On the spacetime of the extreme Kerr throat

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Abstract

This thesis explores the application of Effective Field Theory (EFT) as an extension of General Relativity (GR), focusing on the Near-Horizon Extreme Kerr (NHEK) geometry. This study begins by introducing the essential concepts of Kerr spacetime in Chapter 1, such as symmetries, Killing Fields and singularities. Then, it proceeds in a brief presentation on some exotic features associated with the rotating nature of Kerr. Chapter 2 is focused on deriving the NHEK metric and studying its key aspects concluding with the demonstration of the wave equation in that background. Finally, in Chapter 3 lies the motivation for utilizing purely gravitational EFT in GR. This work formulates an EFT which incorporates higher order corrections in the Einstein-Hilbert Lagrangian based on the work of [1]. Further on, it investigates the EFT-corrected NHEK geometry along with the impact of these corrections in the wave equation. It concludes with the derivation of the EFT-corrected radial and angular wave equations. The solution of these equations appears to be non-trivial and therefore further work on them is in order.

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Chapter 1

1 Kerr spacetime

In this first section of this work we make an attempt to present briefly a well-known spacetime, the Kerr spacetime. We shall exhibit features of this spacetime such as symmetries, Killing Fields, event horizons and singularities. Moreover, there exists a compact introduction to the Frame Dragging Effect and Penrose Process.

1.1 Physical significance of the Kerr geometry

Before we proceed to presenting any piece of information considering this spacetime, it would be a interesting idea to take a moment and explain its significance and the physical structure it represents. Let's start off by reviewing its physical meaning.

Nowadays, we know the existence of black holes as a physical system in the universe. This piece of knowledge has risen from both mathematics (Einstein's Theory of General Relativity predicted them) and physical observations. Perhaps, the most well-known feature that they possess is that near black holes the gravitational force is extremely powerful so that not even photons can escape it. Despite their mysterious and enigmatic nature, we know that they play a crucial role in the cosmos as they influence the motion of stars and galaxies and that they are characterized by their mass and angular momentum. Naturally, that means that black holes have rotational energy and thus that they are rotating around one of their axis. For many years, the metric that described such a physical system remained unknown until 1963 Roy Kerr found the solution to this mystery; The Kerr spacetime. Below we shall begin exploring some intriguing features that this spacetime illustrates.

1.2 Kerr metric

We kick off by writing the Kerr metric in the establised Boyer-Lindquist coordinates following the form used by [2] so as to denote the components of the $g_{\mu\nu}$ more clearly:

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}(dtd\varphi + d\varphi dt) \\ &+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right]d\varphi^2 \end{aligned} \tag{1.1}$$

where:

$$\Delta(r) = r^2 - 2GMr + a^2, \quad \rho^2(r,\theta) = r^2 + a^2\cos^2\theta, \quad c \equiv 1 \tag{1.2}$$

The coordinates r, θ, φ are called Ellipsoidal coordinates and their relation to the Cartesian is the latter:

$$\begin{aligned} x &= \sqrt{r^2+a^2}\sin\theta\cos\varphi \\ y &= \sqrt{r^2+a^2}\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned} \tag{1.3}$$

The first thing that one observes is the existence of two constants: M, a which parameterise the metric. G is Newton's constant considering gravity, M is the mass of the black hole and

a is the angular momentum per unit mass: $a \equiv J/M$ with the constraint $a \leq GM$. The Kerr metric is, of course, a solution to the vacuum Einstein Field Equations $R_{\mu\nu} = 0$ and describes an uncharged rotating axisymmetric black hole with mass M and angular momentum J.

Let us now consider the case of $a = 0$ and see the Kerr metric's behaviour near that limit. We begin by taking the limit of the functions of eq.(1.2) which transform to:

$$\Delta(r; a = 0) = r^2 - 2GMr, \quad \rho^2(r, \theta; a = 0) = r^2 \tag{1.4}$$

Substituting eq.(1.4) and the restriction $a = 0$ into the metric eq.(1.1), one can straight-forward obtain:

$$(1.1) \xrightarrow{a=0} ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \tag{1.5}$$

which is the renowned Schwarzschild metric that describes a non-rotating, uncharged spherically symmetric black hole. The fact that the Kerr metric reduces to the Schwarzschild one when $a = 0$ indicates that Kerr metric is a generalization of Schwarzschild.

Before concluding this subsection we would rather investigate one more limit, that of $r \to \infty$. Basically, this limit will reveal the behaviour of the spacetime at very large distances. For instance, we know that the Schwarzschild geometry is asymptotically flat (i.e. it reduces to the Minkowski spacetime). Let's derive this limit:

$$(1.1) \xrightarrow{r\to\infty} ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \tag{1.6}$$

Which, of course, is the Minkowski spacetime, thus we determine that Kerr is asymptotically flat. This result was expected as Kerr is a generalization of Schwarzschild, which is asymptotically flat.

1.3 Basic features of Kerr spacetime

At this moment, it is time to start discussing some of the basic concepts of the Kerr spacetime that seem to be of interest. Symmetries in physics are of utmost importance, so when studying a new metric one of the first concepts that one attempts to figure out is which symmetries (i.e. Killing fields) hold true for it.

1.3.1 Symmetries and Killing Fields

Let's kick off exploring the symmetries obeyed by the Kerr spacetime. Due to its rotation Kerr spacetime has given up the spherical symmetry thus making the spacetime highly non-trivial to derive from the vacuum Einstein Field Equations. In this subsection we will present the symmetries and the Killing vectors of the spacetime in a more intuitive way as they are straight-forward enough to be observed. However, in §2.3 there exists a more complete discussion considering symmetries, Killing Fields and conserved quantities.

We begin to observe eq.(1.1), almost immediately one can notice that none of the metric's components are explicitly dependent of neither of t nor of φ. The latter statement indicates the existence of the two following Killing vectors, $K^\mu = \partial_t$ and $R^\mu = \partial_\varphi$. Similarly to Schwarzschild geometry, the Killing field K^μ is connected with the conservation of energy, while R^μ is associated with the conservation of angular momentum. For now, we shall just state that for massive particles the conserved quantities are: $K_\mu u^\mu$ and $R_\mu u^\mu$. Let's compute the norms of the Killing Fields K^μ and R^μ:

$$K^\mu K_\mu = g_{\mu\nu} K^\mu K^\nu = g_{00}(K^0)^2 = -\left(1 - \frac{2GMr}{\rho^2}\right) = -\frac{1}{\rho^2}\left(\Delta - a^2 \sin^2\theta\right) \tag{1.7}$$

$$R^\mu R_\mu = g_{\mu\nu} R^\mu R^\nu = g_{33}(R^3)^2 = \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2\theta\right] \tag{1.8}$$

The last equations shall come in handy later on when discussing the singularities of the Kerr spacetime. At this point we move to the symmetries of Kerr. Once more, we will see that these symmetries are direct. It is quite trivial to confirm that time and φ-translations:

$$t \to t + \text{const}, \quad \varphi \to \varphi + \text{const} \tag{1.9}$$

are symmetries obeyed by the spacetime. Naturally, this does not come as a surprise as the only dependence of the metric with respect to t, φ lies in their differentials, dt and $d\varphi$, which are invariant under such translations. Finally, there exists one discrete symmetry that is fairly simple to be observed in eq.(1.1), that is:

$$(t, \varphi) \to -(t, \varphi) \tag{1.10}$$

Which is only valid, due to the fact that dt^2, $dtd\varphi$, $d\varphi^2$ remain invariant under the above transformation. This concludes the subsection concerning symmetries and Killing Fields in Kerr spacetime.

1.3.2 Event Horizons and Singularities

Having clarified the concept of coordinates in the Kerr spacetime, it is time to start exploring some of the more interesting features that it exhibits. Upon inspecting the metric one notable question that might arise is what happens when the metric diverges. First let's identify in which cases eq.(1.1) diverges; this occurs when: $\Delta = 0$ or/and $\rho = 0$. That implies that singularities can emerge in these occasions. Solving these equations we obtain the following results:

$$\Delta = 0 \Rightarrow r^2 - 2GMr + a^2 = 0 \Rightarrow r_\pm = GM \pm \sqrt{G^2M^2 - a^2} \tag{1.11}$$

$$\rho = 0 \Rightarrow \rho^2 = 0 \Rightarrow r^2 + a^2\cos^2\theta = 0 \Rightarrow r = 0 \text{ and } \theta = \pi/2 \tag{1.12}$$

Naturally, the next step would be to consider a way to determine whether some of the above singularities are coordinate singularities, not physical ones. In that case a proper coordinate transformation can be found that eliminates these singularities. Of course, there exists such a way and is by examining the Kretschmann (see p. 351 of [3]) scalar of the spacetime. The result follows:

$$K = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48M^2\left(r^6 - 15a^2r^4\cos^2\theta + 15a^4r^2\cos^4\theta - a^6\cos^6\theta\right)}{\left(r^2 + a^2\cos^2\theta\right)^6}, \quad G \equiv 1 \tag{1.13}$$

Clearly, this astonishingly simple result (given the complexity of its calculation) states that the singularities at $r_\pm$ are likely coordinate singularities, to know for sure one has to find regular coordinates near $r_\pm$, while there exists a physical one at $\rho = 0$. However, the solution of $\rho = 0$ is not a point in spacetime, let's proceed to explore its geometry. To do so, we shall substitute eq.(1.12) into eq.(1.3):

$$x^2 + y^2 = a^2\cos^2\varphi + a^2\sin^2\varphi = a^2 \quad \text{and} \quad z = 0 \tag{1.14}$$

Therefore, the $\rho = 0$ singularity is a ring of radius a on the equatorial plane. From a physical perspective, one could argue that the angular momentum that Kerr contains has softened the Schwarzschild singularity, reshaping it into a ring of radius a. That completes our discussion considering the black holes singularity.

Let's embark on exploring a new topic; the event horizon. It turns out that, similarly to the Schwarzschild spacetime, the event horizon occur when $g^{rr} = 0$. Since $g^{rr} = \Delta/\rho^2$ this reduces $\Delta = 0$. The solutions to this equation are given in eq.(1.11) which proved to be of much more importance than one may have thought initially. The fact that the equation $\Delta = 0$ has two solutions indicates that in Kerr geometry there exist two separate horizons r_+ and r_- with $r_+ > r_-$. Sometimes these horizons are called outer and inner horizon respectively. Notice that in the case of the Extreme Kerr, where $a = GM$, the two horizons emerge into a single one at $r = GM$. Now, we would like to calculate the norm of the Killing Field K^μ on the horizon $r = r_+$. By substitution in eq.(1.7) we get:

$$K^\mu K_\mu = \frac{a^2}{\rho^2} \sin^2\theta \geq 0 \tag{1.15}$$

The latter equation shows that the Killing vector K^μ is not null at the horizon $r = r_+$, but spacelike; with the exception at the north and south pole (i.e. at $\theta = 0$ and $\theta = \pi$). The set of points that satisfy the equation $K^\mu K_\mu = 0$ is called the stationary limit surface and is given by:

$$r^2 - 2GMr + a^2\cos^2\theta = 0 \tag{1.16}$$

We explained that the last equation is also valid for the outer horizon at $r = r_+$ at $\theta = 0, \pi$, so by substituting these conditions in the latter equation we acquire:

$$r_+^2 - 2GMr_+ + a^2 = 0 \tag{1.17}$$

Finally, one can rewrite the last equations in the form:

$$(r - GM)^2 = G^2M^2 - a^2\cos^2\theta, \quad (r_+ - GM)^2 = G^2M^2 - a^2 \tag{1.18}$$

in order to see that $r > r_+$. Which implies that this new surface called stationary limit surface is on the outside of the outer event horizon. The region between these two surfaces is an intriguing and well-known one and it's called ergosphere. Since it lies outside the horizon an observer can move toward or away from it without any problem. These features are shown in fig.(2a)

1.3.3 Dragging of inertial frames

After this brief discussion on event horizons and singularities, it is time to commence into exhibiting another interesting feature of the Kerr spacetime which is present due to the non-diagonal form of the metric (i.e. the $dt\,d\varphi$ component).

The easiest way to demonstrate this effect is through the subsequent scenario. Let's consider a photon on the equatorial plane (i.e. $\theta = \pi/2$) at some constant radius which is emitted along $\pm\hat{\varphi}$-direction. We want to understand what its initial trajectory would be. Because of its initially constant radius and θ it holds true: $dr = d\theta = 0$ and because of its nature it follows a null trajectory and therefore: $ds^2 = 0$, so we have:

$$g_{tt}\,dt^2 + 2g_{t\varphi}\,dt\,d\varphi + g_{\varphi\varphi}\,d\varphi^2 = 0 \tag{1.19}$$

By dividing with dt^2 we obtain:

$$g_{\varphi\varphi}\left(\frac{d\varphi}{dt}\right)^2 + 2g_{t\varphi}\frac{d\varphi}{dt} + g_{tt} = 0 \tag{1.20}$$

Now keep in mind that the last equations are only valid at the initial instant (say $t = t_0$) and do not represent the full solution to the photon's trajectory. Since the $d\varphi/dt$ is just a value we can solve eq.(1.20) as follows:

$$\left.\frac{d\varphi}{dt}\right|_{t=t_0} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} \pm \sqrt{\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \tag{1.21}$$

Finally, for the sake of the argument, we assume that photon is emitted in a radius such that it lies right upon the stationary limit surface, where $g_{tt} = 0$, then:

$$\left.\frac{d\varphi}{dt}\right|_{t=t_0} = 0, \quad \left.\frac{d\varphi}{dt}\right|_{t=t_0} = \frac{a}{2G^2M^2 + a^2} \tag{1.22}$$

We should think about the result that we obtained. We begin from the non-zero solution, clearly this solution has the same sign with the parameter a and therefore can be interpreted as the photon emitted along the $\hat{\varphi}$-direction. Then the vanishing solution must be taken as the photon emitted in the opposite way. It turned out that its instantaneous velocity is zero, meaning that the photon cannot move at all at the direction opposite to the rotation of the black hole. This effect is thus called dragging frame effect and is illustrated in fig.(2b). Naturally, any massive particle which moves slower than a photon has no other option but to move to the direction of the rotation.

There are also other ways to demonstrate this effect of the Kerr spacetime. In [4] it is presented in a more arbitrary scenario which considers a massive particle with zero angular momentum and shows that this particle has an angular velocity $\omega(r, \theta) \neq 0$ (see eq.(11.90) of [4]) due to the rotation of the black hole. Moreover, from this equation we understand that is effect scales off with the radius r as $\sim 1/r^3$ and, in principle, can be used to determine the angular momentum of the source.

Let's present a brief synopsis of how this works. We know that the Kerr geometry has the Killing Field $R^\mu = \partial_\varphi$ and therefore the conserved quantity is the following:

$$L \equiv R^\mu p_\mu = \text{const} \Rightarrow p_\varphi = \text{const} \tag{1.23}$$

where $p^\mu = m\, dx^\mu/d\tau$ is the four-momentum. By the last definition it follows:

$$\frac{d\varphi}{dt} = \frac{p^\varphi}{p^t} = \frac{g^{\varphi\varphi}\, p_\varphi + g^{\varphi t}\, p_t}{g^{tt}\, p_t + g^{t\varphi}\, p_\varphi} \tag{1.24}$$

the last equation stands true for the trajectory of a arbitrary particle. Now, we assume the existence of a zero angular-momemtum massive particle, with $L = p_\varphi = 0$. Demanding this into eq.(1.24) we obtain:

$$\frac{d\varphi}{dt} = \frac{g^{\varphi t}}{g^{tt}} \equiv \omega(r, \theta) \tag{1.25}$$

This scenario states the exact same result as before. We drop a massive particle straight into the black hole with zero angular-momentum and somehow its trajectory ends up with non-zero angular velocity (i.e. $\omega(r, \theta) \neq 0$), which is due to the rotation of the spacetime. One final comment on the dragging frame effect is that the last scenario makes even clearer the fact that this effect is a result to the non-vanishing $g_{\varphi t}$ component, otherwise eq.(1.25) would be zero.

1.3.4 Penrose Process

Before concluding this chapter, there is another intriguing aspect of the Kerr spacetime that we would like to illustrate in a simplified manner; the Penrose Process. In 1969, Roger Penrose proposed a way to extract energy from a rotating black hole under some specific conditions. From eq.(1.7) it becomes apparent that at infinity $r \to \infty$ the Killing vector $K^\mu = \partial_t$ is timelike. Let's consider a massive particle, with four-momentum p^μ, which is also timelike, and concentrate on its conserved energy. Typically one would suggest that its energy is:

$$E = K^\mu p_\mu \tag{1.26}$$

However, it can be proved that the inner product of two timelike vectors for any metric $g_{\mu\nu}$ with signature $(-,+,+,+)$ is negative and thus $K^\mu p_\mu < 0$ but we want the energy to be positive, so we redefine it as:

$$E = -K^\mu p_\mu > 0 \tag{1.27}$$

In order to understand the Penrose Process we consider the following scenario: The system of you and your brother are leaping towards the black hole. We denote the initial four-momentum of the system (yourself + your brother) as p_0^μ and your energy as $E_0 > 0$. After you have crossed the stationary limit surface and have entered the ergosphere you decide to get rid of your brother, as he's getting on your nerves, so the conservation of momentum implies:

$$p_0^\mu = p_1^\mu + p_2^\mu \tag{1.28}$$

where p_1, p_2 is your and your brother's four-momentum respectively. By contracting eq.(1.28) is the Killing Field K^μ we get:

$$E_0 = E_1 + E_2 \tag{1.29}$$

You are in the advantageous position to have a basic understanding of the Kerr spacetime and therefore you know that the Killing Field K^μ is spacelike inside the ergosphere, leading to the fact that inside the ergosphere there is a way such that $E = -K^\mu p_\mu < 0$. Given the latter argument and assuming that you are accurate enough you can throw your brother in a way such that $E_2 < 0$, then from eq.(1.29) it follows that $E_1 > E_0$, meaning that you extracted energy from the black hole. In fact, Penrose showed that there is a way to arrange the throw such that afterwards you follow a geodesic out of the ergosphere into the universe. Finally, it can be proved that in order for this process to work the angular momentum of the object with energy E_2 has to be thrown against the black hole's rotation as the energy is extracted by decreasing the rotational energy of the black hole. This whole process described is illustrated in the figure beside.

