Vorticity production and survival in radiative Friedman universes

Fani Dosopoulou

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Astrophysics Laboratory
Physics Department
University of Crete

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Abstract

Although most astrophysical objects rotate, whether the universe itself rotates or not is as yet unknown. Is is conceivable that some observational data, such as the large-scale asymmetry observed in the CMB as well as several anomalies detected on large angular scales, could be attributed to rotation. Nevertheless, the issue is still open to discussion. In this thesis, we examine the effect the presence of a magnetic field has on vorticity in a Friedman universe during the radiation period. We select a Friedman background with a weak random B-field and perturbe it. We derive the linear form of the vorticity evolution equation and analyze what effect the matter and magnetic energy densities, as well the magnetic tension have on vorticity production and survival. We find that both the matter and the magnetic pressure help vorticity to survive, acting against its reduction caused by the expansion of the universe. We then find a linear analytical solution for the evolution of the vorticity vector during the radiation era. The solution shows that the presence of a magnetic field reduces the standard decay rate of the vorticity and in the long term tends to keep it constant. Finally, we examine the effect of magnetic perturbations on vortex-like distortions in the density distribution of the radiative matter and conclude again that the B-field does help the latter to survive.

1 Introduction

Rotation is an ubiquitous phenomenon in the physical world. From the microscopical scale and the atom to the macroscopical scale and the whole universe, various different objects spin. Electrons orbit around the nucleus, satellites orbit planets, planets revolve around a star, planetary systems orbit the central nucleus of a galaxy. Satellite galaxies, rotating clusters of stars and galaxies, e.t.c are also good paradigms. Although the question of a rotating universe emerges then effortlessly, the research that has been done to answer the global rotation subject is very little and rare. However, over the years, new experimental data, mainly from the detailed study of the cosmic microwave background radiation (CMB), gave rise to new approaches to the cosmic rotation unanswered question.

The first idea of a rotating universe should be attributed to G.Gamow, who claimed that the rotation of galaxies is due to the turbulent motion of masses in the Universe. Since a rotating universe is physically allowed as a solution to the Einstein field equations, Goedel proposed a homogeneous rotating universe. The first observational evidence of global rotation was presented by Birch. He studied the position angles and polarisation of classical high-luminosity double radio-sources and found that the difference between the position angle of the source elongation and of the polarisation are correlated with the source position in the sky. It was argued that such an assymetry can also be explained by the existence of a universal vorticity. However, Barrow, Juszkiewicz and Sonoda found two years later that the asymmetries in radio-source orientations measured by Birch cannot be due to universal rotation as a universal vorticity at the level claimed by Birch is incompatible with the existing observations of the microwave background radiation. They assumed spatially homogeneous vorticity and constraint it based on the current observations of the CMB dipole and quadrupole moments. Some years later, Nodland and Ralston investigated the correlation between the direction and distance to a galaxy and the angle between the polarisation direction and the major galaxy axis. They found an effect of a systematic rotation of the plane of polarisation of electromagnetic radiation. A different explanation to these data was then proposed by Obukhov, Korotky and Hehl and was that of a global cosmic rotation. This explanation was within conventional general relativity and was not in conflict with other observational data. The rotation of the polarisation of an electromagnetic wave is a typical effect of the cosmic rotation and these new data by Nodland and Ralston provided a substantially improved estimate of the magnitude and direction of cosmic vorticity.

The basic method to estimate the amount of cosmic vorticity and test the various rotating models of the Universe is based on the experimental data for the CMB. Observations of the dipole and quadrupole fluctuations in the microwave background radiation are compatible with a global rotation. These include low quadrupole amplitude (Efstathiou), anomalies on large angular scales, curious planarity and alignment of the quadrupole and octopole (De Oliveira-Costa) and a localised source of non-Gaussianity in the form of a very cold spot on the sky (Vielva,Cruz, et al.). Of particular interest also was a remarkable asymmetry in large-scale power measured in the two hemispheres of a specific reference frame (Eriksen, et al., Hansen et al.). Recent analyses of the first-year WMAP data claim large-scale asymmetry and anisotropy of the CMB fluctuations consistent with an angular momentum of the Universe. Even more recent studies of Kashlinsky, et al, on the fluctuations in the cosmic microwave background have revealed a coherent bulk flow of clusters of galaxies on a large cosmic scale. Invoking a primordial rotation in models of the Universe can account for the observed peculiar velocities. Furthermore, models with global rotation exhibit a spiral pattern of temperature anisotropy due to handedness of the geodesics propagating through an anisotropic spacetime. Open models can account for the hot spots of the CMB while closed models exhibit a pure quadrupole pattern.

Apart from the experimental data for the CMB radiation, a primordial rotation is also supported by the close relation between the latter and the angular momentum of a wide range of celestial objects. Specifically, cosmological models which also contain a term involving the primordial spin of the universe can possibly impart rotation to galaxies, clusters, stellar systems, etc. and give rise to the observed rotation angular momenta of the latter. Recent cosmological models of a rotating homogeneous and isotropic Universe (Godlowski, Szydlowski, Flin, et al.) predict the presence of a minimum in the relation between the mass of an astronomical object and its angular momentum. Testing the agreement comparing it with empirical relations between masses of structures and minimal angular momenta provides another way of estimating the amount of cosmic rotation. It is worth mentioning here that closely related to cosmic vorticity is according to Biermann the generation of primordial magnetic fields.

The next question that arises regarding a rotating Universe is what could source this cosmic rotation. Since the experimental data on the CMB support an isotropic and homogeneous Universe, vorticity generation could be obtained applying perturbation theory around a Friedmann-Robertson-Walker background. There are different types of perturbations, classified as scalar, vector and tensor perturbations, depending on the way they transform on three dimensional hypersurfaces. At linear order, vorticity cannot be constructed from scalar quantities and the vorticity tensor constructed from vector perturbations is sourced by anisotropic stress and decays in its absence due to the expansion of the Universe. Consequently, vorticity can be generated at second order by first order scalar and vector perturbations. While at linear order the different types of perturbations decouple, at second order it is this coupling that exists that can source vorticity generation. In fluid dynamics, vorticity is sourced by entropy gradients. Christopherson, et al. showed that at second order gradients in the non-adiabatic pressure perturbation coupled to gradients in the density can source vorticity. These entropy perturbations arise in the case of a single fluid with non-zero intrinsic entropy or in the case of multiple fluids stemming from the mixing of the fluids even if there exists no energy and momentum transfer between them (this could source vorticity as well). A possible observational signature of vorticity in the early universe is that of the B-mode polarisation of the CMB radiation. At linear order, only tensor perturbations can produce B-mode polarisation since scalar perturbations will produce only E-modes and vector perturbations will become subdominant during inflation and will decay with the expansion of the Universe. However, based on what was mentioned above, at second order even density perturbations could generate vorticity and thus lead to B-mode polarisation large enough to be observed by future CMB experiments such as Planck. In interstellar medium, vorticity production in a potential velocity field (spherical expansion wave) could be generated from shear (not significant amount) or the baroclinic term (non-parallel gradients of pressure and density) which is more effective. In addition, spatial gradients of geometry and electromagnetic sources could source large-scale vorticity. As an example, in a strongly coupled electromagnetic plasma, angular momentum exchange between ions, electrons and photons in an expanding spacetime geometry could lead to the formation of large scale vortices. Thus, inhomogeneous geometry and electromagnetic sources could generate vorticity through both divergence-free (solenoidal) and compressive (curl-free) forcing. The former is more efficient due to the stronger tangling of the magnetic field.

The presence of a magnetic field is closely related to vorticity generation. Even in the absence of an initial rotation, inhomogeneities in the magnetic field or non-zero spatial derivative of the curvature of the magnetic field lines could source rotation. The effect of magnetic fields on vorticity generation and survival consists the basic subject of this thesis.

We begin with a general introduction in relativistic cosmological media and their description within the $1+3$ covariant formalism. This includes how the gravitational field, matter kinematics and conservation laws are described in this formalism. Since the basic subject in this thesis is the magnetic field, in the next section we provide the general formalism for describing the electromagnetic field, namely Maxwell's equations. In the fourth section, we describe non-linear ideal magnetohydrodynamics while in the fifth section we provide a brief introduction to Friedman models. As stated above, rotation could be generated using perturbation theory around a Friedman model background. This is why, in the sixth section, we give the basic characteristics and equations of perturbed Friedman models.

The final section includes all the new work that has been done in this thesis.
We discuss the effect of the presence of a magnetic field in a Friedman Universe during the radiation period. We choose as a background a Friedman model with a weak random magnetic field ($< B^2 >\neq 0, < B^a >= 0, B^2 << \rho$). We perturb this model by adding a perturbation in the magnetic field ($< B_a >\neq 0$). We examine how this affects vorticity production and survival. Firstly, we derive the linear vorticity evolution equation. This is our basic equation from which we can get information about what could or could not source cosmic rotation. We analyze the various terms of this equation, which have to do with matter and magnetic energy densities as well as the magnetic tension of the field. What we find is that both matter and magnetic pressure could generate vorticity. They both contribute to the survival of the vorticity, acting against its decay, due to the expansion of the Universe in the non-magnetized case. We then find a linear analytical solution for the vorticity vector during the radiation period. This solution indicates that the presence of a magnetic field reduces the decay rate of the vorticity that is otherwise expected and in the long term tends to keep it constant. In the last paragraph of this section, we examine the effect of magnetic perturbations on vortex-like distortions in the density distribution of the radiative matter. Our main goal is to see if the magnetic field will help them to survive or not. The answer turns out to be affirmative.

2 Relativistic cosmological media

2.1 The 1 + 3 covariant description

The covariant formalism is an approach to general relativity, which uses the kinematic quantities of the fluid, its energy-momentum tensor and the gravito-electromagnetic parts of the Weyl tensor, instead of the metric. In this approach, we employ the Ricci and Bianchi identities, applied to the 4-velocity vector.

2.2 Local spacetime splitting

We consider a general spacetime with a Lorentzian metric g_{ab} of signature $(-,+,+,+)$ and introduce a family of fundamental observers living along a time-like congruence of worldlines tangent to the 4-velocity vector [1]

$$u^a = \frac{dx^a}{d\tau}\,, \tag{2.2.1}$$

where τ is the observer's proper time and $u^a u_a = -1$. This fundamental velocity field introduces a local $1+3$ "threading" of the spacetime into time and space. The vector u_a determines the time direction, while the tensor $h_{ab} = g_{ab} + u_a u_b$ projects orthogonal to the 4-velocity field into what is known as the observer's instantaneous rest space at each event. In the absence of rotation, the 4-velocity vector u_a is hyperface-orthogonal and h_{ab} is the metric of the 3-dimensional spatial sections orthogonal to u_a.

The vector field u_a and its tensor counterpart h_{ab} allow for a unique decomposition of every spacetime quantity into it's irreducible timelike and spacelike parts. By employing u_a and h_{ab}, we define the covariant time derivative and the orthogonally projected spatial derivative of any given tensor field $S_{ab\cdots}{}^{cd\cdots}$ according to

$$\dot{S}_{ab\cdots}{}^{cd\cdots} = u^e \nabla_e S_{ab\cdots}{}^{cd\cdots}\,, \tag{2.2.2}$$

and

$$D_e S_{ab\cdots}{}^{cd\cdots} = h_e{}^s h_a{}^f h_b{}^p h_q{}^c h_r{}^d \cdots \nabla_s S_{fp\cdots}{}^{qr\cdots}\,. \tag{2.2.3}$$

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[1] Latin indices vary between 0 and 3 and refer to arbitrary coordinate frame and Greek indices vary from 1 to 3. We use geometrized units with $c = 1 = 8\pi G$, which means that all geometrical variables have physical dimensions that are integer powers of length.

