Conformal Regge Theory

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Bachelor Thesis

Abstract

In this bachelor thesis we are going to review a paper published by Miguel S. Costa, Vasco Goncalves, João Penedone with the tittle conformal Regge theory. Firstly we introduce some fundamental concepts of Regge theory from non-relativistic scattering in quantum mechanics. Then we review some preliminaries of conformal field theory and present some useful relations necessary for the following text. In addition to that, we review a very important tool for studying conformal regge theory, which is the Mellin amplitude. Furthermore, we are investigating the correlation of Mellin amplitudes with operator product expansion and present a very similar approach with conformal partial waves. With these tools we develop the conformal Regge theory for the four-point function. The ultimate goal of conformal Regge theory is to make prediction for OPE coefficients. In this text we are focusing on $N = 4$ supersymmetric Yang Mills theory in the large N-limit when the theory is weak coupled. More specific, we predict OPE coefficients in leading order.

Contents

1 Introduction

The main goal of this review is to showcase the usage of conformal Regge theory, but before we move on the actual development we should have some kind of understanding of what these words mean.

For starters, Regge theory is a technique to study high energy scattering. It was firstly developed by Tullio Regge in 1959, when he was studying scattering from Coulomb potential. Before quantum chromodynamics (QCD), Regge theory was very important for organizing the phenomenology of hardons. The foundations of this theory are mainly the S-matrix theory and several techniques from complex analysis. As we will see in the Regge theory review section the most important tools are namely, the pole structure of the S-matrix, the Regge trajectory and the Sommerfeld-Watson transformation of the scattering amplitude.

On the other hand we have the conformal part. It refers to conformal field theory (CFT). This is a quantum field theory (QFT), which invariant under conformal transformations. In particular this means that our theory is a scale-invariant theory, meaning that you can expand/shrink the underlying spacetime without changing the theory. In addition to that our theory, well being a QFT, possesses Poincare symmetry. CFT has important applications to condensed matter physics, for example when a system is undergoing a continuous phase transitions (critical points), statistical mechanics, quantum statistical mechanics and string theory. Also, CFT's described QFT's in ultraviolet and infrared scales. An example of a CFT is $N = 4$ supersymmetric Yang Mills theory (SYM), which will be explored in section 6. Probably one of the most interesting features of a CFT are the conformal correlation functions, which due to conformal transformations are constrained to a certain form as we will see in section 3.

In conformal Regge theory we are going to incorporate the tools for Regge theory in a CFT. In particular, we are going to use these tools to predict operator product expansion (OPE) coefficients. Being more specific, our theory will be developed around four-point functions, but the main idea is very useful. This is because calculating OPE coefficients is in general a difficult task and requires explicitly solving the theory.[1].

The main ingredient for developing conformal Regge theory are the Mellin amplitudes. Mellin amplitudes were firstly introduced by Mack [16] and they are essential for our theory. In general, Mellin amplitudes are associated with the n-point functions through Mellin transformation in Mellin space. In addition to that for our case where $n = 4$, we explore through conformal block expansion the pole structure of the Mellin amplitudes and find that these poles are associated with operators appearing in the OPE expansion between the scalars O_1O_2 and O_3O_4. This fact allow us to find a useful expansion for the conformal block. Then we introduce another expansion very similar to conformal block expansion, the conformal partial wave expansion, which will be the main tool for our task.

After that, we develop conformal Regge theory for the four point correlation function

[1]For a QFT, calculating a correlation function in pertubation theory requires all possible Wick contractions, which for even small powers in the interaction terms, is a difficult task to achieve.

in the Regge limit. More specific, we will focus on the leading Regge trajectory. which contains operators with twist equal to two. Then we use it to predict OPE coefficients of the four point correlation function in the N=4 SYM when it is weak coupled with interactions. We do this by considering as external operators single trace operators and restrict our prediction around the free theory and spin $J = 1$. We find the coefficients by using past works about anomalous dimensions together with the results from conformal Regge. The result matches the result from the expression of OPE coefficients calculated directly from the actual free theory.

So at the end of day conformal Regge theory is a technique for calculating OPE coefficients much easier than calculate them from actual theory and hopefully this review provide us a way to showcase that.

2 Regge Theory Review

In this section we are going to introduce Regge theory from the prospective of non-relativistic scattering and consider an application to Coulomb scattering. Regge theory is the study of the analytic properties of scattering amplitudes as a function of angular momentum, where the angular momentum is not restricted to be an integer multiple of $\hbar$ but is allowed to take any complex value. The non-relativistic theory was developed by Tullio Regge in 1959. Through this section we are going to work with natural units and it is mostly based at [6] and [7].

2.1 Scattering Amplitudes

In physics, when we studying particle scattering one of the most important quantity to calculate is the **cross section** σ. This quantity measures the probability of particles to be scattered from a localized potential and it has units of area.

To understand this better, consider an incident particle beam with energy E arriving near the scattering center. A part of this beam is going to be scattered at some angle, thus the incident beam within the area $d\sigma$ is going to be scattered at some solid angle $d\Omega$. When $d\sigma$ gets bigger then $d\Omega$ gets too. In that sense we define the differential cross section $D(\theta) = \frac{d\sigma}{d\Omega}$ and the cross section is given by:

$$\sigma = \int D(\theta) d\Omega \qquad (2.1)$$

The particle beam is a quantum mechanical object, therefore the spatial wavefunction must satisfies the time-independent Schrondinger equation. For a spherical symmetric potential $V(r)$ the angular part is[2] $\sim P_l(cos\theta)$ and our radial part $R(r)$ must be determined from the radial Schrondinger equation:

$$-\frac{1}{2}\frac{d^2u}{dr^2} + [V(r) + \frac{1}{2}\frac{l(l+1)}{r^2}]u = Eu \qquad (2.2)$$

$$u(r) = rR(r) \qquad (2.3)$$

[2] l is the quantum number of the angular momentum, with $l \in \mathbb{N}$.

Now we let the incident beam fixed at the z-direction, thus at $r \to \infty$ the incident wavefunction is $\psi_{inc} \sim e^{ikz}$, where $k^2 = 2E$. The scattered beam is going to be scattered at some angle θ due to spherical symmetry and at $r \to \infty$ the scattered wavefunction[3] is $\psi_{sc} \sim f(\theta)\frac{e^{ikr}}{r}$. Add these two wavefunctions and we obtain the total wavefunction:

$$\psi(r,\theta) \sim e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \ , \quad r \to \infty \qquad (2.4)$$

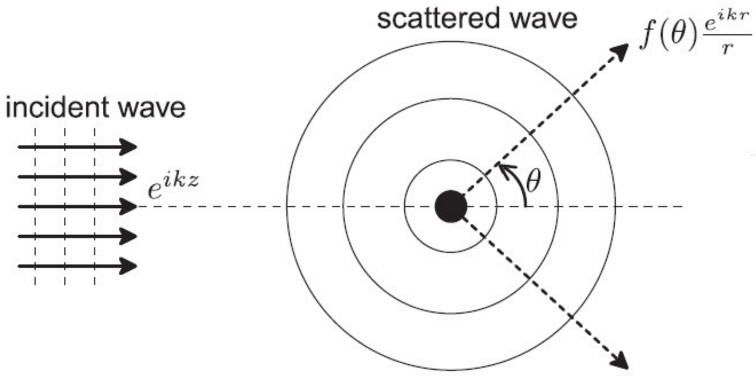

Figure 1: Incoming and outgoing waves as $r \to \infty$

$f(\theta)$ accounts the amount of the scattered wave and we call it **scattering amplitude**. From the discussion about the cross section it should be clear that:

$$D(\theta) = |f(\theta)|^2 \qquad (2.5)$$

2.2 Partial Wave analysis and S-matrix

Our main goal when we studying scattering is to calculate (2.5). To do this requires to solve the Schrondiger equation and this tasks turns out to be very hard or even impossible to do. For this reason we develop other ways to approach this task. One of them is the **partial wave analysis**.

As we saw at (2.4) $f(\theta)$ depends only from θ. This suggests that we can expand this function as series of Legendre polynomials $P_l(cos\theta)$, which form a complete orthogonal set on $[-1, 1]$ and satisfy the orthogonality relation $\langle l|m\rangle = \delta_{mn}\frac{2}{2l+1}$. The expansion goes as:

$$f(\theta) = \sum_{l=0}^{\infty} a_l(k) P_l(cos\theta) \qquad (2.6)$$

Where $a_l(k)$ is the partial wave-l amplitude and depends from the energy through k. For a given potential $V(r)$ we want to calculate $a_l(k)$.

First let's calculate it for the incoming free particle wavepacket. The expansion goes as:

[3]The $\frac{1}{r}$ behavior is fixed from the Schrondiger equation.

$$e^{ik\cdot z} = e^{ikrcos\theta} = \sum_{l=0}^{\infty} b_l(k,r)P_l(cos\theta) \Rightarrow$$

$$e^{ikrcos\theta}P_l(cos\theta)sin\theta = \sum_{l'=0}^{\infty} b_{l'}(k,r)P_{l'}(cos\theta)P_l(cos\theta)sin\theta \Rightarrow$$

$$\int_0^{\pi} e^{ikrcos\theta}P_l(cos\theta)sin\theta d\theta = \int_0^{\pi}\sum_{l'=0}^{\infty} b_{l'}(k,r)P_{l'}(cos\theta)P_l(cos\theta)sin\theta d\theta \Rightarrow^4$$

$$\int_{-1}^{1} P_l(x)e^{ikrx}dx = \sum_{l'=0}^{\infty} \delta_{ll'}\tfrac{2}{2l+1}b_{l'}(k,r) \Rightarrow$$

$$b_l(k,r) = \tfrac{2l+1}{2}\int_{-1}^{1} P_l(x)e^{ikrx}dx \qquad (2.7)$$

To evaluate (2.7) simply consider that $P_l(1) = 1$ and $P_l(-x) = (-1)^l P_l(x)$, so after integrating by parts and consider that $r \to \infty$, we obtain:

$$b_l(k,r) = \tfrac{2l+1}{2}\tfrac{1}{ikr}[e^{ikr} - e^{-i(kr-l\pi)}] = \tfrac{i^l(2l+1)}{kr}sin(kr - l\pi/2) \qquad (2.7)$$

Thus by using (2.7) the partial wave expansion of the free particle is:

$$e^{ikz} = \tfrac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)[e^{ikr} - e^{-i(kr-l\pi)}]P_l(cos\theta) \qquad (2.8)$$

Now let's turn the potential on. As $r \to \infty$ we expect the radial wavefunction to be phase shifted compared with the radial wavefunction of the incoming particle. In that sense the radial part is:

$$R(r) \sim c_l\tfrac{1}{kr}sin(kr - l\pi/2 + \delta_l(k)) \qquad (2.9)$$

Where $\delta_l(k)$ is the phase shift.
The total wavefunction now is:

$$\psi(r,\theta) \sim \tfrac{1}{2ikr}\sum_{l=0}^{\infty} c_l[e^{i(kr-l\pi/2+\delta_l)} - e^{-i(kr-l\pi/2+\delta_l)}]P_l(cos\theta) \qquad (2.10)$$

Compare the coefficient of e^{-ikr} in (2.10) with the one from (2.8) and we get:

$$c_l = (2l+1)e^{i(l\pi/2+\delta_l)} \qquad (2.11)$$

Thus we get for (2.10):

$$\psi(r,\theta) \sim \tfrac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)[e^{i(kr+2\delta_l)} - e^{-i(kr-\pi/2)}]P_l(cos\theta) =$$

$$\tfrac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)[e^{i(kr+2\delta_l)} - e^{-i(kr-\pi/2)} + e^{ikr} - e^{ikr}]P_l(cos\theta) =$$

[4] Let $x = cos\theta$

$$\frac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)[e^{ikr}-e^{-i(kr-l\pi)}]P_l(cos\theta)+\sum_{l=0}^{\infty}(2l+1)\frac{e^{2i\delta_l}-1}{2ik}P_l(cos\theta)\frac{e^{ikr}}{r}==$$

$$e^{ikz}+\sum_{l=0}^{\infty}(2l+1)\frac{e^{2i\delta_l}-1}{2ik}P_l(cos\theta)\frac{e^{ikr}}{r}=e^{ikz}+\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}sin\delta_l P_l(cos\theta)\frac{e^{ikr}}{r}$$

(2.12)

We call the phase factor $e^{2i\delta_l}$ of outgoing wave **S-matrix** and denote it as $S_l(k)$. Compare (2.12) with (2.4) and we obtain:

$$f(\theta)=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}sin\delta_l P_l(cos\theta) \qquad (2.13)$$

We continue with (2.1), (2.5) and we obtain:

$$\sigma=\int|\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}sin\delta_l P_l(cos\theta)|^2 d\Omega=$$

$$\int\sum_{l,m}^{\infty}(2l+1)(2m+1)P_l(cos\theta)P_m(cos\theta)e^{i(\delta_l-\delta_m)}sin\delta_l sin\delta_m d\Omega=$$

$$\frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)sin^2\delta_l=\sum_{l=0}^{\infty}\sigma_l \qquad (2.14)$$

Where σ_l is called partial cross section.

The partial cross section can have many behaviors as a function of energy. When the phase shift passes through $\pi/2$ it appears a sharp peak against a smooth background. Thus, for specific energies around $E\approx E_{res}$, then $\delta_l=\pi/2$ and we write it as:

$$\delta_l=arctan[\frac{\Gamma/2}{E_{res}-E}] \qquad (2.15)$$

Where Γ is a constant which depends on the energy.

We can use the relation $sin^2(arctan(x))=\frac{x^2}{1+x^2}$ with (2.14) to show that σ_l takes the form:

$$\sigma_l=\frac{4\pi}{k^2}(2l+1)\frac{(\frac{\Gamma/2}{E_{res}-E})^2}{1+(\frac{\Gamma/2}{E_{res}-E})^2}=\frac{4\pi}{k^2}(2l+1)\frac{(\Gamma/2)^2}{(E_{res}-E)^2+(\Gamma/2)^2} \qquad (2.16)$$

This form of σ_l around $E\approx E_{res}$ is called Breit-Wigner form. When $\sigma_l^{max}=\frac{4\pi}{k^2}(2l+1)$ we say that we have resonance and $E=E_{res}$.

It is more useful to study resonances form the S-matrix perspective. Near resonance S-matrix takes the form:

$$S_l(k)=e^{2i\delta_l}=e^{i\delta_l}/e^{-i\delta_l}=\frac{cos\delta_l+isin\delta_l}{cos\delta_l-isin\delta_l}=\frac{1+itan\delta_l}{1-itan\delta_l}=\frac{1+i(\Gamma/2)/(E_{res}-E)}{1-i(\Gamma/2)/(E_{res}-E}\Rightarrow$$

$$S_l(k)=\frac{E-E_{res}-i\Gamma/2}{E-E_{res}+i\Gamma/2} \qquad (2.17)$$

If we analytically continue the wavefunction to complex energy, then the energy $E=E_{res}-i\Gamma/2$ corresponds to a pole of the S-matrix (2.16). Moreover for energy $k^2>0$ at $r\to\infty$ we have:

$$R(r) \sim \frac{1}{r}(Ae^{ikr} + Be^{-ikr})$$

Compare it with (2.12) and we get:

$$S_l(k) \sim \frac{A}{B} \qquad (2.18)$$

We can define the S-matrix for any complex k and first we solve the radial equation for complex k. Then at the limit $r \to \infty$ we simply compute the ratio of the outgoing and incoming wave amplitudes, as we saw at (2.18). If $k = i\kappa$ then the energy is $E = -\kappa^2/2 < 0$ and the radial equation is:

$$R(r) \sim \frac{1}{r}(Ae^{-\kappa r} + Be^{\kappa r})$$

If k is such that $B = 0$, it also means that $S_l(i\kappa) = \infty$ and we have a bound state. Thus, we found a way to characterize the bound states, they are poles of the S-matrix. Bound states can be characterized as poles of the S-matrix on the negative real axis in complex energy plane. This means resonances, which are poles near the positive real axis are some kind of bound state. This occurs for $E = E_{res} - i\Gamma/2$, and the time evolution of the state $e^{-iEt} = e^{-iE_{res}t}e^{-\Gamma t/2}$. From this understand that the particle has lifetime of order $1/\Gamma$, so this kind of pole describes some kind of semi-bound state with energy E_{res}. If Γ is large, then the particle is short-lived and it is unstable, but if Γ is small then the particle is long-lived and more stable.

2.3 Complex Angular Momentum and Regge Trajectories

From quantum mechanics we know that angular momentum since it is a quantum number, it can take specific values. But when we study equations like (2.2) l, that is the quantum number of angular momentum, it's just a parameter of the equation and there is no reason to have specific values from a mathematical prospective.

That is what Tullio Regge did in 1959 when he studied non-relativistic scattering by Coulomb potential. In particular he let l to take complex values. Before we jump to Coulomb potential, let's study (2.2) in more general way for $l \in \mathbb{C}$ and study the problem in the notion of the **complex angular momentum plane**. We write (2.2) as:

$$-\frac{d^2u}{dr^2} + [2V(r) + \frac{l(l+1)}{r^2} - 2E]u = 0 \Rightarrow -\frac{d^2u}{dr^2} + [U(r) + \frac{l(l+1)}{r^2} - k^2]u = 0 \qquad (2.19)$$

Where $U(r) = 2V(r)$.

For $r \to 0$ we have $u(r) \sim r^a$ and (2.19) gives:

$$-a(a-1)r^{a-2} + l(l+1)r^{a-2} + U(r)r^a - k^2r^a \sim [-a(a-1) + l(l+1)]r^{a-2} = 0$$

Thus $a = -l$ or $a = l + 1$ and we want the second solution to dominate because we want the solution at 0 to be regular.

On the other hand at $r \to \infty$ we are looking for a purely outgoing wave:

$$u(r) \sim e^{ikr} \qquad (2.20)$$

The total wavefunction is given by (2.12), if we ignore the angular part, then we write it above the threshold ($k^2 > 0$) as:

$$u(r) \sim \frac{(2l+1)}{2ir} e^{i\pi l/2}[e^{-i\pi l/2} e^{2i\delta_l} e^{ikr} - e^{i\pi/2} e^{-ikr}] \Rightarrow$$

$$u(r) \sim N_l [e^{-i(kr - l\pi/2)} - S e^{i(kr - \pi l/2)}] \qquad (2.21)$$

We define the total wavefunction, up to a normalization factor N_l in terms of the S-matrix. At a pole of S the second term becomes infinity large and the result is purely outgoing wave.
Bellow the threshold ($k^2 < 0 \Rightarrow k = i|k|$), (2.21) becomes:

$$u(r) \sim N_l [e^{|k|r + i\pi l/2} - S e^{-|k|r - i\pi l/2}] \qquad (2.22)$$

For the case of (2.22), poles of S yield a bound state. Also below threshold we can have a bound state for $k = -i|k|$, since it gives $k^2 < 0$. The radial equation (2.21) becomes:

$$u(r) \sim N_l [e^{-|k|r + i\pi l/2} - S e^{+|k|r - i\pi l/2}] \qquad (2.23)$$

(2.23) tells us that a bound state is provided by a 0 of S. (2.19) is invariant under transformations $k \to -k$, thus both zero and a pole occur. Therefore, the S-matrix for $l = 0$ has a pole as a function of the energy, in general a family of bound states with $l \in \mathbb{N}$ is represented by a family of energy poles of S.

Bellow threshold bound states have radial wavefunctions $u(r) \sim e^{-|k|r}$ for $r \to \infty$. This is called bound state condition. This condition defines a whole family of solutions as the l increases continuously with physicals values of l corresponding to bound states. We saw that bound states can represented as poles of the S-matrix, so as the energy $E \in \Re$ varies, these solutions can be represented as the continuous movement of an l-plane pole of the S-matrix.

We can generalize this to resonance states. Since bound states and resonances are both poles of the S-matrix, it is only natural unify them in some way. We unify them as properties of the trajectory function $a(E)$. This function describes the movement of S-matrix poles in CAM and it can be obtained from the pole structure of the S-matrix. Therefore, various energies for different complex angular momentum can be described by the function $a(E)$. This trajectory function is called **Regge Trajectory** and we can obtain some useful information about the physical problem by considering the physical values of angular momentum. Bound states all have negative energies and resonance states which are poles of the S-matrix when $E = E_{res} - i\Gamma/2$, have positive energies $Re(E) = E_{res}$. For example by taking the physical values for two bound states and one resonance state we can organize them through one entity in figure 2. All these ideas will be more clear when we consider the Coulomb scattering example.

