} & J_2\rho_{ec} - g\rho_{cb} \\ \frac{g(\rho_{cc}-\rho_{ee})}{-J_1\rho_{ae}-J_2\rho_{be}} & \frac{g(\rho_{ce}-\rho_{ec})}{+J_1(\rho_{ca}-\rho_{ac})+J_2(\rho_{cb}-\rho_{bc})} & \frac{J_1(\rho_{cc}-\rho_{aa})}{-J_2\rho_{ba}-g\rho_{ea}} & \frac{J_2(\rho_{cc}-\rho_{bb})}{-J_1\rho_{ab}-g\rho_{eb}} \\ g\rho_{ac} - J_1\rho_{ce} & J_1(\rho_{aa} - \rho_{cc}) + J_2\rho_{ab} + g\rho_{ae} & J_1(\rho_{ac} - \rho_{ca}) & J_2\rho_{ac} - J_1\rho_{cb} \\ g\rho_{bc} - J_2\rho_{ce} & J_2(\rho_{bb} - \rho_{cc}) + J_1\rho_{ba} + g\rho_{be} & J_1\rho_{bc} - J_2\rho_{ca} & J_2(\rho_{bc} - \rho_{cb}) \end{pmatrix}. \quad (222)$$

So, the system of ODE's we need to solve is

$$\begin{aligned}
\dot{\rho}_{ee} &= ig(\rho_{ec} - \rho_{ce}) - \gamma\rho_{ee}, \\
\dot{\rho}_{ec} &= ig(\rho_{ee} - \rho_{cc}) + iJ_2\rho_{eb} + iJ_1\rho_{ea} - \frac{\gamma + \kappa + \gamma^*}{2}\rho_{ec}, \\
\dot{\rho}_{ea} &= iJ_1\rho_{ec} - ig\rho_{ca} - \frac{\gamma + \kappa_1 + \gamma^*}{2}\rho_{ea}, \\
\dot{\rho}_{eb} &= iJ_2\rho_{ec} - ig\rho_{cb} - \frac{\gamma + \kappa_2 + \gamma^*}{2}\rho_{eb}, \\
\dot{\rho}_{ce} &= -ig(\rho_{ee} - \rho_{cc}) - iJ_1\rho_{ae} - iJ_2\rho_{be} - \frac{\gamma + \kappa + \gamma^*}{2}\rho_{ce}, \\
\dot{\rho}_{cc} &= -ig(\rho_{ec} - \rho_{ce}) + iJ_1(\rho_{ca} - \rho_{ac}) + iJ_2(\rho_{cb} - \rho_{bc}) - \kappa\rho_{cc}, \\
\dot{\rho}_{ca} &= iJ_1(\rho_{cc} - \rho_{aa}) - iJ_2\rho_{ba} - ig\rho_{ea} - \frac{\kappa + \kappa_1}{2}\rho_{ca}, \\
\dot{\rho}_{cb} &= iJ_2(\rho_{cc} - \rho_{bb}) - iJ_1\rho_{ab} - ig\rho_{eb} - \frac{\kappa + \kappa_2}{2}\rho_{cb}, \\
\dot{\rho}_{ae} &= -iJ_1\rho_{ce} + ig\rho_{ac} - \frac{\gamma + \kappa_1 + \gamma^*}{2}\rho_{ae}, \\
\dot{\rho}_{ac} &= -iJ_1(\rho_{cc} - \rho_{aa}) + iJ_2\rho_{ab} + ig\rho_{ae} - \frac{\kappa + \kappa_1}{2}\rho_{ac}, \\
\dot{\rho}_{aa} &= iJ_1(\rho_{ac} - \rho_{ca}) - \kappa_1\rho_{aa}, \\
\dot{\rho}_{ab} &= iJ_2\rho_{ac} - iJ_1\rho_{cb} - \frac{\kappa_1 + \kappa_2}{2}\rho_{ab}, \\
\dot{\rho}_{be} &= -iJ_2\rho_{ce} + ig\rho_{bc} - \frac{\gamma + \kappa_2 + \gamma^*}{2}\rho_{be}, \\
\dot{\rho}_{bc} &= -iJ_2(\rho_{cc} - \rho_{bb}) + iJ_1\rho_{ba} + ig\rho_{be} - \frac{\kappa + \kappa_2}{2}\rho_{bc}, \\
\dot{\rho}_{ba} &= -iJ_2\rho_{ca} + iJ_1\rho_{bc} - \frac{\kappa_1 + \kappa_2}{2}\rho_{ba}, \\
\dot{\rho}_{bb} &= iJ_2(\rho_{bc} - \rho_{cb}) - \kappa_2\rho_{bb}.
\end{aligned} \tag{223}$$

6.2 Calculation of the Time Correlation functions

The Green functions of our system are

$$G^<(t, t + \tau) = \begin{pmatrix} \langle \sigma^+(t + \tau)\sigma^-(t) \rangle & \langle \sigma^+(t + \tau)a(t) \rangle & \langle \sigma^+(t + \tau)a_1(t) \rangle & \langle \sigma^+(t + \tau)a_2(t) \rangle \\ \langle a^\dagger(t + \tau)\sigma^-(t) \rangle & \langle a^\dagger(t + \tau)a(t) \rangle & \langle a^\dagger(t + \tau)a_1(t) \rangle & \langle a^\dagger(t + \tau)a_2(t) \rangle \\ \langle a_1^\dagger(t + \tau)\sigma^-(t) \rangle & \langle a_1^\dagger(t + \tau)a(t) \rangle & \langle a_1^\dagger(t + \tau)a_1(t) \rangle & \langle a_1^\dagger(t + \tau)a_2(t) \rangle \\ \langle a_2^\dagger(t + \tau)\sigma^-(t) \rangle & \langle a_2^\dagger(t + \tau)a(t) \rangle & \langle a_2^\dagger(t + \tau)a_1(t) \rangle & \langle a_2^\dagger(t + \tau)a_2(t) \rangle \end{pmatrix}, \tag{224a}$$