2.3 The Levi-Civita tensor

In three dimensions, the Levi-Civita symbol is a totally antisymmetric pseudotensor and is defined as follows :

$$\epsilon_{abc} = \begin{cases} 1 & \text{if (i,j,k) is an even permutation} \\ -1 & \text{if (i,j,k) is an odd permutation} \\ 0 & \text{if any index repeated} \end{cases}$$

where $i, j, k = 1, 2, 3$.

The effective volume element in the observer's instantaneous rest space is given by the 3-dimensional Levi-Civita symbol defined above. The latter can be derived by contracting the spacetime volume element, which is given by the 4-dimensional Levi-Civita tensor η_{abcd}, along the time direction :

$$\epsilon_{abc} = \eta_{abcd} u^d \ . \tag{2.3.1}$$

The totally antisymmetric pseudotensor η_{abcd} has $\eta^{0123} = [-det(g_{ab})]^{-1/2}$, it is covariantly constant and satisfies the identities

$$\eta_{abcd}\eta^{efpq} = -4!\delta_{[a}{}^{e}\delta_{b}{}^{f}\delta_{c}{}^{p}\delta_{d]}{}^{q} \ , \tag{2.3.2}$$

where $\delta_a{}^b$ is the Kronecker symbol.

It can be easily proved that for the also totally antisymmetric tensor ϵ_{abc} we have the identities

$$\epsilon_{abc} u^a = 0 \ , \tag{2.3.3}$$

and

$$\epsilon_{abc}\epsilon^{def} = 3! h_{[a}{}^{d} h_{b}{}^{e} h_{c]}{}^{f} \ . \tag{2.3.4}$$

It also holds that $D_c h_{ab} = D_d \epsilon_{abc} = 0$, while

$$\dot{\epsilon}_{abc} = 3 u_{[a}\epsilon_{bc]d} A^d \ , \tag{2.3.5}$$

where $A_a = \dot{u}_a$ is the 4-acceleration vector (see §2.6).

2.4 The gravitational field

In general theory of relativity, gravity is an expression of the geometry of the spacetime. Matter determines the spacetime curvature while the latter controls the motion of the matter. This interpretation is given clearly through the Einstein field equations :

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab} - \Lambda g_{ab}, \tag{2.4.1}$$

where G_{ab} is the Einstein tensor, $R_{ab} = R^c_{acb}$ is the spacetime Ricci tensor (with trace R), T_{ab} is the total energy-momentum tensor of the matter fields and Λ is the cosmological constant.

The Ricci tensor R_{ab} describes the local gravitational field that is produced by the presence of matter there. The non-local, long-range gravitational field, conveyed via gravitational waves and tidal forces, is decribed by the Weyl curvature tensor C_{abcd}. This splitting of the gravitational field in local and non-local components is given by the decomposition of the Riemann tensor

$$R_{abcd} = C_{abcd} + \frac{1}{2}(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc}) - \frac{1}{6}Rg_{ac}g_{bd} - g_{ad}g_{bc}\ , \tag{2.4.2}$$

where C_{abcd} is the Weyl curvature tensor which shares all the symmetries of the Riemann tensor and is also trace-free, $C^c{}_{acb} = 0$. Relative to the fundamental observers, the conformal curvature tensor decomposes further into it's irreducible electric

$$E_{ab} = C_{abcd}u^c u^d\ , \tag{2.4.3}$$

and magnetic

$$H_{ab} = \frac{1}{2}\epsilon_a{}^{cd}C_{cdbe}u^e\ , \tag{2.4.4}$$

parts. Then, we have the decomposition

$$C_{abcd} = (g_{abqp}g_{cdsr} - \eta_{abqp}\eta_{cdsr})u^q u^s E^{pr} - (\eta_{abqp}g_{cdsr} + g_{abqp}\eta_{cdsr})u^q u^s H^{pr}, \tag{2.4.5}$$

where g_{abcd} is defined as follows :

$$g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}. \tag{2.4.6}$$

The spatial, symmetric and trace-free tensors E_{ab} and H_{ab} are known as the electric and magnetic Weyl components. The electric part generalizes the tidal tensor of the Newtonian gravitational potential while the magnetic part has no Newtonian analogue. However, both tensors must be present for the propagation of gravitational waves.

The once-contacted Bianchi identities act as the field equations for the Weyl tensor and give

$$\nabla^d C_{abcd} = \nabla_{[b}R_{a]c} + \frac{1}{6}g_{c[b}\nabla_{a]}R. \tag{2.4.7}$$

2.5 Matter fields

With respect to the fundamental observers, the energy-momentum tensor of a general imperfect fluid decomposes into it's irreducible parts as

$$T_{ab} = \rho u_a u_b + p h_{ab} + 2q_{\langle a}u_{b\rangle} + \pi_{ab} \ , \tag{2.5.1}$$

where $\rho = T_{ab}u^a u^b$ is the matter energy density, $p = T_{ab}h^{ab}/3$ is the effective isotropic pressure of the fluid (the sum between the equilibrium pressure and the associated bulk viscosity), $q_a = -h_a{}^b T_{bc}u^c$ is the total energy-flux vector relative to u_a and $\pi_{ab} = h_{\langle a}{}^c h_{b\rangle}{}^d T_{cd}$ is the symmetric and trace-free tensor that describes the anisotropic pressure of the fluid. [2] It follows that $q_a u^a = \pi_{ab}u^a = 0$.

When the fluid is perfect, there is a unique hydrodynamic 4-velocity, relative to which q_a, π_{ab} are identically zero and the effective pressure reduces to the equilibrium one. As a result,

$$T_{ab} = \rho u_a u_b + p h_{ab} \ . \tag{2.5.4}$$

[2] Angled brackets denote the symmetric and trace-free part of spatially projected second-rank tensors and the projected part of vectors according to

$$S_{\langle ab\rangle} = h_{\langle a}{}^c h_{b\rangle}{}^d S_{cd} = h_{(a}{}^c h_{b)}{}^d S_{cd} - \frac{1}{3}h^{cd}S_{cd}h_{ab} \ , \tag{2.5.2}$$

and

$$v_{\langle a\rangle} = h_a{}^b v_b \ , \tag{2.5.3}$$

respectively (with $S_{\langle ab\rangle}h^{ab} = 0$).

If we additionally assume that $p = 0$, we have the simplest case of pressure-free matter, namely 'dust'. In cosmology, dust also includes baryonic matter (after decoupling) and cold dark matter. Otherwise, we need to determine p as a function of ρ and potentially of other thermodynamic variables. In general, the equation of state takes the form $p = p(\rho, s)$, where s is the specific entropy. Finally, for a barotropic medium we have $p = p(\rho) = w\rho$, with $w = constant$ being the barotropic index of the fluid and $c_s^2 = (\partial\rho/\partial p)_s$ giving the square of the associated adiabatic sound speed.

Note that, when dealing with a multi-component fluid, or when allowing for peculiar velocities, one needs to account for the velocity 'tilt' between the matter components and the fundamental observers. Here, we will consider a single-component fluid and we will assume that the fundamental observers move along with it.

Taking the trace of equation (2.4.1), one is led to the relation $R = 4\Lambda - T$, with $T = T^a_a$. Then, Einstein equations cab be written as

$$R_{ab}u^a u^b = \frac{1}{2}(\rho + 3p) - \Lambda \tag{2.5.5}$$

$$h_a^b R_{bc} u^c = -q_a \tag{2.5.6}$$

$$h_a^c h_b^d R_{cd} = \frac{1}{2}(\rho - p)h_{ab} + \Lambda h_{ab} + \pi_{ab}. \tag{2.5.7}$$

It worths noting that the term $\rho + 3p$ is the right hand side of equation (2.5.5) determines the total gravitational energy density of the matter. Matter with $\rho + 3p > 0$ is conventional matter and attracts gravitationally. On the other hand, matter with $\rho + 3p > 0$ is non-conventional (including dark energy) and repulses gravitationally. We will meet again the total gravitational energy of the matter in Raychaudhuri's formula (2.6.3) described in paragraph 2.6 below.

2.6 Covariant kinematics

The observer's motion is characterized by the irreducible kinematical quantities of the u_a-congruence, which emerge from the covariant decomposition of the 4-velocity gradient

$$\nabla_b u_a = \sigma_{ab} + \omega_{ab} + \frac{1}{3}\Theta h_{ab} - A_a u_b, \tag{2.6.1}$$

where $\sigma_{ab} = D_{\langle b}u_{a\rangle}$, $\omega_{ab} = D_{[b}u_{a]}$, $\Theta = \nabla^a u_a = D^a u_a$ and $A_a = \dot{u}_a = u^b \nabla_b u_a$ are, respectively, the shear and vorticity tensors, the volume expansion (or contraction) scalar and the 4-acceleration vector. By construction we have $\sigma_{ab}u^a = \omega_{ab}u^a = A_a u^a = 0$. The 4-acceleration vector A_a represents non-gravitational forces and vanishes when matter moves under gravity alone. We can define using the volume scalar Θ a characteristic scale factor a by the relation $\Theta = \dot{a}/a$. Then, positive $\Theta > 0$ means expansion of the fluid, while negative $\Theta < 0$ denotes a contracting fluid. Also, on using the orthogonally projected alternating tensor ϵ_{abc} (with $\dot{\epsilon}_{abc} = 3u_{[a}\epsilon_{bc]d}A^d$), one defines the vorticity vector $\omega_a = \epsilon_{abc}\omega^{bc}/2$ (with $\omega_{ab} = \epsilon_{abc}\omega^c$). The latter actually defines the rotational axis of the fluid.

We note here that volume scalar represents changes in the volume of the fluid, namely expansions or contractions, while the effect of the vorticity is to change the orientation of a given fluid element without modifying its volume or shape. The shear, on the other hand, changes the shape but leaves the volume unaffected.

The non-linear covariant kinematics are determined by a set of propagation and constraint equations, which are purely geometrical in origin. Both sets emerge after applying the Ricci identities

$$2\nabla_{[a}\nabla_{b]}u_c = R_{abcd}u^d \,, \tag{2.6.2}$$

to the fundamental 4-velocity vector defined in (2.2.1).

Substituting into (2.6.2) the relation (2.6.1), using decompositions (2.4.2) and (2.4.5), as well as the auxiliary relations (2.5.5)-(2.5.7), the timelike and spacelike part of the resulting expression leads to three propagation and three constraint equations.

The first propagation equation is Raychaudhuri's formula

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\rho + 3p) - 2(\sigma^2 - \omega^2) + D^aA_a + A_aA^a + \Lambda, \tag{2.6.3}$$

which gives the time evolution of the volume scalar Θ. We note here that terms with a negative sign in the righthand side of equation (2.6.3) like square magnitudes of volume scalar and shear Θ^2, σ^2 as well as matter with total gravitational energy $\rho + 3p > 0$ leads to a reduction in the expansion rate of the fluid or in an acceleration of the fluid contraction. On the other hand, terms with a positive sign like the square magnitudes of vorticity and acceleration ω^2, A_aA^a as well as non-conventional matter with $\rho + 3p < 0$ accelerate the fluid's expansion or decelerate its contraction. Finally, the term D^aA_a has not a definite sign so its result is not explicit.

The second propagation equation is the shear propagation equation

$$\dot{\sigma}_{<ab>} = -\frac{2}{3}\Theta\sigma_{ab} - \sigma_{c<a}\sigma^c_{b>} - \omega_{<a}\omega_{b>} + D_{<a}A_{b>} - E_{ab} + \frac{1}{2}\pi_{ab}, \tag{2.6.4}$$

which actually describes kinematical anisotropies. It is clear that equation (2.6.4) is a rather complicated evolution equation. They only term that has a distinct result on shear is the term including the volume scalar Θ which states that expansion of the fluid $\Theta > 0$ acts against the increase of the shear, while fluid's contraction $\Theta < 0$ supports kinematical anisotropies.