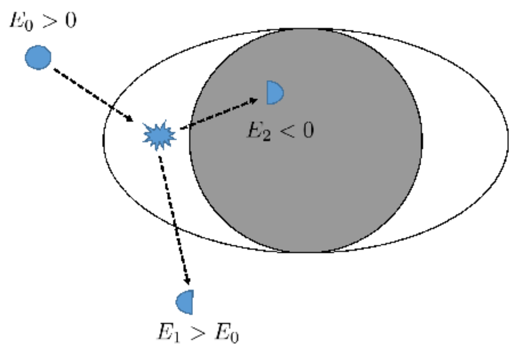
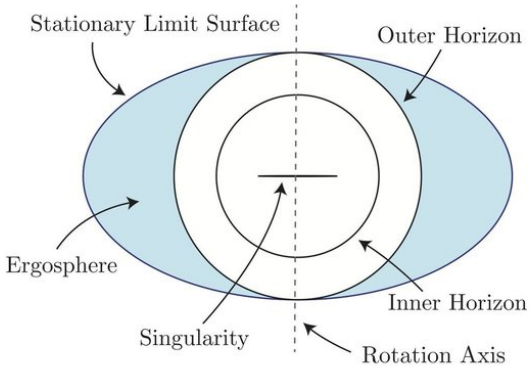
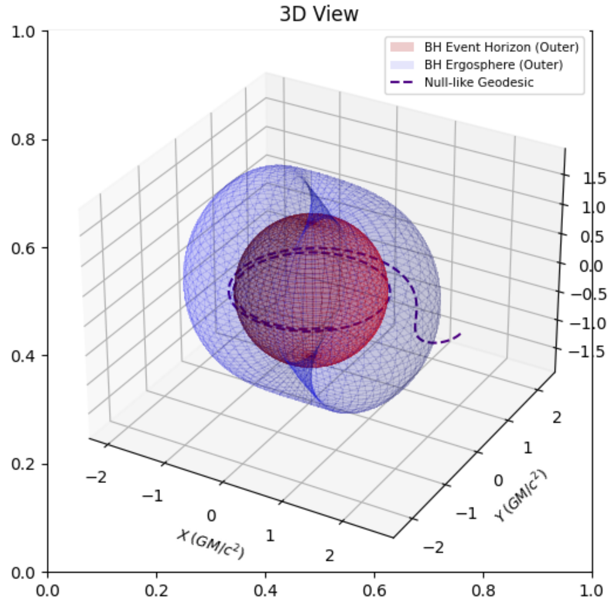


Figure 1: The Penrose Process: A part of a system falls into the horizon with negative energy, thus the other is able to escape the ergosphere. Credits to [5].



(a) This is a demonstration of the features illustrated in §1.3.2 that characterize the Kerr geometry. It depicts the black hole in the yz-plane, the vertical dashed line represents the rotation axis and the horizontal line in the center exhibits the ring-shaped singularity. Credits to Quora.



(b) This is a demonstration of the frame dragging effect in the Kerr spacetime when working in M-units ($G = c = M = 1$). The dashed line represents a null-geodesic initially emitted towards the $-\hat{\varphi}$-direction. The moment it crosses the ergosphere limit its trajectory has been reversed. Credits to EinsteinPy.

Chapter 2

2 Near Horizon Extreme Kerr spacetime in GR

In this section we shall rewrite the Kerr metric in a slightly different form and use it to derive the Near Horizon Extreme Kerr (NHEK) metric. Afterwards, we will press on to demonstrating the fundamental aspects of this spacetime, such as Killing Fields and Symmetries. Further on, we shall explore the existence or not of a conical singularity in that geometry and finally attempt to address the wave equation in this background.

2.1 Extreme Kerr metric

The Kerr metric (eq.(1.1)) in hatted coordinates, assuming $G_N = c = 1$, is:

$$ds^2 = -\frac{\Delta}{\hat{\rho}^2}\Big(d\hat{t}^2 - a\sin^2\theta d\hat{\varphi}\Big)^2 + \frac{\sin^2\theta}{\hat{\rho}^2}\Big((\hat{r}^2+a^2)d\hat{\varphi} - ad\hat{t}\Big)^2 + \frac{\hat{\rho}^2}{\Delta}d\hat{r}^2 + \hat{\rho}^2 d\theta^2 \tag{2.1}$$

where:

$$\Delta = \hat{r}^2 - 2M\hat{r} + a^2, \quad \hat{\rho}^2 = \hat{r}^2 + a^2\cos^2\theta$$

The above is parameterised by two constants, M which is the mass of the Kerr black hole and $J = Ma$ which is the angular momentum. In this section we consider the extreme Kerr limit in which:

$$a = M \Rightarrow J = M^2$$

Therefore the Kerr metric now becomes:

$$ds^2 = -\frac{\Delta}{\hat{\rho}^2}\Big(d\hat{t}^2 - M\sin^2\theta d\hat{\varphi}\Big)^2 + \frac{\sin^2\theta}{\hat{\rho}^2}\Big((\hat{r}^2+M^2)d\hat{\varphi} - Md\hat{t}\Big)^2 + \frac{\hat{\rho}^2}{\Delta}d\hat{r}^2 + \hat{\rho}^2 d\theta^2 \tag{2.2}$$

where:

$$\Delta = (\hat{r} - M)^2, \quad \hat{\rho}^2 = \hat{r}^2 + M^2\cos^2\theta$$

Eq.(2.2) is often called the "Extreme Kerr" metric and we shall adopt it in this work. The horizon of the extreme Kerr is at $\hat{r}_\star = M$. The reason for demanding $a \leq M$ is to ensure that the horizon $r_\star \in \mathbb{R}$ and that it shields the singularity (following the cosmic censorship conjecture). This will be of assistance when it comes to choosing a set of transformations, whose small r limit will zoom into the near horizon area of the extreme Kerr geometry.

2.2 Derivation of Near Horizon Extreme Kerr (NHEK) metric

In this subsection we would like to roughly present the derivation of the NHEK. We shall start with eq.(2.2) and then impose the following transformations:

$$t = \frac{\hat{t}}{2M}, \quad r = \frac{\hat{r} - M}{M}, \quad \varphi = \hat{\varphi} - \frac{\hat{t}}{2M} \tag{2.3}$$

and then obtain the small r limit. Based on the comment of the previous subsection it is clear why the transformations (2.3) are the proper ones for the near horizon area. The last transformations give:

$$dt = \frac{d\hat{t}}{2M}, \quad dr = \frac{d\hat{r}}{M}, \quad d\varphi = d\hat{\varphi} - \frac{d\hat{t}}{2M} \tag{2.4}$$

Now substituting the eq.(2.3), eq.(2.4) into eq.(2.2) one can obtain:

$$\begin{aligned} ds^2 &= -\frac{r^2M^2}{(r+1)^2+\cos^2\theta}\Big[(1+\cos^2\theta)dt - \sin^2\theta d\varphi\Big]^2 \\ &+ \frac{M^2\sin^2\theta}{(r+1)^2+\cos^2\theta}\Big[r(r+2)dt + ((r+1)^2+1)d\varphi\Big]^2 \\ &+ \frac{M^2}{r^2}\Big[(r+1)^2+\cos^2\theta\Big]dr^2 + M^2\Big[(r+1)^2+\cos^2\theta\Big]d\theta^2 \end{aligned} \tag{2.5}$$

Let's keep in mind that the above equation is the Extreme Kerr metric in the new transformed coordinates (t, r, θ, φ). In order to obtain the NHEK metric we must impose the small r limit in eq.(2.5), that is:

$$r \ll 1 \Rightarrow 1 + r \simeq 1$$

This limit is equivalent to $\hat{r} \to M$ (see eq.(2.3)) and thus it allows us to zoom into the near horizon area of the spacetime.

Small r limit

By applying this limit to eq.(2.5) one acquires:

$$\begin{aligned} ds^2 &= -\frac{r^2M^2}{1+\cos^2\theta}\Big[(1+\cos^2\theta)dt - \sin^2\theta d\varphi\Big]^2 + \frac{M^2\sin^2\theta}{1+\cos^2\theta}\Big[2rdt + 2d\varphi\Big]^2 \\ &+ \frac{M^2}{r^2}\Big[1+\cos^2\theta\Big]dr^2 + M^2\Big[1+\cos^2\theta\Big]d\theta^2 \end{aligned} \tag{2.6}$$

This equation is clearly in an unrefined and impractical form. For example, the variables $d\varphi$ and dt are interconnected, making them more challenging to manipulate independently. This statement lead us to start rearranging terms in order to bring the NHEK metric in a more practical form. So:

$$ds^2 = \frac{2M^2(1+\cos^2\theta)}{2}\left\{\frac{dr^2}{r^2} + d\theta^2 + \frac{4\sin^2\theta}{(1+\cos^2\theta)^2}(d\varphi + rdt)^2 - \Big[rdt - \frac{r\sin^2\theta}{1+\cos^2\theta}d\varphi\Big]^2\right\} \tag{2.7}$$

We find it convenient to define the following functions:

$$\mathcal{F}(\theta) \equiv \frac{1+\cos^2\theta}{2}, \quad \Lambda(\theta) \equiv \frac{2\sin\theta}{1+\cos^2\theta}$$

Let's rewrite eq.(2.7) and expand it:

$$\begin{aligned} ds^2 = 2M^2\mathcal{F}(\theta)\Bigg\{&\frac{dr^2}{r^2} + d\theta^2 + \frac{4\sin^2\theta}{(1+\cos^2\theta)^2}(d\varphi^2 + 2rd\varphi dt + r^2dt^2) - r^2dt^2 \\ &+ 2r^2\frac{\sin^2\theta}{1+\cos^2\theta}d\varphi dt - \frac{r^2\sin^4\theta}{(1+\cos^2\theta)^2}d\varphi^2\Bigg\} \end{aligned} \tag{2.8}$$

We can see that now we have distinguished a part of the g_{00} term of the NHEK metric. Now we would like to substitute the $\Lambda(\theta)$ function in order to simplify our expression:

$$\begin{aligned} ds^2 = 2M^2\mathcal{F}(\theta)\Bigg\{&-r^2dt^2 + \frac{dr^2}{r^2} + d\theta^2 + \Lambda^2(\theta)(d\varphi^2 + 2rd\varphi dt + r^2dt^2) \\ &+ \frac{1}{2}r^2(1+\cos^2\theta)\Lambda^2(\theta)d\varphi dt - \frac{1}{4}r^2\sin^2\theta\Lambda^2(\theta)d\varphi^2\Bigg\} \end{aligned} \tag{2.9}$$

In eq.(2.9) one could think to group together the $d\varphi^2$ and $d\varphi dt$ terms and then proceed to examine whether there are any other simplifications to be done under the small r limit. We proceed to do so and in order to avoid extra computations we will examine these terms separately.

We begin by writing these terms as groups:

$$\left[2r + \frac{1}{2}r^2(1+\cos^2\theta)\right]\Lambda^2(\theta)d\varphi dt = \left[2 + \frac{1}{2}r(1+\cos^2\theta)\right]r\Lambda^2(\theta)d\varphi dt \simeq 2r\Lambda^2(\theta)d\varphi dt + \mathcal{O}(r^2)$$

$$\left[1 - \frac{1}{4}r^2\sin^2\theta\right]\Lambda^2(\theta)d\varphi^2 \simeq \Lambda^2(\theta)d\varphi^2 + \mathcal{O}(r^2)$$

The last two equations show that the last two terms of eq.(2.9) are negligible when interested in the near horizon geometry of Kerr spacetime. Thus now we have derived the final form for the NHEK metric which is:

$$ds^2 = 2M^2\mathcal{F}(\theta)\left\{-r^2dt^2 + \frac{dr^2}{r^2} + d\theta^2 + \Lambda^2(\theta)\Big(rdt + d\varphi\Big)^2\right\}, \quad r \ll 1 \tag{2.10}$$

first introduced by [6]. Another useful form of eq.(2.10), which we shall also use, is given if we define:

$$2M^2 \equiv 1, \quad \mathcal{A}(\theta) \equiv \frac{1+\cos^2\theta}{2} = \mathcal{F}(\theta), \quad \mathcal{B}(\theta) \equiv \frac{2\sin^2\theta}{1+\cos^2\theta} = \mathcal{F}(\theta)\Lambda^2(\theta)$$

Then the NHEK metric is given by:

$$ds^2 = \mathcal{A}(\theta)\left[-r^2dt^2 + \frac{dr^2}{r^2} + d\theta^2\right] + \mathcal{B}(\theta)\Big(d\varphi + rdt\Big)^2 \tag{2.11}$$

which is adopted in [6]. The reason why we are presenting both formats of the NHEK metric is because both of them are going to be of use to this particular work. Naturally, one might ask how are we sure that NHEK metric is indeed a vacuum solution? The answer to this question can be expressed in two parts. One way to convince ourselves is, of course, to show that it satisfies $R_{\mu\nu} = 0$ via calculations. However, there is another way to derive the NHEK metric - as it was originally derived by [6]; by applying the transformations given in Eq. 2.5 of [6] in the extreme Kerr metric and then take the limit $\lambda \to 0$. The advantage of this approach is that it clearly results in a vacuum solution. Since the extreme Kerr metric is already a vacuum solution, a coordinate transformation with an arbitrary parameter λ will also yield another solution. Finally, demanding $\lambda \to 0$ selects one specific member of this family of coordinate transformations and hence it is expected to acquire a vacuum solution as well.

2.3 Symmetries and Killing Vectors of NHEK

Now we would like to discuss the symmetries that NHEK obeys and how these symmetries are correlated to Killing vectors. We present both the Killing vectors and some symmetries of the NHEK metric.

2.3.1 Killing Vectors

Let's start off by recalling the definition for a Killing vector.

Definition 1 *A vector field ξ^μ is a Killing field if the Lie derivative with respect to ξ^μ of the metric $g_{\mu\nu}$ vanishes:*

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \tag{2.12}$$

Every Killing vector is implying the existence of a conserved quantity, that is because the metric $g_{\mu\nu}$ remains unchanged along the direction of ξ^μ, as eq.(2.12) indicates[1]. As a matter of fact, if one would like to find all the symmetries of a metric $g_{\mu\nu}$, one would have to find all the solutions to eq.(2.12).

A quantity that is conserved along the direction of a Killing vector is the:

$$\xi^\mu u_\mu = \text{const} \tag{2.13}$$

where $u^\mu = dx^\mu/d\lambda$ is the tangent vector to a geodesic $x^\mu(\lambda)$, with λ an affine parameter. The next topic that we would rather discuss is the notation used in Killing vectors. In general relativity it is very common to see Killing vectors like these: ∂_t, ∂_φ, which correspond to $\xi_1^\mu = (1,0,0,0)$, $\xi_2^\mu = (0,0,0,1)$ respectively. The general format of this notation is this:

$$\xi^\mu = \xi^t\partial_t + \xi^r\partial_r + \xi^\theta\partial_\theta + \xi^\varphi\partial_\varphi$$

where the ∂_σ can be thought as the basis vectors. We shall present the Killing vectors in the same notation.

2.3.2 Symmetries

At this point we would like to continue the discussion with the symmetries of NHEK. First of all, we should at least explain what one means when referring to a "symmetry" of the metric.

We shall begin from a random infinitesimal coordinate transformation of the form:

$$x^\mu \to x^\mu + \varepsilon V^\mu \tag{2.14}$$

where ε a small book-keeping parameter. When the above transformation is applied on the metric $g_{\mu\nu}$, the new metric will be correlated to the old one in the following way:

$$g_{\mu\nu} \to g_{\mu\nu} - \mathcal{L}_V g_{\mu\nu} \tag{2.15}$$

at an infinitesimal level. Upon encountering equation (2.15), one may find oneself asking the following: *Are there any vectors V^μ such that $\mathcal{L}_V g_{\mu\nu} = 0$?*

Because if there are such vectors then LHS and the RHS of eq.(2.15) would be equal and thus the metric would remain invariant under the transformation eq.(2.14) for these particular vectors. Certainly, the answer is affirmative, and vectors capable of achieving this are called Killing vectors, as the definition of the previous subsection demands. We hope that now, the proceeding statements are clear:

1. *Symmetry of a metric is a coordinate transformation that leaves the metric invariant.*
2. *The presence of a Killing vector implies both the existence of a conserved quantity and a symmetry.*

The last statement, of course, does not surprise us as it is a well known one, particularly in physics (remember Noether's theorem).

[1]Intuitively it is quite clear that if the derivative of a quantity along a direction (i.e. ξ^μ in our case) is vanishing then there should exist an conserved quantity along that direction.

2.3.3 Killing Fields and Symmetries of NHEK

Having presented the Killing vectors and symmetries in general we would rather proceed to presenting them in the NHEK geometry. We start off with the Killing vectors of NHEK. If we solve eq.(2.12) for the NHEK metric we will find the following solutions (see eq. 2.7 of [7]):

$$H_{-1} = \partial_t, \quad H_0 = t\partial_t - r\partial_r, \quad H_1 = \left(\frac{1}{2r^2} + \frac{t^2}{2}\right)\partial_t - tr\partial_r - \frac{1}{r}\partial_\varphi, \quad Q_0 = \partial_\varphi \tag{2.16}$$

Having concluded with the Killing Fields in NHEK, we would like to carry on and observe some of its symmetries. When observing eq.(2.10), eq.(2.11) it is clear that none of the coefficients are dependent of the t and φ coordinate, hence if one applies the following transformations on the NHEK metric:

$$t \to t + \text{const}, \quad \varphi \to \varphi + \text{const} \tag{2.17}$$

the metric to remains invariant. Therefore, they comprise two separate symmetries of NHEK that are often called translations. Another thing that we observe with a small amount of effort is the fact that eq.(2.10), eq.(2.11) consist only of terms rdt, r^2dt^2, dr^2/r^2 with respect to t and r coordinates (i.e. the metric is comprised solely by products of rdt, dr/r to some power). Hence the following re-scale as a transformation:

$$t \to ct, \quad r \to \frac{r}{c} \tag{2.18}$$

is also a symmetry, just because the products that we listed above will remain unchanged. Notable is that, in contrast to eq.(2.17), the transformations of eq.(2.18) must be imposed simultaneously, and thus they embody a single symmetry.