Moreover, there is way to find the region of the l-plane poles of the S-matrix. To do this we conjugate (2.19):

$$-\frac{d^2u^*}{dr^2} + (\frac{l^*(l^*+1)}{r^2} + U(r) - k^2)u^* = 0 \Rightarrow$$

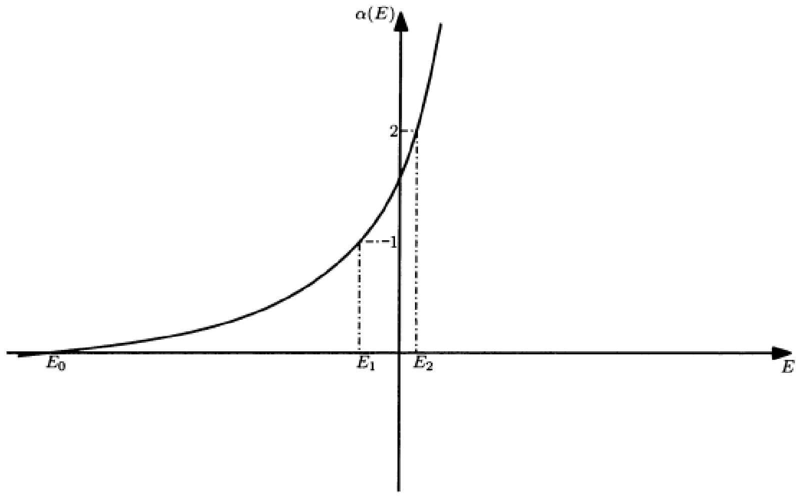

Figure 2: Bound states at E_0, E_1 and a resonance E_2

$$-u\frac{d^2u^*}{dr^2}+(\frac{l^*(l^*+1)}{r^2}+U(r)-k^2)|u|^2=0 \qquad (2.24)$$

Subtracting (2.24) from (2.19) multiplied by u^* and we obtain:

$$-\frac{d^2u}{dr^2}u^*+u\frac{d^2u^*}{dr^2}+(\frac{l(l+1)-l^*(l^*+1)}{r^2})|u|^2=0 \Rightarrow$$

$$\frac{d}{dr}(u\frac{du^*}{dr}-\frac{du}{dr}u^*)+(\frac{l(l+1)-l^*(l^*+1)}{r^2})|u|^2=0 \qquad (2.25)$$

Let $l=l_R+il_I$ and $l(l+1)-l^*(l^*+1)=l_R^2+2il_Il_R+l_R-l_I^2+il_I+2il_I-l_R^2+2il_Rl_I+l_I^2+il_I-l_R=4il_Il_R+2il_I=2il_I(2l_R+1)$. (2.25) becomes:

$$\frac{d}{dr}(u\frac{du^*}{dr}-\frac{du}{dr}u^*)+2il_I(2l_R+1)\frac{|u|^2}{r^2}=0 \Rightarrow$$

$$[u\frac{du^*}{dr}-\frac{du}{dr}u^*]_0^\infty+2il_I(2l_R+1)\int_0^\infty\frac{|u|^2}{r^2}dr=0 \qquad (2.26)$$

Bellow threshold, u satisfies, $u(r)\sim r^{l+1}$ as $r\to 0$ and as $r\to\infty$, $u(r)\sim e^{-|k|r}$. Apply these relations to (2.26) and we obtain:

$$2l_Ir^{2(l_R+1/2)}+2il_I(2l_R+1)\int_0^\infty\frac{|u|^2}{r^2}dr=0$$

The left term vanishes if $l_R>-1/2$ and the integral part is positive, which means that $l_l=0$ for bound states and the S-matrix poles.

Above the threshold it holds true the relation (2.20) and for small r the asymptotic behavior remains the same. (2.26) gives:

$$-2ik+2il_I(2l_R+1)\int_0^\infty\frac{|u|^2}{r^2}dr=0$$

This implies that $l_I>0$.

We conclude that in the notion of complex angular momentum plane, bound states correspond to S-matrix poles for $Re(l)>-1/2$ with $Im(l)=0$ and resonances correspond to S-matrix poles with $Im(l)>0$.

2.4 Coulomb Scattering

At this point we are going to study scattering for the attractive Coulomb potential. The potential for this example is $V(r) = -1/r$. Our first task is to determine the phase-shift δ_l. We substitute Coulomb potential with (2.3) to (2.2):

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + [2E + \frac{2}{r} - \frac{l(l+1)}{r^2}]R = 0 \qquad (2.27)$$

We let $\rho = 2r/n$ with $n = (-2E)^{-1/2}$ and thus $\frac{dR}{dr} = \frac{d\rho}{dr}\frac{dR}{d\rho} = \frac{2}{n}\frac{dR}{d\rho}$, $\frac{d^2R}{dr^2} = \frac{d}{dr}\frac{dR}{dr} = (\frac{d\rho}{dr})^2\frac{d^2R}{d\rho^2} = \frac{4}{n^2}\frac{d^2R}{d\rho^2}$ and (2.27) becomes:

$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + [-\frac{1}{4} + \frac{n}{\rho} - \frac{l(l+1)}{\rho^2}]R = 0 \qquad (2.28)$$

The solution of (2.28) at $\rho \to 0$ is $R \sim \rho^a$, substituting it and we obtain:

$$a(a-1)\rho^{a-2} + 2a\rho^{a-2} - \frac{1}{4}\rho^a + n\rho^{a-1} - l(l+1)\rho^{a-2} \sim [a(a-1) + 2a - l(l+1)]\rho^{a-2} = 0 \Rightarrow$$

$$a_\pm = \frac{\pm(2l+1)-1}{2}$$

Thus we have regular solution around the origin, $R(\rho) \sim \rho^l$. On the other hand at $\rho \to \infty$, (2.28) becomes:

$$\frac{d^2R}{dr^2} - \frac{1}{4}R = 0 \Rightarrow^5 R(\rho) \sim e^{-\rho/2}$$

Putting together these solutions so that our total solution has the appropriate asymptotic forms and we introduce the function $y(\rho)$ such that $R(\rho) = e^{-\rho/2}\rho^l y(\rho)$. Next, we transform (2.28) with $R(\rho) = e^{-\rho/2}\rho^l y(\rho)$ and:

$$\frac{dR}{d\rho} = \rho^l e^{-\rho/2}\frac{dy}{d\rho} + \rho^{l-1}e^{-\rho/2}y(l-\rho/2)$$

$$\frac{d^2R}{d\rho^2} = \rho^l e^{-\rho/2}\frac{d^2y}{d\rho^2} + \rho^{l-1}e^{-\rho/2}\frac{dy}{d\rho}(l-\rho/2) +$$

$$[\rho^{l-1}e^{-\rho/2}\frac{dy}{d\rho} + \rho^{l-2}e^{-\rho/2}y(l-1-\rho/2)](l-\rho/2) - \rho^{l-1}e^{-\rho/2}y/2$$

And we obtain:

$$\rho\frac{d^2y}{d\rho} + (2l+2-\rho)\frac{dy}{d\rho} - (l+1-n)y = 0 \qquad (2.29)$$

This equation has the form of $zy''(z) + (\gamma - z)y'(z) - ay = 0$, which is solved by confluent hypergeometric function[6] $y(z) = F(a;\gamma;z) = \sum_{j=0}^{\infty}\frac{\Gamma(a+j)\Gamma(\gamma)}{\Gamma(a)\Gamma(\gamma+j)}\frac{z^j}{j!}$. Therefore, the equation (2.29) is solved by:

$$y(\rho) = F(l+1-n; 2l+2; \rho) \qquad (2.30)$$

[5]The solution $e^{\rho/2}$ is irregular at $\rho \to \infty$

[6]It exists a second kind of solution $y(z) = z^{1-\gamma}F(a-\gamma+1, 2-\gamma, z)$, which is singular at 0. It can be obtain by transforming the original equation with $y \to z^{1-\gamma}y$.

When we are studying scattering it requires to have energy $E > 0$ and $n = -i(2E)^{-1/2} = -i/k.$, $\rho = 2ikr$. With (2.30) the radial wavefunction R is known up to a normalization factor :

$$R_{kl}(r) = C_{kl}(2ikr)^l F(l+1+i/k; 2l+2; 2ikr) \qquad (2.31)$$

For $r \to \infty$, (2.31) has the following asymptotic form:

$$R_{kl}(r) = \frac{2}{r}|\Gamma(l+1-i/k)|Re[\frac{e^{-i(kr-\pi(l+1)/2+log(kr)/k)}}{\Gamma(l+1-i/k)}G(l+1+i/k, i/k-l, -2ikr)]$$

$$(2.32)$$

Where G has the asymptotic series, $G(a,b,z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(b)} \frac{1}{n! z^n}$. We keep the first term of (2.32) and we get:

$$R_{kl}(r) \sim \frac{2}{r}|\Gamma(l+1-i/k)|Re[\frac{e^{-i(kr-\pi(l+1)/2+log(kr)/k)}}{\Gamma(l+1-i/k)}] =$$

$$\frac{2}{r}|\Gamma(l+1-i/k)|Re[\frac{e^{-i(kr-\pi(l+1)/2+log(kr)/k)}}{\Gamma^*(l+1-i/k)\Gamma(l+1-i/k)}\Gamma^*(l+1-i/k)] =$$

$$\frac{2}{r}\frac{|\Gamma(l+1-i/k)|}{|\Gamma(l+1-i/k)|^2}|Re[e^{-i(kr-\pi(l+1)/2+log(kr)/k)}\Gamma^*(l+1-i/k)] =$$

$$\frac{1}{\Gamma(l+1-i/k)}\frac{2}{r}Re[e^{-i(kr-\pi(l+1)/2+log(kr)/k)}\Gamma^*(l+1-i/k)] \qquad (2.33)$$

We let $\xi = kr - \pi l/2 + log(kr)/k$, $\phi = l+1-i/k$ and $\Gamma^*(l+1-i/k) = Re[\Gamma^*(l+1-i/k)] + iIm[\Gamma^*(l+1-i/k)]$ and we write:

$$Re[ie^{-i\xi}\Gamma(\phi)^*] = sin(\xi)Re[\Gamma^*(\phi)] - cos(\xi)Im[\Gamma^*(\phi)] =$$

$$|\Gamma^*(\phi)|[sin(\xi)cos(arctan(\frac{Im[\Gamma^*(\phi)]}{Re[\Gamma^*(\phi)]}) + cos(\xi)sin(arctan(\frac{Im[\Gamma^*(\phi)]}{Re[\Gamma^*(\phi)]})] =$$

$$|\Gamma^*(\phi)|[sin(\xi)cos(arctan(arg[\Gamma^*(\phi)]) + cos(\xi))sin(arctan(arg[\Gamma^*(\phi)])] =$$

$$|\Gamma(\phi)|sin(\xi + arctan(arg[\Gamma(\phi)])) =$$

$$|\Gamma(l+1-i/k)|sin(\xi + arctan(arg\Gamma(l+1-i/k)]))$$

$$(2.34)$$

Put (2.34) and (2.33) together and we get:

$$R_{kl}(r) \sim \frac{2}{r}sin(\xi + arctan[arg\Gamma(l+1-i/k)])$$

From this it is clear that the phase shift is:

$$\delta_l(k) = arg[\Gamma(l+1-i/k)] \qquad (2.35)$$

Now we can calculate the S-matrix with (2.35) and we get[7]

$$S_l(k) = e^{2i\delta_l} = e^{i\delta_l}/e^{-i\delta_l} = \frac{\Gamma(l+1-i/k)}{\Gamma(l^*+1+i/k)} \qquad (2.36)$$

The poles of the S-matrix are given from gamma matrix in the numerator[8]. In particular, $\Gamma(z)$ has poles at $z = 0, -1, -2, -3, ..$ and thus S-matrix has poles at:

$$l_j + 1 - i/k = -j, \qquad j = 0, 1, 2, 3, .. \qquad (2.37)$$

These poles are known as **Regge poles**. And this is means that the Regge trajectory, since $k = \sqrt{2E}$, is:

$$a_j(E) = -j - 1 + \frac{i}{\sqrt{2E}} \qquad (2.38)$$

If the energy is varying, we see that if $E \to -\infty$, the j-th Regge pole tends to $l = -j - 1$. For $\epsilon > 0$ be small, as the energy winds up to $-\epsilon$ from $-\infty$, all poles reside on the real l-axis and move towards to $Re(l) = \infty$ with $Im(l) = 0$. If (2.38) is equal to physical values $0, 1, 2, 3, ..$ then we obtain Bohr's atom formula for bound state energies:

$$E_{jl} = -\frac{1}{2(j+l+1)^2} \qquad (2.39)$$

For $E \to \epsilon$ the $j-$Regge pole is described from the line $Re(l) = -j - 1$ and tends to $l = -j - 1$ as $E \to \infty$. From this description we can draw Regge trajectory for the Coulomb potential at complex angular momentum plane in figure 3.

2.5 Regge Representation of the Scattering Amplitudes

Our starting point is (2.13). We want to write it in terms of the S-matrix and we have:

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\frac{e^{i\delta_l}-e^{-i\delta_l}}{2i}P_l(cos\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l}-1)P_l(cos\theta) =$$

$$\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(S_l-1)P_l(cos\theta) \qquad (2.40)$$

We want to write (2.40) so that the scattering amplitude is equal to a infinity sum of residues of a contour integral. This conversion is a technique known as **Sommerfeld-Watson transformation**.

We define $z = l + 1/2$ and the function $F(z) = -\frac{\pi f(z)}{cos\pi z}$ with the assumption that $f(z)$ is an analytic function in z. This means that $F(z)$ has simple poles for $z_n = \frac{2n+1}{2}$ with $n = 0, 1, 2, 3, ..$. The residues of such poles are:

$$Re(F(z_n), z_n) = \lim_{z \to z_n}[(z - z_n)\frac{-\pi f(z)}{(z-z_n)(-\pi sin(\pi z_n)) + ..}] = \frac{f(z_n)}{sin(\pi z_n)} = (-1)^n f(z_n)$$

Thus we have the contour integral:

$$\frac{1}{2i}\oint \frac{f(z)}{cos(\pi z)}dz = \sum_{n=0}^{\infty}(-1)^n f(n + 1/2) \qquad (2.41)$$

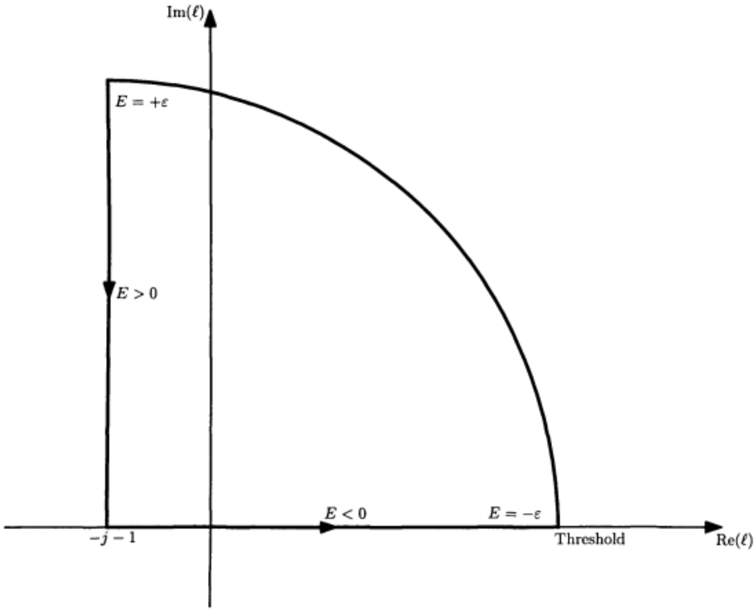

Figure 3: Regge trajectory moving to the right from j-1 for the attractive Coulomb potential

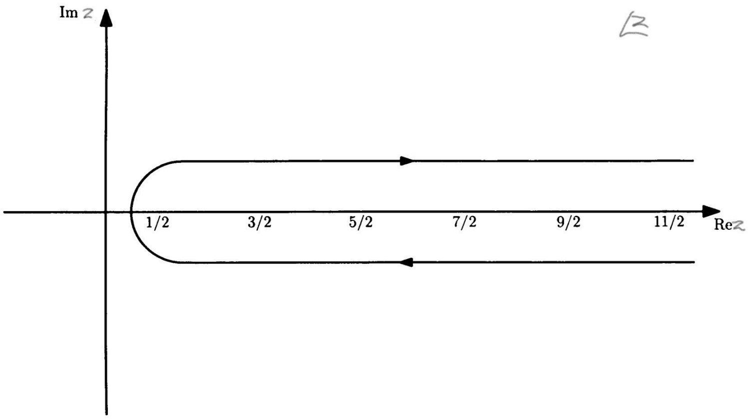

Figure 4: Contour in the complex plane

Where the contour can be seen in figure 4.

We want to use (2.41) to perform the Sommerfeld-Watson transformation for our scattering amplitude (2.40). To do this we assume the S-matrix is analytic function of l and we have:

[7]We also use $e^{iarg(z)} = z$ and $e^{-iarg(z)} = z^*$

[8]Gamma function is never equal to zero

$$\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(S_l-1)P_l(-cos\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(-1)^l(2l+1)(S_l-1)P_l(cos\theta) = \sum_{l=0}^{\infty}(-1)^l f(l+1/2)$$

Thus just as we saw at (2.41), we have the expression:

$$f(\theta) = -\frac{1}{2k}\oint \frac{z(S_{z-1/2}-1)P_{z-1/2}(-cos\theta)}{cos(\pi z)}dz = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(S_l-1)P_l(-cos\theta) \qquad (2.42)$$

Since Legendre polynomials are free of poles and by assuming that the integral is well behaved for $|z| \to \infty$ we can deform the contour so that the new one encloses singularities of the S-matrix in the first quadrant. Then (2.42) becomes:

$$f(\theta) = -\frac{1}{2k}\oint \frac{z(S_{z-1/2}-1)P_{z-1/2}(-cos\theta)}{cos(\pi z)}dz - \frac{i\pi}{k}\sum_{n=0}^{\infty}\frac{z_n r_n}{cos\pi z_n}P_{z_n-1/2}(-cos\theta) \qquad (2.43)$$

Where z_n and r_n are the positions and residues of n-pole respectively. This expression is known as the **Regge representation** of the scattering amplitude and it's very important. The reason why is that it provide us an alternative way to compute scattering amplitudes which is simpler than calculate the partial wave sums for cases which S-matrix has finite poles.

3 Conformal Field Theory Preliminaries

In this section we are going to review some useful relations from conformal field theory that we are going to use later in this text. Although the purpose of this section is to provide a further understanding to conformal Regge theory, it also provides a nice review for some preliminaries of the conformal field theory. This section us mostly based on [2] and [5].

3.1 Conformal Transformation

Conformal field theory is a quantum field theory which is invariant under conformal transformations. From this statement follows that the flat metric is invariant up to factor $\Omega^2(x)$. Consider a conformal transformation in our coordinates $x'^{\mu}x'_{\mu} = \Omega^2(x)x^{\mu}x_{\mu}$. This follows from the fact that if conformal transformations leave our theory invariant then the inner product changes by a factor $\Omega^2(x)$. Then by the definition of $x_{\mu} = \eta_{\mu\nu}x^{\nu}$ and that of the general coordinate transformation we have

$$x'^{\mu}x'_{\mu} = \Omega^2(x)x^{\mu}x_{\mu} \Rightarrow$$

$$\eta'_{\alpha\beta}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}} = \eta_{\mu\nu}\Omega^2(x) \qquad (3.1)$$

We should also note that from the definition of the metric space conformal transformations leave the angles invariant. In addition to that conformal transformations are elements of a Lie group which is called conformal group. This group has the following generators and finite transformations.