Finally, we have the vorticity propagation equation, which will be our basic equation under consideration in the following sections,

$$\dot{\omega}_{\langle a\rangle} = -\frac{2}{3}\Theta\omega_a - \frac{1}{2}curlA_a + \sigma_{ab}\omega^b \,. \tag{2.6.5}$$

We note that $curlv_a = \epsilon_{abc}D^b v^c$ for any orthogonally projected vector v_a, which means that $D^b\omega_{ab} = curl\omega_a$.

The first term in the righthand side of equation (2.6.5) states that the expansion of a fluid ($\Theta > 0$) acts against the vorticity and increases its decay rate while a contracting fluid ($\Theta < 0$) supports its enhancement.

In addition, the terms $-2\Theta\omega_a/3, \sigma_{ab}\omega^b$ on the righthand side of equation (2.6.5) include only the kinematical quantities Θ, σ_{ab}. As a result, they represent inertial Coriolis forces driven by changes in the relative motion between two neighbouring fundamental observers. On the other hand, the term with the curl of the 4-acceleration vector A_a includes external forces that act on the fluid.

Finally, from the form of equation (2.6.5) we can see that in the absence of an initial vorticity vector ω_a the terms $-2\Theta\omega_a/3$ and $\sigma_{ab}\omega^b$ could not generate vorticity. Nevertheless, once an initial rotation of the fluid is created, these terms have an effect on the vorticity vector. For example, we have stated before that in the absence of shear stresses of the fluid and with a zero 4-acceleration vector the term $-2\Theta\omega_a/3$ denotes that an expansion of the fluid does not enhance vorticity but acts against it. On the other hand, only the term $-curlA_a/2$ could act as a possible source of rotation.

Equation (2.6.5) is going to be the focal point of this thesis. We will examine to linear order the form of the term $-curlA_a/2$ in the presence of a magnetic field in the fluid, where we have the act of Lorentz force on the fluid.

The spacelike component of (2.6.2) results to a set of three additional constraints. These include the shear constraint

$$D^b\sigma_{ab} = \frac{2}{3}D_a\Theta + curl\omega_a + 2\epsilon_{abc}A^b\omega^c - q_a, \tag{2.6.6}$$

the vorticity-divergence identity

$$D^a\omega_a = A_a\omega^a, \tag{2.6.7}$$

and the magnetic Weyl equation

$$H_{ab} = curl\sigma_{ab} + D_{<a}\omega_{b>} + 2A_{<a}\omega_{b>}. \tag{2.6.8}$$

2.7 Conservation laws

The twice contacted Bianchi identities guarantee the conservation of the total energy momentum tensor, namely that $\nabla^b T_{ab} = 0$. This condition splits into a timelike and spacelike part, which respectively lead, when dealing with an general imperfect fluid, to the energy density conservation law

$$\dot{\rho} = -\Theta(\rho + p) - D^a q_a - 2A^a q_a - \sigma_{ab}\pi_{ab}, \tag{2.7.1}$$

and the momentum energy conservation law

$$(\rho + p)A_a = -D_a p - \dot{q}_{<a>} - \frac{4}{3}\Theta q_a - (\sigma_{ab} + \omega_{ab})q^b - D^b\pi_{ab} - \pi_{ab}A^b. \tag{2.7.2}$$

We note here that $\rho + p$ describes the relativistic total inertial mass of the medium. The latter indicates that in general relativity the pressure of the matter contributes to the total inertial mass in the same manner as the matter density does.

3 Electromagnetic fields

The covariant description of electromagnetic fields in addition to its inherent mathematical compactness and clarity, provides a physically intuitive fluid description for the Maxwell field. This is represented as an imperfect fluid, with properties specified by its electric and magnetic components.

3.1 Electric and magnetic components

The Maxwell field is covariantly determined by the antisymmetric electromagnetic (Faraday) tensor F_{ab}, which relative to a fundamental observer decomposes into an electric and magnetic component as

$$F_{ab} = 2u_{a[}E_{b]} + \epsilon_{abc}B^c \,, \tag{3.1.1}$$

where E_a, B_a are respectively the electric and magnetic field experienced by the fundamental observer and are defined by

$$E_a = F_{ab}u^b \,, \tag{3.1.2}$$

$$B_a = \frac{\epsilon_{abc}F^{bc}}{2} \,. \tag{3.1.3}$$

It follows that $E_a u^a = B_a u^a = 0$, which ensures that both E_a and B_a are space-like vectors.

The Faraday tensor also determines the energy-momentum tensor of the Maxwell field by means of

$$T^{(em)}_{ab} = -F_{ac}F^c{}_b - \frac{1}{4}F_{cd}F^{cd}g_{ab} \,. \tag{3.1.4}$$

If we combine the above expression with (3.1.1), we take the irreducible decomposition for $T^{(em)}_{ab}$ relative to the u_a-frame

$$T^{(em)}_{ab} = \frac{1}{2}(E^2 + B^2)u_a u_b + \frac{1}{6}(E^2 + B^2)h_{ab} + 2\mathcal{L}_{(a}u_{b)} + \mathcal{P}_{ab} \,, \tag{3.1.5}$$

where $E^2 = E_a E^a$ and $B^2 = B_a B^a$ are the square magnitudes of the two fields, $\mathcal{L}_a = \epsilon_{abc}E^b B^c$ is the electromagnetic Poynting vector and $\mathcal{P}_{ab}$ is a symmetric, trace-free tensor given by

$$\mathcal{P}_{ab} = \mathcal{P}_{\langle ab\rangle} = \frac{1}{3}(E^2 + B^2)h_{ab} - E_a E_b - B_a B_b \,. \tag{3.1.6}$$

Expression (3.1.5) provides a fluid description of the electromagnetic field and manifests it's general anisotropic nature. In particular, the Maxwell field corresponds to an imperfect fluid with energy density $(E^2 + B^2)/2$, isotropic pressure $(E^2 + B^2)/6$, anisotropic stresses given by $\mathcal{P}_{ab}$ and an energy-flux vector represented by $\mathcal{L}_a$. Equation (3.1.5) also ensures that $T^{(em)a}_a = 0$, in agreement with the trace-free nature of the radiation stress-energy tensor.

Finally, we note that putting the isotropic and anisotropic pressure together one arrives at the familiar Maxwell tensor, which assumes the covariant form

$$\mathcal{M}_{ab} = \frac{1}{2}(E^2 + B^2)h_{ab} - E_a E_b - B_a B_b \,. \tag{3.1.7}$$

3.2 Maxwell's equations

We follow the evolution of the electromagnetic field by means of Maxwell's equations. In their standard tensor form these are

$$\nabla_{[c}F_{ab]} = 0 \,, \tag{3.2.1}$$

and

$$\nabla^b F_{ab} = J_a \,, \tag{3.2.2}$$

where (3.2.1) reflects the existence of a 4-potential and J_a is the 4-current that sources the electromagnetic field. With respect to the u_a-congruence, the 4-current splits into it's irreducible parts according to

$$J_a = \mu u_a + \mathcal{J}_a \,, \tag{3.2.3}$$

with $\mu = -J_a u^a$ representing the charge density and $\mathcal{J}_a = h_a{}^b J_b$ the spatial current, for which $\mathcal{J}_a u^a = 0$.

Relative to a fundamental observer, each one of the Maxwell' equations decomposes into a timelike and a spacelike component. Projecting (3.2.1) and (3.2.2) along to the 4-velocity vector u_a one takes the timelike parts, namely the two propagation equations

$$\dot{E}_{\langle a\rangle} = -\frac{2}{3}\Theta E_a + (\sigma_{ab} + \epsilon_{abc}\omega^c)E^b + \epsilon_{abc}A^b B^c + curl B_a - \mathcal{J}_a \,, \tag{3.2.4}$$

and

$$\dot{B}_{\langle a\rangle} = -\frac{2}{3}\Theta B_a + (\sigma_{ab} + \epsilon_{abc}\omega^c)B^b - \epsilon_{abc}A^b E^c - curl E_a \,, \tag{3.2.5}$$

while by projecting (3.2.1) and (3.2.2) orthogonal to the 4-velocity vector u_a one takes the spacelike parts, namely the pair of constraints

$$D^a E_a + 2\omega^a B_a = \mu \,, \tag{3.2.6}$$

and

$$D^a B_a - 2\omega^a E_a = 0 \,. \tag{3.2.7}$$

Equations (3.2.4)-(3.2.7) represent the $1+3$ covariant versions of Ampere's law, Faraday's law, Coulomb's law and Gauss's law respectively. We note that in addition to the usual 'curl' and 'divergence' terms, the expressions (3.2.4)-(3.2.5) also contain terms generated by the relative motion of neighbouring observers (with the same 4-velocity) carried by the kinematic terms in the right-hand side of them. Thus, expression (3.2.6) shows that $\rho_e = -2\omega_a B^a$ is an effective electric charge caused by the relative motion of the magnetic field, while $2\omega_a E^a$ acts as an effective magnetic charge triggered by the relatively moving E-field (see (3.2.7)). The acceleration terms in (3.2.4) and (3.2.5) , on the other hand, reflect the fact that the spacetime is treated as a single entity. Finally, according to (3.2.7) the magnetic vector is not solenoidal unless $\omega^a E_a = 0$.

3.3 Ohm's law

The relation between the 4-current and the electric field, as measured by the fundamental observers, is determined by Ohm's law. Its simplest from satisfies the covariant form

$$J_a = \mu u_a + \varsigma E_a \,, \tag{3.3.1}$$

where ς is the scalar conductivity of the medium. As a result, Ohm's law splits the 4-current into a time-like convective component and a conducting space-like counterpart. Projecting (3.3.1) into the observer's rest space gives

$$\mathcal{J}_a = \varsigma E_a \,. \tag{3.3.2}$$

This form of Ohm's law covariantly describes the resistive magnetohydrodynamic (MHD) approximation in the single-fluid approach. Note that the absence of the induced electric field from the above reflects the fact that the covariant form of Maxwell's equations (3.2.4)-(3.2.7) already incorporates the effects of relative motion. According to (3.3.2), non-zero spatial currents are compatible with a vanishing electric field as long as the conductivity of the medium is infinite (for $\varsigma \to \infty$). Thus, at the limit of ideal magnetohydrodynamics (ideal MHD), the electric field vanishes in the frame of the fluid. On the other hand, zero electrical conductivity implies that the spatial currents vanish, even when the electric field is non-zero.

3.4 Conservation laws

The energy momentum tensor of the electromagnetic field obeys the condition $\nabla^b T^{em}_{ab} = -F_{ab}J^b$, where F_{ab} is the Faraday tensor defined before in equation (3.1.1) and the quantity in the right-hand side represents the Lorentz 4-force. Therefore, for charged matter the conservation of the total energy momentum tensor $T_{ab} = T^m_{ab} + T^{em}_{ab}$ leads to the relation

$$\dot{\rho} = -\Theta(\rho + p) - D^a q_a - 2A^a q_a - \sigma_{ab}\pi_{ab} + E_a \mathcal{J}^a \tag{3.4.1}$$

giving the energy density conservation law and

$$(\rho + p)A_a = -D_a p - \dot{q}_{<a>} - \frac{4}{3}\Theta q_a - (\sigma_{ab} + \omega_{ab})q^b - D^b\pi_{ab} - \pi_{ab}A^b + \mu E^a + \epsilon_{abc}\mathcal{J}^b B^c \tag{3.4.2}$$

representing the momentum density conservation law. The last two terms in the right-hand side of equation represent, in particular, one familiar form of the Lorent force.