So far we have presented 3 different symmetries and 4 Killing vectors, but we established that for each Killing there is a different symmetry. Indeed, there is one more symmetry which is fairly non-trivial to observe in the NHEK metric and we shall present it only for reasons regarding completeness. The fourth and final symmetry is the subsequent (for the t, r components see p. 5 Eq. 11 of [8]):

$$t \to \frac{t - c(t^2 - 1/r^2)}{1 - 2ct + c^2(t^2 - 1/r^2)}, \quad r \to r\Big[1 - 2ct + c^2(t^2 - 1/r^2)\Big], \quad \varphi \to \varphi + \ln\frac{c - r + crt}{-c - r + crt} \tag{2.19}$$

That should conclude the discussion concerning symmetries of NHEK spacetime.

2.3.4 Relationship between symmetries and Killing vectors

At this moment we have explained both Killing vectors and symmetries of a metric and that a Killing vector implies the existence of a symmetry and vice versa. When addressing the Killing vectors we said that if one wishes to find all the symmetries of a metric one should find all solutions of eq.(2.12). That is correct; however, eq.(2.12) is a system of differential equations and therefore it can be rather challenging to solve. It would be quite efficient to put to use the fact that if we observe a symmetry then immediately we know the existence of a Killing vector. Hence, we would like to seize this opportunity to review the process of determining the Killing vector when its corresponding symmetry is known[2], through the example of NHEK.

Let ξ^μ be the Killing vector that we want to identify. We assume that we make the following infinitesimal transformation:

$$x^\mu \to x^\mu + \varepsilon\xi^\mu \tag{2.20}$$

[2]The reverse process (i.e. determining the symmetry when its Killing vector is known) is significantly challenging at the non-linear level and we shall not present it. At the linear level eq.(2.14)-eq.(2.15) are the answer.

where ε is a small book-keeping parameter. The main idea is to expand our known transformation (the symmetry) and bring it in the form of eq.(2.20) from which the identification of the Killing vector can be done effortlessly. Let's start off with the symmetry:

$$t \to t + \text{const}$$

which is in the desired form if we assume that const is small and $\xi^0 = 1$, resulting in the $H_{-1} = \partial_t$ Killing vector. Similarly one can show that the Killing vector corresponding to $\varphi \to \varphi + \text{const}$ is the $Q_0 = \partial_\varphi$. The next symmetry that we are interested in is that of eq.(2.18). We have to bring this transformation into the form of eq.(2.20) so one would propose to set $c = 1 + \varepsilon$, $\varepsilon \ll 1$, then eq.(2.18) becomes:

$$t \to t + \varepsilon t, \quad r \to \frac{r}{1+\varepsilon} \simeq r(1-\varepsilon) = r - \varepsilon r \tag{2.21}$$

Comparing eq.(2.20) with eq.(2.21) it is clear that the Killing vector associated with this symmetry is the $H_0 = t\partial_t - r\partial_r$. Finally, we shall proceed to derive the Killing vector H_1 based on the symmetry given by eq.(2.19), in this case we assume that $c \ll 1$ and expand eq.(2.19) to first order with respect to c:

$$\begin{aligned} t &\to \frac{t - c(t^2 - 1/r^2)}{1 - 2ct + c^2(t^2 - 1/r^2)} \simeq t + \left(\frac{1}{r^2} + t^2\right) c, \\ r &\to r\Big[1 - 2ct + c^2(t^2 - 1/r^2)\Big] \simeq r - (2rt)c, \\ \varphi &\to \varphi + \ln \frac{c - r + crt}{-c - r + crt} \simeq \varphi - \frac{2}{r}c \end{aligned} \tag{2.22}$$

From the above equation it becomes apparent that the Killing vector associated with these symmetries is the H_1 that we mentioned before.

2.3.5 $\mathfrak{sl}(2) \times \mathfrak{u}(1)$ Lie Algebra

After having demonstrated the connection between Killing vectors and symmetries it is time to push through to another interesting matter. We would like to show that the Killing vectors of NHEK do form a Lie Algebra called $\mathfrak{sl}(2) \times \mathfrak{u}(1)$, we would rather not proceed to discuss as to why the Killing vectors form this particular Algebra; one only has to remember that the structure coefficients are those that determine the specific Algebra. Moreover, we shall not advance to discussing the connection between the Lie group and its respective Lie Algebra. We will consider as a fact that the Killing vectors of NHEK are the generators of the SL(2)×U(1) group. Let us begin by giving the definition of a Lie Algebra:

Definition 2 *A Lie Algebra is a vector space $\mathfrak{g}$ over a field F together with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which we call a Lie Bracket, such that the following axioms are satisfied:*

1. *Bilinearity:*
   $[ax + by, z] = a[x, z] + b[y, z]$
   $[z, ax + by] = a[z, x] + b[z, y]$
   for all scalars $a, b \in F$ and all elements $x, y, z \in \mathfrak{g}$.

2. *The Alternating property:*
   $[x, x] = 0$
   for all $x \in \mathfrak{g}$

3. *The Jacobi identity:*
   $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$
   for all $x, y, z \in \mathfrak{g}$

To begin with, we would like to define the Lie Bracket binary operation that is stated in the above definition. In our case we choose the following operation as our Lie Bracket:

$$[A, B]^\mu \equiv \mathcal{L}_A B^\mu = A^\nu \partial_\nu B^\mu - B^\nu \partial_\nu A^\mu \tag{2.23}$$

where $\mathcal{L}_A$ denotes the Lie derivative along the vector A. Also we have added a μ superscript on the bracket that indicates that the result of the commutator is another vector. The linear character of the Lie derivative -and the derivative in general- ensures that the first axiom is satisfied. Obviously, by definition, eq.(2.23) satisfies Alternating property of the Lie Algebra.

Let's consider the Killing vectors given by eq.(2.16). We find it convenient to rename the following two Killing vectors $H_{-1} \to H_-$ and $H_{+1} \to H_+$. In order to show that they form an Algebra we should begin to calculate the commutators of the Killing vectors and hopefully we anticipate that the commutator of two vectors yields the third one with some structure constant. We start off by computing the $[H_+, H_-]^\mu$:

$$[H_+, H_-]^\mu = H_+^\nu \partial_\nu H_-^\mu - H_-^\nu \partial_\nu H_+^\mu \tag{2.24}$$

$$= \left\{ \left(\frac{1}{2r^2} + \frac{t^2}{2} \right) \partial_t - tr\partial_r - \frac{1}{r}\partial_\varphi \right\} H_-^\mu - \partial_t H_+^\mu \tag{2.25}$$

Now one has to substitute $\mu = 0, \ldots, 3$ to the previous equation in order to obtain that

$$[H_+, H_-] = -H_0$$

We proceed to the following one:

$$[H_0, H_+]^\mu = H_0^\nu \partial_\nu H_+^\mu - H_+^\nu \partial_\nu H_0^\mu \tag{2.26}$$

$$= \Big(t\partial_t - r\partial_r \Big) H_+^\mu - \left\{ \left(\frac{1}{2r^2} + \frac{t^2}{2} \right) \partial_t - tr\partial_r - \frac{1}{r}\partial_\varphi \right\} H_0^\mu \tag{2.27}$$

Once again when substituting all the values for μ the latter equation yields:

$$[H_0, H_+] = H_+$$

Finally:

$$[H_0, H_-]^\mu = H_0^\nu \partial_\nu H_-^\mu - H_-^\nu \partial_\nu H_0^\mu \tag{2.28}$$

$$= \Big(t\partial_t - r\partial_r \Big) H_-^\mu - \partial_t H_0^\mu \tag{2.29}$$

Once more, upon substituting all the values for μ, the latter equation produces:

$$[H_0, H_-] = -H_-$$

Next, we will turn our attention to verifying that the Killing vectors $H_0, H_\pm$ satisfy the Jacobi identity:

$$[H_0,[H_+,H_-]]+[H_+,[H_-,H_0]]+[H_-,[H_0,H_+]] = [H_0,-H_0]+[H_+,H_-]+[H_-,H_+] \quad (2.30)$$
$$= -[H_0,H_0]+[H_+,H_-]-[H_+,H_-] \quad (2.31)$$
$$= 0 \quad (2.32)$$

where in the second line we used that $[A,B] = -[B,A]$[3]. This concludes the argument that the $H_0, H_\pm$ form the $\mathfrak{sl}(2)$ Lie Algebra. Now, we have to show that all the Killing vectors $Q_0, H_0, H_\pm$ form the $\mathfrak{sl}(2)\times\mathfrak{u}(1)$ Lie Algebra and in order to do so one would have to prove that the generator of the U(1) group (i.e. Q_0) complies with the successive relations:

$$[Q_0,H_0] = [Q_0,H_\pm] = 0 \quad (2.33)$$

Indeed the preceding relations are valid and demonstrating their validity is fairly straightforward, so we will refrain from providing the proof. Lastly, before we complete this subsection, we will gather the commutative relations that hold true for the $\mathfrak{sl}(2)\times\mathfrak{u}(1)$ Lie Algebra:

$$[H_+,H_-] = -H_0, \quad [H_0,H_\pm] = \pm H_\pm, \quad [Q_0,H_0] = [Q_0,H_\pm] = 0 \quad (2.34)$$

2.4 NHEK metric at $\theta \to 0,\ \pi$

Moving forward, we will examine the behaviour of the NHEK metric in the limit of $\theta \ll 1$. But we before we proceed to this task let us explain the reason why we are interested in this particular behaviour. The main cause as to why we are discussing this specific limit is to determine whether there exists a conical singularity near $\theta = 0$. The existence of such a singularity would mean that there the Ricci scalar diverges at that point. Similarly at $\theta = \pi$.

2.4.1 Conical Singularity

Let us begin by presenting what is a conical singularity. Imagine the xy-plane in polar coordinates (r,φ) with metric $ds^2 = dr^2 + r^2 d\varphi^2$, it is established that $\varphi = \varphi + 2\pi$, indicating that the azimuth angle φ remains unchanged when completing a full circle. Now let us remove a slice of the xy-plane from φ_0 to φ_1, as depicted in the following image.

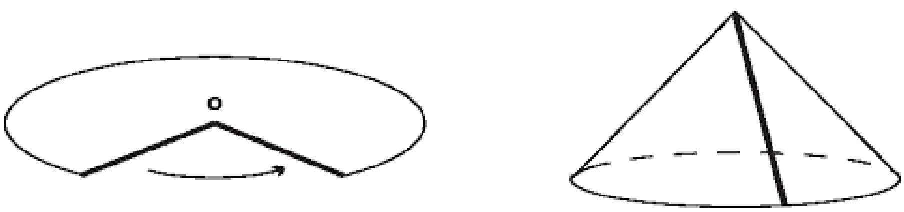


Figure 3: Demonstration of a Conical Singularity. Credits to `StackExchange Physics`.

Then what happens if we identify the two sides of the missing piece is what is called a conical singularity. The name is given, of course, due to the fact that the plane now transforms into a cone.

[3]This can be shown trivially from the definition of the Lie Bracket.

2.4.2 The NHEK limit

At this point, we are interested in exploring how to determine if the NHEK background presents a conical singularity. We focus near $\theta = 0$, π because there is something funny there: $d\varphi$ drops out, leading to either a coordinate singularity or a physical one. The way to extract this piece of information comes naturally as all we need to do is check whether, when $\theta \ll 1$, the NHEK metric reduces to the same limit as the S^2 metric does. We know that the S^2 metric is regular at all points except for $\theta = 0, \pi$. However, we know that this divergent behaviour is only a result of coordinates and that these points are regular hence, if the NHEK background reduces to the same limit as S^2, the absence of such a singularity is indicated. Let's begin by writing the S^2 metric and deriving its small θ limit:

$$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2 \xrightarrow[\theta\ll 1]{} ds^2 = d\theta^2 + \theta^2 d\varphi^2 \tag{2.35}$$

so with $\varphi = \varphi + 2\pi$ the S^2 metric is locally like the plane near $\theta = 0$ and thus it is regular. Now we have to derive the respective limit for the NHEK metric, we find it more convenient to use eq.(2.10) as the desired form of the metric. We start by isolating the $d\theta^2$ and $d\varphi^2$ terms of the metric:

$$ds^2 = 2M^2\left[\mathcal{F}(\theta)\left\{-r^2dt^2 + \frac{dr^2}{r^2}\right\} + \mathcal{F}(\theta)d\theta^2 + \mathcal{F}(\theta)\Lambda^2(\theta)\Big(rdt + d\varphi\Big)^2\right] \tag{2.36}$$

We expand the $\mathcal{F}(\theta)$ and $\mathcal{F}(\theta)\Lambda^2(\theta)$ functions around $\theta = 0$ and keep only the dominant term of their behaviour near $\theta \ll 1$ (i.e. first term of their Taylor expansion):

$$\mathcal{F}(\theta) \simeq 1 + \mathcal{O}(\theta^2)$$

$$\mathcal{F}(\theta)\Lambda^2(\theta) \simeq \theta^2 + \mathcal{O}(\theta^4)$$

Substituting the last two equations into eq.(2.36) we acquire:

$$\begin{aligned} ds^2 &= 2M^2\left[-r^2dt^2 + \frac{dr^2}{r^2} + d\theta^2 + \theta^2\Big(rdt + d\varphi\Big)^2\right] \\ &= 2M^2\left[-r^2dt^2 + \frac{dr^2}{r^2} + \theta^2r^2dt^2 + 2r\theta^2dtd\varphi + d\theta^2 + \theta^2d\varphi^2\right] \\ &= 2M^2\left[-r^2dt^2 + \frac{dr^2}{r^2} + \theta^2r^2dt^2 + 2r\theta^2dtd\varphi\right] + 2M^2\left[d\theta^2 + \theta^2d\varphi^2\right] \end{aligned}$$

Clearly from the last equation the θ and φ dimensions of the NHEK metric reduce to the proper limit when $\theta \ll 1$, which is the limit denoted from eq.(2.35). Meaning that the NHEK background does not exhibit a conical singularity around $\theta = 0$.

2.5 The wave equation on the NHEK background

Having discussed a few things about the NHEK background we shall proceed to the wave equation in that background. In general relativity it is rather common to deal with wave equations, i.e. equations that involve the d' Alembertian operator $\Box = \nabla^\nu\nabla_\nu$ which reduces to $\Box = -\partial_t^2 + \vec{\nabla}^2$ in the Minkowski spacetime. The most simple case of a wave equation that one can imagine is the subsequent:

$$\Box\phi = 0 \tag{2.37}$$

and thus it serves as a toy model for studying these types of equations. Naturally, this is a partial differential equation, our best chance of tackling it is by separating variables, which result in two ordinary differential equations. In our case ϕ is a scalar field of the form $\phi(x^\mu)$. Before we begin solving eq.(2.37) we should clarify what is our objective. Our aim is to try and write an ansatz for our scalar field $\phi(x^\mu)$ so that eq.(2.37) adoptes a separable form. We shall consider the NHEK metric given by the eq.(2.11) and for simplicity we shall not substitute $\mathcal{A}(\theta)$, $\mathcal{B}(\theta)$ with their respective values until it is necessary. The ansatz for our scalar field $\phi(x^\mu)$ is:

$$\phi = e^{-i\omega t+im\varphi}\, R(r)\, S(\theta) \tag{2.38}$$

where $m \in \mathbb{Z}$. Of course, every superposition of eq.(2.38) will be a solution of eq.(2.37) due to its linearity. Now, we progress to start tackling the wave equation of eq.(2.37). We start off by rewritting it in the form:

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0 \tag{2.39}$$

As already stated ϕ is a scalar field and therefore $\nabla_\mu\phi = \partial_\mu\phi$ and thus we find it convenient to define a covariant vector[4] $\omega_\nu \equiv \nabla_\nu\phi$ with the following components:

$$\omega_\nu = \left(-i\omega e^{-i\omega t+im\varphi}\, R(r)\, S(\theta), e^{-i\omega t+im\varphi}\,\frac{dR(r)}{dr}\, S(\theta), e^{-i\omega t+im\varphi}\, R(r)\,\frac{dS(\theta)}{d\theta}, ime^{-i\omega t+im\varphi}\, R(r)\, S(\theta)\right) \tag{2.40}$$

By rewritting eq.(2.39) one obtains:

$$g^{\mu\nu}\nabla_\mu\omega_\nu = 0 \tag{2.41}$$

The only non-vanishing metric components yield:

$$g^{00}\nabla_0\omega_0 + g^{11}\nabla_1\omega_1 + g^{22}\nabla_2\omega_2 + g^{33}\nabla_3\omega_3 + g^{30}\nabla_3\omega_0 + g^{03}\nabla_0\omega_3 = 0 \tag{2.42}$$

By expanding the covariant derivative and keeping only the non-vanishing Christoffels one obtains:

$$\begin{aligned} g^{00}\left(\partial_0\omega_0 - \Gamma^1_{00}\,\omega_1 - \Gamma^2_{00}\,\omega_2\right) + g^{11}\left(\partial_1\omega_1 - \Gamma^1_{11}\,\omega_1 - \Gamma^2_{11}\,\omega_2\right) + g^{22}\left(\partial_2\omega_2 - \Gamma^2_{22}\,\omega_2\right) \\ + g^{33}\left(\partial_3\omega_3 - \Gamma^2_{33}\,\omega_2\right) + 2g^{30}\left(\partial_0\omega_3 - \Gamma^2_{30}\,\omega_2 - \Gamma^1_{30}\,\omega_1\right) = 0 \end{aligned} \tag{2.43}$$