Momentum generator, $P_\mu = \partial_\mu$, creates spacetime translations a^μ, with finite transformation:

$$x'^\mu = x^\mu + a^\mu. \qquad (3.2)$$

Lorentz generator, $L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$, creates boosts and rotations $\omega_{\mu\nu}$, with finite transformation:

$$x'^\mu = e^{\frac{\omega_b^a L_a^b}{2}} x^\mu \qquad (3.3)$$

Dilation generator, $D = x^a \partial_a$, creates scale transformations σ, with finite transformation:

$$x'^\mu = e^\sigma x^\mu \qquad (3.4)$$

Special conformal generator $K_\mu = x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu$, creates parameter b^μ, with finite transformation:

$$x'^\mu = \frac{x^\mu + x^2 b^\mu}{1 + 2b_\nu x^\nu + b^2 x^2} \qquad (3.5)$$

We want to know how scalar operators transform. Dilations are scalings and thus anything that has dimensions is affected by it. Consider the action in d-dimensions:

$$S = \int dx^d \partial_\mu O(x) \partial_\mu O(x)$$

From dimensional analysis follows (in natural units) $[S] = 0 \Rightarrow 2[O(x)] + 2 - d = 0 \Rightarrow [O(x)] = \frac{d-2}{2}$. Also, since $\hbar = c = 1$ it follows for mass and length units that $[M] = [L]^{-1}$.

We define now the dilation weight Δ to be equal to the mass dimension. Therefore, for the scalar operator $O(x)$, $\Delta = \frac{d-2}{2}$.

Under dilation transformation $x'^\mu = e^\sigma x^\mu = \lambda x^\mu$, then since $[M] = [L]^{-1}$ we have:

$$O'(x') = \lambda^{-\Delta} O(x)$$

This can be generalized for any conformal transformation. Moreover, for any scalar primary operator it holds true:

$$O'(x') = \Omega^{-\Delta}(x) O(x) \qquad (3.6)$$

If we insert dilation transformation (3.4) into (3.1) we obtain:

$$\eta'_{ab} \delta_\mu^a \delta_\nu^b e^{-2\sigma} = \Omega^2 \eta_{\mu\nu} \Rightarrow \Omega = e^\sigma$$

Thus $O(x)$ transforms under dilation:

$$O'(x') = e^{-\Delta\sigma} O(x) \qquad (3.7)$$

For translation transformation (3.2) it's easy to show from (3.1) that:

$$O'(x') = O(x) \qquad (3.8)$$

For the case of inversions $x'^{\mu} = \frac{x^{\mu}}{x^2}$, since $\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \frac{1}{x^2}(\delta^{\mu}_{\nu} - 2\frac{x^{\mu}x_{\nu}}{x^2})$ we can use (3.1) to show that $\Omega(x) = \frac{1}{x^2}$ and thus:

$$O'(x') = \frac{1}{x^{2\Delta}}O(x) \qquad (3.9)$$

We consider the infinitesimal version of (3.6) for a general conformal transformation. Infinitesimally, the general conformal transformation is $x'_{\mu} = x_{\mu} + \xi_{\mu}$ and $\Omega^{-\Delta} = 1 - k(x)\Delta$, where $k(x)$ and ξ are functions of x^{μ} associated with conformal transformations. So (3.6) becomes

$$O'(x') = O(x+\xi) \approx O'(x) + \xi^{\mu}\partial_{\mu}O(x) = O(x) - \Delta k(x)O(x) \Rightarrow$$

$$\delta O(x) = O'(x) - O(x) = -\Delta k(x)O(x) - \xi^{\mu}\partial_{\mu}O(x) \qquad (3.10)$$

It can be shown from the infinitesimal version of (3.1) that:

$$\xi_{\mu}(x) = \alpha_{\mu} + \omega_{\mu}{}^{\nu}x_{\nu} + \sigma x_{\mu} + b_{\mu}x^2 - 2b\cdot x x_{\mu} \qquad (3.11)$$

$$k(x) = \sigma - 2b\cdot x \qquad (3.12)$$

So far we have discuss only scalars operators, but there are also operators with indices. The infinitesimal version transformation of an operator with one index is:

$$\delta O_a(x) = -\Delta k(x)O_a(x) - \xi^{\mu}\partial_{\mu}O_a(x) + \rho^{\mu}{}_{\nu}(S^{\nu}{}_{\mu})_a{}^{b}O_b(x) \qquad (3.13)$$

$$\rho^{\mu}{}_{\nu} = \omega^{\mu}{}_{\nu} + 2(b^{\mu}x_{\nu} - x^{\mu}b_{\nu}) \qquad (3.14)$$

$S^{\nu}{}_{\mu}$ is a matrix of the appropriate Lorentz representation.

3.2 Conformal Correlation Functions

The key objects in a CFT are local operators and the key data defining a CFT are the correlation functions. If we obtain these functions then we have solved the theory. So far we have seen conformal symmetries. These symmetries can constrain correlation functions and give us their form without solving the actual theory. We are going to review some cases without showing the calculations from the constrains.

a) **2-point functions** are fixed up to a normalization. The special conformal transformation constrains the two point function to be non-zero for equal weight operators, thus for scalar operators we have:

$$\langle O_{\Delta_1}(x)O_{\Delta_2}(y)\rangle = \frac{C_{12}}{|x-y|^{2\Delta_1}} \qquad (3.15)$$

b) **3-point functions** are fixed up to a constant, thus for scalars:

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}} \qquad (3.16)$$

Where $x_{ij} = x_i - x_j$.

c) **4-point functions** can not be fixed by conformal symmetry and there is a limited number of constrains we can put on the system. The result for scalars is:

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = f(u,v)\prod_{i<j}^4 |x_{ij}|^{\frac{\Delta_1+..\Delta_4}{3}-\Delta_i-\Delta_j} \qquad (3.17)$$

Where $u = \frac{(x_{12})^2(x_{34})^2}{(x_{13})^2(x_{24})^2}$ $v = \frac{(x_{14})^2(x_{23})^2}{(x_{13})^2(x_{24})^2}$ are the cross ratios and $f(u,v)$ is a general function of u, v.

So far we have consider only correlation functions with scalar operators, but it is possible to have spin operators which are tensors. The simplest of them the 2-point function of spin-1 operators with the same weight Δ:

$$\langle O_\mu(x)O_\nu(y)\rangle = C_J\frac{I_{\mu\nu}(x-y)}{(x-y)^{2\Delta}} \qquad (3.18)$$

With $I_{\mu\nu}(x) = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}$ and C_J it's a constant. $I_{\mu\nu}$ its an orthogonal matrix associated with inversions as $\frac{\partial x'^\mu}{\partial x^\nu} = \frac{1}{x^2}(\delta^\mu_\nu - 2\frac{x^\mu x_\nu}{x^2}) = \frac{1}{x^2}I^\mu{}_\nu$ as we saw from deriving (3.9). We can check this definition for a Lorentz transformation. For such transformation (3.13) with (3.14) give us the familiar infinitesimal Lorentz transformation, so we conclude that $O'_\mu(x') = \Lambda_\mu{}^\nu O_\nu(x)$. On the other hand, $I'_{\mu\nu}(x') = \Lambda_\mu{}^a\Lambda_\nu{}^b\eta_{ab} - 2\Lambda_\mu{}^a\Lambda_\nu{}^b\frac{x_a x_b}{x^2} = \Lambda_\mu{}^a\Lambda_\nu{}^b I_{ab}(x)$. With that in mind (3.18) for $y = 0$ gives:

$$\langle O'_\mu(x')O'_\nu(0)\rangle = \Lambda_\mu{}^a\Lambda_\nu{}^b\langle O_a(x)O_b(0)\rangle = C_J\frac{I'_{\mu\nu}(x')}{(x)^{2\Delta}} = \Lambda_\mu{}^a\Lambda_\nu{}^b C_J\frac{I_{ab}(x)}{(x)^{2\Delta}}$$

Thus the RHS and LHS agree.

The form of (3.18) can be generalized to l spin operators which are represented from traceless symmetric tensors that it:

$$\langle O_{\mu_1..\mu_l}(x)O_{\nu_1..\nu_l}(0)\rangle = C_J(\frac{I_{\mu_1\nu_1}..I_{\mu_l\nu_l}}{x^{2\Delta}} - traces) \qquad (3.19)$$

Where "traces" are terms associated with $\delta_{\mu_i\mu_j}$ and $\delta_{\nu_i\nu_j}$. We want such terms, since $I_{\mu\nu}$ has mixed indices from the 2 tensors and we mention that we are interesting in traceless tensors (that's it $O_{\mu_1..\mu_i\mu_i..\mu_l} = 0$).

3.3 Operation Product Expansion and Conformal Blocks

As we mention above conformal constrains are not enough to fix higher point functions. In CFT, we can write the product of two local operators as a sum of local primary operators. This is called operator product expansion (OPE) and it's a very useful tool. So, the product of two operators $O_1(x_1), O_2(x_2)$ can be written as:

$$O_1(x_1)O_2(x_2) = \sum_k C_{12k,\mu_1..\mu_{J_k}}(x_{12},\partial_2)O_k^{\mu_1..\mu_{J_k}}(x_2) \qquad (3.20)$$

Where $C_{12k,\mu_1..\mu_{J_k}}$ is an operator which is a power series in ∂_2 and generates descendants[9]. For the operators $O_k^{\mu_1..\mu_{J_k}}$ inside the OPE with mass dimension Δ_k and J_k we label them with their **twist**, which is $\Delta_k - J_k$.

Consider the 4 point-function:

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle$$

And we take OPE between O_1O_2 and O_3O_4:

$$\begin{gathered}\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \\ \sum_{k,k'} C_{12k,\mu_1..\mu_{J_k}}(x_{12},\partial_2)C_{34k',\nu_1..\nu_{J_k}}(x_{34},\partial_4)\,\langle O_k^{\mu_1..\mu_{J_k}}(x_2)O_{k'}^{\prime\nu_1..\nu_{J_k}}(x_4)\rangle = \\ \sum_k C_{12k}C_{34k}G_{J_k,\Delta_k}(x_1,x_2,x_3,x_4)\end{gathered} \tag{3.21}$$

Where in the last step we have diagonalize the 2-point function thus we get terms for $O = O'$. $C_{12k}C_{34k}$ are the leading coefficients from the power series we saw above and $G_{J_k,\Delta_k}(x_1,x_2,x_3,x_4)$ is called **conformal block**.

3.4 Radial Quantization and State-Operator Correspondence

When we first encounter quantum field theory we proceed with canonical quantization where we quantize on equal time slicing. In CFT it's useful to quantize on equal radial slices. This process is called **radial quantization**.

Consider an Euclidean CFT on a D-dimensional cylinder. This CFT has $D-1$ spatial dimensions and 1 time dimensions. To understand this better consider the metric of this space for $D = 2$:

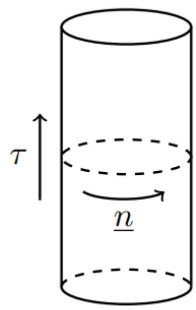

Figure 5: $D = 2$ cylinder

[9]Descendants are operators that can written as a total derivative. An example of them is $\partial_\mu O(x)$ with mass dimension $\Delta + 1$.

$$ds^2 = -dt^2 + dn^2 \Rightarrow ds^2 = d\tau^2 + dn^2 \qquad (3.22)$$

Where in the last step we wick rotate the time variable ($\tau = -it$). From this we see that the time τ runs from the bottom of cylinder (infinity past) to the top of the cylinder (infinity future) and the one spatial variable runs across the cylinder surface for a given τ. Thus a spatial slice is a circular slice of the cylinder which is parameterized by the unit vector n.

Consider the coordinate transformation:

$$\tau \to r = e^{\tau} \qquad (3.23)$$

$$dr = e^{\tau} d\tau = r d\tau \qquad (3.24)$$

The metric (3.22) according to (3.23) and (3.24) becomes:

$$ds^2 = \frac{dr^2}{r^2} + dn^2 = \frac{1}{r^2} \underbrace{(dr^2 + r^2 dn^2)}_{\text{"flat space metric"}}$$

Thus transformation has scaled the metric by an overall factor. This means that it exists a conformal transformation such that, Cylinder $\to$ Flat space. The infinity past on the bottom of the cylinder is mapped at the origins of the flat space as for the infinity future on the top of the cylinder is mapped at the infinity radius circle of the flat space.

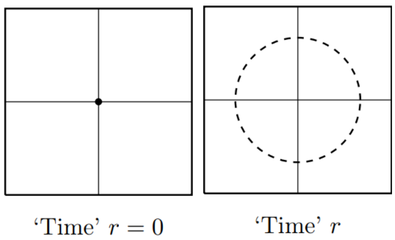

Figure 6: flat space map

In a CFT, there is a 1-1 correspondence between states and local operators. This is called **state-operator correspondence**. In this text is important to us to understand better the **state** $\Rightarrow$ **local operator** correspondence, so in the following we are going to motivate it.

Our first encounter with quantum mechanics, the states that we consider are equivalent to the Schrondinger wavefunction $\psi(x, t_0)$ in a particular time t_0. This state is defined across a spatial side. The intial state $\psi_I(x)$ is defined at $t_0 \to -\infty$, but in quantum field theories we have wavefunctionals rather than wavefunctions. Thus for the intial state rather than x we have $\phi(x)$ if we are talking about a scalar field theory and $\psi_I = \psi_I[\phi(x)]$. When we go to radial quantization the limit $t_0 \to -\infty$ in the intial state becomes $r \to 0$. Therefore, ψ_I is a function of $\phi(0), \partial_\mu \phi(0), \partial_\mu \partial_\nu \phi(0)$.. all local at 0 and we see that for a given state we can obtain a local operator, by taking infinity dilation backwards, that is is shrink it down to the origins.

Thus we can obtain a local operator at the origin for a given state. We label our states by their corresponding operator at the origins:

$$|O\rangle = O(0)\,|0\rangle$$

We said that $|O\rangle$ is dual to the operator O. In this sense we say that the vacuum is dual to identity:

$$|0\rangle = |1\rangle$$

3.5 Conformal Blocks in the Dolan-Osborn coordinates

We already introduced the conformal block $G_{J,\Delta}$ of the 4-point function in the text. In this subsection we introduce the Dolan-Osborn coordinates and a useful expansion for the conformal block.

We start from (3.17) and we consider operators of the same weight Δ for simplicity:

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle = f(u,v)\frac{|x_{24}|^{\frac{-2\Delta}{3}}|x_{14}|^{\frac{-2\Delta}{3}}}{|x_{12}|^{\frac{2\Delta}{3}}|x_{34}|^{\frac{2\Delta}{3}}|x_{13}|^{\frac{2\Delta}{3}}|x_{23}|^{\frac{2\Delta}{3}}} = g(u,v)\frac{1}{(x_{12}^2)^{\Delta}(x_{34}^2)^{\Delta}}$$

(3.25)

Where $g(u,v)$ is a function of conformally invariant cross ratios. We can decompose $g(u,v)$ in terms of conformal blocks $G_O(u,v)$ of the primary operators O appearing in OPE. Since all four operators have the same weight we write:

$$g(u,v) = \textstyle\sum_O C_O^2 G_O(u,v) \qquad (3.26)$$

At this point we introduce the **Dolan-Osborn** coordinates:

$$u = z\bar{z}, \qquad v = (1-z)(1-\bar{z}) \qquad (3.27)$$

The conformal blocks now are parametrized by the complex variable z and (3.8) allow us to set $x_1 = 0$, $x_3 = (1,0,..,0,0)$, $x_4 \to \infty$. At this point we are working on the z-plane, we can use the radial quantization by using the inverse transformation of (3.23), which is $\tau = log(r)$. The operators at $x_1 = 0$ and $x_4 \to \infty$ are mapped at the bottom and top of cylinder respectively and by using the state-operator correspondence we have $O(x_1 = 0)\,|1\rangle = |O\rangle$ and $\langle O| = \langle 1|\,O(x_4 \to \infty)$. Also, $\tau_3 = log(r_3) = log(1) = 0$ and $\tau_2 = log|z|$. Therefore we write (3.25) as:

$$\langle O|\,O(0,n_3)O(\tau_2,n_2)\,|O\rangle \qquad (3.28)$$

The units vectors n_i don't change under the transformation and we can think them as the unit vector $\hat{\theta}$ in our familiar 2 dimensional plane (r,θ) when we refer to the d-dimensional Euclidean plane. In that sense, since $x_3 = (1,0,0,0..,0)$ and $x_2 = z$ it holds true that:

$$n_2 \cdot n_3 = cos\theta, \qquad \theta = arg z \qquad (3.29)$$

We are moving on by inserting a complete basis energy eigenstates on $d-1$-dimensional sphere of the d-dimensional cylinder[10] to (3.27). Thus if we ignore the leading term of (3.25) we get:

$$g(u,v) = \sum_E \langle O| O(0,n_3) |E\rangle \langle E| O(\tau_2, n_2) |O\rangle =$$

$$\sum_E \langle O| O(0,n_3) |E\rangle \langle E| e^{-iHt} O(0,n_2) |O\rangle =$$

$$\sum_E \langle O| O(0,n_3) |E\rangle \langle E| e^{H\tau} O(0,n_2) |O\rangle =$$

$$\sum_E e^{E\tau_2} \langle O| O(0,n_3) |E\rangle \langle E| O(0,n_2) |O\rangle = \sum_E \langle O| O(0,n_3) |E\rangle \langle E| e^{H\tau} O(0,n_2) |O\rangle \Rightarrow$$

$$g(u,v) = \sum_E |z|^E \langle O| O(0,n_3) |E\rangle \langle E| O(0,n_2) |O\rangle \qquad (3.30)$$

The energy states $|E\rangle$ are in 1-1 correspondence with the local operators appearing in OPE. Every state will come in a multiplet of $SO(d)$ and it turns out that only the symmetric traceless tensors with $j \geq 0$ survive, since antisymmetric tensors can not be contracted by a single vector n. Therefore the element $\langle E| O(0,n_2) |O\rangle$ is:

$$\langle E, \mu_1..\mu_J| O(0,n_2) |O\rangle \approx n_2^{\mu_1}..n_2^{\mu_J} - traces$$

And (3.30) is approximately:

$$(n_2^{\mu_1}..n_2^{\mu_J} - traces)(n_3^{\mu_1}..n_3^{\mu_J} - traces) \approx C_J^\nu(n_2 \cdot n_3) = C_J^\nu(cos\theta) \qquad (3.31)$$

Where $\nu = d/2-1$ and C_J^ν are the Gegenbauer polynomials and for $d = 3$ ($\nu = 1/2$) they are the Legendre polynomials $P_J(cos\theta)$. Gegenbauer polynomials can be viewed as d-dimensional Legendre polynomials, thus we understand their appearance, since Legendre polynomials are appearing in amplitudes when we study scattering.

We put together (3.30) with (3.31) and we obtain:

$$g(u,v) \approx \sum |z|^E C_J^\nu(cos\theta) \qquad (3.32)$$

Consider the primary operator O of dimension Δ and l occurring in the OPE between 2 operators, then if the sum is restricted to its conformal multiplet it must match the conformal block at (3.26). The conformal multiplet has descendants of integer spaced dimension $\Delta + n$ with spins at level n taking values:

$$J = l+n, l+n-2, ..max(J-n, J+n \ mod \ 2) \qquad (3.33)$$

Thus the conformal block must have the following expansion:

$$G_{\Delta,J}(u,v) \approx \sum_{n=0}^\infty |z|^{\Delta+n} \sum_J \frac{C_J^\nu(cos\theta)}{C_J^\nu(1)} \qquad (3.34)$$

where we have used the Gengebauer normalization factor:

$$C_J^\nu = \frac{(2\nu)_J}{J!} \qquad (3.35)$$

[10] For example in $d = 3$ where our familiar cylinder "lives", these eigenstates are "living" on the equal radius circles of the cylinder.

4 Mellin Amplitudes

In this section we are going to introduce **Mellin amplitudes** and some of them properties. These amplitudes are very useful for the analogy between scattering amplitudes and conformal correlation functions. In quantum mechanics and quantum field theory we learn that states can be expressed over momentum or position space and so probability amplitudes can be calculated in terms of momentum or position. In the same sense Mellin amplitudes are expressed in Mellin space, which is characterized by the variables δ_{ij}.