The antisymmetry of the Faraday tensor as well as the second of the Maxwell equations (equation (3.2.2)) guarantee the condition $\nabla^a J_a = 0$ and thus the conservation law of the 4-current. Then, using the decomposition (3.3.1) one is led to the covariant charge-density conservation law

$$\dot{\mu} = -\Theta\mu - D^a \mathcal{J}_a - A_a \mathcal{J}_a. \tag{3.4.3}$$

This form of charge density conservation indicates that in the absence of spatial currents, the charge density evolution law depends entirely on the expansion (or contraction) of the fluid.

4 Non-linear ideal magnetohydrodynamics

4.1 The ideal MHD approximation

We are dealing now with the ideal MHD approximation, which will be under consideration from now on. For this reason, we describe in the section below its basic characteristics that will prove very useful in the following sections.

4.1.1 The energy-momentum tensor of the magnetic field

We consider a general spacetime filled with a magnetized single barotropic fluid of very high conductivity. As we have already mentioned, Ohm's law (3.3.2) guarantees that in the frame of the fundamental observer the electric field vanishes despite the presence of non-zero currents. Therefore, in the ideal MHD limit the energy momentum tensor of the residual magnetic field simplifies to

$$T_{ab}^{(B)} = \frac{1}{2}B^2 u_a u_b + \frac{1}{6}B^2 h_{ab} + \Pi_{ab} \,, \tag{4.1.1}$$

with

$$\Pi_{ab} = \frac{1}{3}B^2 h_{ab} - B_a B_b \,. \tag{4.1.2}$$

Accordingly, the B-field corresponds to an imperfect fluid with energy density $\rho_B = B^2/2$, isotropic pressure $p_B = B^2/6$ and anisotropic stresses represented by the symmetric and trace-free tensor Π_{ab}.

If we project equation (4.1.2) parallel to the magnetic force-lines (by contracting it with B^b) we take

$$\Pi_{ab} B^b = -\frac{2}{3}B^2 B_a \,. \tag{4.1.3}$$

Equation (4.1.3) shows that the magnetic field B^a is an eigenvector of the anisotropic stress tensor Π_{ab}, with an eigenvalue $-2B^2/3$. The negative sign shows that the magnetic pressure along the field lines is negative and this reflects the tension properties of the latter. This magnetic tension is the reason for the elasticity of the field lines and of their tendency to remain as"straight" as possible and also to react to any effect that distorts them from equilibrium.

On the other hand, if we project equation (4.1.2) orthogonal to the magnetic-force lines ($\Pi_{ab}v^b = B^2 v^b/3$, where $v^b B_b = 0$) we find a positive eigenvalue $B^2/3$, which shows that the magnetic field exerts a positive pressure orthogonal to its own direction. Thus, neighbouring magnetic field lines tend to push each other apart.

We will see in the following paragraph 4.3 that Lorentz force splits into two components. One them is related to the pressure of the B-field mentioned before and the other with the magnetic tension of the field lines. The latter worths noting since we will see that all the effects of the magnetic field emerge from one or the other component or from both of them.

4.1.2 Maxwell's equations

In the absence of an electric field, Maxwell's equations reduce to a single propagation formula, namely the covariant magnetic induction equation

$$\dot{B}_{\langle a\rangle} = (\sigma_{ab} + \epsilon_{abc}\omega^c - \frac{2}{3}\Theta h_{ab})B^b \,, \tag{4.1.4}$$

and the following three constraints:

$$curl B_a = \mathcal{J}_a - \epsilon_{abc}A^b B^c \,, \tag{4.1.5}$$

$$\omega^a B_a = \frac{1}{2}\mu \,, \tag{4.1.6}$$

$$D^a B_a = 0 \,. \tag{4.1.7}$$

The right-hand side of (6.4.2) is due to the relative motion of the neighbouring observers and guarantees that the magnetic field lines always connect the same matter particles. This means that the field remains frozen-in with the highly conducting fluid. Expression (4.1.5) provides a direct relation between the spatial currents, which are responsible for keeping the field lines frozen-in with the matter, and the magnetic field itself. Equation (4.1.6) shows that the rotating observers will measure a non-zero charge density, triggered by their relative motion, unless $\omega^a B_a = 0$. Finally, (4.1.7) demonstrates that in the absence of magnetic monopoles the field lines remain closed and B_a is a solenoidal vector.

4.2 Magnetic evolution

The magnetic induction equation also provides the evolution law for the energy density of the field. In particular, contracting (6.4.2) with B_a and then using (4.1.2), we arrive at

$$(B^2)^{\cdot} = -\frac{4}{3}\Theta B^2 - 2\sigma_{ab}\Pi^{ab} \,. \tag{4.2.1}$$

For a nearly isotropic fluid the first term in the right-hand side of equation (4.1.2) denotes that an expanding fluid tends to reduce the energy density of the magnetic field. For example, in cosmology, this terms reflects that an expanding universe reduces the energy density of the B-field. The latter shows how difficult is to generate considerable cosmological magnetic fields, for example magnetic fields that could source the galactic dynamo. The non-linear evolution of the anisotropic magnetic stresses comes from the time derivative of (4.1.2), which by means of (6.4.2) and (6.4.3) leads to

$$\dot{\Pi}_{ab} = -\frac{4}{3}\Theta\Pi_{ab} - \frac{2}{3}B^2\sigma_{ab} + 2\sigma_{c\langle a}\Pi^c{}_{b\rangle} - 2\omega_{c\langle a}\Pi^c{}_{b\rangle} \ . \tag{4.2.2}$$

4.3 Conservation laws

In the case of a perfect fluid, the energy momentum tensor is given by

$$T_{ab} = \rho u_a u_b + p h_{ab} \tag{4.3.1}$$

and the conservations laws (2.7.1),(2.7.2) reduce to

$$\dot{\rho} = -\Theta(\rho + p) \tag{4.3.2}$$

$$(\rho + p)A_a = -D_a p. \tag{4.3.3}$$

Similarly, the magnetic energy momentum tensor (3.1.5) for a single magnetized perfect fluid of infinite conductivity reduces to

$$T^{(em)}_{ab} = \frac{1}{2}B^2 u_a u_b + \frac{1}{6}B^2 h_{ab}. \tag{4.3.4}$$

Thus, for a single magnetized perfect fluid of infinite conductivity the total energy-momentum tensor is given by

$$T_{ab} = (\rho + \frac{1}{2}B^2)u_a u_b + (p + \frac{1}{6}B^2)h_{ab} + \Pi_{ab} \ . \tag{4.3.5}$$

Accordingly, the medium corresponds to an imperfect fluid with effective density equal to $\rho + B^2/2$, isotropic pressure given by $p + B^2/6$, zero heat flux and solely magnetic anisotropic stresses represented by Π_{ab}.

If we apply this energy-momentum tensor (4.3.5) to the general conservation law $\nabla^b T_{ab} = 0$, using the MHD form of Maxwell's equations, then the standard conservation law decomposes into the following expressions that, respectively, describe the energy-density

$$\dot{\rho} = -(\rho + p)\Theta, \tag{4.3.6}$$

and the momentum-density

$$(\rho + p + \frac{2}{3}B^2)A_a = -D_a p - \epsilon_{abc}B^b curl B^c - \Pi_{ab}A^b. \tag{4.3.7}$$

conservation.

We note the absence of magnetic terms in (4.3.6). This is due to the magnetic induction equation (4.3.7), which guarantees that the magnetic energy is separately conserved. Also, if there are no pressure gradients, (4.3.7) gives $A_a B^a = 0$. Finally, the left-hand side of (4.3.7) shows that the magnetic contribution of the B-field to the total inertial mass is $2B^2/3$.

If we contract (4.3.7) along B_a we find that the contribution of the B-field to the momentum density vanishes, thus reflecting the fact that the magnetic Lorentz force is always normal to the field lines.

4.4 The magnetic lorentz force

Finally, the second term in the right-hand side of (4.3.7) decomposes as

$$\epsilon_{abc}B^b curl B^c = \frac{1}{2}D_a B^2 - B^b D_b B_a \,. \tag{4.4.1}$$

Equation (4.4.1) shows that the magnetic Lorentz force splits into two stresses. The first term on the right-hand side of the above equation, is due to the magnetic pressure (see equation (4.1.1), while the second term reflects the tension of the field lines. From the latter, we see that if these two stresses balance each other the magnetic field reaches equilibrium and no Lorentz force is present. In any other case, there exists a Lorentz force acting on the particles of the magnetized fluid in a plane orthogonal to the field lines.

It can be proved that if we move along geodesics of the magnetic field B_a the second term in the right-hand side of Equation (4.4.1) gives the constraint

$$B^b D_b B_a = 0 \,, \tag{4.4.2}$$

which shows that in this case any Lorentz force acting on the particles of the magnetized fluid emerges only from the pressure term. This also reflects the fact that the tension component of the Lorentz force appears when we try to deform the field lines from their equilibrium. In other words, the magnetic tension expresses the reaction of the field lines to deformations. These deformations can be caused for example by the kinematical behaviour of the fluid or by charged particles trying to deform the field lines. Additionally, in the general relativistic case such deformations can emerge from spacetime curvature distortions. In any case, the field lines react to the cause of these deformations through magnetic tension.

We will see in the coming sections that in the presence of a magnetic field all the effects on vorticity emerge from the action of the Lorentz force. Decomposition (4.4.1) is extremely important since from it we can each time examine the contribution of magnetic pressure or tension on various effects (i.e on the vorticity evolution equation in this thesis).

5 Friedman models

Cosmic microwave background (CMB) radiation which fills the observable universe almost uniformly as well as the cosmological principle which assumes that observers on Earth do not occupy an unusual or privileged location within the universe as a whole, have eventually led to the belief that the universe is both homogeneous and isotropic. FRW models advocate this conviction and are the simplest cosmological solutions of the Einstein equations.

5.1 General properties

The Friedman models are described by the *Robertson-Walker* line element, which has the form

$$ds^2 = -dt^2 + a^2 dl^2 = -dt^2 + a^2 \left[(1 - Kr^2)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right] . \tag{5.1.1}$$

The second term a^2dl^2 in the righthand side of equation (5.1.1) is the line element of the three dimensional space expressed in a spherical coordinate system (r, θ, ϕ) co-moving with the expansion of the Universe. The latter is denoted by the scale factor a which due to the homogeneity and isotropy of the model is only a function of time t $a = a(t)$. Parameter K is the curvature index and in this case is constant and takes the values $K = 0, \pm 1$. Parameter K determines the geometry of the three dimensional space. Specifically, $K = 0$ denotes an Euclidean geometry, $K = 1$ a spherical and $K = -1$ a hyperbolic one. Based on the value of the curvature index K FRW models are called flat,closed and open respectively.

The three dimensional line element $d\tilde{s}^2 = a^2dl^2$ mentioned before denotes the actual distance $d\tilde{s}$ between two co-moving neighbouring observers. This distance can be written as $d\tilde{s} = adl$, with dl expressing the distance between two co-moving observers (which have no self motion) and a indicating the expansion of the Universe. As a result, the velocity υ at which two neighboring observers are moving away from each other due to the expansion of the Universe is given by

$$\upsilon = \frac{d\tilde{s}}{dt} = Hd\tilde{s}, \tag{5.1.2}$$

where $H = H(t) = \dot{a}/a$ is the Hubble parameter. Equation (5.1.2) is named Hubble's law and states that the expansion velocity between two neighbouring observers is proportional to their mutual distance.