We compactified the last two terms of eq.(2.42) due to the definition of the covariant vector ω_ν and the symmetric nature of the Christoffel symbols under permutation of their lower indices. Moreover, we detect that all the terms of eq.(2.43) contain the term $\exp(-i\omega t + im\varphi)$ and thus we eliminate them at this moment:

$$\begin{aligned} \frac{\omega^2 RS}{\mathcal{A}r^2} + \frac{\mathcal{A}-\mathcal{B}}{\mathcal{A}^2} rS\frac{dR}{dr} + \frac{1}{2\mathcal{A}^2}\left[\frac{d\mathcal{A}}{d\theta} - \frac{d\mathcal{B}}{d\theta}\right] R\frac{dS}{d\theta} + \frac{r^2}{\mathcal{A}} S\frac{d^2R}{dr^2} + \frac{r}{\mathcal{A}} S\frac{dR}{dr} + \frac{1}{\mathcal{A}} R\frac{d^2S}{d\theta^2} \\ - \frac{\mathcal{A}-\mathcal{B}}{\mathcal{A}\mathcal{B}} m^2 RS + \frac{\mathcal{A}-\mathcal{B}}{2\mathcal{B}\mathcal{A}^2}\frac{d\mathcal{B}}{d\theta} R\frac{dS}{d\theta} + \frac{2m\omega}{r\mathcal{A}} RS + \frac{1}{\mathcal{A}^2}\frac{d\mathcal{B}}{d\theta} R\frac{dS}{d\theta} + \frac{\mathcal{B}r}{\mathcal{A}^2} S\frac{dR}{dr} = 0 \end{aligned} \tag{2.44}$$

By rearranging the terms with respect to R, S and multiplying by $\times\mathcal{A}$ we derive:

$$r^2\frac{d^2R}{dr^2}S + R\frac{d^2S}{d\theta^2} + 2rS\frac{dR}{dr} + \frac{1}{2}\left[\frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{d\theta} + \frac{1}{\mathcal{B}}\frac{d\mathcal{B}}{d\theta}\right] R\frac{dS}{d\theta} + \left[\frac{\omega^2}{r^2} + \left(1 - \frac{\mathcal{A}}{\mathcal{B}}\right) m^2 + \frac{2m\omega}{r}\right] RS = 0 \tag{2.45}$$

Let's pause for a moment to discuss the pattern that we observe in the above equation. In each term it is apparent that the coefficient is either dependent on r or it is dependent on θ, but not on both of them. That is encouraging as it suggests that our equation is obtaining a

[4]we know that the partial derivatives of a scalar field are a well-defined covariant vector.

separable form. The next step would be, of course, to multiply $\times 1/RS$ hoping that eq.(2.45) will separate, from now on we shall denote with prime and dot the derivative with respect to r and θ respectively:

$$r^2\frac{R''}{R}+\frac{\ddot{S}}{S}+2r\frac{R'}{R}+\frac{1}{2}\left[\frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{d\theta}+\frac{1}{\mathcal{B}}\frac{d\mathcal{B}}{d\theta}\right]\frac{\dot{S}}{S}+\frac{\omega^2}{r^2}+\left(1-\frac{\mathcal{A}}{\mathcal{B}}\right)m^2+\frac{2m\omega}{r}=0 \tag{2.46}$$

Separating the r-dependent components in the LHS and the θ-dependent components in the RHS, and then setting it equal to a separation constant $\lambda\in\mathbb{C}$:

$$r^2\frac{R''}{R}+2r\frac{R'}{R}+\frac{\omega^2}{r^2}+\frac{2m\omega}{r}=-\frac{\ddot{S}}{S}-\frac{1}{2}\left[\frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{d\theta}+\frac{1}{\mathcal{B}}\frac{d\mathcal{B}}{d\theta}\right]\frac{\dot{S}}{S}+\left(\frac{\mathcal{A}}{\mathcal{B}}-1\right)m^2\equiv-\lambda \tag{2.47}$$

So the two ordinary differential equations for the radial and angular part of the scalar are the proceeding:

$$r^2R''+2rR'+\left(\frac{\omega^2}{r^2}+\frac{2m\omega}{r}+\lambda\right)R=0 \tag{2.48}$$

$$\ddot{S}+\frac{1}{2}\left[\frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{d\theta}+\frac{1}{\mathcal{B}}\frac{d\mathcal{B}}{d\theta}\right]\dot{S}-\left[\left(\frac{\mathcal{A}}{\mathcal{B}}-1\right)m^2+\lambda\right]S=0 \tag{2.49}$$

Note that the equation (2.49) involving angular quantities is not yet in its final form. In order to proceed with its solution, we need to substitute the expressions for $\mathcal{A}(\theta)$ and $\mathcal{B}(\theta)$ into it. This is pretty straight forward and should one do it one deduces:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right)+\left[K-\frac{m^2}{\sin^2\theta}-\frac{1}{4}m^2\sin^2\theta\right]S=0 \tag{2.50}$$

where now we redefined $K\equiv 2m^2-\lambda$. The last equation was written in this particular form due to its convenience and is in agreement with eq.(3.5) by [6]. Finally, we have obtained the ordinary differential equations satisfied by the radial and angular part of the scalar field ϕ. Clearly the eq.(2.48) and eq.(2.50) represent a Sturm-Liouville problem, and in order to tackle it we should specify boundary conditions, find the spectrum (discrete or continuous) of eigenvalues and eigenfunctions.

Notable is the fact that for $m=0$ the eq.(2.50) reduces to the usual Legendre equation with $K=\ell(\ell+1)$ and general solution of the form: $c_1P_\ell(\cos\theta)+c_2Q_\ell(\cos\theta)$

2.5.1 The Angular Equation

We should now turn our attention to exploring the solution of the angular equation eq.(2.50). This equation presents a generalization of the Legendre equation, implying that identifying regular solutions is not a straightforward task. But before we start discussing its solutions let us take a moment to set up the problem properly, i.e. specify the boundary conditions needed to tackle this Sturm-Liouville problem. Notable is the fact that this equation has singular points at $\theta=0$ and $\theta=\pi$ and as there is no physical reason for the solution to diverge at any point of the interval $\theta\in[0,\pi]$ we set the following boundary conditions:

$$S(0)=\text{finite},\quad S(\pi)=\text{finite} \tag{2.51}$$

Having presented that we have set up the whole Sturm-Liouville problem. The fact that eq.(2.50) has singular points complicates the problem dearly. However luckily for us this is a well-studied equation named "angular spheroidal harmonic equation" with well-defined regular solutions (at least one of the two kinds of solutions). The extraction of the solution of this equation demands the usage of three-term recurrence relation theory ([9], [10]) and therefore we shall not engage in this process. However the calculation of the eigenvalues can be done via numerical methods. Below we present some of them given by [6]:

m	$\ell = m$	$\ell = m+1$	$\ell = m+2$
1	2.200	6.143	12.133
2	6.855	12.664	20.597
3	13.995	21.629	31.459

Table 1: Eigenvalues K of eq.(2.50).

Let us transform the eq.(2.50) by setting a new independent variable $\xi = \cos\theta$. Then the transformed form of eq.(2.50) will be:

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dS}{d\xi}\right] + \left[K - \frac{m^2}{1-\xi^2} - \frac{1}{4}m^2(1-\xi^2)\right] S = 0, \quad S(\pm 1) = \text{finite} \tag{2.52}$$

By absorbing the term $-m^2/4$ into the separation constant K, i.e. $\widetilde{K} \equiv K - m^2/4$, eq.(2.52) can be rewritten into:

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dS}{d\xi}\right] + \left[\widetilde{K} - \frac{m^2}{1-\xi^2} + \frac{1}{4}m^2\xi^2\right] S = 0, \quad S(\pm 1) = \text{finite} \tag{2.53}$$

The latter equation is of a specific form, it represents eq.(A8) with $c = im/2$ and therefore its solution and eigenvalues are $S_{mn}(\xi; im/2)$, $\lambda_{mn}(im/2)$.

2.5.2 The Radial Equation

In the upcoming part, we will examine the radial equation eq.(2.48). The first thing that one would observe in this equation would probably evolve around its irregular singular point at $r = 0$. This singularity is termed as "irregular"[5] due to the fact that around $r = 0$ the R coefficient diverges as $\sim 1/r^4$. Based on the latter observations, once again, this equation does not seem to have a straightforward solution and thus needs further examination.

One reasonable next step would be to attempt to use Fuchsian Theory, express the two linear independent solutions as series and determine whether these solutions converge $\forall r \in [0, +\infty)$. Naturally, we expect that these series will converge $\forall r \in (0, +\infty)$ as the differential equation is perfectly regular in that interval. However, before we proceed to this complicated calculation let us explore a different path. One of the most effective ways to deal with an ODE is to find a transformation -either of independent or dependent variable- to apply on it that will modify it into a known or easier ODE. We shall attempt to follow that strategy.

We start off by making the leading observations:

1. the $r^2R'' + 2rR'$ terms will transform to $u^2\ddot{R}$ when $u = 1/r$.

2. the R coefficient will be simplified under the same transformation.

So we start off by using the $u = 1/r$ transformation to eq.(2.48). The relations that govern the derivatives of this transformation are the following:

$$\frac{d}{dr} = -u^2\frac{d}{du}, \quad \frac{d^2}{dr^2} = u^4\frac{d^2}{du^2} + 2u^3\frac{d}{du} \tag{2.54}$$

[5]The terminology is based on Fuchs' Theorem considering the series convergence around a singularity.

Now by substituting eq.(2.54) and $u = 1/r$ into eq.(2.48) we obtain the transformed radial differential equation:

$$\ddot{R} + \left(\omega^2 + \frac{2m\omega}{u} + \frac{\lambda}{u^2}\right) R = 0 \tag{2.55}$$

We should take a moment to discuss tha last equation. Using the $u = 1/r$ transformation we have achieved two very important objectives. The first one is that, despite the fact that our equation continues to present a singular point at $u = 0$, the singularity is currently regular (scales as $\sim 1/u^2$) which allows to be certain that at least one of the two solutions can be expressed as a convergent series around $u = 0$. The second -and most important- one is that the form of eq.(2.55) is similar to the Whittaker[6] differential equation with regular solutions $M_{\kappa,\mu}(z), W_{\kappa,\mu}(z)$.

Given the fact that we have not yet transformed our radial equation into a Whittaker one we have to make a reparametrization of the $u = 1/r$ transformation so that the ω^2 term can be reduced to $-1/4$. This procedure is mostly a trial and error one, however in this particular case it is fairly plain to see that the proper transformation is $u = -2i\omega/r$. We move forward to apply $u = -2i\omega/r$ into eq.(2.48). The relations that govern this transformation are the successive:

$$\frac{d}{dr} = \frac{u^2}{2i\omega}\frac{d}{du}, \quad \frac{d^2}{dr^2} = -\frac{1}{4\omega^2}\left(u^4\frac{d^2}{du^2} + 2u^3\frac{d}{du}\right) \tag{2.56}$$

By substituting eq.(2.56) into eq.(2.48) we derive:

$$\ddot{R} + \left(-\frac{1}{4} + \frac{im}{u} + \frac{\lambda}{u^2}\right) R = 0 \tag{2.57}$$

Finally the last equation has the Whittaker equation form. The general solution of eq.(2.57) is:

$$R = c_1 M_{im,\sqrt{1/4-\lambda}}(u) + c_2 W_{im,\sqrt{1/4-\lambda}}(u), \quad u = -\frac{2i\omega}{r} \tag{2.58}$$

one more thing that could be done in the above equation is an arbitrary choice of either c_1 or c_2. This statement holds true due to the nature of the equation that we attempt to tackle; $\Box\phi = 0$ is a linear equation and thus the amplitude of the wave is not important.

Naturally, in order to determine the constants c_1, c_2 we have to impose boundary conditions. There is a variety of physically interesting boundary conditions that can be imposed. However, we choose to proceed with the following ones:

Boundary Condition: *ingoing* wave at $r = 0$ (horizon) $\Rightarrow$ *ingoing* + *outgoing* wave at $r = \infty$

Which can be thought as the following scattering problem. We send radially a wave from infinity (i.e. ingoing at $r = \infty$) that travels towards the horizon (i.e. ingoing at $r = 0$), then a part of it gets scattered and returns to infinity (i.e. the outgoing part at $r = 0$). Due to the fact that our boundary conditions refer to $R(r \to 0)$ and $R(r \to +\infty)$ one would have to study the asymptotic behaviours of the Whittaker functions $M_{\kappa,\mu}(z), W_{\kappa,\mu}(z)$. The asymptotic behaviours (see [11]) with respect to z are the following:

$$\begin{aligned} M_{\kappa,\mu}(z) &\overset{z\to 0}{\sim} z^{\mu+1/2}, \\ &\overset{z\to\infty}{\sim} \frac{\Gamma(1+2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)} e^{z/2} z^{-\kappa} + \frac{\Gamma(1+2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)} e^{-z/2\pm(1/2+\mu-\kappa)\pi i} z^{\kappa} \end{aligned}$$

[6]More information consider this ODE are presented in Appendix B

and for $W_{\kappa,\mu}(z)$:

$$W_{\kappa,\mu}(z) \overset{z\to 0}{\sim} \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)} z^{1/2-\mu} + \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)} z^{1/2+\mu},$$
$$\overset{z\to\infty}{\sim} e^{-z/2}\, z^{\kappa}$$

The above asymptotic behaviours are valid with respect to z and by extension for our $u = -2i\omega/r$ variable. Naturally, in order to solve the radial problem one should convert the asymptotic behaviours with respect to r and then impose the boundary conditions described above.

Chapter 3

3 Effective Field Theory in the NHEK background

In this section we would like to introduce the concept of the Effective Field Theory (EFT) both in a general sense and within its application in General Relativity. Afterwards, we shall probe into discussing the EFT-corrected NHEK in that particular regime and push through to exploring more aspects of it. Once more, we shall investigate the existence of conical singularity and afterwards the wave equation in this background.

3.1 Effective Field Theory

When physicists refer to the term of "Effective Field Theory" they refer to a type of approximation corresponding to a intrinsic complete physical theory. Typically EFTs are of apparent use in situations where all the interactions of a dynamical system are not understood, known or feasible. Instead of taking into consideration all the microscopic degrees of freedom and interactions an EFT focuses on the most relevant ones.

All the physical theories have a certain region of validity, after which they break down and thus require a new theory to replace them; a great example, of course, would be Newtonian Gravity and General Relativity or even General Relativity and Quantum Gravity. It is well established that these theories are applied in different regimes (or scales) and therefore depending on the dynamical system studied one has to choose which theory to employ. However, in a wide number of cases there are scales where it is not perfectly clear which theory is a proper to utilize. In such cases an EFT could be used to "extent" the regime of validity of a theory in order to avoid taking into consideration all the interactions of the system (i.e. using the complete theory) but only those of utmost importance. Naturally, EFTs are expected to be valid in a certain regime (or scale) as they attempt to stretch one theory towards the other.

3.2 Motivation for an EFT-corrected GR

In this context, let's discuss what is the drive to consider Effective Field Theories in General Relativity. Today, it is established that there are some regimes where Einstein's General Relativity breaks down, a well-known example is, of course, the case of Black Holes. Near the singularity the curvature diverges indicating that in this particular regime our theory is inadequate to describe the physical interactions of the system. As a result, alternative theories are proposed by scientists that take into consideration additional features of the dynamical system.

Within this specific context, General Relativity is expected to be an low-energy limit of a more complete theory (such as String Theory). However, in some cases where we aim to address problems near the borders of GR's validity it could be advantageous to incorporate some small terms in the Lagrangian in order to extend the theory. The reason why adding these terms is an insightful concept due to our expectation of existence of higher order terms in the Lagrangian in the completed theory.

3.3 A purely Gravitational Effective Field Theory

3.3.1 Formulating the Effective Field Theory

The Effective Field Theory that we are interested in utilizing is a purely gravitational one (i.e. we are interested in vacuum solutions). In such cases the Lagrangian can be formulated in terms of the Riemann tensor alone. The Gravitational Effective Field Theory that we would like to employ is the following (see p. 1 Eq. 1 of [1]):

$$\mathcal{L} = \frac{1}{2\kappa^2}\Big(R + \eta\,\kappa^4\,\mathcal{R}^3 + \lambda\,\kappa^6\,\mathcal{C}^2 + \tilde{\lambda}\,\kappa^6\,\tilde{\mathcal{C}}^2\Big) \tag{3.1}$$

where $\kappa^2 \equiv 8\pi G$, $\mathcal{R}^3 \equiv R_{ab}{}^{cd}\,R_{cd}{}^{ef}\,R_{ef}{}^{ab}$ and $\mathcal{C} \equiv R_{abcd}\,R^{abcd}$, $\tilde{\mathcal{C}} \equiv \tilde{R}_{abcd}\,R^{abcd} = \varepsilon_{ab}{}^{pq}\,R_{pqcd}\,R^{abcd}$. In eq.(3.1) the $\eta, \lambda, \tilde{\lambda}$ are considered to be small dimensionless constants that multiply the higher order terms. These corrections are considered to be independent and of $\mathcal{O}(\varepsilon)$. The form of the eq.(3.1) clearly states that this is an EFT as the first term is the classical GR Lagrangian that leads to the Einstein-Hilbert action; plus some higher order corrections that are expected to be non-negligible.

In this work we shall not explore why this particular corrections are the proper ones that should be added, we will simply state that when attempting to write such a theory the terms added should respect a number of conditions and symmetries. The equations of motion that follow from the above Lagrangian -which can be obtained using the stationary action principle- are (see p. 1 Eq. 2, 3 of [1]):

$$R_{ab} - \frac{1}{2}Rg_{ab} = T^{\text{cubic}}_{ab} + T^{\text{quartic}}_{ab} \tag{3.2}$$

with

$$T^{\text{cubic}}_{ab} = \eta\,\kappa^4\left[3R_a{}^{cde}R_{de}{}^{gh}R_{ghcb} + \frac{1}{2}g_{ab}R_{gh}{}^{cd}R_{cd}{}^{ef}R_{ef}{}^{gh} - 6\nabla^c\nabla^d\Big(R_{acgh}R_{bd}{}^{gh}\Big)\right] \tag{3.3}$$

$$T^{\text{quartic}}_{ab} = -\lambda\,\kappa^6\left(8R_{acbd}\nabla^c\nabla^d\mathcal{C} + \frac{1}{2}g_{ab}\mathcal{C}^2\right) - \tilde{\lambda}\,\kappa^6\left(8\tilde{R}_{acbd}\nabla^c\nabla^d\tilde{\mathcal{C}} + \frac{1}{2}g_{ab}\tilde{\mathcal{C}}^2\right) \tag{3.4}$$

Naturally, as we are discussing perturbations in our Lagrangian, background solutions satisfy the vacuum Einstein equations $R_{ab} = 0$. The notion itself of a metric $g_{\mu\nu}$ that contains EFT corrections with respect to η, λ, $\tilde{\lambda}$ leads to formulating it in the following form:

$$g^{\text{EFT}}_{\mu\nu} = g^{(0)}_{\mu\nu} + \eta\,h^{(6)}_{\mu\nu} + \lambda\,h^{(8)}_{\mu\nu} + \tilde{\lambda}\,\tilde{h}^{(8)}_{\mu\nu} \tag{3.5}$$

where the corrections solve eq.(3.2)-(3.4) to linear order.