In this analysis we are not going to be restricted by a proper definition of Mellin space, but we are going to start from the Mellin transformation of the n-point function:

$$A_n(x_1, ..x_n) = \int [d\delta] M_n(\delta_{ij}) \prod_{i<j}^{n} (x_{ij})^{-2\delta_{ij}} \Gamma(\delta_{ij}) \qquad (4.1)$$

Where $x_{ij} = (x_i - x_j)$ and $[d\delta] = \frac{d\delta_{12} d\delta_{13}..}{2\pi i 2\pi i..}$. $M_n(\delta_{ij})$ is the Mellin amplitude and for $n = 4$ it's associate with the connected part of the correlation functions. The integration for each $d\delta_{ij}$ runs parallel to the imaginary axis in the complex plane and they are in total $\frac{n(n-3)}{2}$ of them which are independent. In analogy we can think δ_{ij} as the kinematic invariant $p_i \cdot p_j$ in an n-particle scattering and Δ_i as the masses of these particles. When we consider this we can derive from momentum conservation:

$$\sum_j p_j = 0 \Rightarrow \sum_j p_i \cdot p_j = 0 \Rightarrow \sum_j \delta_{ij} = 0 \qquad (4.2)$$

Then if we consider Einstein's dispersion relation we have:

$$p_i^2 = -\Delta_i \Rightarrow \delta_{ii} = -\Delta_i \qquad (4.3)$$

Also, its clear from (4.2) and (4.3) that the total number of independent $d\delta_{ij}$ are $n(n-3)/2$. For example, take $n = 4$, (4.2) gives $\delta_{i1} + \delta_{i2} + \delta_{i3} + \delta_{i4} = 0$, then for i=1 and with (4.3) it is $-\Delta_1 + \delta_{12} + \delta_{13} + \delta_{14} = 0$. From this we see that only 2 can be independent.

We can use the definition of δ_{ij} from (4.3) to relate them with the Mandelstam variables t, s for the case $n = 4$:

$$t = -(p_1+p_2)^2 = -(p_3+p_4)^2 = -p_1^2 - p_2^2 - 2p_1p_2 = \Delta_1 + \Delta_2 - 2\delta_{12} = \Delta_3 + \Delta_4 - 2\delta_{34} \Rightarrow$$

$$t = \Delta_1 + \Delta_2 - 2\delta_{12} = \Delta_3 + \Delta_4 - 2\delta_{34} \qquad (4.4)$$

$$s = -(p_1+p_3)^2 = -(p_2+p_4)^2 = -p_1^2 - p_3^2 - 2p_1p_3 = \Delta_1 + \Delta_3 - 2\delta_{13} = \Delta_2 + \Delta 4 = 2\delta_{24} \Rightarrow$$

$$s' = s - \Delta_1 - \Delta_4 = -(p_1 + p_3)^2 - \Delta_1 - \Delta_4 = \Delta_3 - \Delta_4 - 2\delta_{13} = \Delta_2 - \Delta_1 - 2\delta_{24}$$

$$s' = \Delta_2 - \Delta_1 - 2\delta_{24} = \Delta_3 - \Delta_4 - 2\delta_{13} \qquad (4.5)$$

Where s' is the shifted Mandelstam variable, which is more convenient. As we continue we simply let $s' \Rightarrow s$.

We can expand (3.17):

$$A(x_1,x_2,x_3,x_4) =$$
$$f(u,v)\frac{|x_{24}|^{\frac{-2\Delta_2-2\Delta_4+\Delta_3+\Delta_1}{3}}|x_{14}|^{\frac{-2\Delta_4-2\Delta_1+\Delta_3+\Delta_2}{3}}}{|x_{12}|^{\frac{2\Delta_1+2\Delta_2-\Delta_3-\Delta_4}{3}}|x_{34}|^{\frac{2\Delta_3+2\Delta_4-\Delta_1-\Delta_2}{3}}|x_{13}|^{\frac{2\Delta_1+2\Delta_3-\Delta_2-\Delta_4}{3}}|x_{23}|^{\frac{2\Delta_2+2\Delta_3-\Delta_1-\Delta_4}{3}}} \Rightarrow$$

$$A(x_1,x_2,x_3,x_4) = A(u,v)\frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_1-\Delta_2}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}} \tag{4.6}$$

Where $A(u,v)$ is the reduced correlator A, which is a complicate function of u,v and its current form will not bother us.

We want to find the Mellin transformation of $A(u,v)$. To do this we expand (4.1) for $n=4$. First we focus on the product term:

$$\prod_{i<j}^4 (x_{ij})^{-2\delta_{ij}}\Gamma(\delta_{ij}) =$$
$$\Gamma(\delta_{14})(x_{14}^2)^{-\delta_{14}}\Gamma(\delta_{24})(x_{24}^2)^{-\delta_{24}}\Gamma(\delta_{34})(x_{34}^2)^{-\delta_{34}}\Gamma(\delta_{13})(x_{13}^2)^{-\delta_{13}}\Gamma(\delta_{23})(x_{23}^2)^{-\delta_{23}}\Gamma(\delta_{12})(x_{12}^2)^{-\delta_{12}} \tag{4.7}$$

From (4.4) (4,5) we get:

$$\begin{gathered}\delta_{12} = \frac{\Delta_1+\Delta_2-t}{2}\\ \delta_{13} = \frac{\Delta_3-\Delta_4-s}{2} = \frac{\Delta_{34}-s}{2}\\ \delta_{34} = \frac{\Delta_3+\Delta_4-t}{2}\\ \delta_{24} = \frac{\Delta_2-\Delta_1-s}{2} = \frac{-\Delta_{12}-s}{2}\end{gathered} \tag{4.8}$$

We combine them as (4.4)+(4.5) and use also (4.2),(4.3) to obtain:

$$t+s = \Delta_1+\Delta_2+\Delta_3-\Delta_4-2\delta_{12}-2\delta_{13} = 2\delta_{12}+2\delta_{13}+2\delta_{23}-2\delta_{12}-2\delta_{13} = 2\delta_{23}$$
$$\Rightarrow \delta_{23} = \frac{t+s}{2} \tag{4.9}$$

We can obtain also by (4.9)/2+$\Delta_{12}-\Delta_{34}$ with (4.2),(4.3):

$$t+s+\Delta_{12}-\Delta_{34} = 2\delta_{23}+\Delta_1-\Delta_2-\Delta_3+\Delta_4 = 2\delta_{23}+2\delta_{14}-2\delta_{23} = 2\delta_{14}$$
$$\Rightarrow \delta_{14} = \frac{t+s+\Delta_{12}-\Delta_{34}}{2} \tag{4.10}$$

Now consider the integration measure $\int[d\delta]$ for $n=4$. We want to transform it from δ_{ij} to s and t . To do this we differentiate (4.4) (4.5) and we get:

$$d\delta_{12}d\delta_{13} = dtds/4$$
$$\Rightarrow \int[d\delta] = \int\frac{dtds}{16(\pi i)^2} = \int\frac{dtds}{(4\pi i)^2} \tag{4.11}$$

Put together (4.6), (4.7), (4.8), (4.9), (4.10), (4.11), and we have:

$$A(u,v)\frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_1-\Delta_2}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}} =$$
$$\int_{-i\infty}^{i\infty}\frac{dtds}{(4\pi i)^2}\Gamma\left(\frac{t+s}{2}\right)\Gamma\left(\frac{t+s+\Delta_{12}-\Delta_{34}}{2}\right)\Gamma\left(\frac{-\Delta_{12}-s}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-t}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-t}{2}\right)\Gamma\left(\frac{\Delta_{34}-s}{2}\right)(x_{14}^2)^{-\frac{t+s}{2}}$$
$$(x_{14}^2)^{\frac{\Delta_{34}-\Delta_{12}}{2}}(x_{24}^2)^{\frac{\Delta_{12}}{2}}(x_{24}^2)^{\frac{s}{2}}(x_{34}^2)^{\frac{t}{2}}(x_{34}^2)^{\frac{-\Delta_{34}}{2}}(x_{13}^2)^{\frac{-\Delta_{34}}{2}}(x_{13}^2)^{\frac{s}{2}}(x_{12}^2)^{\frac{t}{2}}(x_{12}^2)^{-\frac{-\Delta_1+\Delta_2}{2}}(x_{23}^2)^{-\frac{t+s}{2}}M(s,t) \Rightarrow$$

$$A(u,v) = \int_{-i\infty}^{i\infty} M(s,t)\frac{dtds}{(4\pi i)^2}\Gamma(\frac{t+s}{2})\Gamma(\frac{t+s+\Delta_{12}-\Delta_{34}}{2})\Gamma(\frac{-\Delta_{12}-s}{2})\Gamma(\frac{\Delta_3+\Delta_4-t}{2})\Gamma(\frac{\Delta_1+\Delta_2-t}{2})\Gamma(\frac{\Delta_{34}-s}{2}) \frac{(x_{14}^2x_{23}^2)^{-\frac{t+s}{2}}}{(x_{13}^2x_{24}^2)^{-\frac{s}{2}}}\frac{(x_{12}^2x_{34}^2)^{\frac{t}{2}}}{(x_{13}^2x_{24}^2)^{\frac{t}{2}}(x_{13}^2x_{24}^2)^{-\frac{t}{2}}} \Rightarrow$$

$$A(u,v) = \int_{-i\infty}^{+i\infty}\frac{dtds}{(4\pi i)^2}M(s,t)u^{\frac{t}{2}}v^{-\frac{s+t}{2}} \Gamma(\frac{t+s}{2})\Gamma(\frac{t+s+\Delta_{12}-\Delta_{34}}{2})\Gamma(\frac{-\Delta_{12}-s}{2})\Gamma(\frac{\Delta_3+\Delta_4-t}{2})\Gamma(\frac{\Delta_1+\Delta_2-t}{2})\Gamma(\frac{\Delta_{34}-s}{2}) \qquad (4.12)$$

Note that the integration contours should be placed such that the infinity series of poles produced from each Gamma function to stays to one side of contour. This also holds true and for the pole structure of $M(s,t)$. This representation for the reduced correlation function is called **Mellin space representation**.

4.1 Operator Product Expansion

In this section we are going to use the operator product expansion (3.16) to investigate the analytic structure of Mellin amplitudes.

We want to find the leading behavior of the OPE. To do this we set in (3.16) $x_2 = 0$ and we expand the power series $C_{12k,\mu_1..\mu_{J_k}}(x,\partial_2)$. We get:

$$O_1(x)O_2(0) = \sum_k a_{12k}[b_{\mu_1..\mu_{J_k}} + cx^\mu\partial_\mu + dx^\mu x^\nu\partial_\mu\partial_\nu + ..]O_k^{\mu_1..\mu_{J_k}}(0) = \sum_k a_{12k}[b_{\mu_1..\mu_{J_k}}O_k^{\mu_1..\mu_{J_k}}(0) + descendants] \qquad (4.13)$$

Where a_{12k} and the b are the leading coefficients that we want to determine. To determine a_{12k} we are going to use dimensional analysis. The left side has mass dimensions $\Delta_1 + \Delta_2$, as for the right side mass dimensions are $[a_{12k}] - J_k + \Delta_k$[11]. From this it is clear that $b_{\mu_1..\mu_{J_k}}$ must be dimensionless, thus $b_{\mu_1..\mu_{J_k}} = \frac{x_{\mu_1}..x_{\mu_{J_k}}}{(x^2)^{\frac{J_k}{2}}}$ and then $[a_{12k}] + \Delta_k = \Delta_1 + \Delta_2 \Rightarrow [a_{12k}] = -\Delta_k + \Delta_1 + \Delta_2$. Therefore, $a_{12k} = \frac{C_{12k}}{(x^2)^{\frac{\Delta_1+\Delta_2-\Delta_k}{2}}}$ and (4.13) becomes:

$$O_1(x)O_2(0) = \sum_k \frac{C_{12k}}{(x^2)^{\frac{\Delta_1+\Delta_2-\Delta_k}{2}}}[\frac{x_{\mu_1}..x_{\mu_{J_k}}}{(x^2)^{\frac{J_k}{2}}}O_k^{\mu_1..\mu_{J_k}}(0) + descendants] \qquad (4.14)$$

We normalize all operators to have 2-points functions such that (3.14) becomes:

$$\langle O_{\mu_1..\mu_J}O_{\nu_1..\nu_J}\rangle = \frac{1}{J!}\sum_{perm,\sigma}\frac{I_{\mu_1\nu_{\sigma(1)}}I_{\mu_J\nu_{\sigma(J)}}}{(x^2)^\Delta} - traces \qquad (4.15)$$

In order to find the analytic structure, we continue with OPE of the reduced correlation function (4.6) as we proceed with (3.21):

$$A(u,v) = \sum_k C_{12k}C_{34k}G_{\Delta_k,J_k}(u,v) \qquad (4.16)$$

[11] Δ_k is the mass dimension of $O^{\mu_1..\mu_{J_k}}$.

We want to find the pole structure of $M(s,t)$. Consider (3.20), we can set $x_2 = 0$, since operators are invariant under translations (3.8) and we get:

$$O_1(x_1)O_2(0) = \sum_k C_{12k,\mu_1..\mu_{J_k}}(x_1,\partial_2)O_k^{\nu_1..\nu_{J_k}}(0) =$$
$$\sum_k \sum_{m=0}^{\infty} C_{k,\nu_1..\nu_{J_k}}^{\mu_1..\mu_m}(x_1,0)\partial_{\mu_1}..\partial_{\mu_m}O_k^{\nu_1..\nu_{J_k}}(0)$$

(4.17)

Then we perform a dilation and according to (3.4) and (3.7), (4.17) becomes:

$$O_1(e^{-\sigma}x_1)O_2(0) = \sum_k \sum_{m=0}^{\infty} e^{-\sigma(\Delta_k+m)+\sigma(\Delta_1+\Delta_2)}C_{k,\nu_1..\nu_{J_k}}^{\mu_1..\mu_m}(x_1,0)\partial_{\mu_1}..\partial_{\mu_m}O_k^{\nu_1..\nu_{J_k}}(0)$$

(4.18)

where the term $e^{\sigma(\Delta_1+\Delta_2)}$ comes from the leading term shown at (4.16). Thus, we write the 4-point function according to (4.18):

$$\langle O_1(e^{-\sigma}x_1)O_2(0)O_3(x_3)O_4(x_4)\rangle =$$

$$\sum_k \sum_{m=0}^{\infty} e^{-\sigma(\Delta_k+m)+\sigma(\Delta_1+\Delta_2)}C_{k,\nu_1..\nu_{J_k}}^{\mu_1..\mu_m}(x_1,0)\,\langle\partial_{\mu_1}..\partial_{\mu_m}O_k^{\nu_1..\nu_{J_k}}(0)O_3(x_3)O_4(x_4)\rangle =$$

$$\sum_k \sum_{m=0}^{\infty} e^{-\sigma(\Delta_k+m)+\sigma(\Delta_1+\Delta_2)}F_{k,m}(x_1,0,x_3,x_4)$$

(4.19)

Now we rescale Mellin transformation (4.1) of the 4-point function with $x_i' = e^{-\sigma}x_i$ for $i \leq 2$:

$$\int_{-i\infty}^{i\infty}[d\delta]M(\delta_{ij})\prod_{i<j}^{4}\Gamma(\delta_{ij})(x_{12}')^{-2\delta_{12}}\prod_{i\leq 2<j}^{4}(x_{ij}')^{-2\delta_{ij}}\prod_{2<i<j}^{4}(x_{ij}^2)^{-\delta_{ij}} =$$

$$\int_{-i\infty}^{i\infty}[d\delta]M(\delta_{ij})e^{2\sigma\delta_{12}}\prod_{i<j}^{4}\Gamma(\delta_{ij})(x_{12})^{-2\delta_{12}}$$
$$\prod_{i\leq 2<j}^{4}(x_j^2 - e^{-\sigma}2x_i\cdot x_j + e^{-2\sigma}x_i^2)^{-\delta_{ij}}\prod_{2<i<j}^{4}(x_{ij}^2)^{-\delta_{ij}} =$$

$$\int_{-i\infty}^{i\infty}[d\delta]M(\delta_{ij})e^{2\sigma\delta_{12}}\prod_{i<j}^{4}\Gamma(\delta_{ij})(x_{12})^{-2\delta_{12}}\sum_{q=0}^{\infty}e^{-q\sigma}Q_q(x_1,x_2,x_3,x_4)(x_{34})^{-2\delta_{34}}$$

(4.20)

In the last line we simply used the Taylor expansion for the product term $\prod_{i\leq 2<j}^{4}(x_j^2 - e^{-\sigma}2x_i\cdot x_j + e^{-2\sigma}x_i^2)^{-\delta_{ij}}$ and Q_q is a polynomial of degree q in x_i for $i=1,2$.

Next we match (4.20) with (4.19) and we get:

$$\int_{-i\infty}^{i\infty}[d\delta]M(\delta_{ij})e^{2\sigma\delta_{12}}\prod_{i<j}^{4}\Gamma(\delta_{ij})(x_{12})^{-2\delta_{12}}\sum_{q=0}^{\infty}e^{-q\sigma}Q_q(x_1,x_2,x_3,x_4)(x_{34})^{-2\delta_{34}} =$$

$$\sum_k\sum_{m=0}^{\infty}e^{-\sigma(\Delta_k+m)+\sigma(\Delta_1+\Delta_2)}F_{k,m}(x_1,0,x_3,x_4) \tag{4.21}$$

The right side contains all residues produced by the integral from the left side, but it does not exist a pole structure in the integral side so that we get these residues. For $q=J_k$ we get contribution of a spin l operator, therefore we conclude that $M(\delta_{ij})$ must have poles at $2\delta_{12}-J_k=\Delta_1+\Delta_2-\Delta_k-m \Rightarrow t=\Delta_k-J_k+m$ and the pole structure of the Mellin amplitude is:

$$M(s,t)\approx\sum_k\sum_{m=0}^{\infty}\frac{1}{t-\Delta_k+J_k-m} \tag{4.22}$$

We can write (4.21) by using (4.16) and (4.6) as:

$$\int_{-i\infty}^{i\infty}[d\delta]M(\delta_{ij})e^{2\sigma\delta_{12}}\prod_{i<j}^{4}\Gamma(\delta_{ij})(x_{12})^{-2\delta_{12}}\sum_{q=0}^{\infty}e^{-q\sigma}Q_q(x_1,x_2,x_3,x_4)(x_{34})^{-2\delta_{34}} =$$

$$\sum_k\sum_{m=0}^{\infty}e^{-\sigma(\Delta_k+m)+\sigma(\Delta_1+\Delta_2)}F_{k,m}(x_1,0,x_3,x_4) =$$

$$\frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_1-\Delta_2}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}}\sum_k C_{12k}C_{34k}G_{\Delta_k,J_k}(u,v)$$

We can use this to determine $M(s,t)$ even further. The last line contains coefficients C_{12k} and C_{34k}, therefore they must originate from $M(\delta_{ij})$. Also, since $F_{k,m}\approx C^{\mu_1..\mu_m}_{k,\nu_1..\nu_J}(x_1,0)$ and $C^{\mu_1..\mu_m}_{k,\nu_1..\nu_J}(x_1,0)$ is a polynomial we conclude $M(\delta_{ij})$ must contain polynomials $Q_{J,m}(s)$, where J is the degree of the polynomials. These are residues of $M(s,t)$ from the t-plane and they must be a function of s, so that the integration in s will give $C^{\mu_1..\mu_m}_{k,\nu_1..\nu_J}(x_1,0)$.Thus, we conclude (4.22) must be:

$$M(s,t)\approx\sum_k\sum_{m=0}^{\infty}\frac{C_{12k}C_{34k}Q_{J,m}(s)}{t-\Delta_k+J_k-m} \tag{4.23}$$

It's convenient to write $Q_{J_k,m}(s)$ as:

$$Q_{J_k,m}(s) = -\frac{2\Gamma(\Delta_k+J_k)(\Delta_k-1)_{J_k}}{4^{J_k}\Gamma(\frac{\Delta_k+J_k+\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k-\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k+\Delta_{34}}{2})\Gamma(\frac{\Delta_k+J_k-\Delta_{34}}{2})}$$
$$\frac{\tilde{Q}_{J_k,m}(s)}{m!(\Delta_k-h+1)_m\Gamma(\frac{\Delta_1+\Delta_2-\Delta_k+J_k}{2}-m)\Gamma(\frac{\Delta_3+\Delta_4-\Delta_k+J_k}{2}-m)}$$

(4.24)

Where we used the Pochhammer symbol, $(a)_{J_k} = \frac{\Gamma(a+J_k)}{\Gamma(a)}$ and $h = d/2$.