$$G^>(t, t + \tau) = \begin{pmatrix} \langle \sigma^-(t)\sigma^+(t + \tau) \rangle & \langle a(t)\sigma^+(t + \tau) \rangle & \langle a_1(t)\sigma^+(t + \tau) \rangle & \langle a_2(t)\sigma^+(t + \tau) \rangle \\ \langle \sigma^-(t)a^\dagger(t + \tau) \rangle & \langle a(t)a^\dagger(t + \tau) \rangle & \langle a_1(t)a^\dagger(t + \tau) \rangle & \langle a_2(t)a^\dagger(t + \tau) \rangle \\ \langle \sigma^-(t)a_1^\dagger(t + \tau) \rangle & \langle a(t)a_1^\dagger(t + \tau) \rangle & \langle a_1(t)a_1^\dagger(t + \tau) \rangle & \langle a_2(t)a_1^\dagger(t + \tau) \rangle \\ \langle \sigma^-(t)a_2^\dagger(t + \tau) \rangle & \langle a(t)a_2^\dagger(t + \tau) \rangle & \langle a_1(t)a_2^\dagger(t + \tau) \rangle & \langle a_2(t)a_2^\dagger(t + \tau) \rangle \end{pmatrix}, \tag{224b}$$

$$G^R(t, t + \tau) = \Theta(\tau) (G^<(t, t + \tau) + G^>(t, t + \tau)). \tag{224c}$$

Also, the total retarder self-energies of the system are

$$\Sigma_{tot}^R(t, t + \tau)\delta(\tau) \begin{pmatrix} \frac{\gamma + \gamma^*}{2} & 0 & 0 & 0 \\ 0 & \frac{\kappa}{2} & 0 & 0 \\ 0 & 0 & \frac{\kappa_1}{2} & 0 \\ 0 & 0 & 0 & \frac{\kappa_2}{2} \end{pmatrix}. \quad (225)$$

So, in matrix form the retarded Green function in the frequency domain is

$$G^R(\omega) = \begin{pmatrix} \omega + \frac{i}{2}(\gamma + \gamma^*) & g & 0 & 0 \\ g & \omega + \frac{i}{2}\kappa & J_1 & J_2 \\ 0 & J_1 & \omega + \frac{i}{2}\kappa_1 & 0 \\ 0 & J_2 & 0 & \omega + \frac{i}{2}\kappa_2 \end{pmatrix}^{-1}. \quad (226)$$

So the time evolution of the retarded Green function is

$$i \begin{pmatrix} \dot{G}_{ee}^R(\tau) & \dot{G}_{ec}^R(\tau) & \dot{G}_{ea}^R(\tau) & \dot{G}_{eb}^R(\tau) \\ \dot{G}_{ce}^R(\tau) & \dot{G}_{cc}^R(\tau) & \dot{G}_{ca}^R(\tau) & \dot{G}_{cb}^R(\tau) \\ \dot{G}_{ae}^R(\tau) & \dot{G}_{ac}^R(\tau) & \dot{G}_{aa}^R(\tau) & \dot{G}_{ab}^R(\tau) \\ \dot{G}_{be}^R(\tau) & \dot{G}_{bc}^R(\tau) & \dot{G}_{ba}^R(\tau) & \dot{G}_{bb}^R(\tau) \end{pmatrix} = i\delta(\tau)\mathbb{1} + \begin{pmatrix} -i\frac{\gamma + \gamma^*}{2} & g & 0 & 0 \\ g & -i\frac{\kappa}{2} & J_1 & J_2 \\ 0 & J_1 & -i\frac{\kappa_1}{2} & 0 \\ 0 & J_2 & 0 & -i\frac{\kappa_2}{2} \end{pmatrix} \begin{pmatrix} G_{ee}^R(\tau) & G_{ec}^R(\tau) & G_{ea}^R(\tau) & G_{eb}^R(\tau) \\ G_{ce}^R(\tau) & G_{cc}^R(\tau) & G_{ca}^R(\tau) & G_{cb}^R(\tau) \\ G_{ae}^R(\tau) & G_{ac}^R(\tau) & G_{aa}^R(\tau) & G_{ae}^R(\tau) \\ G_{be}^R(\tau) & G_{bc}^R(\tau) & G_{ba}^R(\tau) & G_{bb}^R(\tau) \end{pmatrix}. \quad (227)$$

Finally, we will study the efficiency and the indistinguishability of all the cavities, the corresponding terms will be

$$\beta_a = \kappa_1 \int_0^\infty dt \rho_{aa}(t) \quad , \quad \beta_b = \kappa_2 \int_0^\infty dt \rho_{bb}(t), \quad (228a)$$

$$I_a = \frac{\int_0^\infty \int_0^\infty dt d\tau \left| \langle a_1^\dagger(t + \tau) a_1(t) \rangle \right|^2}{\int_0^\infty \int_0^\infty dt d\tau \langle a_1^\dagger(t + \tau) a_1(t + \tau) \rangle \langle a_1^\dagger(t) a_1(t) \rangle}, \quad I_b = \frac{\int_0^\infty \int_0^\infty dt d\tau \left| \langle a_2^\dagger(t + \tau) a_2(t) \rangle \right|^2}{\int_0^\infty \int_0^\infty dt d\tau \langle a_2^\dagger(t + \tau) a_2(t + \tau) \rangle \langle a_2^\dagger(t) a_2(t) \rangle}. \quad (228b)$$

6.3 Adiabatic Elimination

In our problem, the cavities a and b don't interact so we can adiabatically eliminate their coherence terms

$$\dot{\rho}_{ab} = iJ_2\rho_{ac} - iJ_1\rho_{cb} - \frac{\kappa_1 + \kappa_2}{2}\rho_{ab} \rightarrow \rho_{ab} = \frac{2i}{\kappa_1 + \kappa_2}(J_2\rho_{ac} - J_1\rho_{cb}). \quad (229)$$

The dissipative term due to the atom's pure dephasing is much larger than the other dissipative terms so initially we can eliminate the coherences of the atom with the cavities a and b .