3.3.2 EFT-corrected NHEK

Given the above context, [1] attempted to correct the NHEK geometry within this particular Effective Field Theory. To begin with, they proposed the most general metric which respects the symmetries satisfied by the NHEK metric (see §2.3). Imposing the SL(2)×U(1) symmetry they wrote down the following ansatz for the EFT-corrected line element of NHEK:

$$ds^2_{\text{EFT}} = 2J\Omega^2_{\text{NH}}(\theta)\left[-r^2dt^2 + \frac{dr^2}{r^2} + \Gamma^2_{\text{NH}}d\theta^2 + B^2_{\text{NH}}(\theta)\Big(d\varphi + r\omega_{\text{NH}}dt\Big)^2\right] \tag{3.6}$$

where J is the Kerr angular momentum and as eq.(3.6) is an ansatz for EFT-corrected NHEK reduces to $J = M^2$. The functions $\Omega^2_{\rm NH}(\theta)$, $B^2_{\rm NH}(\theta)$ are anticipated to arise due to the structure of the background metric (see eq.(2.10)). However, the restriction of the SL(2)×U(1) symmetry permits the existence of two more constants $\Gamma_{\rm NH}$, $\omega_{\rm NH}$. By observing both eq.(3.5) and eq.(3.6) it becomes apparent that EFT corrections should exist within the quantities $\Omega^2_{\rm NH}(\theta)$, $B^2_{\rm NH}(\theta)$, $\Gamma^2_{\rm NH}$, $\omega_{\rm NH}$. The previous statement leads to the following expansions (see p. 2 Eq. 7 of [1]):

$$\Omega_{\rm NH}(\theta) = \Omega^{(0)}(\theta)\Big[1 + \eta\,\Omega^{(6)}(\theta) + \lambda\,\Omega^{(8)}(\theta) + \tilde{\lambda}\,\tilde{\Omega}^{(8)}(\theta)\Big] \tag{3.7}$$

$$B_{\rm NH}(\theta) = B^{(0)}(\theta)\Big[1 + \eta\,B^{(6)}(\theta) + \lambda\,B^{(8)}(\theta) + \tilde{\lambda}\,\tilde{B}^{(8)}(\theta)\Big] \tag{3.8}$$

$$\Gamma_{\rm NH} = \Gamma^{(0)}\Big[1 + \eta\,\Gamma^{(6)} + \lambda\,\Gamma^{(8)} + \tilde{\lambda}\,\tilde{\Gamma}^{(8)}\Big] \tag{3.9}$$

$$\omega_{\rm NH} = \omega^{(0)}\Big[1 + \eta\,\omega^{(6)} + \lambda\,\omega^{(8)} + \tilde{\lambda}\,\tilde{\omega}^{(8)}\Big] \tag{3.10}$$

Let us take a brief moment to appreciate the form of the leading expansions; by a quick look at the Lagrangian that defines the Effective Field Theory it is straightforward to see that if $\eta = \lambda = \tilde{\lambda} = 0$ then the EFT reduces to classical General Relativity. That very property should extend to both the EFT-corrected Einstein equations -and it does- and the expansions provided above. The latter condition leads to the consecutive remark:

$$\Omega^{(0)}(\theta) = \sqrt{\frac{1+\cos^2\theta}{2}}, \quad B^{(0)}(\theta) = \frac{2\sin\theta}{1+\cos^2\theta}, \quad \Gamma^{(0)} = \omega^{(0)} = 1 \tag{3.11}$$

which now ensures that when $\eta = \lambda = \tilde{\lambda} = 0$ eq.(3.6) reduces to the NHEK metric in classical GR. The next step, of course, would be to determine the higher order corrections that come up in equations eq.(3.7)-(3.10). This calculation was performed by [1] and some of their results are provided below:

$$\Gamma^{(6)} = -\frac{15\kappa^4}{32\sqrt{2}J^2}, \quad \Gamma^{(8)} = -\frac{366435\kappa^6}{256\sqrt{2}J^3}, \quad \tilde{\Gamma}^{(8)} = -\frac{368829\kappa^6}{64\sqrt{2}J^3} \tag{3.12}$$

$$\omega^{(6)} = \frac{\kappa^4}{7J^2}, \quad \omega^{(8)} = \frac{(4864+1575\pi)\kappa^6}{20J^3}, \quad \tilde{\omega}^{(8)} = \frac{(4736+1575\pi)\kappa^6}{5J^3} \tag{3.13}$$

The explicit expressions for the EFT corrections concerning the $\Omega_{\rm NH}(\theta)$ and $B_{\rm NH}(\theta)$ are displayed in Appendix C (see §Sec 6). Taking into account eq.(3.7)-eq.(3.13) plus the expressions given in Appendix C one has formulated the complete EFT-corrected NHEK spacetime. Let's us start exploring some of its concepts.

3.4 Verification of the EFT-corrections

We would like to press on to our next topic of interest. In this subsection we direct our concentration in the EFT-corrections themselves (§Appendix C), provided by [1]. Our next goal, of course, would be to verify that these corrections do indeed satisfy the Einstein EFT-corrected equation (i.e. eq.(3.2)-eq.(3.4)). The form of these equations indicates that this task is not trivial and thus we would like to explain the way that we intent to tackle this task.

We begin by trying to find the optimal way to simplify our problem, because both our equations and the corrections are too complicated. First thing that comes to mind is that our Effective Field Theory Lagrangian contains three independent perturbations with book-keeping parameters η, λ, $\tilde{\lambda}$ and attempting to verify all of them at once (i.e. substitute all EFT-corrections in eq.(3.7)-eq.(3.10) and then substitute in the EFT-corrected Einstein equation) would be extremely computationally expensive and unwise; it only makes sense to attempt the verification separately for each set of perturbations. One thing that needs caution in this strategy is that when dealing with a particular set of perturbations we should write the Einstein equation with its corresponding stress-energy tensor (or part of it).

The piece of software utilized in this particular task is `Mathematica` and more specifically Mathematica's `RGTC` [12] package which due to its built-in functions makes the tensorial calculations quite straight-forward. Let's proceed to present the optimal way to verify the EFT-corrections.

The plan of attack is as follows: Firstly, we introduce the EFT-corrected metric as $g_{\mu\nu}^{\text{EFT}} = g_{\mu\nu}^{\text{EFT}}(\Omega_{\text{NH}}, B_{\text{NH}}, \Gamma_{\text{NH}}, \omega_{\text{NH}})$ (similar to Eq.(6) of [1]). Then, we compute all the necessary quantities for the EFT-corrected Einstein equations ($R_{\mu\nu}$, R, $T_{\mu\nu}^{\text{cubic}}$, $T_{\mu\nu}^{\text{quartic}}$) as a function of the "NH" quantities. Putting together all the previous components yields the Einstein equations. Next, we decide which perturbations we would like to verify, say η-corrections, and thus we make the expansion of all the "NH" quantities with respect to the corrections chosen (just like we did in eq.(3.14) but now we include ω_{NH} as well). At this point, Mathematica cannot distinguish whether the expansion that we made is a perturbative one or not and hence it will store all the terms of all the orders with respect to η that appear. Following that argument, we expand all the Einstein's equations up to $\mathcal{O}(\eta)$, leading to linear equations with respect to η. Finally, we substitute the values of the corrections, as denoted in §Appendix C, and hopefully verify that the Einstein's equations are satisfied (see §Appendix D).

After repeating the above process for each perturbation -η, λ, $\tilde{\lambda}$- we obtain our results. Indeed, we were able to verify that the η and λ-corrections provided by [1] do satisfy the EFT-corrected Einstein equations. However, we believe that we have found a typo in the $\tilde{\lambda}$-corrections in the Supplementary material VA of [1]. More specifically, the typo is located on the $\tilde{\Omega}^{(8)}(x)$ function; term multiplied by $K(x)$ should have a $(+)$ sign, not a $(-)$, this correction can also be seen in §Appendix C.

3.5 Absence of conical singularity in the EFT-corrected NHEK

The next notion that we are interested in the EFT-corrected NHEK spacetime is its behaviour near the poles $\theta = 0, \pi$. As explained in §2.4, from the investigation of this limit will emerge whether there exists a conical singularity (see §2.4.1) in the spacetime itself which could lead to curvature divergence. We presented that the proper way to answering this question is to examine if our metric reduces to the same limit as S^2 does when $\theta \ll 1$ (see eq.(2.35)).

Before we engage in this calculation we remind ourselves that in §2.4.2 we showed that the NHEK spacetime in GR does not present with a conical singularity and thus we expect that the background metric $g_{\mu\nu}^{(0)}$ will not illustrate such a singularity. Therefore, our main aim is to determine whether the EFT-corrections introduce one in the overall spacetime. We shall treat each perturbation (i.e. η, λ, $\tilde{\lambda}$) separately, we begin from the η perturbations.

3.5.1 EFT η-corrections

We write our perturbations with respect to η as:

$$\Omega_{\text{NH}}(\theta) = \Omega^{(0)}(\theta)\Big[1 + \eta\, \Omega^{(6)}(\theta)\Big], \quad B_{\text{NH}}(\theta) = B^{(0)}(\theta)\Big[1 + \eta\, B^{(6)}(\theta)\Big], \quad \Gamma_{\text{NH}} = \Gamma^{(0)}\Big[1 + \eta\, \Gamma^{(6)}\Big] \tag{3.14}$$

However, we do not need worry about the corrections on ω_{NH} because they are not engaged in the θ or φ components of the metric. We kick off by writing the EFT-corrected metric as displayed in eq.(3.6) and then substitute eq.(3.14) in it:

$$ds^2_{\text{EFT}} = 2J\Big(\Omega^{(0)} + \eta\,\Omega^{(0)}\Omega^{(6)}\Big)^2\Big[-r^2dt^2 + \frac{dr^2}{r^2} + \Big(\Gamma^{(0)} + \eta\Gamma^{(0)}\Gamma^{(6)}\Big)^2 d\theta^2 + \Big(B^{(0)} + \eta B^{(0)}B^{(6)}\Big)^2\Big(d\varphi + \omega_{\text{NH}}r dt\Big)^2\Big] \tag{3.15}$$

Now, we linearize the corrections up to $\mathcal{O}(\eta)$ as we remind ourselves that these are perturbations and we are interested in keep up to linear order:

$$ds^2_{\text{EFT}} = 2J\Big(\Omega^{(0)\,2} + 2\eta\,\Omega^{(0)\,2}\Omega^{(6)}\Big)\Big[-r^2dt^2 + \frac{dr^2}{r^2} + \Big(\Gamma^{(0)\,2} + 2\eta\Gamma^{(0)\,2}\Gamma^{(6)}\Big)d\theta^2 + \Big(B^{(0)\,2} + 2\eta B^{(0)\,2}B^{(6)}\Big)\Big(d\varphi + \omega_{\text{NH}}r dt\Big)^2\Big] \tag{3.16}$$

The next step is to massage eq.(3.16) in order to bring it in a more convenient form:

$$\begin{aligned} ds^2_{\text{EFT}} = 2J\Omega^{(0)\,2}\Big[-r^2dt^2 \quad &+ \frac{dr^2}{r^2} + \Big(\Gamma^{(0)\,2} + 2\eta\Gamma^{(0)\,2}\Gamma^{(6)}\Big)d\theta^2 \\ &+ \Big(B^{(0)\,2} + 2\eta B^{(0)\,2}B^{(6)}\Big)\Big(d\varphi + \omega_{\text{NH}}r dt\Big)^2\Big] \\ + 4J\eta\Omega^{(0)\,2}\Omega^{(6)}\Big[-r^2dt^2 &+ \frac{dr^2}{r^2} + \Big(\Gamma^{(0)\,2} + \cancelto{0}{2\eta\Gamma^{(0)\,2}\Gamma^{(6)}}\Big)d\theta^2 \\ &+ \Big(B^{(0)\,2} + \cancelto{0}{2\eta B^{(0)\,2}B^{(6)}}\Big)\Big(d\varphi + \omega_{\text{NH}}r dt\Big)^2\Big] \end{aligned} \tag{3.17}$$

where the two terms are canceled because they will be of order $\mathcal{O}(\eta^2)$ when multiplied by the exterior product. Following we would like to separate the term of zeroth-order and of order η.

$$\begin{aligned} ds^2_{\text{EFT}} &= 2J\Omega^{(0)\,2}\Big[-r^2dt^2 + \frac{dr^2}{r^2} + \Gamma^{(0)\,2}d\theta^2 + B^{(0)\,2}\Big(d\varphi + \omega_{\text{NH}}r dt\Big)^2\Big] \\ &+ 4J\eta\Big\{\Omega^{(0)\,2}\,\Gamma^{(0)\,2}\,\Gamma^{(6)}d\theta^2 + \Omega^{(0)\,2}\,B^{(0)\,2}\,B^{(6)}\Big(d\varphi + \omega_{\text{NH}}r dt\Big)^2 \\ &+ \Omega^{(0)\,2}\,\Omega^{(6)}\Big[-r^2dt^2 + \frac{dr^2}{r^2} + \Gamma^{(0)\,2}d\theta^2 + B^{(0)\,2}\Big(d\varphi + \omega_{\text{NH}}r dt\Big)^2\Big]\Big\} \end{aligned} \tag{3.18}$$

There is one more step to take, to gather all the terms of order $\mathcal{O}(\eta)$ that correspond to $d\theta^2$ and $(d\varphi + \omega_{\text{NH}}r dt)^2$. But before we proceed to it, let's observe the first term of eq.(3.18); clearly this term is -almost- of zeroth order. Its angular components (i.e. $d\varphi, d\theta$) are multiplied by the $\Gamma^{(0)}$, $B^{(0)}$ functions of the background NHEK metric and thus implying that when taking the limit $\theta \ll 1$ no conical singularity will be raised by this term. This argument leads us to channel our focus to the second term of eq.(3.18). Taking the final step we obtain:

$$\begin{aligned} ds^2_{\text{EFT}} &= 2J\Omega^{(0)\,2}\Big[-r^2dt^2 + \frac{dr^2}{r^2} + \Gamma^{(0)\,2}d\theta^2 + B^{(0)\,2}\Big(d\varphi + \omega_{\text{NH}}r dt\Big)^2\Big] \\ &+ 4J\eta\Big\{\Big(\Omega^{(0)\,2}\,\Gamma^{(0)\,2}\,\Gamma^{(6)} + \Omega^{(0)\,2}\,\Omega^{(6)}\Gamma^{(0)\,2}\Big)d\theta^2 + \Omega^{(0)\,2}\,\Omega^{(6)}\Big[-r^2dt^2 + \frac{dr^2}{r^2}\Big] \\ &+ \Big(\Omega^{(0)\,2}\,B^{(0)\,2}\,B^{(6)} + \Omega^{(0)\,2}\,\Omega^{(6)}B^{(0)\,2}\Big)\Big(d\varphi + \omega_{\text{NH}}r dt\Big)^2\Big\} \end{aligned} \tag{3.19}$$

From the last equation it becomes apparent that the question of whether these perturbations introduce a conical singularity in the spacetime comes down to how the following two terms behave near $\theta \ll 1$:

$$\Big(\Omega^{(0)\,2}\,\Gamma^{(0)\,2}\,\Gamma^{(6)} + \Omega^{(0)\,2}\,\Omega^{(6)}\Gamma^{(0)\,2}\Big)d\theta^2 + \Big(\Omega^{(0)\,2}\,B^{(0)\,2}\,B^{(6)} + \Omega^{(0)\,2}\,\Omega^{(6)}B^{(0)\,2}\Big)d\varphi^2 \tag{3.20}$$

Finally, we can expand the above expressions to their dominant behaviour with respect to θ and determine whether they reduce to the same limit as S^2 metric does when $\theta \ll 1$ (see §Appendix C). Using Wolfram Mathematica we expand them and achieve the following result:

$$(3.20) \xrightarrow[\theta \ll 1]{\text{dominant term}} c_1 d\theta^2 + c_1 \theta^2 d\varphi^2 = c_1(d\theta^2 + \theta^2 d\varphi^2) \tag{3.21}$$

where:

$$c_1 = -\frac{(-214 + 105\sqrt{2} - 448C^{(6)})\kappa^4}{112J} \tag{3.22}$$

Clearly, from eq.(3.21) one can see that the η-corrections introduced by the EFT converge to the same limit as S^2 multiplied by a constant c_1, thus these corrections do not raise any conical singularity.