In (3.34) it has been shown that the conformal block of the 4-point function has a useful expansion in terms of Gegenbauer polynomials. This expansion can viewed as partial wave expansion in d dimensions, where the familiar Legendre polynomials are the Gegenbauer polynomials. The expansion is:

$$G_{\Delta_k,J_k}(u,v) \approx \sum_{n=0}^{\infty} |z|^{\Delta+n} \sum_j \frac{C_j^{h-1}(cos\theta)}{C_j^{h-1}(1)} \qquad (4.25)$$

With $u = z\bar{z}$, $v = (1-z)(1-\bar{z})$ and z is a complex number with $argz = \theta$. $C_j^{h-1}(x)$ is the Gegenbauer polynomial with $C_j^{h-1}(1) = \frac{(2h-2)_j}{j!}$ being the Gegenbauer normalization factor. The leading term from (4.25) is given for $n = 0$:

$$G_{\Delta_k,J_k}(u,v) \approx |z|^{\Delta_k} \frac{J_k!}{(2h-2)_{J_k}} C_{J_k}^{h-1}(cos\theta) = |z|^{\Delta_k} \frac{J_k!}{2^{J_k}(h-1)_{J_k}} C_{J_k}^{h-1}(cos\theta) \qquad (4.26)$$

In the limit $u \to 0$ and $v \to 1$, we write $|z| = \sqrt{u}$ and $v = 1 - z - \bar{z} + u \approx 1 - z - \bar{z} \Rightarrow v - 1 = -z - \bar{z} \Rightarrow (v-1)/(2\sqrt{u}) = -(z+\bar{z})/2|z| = -cos\theta$, thus (4.26) becomes[12]

$$G_{\Delta_k,J_k}(u,v) \approx u^{\frac{\Delta_k}{2}} \frac{J_k!}{2^{J_k}(h-1)_{J_k}} C_{J_k}^{h-1}\left(-\frac{v-1}{2\sqrt{u}}\right) = (-1)^{J_k} u^{\frac{\Delta_k}{2}} \frac{J_k!}{2^{J_k}(h-1)_{J_k}} C_{J_k}^{h-1}\left(\frac{v-1}{2\sqrt{u}}\right) \Rightarrow$$

$$G_{\Delta_k,J_k}(u,v) \approx u^{\frac{\Delta_k}{2}} \frac{J_k!}{2^{J_k}(h-1)_{J_k}} C_{J_k}^{h-1}\left(\frac{v-1}{2\sqrt{u}}\right) \qquad (4.27)$$

With (4.27) we can show that m is limited to even numbers. We let the poles $\tau = \Delta_k - J_k + m$. It should be clear that these poles are lying on the real positive axis and with a proper contour we can calculate the complex integral by using the residues theorem[13]

$$\sum_k \sum_{m=0}^{\infty} \int_{-i\infty}^{i\infty} \frac{dt}{(4\pi i)^2} \frac{C_{12k}C_{34k}Q_{J_k,m}(s)}{t-\tau} u^{\frac{t}{2}} v^{-\frac{t}{2}} \Gamma\left(\frac{t+s}{2}\right)\Gamma\left(\frac{t+s+\Delta_{12}-\Delta_{34}}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-t}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-t}{2}\right) \sim$$

$$-2\pi i \sum_k \sum_{m=0}^{\infty} \frac{1}{(4\pi i)^2} C_{12k}C_{34k} u^{\frac{\Delta_k - J_k + m}{2}} \sim$$

$$\sum_k \sum_{m=0}^{\infty} C_{12k}C_{34k} u^{\frac{\Delta_k - J_k + m}{2}} \qquad (4.28)$$

On the other hand this expression is equal to (4.16), which in the limit $u \to 0$ and $v \to 1$ it becomes with (4.27)

[12] We also use that $C_n^a(-x) = (-1)^n C_n^a(x)$, since $C_n^a(x)$ have generating function $\frac{1}{(1-2xt+t^2)^a} = \sum_{n=0}^{\infty} C_n^a(x) t^n$. More information can be found at: https://en.wikipedia.org/wiki/Gegenbauer polynomials

[13] We calculate the complex integral without taking into account the poles in gamma functions. We are going to explain the reason behind it in section 6 .

$$A(u,v) \sim \sum_k C_{12k} C_{34k} u^{\Delta_k/2} C^{h-1}_{J_k}(\tfrac{v-1}{2\sqrt{u}})$$

And by using the expression of the Gegenbauer polynomials

$$C^{h-1}_{J_k}(\tfrac{v-1}{2\sqrt{u}}) \sim \sum_{n=0}^{[J_k/2]} (v-1)^{J-2n} (u)^{-\frac{J}{2}+n} \sim \sum_{n=0}^{[J_k/2]} (u)^{-\frac{J_k}{2}+n}$$

we obtain then

$$A(u,v) \sim \sum_k \sum_{n=0}^{[J/2]} C_{12k} C_{34k} u^{\frac{\Delta_k - J_k}{2}+n} \tag{4.29}$$

Thus by equating (4.29) and (4.28) we have $m = 2n$.

Now we are going to use (4.23) to study the polynomials $Q_{J_k,m}$. We focus on the integral of the t-variable from (4.12). By letting the poles $\tau = \Delta_k - J_k + 2m$. We calculate again the complex integral by using the residues theorem:

$$\sum_k \sum_{m=0}^{\infty} \int_{-i\infty}^{i\infty} \frac{dt}{(4\pi i)^2} \frac{C_{12k}C_{34k}Q_{J_k,m}(s)}{t-\tau} u^{\frac{t}{2}} v^{-\frac{t}{2}} \Gamma(\tfrac{t+s}{2}) \Gamma(\tfrac{t+s+\Delta_{12}-\Delta_{34}}{2}) \Gamma(\tfrac{\Delta_3+\Delta_4-t}{2}) \Gamma(\tfrac{\Delta_1+\Delta_2-t}{2}) =$$

$$-2\pi i \sum_k \sum_{m=0}^{\infty} \frac{1}{(4\pi i)^2} C_{12k}C_{34k}Q_{J_k,m}(s) u^{\tau/2} v^{-\tau/2} \Gamma(\tfrac{\tau+s}{2}) \Gamma(\tfrac{\tau+s+\Delta_{12}-\Delta_{34}}{2}) \Gamma(\tfrac{\Delta_3+\Delta_4-\tau}{2}) \Gamma(\tfrac{\Delta_1+\Delta_2-\tau}{2})$$

substituting (4.24) we get:

$$\frac{1}{8\pi i} \sum_k \sum_{m=0}^{\infty} \frac{2\Gamma(\Delta_k+J_k)(\Delta_k-1)_{J_k}}{4^J \Gamma(\frac{\Delta_k+J_k+\Delta_{12}}{2}) \Gamma(\frac{\Delta_k+J_k-\Delta_{12}}{2}) \Gamma(\frac{\Delta_k+J_k+\Delta_{34}}{2}) \Gamma(\frac{\Delta_k+J_k-\Delta_{34}}{2})} C_{12k}C_{34k} \frac{\tilde{Q}_{J_k,m}(s) u^{\tau/2} v^{-\tau/2}}{m!(\Delta_k-h+1)_m} \tag{4.30}$$

Then we consider the integral of the s-variable from (4.12) and substituting (4.30):

$$\sum_k \sum_{m=0}^{\infty} \frac{2\Gamma(\Delta_k+J_k)(\Delta_k-1)_{J_k}}{4^{J_k} \Gamma(\frac{\Delta_k+J_k+\Delta_{12}}{2}) \Gamma(\frac{\Delta_k+J_k-\Delta_{12}}{2}) \Gamma(\frac{\Delta_k+J_k+\Delta_{34}}{2}) \Gamma(\frac{\Delta_k+J_k-\Delta_{34}}{2})} C_{12k}C_{34k} u^{\tau/2}$$
$$\int_{-i\infty}^{i\infty} \frac{ds}{8\pi i} v^{-\frac{\tau+s}{2}} \frac{\tilde{Q}_{J_k,m}(s)}{m!(\Delta_k-h+1)_m} \Gamma(\tfrac{\tau+s}{2}) \Gamma(\tfrac{\tau+s+\Delta_{12}-\Delta_{34}}{2}) \Gamma(\tfrac{\Delta_{34}-s}{2}) \Gamma(\tfrac{-\Delta_{12}-s}{2}) \tag{4.31}$$

(4.31) is equal to (4.12), but we saw that $A(u,v)$ can be written in terms of the conformal blocks in (4.16), thus by equating (4.31) with (4.16) we get:

$$\sum_k \sum_{m=0}^{\infty} \frac{2\Gamma(\Delta_k+J)(\Delta_k-1)_{J_k}}{4^{J_k} \Gamma(\frac{\Delta_k+J_k+\Delta_{12}}{2}) \Gamma(\frac{\Delta_k+J_k-\Delta_{12}}{2}) \Gamma(\frac{\Delta_k+J_k+\Delta_{34}}{2}) \Gamma(\frac{\Delta_k+J_k-\Delta_{34}}{2})} C_{12k}C_{34k} u^{\tau/2}$$

$$\int_{-i\infty}^{i\infty} \frac{ds}{8\pi i} v^{-\frac{\tau+s}{2}} \frac{\tilde{Q}_{J_k,m}(s)}{m!(\Delta_k-h+1)_m} \Gamma(\tfrac{\tau+s}{2})\Gamma(\tfrac{\tau+s+\Delta_{12}-\Delta_{34}}{2})\Gamma(\tfrac{\Delta_{34}-s}{2})\Gamma(\tfrac{-\Delta_{12}-s}{2}) =$$

$$\sum_k C_{12k}C_{34k}G_{\Delta_k,J_k}(u,v)$$

This implies that the conformal block $G_{\Delta_k,J_k}(u,v)$ can be expressed as:

$$G_{\Delta_k,J_k}(u,v) = \sum_{m=0}^{\infty} \frac{2\Gamma(\Delta_k+J_k)(\Delta_k-1)_{J_k}}{4^{J_k}\Gamma(\frac{\Delta_k+J_k+\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k-\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k+\Delta_{34}}{2})\Gamma(\frac{\Delta_k+J_k-\Delta_{34}}{2})} u^{\tau/2}$$

$$\int_{-i\infty}^{i\infty} \frac{ds}{8\pi i} v^{-\frac{\tau+s}{2}} \frac{\tilde{Q}_{J_k,m}(s)}{m!(\Delta_k-h+1)_m} \Gamma(\tfrac{\tau+s}{2})\Gamma(\tfrac{\tau+s+\Delta_{12}-\Delta_{34}}{2})\Gamma(\tfrac{\Delta_{34}-s}{2})\Gamma(\tfrac{-\Delta_{12}-s}{2}) =$$

$$u^{\frac{\Delta_k-J_k}{2}} \sum_{m=0}^{\infty} u^m \frac{2\Gamma(\Delta_k+J_k)(\Delta_k-1)_{J_k}}{4^{J_k}\Gamma(\frac{\Delta_k+J_k+\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k-\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k+\Delta_{34}}{2})\Gamma(\frac{\Delta_k+J_k-\Delta_{34}}{2})}$$

$$\int_{-i\infty}^{i\infty} \frac{ds}{8\pi i} v^{-\frac{\tau+s}{2}} \frac{\tilde{Q}_{J_k,m}(s)}{m!(\Delta_k-h+1)_m} \Gamma(\tfrac{\tau+s}{2})\Gamma(\tfrac{\tau+s+\Delta_{12}-\Delta_{34}}{2})\Gamma(\tfrac{\Delta_{34}-s}{2})\Gamma(\tfrac{-\Delta_{12}-s}{2}) \Rightarrow$$

$$G_{\Delta_k,J_k}(u,v) = u^{\frac{\Delta_k-J_k}{2}} \sum_{m=0}^{\infty} u^m g_{m,k}(v) \qquad (4.32)$$

$$g_{m,k}(v) = \frac{2\Gamma(\Delta_k+J_k)(\Delta_k-1)_{J_k}}{4^{J_k}\Gamma(\frac{\Delta_k+J_k+\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k-\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k+\Delta_{34}}{2})\Gamma(\frac{\Delta_k+J_k-\Delta_{34}}{2})}$$

$$\int_{-i\infty}^{i\infty} \frac{ds}{8\pi i} v^{-\frac{\tau+s}{2}} \frac{\tilde{Q}_{J_k,m}(s)}{m!(\Delta_k-h+1)_m} \Gamma(\tfrac{\tau+s}{2})\Gamma(\tfrac{\tau+s+\Delta_{12}-\Delta_{34}}{2})\Gamma(\tfrac{\Delta_{34}-s}{2})\Gamma(\tfrac{-\Delta_{12}-s}{2})$$

$$(4.33)$$

The first term $g_{0,k}(v)$ of (4.32) can be expressed in terms of hypergeometric function:

$$g_{0,k}(v) = (\tfrac{v-1}{2})^{J_k} {}_2F_1(\tfrac{\Delta_k+J_k-\Delta_{12}}{2}, \tfrac{\Delta_k+J_k+\Delta_{34}}{2}, \Delta_k+J_k, 1-v) \qquad (4.34)$$

Next we expand (4.34) as powers of $1-v$. To do this we use the series $\frac{1}{v^{\frac{\tau+s}{2}}} = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\tau+s}{2}+n)}{\Gamma(\frac{\tau+s}{2})} \frac{(1-v)^n}{n!}$ and the series of the hypergeometric function ${}_2F_1(\frac{\Delta_k+J_k-\Delta_{12}}{2}, \frac{\Delta_k+J_k+\Delta_{34}}{2}, \Delta_k+J_k, 1-v) = \sum_{n=0}^{\infty} \frac{(\frac{\Delta_k+J_k-\Delta_{12}}{2})_n(\frac{\Delta_k+J_k+\Delta_{34}}{2})_n}{(\Delta_k+J_k)_n} \frac{(1-v)^n}{n!}$, so (4.32) with (4.33) gives:

$$(\tfrac{1-v}{-2})^{J_k} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\Delta_k+J_k-\Delta_{12}}{2}+n)\Gamma(\frac{\Delta_k+J_k+\Delta_{34}}{2}+n)\Gamma(\Delta_k+J_k)}{\Gamma(\frac{\Delta_k+J_k-\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k+\Delta_{34}}{2})\Gamma(\Delta_k+J_k+n)} \frac{(1-v)^n}{n!} =$$

$$\frac{2\Gamma(\Delta_k+J_k)(\Delta_k-1)_{J_k}}{4^{J_k}\Gamma(\frac{\Delta_k+J_k+\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k-\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k+\Delta_{34}}{2})\Gamma(\frac{\Delta_k+J_k-\Delta_{34}}{2})}$$

$$\int_{-i\infty}^{i\infty} \frac{ds}{8\pi i} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\tau+s}{2}+n)}{\Gamma(\frac{\tau+s}{2})} \frac{(1-v)^n}{n!} \tilde{Q}_{J_k,0}(s)\Gamma(\tfrac{\tau+s}{2})\Gamma(\tfrac{\tau+s+\Delta_{12}-\Delta_{34}}{2})\Gamma(\tfrac{\Delta_{34}-s}{2})\Gamma(\tfrac{-\Delta_{12}-s}{2}) \Rightarrow$$

$$(-2)^{J_k}(1-v)^{J_k} \frac{\Gamma(\frac{\Delta_k+J_k-\Delta_{12}}{2}+n)\Gamma(\frac{\Delta_k+J_k+\Delta_{34}}{2}+n)}{\Gamma(\Delta_k+J_k+n)} \frac{\Gamma(\frac{\Delta_k+J_k+\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k-\Delta_{34}}{2})}{(\Delta_k-1)_{J_k}} =$$

$$\int_{-i\infty}^{i\infty} \frac{ds}{4\pi i} \tilde{Q}_{J_k,0}(s)\Gamma(\tfrac{\tau+s}{2}+n)\Gamma(\tfrac{\tau+s+\Delta_{12}-\Delta_{34}}{2})\Gamma(\tfrac{\Delta_{34}-s}{2})\Gamma(\tfrac{-\Delta_{12}-s}{2}) \Rightarrow$$

$$\frac{(-2)^{J_k} n!\Gamma(\frac{\Delta_k+J_k-\Delta_{12}}{2}+n)\Gamma(\frac{\Delta_k+J_k+\Delta_{34}}{2}+n)}{(n-J_k)!\Gamma(\Delta_k+n)} \frac{\Gamma(\frac{\Delta_k+J_k+\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k-\Delta_{34}}{2})}{(\Delta_k-1)_{J_k}} =$$

$$\int_{-i\infty}^{i\infty} \frac{ds}{4\pi i} \tilde{Q}_{J_k,0}(s)\Gamma(\tfrac{\tau+s}{2}+n)\Gamma(\tfrac{\tau+s+\Delta_{12}-\Delta_{34}}{2})\Gamma(\tfrac{\Delta_{34}-s}{2})\Gamma(\tfrac{-\Delta_{12}-s}{2}) \tag{4.35}$$

For $n < J_k$ the left hand side vanishes[14], so we get:

$$0 = \int_{-i\infty}^{i\infty} \frac{ds}{4\pi i} \tilde{Q}_{J_k,0}(s)\Gamma(\tfrac{\tau+s}{2}+n)\Gamma(\tfrac{\tau+s+\Delta_{12}-\Delta_{34}}{2})\Gamma(\tfrac{\Delta_{34}-s}{2})\Gamma(\tfrac{-\Delta_{12}-s}{2})$$

We take a linear combinations of this equation for $n < J$, we conclude that it defines an inner product under which $\tilde{Q}_{J_k,0}(s)$ is orthogonal to all polynomials of s with degree less than J_k. This means that the polynomials $\tilde{Q}_{J_k,0}(s)$ satisfy:

$$0 = \int_{-i\infty}^{i\infty} \frac{ds}{4\pi i} \sum_{n=0}^{\infty} [\tilde{Q}_{J_k,0}(s) c_n (\tfrac{\tau+s}{2})_n] \Gamma(\tfrac{\tau+s}{2})\Gamma(\tfrac{\tau+s+\Delta_{12}-\Delta_{34}}{2})\Gamma(\tfrac{\Delta_{34}-s}{2})\Gamma(\tfrac{-\Delta_{12}-s}{2}) \Rightarrow$$

$$\delta_{J_k J_k'} \approx \int_{-i\infty}^{i\infty} \frac{ds}{4\pi i} [\tilde{Q}_{J_k,0}(s)\tilde{Q}_{J_k',0}(s)] \Gamma(\tfrac{\tau+s}{2})\Gamma(\tfrac{\tau+s+\Delta_{12}-\Delta_{34}}{2})\Gamma(\tfrac{\Delta_{34}-s}{2})\Gamma(\tfrac{-\Delta_{12}-s}{2}) \tag{4.36}$$

If we let $s = -2ix \Rightarrow ds = -2idx$ and $a = \Delta_{34}/2, b = -\Delta_{12}/2, c = \tau/2, d = \frac{\tau+\Delta_{12}-\Delta_{34}}{2}$, (4.36) becomes:

$$\delta_{J_k,J_k'} \approx \int_{-\infty}^{\infty} \frac{dx}{2\pi} \tilde{Q}_{J_k,0}(x)\tilde{Q}_{J_k',0}(x)\Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix)$$

This normalization can be fixed by imposing (4.30) for any $n \geq J$. This orthogonality expression is satisfied by the continuous Hanh polynomials[15]. This suggests us to write $Q_{J_k,0}(s)$ polynomials as:

$$Q_{J_k,0}(s) \sim {}_3F_2(-J_k, J_k+\tau-1, \tfrac{\Delta_{34}-s}{2}, \tfrac{\tau+\Delta_{12}}{2}, \tfrac{\Delta_{34}+\tau}{2}, 1) \tag{4.37}$$

[14] $n - J_k$ is an integer number since n and J_k are, thus if $n - J_k < 0$ then $(n-J_k)!$ is infinity and the LHS vanishes.