$$\dot{\rho}_{ea} = iJ_1\rho_{ec} - ig\rho_{ca} - \frac{\gamma + \kappa_1 + \gamma^*}{2}\rho_{ea} \rightarrow \rho_{ea} = \frac{2i(J_1\rho_{ec} - g\rho_{ca})}{\gamma + \kappa_1 + \gamma^*}, \quad (230)$$

$$\dot{\rho}_{eb} = iJ_2\rho_{ec} - ig\rho_{cb} - \frac{\gamma + \kappa_2 + \gamma^*}{2}\rho_{eb} \rightarrow \rho_{eb} = \frac{2i(J_2\rho_{ec} - g\rho_{cb})}{\gamma + \kappa_2 + \gamma^*}. \quad (231)$$

So, the other coherence terms become

$$\dot{\rho}_{ec} = ig(\rho_{ee} - \rho_{cc}) - \frac{2J_2(J_2\rho_{ec} - g\rho_{cb})}{\gamma + \kappa_2 + \gamma^*} - \frac{2J_1(J_1\rho_{ec} - g\rho_{ca})}{\gamma + \kappa_1 + \gamma^*} - \frac{\gamma + \kappa + \gamma^*}{2}\rho_{ec}, \quad (232)$$

$$\dot{\rho}_{ca} = iJ_1(\rho_{cc} - \rho_{aa}) - \frac{\kappa + \kappa_2}{2}\rho_{ca} + \frac{2g}{\gamma + \kappa_1 + \gamma^*}(J_1\rho_{ec} - g\rho_{ca}) - \frac{2J_2}{\kappa_1 + \kappa_2}(J_2\rho_{ca} - J_1\rho_{bc}), \quad (233)$$

$$\dot{\rho}_{cb} = iJ_2(\rho_{cc} - \rho_{bb}) - \frac{\kappa + \kappa_2}{2}\rho_{cb} + \frac{2g}{\gamma + \kappa_2 + \gamma^*}(J_2\rho_{ec} - g\rho_{cb}) - \frac{2J_1}{\kappa_1 + \kappa_2}(J_2\rho_{ac} - J_1\rho_{cb}). \quad (234)$$

We define, as in the case of the two cavity system, the population exchange rate $\tilde{R} = \frac{4g^2}{\gamma + \gamma^* + \kappa_1}$ between the atom and cavity a , the population exchange rate $\tilde{R}' = \frac{4g^2}{\gamma + \gamma^* + \kappa_2}$ between the atom and the cavity b . We also define the quantities $X_1 = \frac{4J_1^2}{\kappa_1 + \kappa_2}$ and $X_2 = \frac{4J_2^2}{\kappa_1 + \kappa_2}$. Since $J_1\rho_{ec} \ll g\rho_{ca}$ and $J_2\rho_{ec} \ll g\rho_{cb}$ we can ignore these terms. So, for times $t \gg 1\tilde{R}, \tilde{R}'$ we can adiabatically eliminate the coherence terms between the cavities c and a, b . So we have

$$\dot{\rho}_{ca} = -\frac{\kappa + \kappa_1 + \tilde{R} + X_2}{2}\rho_{ca} + iJ_1(\rho_{cc} - \rho_{aa}) + \frac{2J_1J_2}{\kappa_1 + \kappa_2}\rho_{ba} \rightarrow \rho_{ca} \approx \frac{2iJ_1}{\kappa + \kappa_1 + \tilde{R} + X_2}(\rho_{cc} - \rho_{aa}), \quad (235)$$

$$\dot{\rho}_{cb} = -\frac{\kappa + \kappa_2 + \tilde{R}' + X_1}{2}\rho_{cb} + iJ_2(\rho_{cc} - \rho_{bb}) + \frac{2J_1J_2}{\kappa_1 + \kappa_2}\rho_{ac} \rightarrow \rho_{cb} \approx \frac{2iJ_2}{\kappa + \kappa_2 + \tilde{R}' + X_1}(\rho_{cc} - \rho_{bb}). \quad (236)$$

We substitute the above expression in ρ_{ec} and find

$$\begin{aligned} \dot{\rho}_{ec} &= ig(\rho_{ee} - \rho_{cc}) - \frac{\gamma + \kappa + \frac{4J_1^2}{\gamma + \kappa_1 + \gamma^*} + \frac{4J_2^2}{\gamma + \kappa_2 + \gamma^*}}{2}\rho_{ec} \\ &+ \frac{4igJ_1^2}{(\gamma + \kappa_1 + \gamma^*)(\kappa + \kappa_1 + \tilde{R} + X_2)}(\rho_{cc} - \rho_{aa}) + \frac{4igJ_2^2}{(\gamma + \kappa_2 + \gamma^*)(\kappa + \kappa_2 + \tilde{R}' + X_1)}(\rho_{cc} - \rho_{bb}) \\ &\approx ig(\rho_{ee} - \rho_{cc}) - \frac{\gamma + \kappa + \gamma^*}{2}\rho_{ec} \rightarrow \rho_{ec} = \frac{2ig}{\gamma + \kappa + \gamma^*}(\rho_{ee} - \rho_{cc}). \end{aligned} \quad (237)$$

After the adiabatic elimination of all the coherences, the population rate equations are

$$\dot{\rho}_{ee} = -\gamma\rho_{ee} - \frac{4g^2}{\gamma + \kappa + \gamma^*}(\rho_{ee} - \rho_{cc}), \quad (238)$$

$$\begin{aligned} \dot{\rho}_{cc} &= -\kappa\rho_{cc} - \frac{4g^2}{\gamma + \kappa + \gamma^*}(\rho_{cc} - \rho_{ee}) \\ &- \frac{4J_1^2}{\kappa + \kappa_1 + \tilde{R} + X_2}(\rho_{cc} - \rho_{aa}) - \frac{4J_2^2}{\kappa + \kappa_2 + \tilde{R}' + X_1}(\rho_{cc} - \rho_{bb}), \end{aligned} \quad (239)$$

$$\dot{\rho}_{aa} = -\kappa_1\rho_{aa} - \frac{4J_1^2}{\kappa + \kappa_1 + \tilde{R} + X_2}(\rho_{aa} - \rho_{cc}), \quad (240)$$

$$\dot{\rho}_{bb} = -\kappa_2\rho_{bb} - \frac{4J_2^2}{\kappa + \kappa_2 + \tilde{R}' + X_1}(\rho_{bb} - \rho_{cc}). \quad (241)$$

In matrix form, the rate equations are

$$\begin{pmatrix} \dot{\rho}_{ee} \\ \dot{\rho}_{cc} \\ \dot{\rho}_{aa} \\ \dot{\rho}_{bb} \end{pmatrix} = \begin{pmatrix} -\gamma - R & R & 0 & 0 \\ R & -\kappa - R - R_1 - R_2 & R_1 & R_2 \\ 0 & R_1 & -\kappa_1 - R_1 & 0 \\ 0 & R_2 & 0 & -\kappa_2 - R_2 \end{pmatrix} \begin{pmatrix} \rho_{ee} \\ \rho_{cc} \\ \rho_{aa} \\ \rho_{bb} \end{pmatrix}, \quad (242)$$

where $R_1 = \frac{4J_1^2}{\kappa + \kappa_1 + R + X_2}$ and $R_2 = \frac{4J_2^2}{\kappa + \kappa_2 + R' + X_1}$.