Friedman models are uniform since they satisfy both homogeneity and isotopy. Homogeneity requires only time dependence for the model parameters while isotropy demands the existence of only scalar variables. Thus, by definition in FRW models we have $A_a = \sigma_{ab} = \omega_{ab} = 0$. This means that thE matter that can fitt in an FRW model are perfect fluids only.

These are described by the energy-momentum tensor we have discussed in paragraph 4.3

$$T_{ab} = \rho u_a u_b + p h_{ab}, \tag{5.1.3}$$

where ρ is the matter energy density and p is the isotropic pressure of the fluid (equilibrium pressure in this case). We note here that due to the aforementioned homogeneity of the model ρ, p are only functions of time ($\rho = \rho(t), p = p(t)$).

We have seen in paragraph that in the case of perfect fluids the momentum density conservation law degenerates in a trivial identity while the 'energy density conservation law is written as

$$\dot{\rho} = -3H(\rho + p), \tag{5.1.4}$$

where Hubble's parameter H satisfies by definition the relation $H = \Theta/3$. It is worth noting that the energy density conservation law does not include the curvature index (K) and is identical in all FRW models.

The evolution of FRW models is described by Friedman equations which are given in paragraph 5.2 below and have different solutions based on the type of matter that fills the Universe. We are going to discuss two different types of conventional matter which include non-relativistic matter (satisfying the condition $k_B T \ll mc^2$) with zero pressure $p = 0$ and relativistic matter with $p = \rho/3$ (in this case it holds $k_B T \gg mc^2$). In cosmology, dust represents matter after the recombination while radiation dominates the the epoch after inflation and before recombination (radiation era).

When dealing with zero-pressure matter, namely "dust", the energy density conservation law reduces to the continuity equation

$$\dot{\rho} = -3H\rho, \tag{5.1.5}$$

which accepts the solution

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^3, \tag{5.1.6}$$

where $\rho_0 = \rho(a_0)$. Thus, the matter density of 'dust' ρ is inversely proportional to the volume of the Universe expressed by a^3.
However, things change when dealing with relativistic matter where the equation of state is expressed by $p = \rho/3$ and the energy density conservation law becomes

$$\dot{\rho} = -4H\rho, \tag{5.1.7}$$

with

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^4, \tag{5.1.8}$$

which denotes that in case of relativistic matter energy density ρ decays faster due to the expansion of the Universe than in the dust case.
Relations (5.1.6)(5.1.8) remain the same for all FRW models independent of the curvature index K. Using these relations in the next section we are going to discuss evolution of FRW models during the dust and the radiation epochs.

5.2 The Friedman equations

The Friedman models are the simplest solutions of the Einstein equations (2.4.1) we discussed in paragraph 2.4.

Using now the energy-momentum tensor (5.1.3) and the *Robertson-Walker* line element (5.1.1) that apply in the case of FRW models we derive Friedman equations. The latter are differential evolution equations of the scale factor $a(t)$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\kappa\rho - \frac{K}{a^2} + \frac{1}{3}\Lambda \tag{5.2.1}$$

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -\frac{1}{2}\kappa(\rho + p) + \frac{K}{a^2}, \tag{5.2.2}$$

which can be combined in one equation known as Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\kappa(\rho + 3p) + \frac{1}{3}\Lambda. \tag{5.2.3}$$

The above equation (5.2.1) can be derived also from the general Raychaudhuri equation discussed in paragraph by applying the symmetries of FRW models which lead to

$$H^2 = \frac{1}{3}\kappa\rho - \frac{K}{a^2} + \frac{1}{3}\Lambda. \tag{5.2.4}$$

By taking the derivative of equation (5.2.4) owe are taking again equation (5.2.3)

$$\dot{H} = -H^2 - \frac{1}{6}\kappa(\rho + 3p) + \frac{1}{3}\Lambda. \tag{5.2.5}$$

We remind here that in equations (5.2.4),(5.2.5) H is the Hubble parameter satisfying $H = H(t) = \Theta/3 = \dot{a}/a$. Finally, combining equations (5.2.4),(5.2.5) one is led to the following equation

$$\dot{H} = -\frac{1}{2}\kappa(\rho + p) + \frac{K}{a^2}, \tag{5.2.6}$$

which is identical to equation (2.2.2).

5.3 The deceleration parameter

The acceleration or deceleration of the expanding Universe is determined by the second derivative of the scale factor. This defines the deceleration parameter as

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -\left(1 + \frac{\dot{H}}{H^2}\right), \tag{5.3.1}$$

where a positive sign denotes a decelerating expanding Universe and a negative sign indicates an accelerating expanding Universe.
Rewriting Raychaudhuri equation (5.2.5) with the deceleration parameter one gets

$$qH^2 = \frac{1}{6}\kappa(\rho + 3p) - \frac{1}{3}\Lambda, \tag{5.3.2}$$

where it clear that a positive cosmological constant tends to accelerate the expansion of the Universe while conventional matter with $\rho + 3p < 0$ leads to a deceleration in the expansion.

5.4 The critical density

When we deal with a Euclidean space $K = 0$ considering a zero cosmological constant the first Friedman equation (5.2.4) ia written as

$$H^2 = \frac{1}{3}\kappa\rho_c, \tag{5.4.1}$$

where ρ_c is the critical density for which $K = 0$.
We define now based on the critical density ρ_c the dimensionless density parameter as

$$\Omega = \frac{\rho}{\rho_c} = \frac{\kappa\rho}{3H^2}, \tag{5.4.2}$$

from which we see that $\Omega = 1$ when $K = 0$. Also, density ρlarger than the critical density ρ_c means $\Omega > 1$ and thus a positive curvature index $K = +1$ and a closed FRW model with spherical geometry. On the other hand, $\Omega < 1$ denotes an open FRW model with hyperbolic geometry $K = -1$. There is actually a relation between the density parameter and the geometry of Friedman models.
Taking the time derivative of equation (5.4.2) using the definition of deceleration parameter (5.3.1) and Raychaudhuri equation in the form (5.3.2) one is led to the evolution equation of the density parameter

$$\dot{\Omega} = -(1 + 3w)(1 - \Omega)H\Omega, \tag{5.4.3}$$

where $w = p/\rho$ is the barotropic index of the fluid which appears after we apply in the above equation the energy density conservation law (5.1.4).
Equation (5.4.3) states that the density parameter remains constant when we have a flat Universe with $\Omega = 1$ or when we have matter with $w = -1/3$. In the first case, this actually means that a flat Universe remains flat during it's evolution. In the second case, it denotes that due to the total gravitational energy $\rho + 3p$ remaining 0 the curvature index remains constant although generally non-zero. Finally, it is clear the conventional matter with $\rho + 3p > 0$ leads to an increase in the curvature of the model.

5.5 Solutions to Friedman equations with $K = 0$

In this section we assume flat Euclidean three dimensional space with $K = 0$. We have already derived in a previous section the density relations (5.1.5),(5.1.7) for dust and radiation respactively. We will use these relations to solve the Friedman equations in the case of dust and radiation in a flat universe.

5.5.1 The case of 'dust' with $p = 0$

When dealing with dust, we consider non-relativistic particles of low energy where $p = 0$. With this assumption and based on the fact that we consider a flat Universe with $K = 0$ Friedman equations (5.2.1), (5.2.2),(5.2.3) can be written based on equation (5.1.5) as :

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\kappa\rho_0\left(\frac{a_0}{a}\right)^3 \tag{5.5.1}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = 0 \tag{5.5.2}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\kappa\rho_0\left(\frac{a_0}{a}\right)^3 . \tag{5.5.3}$$

We can integrate equation (5.5.6) and get the relation of the scale factor with the time t in the case of 'dust' :

$$a = a_0\left(\frac{t}{t_0}\right)^{2/3} . \tag{5.5.4}$$

Thus, in the case of 'dust' we have a spatial expansion as the time goes by due to the scale factor increasing with the time. Finally, Hubble's parameter is given in the case of $p = 0$ by the relation

$$H = \frac{\dot{a}}{a} = \frac{2}{3t}. \tag{5.5.5}$$

5.5.2 The radiation era with $p = \rho/3$

In radiation era the energy density of the Universe is dominated by radiation, which is actually relativistic high-energy particles for which $p = \rho/3$. For this case we have derived the density relation (5.1.7) based on which Friedman equations for a flat Universe can be written as :

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\kappa\rho_0\left(\frac{a_0}{a}\right)^4 \tag{5.5.6}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\frac{1}{3}\kappa\rho_0\left(\frac{a_0}{a}\right)^4 \tag{5.5.7}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{3}\kappa\rho_0\left(\frac{a_0}{a}\right)^4 . \tag{5.5.8}$$

Similarly, by integrating the first of the above equations we get the following scale factor relation with time t :

$$a = a_0 \left(\frac{t}{t_0}\right)^{1/2}, \tag{5.5.9}$$

and the relative relation for Hubble's parameter

$$H = \frac{\dot{a}}{a} = \frac{1}{2t}. \tag{5.5.10}$$

Comparing equations —(5.5.5),(5.5.10) it is clear that in radiation era the expansion of the Universe is slower that in the dust era.
In the next section, we are going to perturbe a FRW model using as a background a FRW model in the radiation era including some other characteristics that will be discussed below. Thus, equations (5.5.9),(5.5.10) as well as Friedman equations as written for this era will be very useful in the next sections.

6 Perturbed Friedman models

In the previous section, we have stated the basic characteristics of FRW models. The latter describe in a simple way expanding universes that are homogeneous, isotropic and have no initial vorticity. The main subject under discussion in this thesis is magnetic fields and how they affect vorticity production and survival. In this section, we deal with a perturbed magnetized FRW model. Specifically, we examine the case of a random magnetic field with non-zero pressure in the background and a linear perturbation of it in the perturbed model. As we want to examine the behaviour of the vorticity, we derive firstly the linear vorticity propagation equation. Then, in the next section we discuss the effects of the presence of the B-field on vorticity generation and survival as well as on the evolution of density perturbations.

6.1 Background setup

We assume as a background a FRW model in the radiation era $\bar{p} = w\bar{\rho}$,[3] with w being the barotropic index of the fluid which in the radiation era is equal to $1/3$. As we consider both homogeneity and isotropy for the background model the density, the pressure and the Hubble parameter of the model are only functions of time and have no spatial dependence :

$$\bar{\rho} = \bar{\rho}(t), \bar{p} = \bar{p}(t), \bar{H} = \bar{H}(t) \tag{6.1.1}$$

[3] Bared variables denote background variables.

In addition, we assume that in the background model we have a random magnetic field such that

$$\langle \bar{B}_a \rangle = 0 \tag{6.1.2}$$
$$\langle \bar{B}^2 \rangle \neq 0, \tag{6.1.3}$$

where we consider that $\bar{B}^2$ is only a function of time t and is such that $\bar{B}^2 \ll \rho$. The latter condition indicates the small contribution of the B-field to the total energy of the system. As a result, the presence of the magnetic field does not influence the time evolution of the system.

6.2 Alfven speed

We define as the Alfven speed the dimensionless following ratio

$$c_a^2 = \frac{B^2}{\rho(1+w)}, \tag{6.2.1}$$

where based on the condition $B^2 \ll \rho$ we have $c_a^2 \ll 1$ (recall that we assumed c=1).
The Alfven speed actually describes how strong is the magnetic energy density B^2 relative to the matter energy density.

6.3 Linearization scheme

Before proceeding to the linearization of the equations given in the above sections, we will describe in brief the linearization scheme.

Terms with nonzero background value are assigned zero perturbative order, while those that vanish in the background are treated as first order perturbations. Finally, terms of perturbative order higher than one will be ignored. Based on all these, the parameters $\bar{\rho}, \bar{p}, \bar{H}$ that have nonzero value in the unperturbed model are zero-order variables while the vorticity and the 4-acceleration vector ω_a, A_a which vanish in the background are treated as first-order variables.