[15] $https://en.wikipedia.org/wiki/ContinuousHahnpolynomials$

For the $s \to \infty$ we can find a boundary condition for $Q_{J_k,0}(s)$ from[16] (4.37):

$$Q_{J_k,0}(s) \approx \sum_{J_k=0}^{\infty} (\tfrac{\Delta_{34}-s}{2})_{J_k} \to s^{J_k} + \mathcal{O}(s^{J_k-1}) \qquad (4.38)$$

In [8], it has been shown the conformal block satisfies the differential equation:

$$DG_{\Delta_k,J_k} = \tfrac{1}{2} C_{\Delta_k,J_k} G_{\Delta_k,J_k} \qquad (4.39)$$

where,

$$D = (1-u-v)\frac{\partial}{\partial v}(v\frac{\partial}{\partial v} + \tfrac{\Delta_{34}-\Delta_{12}}{2} + u\frac{\partial}{\partial u}(2u\frac{\partial}{\partial u} - d) -$$
$$(1+u-v)(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} - \tfrac{\Delta_{12}}{2})(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + \tfrac{\Delta_{34}}{2})$$

and

$$C_{\Delta_k,J_k} = \Delta_k(\Delta_k - d) + J_k(J_k + d + 2)$$

(4.40)

These equations can be obtained from the eigenvalue problems of the Casimir operators of the conformal group, where the operators O^l_Δ with spin l and dimension Δ and its descendants are the eigenvectors of such operators.

Then if we use (4.39) with (4.40) and substitute (4.32) with (4.33), we obtain a recursion relation for the $Q_{J_k,m}$:

$$(D_s - \lambda_{J_k})Q_{J_k,m}(s) = 4m(h-\Delta_k-m)(2Q_{J_k,m} - Q_{J_k,m}(s) - Q_{J_k,m-1}(s+2) - Q_{J_k,m-1}(s))$$

(4.41)

Where,

$$D_s Q(s) = (s+\tau+\Delta_{12}-\Delta_{34})[(s+\tau)Q(s+2) - 2sQ(s)] + (s+\Delta_{12})(s-\Delta_{34})Q(s-2)$$

and

$$\lambda_{J_k} = 4J_k^2 + 4J_k(\tau-1) + (\tau+\Delta_{12})(\tau-\Delta_{34})$$

(4.42)

(4.41) suggests that (4.38) becomes:

$$Q_{J_k,m}(s) \to s^{J_k} + \mathcal{O}(s^{J_k-1}) \qquad (4.43)$$

This follows from the that the LHS (4.41) is automatically a polynomial of degree $(J-1)$, if we assume that $Q_{J_k,m}(s)$ is a polynomial of degree J. Imposing the same condition to RHS implies that $Q_{J_k,m}(s)$ and $Q_{J_k,m-1}(s)$ have the same leading behavior.

When we calculate the contour integral (4.12), we only consider poles from the Mellin amplitude. These poles are associated with specific operator's twist. But, we see that it also have poles from Gamma functions. These are associated with operator's twist, which will not bother us. This matter will be explained in the section 6.

[16]We use: ${}_3F_2(a,b,c,d,e,1) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\frac{1}{n!}$

4.2 Conformal Partial Waves

Our next step to derive Regge theory for conformal fields is the partial wave expansion.

In the Regge theory review we saw that we can expand the scattering amplitude as an infinity sum of Legendre polynomials, since the scattering problem had a spherical symmetric potential. But, in the case of non-spherical symmetric potentials the sum contains spherical harmonics instead of Legendre polynomials. When we first encounter a group theory course we learn about the $SO(3)$ group and that it has discrete representations, contracted by the spherical harmonics. The conformal group on the other hand has both discrete and continuous representations. Moreover, conformal blocks are representations of the conformal group and it turns out there is an expansion just like the expansion we saw for the non-relativistic scattering, but this time the expansion hold true for the reduced correlation function introduced at (4.6). The expansion is:

$$A(u,v) = 2\sum_{J=0}^{\infty}\int_{-\infty}^{\infty} b(\nu^2)\kappa_{\nu,J}G_{h+i\nu,J}(u,v)d\nu \qquad (4.44)$$

We write the normalization constant as:

$$\kappa_{\nu,J} = \frac{i\nu}{2\pi K_{h+i\nu,J}} \qquad (4.45)$$

We can write (4.44) as:

$$A(u,v) = 2\sum_{J=0}^{\infty}\int_{-\infty}^{\infty} b(\nu^2)\kappa_{\nu,J}G_{h+i\nu,J}(u,v)d\nu =$$

$$\sum_{J=0}^{\infty}\int_{-\infty}^{\infty} b(\nu^2)\kappa_{\nu,J}G_{h+i\nu,J}(u,v)d\nu + \sum_{J=0}^{\infty}\int_{-\infty}^{\infty} b(\nu^2)\kappa_{-\nu,J}G_{h-i\nu,J}(u,v)d\nu =$$

$$\sum_{J=0}^{\infty}\int_{-\infty}^{\infty} b(\nu^2)[\kappa_{\nu,J}G_{h+i\nu,J}(u,v) + \kappa_{-\nu,J}G_{h-i\nu,J}(u,v)]d\nu \Rightarrow$$

$$A(u,v) = \sum_{J=0}^{\infty}\int_{-\infty}^{\infty} d\nu b(\nu^2)F_{\nu,J}(u,v) \qquad (4.46)$$

with

$$F_{\nu,J}(u,v) = \kappa_{\nu,J}G_{h+i\nu,J}(u,v) + \kappa_{-\nu,J}G_{h-i\nu,J}(u,v) \qquad (4.47)$$

We derived (4.46) by using $\kappa_{-\nu,J} = -\kappa_{\nu,J}$.

(4.47) is written in position space and we saw at (4.12) that we can write $A(u,v)$ in the Mellin space. $F_{\nu,J}(u,v)$ can be expressed in the same representation, in particular we can write in terms of a single conformal partial wave $M_{\nu,J}(s,t)$, just like we wrote $A(u,v)$ in terms of the Mellin amplitude. Thus, we got:

$$F_{\nu,J}(u,v) = \int_{-i\infty}^{+i\infty}\frac{dtds}{(4\pi i)^2}M_{\nu,J}(s,t)u^{\frac{t}{2}}v^{-\frac{s+t}{2}}\Gamma(\tfrac{t+s}{2})\Gamma(\tfrac{t+s+\Delta_{12}-\Delta_{34}}{2})$$

$$\Gamma(\tfrac{-\Delta_{12}-s}{2})\Gamma(\tfrac{\Delta_3+\Delta_4-t}{2})\Gamma(\tfrac{\Delta_1+\Delta_2-t}{2})\Gamma(\tfrac{\Delta_{34}-s}{2}) \qquad (4.48)$$

Putting together (4.48) with (4.46) (4.12) and we obtain:

$$\int_{-i\infty}^{+i\infty} \frac{dtds}{(4\pi i)^2} M(s,t) u^{\frac{t}{2}} v^{-\frac{s+t}{2}} \Gamma(\tfrac{t+s}{2})\Gamma(\tfrac{t+s+\Delta_{12}-\Delta_{34}}{2})\Gamma(\tfrac{-\Delta_{12}-s}{2})\Gamma(\tfrac{\Delta_3+\Delta_4-t}{2})\Gamma(\tfrac{\Delta_1+\Delta_2-t}{2})\Gamma(\tfrac{\Delta_{34}-s}{2}) =$$

$$\sum_{J=0}^{\infty} \int_{-\infty}^{\infty} d\nu b(\nu^2) \int_{-i\infty}^{+i\infty} \frac{dtds}{(4\pi i)^2} M_{\nu,J}(s,t) u^{\frac{t}{2}} v^{-\frac{s+t}{2}} \Gamma(\tfrac{t+s}{2})\Gamma(\tfrac{t+s+\Delta_{12}-\Delta_{34}}{2})\Gamma(\tfrac{-\Delta_{12}-s}{2})$$

$$\Gamma(\tfrac{\Delta_3+\Delta_4-t}{2})\Gamma(\tfrac{\Delta_1+\Delta_2-t}{2})\Gamma(\tfrac{\Delta_{34}-s}{2}) \Rightarrow$$

$$M(s,t) = \sum_{J=0}^{\infty} \int_{-\infty}^{\infty} d\nu b(\nu^2) M_{\nu,J}(s,t) \qquad (4.49)$$

(4.49) is called partial wave expansion of the Mellin amplitude. Partial waves can be expressed in terms of Mack polynomials :

$$M_{\nu,J}(s,t) = \omega_{\nu,J}(t) P_{\nu,J}(s,t) \qquad (4.50)$$

With:

$$\omega_{\nu,J}(t) = \frac{\Gamma(\frac{\Delta_1+\Delta_2+J+i\nu-h}{2})\Gamma(\frac{\Delta_3+\Delta_4+J+i\nu-h}{2})\Gamma(\frac{\Delta_1+\Delta_2+J-i\nu-h}{2})\Gamma(\frac{\Delta_3+\Delta_4+J-i\nu-h}{2})}{8\pi\Gamma(i\nu)\Gamma(-i\nu)}$$

$$\frac{\Gamma(\frac{h+i\nu-J-t}{2})\Gamma(\frac{h-i\nu-J-t}{2})}{\Gamma(\frac{\Delta_1+\Delta_2-t}{2})\Gamma(\frac{\Delta_3+\Delta_4-t}{2})}$$

$$(4.51)$$

In (A.6) we have shown that the leading term of $P_{\nu,J}$ in s is s^J, therefore we normalized the polynomials as $P_{\nu,J}(s,t) = s^J + \mathcal{O}(s^{J-1})$.

We can fix $K_{\nu,J}$ by calculating the complex integral (4.48) at the poles $t = h \pm i\nu - J + 2m$ (m integer), which are obtained from (4.51). Then compare it through (4.47) with the general expression of the conformal block at (4.32). The expression is[17]

$$K_{\Delta,J} = \frac{\Gamma(\Delta+J)\Gamma(\Delta-h+1)(\Delta-1)_J}{4^{J-1}\Gamma(\frac{\Delta+J+\Delta_{12}}{2})\Gamma(\frac{\Delta+J-\Delta_{12}}{2})\Gamma(\frac{\Delta+J+\Delta_{34}}{2})\Gamma(\frac{\Delta+J-\Delta_{34}}{2})}$$
$$\frac{1}{\Gamma(\frac{\Delta_1+\Delta_2-\Delta+J}{2})\Gamma(\frac{\Delta_3+\Delta_4-\Delta+J}{2})\Gamma(\frac{\Delta_1+\Delta_2-\Delta+J-d}{2})\Gamma(\frac{\Delta_3+\Delta_4-\Delta+J-d}{2})}$$

$$(4.52)$$

At this point we are going to investigate the pole structure of the partial waves in the ν variable. To do this we are going to compare two representations of the Mellin amplitude at the Regge limit $s \to \infty$ and t fixed. We have seen so far the representation (4.49) with the partial waves and (4.16) with the conformal blocks.

Moreover when we were discussing the conformal block representation we saw that the Mellin amplitude takes the form (4.23) and at the Regge limit it becomes with the usage of (4.43):

[17] We should also note that to obtain this expression it is necessary to use the expression ($A7$), because it allows us to match Mack polynomials with the polynomials $\tilde{Q}_{J,m}(s)$

$$M(s,t) \approx \sum_k C_{12k} C_{34k} \sum_{m=0}^{\infty} s^{J_k} f(t) \frac{1}{t-\Delta_k+J_k-2m} \qquad (4.53)$$

$$f_k(t) = -\sum_{m=0} \frac{2\Gamma(\Delta_k+J_k)(\Delta_k-1)_{J_k}}{4^{J_k}\Gamma(\frac{\Delta_k+J_k+\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k-\Delta_{12}}{2})\Gamma(\frac{\Delta_k+J_k+\Delta_{34}}{2})\Gamma(\frac{\Delta_k+J_k-\Delta_{34}}{2})}$$
$$\frac{1}{m!(\Delta_k-h+1)_m\Gamma(\frac{\Delta_1+\Delta_2-\Delta_k+J_k}{2}-m)\Gamma(\frac{\Delta_3+\Delta_4-\Delta_k+J_k}{2}-m)} \frac{1}{t-\Delta_k+J_k-2m}$$
$$(4.54)$$

In [10] can be found an integral representation for $f_k(t)$:

$$f_k(t) = K_{\Delta_k,J} \int d\nu \frac{\omega_{\nu,J_k}(t)}{(\Delta_k-h)^2+\nu^2} \qquad (4.55)$$

Thus, we write (4.53) as:

$$M(s,t) \approx \sum_k C_{12k} C_{34k} s^{J_k} K_{\Delta_k,J_k} \int d\nu \frac{\omega_{\nu,J_k}(t)}{(\Delta_k-h)^2+\nu^2} \qquad (4.56)$$

On the other hand in the Regge limit, (4.49) becomes with the usage of (A.6) :

$$M(s,t) \approx \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} d\nu b_{J_k}(\nu^2) s^{J_k} \omega_{\nu,J_k}(t) \qquad (4.57)$$

Compare (4.56) with (4.57) and it suggests that $b_J(\nu^2)$ has the following form:

$$b_J(\nu^2) \approx C_{12k} C_{34k} \frac{K_{\Delta,J}}{\nu^2+(\Delta-h)^2} \qquad (4.58)$$

Therefore we found the pole structure of the partial waves in ν.

5 Conformal Regge Theory

We are now ready to derive Regge theory for the Mellin amplitudes. For simplicity we focus on the case $\Delta_{12} = \Delta_{34} = 0$. It's convenient to let $z = 1 + \frac{2s}{t} \Rightarrow s = \frac{zt-1}{2}$, thus the Mack polynomials become $P_{\nu,J}(\frac{z(t-1)}{2}, t)$ and we define:

$$P_{\nu,J}(z,t) = P_{\nu,J}(\tfrac{t(z-1)}{2}, t) \qquad (5.1)$$

As we explained in (A.6), we have the leading behavior $P_{\nu,J}(s,t) \approx s^J$, therefore for J being integer we have the property:

$$P_{\nu,J}(-z,t) = (-1)^J P_{\nu,J}(z,t) \qquad (5.2)$$

When we were studying Regge theory for non-relativistic scattering we analytic continue the scattering amplitude to complex values of l. Moreover, we saw that we can express the scattering amplitude as a contour integral in the complex plane through Sommerfeld-Watson transformation. We are going to do the same for the Mellin amplitude with the usage of the partial wave expansion just like the non-relativistic scattering. To do this we are going to analytic continue the partial amplitudes $b_J(\nu^2)$ to complex values of J. So we start from (4.49) and we write the sum in terms of even (+) and odd (-) $J's$.

$$M(s,t) = M^{+}(s,t) + M^{-}(s,t) =$$

$$\sum_{J_{even}}^{\infty} \int_{-\infty}^{\infty} d\nu b_J^{+}(\nu^2)\omega_{\nu,J}(t)P_{\nu,J}(z,t) + \sum_{J_{odd}}^{\infty} \int_{-\infty}^{\infty} d\nu b_J^{-}(\nu^2)\omega_{\nu,J}(t)P_{\nu,J}(z,t) \qquad (5.3)$$

To perform Sommerfeld-Watson transformation we are going to use the fact that:

$$\oint \frac{dJ'}{2\pi i}\frac{\pi F(J')}{sin(\pi J')} = 2\pi i \sum_{J=0}^{\infty} \frac{(J-J')}{2\pi i}\frac{\pi F(J)}{(-1)^J(J-J')} = \sum_{J=0}^{\infty} \frac{F(J)}{(-1)^J \pi} = \sum_{J_{even}}^{\infty} F(J) - \sum_{J_{odd}}^{\infty} F(J) \qquad (5.4)$$

Where $F(J')$ is an analytic function in J' and the contour integral is such it is picking poles of the function $\frac{1}{sin\pi J}$ at $J \in \mathbb{N}$.

We can write (5.3) by using (5.4) and (5.2) as:

$$M(s,t) =$$

$$\sum_{J_{even}}^{\infty} \int_{-\infty}^{\infty} d\nu b_J^{+}(\nu^2)\omega_{\nu,J}(t)\frac{P_{\nu,J}(z,t)+P_{\nu,J}(z,t)}{2} + \sum_{J_{odd}}^{\infty} \int_{-\infty}^{\infty} d\nu b_J^{-}(\nu^2)\omega_{\nu,J}(t)\frac{P_{\nu,J}(z,t)+P_{\nu,J}(z,t)}{2} =$$

$$\sum_{J_{even}}^{\infty} \int_{-\infty}^{\infty} d\nu b_J^{+}(\nu^2)\omega_{\nu,J}(t)\frac{P_{\nu,J}(-z,t)+P_{\nu,J}(z,t)}{2} -$$
$$\sum_{J_{odd}}^{\infty} \int_{-\infty}^{\infty} d\nu b_J^{-}(\nu^2)\omega_{\nu,J}(t)\frac{P_{\nu,J}(-z,t)-P_{\nu,J}(z,t)}{2} =$$

$$\int_{-\infty}^{\infty} d\nu \oint \frac{dJ}{2\pi i}\frac{\pi}{2sin(\pi J)} b_J^{+}(\nu^2)\omega_{\nu,J}(t)[P_{\nu,J}(-z,t) + P_{\nu,J}(z,t)] +$$

$$\int_{-\infty}^{\infty} d\nu \oint \frac{dJ}{2\pi i}\frac{\pi}{2sin(\pi J)} b_J^{-}(\nu^2)\omega_{\nu,J}(t)[P_{\nu,J}(-z,t) - P_{\nu,J}(z,t)]$$

Therefore we conclude that:

$$M^{\pm}(s,t) = \int_{-\infty}^{\infty} d\nu \oint \frac{dJ}{2\pi i}\frac{\pi}{2sin(\pi J)} b_J^{\pm}(\nu^2)\omega_{\nu,J}(t)[P_{\nu,J}(-z,t) \pm P_{\nu,J}(z,t)] \qquad (5.5)$$

This is the Regge representation of the Mellin amplitude. Just like scattering amplitude in the non-relativistic case we can deform the contour to pick up poles from the partial wave $b_J(\nu^2)$. In particular we are interesting in poles associated with the leading **Regge trajectory** $\Delta(J)$ for $J = 2, 4, ...$ These are the operators of lowest dimension for each even spin. One of the main tools in Regge theory was the Regge trajectory which we saw contains bound and resonance states. One the other hand, in conformal Regge theory the Regge trajectory $\Delta(J)$ contains operators twist in the OPE. The expressions in this section derived at the Regge limit, which corresponds to the leading Regge Trajectory. That is the trajectory with operators of twist 2. So in the leading Regge trajectory physical operators are with $\Delta = 4$ $J = 2$, $\Delta = 6$ $J = 4$ etc. The first one actually corresponds to the energy momentum tensor.

We continue with the pole structure of the partial waves in ν. For even spins (4.58) gives

$$b_J^{+}(\nu^2) \approx C_{12k}C_{34k}\frac{K_{\Delta(J),J}}{\nu^2+(\Delta(J)-h)^2} = \frac{r(J)}{\nu^2+(\Delta(J)-h)^2} \qquad (5.6)$$

After analytic continuation in J this pole structure in ν becomes in J. Moreover $j(\nu)$ is the inverse function of $\Delta(J)$ defined by

$$\nu^2 + (\Delta(j(\nu)) - h)^2 = 0 \qquad (5.7)$$

We expand (5.6) in J near $j(\nu)$ as

$$b_J^+(\nu^2)[2(\Delta(j(\nu)) - h)\tfrac{d\Delta(j(\nu))}{dJ}] \approx r(j(\nu))$$

By taking the derivative of (5.7) with respect the ν and we have

$$\nu + (\Delta(j(\nu)) - h)\tfrac{d\Delta}{d\nu} = 0 \Rightarrow$$

$$\nu + (\Delta(j(\nu)) - h)\tfrac{d\Delta}{dJ}\tfrac{1}{j'(\nu)} = 0 \Rightarrow$$

$$\tfrac{d\Delta(j(\nu))}{dJ} = -\tfrac{\nu j'(\nu)}{\Delta(j(\nu))-h}$$

With this expression we obtain

$$b_J^+(\nu^2)[2(\Delta(j(\nu)) - h)\tfrac{\nu j'(\nu)}{\Delta(j(\nu))-h}] \approx -r(j(\nu))$$

$$b_J^+(\nu^2) \approx -\tfrac{j'(\nu)r(j(\nu))}{2\nu(J-j(\nu))}$$

$$(5.8)$$

The function $j(\nu)$ is called **Reggeon spin** and is defined from the equation (5.7). Since the analytic continuation is in J it means that we have also analytic continue the OPE coefficients, which it is the key feature in conformal Regge theory.