3.5.2 EFT λ-corrections

We move on to the next EFT corrections, those of λ. We want once more to check if these corrections give rise to a conical singularity. In order to do so we shall repeat the process presented in §3.4.1. We write down the corrections with respect to λ:

$$\Omega_{\rm NH}(\theta) = \Omega^{(0)}(\theta)\Big[1 + \lambda\, \Omega^{(8)}(\theta)\Big], \quad B_{\rm NH}(\theta) = B^{(0)}(\theta)\Big[1 + \lambda\, B^{(8)}(\theta)\Big], \quad \Gamma_{\rm NH} = \Gamma^{(0)}\Big[1 + \lambda\, \Gamma^{(8)}\Big] \tag{3.23}$$

As before the coming step is to substitute eq.(3.23) into eq.(3.6) and then linearize. However, one can easily observe that the perturbations displayed in eq.(3.23) are of the same form as those of eq.(3.14) suggesting that the final result will be identical to eq.(3.19) with the substitutions $\eta \to \lambda$ and the superscript $(6) \to (8)$ on the corresponding functions. We write down the final result:

$$\begin{aligned} ds^2_{\rm EFT} &= 2J\Omega^{(0)\,2}\Big[-r^2dt^2 + \frac{dr^2}{r^2} + \Gamma^{(0)\,2}d\theta^2 + B^{(0)\,2}\Big(d\varphi + \omega_{\rm NH} r dt\Big)^2\Big] \\ &+ 4J\lambda\Big\{\Big(\Omega^{(0)\,2}\,\Gamma^{(0)\,2}\,\Gamma^{(8)} + \Omega^{(0)\,2}\,\Omega^{(8)}\Gamma^{(0)\,2}\Big)d\theta^2 + \Omega^{(0)\,2}\,\Omega^{(8)}\Big[-r^2dt^2 + \frac{dr^2}{r^2}\Big] \\ &+ \Big(\Omega^{(0)\,2}\,B^{(0)\,2}\,B^{(8)} + \Omega^{(0)\,2}\,\Omega^{(8)}B^{(0)\,2}\Big)\Big(d\varphi + \omega_{\rm NH} r dt\Big)^2\Big\} \end{aligned} \tag{3.24}$$

Once more, the question to whether these corrections result in the emergence of a conical singularity boils down to the behaviour of the two following terms:

$$\Big(\Omega^{(0)\,2}\,\Gamma^{(0)\,2}\,\Gamma^{(8)} + \Omega^{(0)\,2}\,\Omega^{(8)}\Gamma^{(0)\,2}\Big)d\theta^2 + \Big(\Omega^{(0)\,2}\,B^{(0)\,2}\,B^{(8)} + \Omega^{(0)\,2}\,\Omega^{(8)}B^{(0)\,2}\Big)d\varphi^2 \tag{3.25}$$

Now, we take the Taylor expansion these terms with respect to θ up to dominant term (see §Appendix C). Below we present the limit $\theta \ll 1$ of eq.(3.25):

$$(3.25) \xrightarrow[\theta \ll 1]{\text{dominant term}} c_2 d\theta^2 + c_2 \theta^2 d\varphi^2 = c_2(d\theta^2 + \theta^2 d\varphi^2) \tag{3.26}$$

where:

$$c_2 = -\frac{\kappa^6(-2560C^{(8)} + 3(-655678 + 610725\sqrt{2} + 33600\pi))}{640J^2} \tag{3.27}$$

Evidently from eq.(3.26), the λ-corrections introduced by the Effective Field Theory converge to the identical limit as S^2 metric, multiplied by a constant which we denote as c_2. Therefore, these corrections do not give rise to a conical singularity in the EFT-corrected spacetime.

3.5.3 EFT $\tilde{\lambda}$-corrections

So far we have studied the effect of the EFT-corrections with respect to η and λ. At this point, we would like to move on to exploring the effect of the last EFT-introduced corrections; those of $\tilde{\lambda}$. One more time, we progress to writing these perturbations:

$$\Omega_{\rm NH}(\theta) = \Omega^{(0)}(\theta)\Big[1+\tilde{\lambda}\,\tilde{\Omega}^{(8)}(\theta)\Big], \quad B_{\rm NH}(\theta) = B^{(0)}(\theta)\Big[1+\tilde{\lambda}\,\tilde{B}^{(8)}(\theta)\Big], \quad \Gamma_{\rm NH} = \Gamma^{(0)}\Big[1+\tilde{\lambda}\,\tilde{\Gamma}^{(8)}\Big] \tag{3.28}$$

The procedure that should be followed in order to determine if the $\tilde{\lambda}$-corrections lead to a conical singularity is the exact one as demonstrated in §3.4.1 and §3.4.2. Fortunately for us, eq.(3.28) suggests that, once again, these perturbations acquire the same form as those of η and λ, hence we are able to pull off the same trick we used before. That is, claim that $\tilde{\lambda}$-corrected NHEK metric will acquire the form of eq.(3.19) with the substitutions $\eta \to \tilde{\lambda}$, the superscript $(6) \to (8)$ along with the bearing of a tilde in the corresponding functions. We present our final result:

$$\begin{aligned} ds^2_{\rm EFT} &= 2J\Omega^{(0)\,2}\Big[-r^2dt^2 + \frac{dr^2}{r^2} + \Gamma^{(0)\,2}d\theta^2 + B^{(0)\,2}\Big(d\varphi + \omega_{\rm NH} r dt\Big)^2\Big] \\ &+ 4J\tilde{\lambda}\Big\{\Big(\Omega^{(0)\,2}\,\Gamma^{(0)\,2}\,\tilde{\Gamma}^{(8)} + \Omega^{(0)\,2}\,\tilde{\Omega}^{(8)}\Gamma^{(0)\,2}\Big)d\theta^2 + \Omega^{(0)\,2}\,\tilde{\Omega}^{(8)}\Big[-r^2dt^2 + \frac{dr^2}{r^2}\Big] \\ &+ \Big(\Omega^{(0)\,2}\,B^{(0)\,2}\,\tilde{B}^{(8)} + \Omega^{(0)\,2}\,\tilde{\Omega}^{(8)}B^{(0)\,2}\Big)\Big(d\varphi + \omega_{\rm NH} r dt\Big)^2\Big\} \end{aligned} \tag{3.29}$$

Again, as we expected, the question that we have asked considering the conical singularity simplifies to which behaviour that the following terms adopt near $\theta \ll 1$:

$$\Big(\Omega^{(0)\,2}\,\Gamma^{(0)\,2}\,\tilde{\Gamma}^{(8)} + \Omega^{(0)\,2}\,\tilde{\Omega}^{(8)}\Gamma^{(0)\,2}\Big)d\theta^2 + \Big(\Omega^{(0)\,2}\,B^{(0)\,2}\,\tilde{B}^{(8)} + \Omega^{(0)\,2}\,\tilde{\Omega}^{(8)}B^{(0)\,2}\Big)d\varphi^2 \tag{3.30}$$

Finally, we take the Taylor expansion, with respect to θ, of the previous terms (see §Appendix C) and retrieve the subsequent result:

$$(3.30) \xrightarrow[\theta \ll 1]{\text{dominant term}} c_3 d\theta^2 + c_3\theta^2 d\varphi^2 = c_3(d\theta^2 + \theta^2 d\varphi^2) \tag{3.31}$$

where:

$$c_3 = -\frac{\kappa^6(-640\tilde{C}^{(8)} + 3(-663554 + 614715\sqrt{2} + 33600\pi))}{160J^2} \tag{3.32}$$

Surely, the form of eq.(3.31) assures that the $\tilde{\lambda}$-corrections do not lead to introducing a conical singularity as well. We have showed that neither of the EFT introduced corrections nor the NHEK bakground (see §2.4.2) give rise to such a singularity. Therefore, we can be certain that the EFT-corrected NHEK spacetime does not present any conical singularity.

It's worth making a note considering the number of arbitrary constants in the solutions presented in §Appendix C. The differential equations yielded from the EFT-corrected Einstein equations with respect to the perturbations of $\Omega^{(i)}(\theta)$, $B^{(i)}(\theta)$ are second order ones, thus one would expect that the general solutions would contain two arbitrary constants. However, in §Appendix C the solutions contain only one ($C^{(6)}$, $C^{(8)}$, $\tilde{C}^{(8)}$), the reason for this is that in the work of [1] they demanded the absence of a conical singularity giving up one of their arbitrary constants in order to ensure that.

3.6 The wave equation on the EFT-corrected NHEK background

At this stage, we would like to press on and explore how the wave equation eq.(2.37) behaves on the EFT-corrected NHEK background. The procedure followed in this subsection is identical to the one presented in §2.5, where we studied how this partial differential equation is solved in General Relativity for the NHEK background. Once more, our ansatz is, the same as §2.5:

$$\phi = e^{-i\omega t + im\varphi} R(r)\, S(\theta) \tag{3.33}$$

where $m \in \mathbb{Z}$. The main difference is that now our background is changed and therefore we shall use EFT-corrected NHEK metric of eq.(3.6) with $J = M^2$. Before we start tackling our equation of interest we find convenient to define a covariant vector $\xi_\nu \equiv \nabla_\nu \phi$ with components:

$$\xi_\nu = \left(-i\omega e^{-i\omega t + im\varphi} R(r)\, S(\theta), e^{-i\omega t + im\varphi} \frac{dR(r)}{dr} S(\theta), e^{-i\omega t + im\varphi} R(r) \frac{dS(\theta)}{d\theta}, ime^{-i\omega t + im\varphi} R(r)\, S(\theta)\right) \tag{3.34}$$

By rewritting the wave equation one obtains:

$$g^{\mu\nu} \nabla_\mu \xi_\nu = 0 \tag{3.35}$$

The only non-vanishing metric components yield:

$$g^{00}\nabla_0\xi_0 + g^{11}\nabla_1\xi_1 + g^{22}\nabla_2\xi_2 + g^{33}\nabla_3\xi_3 + 2g^{30}\nabla_3\xi_0 = 0 \tag{3.36}$$

We expand the covariant derivatives of the last equation and keep only the non-vanishing Christoffel symbols:

$$\begin{aligned} g^{00}\left(\partial_0\xi_0 - \Gamma^1_{00}\,\xi_1 - \Gamma^2_{00}\,\xi_2\right) + g^{11}\left(\partial_1\xi_1 - \Gamma^1_{11}\,\xi_1 - \Gamma^2_{11}\,\xi_2\right) + g^{22}\left(\partial_2\xi_2 - \Gamma^2_{22}\,\xi_2\right) \\ + g^{33}\left(\partial_3\xi_3 - \Gamma^2_{33}\,\xi_2\right) + 2g^{30}\left(\partial_0\xi_3 - \Gamma^2_{30}\,\xi_2 - \Gamma^1_{30}\,\xi_1\right) = 0 \end{aligned} \tag{3.37}$$

We observe that all the terms of this equation are multiplied by $\exp(-i\omega t + im\varphi)$ and therefore we can eliminate it. By expanding the Christoffels, the metric and the covariant vector components one acquires the following equation:

$$\begin{aligned} \frac{r^2}{\Omega^2_{\rm NH}} R'' S + \frac{1}{\Gamma^2_{\rm NH}\Omega^2_{\rm NH}} R\ddot{S} + \frac{2r}{\Omega^2_{\rm NH}} R' S + \left[2\dot{\Omega}_{\rm NH} + \frac{\dot{B}_{\rm NH}}{B_{\rm NH}}\Omega_{\rm NH}\right] \frac{R\dot{S}}{\Gamma^2_{\rm NH}\Omega^3_{\rm NH}} \\ + \left[\frac{\omega^2}{r^2} + \frac{m^2(\omega^2_{\rm NH}B^2_{\rm NH} - 1)}{B^2_{\rm NH}} + \frac{2m\omega\,\omega_{\rm NH}}{r}\right] \frac{RS}{\Omega^2_{\rm NH}} = 0 \end{aligned} \tag{3.38}$$

where the prime and dot denote the derivatives with respect to r and θ respectively. Now, we multiply eq.(3.38) by $\times\, \Omega^2_{\rm NH}/RS$ and obtain the upcoming equation:

$$\begin{aligned} r^2\frac{R''}{R} + \frac{1}{\Gamma^2_{\rm NH}}\frac{\ddot{S}}{S} + 2r\frac{R'}{R} + \left[2\dot{\Omega}_{\rm NH} + \frac{\dot{B}_{\rm NH}}{B_{\rm NH}}\Omega_{\rm NH}\right] \frac{\dot{S}}{\Gamma^2_{\rm NH}\Omega_{\rm NH} S} \\ + \frac{\omega^2}{r^2} + \frac{m^2(\omega^2_{\rm NH}B^2_{\rm NH} - 1)}{B^2_{\rm NH}} + \frac{2m\omega\,\omega_{\rm NH}}{r} = 0 \end{aligned} \tag{3.39}$$

Clearly, the above equation has adopted a form that implies that our partial differential equation is separable. Observe that each term of this equation is either dependent of r or of θ.

Separating the r-dependent components in the LHS and the θ-dependent components in the RHS, and then setting it equal to a separation constant $K_{\rm NH} \in \mathbb{C}$:

$$r^2\frac{R''}{R}+2r\frac{R'}{R}+\frac{\omega^2}{r^2}+\frac{2m\omega\,\omega_{\rm NH}}{r} = -\frac{1}{\Gamma_{\rm NH}^2}\frac{\ddot{S}}{S}-\left[2\frac{\dot{\Omega}_{\rm NH}}{\Omega_{\rm NH}}+\frac{\dot{B}_{\rm NH}}{B_{\rm NH}}\right]\frac{\dot{S}}{\Gamma_{\rm NH}^2 S}-\frac{m^2(\omega_{\rm NH}^2B_{\rm NH}^2-1)}{B_{\rm NH}^2} \equiv -K_{\rm NH} \tag{3.40}$$

So the two ordinary differential equations for the radial and angular part of the scalar are the proceeding:

$$r^2R''+2rR'+\left(\frac{\omega^2}{r^2}+\frac{2m\omega\,\omega_{\rm NH}}{r}+K_{\rm NH}\right)R=0 \tag{3.41}$$

$$\ddot{S}+\left[2\frac{\dot{\Omega}_{\rm NH}}{\Omega_{\rm NH}}+\frac{\dot{B}_{\rm NH}}{B_{\rm NH}}\right]\dot{S}+\left[\frac{m^2(\omega_{\rm NH}^2B_{\rm NH}^2-1)}{B_{\rm NH}^2}-K_{\rm NH}\right]\Gamma_{\rm NH}^2 S=0 \tag{3.42}$$

Obviously, the last two equations are somewhat similar to these of the NHEK background. This is to be expected as the EFT-corrected NHEK spacetime consists of the NHEK spacetime with some perturbations. However, the similarity of these equations with respect to those of NHEK does not guarantee that they are solvable. Before we commence into discussing them in more detail we would like to make the following assumption. Due to the fact that these equations contain corrections with respect to η, λ, $\tilde{\lambda}$ one would expect that the functions $R(r)$, $S(\theta)$ would also be expansions to the corrections; for convenience we denoted them with subscript "NH". Therefore we make the subsequent ansatz:

$$R_{\rm NH}(r) = R^{(0)}[1+\eta\,R^{(6)}+\lambda\,R^{(8)}+\tilde{\lambda}\,\tilde{R}^{(8)}] \tag{3.43}$$

$$S_{\rm NH}(\theta) = S^{(0)}[1+\eta\,S^{(6)}+\lambda\,S^{(8)}+\tilde{\lambda}\,\tilde{S}^{(8)}] \tag{3.44}$$

$$K_{\rm NH} = K^{(0)}[1+\eta\,K^{(6)}+\lambda\,K^{(8)}+\tilde{\lambda}\,\tilde{K}^{(8)}] \tag{3.45}$$

The above ansatz means that we expect that the perturbations introduced in the Lagrangian to result to perturbed solutions in both the angular and the radial Sturm-Liouville problem.

3.6.1 The Angular Equation

Let's press on to review the EFT-corrected angular equation eq.(3.42). To determine the complexity of this ODE, one needs to apply perturbations theory. However, once again the idea of substituting all the perturbations (i.e. with respect to η, λ, $\tilde{\lambda}$) in it would lead to a very complicated problem. The trick for simplifying our problem is to recall that all of our perturbations are independent and of order $\mathcal{O}(\varepsilon)$ and therefore all the mixed terms (i.e. $\eta\,\lambda$, $\lambda\,\tilde{\lambda}$ etc are considered to be of order $\mathcal{O}(\varepsilon^2)$ and hence neglected). The latter argument leads to the fact that it suffices to substitute only one perturbation into the angular equation and find the correction of the ODE to first order. The other perturbations will have the exact same form with the respective substitutions (for instance $\eta \to \lambda$, $S^{(6)} \to S^{(8)}$ etc).