As we mentioned in paragraph 6.1 we are dealing with a magnetized FRW unperturbed model during the radiation era, where we have allowed for a nonzero background random magnetic field. Therefore, the energy density $\bar{B}^2$ of the background magnetic field is a zero perturbative order variable. Given that all the magnetic terms in the equations are of order

$\propto B^2$, we will treat the energy density of the perturbed magnetic field B^2 as first order variable. This means that the linear perturbation of the magnetic vector B_a as well as its spatial gradient $D_b B_a$ are of 1/2-order. the magnetic pressure terms (both isotropic and anisotropic) are first-order. The latter means that B^2 and Π_{ab} have perturbative order one. This linerization treatment guarantees the consistency of the linear equations where all the magnetic effects are decribed by firts-order variables.

6.4 Linear equations

We have mentioned before the non-linear equation of the momentum-density conservation law

$$\dot{\rho} = -(\rho + p)\Theta, \tag{6.4.1}$$

the non-linear magnetic induction equation

$$\dot{B}_{\langle a \rangle} = (\sigma_{ab} + \epsilon_{abc}\omega^c - \frac{2}{3}\Theta h_{ab})B^b \,, \tag{6.4.2}$$

the non-linear evolution law for the energy density of the field

$$(B^2)^{\cdot} = -\frac{4}{3}\Theta B^2 - 2\sigma_{ab}\Pi^{ab} \,, \tag{6.4.3}$$

and the non linear equation of the energy-density conservation law

$$(\rho + p + \frac{2}{3}B^2)A_a = -D_a p - \epsilon_{abc}B^b curl B^c - \Pi_{ab}A^b. \tag{6.4.4}$$

Finally, we have the non-linear form of the vorticity propagation equation

$$\dot{\omega}_{\langle a \rangle} = -\frac{2}{3}\Theta\omega_a - \frac{1}{2}curl A_a + \sigma_{ab}\omega^b. \tag{6.4.5}$$

We are going to linearize the above equations around the aforementioned background and use the simpler equations [4]

[4] Under linearization we get $\dot{\omega}_{\langle a \rangle} = \dot{\omega}_a$

$$
\begin{aligned}
\dot{\rho} &= -3\bar{H}(\rho + p) && (6.4.6)\\
(\bar{\rho} + \bar{p} + \frac{2}{3}\bar{B}^2)A_a &= -D_a p - \frac{1}{2}D_a B^2 + B^b D_b B_a && (6.4.7)\\
\dot{B}_a &= -2\bar{H}B_a && (6.4.8)\\
(B^2)^{\cdot} &= -4\bar{H}B^2 && (6.4.9)\\
\dot{\omega}_a &= -2\bar{H}\omega_a - \frac{1}{2}curlA_a && (6.4.10)
\end{aligned}
$$

where we defined the Hubble parameter $H = \Theta/3$ and used the decomposition (4.4.1). We note that the terms $\sigma_{ab}\omega^b, \Pi_{ab}A^b$ are not included in the above equations as they are second-order terms.
Taking the curl of both sides of equation (6.4.7) using the covariant commutation laws described in Appendix C we get

$$
\bar{\beta}curlA_a = 2\dot{\bar{p}}\omega_a + (\bar{B}^2)^{\cdot}\omega_a + \epsilon_{abc}D^b B^d D_d B^c + \epsilon_{abc}B^d D^b D_d B^c, \qquad (6.4.11)
$$

where we have defined the parameter $\bar{\beta}$ as $\bar{\beta} = \bar{\rho} + \bar{p} + \frac{2}{3}\bar{B}^2$. We note here that the magnetic energy density $B^2/2$ as well as the magnetic isotropic pressure $B^2/6$ contribute to the total inertial mass of the system as $B^2/2 + B^2/6 = 2B^2/3$.

The third and forth terms of the equation (6.4.11) decompose as

$$
\begin{aligned}
\epsilon_{abc}D^b B^d D_d B^c &= -curlB^b D_{(b}B_{a)} && (6.4.12)\\
\epsilon_{abc}B^d D^b D_d B^c &= B^d D_d curlB_a + \epsilon_a^{\ bc}R_{scdb}B^d B^s + 2\bar{H}\epsilon_a^{\ bc}\omega_{bd}B^d B_c && (6.4.13)\\
\epsilon_a^{\ bc}R_{scdb}B^d B^s &= \frac{2}{3}\bar{H}\bar{B}^2\omega_a && (6.4.14)\\
2\bar{H}\epsilon_a^{\ bc}\omega_{bd}B^d B_c &= \frac{4}{3}\bar{H}\bar{B}^2\omega_a, && (6.4.15)
\end{aligned}
$$

where for the geometry term (6.4.14) we have used the linear part of the decomposition derived in Appendix A

$$
\epsilon_a^{\ bc}\mathcal{R}_{dcsb}B^s B^d = 2\bar{H}B^2\omega_a + 2\bar{H}\epsilon_a^{\ bc}B^d B_b\omega_{cd}. \qquad (6.4.16)
$$

The decompositions (6.4.12)-(6.4.15) lead equation (6.4.11) to

$$
curlA_a = \frac{2}{\bar{\beta}}\dot{\bar{p}}\omega_a - \frac{(\bar{B}^2)^{\cdot}}{\bar{\beta}}\omega_a - \frac{1}{\bar{\beta}}curlB^b D_{(b}B_{a)} + \frac{1}{\bar{\beta}}B^d D_d curlB_a + \frac{2}{\bar{\beta}}\bar{H}\bar{B}^2\omega_a. \qquad (6.4.17)
$$

We define now the square of the adiabatic sound speed as

$$c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \dot{p}/\dot{\rho}. \tag{6.4.18}$$

We note now that the evolution equation for the barotropic index of the fluid is

$$\dot{w} = -3H(1+w)(c_s^2 - w). \tag{6.4.19}$$

From the above equation it is clear that in our case where the barotropic index of the fluid is constant ($w = 1/3$) we get the relation $w = c_s^2$.

Using the linear equation (6.4.6) for the energy density conservation law and based on the relation we mentioned before between density and pressure is the radiation area $p = c_s^2\rho$, we get for the evolution of pressure the following linear equation

$$\dot{p} = -3\bar{H}c_s^2(\rho + p), \tag{6.4.20}$$

The linear evolution law for the energy density of the magnetic field is (see equation (6.4.3))

$$(B^2)^{\cdot} = -\frac{4}{3}\Theta B^2 \tag{6.4.21}$$

Substituting equations (6.4.20) and (6.4.21) into equation (6.4.17) we get

$$curlA_a = -6\bar{H}\frac{c_s^2(\rho+p)}{\bar{\beta}}\omega_a + \frac{1}{\bar{\beta}}B^d D_d curlB_a - 2\bar{H}\frac{\bar{B}^2}{\bar{\beta}}\omega_a, \tag{6.4.22}$$

where the term $\frac{1}{\bar{\beta}}curlB^b D_{(b}B_{a)}$ does not still appear in the right hand side of equation (6.4.25) regarding that the magnetic field is not so inhomogeneous.
We then calculate the following ratio appearing in equation (6.4.25) as

$$\frac{c_s^2(\rho+p)}{\bar{\beta}} = c_s^2(1 - 2c_a^2/3) \tag{6.4.23}$$

$$\frac{\bar{B}^2}{\bar{\beta}} \simeq c_a^2, \tag{6.4.24}$$

where $c_a \ll 1$ is the Alfven speed.
Using the above calculations (6.4.23) and (6.4.24) in equation (6.4.25) and then substituting

the latter into the initial linear equation for the vorticity (6.4.10), we finally get the linear vorticity propagation equation

$$\dot{\omega}_a = -2\bar{H}\left[1 - \left(\frac{3}{2} - c_a^2\right)c_s^2 - \frac{1}{2}c_a^2\right]\omega_a - \frac{1}{2\bar{\beta}}B^d D_d curl B_a. \tag{6.4.25}$$

7 Linear magnetized vorticity

We have finally derived the linear vorticity propagation equation (6.4.25) and we are going to study various effects on vorticity generation and survival which are and the main subjects under consideration. In particular, we will discuss if the magnetic field can act as a source term for the vorticity. Also, we examine the role of the magnetic tension and pressure in vorticity time evolution. Finally, by forming a second order differential equation for the vorticity we calculate the solution $\omega = \omega(t)$ and discuss the form of the latter. In the last paragraph, we discuss the effect of the magnetic field on the evolution of density perturbations.

7.1 Vorticity generation

Equation (6.4.25) has on the righthand side two terms. The first of them includes vorticity ω_a and thus can not be a source term for vorticity. This means that having no initial vorticity this term can not actually generate vorticity from zero. On the other hand, the second term which comes specifically from the tension part of Lorentz force does not include vorticity and can generate vorticity. Thus, we see that by inserting a magnetic field B_a in our background model vorticity could be generated. The latter is not actually surprising since there is not the first time that the presence of a magnetic field acts as a source term in vorticity production.

The second subject under discussion is what effect has this term on vorticity propagation. From the physical point of view this term has to do with the spatial derivative of the curl of the B field along the field lines. If the latter is positive this term acts against the vorticity leading to its reduction. If it is negative it tends to enhance vorticity and if it is zero it has no effect on vorticity. What is actually interesting in this case, is that in general relativity this term emerges, as we stated, from the tension of the B-field, which describes the tendency of the field lines to remain as "straight" as possible. Specifically, if the field lines are lines or circles this terms becomes zero and does dot affect vorticity at all. Whatever tends to deform the field lines from their equilibrium position (if this could be lines or circular field lines) gives source to this term due to the tension of the field lines. Then, the latter has the afovemetioned effect on vorticity based on its sign.

7.2 The role of the matter and magnetic energy density

The first term on the righthand side of equation (6.4.25) consists of a part that relates to the expansion of the fluid, a part that includes the adiabatic sound speed of the fluid (c_s^2) and a part relating to the Alfven speed (c_a^2).
The first part has a negative sign which means that kinematic vorticies decay with the expansion of the fluid.
The second part has to do with the adiabatic sound speed of the fluid and derives from differences in the pressure of the fluid combined with the Ricci identities. This term indicates that vorticity decays with the expansion of the universe unless the barotropic medium has an equation of state "stiffer" than 2/3. In our case, this term has a positive and constant value so that finally matter density tends to enhance vorticity acting against the result of the expansion of the fluid.
The third part includes the Alfven speed and comes from both the tension and pressure component of Lorentz force combined also with the Ricci identities. We note here that the contribution of the pressure part of Lorentz force finally dominates against the contribution of the tension part of Lorentz force and this term has finally a positive sign. This states that this term has the same effect on vorticity as the matter density of the fluid helping vorticity to survive.

7.3 The vorticity evolution equation

We define the following two parameters

$$A = 1 - \frac{3}{2}c_s^2 - \frac{1}{2}c_a^2 + c_s^2 c_a^2 \qquad (7.3.1)$$

$$B = 2(1+w)\left(1 + \frac{2}{3}c_a^2\right). \qquad (7.3.2)$$

The above parameters are constant due to he fact that $c_s = w = 1/3$ as well as to the fact that in the radiation era we have $B^2 \propto a^{-4}$ (see equation (6.4.3)) and $\rho \propto a^{-4}$ (see equation (5.1.8)). The latter guarantee that the Alfven speed $c_a^2 = B^2/\rho(1+w) = constant$ and thus the aforementioned parameters A and B.