The contribution of this pole in (5.5) for $'+'$ is:

$$\int_{-\infty}^{\infty} d\nu \oint \tfrac{dJ}{2\pi i}\tfrac{\pi}{2sin(\pi J)}[-\tfrac{j'(\nu)r(j(\nu))}{2\nu(J-j(\nu))}]\omega_{\nu,J}(t)[P_{\nu,J}(-z,t) + P_{\nu,J}(z,t)] =$$

$$-\int_{-\infty}^{\infty} d\nu 2\pi i\tfrac{1}{2\pi i}\tfrac{\pi}{2sin(\pi j(\nu))}[-\tfrac{j'(\nu)r(j(\nu))}{2\nu}]\omega_{\nu,j(\nu)}(t)[P_{\nu,j(\nu)}(-z,t) + P_{\nu,j(\nu)}(z,t)]$$

$$\int_{-\infty}^{\infty} d\nu\tfrac{\pi}{2sin(\pi j(\nu))}\tfrac{j'(\nu)r(j(\nu))}{2\nu}\omega_{\nu,j(\nu)}(t)[P_{\nu,j(\nu)}(-z,t) + P_{\nu,j(\nu)}(z,t)]$$

$$(5.9)$$

In the Regge limit $s \to \infty$, (5.9) becomes:

$$\int_{-\infty}^{\infty} d\nu\tfrac{\pi}{4sin(\pi j(\nu))}\tfrac{j'(\nu)r(j(\nu))}{\nu}\omega_{\nu,j(\nu)}(t)[(-s)^{j(\nu)} + s^{j(\nu)}] = \int d\nu\beta(\nu)\omega_{\nu,j(\nu)}(t)\tfrac{(-s)^{j(\nu)}+s^{j(\nu)}}{sin(\pi j(\nu))} \Rightarrow$$

$$M(s,t) \approx \int d\nu\beta(\nu)\omega_{\nu,j(\nu)}(t)\tfrac{(-s)^{j(\nu)}+s^{j(\nu)}}{sin(\pi j(\nu))} \qquad (5.10)$$

$$\beta(\nu) = \tfrac{\pi j'(\nu)r(j(\nu))}{4\nu} = \tfrac{\pi j'(\nu)}{4\nu}K_{h\pm i\nu}C_{12j(\nu)}C_{34j(\nu)} \qquad (5.11)$$

Regge Theory in non-relativistic scattering	Conformal Regge Theory
Scattering Amplitude $f(\theta)$	Mellin Amplitude $M(s,t)$
Partial Wave Expansion $f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(S_l - 1)\, P_l(cos\theta)$	Conformal Partial Wave Expansion $M(s,t) = \sum_{J=0}^{\infty}\int_{-\infty}^{\infty} d\nu\, b_J(\nu^2) M_{\nu,J}(s,t)$
S_l-matrix poles	Partial Wave $b_J(\nu^2)$ poles
Regge Representation $f(\theta) = -\frac{1}{2k}\oint \frac{z\left(S_{z-\frac{1}{2}} - 1\right)P_{z-\frac{1}{2}}(-cos\theta)}{cos\pi z} dz$	Regge Representation $M_{\pm}(s,t) = \int_{-\infty}^{\infty} d\nu \oint \frac{dJ}{2\pi i}\frac{\pi}{2\sin(\pi J)} b_J^{\pm}(\nu^2)\omega_{\nu,J}(t)[P_{\nu,J}(-z,t) \pm P_{\nu,J}(z,t)]$
Regge Trajectory $\alpha(E)$ Contains bound and resonance states in one quantity	Regge Trajectory $\Delta(J)$ Contains operators twist appearing in the OPE
Residues from S-matrix poles Simplify the calculations for the scattering amplitudes	Residues from the partial wave $b_J(\nu^2)$ poles Contain OPE coefficients

Figure 7: Regge Theory vs Conformal Regge Theory

To obtain (5.9) we used (A.6) and the residues (4.10) are written at $\Delta = h \pm i\nu$, since for these values of Δ, (5.7) is satisfied.

The equations (5.10) and (5.11) are main results of this review. In particular we are going to use (5.11) to derive non-trivial results about the OPE coefficients.

By recalling our results from the Regge theory review we can actually see the correspondences between Regge theory and conformal Regge in figure 7

It also exists a position space version of (5.9), which is going to useful later on. To obtain it we start from (4.12), and work at the Regge limit. First, we perform the rotation:

$$v^{-\frac{(s+t)}{2}} \to (e^{2i\pi}v)^{-\frac{(s+t)}{2}} = e^{-i\pi(s+t)}v^{-\frac{(s+t)}{2}} \tag{5.12}$$

Next, we let $s = ix$ and we work at $x \to \infty$. (4.12) becomes for $\Delta_{12} = \Delta_{34} = 0$ with (5.12) :

$$A(u,v) = \int_{-i\infty}^{+i\infty} dt \int_{-\infty}^{\infty} \frac{idx}{(4\pi i)^2} M(ix,t) u^{\frac{t}{2}} e^{-i\pi(s+t)} v^{-\frac{(s+t)}{2}}$$
$$\Gamma(\tfrac{t+ix}{2})\Gamma(\tfrac{t+ix}{2})\Gamma(\tfrac{-ix}{2})\Gamma(\tfrac{2\Delta_3-t}{2})\Gamma(\tfrac{2\Delta_1-t}{2})\Gamma(\tfrac{-ix}{2}) \tag{5.13}$$

We can use the approximation at $x \to \infty$

$$\Gamma(a + i\tfrac{x}{2})\Gamma(b - i\tfrac{x}{2}) \approx 2\pi e^{i\pi(a-b)/2}(\tfrac{x}{2})^{a+b-1}e^{-\pi x/2}$$

To write for the case of $a = t/2$ and $b = 0$

$$[\Gamma(\tfrac{t+ix}{2})\Gamma(-i\tfrac{x}{2})]^2 \approx 4\pi^2 e^{i\pi t/2}(\tfrac{x}{2})^{t-2}e^{-\pi x}$$

And (5.13) becomes

$$A(u,v) \approx \int_{-i\infty}^{+i\infty} dt \int_{-\infty}^{\infty} \frac{idx}{(4\pi i)^2} M(ix,t) u^{\frac{t}{2}} e^{-i\pi(ix+t)} v^{-\frac{(ix+t)}{2}}$$
$$4\pi^2 e^{i\pi t/2} (\tfrac{x}{2})^{t-2} e^{-\pi x} \Gamma(\tfrac{2\Delta_3 - t}{2}) \Gamma(\tfrac{2\Delta_1 - t}{2}) \Rightarrow$$

$$A(u,v) \approx \int_{-i\infty}^{+i\infty} \frac{dt}{4i} \Gamma(\tfrac{2\Delta_3 - t}{2}) \Gamma(\tfrac{2\Delta_1 - t}{2}) u^{t/2} v^{-t/2} e^{-i\pi t/2} \int_{-\infty}^{\infty} dx M(ix,t) (\tfrac{x}{2})^{t-2} v^{-ix/2} \tag{5.14}$$

At this point we introduce the variables

$$u = \sigma^2 \qquad \text{and} \qquad v = (1-\sigma e^{\rho})(1-\sigma e^{-\rho}) \approx 1 - 2\sigma cosh(\rho) \tag{5.15}$$

such that the Regge limit corresponds to $\sigma \to 0$ and ρ fixed. In this limit (5.7) becomes

$$A(u,v) \approx \int_{-i\infty}^{+i\infty} \frac{dt}{4i} \Gamma(\tfrac{2\Delta_3 - t}{2}) \Gamma(\tfrac{2\Delta_1 - t}{2}) (\sigma^2)^{t/2}$$
$$(1 - 2\sigma cosh(\rho))^{-t/2} e^{-i\pi t/2} \int_{-\infty}^{\infty} dx M(ix,t) (\tfrac{x}{2})^{t-2} (1 - 2\sigma cosh(\rho))^{-ix/2} \Rightarrow$$

$$A(u,v) \approx \int_{-i\infty}^{+i\infty} \frac{dt}{4i} \Gamma(\tfrac{2\Delta_3 - t}{2}) \Gamma(\tfrac{2\Delta_1 - t}{2}) (\sigma^2)^{t/2}$$
$$(1 - 2\sigma cosh(\rho))^{-t/2} e^{-i\pi t/2} \int_{-\infty}^{\infty} dx M(ix,t) (\tfrac{x}{2})^{t-2} e^{-i\frac{x}{2} ln(1-2\sigma cosh(\rho))} \approx$$

$$\int_{-i\infty}^{+i\infty} \frac{dt}{4i} \Gamma(\tfrac{2\Delta_3 - t}{2}) \Gamma(\tfrac{2\Delta_1 - t}{2}) (\sigma)^t e^{-i\pi t/2} \int_{-\infty}^{\infty} dx M(ix,t) (\tfrac{x}{2})^{t-2} e^{ix\sigma cosh(\rho)}$$

(5.16)

Where in the last we used the approximation $ln(1-x) \approx -x$ and that $\sigma \to 0$.

On the other hand we write (5.3) for $s = ix$ and we get:

$$M(s,t) \approx \int d\nu \beta(\nu) \omega_{\nu,j(\nu)}(t) x^{j(\nu)} \frac{(-i)^{j(\nu)} + (i)^{j(\nu)}}{sin(\pi j(\nu))} =$$

$$\int d\nu \beta(\nu) \omega_{\nu,j(\nu)}(t) x^{j(\nu)} \frac{(e^{-i\pi/2})^{j(\nu)} + (e^{i\pi/2})^{j(\nu)}}{sin(\pi j(\nu))} =$$

$$\int d\nu \beta(\nu) \omega_{\nu,j(\nu)}(t) x^{j(\nu)} \frac{2cos(\pi j(\nu)/2)}{sin(\pi j(\nu))} =$$

$$\int d\nu \beta(\nu) \omega_{\nu,j(\nu)}(t) x^{j(\nu)} \frac{2cos(\pi j(\nu)/2)}{\pi j(\nu)/2 cos(\pi j(\nu)/2) sin(\pi j(\nu)/2)} =$$

$$\int d\nu \beta(\nu) \omega_{\nu,j(\nu)}(t) \frac{x^{j(\nu)}}{sin(\pi j(\nu)/2)}$$

(5.17)

Putting together (5.17) and (5.16)

$$\int_{-i\infty}^{+i\infty} \frac{dt}{4i}\Gamma(\frac{2\Delta_3 - t}{2})\Gamma(\frac{2\Delta_1 - t}{2})(\sigma)^t e^{-i\pi t/2} \int_{-\infty}^{\infty} dx \int d\nu \beta(\nu)\omega_{\nu,j(\nu)}(t)\frac{x^{j(\nu)}}{sin(\pi j(\nu)/2)}(\frac{x}{2})^{t-2}e^{ix\sigma cosh(\rho)}$$

The integral in x is

$$I = \frac{1}{2^{t-2}} \int_{-\infty}^{+\infty} e^{ix\sigma cosh\rho} x^{j(\nu)+t-2} dx$$

By letting $x' = -ix\sigma cosh\rho \Rightarrow dx = \frac{idx'}{\sigma cosh\rho}$, $a = j(\nu)+t-1$ and noting that the amplitude A is controlled by the large s behavior, then[18]

$$I = \frac{2}{(-i)^{j(\nu)+t-2}2^t} \frac{i}{\sigma cosh\rho(\sigma cosh\rho)^{j(\nu)+t-2}} \int_0^{+\infty} e^{-x'} x'^{a-1} dx' =$$

$$I = \frac{2}{(-i^{j(\nu)+t-2})2^t} \frac{i}{\sigma cosh\rho(\sigma cosh\rho)^{j(\nu)+t-2}} \Gamma(j(\nu)+t-1) \qquad (5.18)$$

Then put together (5.18), (5.17) with some algebra and we obtain

$$A \approx -\pi i \int d\nu \beta(\nu) \frac{e^{i\pi j(\nu)/2}}{sin(\frac{\pi j(\nu)}{2})} \sigma^{1-j(\nu)} 2^{j(\nu)}$$

$$\int_{-i\infty}^{i\infty} \frac{dt}{2\pi i} \Gamma(\frac{2\Delta_1 - t}{2})\Gamma(\frac{2\Delta_3 - t}{2}) \frac{\Gamma(j(\nu)+t-1)}{(2cosh\rho)^{j(\nu)+t-1}} \omega_{\nu,j(\nu)}(t)$$

$$(5.19)$$

The harmonic functions $\Omega_{i\nu}(\rho)$ have the following integral representation on $2h-1$-dimensional hyperbolic space

$$\Omega_{i\nu}(\rho) = \int \frac{dz}{2\pi i} \frac{\Gamma(z)\Gamma(\frac{h+i\nu-z-1}{2})\Gamma(\frac{h-i\nu-z-1}{2})}{8\pi^h \Gamma(i\nu)\Gamma(-i\nu)} (2cosh\rho)^{-z} \qquad (5.20)$$

Also we let

$$\gamma(\nu) = \Gamma(\frac{2\Delta_1 + j + i\nu - h}{2})\Gamma(\frac{2\Delta_3 + j + i\nu - h}{2}) \qquad (5.21)$$

And at the end with (5.21), (5.20), (5.19) becomes

$$A(\sigma,\rho) \approx -2\pi i \int d\nu \beta(\nu) \frac{\pi^{h-1} e^{i\pi j(\nu)/2}}{sin(\frac{\pi j(\nu)}{2})} \sigma^{1-j(\nu)} 2^{j(\nu)-1} \gamma(\nu)\gamma(-\nu)\Omega_{i\nu}(\rho) \Rightarrow$$

$$A(\sigma,\rho) \approx \int d\nu a(\nu) \sigma^{1-j(\nu)} \Omega_{i\nu}(\rho) \qquad (5.22)$$

with

$$a(\nu) = -\frac{\pi^{h-1} e^{i\pi j(\nu)/2}}{sin(\frac{\pi j(\nu)}{2})} 2^{j(\nu)-1} \gamma(\nu)\gamma(-\nu)\beta(\nu) \qquad (5.23)$$

And we obtained the final expression (5.22) for position space of the amplitude A.

[18]This means that the lowest value of interval is a number $e > 0$ which at the limit $\sigma \to 0$ corresponds with it being 0

6 Application to N=4 Supersymmetric Yang–Mills

In this section we are going to use the Conformal Regge theory to find the OPE coefficients in the operator product expansion between O_1O_2 and O_3O_4 for a 4-point correlation function of a CFT. The CFT that we are going to study is the $N = 4$ **Supersymmetric Yang Mills theory** and thus $d = 2h = 4$.

This particular theory describes a universe containing boson and fermion fields which are related by 4 supersymmetries, which are the the kind of symmetries that relate bosons and fermions. This is what the $N = 4$ Sypersymmetric stands for. The Yang Mills part stands for a class of theories known as gauge theories, based on a specific group and they can be thought as a generalized version of electromagnetism[19]. Another example of a gauge theory is the quantum chromodynamics[20], which is a $SU(3)$ Yang Mills theory. Moreover, this field theory contains 6 real scalar fields, 4 complex Weyl fermions and gauge field A_μ with coupling constant g_{YM}. We should note that g_{YM} has no restrictions on its value.

In such theories the boson field or the fermion field are contained in the adjoint representation, which for the $SU(N)$ group are $N \times N$ complex matrices made from the structure constants of the Lie algebra. In addition to this the physical operators in these theories are invariant under gauge transformations and the simplest way to contract those is to multiply the matrices that were mentioned above and calculate their trace. The traces are of course invariant under transformations and are suitable for physical operators. These are the single trace operators. We can go even further and multiply these traces and obtain multi-trace operators.

In particular, we are going to study $N = 4$ SYM in the **large-N limit**[21], where we are interesting in the leading planar diagrams in the OPE expansion. These diagrams are multiplied by $1/N^2$. As external operators[22] we will consider single trace operators.

In section 4 was mentioned that we consider Mellin poles for the complex integration in (4.12). These poles are associated with the twist of single trace operators. On the other hand, Gamma function poles are associated with double trace operators. In the large N-limit correlation functions with double trace operators in the OPE are coming as subleading powers of $1/N^2$ and the leading correlation functions contain single trace operators, which account for the planar part of the four point correlation function.

When the interaction is on and consider weak coupling, the OPE coefficients appearing in Regge trajectory are associated with twist two operators and they are linear combinations of

$$tr(F_{\mu\nu_1}D_{\nu_2}..D_{\nu_{J-1}}F^{\mu}_{\nu_J}), \qquad tr(\phi_{AB}D_{\nu_2}..D_{\nu_J}\phi^{AB}), \qquad tr(\bar{\psi}_A D_{\nu_1}..D_{\nu_{J-1}}\Gamma_{\mu_J}\psi^A)$$

Where F is the generalized field strength tensor in the Yang Mills theory, D is the covariant derivative, ϕ is the complex scalar field, ψ is the spinor field and Γ is associated

[19] Electromagnetism is based on the $U(1)$ group, where gauge transformations are originated from.

[20] Quantum Chromodynamics is the theory of the strong nuclear force and $N = 3$ stands for the total number of colour charges (Red, Blue, Green).

[21] More information can found at the appendix B .

[22] These are the operators that we will insert at the correlation function.

with gamma matrices. The first operator is associated with the gauge field, the second with the complex scalar fields and the third with the fermion fields.

6.1 Weak Coupling

In our application we will consider N=4 SYM in weak coupling, where the shape of Regge trajectory changes from the one in the free theory as we see in figure 8. This is because when interaction is on the dimensions are shifted due to **anomalous dimensions**[23]. More specific in the free theory operators twist in the leading Regge trajectory satisfy $\Delta - J = 2$, but in the interacting theory they satisfy $\Delta - J > 2$.

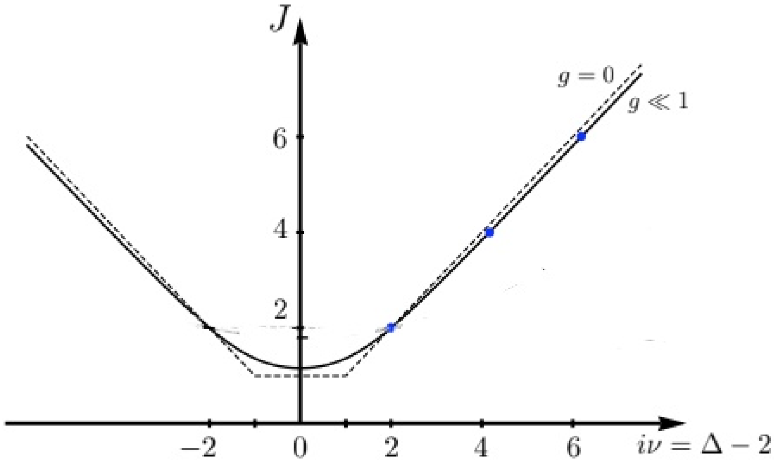

Figure 8: Regge Trajectory $\Delta(J)$. The dotted lines represent the trajectory in the free theory ($g = 0$, $\Delta - J = 2$). The other line represents the trajectory in weak coupling ($g << 1$, $\Delta - J > 2$). The blue points on the other hand represent physical operators with $\Delta > 0$ and J even, with lowest operator with $\Delta = 4$, $J = 2$ being the energy momentum tensor. All operators in the trajectory start around the free theory value $J = 1$.

Now in this part we are going to present some useful expressions associated with anomalous dimension when the theory is weak coupled. The expressions are based on previous works and we are going to apply them around the free theory value $J = 1$. The reason why choose this particular value of J lies on the properties of scattering in the Regge limit for Yang-Mills theories.

The anomalous dimensions of the operators in the leading Regge trajectory is a function of the spin and the 't Hooft coupling. In particular, there is an expansion for weak coupling:

$$\gamma(J) = \Delta(J) - J - 2 = \sum_{n=1}^{\infty} g^{2n} \gamma_n(J) \qquad (6.1)$$

[23]More information about anomalous dimensions can be found at the appendix C .