6.4 Calculation of the efficiency

To find the efficiency, we need the characteristic polynomial

$$\begin{aligned} (s + \gamma + R) [(s + \kappa + R + R_1 + R_2)(s + \kappa_1 + R_1)(s + \kappa_2 + R_2) \\ - R_1^2(s + \kappa_2 + R_2) - R_2^2(s + \kappa_1 + R_1)] - R^2(s + \kappa_1 + R_1)(s + \kappa_2 + R_2) = 0. \end{aligned} \quad (243)$$

This equation is of fourth order, having four solutions $s = s_1, s_2, s_3, s_4$. So, the populations of cavities a and b will be

$$\rho_{aa} = A_a e^{-s_1 t} + B_a e^{-s_2 t} + C_a e^{-s_3 t} + D_a e^{-s_4 t} \quad (244)$$

$$\rho_{bb} = A_b e^{-s_1 t} + B_b e^{-s_2 t} + C_b e^{-s_3 t} + D_b e^{-s_4 t} \quad (245)$$

To find the coefficients A_i, B_i, C_i, D_i , we need to apply the initial conditions of the problem

$$\begin{aligned} \rho_{ee}(0) = 1 \quad \rho_{aa}(0) = 0 \quad \rho_{bb}(0) = 0 \quad \rho_{cc}(0) = 0 \\ \dot{\rho}_{ee}(0) = -\gamma \rho_{ee}(0) - R(\rho_{ee}(0) - \rho_{cc}(0)) = -\gamma - R \\ \dot{\rho}_{aa}(0) = -(\kappa_1 + R_1)\rho_{aa}(0) + R_1\rho_{cc}(0) = 0 \\ \dot{\rho}_{bb}(0) = -(\kappa_2 + R_2)\rho_{bb}(0) + R_2\rho_{cc}(0) = 0 \\ \dot{\rho}_{cc}(0) = R\rho_{ee}(0) - (\kappa + R + R_1 + R_2)\rho_{cc}(0) + R_1\rho_{aa}(0) + R_2\rho_{bb}(0) = R \\ \ddot{\rho}_{ee}(0) = -\gamma\dot{\rho}_{ee}(0) - R(\dot{\rho}_{ee}(0) - \dot{\rho}_{cc}(0)) = (\gamma + R)^2 + R^2 \\ \ddot{\rho}_{aa}(0) = -(\kappa_1 + R_1)\dot{\rho}_{aa}(0) + R_1\dot{\rho}_{cc}(0) = RR_1 \\ \ddot{\rho}_{bb}(0) = -(\kappa_2 + R_2)\dot{\rho}_{bb}(0) + R_2\dot{\rho}_{cc}(0) = RR_2 \\ \ddot{\rho}_{cc}(0) = R\dot{\rho}_{ee}(0) - (\kappa + R + R_1 + R_2)\dot{\rho}_{cc}(0) + R_1\dot{\rho}_{aa}(0) + R_2\dot{\rho}_{bb}(0) = -R(\gamma + \kappa + 2R + R_1 + R_2) \\ \ddot{\rho}_{aa}(0) = -(\kappa_1 + R_1)\ddot{\rho}_{aa}(0) + R_1\ddot{\rho}_{cc}(0) = -RR_1(\gamma + \kappa + 2R + 2R_1 + \kappa_1 + R_2) \\ \ddot{\rho}_{bb}(0) = -(\kappa_2 + R_2)\ddot{\rho}_{bb}(0) + R_2\ddot{\rho}_{cc}(0) = -RR_2(\gamma + \kappa + 2R + R_1 + 2R_2 + \kappa_2) \end{aligned} \quad (246)$$

The efficiency of each cavity is given by

$$\begin{aligned} \beta_a = \kappa_1 \int_0^\infty \rho_{aa}(t) dt = \frac{A_a}{s_1} + \frac{B_a}{s_2} + \frac{C_a}{s_3} + \frac{D_a}{s_4} \\ \beta_b = \kappa_1 \int_0^\infty \rho_{bb}(t) dt = \frac{A_b}{s_1} + \frac{B_b}{s_2} + \frac{C_b}{s_3} + \frac{D_b}{s_4} \end{aligned} \quad (247)$$

And we find the expression

$$\begin{aligned} \beta_a = \frac{\kappa_1 R_1}{\kappa(\kappa_1 + R_1) + R_1 \kappa_1 + R_2 \kappa_2} \\ \beta_b = \frac{\kappa_2 R_2}{\kappa(\kappa_1 + R_1) + R_1 \kappa_1 + R_2 \kappa_2} \end{aligned} \quad (248)$$

6.5 Similar Cavity Limit

The limit of the similar cavities corresponds to the region where the two empty cavities a, b have the same parameters $J_1 = J_2$ and $\kappa_1 = \kappa_2$. In that limit, the constants of the previous section become $\tilde{R} = \tilde{R}'$, $X_1 = X_2$ and $R_1 = R_2$. So the system (242) becomes

$$\begin{pmatrix} \dot{\rho}_{ee} \\ \dot{\rho}_{cc} \\ \dot{\rho}_{aa} \\ \dot{\rho}_{bb} \end{pmatrix} = \begin{pmatrix} -\gamma - R & R & 0 & 0 \\ R & -\kappa - R - 2R_1 & R_1 & R_1 \\ 0 & R_1 & -\kappa_1 - R_1 & 0 \\ 0 & R_1 & 0 & -\kappa_1 - R_1 \end{pmatrix} \begin{pmatrix} \rho_{ee} \\ \rho_{cc} \\ \rho_{aa} \\ \rho_{bb} \end{pmatrix}. \quad (249)$$