Before we proceed the perturbation theory, let's take a step back to set up the Sturm-Liouville problem properly. We have to specify the boundary conditions of it. For simplicity we choose to adopt the same boundary conditions as in the background (§2.5.1) which are:

$$S_{\rm NH}(0) = \text{finite}, \quad S_{\rm NH}(\pi) = \text{finite} \tag{3.46}$$

Now, we proceed to applying the perturbation theory in eq.(3.42) with respect to the η-corrections; meaning that we expand all the quantities ($\Omega_{\rm NH}$, $B_{\rm NH}$, $\Gamma_{\rm NH}$, $\omega_{\rm NH}$, $K_{\rm NH}$, $S_{\rm NH}$) up to that correction. Due to the considerable number of corrected quantities the perturbation theory is a bit involved and therefore we shall just present the result:

$$\ddot{S}^{(0)} + \left[\frac{\dot{B}^{(0)}}{B^{(0)}} + 2\frac{\dot{\Omega}^{(0)}}{\Omega^{(0)}}\right]\dot{S}^{(0)} + \left[\frac{m^2(\omega^{(0)\,2}B^{(0)\,2}-1)}{B^{(0)\,2}} - K^{(0)}\right]\Gamma^{(0)\,2}S^{(0)}$$

$$+ \Bigg\{\Gamma^{(0)\,2}S^{(0)}\Bigg[-K^{(0)}K^{(6)} + 2\Gamma^{(6)}\left(\frac{m^2(\omega^{(0)\,2}B^{(0)\,2}-1)}{B^{(0)\,2}} - K^{(0)}\right) + \frac{2m^2(\omega^{(0)\,2}\omega^{(6)}B^{(0)\,2}+B^{(6)})}{B^{(0)\,2}}$$

$$+ \left(\frac{m^2(\omega^{(0)\,2}B^{(0)\,2}-1)}{B^{(0)\,2}} - K^{(0)}\right)S^{(6)}\Bigg] + 2\dot{S}^{(0)}\dot{S}^{(6)} + (S^{(6)}\dot{S}^{(0)} + S^{(0)}\dot{S}^{(6)})\left[\frac{\dot{B}^{(0)}}{B^{(0)}} + 2\frac{\dot{\Omega}^{(0)}}{\Omega^{(0)}}\right]$$

$$+ \dot{S}^{(0)}(\dot{B}^{(6)} + 2\dot{\Omega}^{(6)}) + S^{(6)}\ddot{S}^{(0)} + S^{(0)}\ddot{S}^{(6)}\Bigg\}\eta + \mathcal{O}(\eta^2) = 0 \tag{3.47}$$

Based on the same method, one can obtain the ODE corrections with respect to λ and $\tilde{\lambda}$. The correction with respect to λ is:

$$\Bigg\{\Gamma^{(0)\,2}S^{(0)}\Bigg[-K^{(0)}K^{(8)} + 2\Gamma^{(8)}\left(\frac{m^2(\omega^{(0)\,2}B^{(0)\,2}-1)}{B^{(0)\,2}} - K^{(0)}\right) + \frac{2m^2(\omega^{(0)\,2}\omega^{(8)}B^{(0)\,2}+B^{(8)})}{B^{(0)\,2}}$$

$$+ \left(\frac{m^2(\omega^{(0)\,2}B^{(0)\,2}-1)}{B^{(0)\,2}} - K^{(0)}\right)S^{(8)}\Bigg] + 2\dot{S}^{(0)}\dot{S}^{(8)} + (S^{(8)}\dot{S}^{(0)} + S^{(0)}\dot{S}^{(8)})\left[\frac{\dot{B}^{(0)}}{B^{(0)}} + 2\frac{\dot{\Omega}^{(0)}}{\Omega^{(0)}}\right]$$

$$+ \dot{S}^{(0)}(\dot{B}^{(8)} + 2\dot{\Omega}^{(8)}) + S^{(8)}\ddot{S}^{(0)} + S^{(0)}\ddot{S}^{(8)}\Bigg\}\lambda + \mathcal{O}(\lambda^2) = 0 \tag{3.48}$$

While the $\tilde{\lambda}$-correction of the ODE is:

$$\Bigg\{\Gamma^{(0)\,2}S^{(0)}\Bigg[-K^{(0)}\tilde{K}^{(8)} + 2\tilde{\Gamma}^{(8)}\left(\frac{m^2(\omega^{(0)\,2}B^{(0)\,2}-1)}{B^{(0)\,2}} - K^{(0)}\right) + \frac{2m^2(\omega^{(0)\,2}\tilde{\omega}^{(8)}B^{(0)\,2}+\tilde{B}^{(8)})}{B^{(0)\,2}}$$

$$+ \left(\frac{m^2(\omega^{(0)\,2}B^{(0)\,2}-1)}{B^{(0)\,2}} - K^{(0)}\right)\tilde{S}^{(8)}\Bigg] + 2\dot{S}^{(0)}\dot{\tilde{S}}^{(8)} + (\tilde{S}^{(8)}\dot{S}^{(0)} + S^{(0)}\dot{\tilde{S}}^{(8)})\left[\frac{\dot{B}^{(0)}}{B^{(0)}} + 2\frac{\dot{\Omega}^{(0)}}{\Omega^{(0)}}\right]$$

$$+ \dot{S}^{(0)}(\dot{\tilde{B}}^{(8)} + 2\dot{\tilde{\Omega}}^{(8)}) + \tilde{S}^{(8)}\ddot{S}^{(0)} + S^{(0)}\ddot{\tilde{S}}^{(8)}\Bigg\}\tilde{\lambda} + \mathcal{O}(\tilde{\lambda}^2) = 0 \tag{3.49}$$

At this instant, it becomes apparent that the ODEs yielded from the EFT-corrections are highly non-trivial to be solved. Naturally, the fact that we know the solution of the zeroth order ODE (i.e. that of the background) is helpful. However, the sole existence of the B, Ω corrections complicates these ODEs dearly. Moreover these equations are inhomogeneous. At this point, we reckon that the most promising method of tackling these equations would be numerically.

3.6.2 The Radial Equation

Let's push through to analyze the EFT-corrected radial equation eq.(3.41). A mere inspection of this equation reveals that it contains a relatively small number of corrected components. Only the solution $R_{\rm NH}(r)$, the constant $\omega_{\rm NH}$ and the eigenvalues $K_{\rm NH}$ produce corrections and therefore one would anticipate that the ODEs yielded to first order will adopt a more straightforward form. Once again, we are going to apply perturbation theory and exploit the same trick as in the angular equation. We shall acquire the correction with respect to η and expect those of λ, $\tilde{\lambda}$ to obtain the same form.

Moreover, for completeness let's consider the same boundary conditions as before. We consider the scattering problem described in §2.5.2; that is:

Boundary Condition: *ingoing* wave at $r = 0$ (horizon) $\Rightarrow$ *ingoing* + *outgoing* wave at $r = \infty$

Now, let's move on to apply the perturbation theory on the EFT-corrected radial equation. In the same context as before, we shall not demonstrate the whole process but only the desired outcome.

$$\begin{aligned} r^2 R^{(0)\prime\prime} + 2r R^{(0)\prime} + \left(\frac{\omega^2}{r^2} + K^{(0)} + \frac{2m\omega\,\omega^{(0)}}{r}\right) R^{(0)} + \Bigg\{ R^{(0)} \bigg[K^{(0)} K^{(6)} + \frac{2m\omega\,\omega^{(0)}\omega^{(6)}}{r} \\ + \left(\frac{\omega^2}{r^2} + K^{(0)} + \frac{2m\omega\,\omega^{(0)}}{r}\right) R^{(6)} \bigg] + 2r(R^{(6)} R^{(0)\prime} + R^{(0)} R^{(6)\prime}) \\ + r^2 (2 R^{(0)\prime} R^{(6)\prime} + R^{(6)} R^{(0)\prime\prime} + R^{(0)} R^{(6)\prime\prime}) \Bigg\} \eta + \mathcal{O}(\eta^2) = 0 \end{aligned} \tag{3.50}$$

Based on the same method, one can obtain the ODE corrections with respect to λ and $\tilde{\lambda}$. The correction with respect to λ is:

$$\begin{aligned} \Bigg\{ R^{(0)} \bigg[K^{(0)} K^{(8)} + \frac{2m\omega\,\omega^{(0)}\omega^{(8)}}{r} + \left(\frac{\omega^2}{r^2} + K^{(0)} + \frac{2m\omega\,\omega^{(0)}}{r}\right) R^{(8)} \bigg] + 2r(R^{(8)} R^{(0)\prime} + R^{(0)} R^{(8)\prime}) \\ + r^2 (2 R^{(0)\prime} R^{(8)\prime} + R^{(8)} R^{(0)\prime\prime} + R^{(0)} R^{(8)\prime\prime}) \Bigg\} \lambda + \mathcal{O}(\lambda^2) = 0 \end{aligned} \tag{3.51}$$

While the $\tilde{\lambda}$-correction of the ODE is:

$$\begin{aligned} \Bigg\{ R^{(0)} \bigg[K^{(0)} \tilde{K}^{(8)} + \frac{2m\omega\,\omega^{(0)}\tilde{\omega}^{(8)}}{r} + \left(\frac{\omega^2}{r^2} + K^{(0)} + \frac{2m\omega\,\omega^{(0)}}{r}\right) \tilde{R}^{(8)} \bigg] + 2r(\tilde{R}^{(8)} R^{(0)\prime} + R^{(0)} \tilde{R}^{(8)\prime}) \\ + r^2 (2 R^{(0)\prime} \tilde{R}^{(8)\prime} + \tilde{R}^{(8)} R^{(0)\prime\prime} + R^{(0)} \tilde{R}^{(8)\prime\prime}) \Bigg\} \tilde{\lambda} + \mathcal{O}(\tilde{\lambda}^2) = 0 \end{aligned} \tag{3.52}$$

In comparison to eq.(3.47)-eq.(3.49) the ODEs generated from the EFT-corrections seem to have obtained a simplified form as they are not dependent on the corrections of Ω, B. However, that does not imply that they are solvable since they involve $R^{(0)}$, which is a superposition of the Whittaker functions, and its first and second derivatives. Also, notable is the fact that these ODEs are inhomogeneous as well, which complicates their solution even more. Following these arguments, it becomes clear that the solutions of these equations are highly complicated. One promising way to tackle them would demand the utilization of numerical methods.

Conclusions

This section will be focused on gathering the results discussed in the EFT-corrected NHEK spacetime. Based on the work of [1] we were able to verify each set of independent corrections (η, λ, $\tilde{\lambda}$) with respect to all the perturbed quantities ($\Omega_{\text{NH}}(\theta)$, $B_{\text{NH}}(\theta)$, Γ_{NH}, ω_{NH}) using Mathematica's `RGTC` package [12] (see §Appendix D). Then, we proceeded into verifying the absence of a conical singularity in that spacetime indicating the regularity of the EFT-corrected spacetime. Finally, we were interested in studying the wave equation $\Box\phi = 0$ under the ansatz of eq.(2.38). We demonstrated the separable character of this partial differential equation and derived the angular and radial equation. Afterwards, we employed perturbation theory on these ordinary differential equations and derived their first order corrections due to the perturbed spacetime. Given their highly non-trivial character we concluded that these equations need further investigation and possibly numerical approaches in order to be tackled.

Appendix A

A Angular Prolate Spheroidal Wave Functions

In this appendix we would like to introduce the prolate spheroidal coordinates (ξ, η, φ). Prolate spheroidal coordinates is a 3-dimensional coordinate system that is orthogonal and results from the rotation of the 2-dimensional elliptic coordinate system about the focal axis. The relation to the Cartesian coordinates (x, y, z) is:

$$x = f\xi\eta, \quad y = f\sqrt{(\xi^2-1)(1-\eta^2)}\cos\varphi, \quad z = f\sqrt{(\xi^2-1)(1-\eta^2)}\sin\varphi \tag{A1}$$

The Laplacian operator in prolate spheroidal coordinates is:

$$\vec{\nabla}^2 = \frac{1}{h_\xi h_\eta h_\varphi}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\varphi}{h_\xi}\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_\xi h_\varphi}{h_\eta}\frac{\partial}{\partial\eta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{h_\eta h_\xi}{h_\varphi}\frac{\partial}{\partial\varphi}\right)\right] \tag{A2}$$

where:

$$h_\xi = f\sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}}, \quad h_\eta = f\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}, \quad h_\varphi = f\sqrt{(\xi^2-1)(1-\eta^2)} \tag{A3}$$

Here we have presented the definitions given by [11]. Let us push through to start discussing the Helmholtz equation:

$$(\vec{\nabla}^2 + k^2)\Phi = 0 \tag{A4}$$

which is the equation for determining the eigenvalues of the Laplacian operator. Substituting eq.(A2) into eq.(A4) we get:

$$\frac{\partial}{\partial\xi}\left((\xi^2-1)\frac{\partial\Phi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial\Phi}{\partial\eta}\right) + \frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\frac{\partial^2\Phi}{\partial\varphi^2} + c^2(\xi^2-\eta^2)\Phi = 0 \tag{A5}$$

with $c \equiv fk/2$. The final equation is a partial differential equation and hence our most effective method of solving it is by separation of variables. Therefore we proceed to the following ansatz for the $\Phi(\xi, \eta, \varphi)$ function:

$$\Phi = R_{mn}(\xi; c)\, S_{mn}(\eta; c)\, e^{im\varphi}, \quad m \in \mathbb{Z} \tag{A6}$$

Substituting our ansatz into eq.(A5) one can obtain the succeeding ordinary differential equations:

$$\frac{d}{d\xi}\left[(\xi^2-1)\frac{dR_{mn}}{d\xi}\right] - \left(\lambda_{mn} - c^2\xi^2 + \frac{m^2}{\xi^2-1}\right)R_{mn} = 0 \tag{A7}$$

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dS_{mn}}{d\eta}\right] + \left(\lambda_{mn} - c^2\eta^2 - \frac{m^2}{1-\eta^2}\right)S_{mn} = 0 \tag{A8}$$

where the eigenvalues λ_{mn} can be determined so that both $R_{mn}(\xi; c)$ and $S_{mn}(\eta; c)$ are regular (i.e. finite) at $\xi = \pm 1$ and $\eta = \pm 1$ respectively. Another intriguing thing that can be observed is that eq.(A7) and eq.(A8) are identical.

The solution of these ordinary differential equations is highly non-trivial and thus we shall not dive into presenting it in this particular work. However, we would like to demonstrate the

Taylor expansions around $c = 0$ up to $\mathcal{O}(c^3)$ in order to give an idea of its behavior at small η. We will denote with a superscript the first and second kind of the $S_{mn}(\eta; c)$.

$$S^{(1)}_{mn}(\eta; c) = P^m_n(\eta) + \left[\frac{(-m+n+1)(-m+n+2)P^m_{n+2}(\eta)}{2(2n+1)(2n+3)^2} - \frac{(m+n-1)(m+n)P^m_{n-2}(\eta)}{2(2n-1)^2(2n+1)}\right] c^2 + \mathcal{O}(c^3) \tag{A9}$$

$$S^{(2)}_{mn}(\eta; c) = Q^m_n(\eta) + \left[\frac{(-m+n+1)(-m+n+2)Q^m_{n+2}(\eta)}{2(2n+1)(2n+3)^2} - \frac{(m+n-1)(m+n)Q^m_{n-2}(\eta)}{2(2n-1)^2(2n+1)}\right] c^2 + \mathcal{O}(c^3) \tag{A10}$$

where $P^m_\ell(\eta)$, $Q^m_\ell(\eta)$ are the associated Legendre functions of the first and second kind respectively.

Surely, there is an abundance of information considering these equations and their solutions, such as relations with other functions of mathematical physics (Bessel, Hermite, etc), asymptotic behaviors or even recurrence relations, that are explored in [11]. It is sufficient in this work to remember that eq.(A7) and eq.(A8) are Sturm-Liouville problems with regular solutions and eigenvalues.

Appendix B

B Whittaker Functions

B.1 Definition

In this appendix we would like to briefly provide some further insight on these special functions called Whittaker Functions. The main source of this information is [11]. Let us begin by writing the respective differential equation:

$$\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1/4 - \mu^2}{z^2}\right) W = 0 \tag{B1}$$

which has a regular singularity at $z = 0$ with starting powers $1/2 \pm \mu$ and an irregular singularity at $z = \infty$. The above equation is often referred as "Whittaker's Equation" and it can be obtained by Kummer's (or confluent hypergeometric) equation:

$$z\frac{d^2w}{dz^2} + (b - z)\frac{dw}{dz} - aw = 0 \tag{B2}$$

via the following substitutions:

$$W = e^{-z/2} z^{1/2+\mu} w, \quad \kappa = \frac{b}{2} - a, \quad \mu = \frac{b}{2} - \frac{1}{2} \tag{B3}$$

The last substitutions indicate that the solutions of eq.(B1) are related to the those of eq.(B2). Luckily for us the solutions of eq.(B2) are well-known and regular (at least one of them); they are the confluent hypergeometric functions $M(a; b; z)$, $U(a; b; z)$ of first and second kind respectively. The two independent solution of eq.(B1) are the subsequent:

$$M_{\kappa,\mu}(z) = e^{-z/2} z^{1/2+\mu} M\left(\frac{1}{2} + \mu - \kappa; 1 + 2\mu; z\right) \tag{B4}$$

$$W_{\kappa,\mu}(z) = e^{-z/2} z^{1/2+\mu} U\left(\frac{1}{2} + \mu - \kappa; 1 + 2\mu; z\right) \tag{B5}$$

In case that $2\mu \neq -1, -2, -3, \ldots$ the Whittaker function of first kind $M_{\kappa,\mu}(z)$ does not exist[7] but beside that restriction it is a well-behaved function $\forall z \in \mathbb{C}$. Considering the fact that we discuss this ODE in the complex plane a quite perceptive question: what is the nature of these complex functions in it? The answer to this can be obtained fairly straightforward from their very definition. Both of the Whittaker functions are multi-valued functions with branch points at $z = 0, z = \infty$ and their principal branches are consistent with those of the $z^{1/2+\mu}$ and $e^{-z/2}$.

Naturally these functions are highly non-trivial and therefore there exists a vast majority of information considering them, such as asymptotic behaviours (see §2.5.2), integral representations, recurrence relations, connection with other functions, etc. We shall not present all of this information as we find it to be purposeless and can be easily found in [11]. However, we intent to demonstrate some relations with other functions of mathematical physics.

[7]However there are some formulas that, even in this case, are valid in their limiting form.

B.2 Relation to other functions

At this instance we proceed to briefly illustrate some relations to elementary functions. Let us consider the case of $M_{0,\frac{1}{2}}(2z)$:

$$\begin{aligned} M_{0,\frac{1}{2}}(2z) &= e^{-2z/2}(2z)^{1/2+1/2} M\left(\frac{1}{2}+\frac{1}{2};1+2\frac{1}{2};2z\right) \\ &= 2z\, e^{-z} M(1;2;2z) \\ &= 2z\, e^{-z}\left(1+z+\frac{2z^2}{3}+\frac{z^3}{3}+\frac{2z^4}{15}+\dots\right) \\ &= 2z\, e^{-z}\frac{e^z}{z}\sinh z \\ &= 2\sinh z \end{aligned}$$

Now, we move on to the case of $M_{\kappa,\kappa-\frac{1}{2}}(z)$ which also consists an interesting case:

$$\begin{aligned} M_{\kappa,\kappa-\frac{1}{2}}(z) &= e^{-z/2} z^{1/2+\kappa-1/2} M\left(\frac{1}{2}+\kappa-\frac{1}{2}-\kappa;2\kappa;z\right) \\ &= e^{-z/2} z^{\kappa} M(0;2\kappa;z) \\ &= e^{-z/2} z^{\kappa} \end{aligned}$$

where in the last we have used the identity $M(0;b;z) = U(0;b;z) = 1$ obeyed by the confluent hypergeometric functions of the first and second kind. Furthermore, from the above identity and the definitions eq.(B4), eq.(B5) it is evident that $W_{\kappa,\kappa-\frac{1}{2}}(z) = M_{\kappa,\kappa-\frac{1}{2}}(z) = e^{-z/2}z^{\kappa}$.