Based on the above parameters we can rewrite the vorticity evolution equation (6.4.25) as

$$\dot{\omega}_a = -2\bar{H}A\omega_a - \frac{1}{B\rho}B^d D_d curl B_a \qquad (7.3.3)$$

We wish to rewrite equation (6.4.25) in a form that will not include directly the magnetic terms. For this reason we take the second derivative of equation (6.4.25) and we get

$$\begin{aligned} \ddot{\omega}_a &= -2A\dot{\bar{H}}\omega_a - 2A\bar{H}\dot{\omega}_a + \frac{1}{B\rho^2}\dot{\bar{\rho}}B^d D_d curl B_a \\ &- \frac{1}{B\rho}\dot{B}^d D_d curl B_a - \frac{1}{B\rho}B^d (D_d curl B_a)^{\cdot} \end{aligned} \tag{7.3.4}$$

In order to continue we need the following linear evolution equations

$$\dot{\bar{H}} = -\frac{\kappa(\bar{\rho}+\bar{p})}{2} + \frac{K}{a^2} \tag{7.3.5}$$

$$\dot{\bar{\rho}} = -3\bar{H}(\bar{\rho}+\bar{p}) \tag{7.3.6}$$

$$\dot{B}^d = -2\bar{H}B^d, \tag{7.3.7}$$

$$\tag{7.3.8}$$

which are the linear expressions of Friedman equation, energy density conservation law equation and magnetic induction equation respectively.

In addition we need the time derivative of the terms $curl B_a, D_d curl B_a$ which we compute analytically in the Appendix B as

$$(curl B_a)^{\cdot} = -3\bar{H} curl B_a \tag{7.3.9}$$

$$(D_d curl B_a)^{\cdot} = -4\bar{H} D_d curl B_a \tag{7.3.10}$$

Finally we reapply the vorticity evolution equation (6.4.25) for $\dot{\omega}_a$ into equation (7.3.3) and finally get the much simpler equation

$$\ddot{\omega}_a + 2\bar{H}(1+A)\dot{\omega}_a - \frac{2K}{a^2}\omega_a = 0, \tag{7.3.11}$$

where we have used the Friedman equation

$$\bar{H}^2 = \frac{\kappa\rho}{3} - \frac{K}{a^2}. \tag{7.3.12}$$

7.4 Solutions

We investigate the solutions of equation (7.3.11) assuming Euclidean space in which the curvature index is equal ro $K = 0$. Then, we get this form of the equation

$$\ddot{\omega}_a + 2\bar{H}(1 + A)\dot{\omega}_a = 0. \tag{7.4.1}$$

We recall that in the radiation era $\bar{H} = 1/2t$. For $w = 1/3, c_a^2 << 1$ the constant A becomes $A = 1/2$ and so we get

$$\ddot{\omega}_a + \frac{3}{2t}\dot{\omega}_a = 0, \tag{7.4.2}$$

which accepts the solution

$$\omega_a(t) = c_1 + c_2\frac{1}{\sqrt{t}}. \tag{7.4.3}$$

We have mentioned in paragraph that in the radiation era we have the relation (see equation 5.5.9)

$$a \propto t^{1/2}, \tag{7.4.4}$$

based on which equation 7.4.3 is written as

$$\omega_a = c_1 + c_2 a^{-1}. \tag{7.4.5}$$

Assuming that we have initial conditions $\omega_0, \dot{\omega}_0, a_0$ equation 7.4.5 can be written as

$$\omega_a = \omega_0 + a_0\dot{\omega}_0 \left(1 - \frac{a_0}{a}\right) \tag{7.4.6}$$

Equation (7.4.6) has two terms on the righthand side. One that is actually constant and one that depends on the scale factor a.

Under time evolution the first term dominates unless there are initial conditions such that condition $\omega_0 + a_0\dot{\omega}_0 = 0$ holds. This means that the presence of the magnetic field helps vorticity to remain constant in comparison with the non-magnetized case where vorticity reduces with the expansion of the Universe.

Even before eventually vanished the second term in the righthand side of equation (7.4.6) has a similar result on vorticity. As we see it is proportional to a^{-1}. If we compare the latter with the relation $\omega_a \propto t^{-4/3} \propto a^{-2}$ (see equation (6.4.5)) that holds in the non-magnetized case we see that the presence of magnetic field actually reduces the decay rate of vorticity and finally helps vorticity to survive.

We note here that in the case of dust $p = 0$ linear perturbations in the magnetic field around a Newtonian analogue of the Einstein-de Sitter universe leads to a relation of the form

$$\omega_a \propto t^{-1} \propto a^{-3/2}, \tag{7.4.7}$$

which again compared with the relation that holds in the non-magnetized case $\omega_a \propto t^{-4/3} \propto a^{-2}$ denotes that the magnetic field indeed help linear vorticity perturbations to remain alive for a longer time interval.
In other words, in both cases there is more residual rotation in the magnetized universe than in it's non-magnetized counterpart.
However, in the radiation era there exists a constant term which surprisingly does not just reduce the decay rate of vorticity but actually tends to keep it constant. This means that the presence of a magnetic field in a FRW Universe both creates linear vorticity perturbations and then help them remain constant.

7.5 Density vortices

In covariant formalism, spatial inhomogeneities in the distribution of a physical quantity are described by the orthogonally projected gradient of the quantity. Based on the latter, the fractional gradient in the matter energy density of a fluid is given by

$$\Delta_a = \frac{a}{\rho} D_a \rho. \tag{7.5.1}$$

The above relation actually determines how two neighbouring observers measure density variations (local density variation). Since $D_a\rho$ is zero in homogeneous spacetimes like FRW models, Δ_a is non-zero within perturbed FRW models like in our case.

There are three types of inhomogeneities. Density perturbations which are scalars, vortices which are vectors and shape distortions which are a combination of scalars and vectors. All of this information can emerge if we compute the spatial derivative of the fractional gradient Δ_a (orthogonally projected). We then have the second-rand tensor $\Delta_{ab} = aD_b\Delta_a$ which by splitting into its irreducible parts becomes

$$\Delta_{ab} = \Delta_{<ab>} + \Delta_{[ab]} = \frac{1}{3}\Delta h_{ab}, \qquad (7.5.2)$$

where $\Delta_{<ab>} = aD_{<b}\Delta_{a>}$, $\Delta_{[ab]} = aD_{[b}\Delta_{a]}$ and $\Delta = aD^a\Delta_a$. The first term in the right-hand side of equation 7.5.2 describes variations in the anisotropy pattern of the gradient field, the second vortex-like distortions and the third one scalar variations in matter energy density.

We are going to examine the behaviour of vortex-like distortions in the matter energy density described by the antisymmetric tensor $\Delta_{[ab]}$. The latter based on Frobenius Theorem (Appendix C) can be written as

$$\begin{aligned} \Delta_{[ab]} &= aD_{[b}\Delta_{a]} \\ &= \frac{a^2}{\rho} D_{[b}D_{a]}\rho \\ &= \frac{a^2}{\rho}\omega_{ab}\dot{\rho} \\ &= -3a^2(1+w)H\omega_{ab}, \end{aligned} \qquad (7.5.3)$$

where we have used the energy density conservation law $\dot{\rho} = -3H(1+w)\rho$.

Using the alternating tensor Levi-Civita ϵ_{abc}, one defines the vector Δ_a as $\Delta_a = \epsilon_{abc}\Delta^{bc}/2$ which finally is written as

$$\begin{aligned} \Delta_a &= \frac{\epsilon_{abc}\Delta^{bc}}{2} \\ &= -\frac{3a^2(1+w)H\epsilon_{abc}\omega^{bc}}{2} \\ &= -3a^2(1+w)H\omega_a, \end{aligned} \qquad (7.5.4)$$

based on the similar relation $\omega_a = \epsilon_{abc}\omega^{bc}/2$. Equations (7.5.3),(7.5.4) denote that linear vorticity perturbations lead to linear vortex-like density perturbations.

In the radiation era we have the analogies $H \propto t^{-1}$ and $a^2 \propto t$. Based on the latter and given that in the radiation era w is constant and equal to $1/3$ we conclude that the factor $-3a^2(1+w)H$ in equations (7.5.3),(7.5.4) is constant. This means that vorticity and matter energy density have the same perturbative behaviour.

It is worth noting also that although in general the presence of a B-field reduces the density perturbations Δ, in the relativistic period tends to enhance them as it does to the vorticity as well.

8 Conclusions-Discussion

Since Einstein proposed his equations of gravitation in 1915, many people explored solutions to them. These solutions contain a description of the geometry of the spacetime involved and how this is produced by the distribution of mass and energy. Among of these solutions, many describe rotating universes with the most famous one being the Godel's rotating Universe in 1949. Godel's Universe was neither expanding nor contracting but instead rotating. In this Universe, a non-rotating observer would see the whole Universe spinning around and conclude that the Universe is rotating. Also, the distances between galaxies do not change with time while a photon sent out would actually execute a wide turn like a boomerang due to the global rotation. But the most amazing fact about Godel's Universe is that he spin Einstein's theory to a new direction, that of time-travel, as it contains closed time-like curves.

Observational data like the expansion of the Universe or the fact that distant galaxies are not rotating relative to the solar system giant gyroscope do not favor Godel's Universe. However, Godel's Universe showed that time-travel is possible in principle.

There have been also other solutions involving rotating Universes as we stated in the Introduction that were able to reproduce the current observational data, mainly that concerning the CMB anisotropies and polarization. Given that we could as well assume that the Universe is rotating, it remains the question of what could have caused this rotation and to which level. In this thesis, we examine a possible source of the global rotation, which is the presence of a magnetic field in the Universe.

We treat the magnetic field as a perturbation around a FRW background model with random magnetic field. We remind here that FRW models describe homogeneous, isotropic expanding Universes that have no initial rotation. In these models, we have added a random magnetic field with non-zero pressure ($< B^2 > \neq 0$, $< B_a >= 0$) in the background. Then, we perturbe that model adding a linear perturbation of the magnetic field ($< B_a > \neq 0$). Our goal is to examine the effects of this perturbation, i.e the effects of the presence of a magnetic field, on vorticity generation and survival as well as on the evolution of density perturbations.

We begin considering a FRW model in the radiation era with the presence of a random magnetic field. This field does not contribute significantly in the system's total energy and thus does not affect the time evolution of the system. In other words, the Alfven speed is much less that one, as the magnetic energy density is much less that the matter energy density. Treating the rotation and the acceleration vector as first order variables we proceed with the linearizarion scheme and give the linear equations that describe the system. From these equations we focus on the what is the main interest of this thesis, namely the vorticity propagation equation.

We use that equation to study the effects of the magnetic field on vorticity generation and survival. In particular, we examine if the magnetic field can source vorticity. It turns out that indeed the magnetic field can generate vorticity, a fact that it is not surprising since it is a well-known fact that the latter can act as a source term for rotation. Then, we study the effect of the field on the vorticity propagation. We find that the relative term in the equation pertains to the spatial derivative of the curl of the field. This term stems from the magnetic tension and specifically the more positive it is the more it acts against vorticity and tend to reduce it. In other words, deforming the magnetic field lines from their straight line position favored by the tension, leads to the decay of the vorticity. As long as the matter and magnetic energy densities are concerned, we find that both of them enhance vorticity and help it to survive against the decay due to the expansion of the Universe.

Next, we study how the magnetic tension and pressure affect the time evolution of rotation. To do this, we derive a second order differential equation for the vorticity that includes only the magnetic terms and solve for $\omega(t)$. The form of the last expression includes two terms. The first term shows that the presence of the magnetic field helps vorticity to survive. This is because the decay rate of vorticity, due to the expansion of the Universe, in the case of the presence of a magnetic field is less than that in the no-magnetic field case. In other words, there is more residual rotation in a magnetized Universe than in its non-magnetized counterpart. The other term is a constant term, which means that the field tends to keep vorticity perturbations constant. In simple words, the magnetic field favors both vorticity production and survival.