We observe that if we swap ρ_{aa} and ρ_{bb} the matrix remains the same. That symmetry, combined with the common initial condition $\rho_{aa}(0) = \rho_{bb}(0)$ leads us to the conclusion that the populations evolve in time the same, $\rho_{aa}(t) = \rho_{bb}(t)$. This condition further simplifies our problem, which now becomes

$$\begin{pmatrix} \dot{\rho}_{ee} \\ \dot{\rho}_{cc} \\ \dot{\rho}_{aa} \end{pmatrix} = \begin{pmatrix} -\gamma - R & R & 0 \\ R & -\kappa - R - 2R_1 & 2R_1 \\ 0 & R_1 & -\kappa_1 - R_1 \end{pmatrix} \begin{pmatrix} \rho_{ee} \\ \rho_{cc} \\ \rho_{aa} \end{pmatrix}. \quad (250)$$

Also, it's easy to show

$$\beta_a = \beta_b, \quad (251)$$

$$I_a = I_b. \quad (252)$$

Where

$$\kappa_a = \frac{\kappa_1 R_1}{\kappa R_1 + 2\kappa_1 R_1 + \kappa \kappa_1} \quad (253)$$

Additionally, we can find the Green Functions of the problem

$$G_{ae}^R(\tau) = G_{be}^R(\tau) \propto e^{-\frac{\gamma+\tilde{\gamma}-R}{2}\tau}, \quad (254a)$$

$$G_{ac}^R(\tau) = G_{bc}^R(\tau) \propto e^{-\frac{\kappa+R-R_1}{2}\tau}, \quad (254b)$$

$$G_{aa}^R(\tau) = G_{bb}^R(\tau) \propto e^{-\frac{\kappa_1+R_1}{2}\tau}. \quad (254c)$$

Thus, the dominating terms are $G_{aa}^R(\tau)$ and $G_{bb}^R(\tau)$ and the two-time correlation function becomes

$$\begin{aligned} \langle a_1^\dagger(t+\tau)a_1(t) \rangle &\approx G_{aa}^R(\tau)\rho_{aa}(t), \\ \langle a_2^\dagger(t+\tau)a_2(t) \rangle &\approx G_{bb}^R(\tau)\rho_{bb}(t). \end{aligned} \quad (255)$$

If we substitute the above expression to the indistinguishability we get

$$I_a \approx \frac{\int_0^\infty dt \rho_{aa}(t)^2 \int_0^\infty d\tau |G_{aa}^R(\tau)|^2}{\frac{1}{2} \left| \int_0^\infty dt \rho_{aa}(t) \right|^2}. \quad (256)$$

We calculate the integrals

$$\int_0^\infty dt \rho_{aa}^2(t) = \frac{1}{2} \frac{\kappa + 2\kappa_1 + 3R_1}{\kappa\kappa_1 + R_1(\kappa + \kappa_1)}, \quad (257a)$$

$$\left| \int_0^\infty dt \rho_{aa}(t) \right|^2 = \left(\frac{\kappa + 2\kappa_1 + 3R_1}{\kappa\kappa_1 + R_1(\kappa + \kappa_1)} \right)^2, \quad (257b)$$

$$\int_0^\infty d\tau |G_{aa}^R|^2(\tau) = \frac{1}{\kappa_1 + R_1}. \quad (257c)$$

and we find

$$\begin{aligned} I_a &= \frac{\frac{1}{2} \frac{\kappa + 2\kappa_1 + 3R_1}{\kappa\kappa_1 + R_1(\kappa + \kappa_1)} * \frac{1}{\kappa_1 + R_1}}{\frac{1}{2} \left(\frac{\kappa + 2\kappa_1 + 3R_1}{\kappa\kappa_1 + R_1(\kappa + \kappa_1)} \right)^2} = \frac{\kappa\kappa_1 + R_1(\kappa + \kappa_1)}{(\kappa_1 + R_1)(\kappa + 2\kappa_1 + 3R_1)} = \frac{\kappa + \frac{\kappa_1 R_1}{\kappa_1 + R_1}}{\kappa + 2\kappa_1 + 3R_1} \\ &= \frac{\kappa + (\kappa_1 || R_1)}{\kappa + 2\kappa_1 + 3R_1}. \end{aligned} \quad (258)$$

Thus, we conclude that by just adding another cavity similar to cavity a we can't improve the indistinguishability of the photon. Also, the efficiency of each external cavity is exactly half of the efficiency of the external cavity of the 2 Cavity System, thus the efficiency isn't improved either.

6.6 The Identical 3 Cavity Limit

Since by just adding another external cavity, we effectively got the same system with regards to its efficiency and photon indistinguishability, we will try another limit. We also engineer our system so that all the involved cavities have the same decay rate and so that all the couplings are the same. We will call this limit the Identical Three Cavity Limit, with parameters which satisfies the conditions

$$\begin{aligned} \kappa &= \kappa_1 = \kappa_2 \\ g &= J_1 = J_2 \end{aligned} \quad (259)$$

Also, the efficiency and the indistinguishability from the external cavities is

$$\begin{aligned} \beta_a &= \beta_b \\ I_a &= I_b \end{aligned} \quad (260)$$

We will calculate numerically the indistinguishability of the photons produced from each cavity.

6.6.1 Results for Cavity c

In Fig.(22) we plot the efficiency of the photons produced from the main Cavity c . The efficiency takes moderate values in the region $g > 50\gamma$, with exceptionally high values at the limit $\kappa \rightarrow \bar{\gamma}$.