Finally, in order to conclude this appendix we would like to present their relation with respect to some of the most common functions that come up in mathematical physics. We are referring to the Bessel functions and in particular to the modified Bessel functions $I_\nu(z)$, $K_\nu(z)$. We begin with $M_{0,\nu}(2z)$:

$$\begin{aligned} M_{0,\nu}(2z) &= e^{-z}(2z)^{\nu+1/2} M\left(\nu+\frac{1}{2};1+2\nu;2z\right) \\ &= e^{-z}(2z)^{\nu+1/2}\,\Gamma(1+\nu)\, e^{z}\left(\frac{z}{2}\right)^{-\nu} I_\nu(z) \\ &= 2^{2\nu+1/2}\,\Gamma(1+\nu)\sqrt{z}\, I_\nu(z) \end{aligned}$$

We proceed to $W_{0,\nu}(2z)$:

$$\begin{aligned} W_{0,\nu}(2z) &= e^{-z}(2z)^{\nu+1/2} U\left(\nu+\frac{1}{2};1+2\nu;2z\right) \\ &= e^{-z}(2z)^{\nu+1/2}\frac{e^z}{\sqrt{\pi}}(2z)^{-\nu} K_\nu(z) \\ &= \sqrt{\frac{2z}{\pi}}\, K_\nu(z) \end{aligned}$$

We conclude based on the above calculations that when $\kappa = 0$ both Whittaker functions of the first and second kind behave as elementary function multiplied by the respective modified Bessel function. Naturally, there are much more information considering relations with other special functions such as Hermite, Laguerre and Airy Polynomials which can be obtained from the literature.

Appendix C

C Supplementary corrections of the EFT-corrected NHEK

In this section we would like to present the rest of the EFT corrections established by [1] concerning the $\Omega_{\rm NH}(\theta)$ and $B_{\rm NH}(\theta)$. They begin by defining:

$$K(\theta) \equiv \arcsin\left(\frac{\sqrt{2}\cos\theta}{\sqrt{1+\cos^2\theta}}\right) - \arcsin\cos\theta \tag{C1}$$

We proceed to exhibit the sixth-order derivative corrections indicated by (6) superscript.

$$B^{(6)}(\theta) = \frac{\kappa^4}{J^2}\Bigg[\frac{2656 - 42885\cos^2\theta + 45895\cos^4\theta - 8130\cos^6\theta - 1218\cos^8\theta + 183\cos^{10}\theta + 139\cos^{12}\theta}{224(1+\cos^2\theta)^6} - \frac{15\sqrt{2}\cos\theta(3-\cos^2\theta)}{64(1+\cos^2\theta)\sin\theta}K(\theta)\Bigg] \tag{C2}$$

and

$$\Omega^{(6)}(\theta) = \frac{\kappa^4}{J^2}\Bigg[C^{(6)} - \frac{3285 - 55449\cos^2\theta + 54210\cos^4\theta - 7058\cos^6\theta - 1527\cos^8\theta - 309\cos^{10}\theta}{224(1+\cos^2\theta)^6} + \frac{15\cos\theta\sqrt{2}\sin\theta}{64(1+\cos^2\theta)}K(\theta)\Bigg] \tag{C3}$$

Further on, we display the eighth-derivative corrections denoted with a superscript (8):

$$B^{(8)}(\theta) = \frac{\kappa^6}{J^3}\Bigg[\frac{832989}{1280} - \frac{315\pi}{4} - \frac{407005 + 32887800\cos^2\theta + 38302380\cos^4\theta + 227158536\cos^6\theta}{1280(1+\cos^2\theta)^9} - \frac{244951182\cos^8\theta + 207667400\cos^{10}\theta + 108083820\cos^{12}\theta + 31954360\cos^{14}\theta + 4114685\cos^{16}\theta}{1280(1+\cos^2\theta)^9} + \frac{630\cos\theta}{1+\cos^2\theta}\arctan\cos\theta - \frac{366435\cos\theta(3-\cos^2\theta)}{256\sqrt{2}\sin\theta(1+\cos^2\theta)}K(\theta)\Bigg] \tag{C4}$$

and

$$\Omega^{(8)}(\theta) = \frac{\kappa^6}{J^3}\Bigg[C^{(8)} + \frac{783837 + 16684758\cos^2\theta + 33602022\cos^4\theta + 119986542\cos^6\theta}{1280(1+\cos^2\theta)^9} + \frac{27639936\cos^8\theta + 23049562\cos^{10}\theta + 11880370\cos^{12}\theta + 3484978\cos^{14}\theta + 445863\cos^{16}\theta}{256(1+\cos^2\theta)^9} - \frac{315\cos\theta}{1+\cos^2\theta}\arctan\cos\theta + \frac{366435\cos\theta\sin\theta}{256\sqrt{2}(1+\cos^2\theta)}K(\theta)\Bigg] \tag{C5}$$

Finally, there is the other family of solutions corresponding to eighth-derivative corrections denoted by the same superscript (8) but with the difference that the quantities bear a tilde:

$$\tilde{B}^{(8)}(\theta) = \frac{\kappa^6}{J^3}\left[\frac{846339}{320} - 315\pi - \frac{1149443 + 5618952\cos^2\theta + 136013268\cos^4\theta + 154320120\cos^6\theta}{320(1+\cos^2\theta)^9}\right.$$
$$- \frac{254641842\cos^8\theta + 208733752\cos^{10}\theta + 108674580\cos^{12}\theta + 32136008\cos^{14}\theta + 4138723\cos^{16}\theta}{320(1+\cos^2\theta)^9}$$
$$\left. + \frac{2520\cos\theta}{1+\cos^2\theta}\arctan\cos\theta - \frac{368829\cos\theta(3-\cos^2\theta)}{64\sqrt{2}\sin\theta(1+\cos^2\theta)}K(\theta)\right] \tag{C6}$$

and

$$\tilde{\Omega}^{(8)}(\theta) = \frac{\kappa^6}{J^3}\left[\tilde{C}^{(8)} + \frac{1018371 + 7724394\cos^2\theta + 67516506\cos^4\theta + 96062418\cos^6\theta + 141833088\cos^8\theta}{320(1+\cos^2\theta)^9}\right.$$
$$+ \frac{115923454\cos^{10}\theta + 59757382\cos^{12}\theta + 17530822\cos^{14}\theta + 2243037\cos^{16}\theta}{320(1+\cos^2\theta)^9}$$
$$\left. - \frac{1260\cos\theta}{1+\cos^2\theta}\arctan\cos\theta + \frac{368829\cos\theta\sin\theta}{64\sqrt{2}(1+\cos^2\theta)}K(\theta)\right] \tag{C7}$$

One more detail that should be added is that $C^{(6)}$, $C^{(8)}$ and $\tilde{C}^{(8)}$ are arbitrary integration constants.

Appendix D

D Supplementary material on verification of EFT-corrected equations

In this appendix we would like to present additional information considering the verification of the EFT-corrections. In §3.4 we described the process that we followed in order to verify the EFT-corrections introduced by [1]. However, we shall present one of the Mathematica notebooks used for the verification. In the next pages, we have included the notebook that verifies the $\tilde{\lambda}$-corrections. Naturally, the process followed for the other EFT-corrections (η, λ) is analogous to the one followed below and therefore there is no need to illustrate them as well.

We believe that it's worth clarifying some aspects of the leading notebook. Let's begin with the `RGtensors[Metric, Coordinates]` command. `RGtensors[]` is a built-in function that receives two arguments, the metric and the coordinates of the spacetime, as lists. Then it progresses into calculating the majority of the spacetime's crucial quantities, such as Christoffel symbols, Riemann, Ricci, Weyl tensors and the Ricci scalar. We move on to the `simpRules = TrigRules;` command. `simpRules` is a built-in variable that stores a list with identities utilized for simplifications within the reach of the package. `TrigRules` is also a built-in variable containing a list with trigonometric identities. Therefore this command is used in order for the package to be able to simplify mathematical expressions using trigonometric identities. Moreover, the tensor `eta[3, 4]` needs a small clarification; `eta[]` is a built-in tensor that denotes the fully covariant Levi-Civita tensor $\varepsilon_{\alpha\beta\gamma\delta\ldots}$ (with the convention $\varepsilon_{0123\ldots} = \sqrt{|g|}$) while the `eta[3, 4]` denotes the ${\varepsilon_{\alpha\beta}}^{\gamma\delta}$. Finally, there exists one more command that we would like to explain; the `FacSimp[]`. Once again, this represents a built-in function that simplifies the components of tensors within this package. It is really useful for optimization.

Naturally, there exists an extended variety of information considering the documentation of the functions that utilizes the `RGTC` package. This piece of information can be found in a notebook within the zip file of the package.

EFT-corrected Einstein Equations Λ-expansion.

To begin with, we like to use a notation close to the paper. This notebook attempts to verify the $\tilde{\lambda}$ corrections. However, here we denote it as Λ for convenience.

```
In[•]:= ClearAll["Global`"]

In[•]:= << RGTC`EDCRGTCcode` (*including the package*)
```

SetDelayed: Tag Classify in Classify[x_] is Protected.

```
In[•]:= Coordinates = {t, r, θ, ϕ};(*introduce the coordinates*)
```

$$\text{Metric} = \left\{\left\{2 M^2 r^2 \Omega_{NH}[\theta]^2 \left(\omega_{NH}{}^2 B_{NH}[\theta]^2 - 1\right), 0, 0, 2 M^2 \omega_{NH}\, r\, B_{NH}[\theta]^2 \Omega_{NH}[\theta]^2\right\},\right.$$
$$\left\{0, 2 M^2 \frac{\Omega_{NH}[\theta]^2}{r^2}, 0, 0\right\}, \left\{0, 0, 2 \Gamma_{NH}{}^2 M^2 \Omega_{NH}[\theta]^2, 0\right\},$$
$$\left.\left\{2 M^2 \omega_{NH}\, r\, B_{NH}[\theta]^2 \Omega_{NH}[\theta]^2, 0, 0, 2 M^2 B_{NH}[\theta]^2 \Omega_{NH}[\theta]^2\right\}\right\};\ \text{(*The NHEK metric*)}$$

```
In[•]:= RGtensors[Metric, Coordinates];
```

$$\text{gdd} = \begin{pmatrix} 2 M^2 r^2 \left(-1 + \omega_{NH}^2 B_{NH}[\theta]^2\right) \Omega_{NH}[\theta]^2 & 0 & 0 & 2 M^2 r\, \omega_{NH} B_{NH}[\theta]^2 \Omega_{NH}[\theta]^2 \\ 0 & \frac{2 M^2 \Omega_{NH}[\theta]^2}{r^2} & 0 & 0 \\ 0 & 0 & 2 M^2 \Gamma_{NH}^2 \Omega_{NH}[\theta]^2 & 0 \\ 2 M^2 r\, \omega_{NH} B_{NH}[\theta]^2 \Omega_{NH}[\theta]^2 & 0 & 0 & 2 M^2 B_{NH}[\theta]^2 \Omega_{NH}[\theta]^2 \end{pmatrix}$$

$$\text{LineElement} = \frac{2 M^2 d[r]^2 \Omega_{NH}[\theta]^2}{r^2} + 2 M^2 d[\theta]^2 \Gamma_{NH}^2 \Omega_{NH}[\theta]^2 + 2 M^2 d[\phi]^2 B_{NH}[\theta]^2 \Omega_{NH}[\theta]^2 +$$
$$4 M^2 r\, d[t] \times d[\phi]\, \omega_{NH} B_{NH}[\theta]^2 \Omega_{NH}[\theta]^2 + 2 M^2 r^2 d[t]^2 \left(-1 + \omega_{NH} B_{NH}[\theta]\right) \left(1 + \omega_{NH} B_{NH}[\theta]\right) \Omega_{NH}[\theta]^2$$

$$\text{gUU} = \begin{pmatrix} -\frac{1}{2 M^2 r^2 \Omega_{NH}[\theta]^2} & 0 & 0 & \frac{\omega_{NH}}{2 M^2 r\, \Omega_{NH}[\theta]^2} \\ 0 & \frac{r^2}{2 M^2 \Omega_{NH}[\theta]^2} & 0 & 0 \\ 0 & 0 & \frac{1}{2 M^2 \Gamma_{NH}^2 \Omega_{NH}[\theta]^2} & 0 \\ \frac{\omega_{NH}}{2 M^2 r\, \Omega_{NH}[\theta]^2} & 0 & 0 & -\frac{\left(-1+\omega_{NH} B_{NH}[\theta]\right)\left(1+\omega_{NH} B_{NH}[\theta]\right)}{2 M^2 B_{NH}[\theta]^2 \Omega_{NH}[\theta]^2} \end{pmatrix}$$

```
gUU computed in 0. sec

Gamma computed in 0.016 sec

Riemann(dddd) computed in 0.016 sec

Riemann(Uddd) computed in 0.031 sec

Ricci computed in 0.016 sec

Weyl computed in 0.015 sec

Einstein computed in 0.016 sec

All tasks completed in 0.109375 seconds

In[•]:=

In[•]:= simpRules = TrigRules;
```

In[•]:=
```
RUUUU = Raise[RUddd, 2, 3, 4];
RddUU = Raise[Rdddd, 3, 4];
RdUUU = Raise[Rdddd, 2, 3, 4];
RddddTilde = multiDot[eta[3, 4], Rdddd, {3, 1}, {4, 2}];
RdUdUTilde = Raise[RddddTilde, 2, 4];
Kretschmann = Simplify[multiDot[Rdddd, RUUUU, {1, 1}, {2, 2}, {3, 3}, {4, 4}]];
thirdorderscalar = Simplify[multiDot[
    multiDot[RddUU, RddUU, {3, 1}, {4, 2}], RddUU, {1, 3}, {2, 4}, {3, 1}, {4, 2}]];
KretschmannTilde = Simplify[multiDot[RddddTilde, RUUUU, {1, 1}, {2, 2}, {3, 3}, {4, 4}]]
```

Out[•]=

$$-\frac{4\,\omega_{NH}\,B_{NH}'[\theta]\,\left(\Gamma_{NH}^2\,B_{NH}[\theta]\,\left(-1+\omega_{NH}^2\,B_{NH}[\theta]^2\right)-B_{NH}''[\theta]\right)}{M^4\,\Gamma_{NH}^3\,B_{NH}[\theta]\,\Omega_{NH}[\theta]^4}$$

In[•]:=

In[•]:=
```
Tquarticdd =
  FacSimp[ -Λ k^6 (8 multiDot[RdUdUTilde, covD[covD[KretschmannTilde]], {2, 2}, {4, 1}] +
      1/2 gdd KretschmannTilde^2)];
```

In[•]:=

In[•]:=
```
EinsteinEqn = FacSimp[Rdd - 1/2 gdd R - Tquarticdd];
```

In[•]:=
```
Γ_NH = Γ_0 (1 + Λ ΓT_8);
ω_NH = ω_0 (1 + Λ ωT_8);
B_NH[θ_] := B_0[θ] (1 + Λ BT_8[θ]);
Ω_NH[θ_] := Ω_0[θ] (1 + Λ ΩT_8[θ]);
```

In[•]:=

First order of Λ-expansion of $R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R - T_{\mu\nu} = 0$

In[•]:=
```
EinsteinEqnExpanded = MatrixForm@Map[Normal@Series[#, {Λ, 0, 1}] &, EinsteinEqn, {2}];
```

In[∘]:=
```
K[θ_] := Simplify[
   ArcSin[(√2 Cos[θ]) / (√(1 + Cos[θ]^2))] - ArcSin[Cos[θ]], Assumptions → 0 ≤ θ ≤ π];

Γ0 = 1;
ΓT8 = -((368829 k^6) / (64 × √2 J^3));

ω0 = 1;
ωT8 = ((4736 + 1575 π) k^6) / (5 J^3);

Ω0[θ_] := Sqrt[(1 + Cos[θ]^2)/2];
ΩT8[θ_] := Simplify[
   1/J^3 k^6 ((368829 Cos[θ] Sqrt[1 - Cos[θ]^2]) K[θ] / (64 √2 (Cos[θ]^2 + 1)) + C8T + 1/(320 (Cos[θ]^2 + 1)^9) (141833088 Cos[θ]^8 +
        96062418 Cos[θ]^6 + 67516506 Cos[θ]^4 + 7724394 Cos[θ]^2 + 1018371) +
      1/(320 (Cos[θ]^2 + 1)^9) (2243037 Cos[θ]^16 + 17530822 Cos[θ]^14 + 59757382 Cos[θ]^12 +
        115923454 Cos[θ]^10) - ((1260 Cos[θ]) ArcTan[Cos[θ]]) / (Cos[θ]^2 + 1)),
   Assumptions → {0 ≤ θ ≤ π, J ≠ 0, k ≠ 0}];(*With the first
  sign as + and not -*)

B0[θ_] := (2 Sin[θ]) / (1 + Cos[θ]^2);
BT8[θ_] := Simplify[
   1/J^3 k^6 (-((368829 Cos[θ] (3 - Cos[θ]^2)) K[θ]) / (64 √2 Sqrt[1 - Cos[θ]^2] (Cos[θ]^2 + 1)) - 1/(320 (Cos[θ]^2 + 1)^9) (254641842 Cos[θ]^8 +
        154320120 Cos[θ]^6 + 136013268 Cos[θ]^4 + 5618952 Cos[θ]^2 + 1149443) -
      1/(320 (Cos[θ]^2 + 1)^9) (4138723 Cos[θ]^16 + 32136008 Cos[θ]^14 + 108674580 Cos[θ]^12 +
        208733752 Cos[θ]^10) + ((2520 Cos[θ]) ArcTan[Cos[θ]]) / (Cos[θ]^2 + 1) +
      846339/320 - 315 π), Assumptions → {0 ≤ θ ≤ π, J ≠ 0, k ≠ 0}];
```

In[◦]:=
```
AbsoluteTiming[res = FullSimplify[EinsteinEqnExpanded,
   TimeConstraint → 600, Assumptions → {k ≠ 0, J = M^2, 0 ≤ θ ≤ π}]]
```

Out[◦]=

$$\left\{456.814, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}$$

The corrections with respect to Λ do indeed satisfy the EFT-corrected Einstein equations.

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