In the final section, we examine the effect of the magnetic field on the evolution of density perturbations. We find out a similar effect, where the presence of the field, although in general reduces density perturbations, in the relativistic period tends to enhance them as it does also, as we said, with the vorticity perturbations.

As a final comment it is worth mentioning here that apart from answering the fundamental question of what could lead in a rotating Universe, this vorticity production and survival could answer a lot of other questions too. Among them is the generation of the primordial magnetic fields. Also, as we mentioned before, the relations between the angular momenta of the various spinning objects and their masses could be also explained. It becomes then clear that the subject of a rotating Universe or not is not just a philosophical question to be answered but can explain specific observational data that we have in hand right now or we will have in future.

Appendices

A The geometry term

The irreducible decomposition of the projected Riemann tensor for a perfect magnetized fluid can be written as

$$
\begin{aligned}
\mathcal{R}_{abcd} = & - \epsilon_{abq}\epsilon_{cds}E^{qs} + \frac{1}{3}\left(\rho - \frac{1}{3}\Theta^2\right)(h_{ac}h_{bd} - h_{ad}h_{bc}) \\
& + \frac{1}{2}(h_{ac}\Pi_{bd} + \Pi_{ac}h_{bd} - h_{ad}\Pi_{bc} - \Pi_{ad}h_{bc}) \\
& - \frac{1}{3}\Theta[h_{ac}(\sigma_{bd} + \omega_{bd}) + (\sigma_{ac} + \omega_{ac})h_{bd} - h_{ad}(\sigma_{bc} + \omega_{bc}) - (\sigma_{ad} + \omega_{ad})h_{bc}] \\
& - (\sigma_{ac} + \omega_{ac})(\sigma_{bd} + \omega_{bd}) + (\sigma_{ad} + \omega_{ad})(\sigma_{bc} + \omega_{bc}) \,, \qquad \text{(A.1)}
\end{aligned}
$$

where E_{ab} is the electric part of the Weyl tensor defined in §2.6 and Π_{ab} the magnetic anisotropic tensor defined in §4.1.1.
We are going to compute the geometry term $-\epsilon_a{}^{bc}\mathcal{R}_{dcsb}B^sB^d$.

For the first term after substituting (A.1) to the geometry term we get

$$
\epsilon_a{}^{bc}\epsilon_{dcq}\epsilon_{sbf}B^sB^dE^{qf} = 0 \,, \qquad \text{(A.2)}
$$

where we used the antisymmetry of the alternating tensors ϵ_{abc} and the symmetry of E_{ab}.

The second term appearing gives

$$
\begin{aligned}
\frac{1}{3}\left(\rho - \frac{1}{3}\Theta^2\right)\epsilon_a{}^{bc}(h_{ds}h_{bc} - h_{db}h_{cs})B^sB^d &= \frac{1}{3}\left(\rho - \frac{1}{3}\Theta^2\right)\left(\underbrace{\epsilon_a{}^{bc}h_{cb}}_{0}B^2 - \underbrace{\epsilon_a{}^{bc}B_cB_b}_{0}\right) \\
&= 0 \,, \qquad \text{(A.3)}
\end{aligned}
$$

where similarly we used the symmetry of h_{ab} and the antisymmetry of ϵ_{abc}.

For the third term we have

$$\frac{1}{2}\epsilon_a{}^{bc}\left(h_{ds}\Pi_{cb}+\Pi_{ds}h_{cb}-h_{db}\Pi_{cs}-\Pi_{db}h_{cs}\right)B^sB^d = \frac{1}{2}(\underbrace{\epsilon_a{}^{bc}\Pi_{cb}}_{0}B^2)+\frac{1}{2}(\underbrace{\epsilon_a{}^{bc}h_{cb}}_{0}\Pi_{ds}B^sB^d)$$
$$\underbrace{(-\Pi_{cs}B^sB_b-\Pi_{db}B^dB_c)}_{0}$$
$$=0\,, \tag{A.4}$$

where we used again the the symmetry of Π_{ab} and h_{ab} and the antisymmetry of ϵ_{abc}.

The fourth term can be written as

$$\begin{aligned}-\frac{1}{3}\Theta\epsilon_a{}^{bc}h_{ds}(\sigma_{cb}+\omega_{cb})B^sB^d =& -\frac{1}{3}\Theta(\underbrace{\epsilon_a{}^{bc}\sigma_{cb}}_{0}h_{ds}B^sB^d)\\ &-\frac{1}{3}\Theta(\underbrace{\epsilon_a{}^{bc}\omega_{cb}}_{-2\omega_a}h_{ds}B^sB^d)\\ &=\frac{2}{3}\Theta B^2\omega_a\,.\end{aligned} \tag{A.5}$$

The fifth term leads to

$$-\frac{1}{3}\Theta\underbrace{\epsilon_a{}^{bc}h_{cb}}_{0}(\sigma_{ds}+\omega_{ds})B^sB^d=0\,, \tag{A.6}$$

for the same (anti)symmetry reasons stated above.

We continue with the sixth and the seventh term

$$\frac{1}{3}\Theta\epsilon_a{}^{bc}h_{db}(\sigma_{cs}+\omega_{cs})B^sB^d = \frac{1}{3}\Theta\epsilon_a{}^{bc}(\sigma_{cs}+\omega_{cs})B^sB_b\,, \tag{A.7}$$

$$\frac{1}{3}\Theta\epsilon_a{}^{bc}h_{cs}(\sigma_{db}+\omega_{db})B^sB^d = \frac{1}{3}\Theta\epsilon_a{}^{bc}(\sigma_{db}+\omega_{db})B^dB_c\,. \tag{A.8}$$

Adding the expressions (A.7) and (A.8), these two terms together give

$$\frac{1}{3}\Theta(\epsilon_a{}^{bc}\sigma_{cs}B^sB_c + \epsilon_a{}^{bc}\sigma_{db}B^dB_c) = 0\;, \tag{A.9}$$

and

$$\frac{1}{3}\Theta(\epsilon_a{}^{bc}\omega_{cs}B^sB_c + \epsilon_a{}^{bc}\omega_{db}B^dB_c) = \frac{2}{3}\Theta\epsilon_a{}^{bc}B^dB_b\omega_{cd}\;. \tag{A.10}$$

Next, the first of the last two terms lead to

$$\begin{aligned}
-\epsilon_a{}^{bc}(\sigma_{ds}+\omega_{ds})(\sigma_{cb}+\omega_{cb})B^sB^d = &-(\underbrace{\epsilon_a{}^{bc}\sigma_{cb}}_{0}\sigma_{ds}B^sB^d + \underbrace{\epsilon_a{}^{bc}\omega_{cb}}_{-2\omega_a}\sigma_{ds}B^sB^d \\
&+\underbrace{\epsilon_a{}^{bc}\sigma_{cb}}_{0}\omega_{ds}B^sB^d + \underbrace{\epsilon_a{}^{bc}\omega_{cb}}_{-2\omega_a}\omega_{ds}B^sB^d) \\
&= -2(\sigma_{ds}+\omega_{ds})B^sB^d\omega_a \\
&= -2\omega_a\left(\frac{1}{3}B^2h^{sd} - \Pi^{sd}\right)(\sigma_{ds}+\omega_{ds}) \\
&= -2\omega_a\left(\frac{1}{3}B^2\underbrace{h^{sd}\sigma_{ds}}_{0} + \frac{1}{3}B^2\underbrace{h^{sd}\omega_{ds}}_{0} - \Pi^{sd}\sigma_{ds} - \underbrace{\Pi^{sd}\omega_{ds}}_{0}\right) \\
&= -2\Pi^{sd}\sigma_{ds}\omega_a\;.
\end{aligned} \tag{A.11}$$

For the second of the last two terms we have

$$\begin{aligned}
&\underbrace{\epsilon_a{}^{bc}B^sB^d\sigma_{db}\sigma_{ds}}_{0} + \underbrace{\epsilon_a{}^{bc}B^sB^d(\sigma_{db}\omega_{cs}+\sigma_{cs}\omega_{db})}_{2\epsilon_a{}^{bc}\sigma_{db}\omega_{cs}B^sB^d} + \epsilon_a{}^{bc}B^sB^d\omega_{db}\sigma_{cs} \\
&= 2\epsilon_a{}^{bc}\sigma_{db}\omega_{cs}B^sB^d + \epsilon_a{}^{bc}B^sB^d\omega_{db}\sigma_{cs}\;.
\end{aligned} \tag{A.12}$$

Finally, putting all together the expressions (A.2), (A.7), (A.4), (A.5), (A.6), (A.9), (A.10) we get the final non-linear decomposition of the geometry term discussed in paragraph 6.4

$$\begin{aligned}
-\epsilon_a{}^{bc}\mathcal{R}_{dcsb}B^sB^d = &-\frac{2}{3}\Theta B^2\omega_a + 2\Pi^{ds}\sigma_{ds}\omega_a - \frac{2}{3}\Theta\epsilon_a{}^{bc}B^dB_b\omega_{cd} - 2\epsilon_a{}^{bc}B^sB^d\sigma_{db}\omega_{cs} \\
&-\epsilon_a{}^{bc}B^sB^d\omega_{db}\omega_{cs}\;.
\end{aligned} \tag{A.13}$$

B Linear commutation laws

In general theory of relativity time and spatial convective derivatives do not commute. Then, when dealing with convective derivatives instead of partial derivatives one needs to apply the following linearized identity

$$(D_a S_{b...})^{\cdot} = D_a \dot{S}_{b...} - \bar{H} D_a S_b, \tag{B.1}$$

which actually gives the linear time evolution of the spatial derivative of a tensor.
Based on the above linear identity we get for the term $(curlB_a)^{\cdot}$ to linear order

$$\begin{aligned}(curlB_a)^{\cdot} &= (\epsilon_{abc} D^b B^c)^{\cdot} \\ &= \epsilon_{abc} (D^b B^c)^{\cdot} \\ &= \epsilon_{abc} (D^b \dot{B}^c - \bar{H} D^b B^c) \\ &= 2\bar{H} \epsilon_{abc} D^b B^c - \bar{H} \epsilon_{abc} D^b B^c \\ &= -3\bar{H} \epsilon_{abc} D^b B^c \\ &= -3\bar{H} curlB_a. \end{aligned} \tag{B.2}$$

Similarly,for the term $(D_d curlB_a)^{\cdot}$ to linear order we get

$$\begin{aligned}(D_d curlB_a)^{\cdot} &= D_d(-3\bar{H} curlB_a) - \bar{H} D_d curlB_a \\ &= -3\bar{H} D_d curlB_a - \bar{H} D_d curlB_a \\ &= -4\bar{H} D_d curlB_a. \end{aligned} \tag{B.3}$$

We note that the final relation B.3 denotes the time evolution of magnetized rotational perturbations.

C Covariant commutation laws-Frobenius theorem

Following Frobenius theorem, when acting on a scalar quantity the orthogonally projected covariant derivative operators commute according to

$$D_{[a} D_{b]} f = -\omega_{ab} \dot{f} \ . \tag{C.1}$$

The above is purely relativistic result and underlines the different behaviour of rotating spacetimes within Einstein theory.

Similarly, the commutation law for the orthogonally projected derivatives of spacelike vectors reads

$$D_{[a}D_{b]}v_c = -\omega_{ab}\dot{v}_{<c>} + \frac{1}{2}\mathcal{R}_{dcba}v^d\,, \tag{C.2}$$

where $v_av^a = 0$ and $\mathcal{R}_{abcd}$ the Riemann tensor of the observer's local rest-space. Note that in the absence of rotation, $\mathcal{R}_{abcd}$ is the Riemann tensor of the 3-D hypersurfaces orthogonal to u_a-congruence.

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