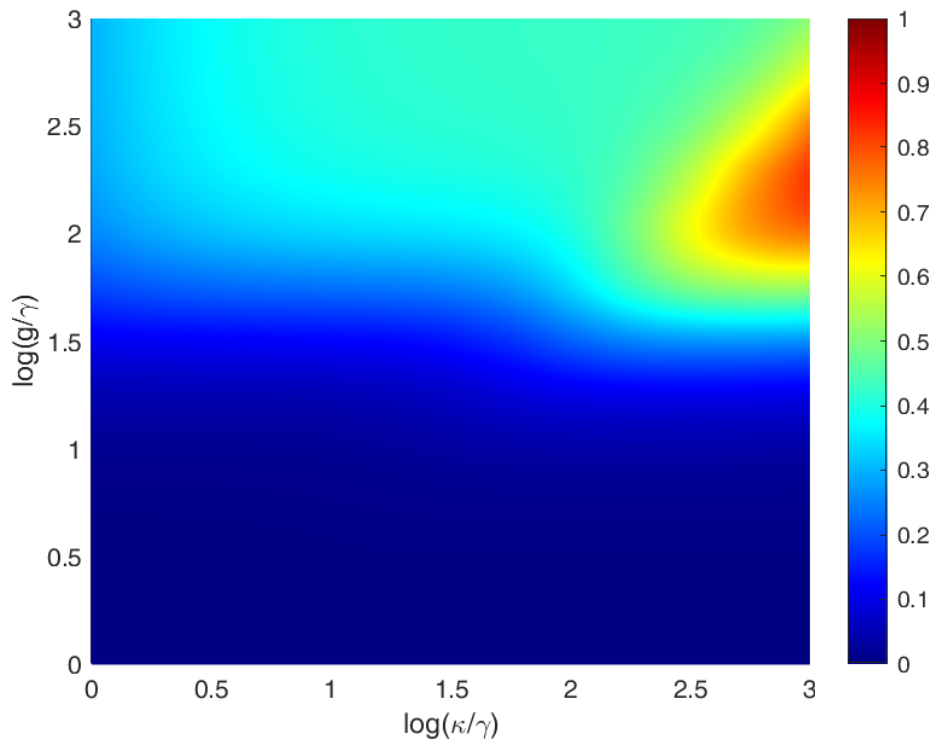


Figure 22: Graph of the Efficiency of the Cavity c in the Identical Cavity Limit.

In Fig.(23) we plot the indistinguishability of the photons produced from the main Cavity c . The indistinguishability takes moderate values in a band region, similar to the one in the Single Cavity System.

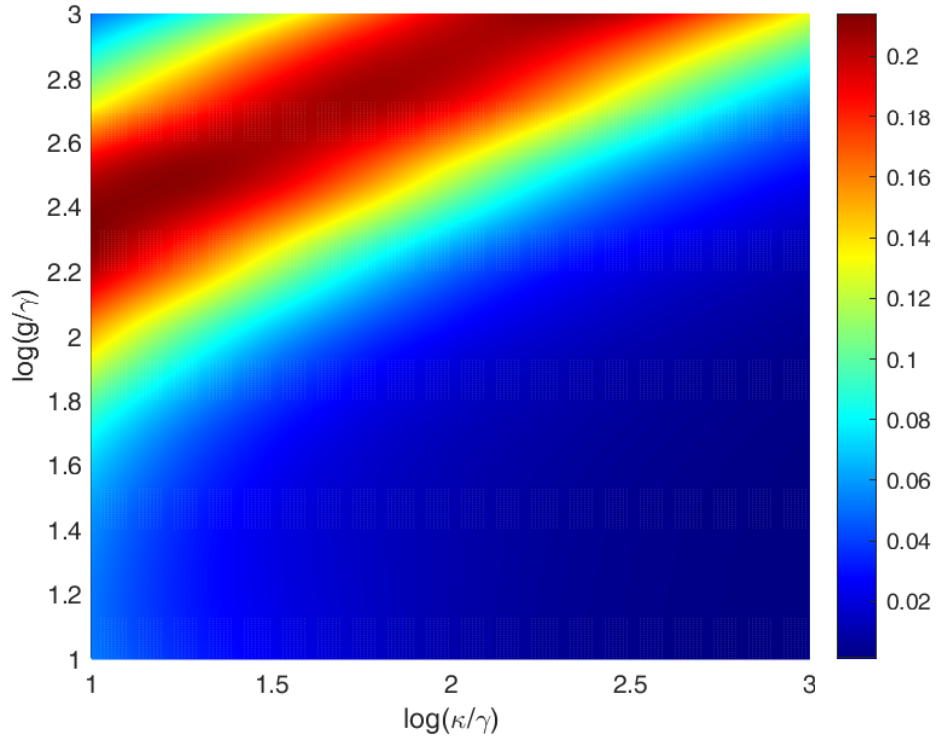


Figure 23: Graph of the Indistinguishability of the photons from Cavity c in the Identical Cavity Limit.

6.6.2 Results for Cavities a and b

In Fig.(24) we plot the total efficiency $\beta_{external} = \beta_a + \beta_b$ of the photons produced from the external Cavities a and b . The efficiency has the same form as the efficiency of the Single Cavity Limit, but takes a smaller maximum value (around $\beta_{max} = 50\%$).

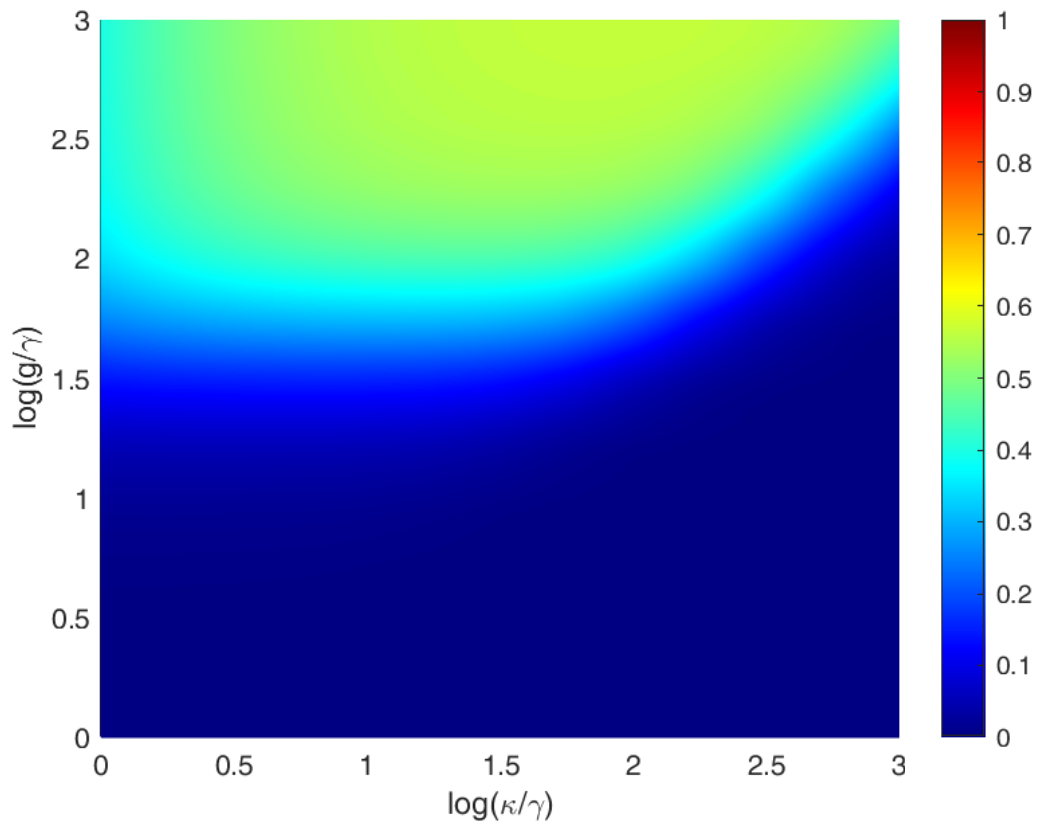


Figure 24: Graph of the Total Efficiency of the Cavities a and b in the Identical Cavity Limit.

In Fig.(25) we plot the indistinguishability of the photons produced from the main Cavity a and b . The indistinguishability takes values close to the indistinguishability of the main cavity c and in the same band region of our systems parameters.

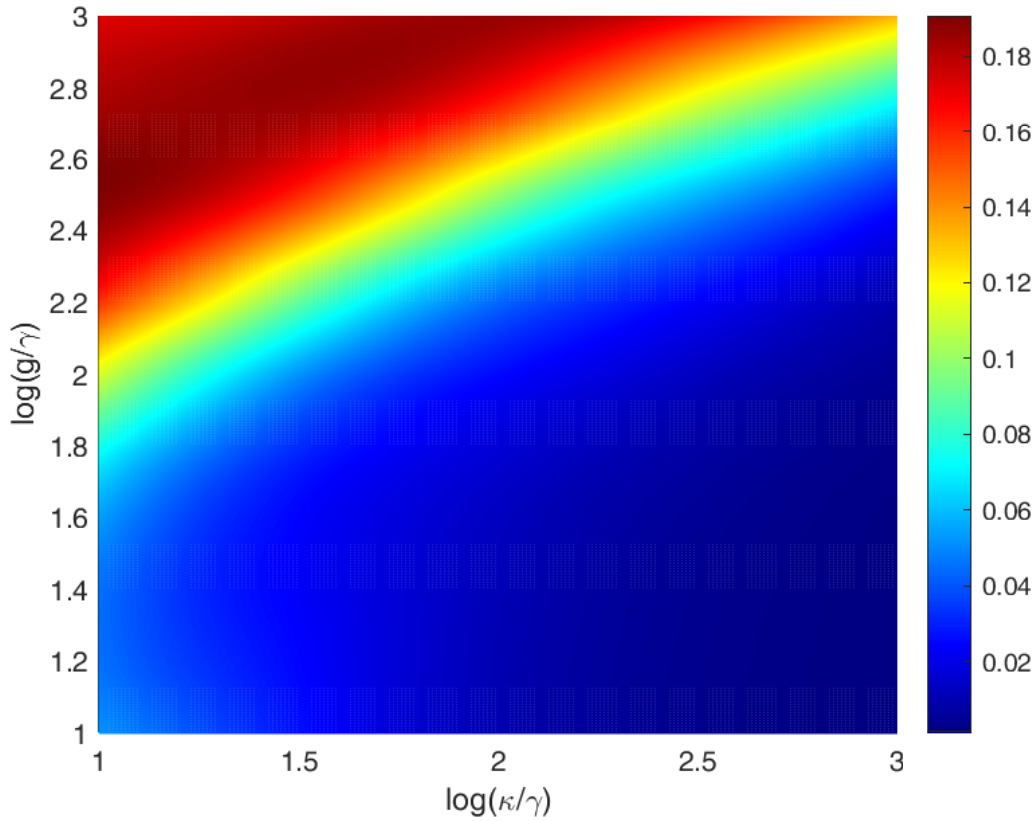


Figure 25: Graph of the Indistinguishability of the photons from Cavities a and b in the Identical Cavity Limit.

6.6.3 Results for the effective system

Because of the close resemblance between the indistinguishability of the photons produced from the main and the external cavities, we employ a strategy to significantly increase the efficiency of our system.

In the previous Chapters, we only collected the photons emitted from a specific cavity, a method when we knew exactly the indistinguishability of the photon. But, in the Identical Cavity Limit, the indistinguishability of the photons from the main cavity is almost equal to the indistinguishability of photons from the external cavities. Thus we could collect the photons from all the cavities, increasing the total efficiency of the system, which will take the form

$$\beta_{eff} = \beta_c + \beta_a + \beta_b \quad (261)$$

Despite the fact that the a photon emitted from an external cavity has a different temporal profile than a photon emitted from the central cavity, before it is actually produced we have no way of knowing from which cavity it will be emitted, thus we can't know the exact value of the indistinguishability beforehand. Thus we also need to define an Effective Indistinguishability of the system

$$I_{eff} = \frac{\beta_c I_c + \beta_a I_a + \beta_b I_b}{\beta_{eff}} \quad (262)$$

which is the weighted average of each cavities emitted photon.

In Fig.(26) we plot the effective efficiency of our system in the Identical Cavity Limit. The efficiency now takes high values in the region where $g > 500\gamma$, thus spanning most of the parameter space we can reach experimentally.

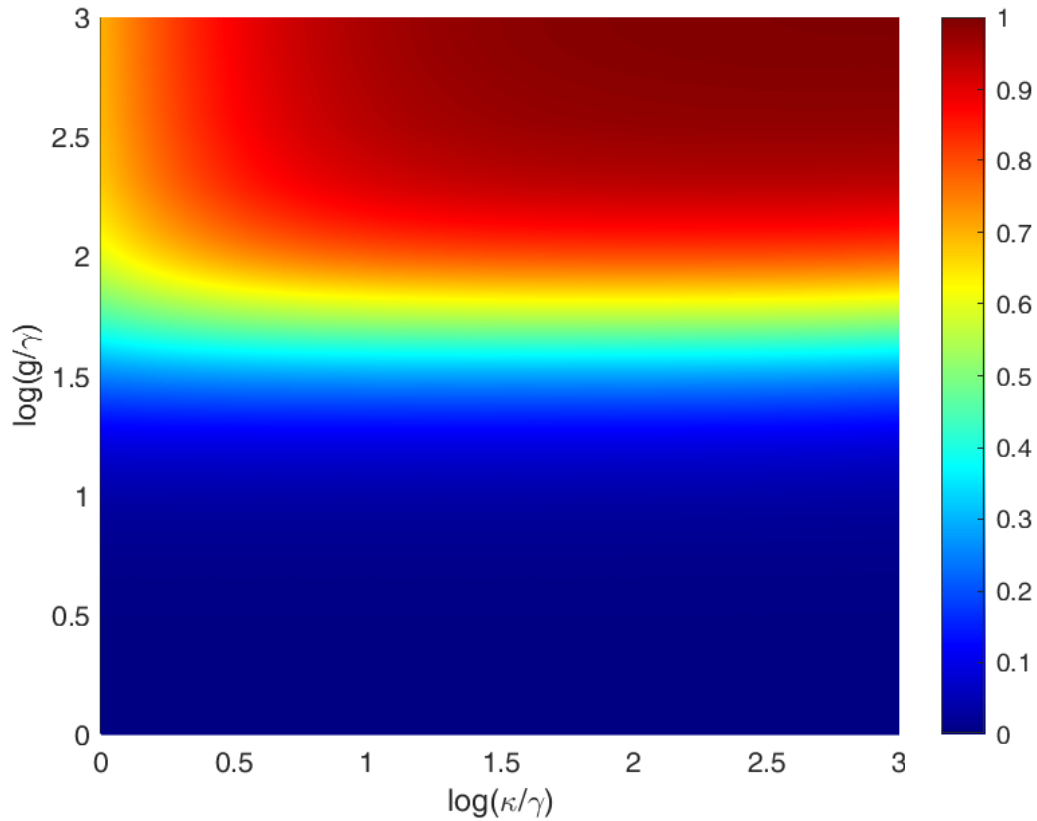


Figure 26: Graph of the Effective Efficiency of the 3 Cavity System in the Identical Cavity Limit.

In Fig.(27) we plot the effective indistinguishability of the photons produced the system in the Identical Cavity Limit. The indistinguishability takes moderate in a band region of our parameter space. In contrast with our previous systems, the efficiency of the system that produces these moderate-indistinguishable photons is close to 1. In the Single and Two Cavity, the efficiency of the system would be $\approx 1\%$ and $\approx 10\%$ respectively.

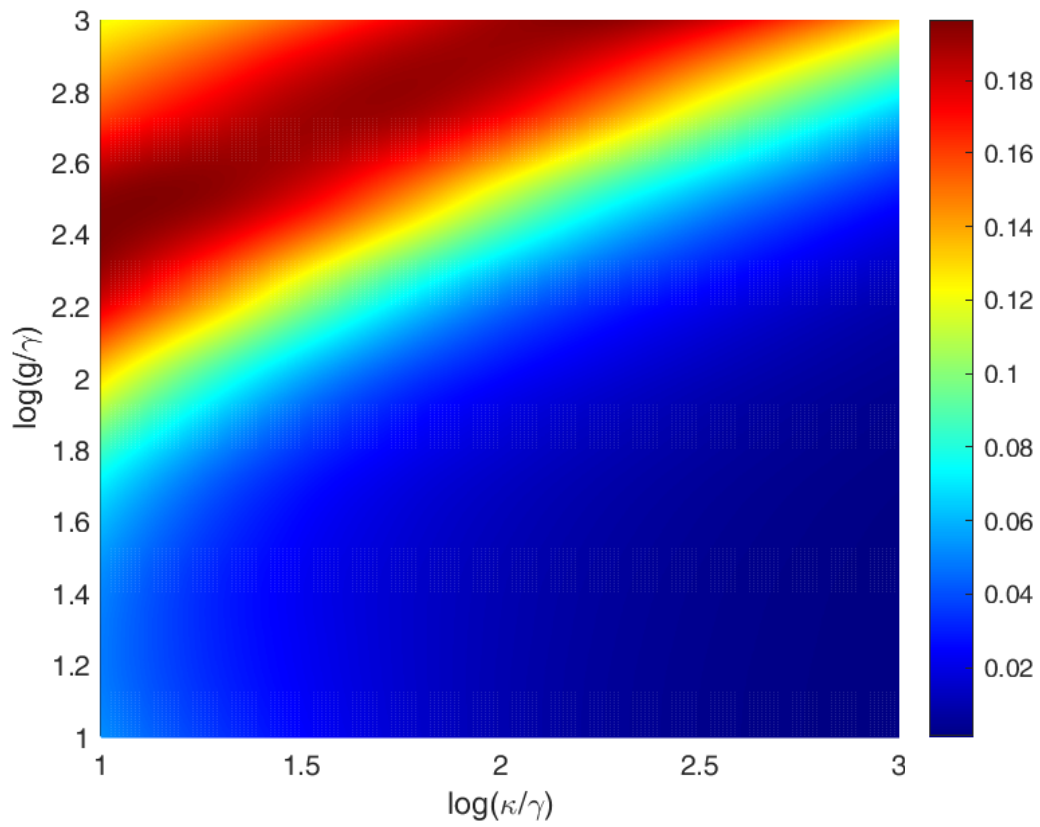


Figure 27: Graph of the Effective Indistinguishability of the 3 Cavity System in the Identical Cavity Limit.

7 References

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