Where the coupling g is related to the 't Hooft coupling $\lambda = g_{YM}^2 N$ by

$$g^2 = \frac{\lambda}{16\pi^2} \qquad (6.2)$$

In general the anomalous dimensions γ_n are very difficult to calculate. The first two terms in the expansion are

$$\gamma_1 = 8S_1(x)$$

$$\gamma_2 = -32(S_3(x) + S_{-3}(x)) + 64S_{-2,1}(x) - 64(S_1(x)S_2(x) + S_1(x)S_{-2}(x))$$

$$(6.3)$$

Where $x = J - 2$ and the functions S are harmonics sums defined by

$$S_{a_1,a_2,..,a_n}(x) = \sum_{y=1}^{x} \frac{(sign(a_1))^y}{y^{|a_1|}} S_{a_2,..,a_n}(y) \qquad (6.4)$$

Starting from the trivial seed $S(y) = 1$. The weak expansion of the function $\Delta(J)$ is around the free theory which corresponds for $g = 0$ and the dimension is $\Delta(J) = J + 2$.

We saw at section 5 that the function $\Delta(J)$ defines the Reggeon spin $j(\nu)$ from (5.7) and since $h = \frac{d}{2} = 2$ (5.7) gives

$$\nu^2 + (\Delta(j(\nu)) - 2)^2 = 0 \Rightarrow$$

$$\Delta(j(\nu)) = 2 \pm i\nu \qquad (6.5)$$

In [12], it is explained that the Reggeon spin can be also be computed directly from the Regge limit of the four point correlation function. At weak coupling, there are methods that provide us an expansion around the free theory value $j = 1$. In [13] the function $j(\nu)$ is known up to next to leading order

$$j(\nu) = 1 + 4g^2(\chi(\nu) + g^2\delta(\nu)) + \mathcal{O}(g^6) \qquad (6.6)$$

With

$$\chi(\nu) = 2\Psi(1) - \Psi(\tfrac{1+i\nu}{2}) - \Psi(\tfrac{1-i\nu}{2}), \qquad (6.7)$$

$$\delta(\nu) = 4\chi''(\nu) + 6\zeta(3) - 2\zeta(2)\chi(\nu) - 2\Phi(\tfrac{1+i\nu}{2}) - 2\Phi(\tfrac{1-i\nu}{2}) \qquad (6.8)$$

where $\Psi(x) = \frac{d(ln\Gamma(x))}{dx}$ is the Digamma function and $\Phi(x)$ is given by

$$\Phi(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Psi'(\frac{k+2}{2}) - \Psi'(\frac{k+1}{2})}{k+x} \qquad (6.9)$$

Also there is another expression of $j(\nu)$, which is

$$j(\nu) = 1 + \sum_{n=1}^{\infty} g^{2n}[F_n(\tfrac{1-i\nu}{2}) + F_n(\tfrac{-1-i\nu}{2})] \qquad (6.10)$$

where

$$F_1(x) = -4S_1(x) \qquad (6.11)$$

As we saw from (6.7) $\Delta(J)$ and $j(\nu)$ are basically the inverse of each other, but as explained in [14] their perturbative expansions contain different information. Therefore, we say that the process of inverting the functions does not commute with perturbation theory.

Now let's consider the limit $g^2 \to 0$ and $j \to 1$ with $\frac{j-1}{g^2}$ fixed, then (6.10) with (6.11) give

$$\frac{j(\nu)-1}{-4g^2} = S_1(\frac{i\nu-1}{2}) + S_1(\frac{-i\nu-1}{2}) \qquad (6.12)$$

We can calculate $S_1(x)$ from (6.4) and we obtain $S_1(x) = \sum_{y=1}^{x} \frac{1}{y} = H_x$. H_x is the harmonic number which is related with digamma function as $\Psi(x+1) - \Psi(1) = H_x$. Thus, $S_1(x) = \Psi(x+1) - \Psi(1)$. It is easy to see that $\Psi(x)$ has the same poles as the gamma function, therefore the RHS in (6.12) has poles at $\pm i\nu = -2n-1$ for $n = 0, 1, 2, 3, ..$. For $n = 0$ we have $\pm i\nu = -1$ and we can expand around $i\nu = 1$ just like in [14]. We obtain

$$\frac{j(\nu)-1}{-4g^2} = \frac{2}{i\nu-1} - 2\sum_{k=1}^{\infty} \zeta(2k+1)(\frac{i\nu-1}{2})^{2k} \Rightarrow \qquad (6.13)$$

$$y = \frac{2}{x} - 2\sum_{k=1}^{\infty} \zeta(2k+1)(\frac{x}{2})^{2k} \qquad (6.14)$$

We write (6.5) for '+' as $\Delta(J) - 3 = i\nu - 1 = x$. Therefore by finding the first order terms of the inverse function of (6.14), just like in [14], we obtain

$$x = 2(\frac{1}{y}) + 0(\frac{1}{y})^2 + 0(\frac{1}{y})^3 - 4\zeta(3)(\frac{1}{y})^4 + .. \Rightarrow$$

$$\Delta(J) - 3 = 2(\frac{-4g^2}{J-1}) + 0(\frac{-4g^2}{J-1})^2 + 0(\frac{-4g^2}{J-1})^3 - 4\zeta(3)(\frac{-4g^2}{J-1})^4 + .. \qquad (6.15)$$

The expression (6.15) it's useful for the prediction of the leading singularities of the anomalous dimension function around $J = 1$ at all orders in perturbation theory. In the next section we will follow a similar procedure to study the behavior of OPE coefficients.

6.2 OPE Coefficients-Leading Order Predictions

In the conformal Regge theory section we saw that we can find the OPE coefficients from the residues $r(j(\nu))$. We should remind that we are considering OPE coefficients with operators normalized as (4.15). After the analytic continuation of the OPE coefficients in J we have for the case $\Delta_{12} = \Delta_{34} = 0$, $C^2(J) \to C^2(j(\nu)) = C_{11j(\nu)}C_{33j(\nu)}$.

In free theory we can compute the OPE coefficient $C(J)$ of the leading Regge trajectory in the OPE between two protected scalar operators of the form $O_1 = tr(\phi_{12}\phi^{12})$, where ϕ_{12} is a complex scalar field of SYM. The result is obtained directly from the theory in the large N-limit and it can be found in [3]. We have

$$C^2(J) = \frac{1}{N^2}\frac{2^{1+J}J(J-1)\Gamma^2(J+1)}{(4J^2-1)\Gamma(2J+1)} + \mathcal{O}(g^2) \qquad (6.16)$$

We can expand (6.16) around $J = 1$ to obtain

$$C^2(J) = \frac{1}{N^2}\frac{2^{1+J}J(J-1)\Gamma^2(J+1)}{(4J^2-1)\Gamma(2J+1)} + \mathcal{O}(g^2) \approx \frac{J-1}{N^2}[\tfrac{4}{2}(\tfrac{1}{3} - \tfrac{8}{9}(J-1))] + \mathcal{O}(g^2) \Rightarrow$$

$$C^2(J) = \frac{J-1}{N^2}(\tfrac{2}{3} + \mathcal{O}(J-1)) + \mathcal{O}(g^2) \qquad (6.17)$$

Next we are going to take look at the Regge residue $r(j(\nu))$. In [12] the Regge residue in the position space $a(\nu)$ firstly introduced at (5.23) was shown to be

$$a(\nu) = i\frac{16\pi^5 g^4 tanh(\frac{\pi\nu}{2})}{\nu cosh^2(\frac{\pi\nu}{2})} + \mathcal{O}(g^6) \qquad (6.18)$$

Then by compare it with our result (5.23) and using the expressions (5.11) with (6.6), we obtain the following expression for $r(j(\nu))$

$$r(j(\nu)) = -\frac{2^8\pi g^2}{N^2}\frac{tanh(\frac{\pi\nu}{2})}{\chi'(\nu)(1+\nu^2)^2} + \mathcal{O}(g^4) \qquad (6.19)$$

For weak coupling around $J = 1$, $\frac{J-1}{g^2}$ is fixed, thus it is clear that (6.19) computes the behavior of the function $r(j(\nu))$ around $J-1$, since it starts as $g^2 \sim J-1$. The same holds true for the equation of (6.17), which was directly computed in free theory. This is true, since $r(J))$ and $C^2(J)$ are related as we saw at (5.6). For $J = j(\nu)$ (5.6) gives

$$r(j(\nu)) = C^2(j(\nu))K_{\Delta(j(\nu)),j(\nu)} \qquad (6.20)$$

The equation (6.20) can be used to compute the OPE coefficients from the Regge residue in the region $J-1 \sim g^2$. Moreover, we are going to reproduce the free theory result (6.17)

To do we are going to follow an analogous procedure with the one described for the anomalous dimensions at the weak limit section. First, from (6.17) we conclude that the continuation of the OPE coefficients $C(J)$ in the region where $g^2 \sim J-1$ admits the following general perturbative expansion

$$C^2(J) = (J-1)a(\tfrac{g^2}{J-1}) + \mathcal{O}(g^4) \qquad (6.21)$$

Where $a(x)$ is some function that we will soon determine. For $J = j(\nu)$ (6.21) becomes with (6.6) at order g^2

$$C^2(j(\nu)) = (j(\nu)-1)a(\tfrac{g^2}{j(\nu)-1}) + \mathcal{O}(g^4) = (1 + 4g^2\chi(\nu) - 1)a(\tfrac{g^2}{1+4g^2\chi(\nu)-1}) + \mathcal{O}(g^4) \Rightarrow$$

$$C^2(j(\nu)) = 4g^2\chi(\nu)a(\tfrac{1}{4\chi(\nu)}) + \mathcal{O}(g^4) \qquad (6.22)$$

Then with (6.20) and (6.19), (6.22) gives

$$C^2(j(\nu)) = 4g^2\chi(\nu)a(\tfrac{1}{4\chi(\nu)})) = -\frac{1}{K_{\Delta(j(\nu)),j(\nu)}}\frac{2^8\pi g^2}{N^2}\frac{tanh(\frac{\pi\nu}{2})}{\chi'(\nu)(1+\nu^2)^2} \Rightarrow$$

$$a(\frac{1}{\chi(\nu)}) = -\frac{1}{K_{\Delta(j(\nu)),j(\nu)}}\frac{2^6\pi}{N^2}\frac{tanh(\frac{\pi\nu}{2})}{\chi(\nu)\chi'(\nu)(1+\nu^2)^2} \qquad (6.23)$$

We want a prediction for the OPE coefficients around $J = 1$ so that the function $a(x)$ can be expanded as

$$a(x) = \frac{1}{N^2}(a_0 + a_1x + a_2x^2 + ...) \qquad (6.24)$$

With $x = \frac{g^2}{(J-1)}$ being small. This procedure is analogous with the one we demonstrate at weak coupling section. In (6.13) we expand around $i\nu = 1$, to obtain an expression for $\frac{j(\nu)-1}{-4g^2}$ in terms of powers of $i\nu - 1$, which lead us at the end to an expansion of anomalous dimension function at leading orders. This time, around $i\nu = \pm 1$ we notice (6.23) is 0, since $\chi(\nu)$ has poles at $i\nu = \pm(2n+1)$ for $n = 0, 1, 2, 3$. This fact tell that an expansion like (6.24) around the origin, it is possible. With the usage of (6.15) we can expand (6.23) in terms of x. After directly computing the leading coefficients of (6.24) we find

$$a_0 = \frac{2}{3}, \qquad a_1 = \frac{64}{9}, \qquad a_2 = \frac{32}{27}(61 - 3\pi^2) \qquad (6.25)$$

The free theory prediction $a_0 = 2/3$ is in agreement with (6.17). This fact showcase us that our formulas of conformal Regge theory work.

Conclusion

In the last section of this review which was the application to N=4 SYM, we tested our formulas and saw that reproduce the same results with that of the usual perturbative calculations in a QFT. This fact showcase that the general idea of incorporating tools from Regge theory to a CFT is great alternative for finding OPE coefficients. The main goal of this project was to show the general idea behind of Regge theory and conformal Regge theory, which is highlighted in figure 7.

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A Mack Polynomials

In this section of the appendix we are going to study Mack polynomials and some of their properties. Mack polynomials are given by the series:

$$P_{\nu,J}(s,t)=\sum_{r=0}^{[J/2]} a_{J,r}\frac{2^{J+2r}(\frac{h+i\nu-J-t}{2})_r(\frac{h-i\nu-J-t}{2})_r(J-2r)!}{(h+i\nu-1)_J(h-i\nu-1)_J}$$
$$\sum_{\sum k_{ij}=J-2r}(-1)^{k_{13}+k_{24}}\prod_{(ij)}\frac{(\delta_{ij})_{k_{ij}}}{k_{ij}!}\prod_{n=1}^{4}(a_n)_{J-r-\sum_j k_{jn}} \qquad \text{(A.1)}$$

The label (ij) run over four possibilities $(13),(14),(23)$ and (24). As for δ_{ij}'s, they are given by (4.8).

The variables (a_n) are given by

$$a_1=1-\frac{h+i\nu+J+\Delta_{12}}{2}, \qquad a_2=1-\frac{h+i\nu+J-\Delta_{12}}{2}$$
$$a_3=1-\frac{h-i\nu+J+\Delta_{34}}{2}, \qquad a_4=1-\frac{h-i\nu+J-\Delta_{34}}{2}$$

(A.2)

Mack polynomials have a particular leading behavior in s. For $r=0$, (A.1) becomes:

$$P_{\nu,J}\sim\frac{J!}{(h+i\nu-1)_J(h-i\nu-1)_J}\sum_{\sum k_{ij}=J}\prod_{(ij)}\frac{(\delta_{ij})_{k_{ij}}}{k_{ij}!}\prod_{n=1}^{4}(a_n)_{J-\sum_j k_{jn}}\Rightarrow$$
$$P_{\nu,J}\approx\frac{J!}{(h+i\nu-1)_J(h-i\nu-1)_J}s^J\sum_{\sum k_{ij}=J}\prod_{(ij)}\frac{1}{k_{ij}!}\prod_{n=1}^{4}(a_n)_{J-\sum_j k_{jn}}$$

(A.3)

In the last we have approximate $(\delta_{ij})_{k_{ij}}\approx s^J$ by using (4.8) and use the fact that $\sum k_{ij}=J$ from[24] (A.1). We calculate the term:

$$\sum_{\sum k_{ij}=J}\prod_{(ij)}\frac{1}{k_{ij}!}\prod_{n=1}^{4}(a_n)_{J-\sum_j k_{jn}}=$$
$$\sum_{\sum k_{ij}=J}\frac{1}{k_{14}!k_{13}!k_{24}!k_{23}!}(a_1)_{J-\sum_j k_{j1}}(a_2)_{J-\sum_j k_{j2}}(a_3)_{J-\sum_j k_{j3}}(a_4)_{J-\sum_j k_{j4}} \qquad \text{(A.4)}$$

We let $q_1=J-k_{13}-k_{14}$, $q_3=J-k_{13}-k_{23}$ and the only dummy variables of the first sum at (A.3) are q_1, q_3 and k_{13}. Thus, $\sum_{\sum k_{ij}=J}=\sum_{q_1=0}^{J}\sum_{q_3=0}^{J}\sum_{k_{13}=0}^{J-q_1}$. We use $\sum k_{ij}=J\Rightarrow k_{13}+k_{23}+k_{14}+k_{24}=J$ to write $k_{14}=J-q_1-k_{13}$, $k_{23}=J-q_3-k_{13}$, $k_{24}=q_1+q_3-J+k_{13}$. Also we write the Pochhammer indices in (A.3) as $J-\sum_j k_{j1}=J-k_{13}-k_{14}=q_1$, $J-\sum_j k_{j2}=J-k_{23}-k_{24}=k_{13}+k_{14}=J-q_1$, $J-\sum_j k_{j3}=J-k_{13}-k_{23}=q_3$, $J-\sum_j k_{j4}=J-k_{14}-k_{24}=k_{13}+k_{23}=J-q_3$. With that in mind (A.4) becomes[25]

[24] For $r=0$.

[25] In the last line we use the expression $(a+b)_n=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(a)_{n-k}(b)_k$ https://en.wikipedia.org/wiki/Falling and rising factorials

$$\sum_{q_1=0}^{J}\sum_{q_3=0}^{J}\sum_{k_{13}=0}^{J-q_1}\frac{1}{(J-q_1-k_{13})!k_{13}!(q_1+q_3-J+k_{13})!(J-q_3-k_{13})!}(a_1)_{q_1}(a_2)_{q_1-J}(a_3)_{q_3}(a_4)_{J-q_3} \Rightarrow$$

$$J!\sum_{q_1=0}^{J}\sum_{q_3=0}^{J}\frac{(a_1)_{q_1}(a_2)_{q_1-J}(a_3)_{q_3}(a_4)_{J-q_3}}{(q_1)!(J-q_1)!(q_3)!(J-q_3)!}=\frac{(a_1+a_2)_J(a_3+a_4)_J}{J!} \qquad (A.5)$$

With (A.5), (A.3) becomes:

$$P_{\nu,J}\approx\frac{J!}{(h+i\nu-J-1)_J(h-i\nu-J-1)_J}s^J\frac{(a_1+a_2)_J(a_3+a_4)_J}{J!}\Rightarrow$$

$$P_{\nu,J}(s,t)\approx s^J \qquad (A.6)$$

Since a_i's in (A.2) haven't dependence from s.

A very useful property of the Mack polynomials for a $t=\Delta-J+2m$ is:

$$P_{i(\Delta-h),J}(s,\Delta-J+2m)=\tilde{Q}_{J,m}(s) \qquad (A.7)$$

B Large N-limit

In this section we are going to introduce the large N-limit and its usage in the correlation functions in Yang-Mills theories, which is extended to $N=4$ SYM.

Yang-Mills theories like QCD in four dimensions have no dimensionless parameters. In fact the only dimensionless parameter associated with the dimensionful is Λ_{QCD}, which is only the only mass scale in these theories. From this follows that there is not an obvious perturbation expansion. But if our theory is based for example on $SU(N)$, there is an additional parameter N, which is an integer. Our hope is that as $N\to\infty$, our theory is simplified and that it exists a perturbation expansion in terms of the parameter N. This is the so called **large N-limit**.

In an asymptotically free theory, like a pure Yang Mills theory, it natural to scale the coupling g_{YM} as $N\to\infty$, so that Λ_{QCD} remains constant. This means $\lambda\equiv g_{YM}^2N$ is fixed as $N\to\infty$ and this limit is called **'t Hooft limit**.

The same behavior is valid even if the theory includes matter fields in the adjoint representation, as long as the theory is asymptotically free. Let's focus on a general theory which has some fields Φ_i^a, where a is an index in the adjoint representation of $SU(N)$ and i is some label of the field, for example spin, flavor, etc. . In Yang-Mills theory the 3-point vertices of all these fields are proportional to g_{YM}, and the 4-point functions to g_{YM}^2. We assume that this statement also holds true for our general theory. With that in mind the langragian of this theory has the form

$$L\sim Tr(d\Phi_i d\Phi_i)+g_{YM}c^{ijk}Tr(\Phi_i\Phi_j\Phi_k)+g_{YM}^2d^{ijkl}Tr(\Phi_i\Phi_j\Phi_k\Phi_l) \qquad (B.1)$$

Where d is the exterior derivative, c^{ijk} and d^{ijkl} are $SU(N)$ invariant. We continue by rescaling the fields by $\tilde{\Phi}_i=g_{YM}\Phi_i$ and (B.1) becomes

$$L\sim\frac{1}{g_{YM}^2}[Tr(d\tilde{\Phi}_i d\tilde{\Phi}_i)+c^{ijk}Tr(\tilde{\Phi}_i\tilde{\Phi}_j\tilde{\Phi}_k)+d^{ijkl}Tr(\tilde{\Phi}_i\tilde{\Phi}_j\tilde{\Phi}_k\tilde{\Phi}_l)] \qquad (B.2)$$

And the coefficient in front of the whole lagrangian is $1/g_{YM}^2 = N/\lambda$.

The question that we want to answer is what happen to the correlation functions in the large N limit? Clearly at this limit the langragian (B.2) diverges, but this is not the case since at this limit the fields Φ_i also goes to infinity. The Feynman diagrams of our theory are written in a different notation called double line notation and the interaction vertices are all consistent with this. In these diagrams an adjoint field Φ_i is represented as direct product of a fundamental and an anti-fundmental field. We should note that our theory based on $SU(N)$ is slightly different than a theory based on $U(N)$, where this direct product of fundamental and anti-fundmental field takes place. This due to the existence of a small mixing term which makes the expansion more complicated, but it involves terms which are only subleading in the large N-limit.

Let us consider vacuum diagrams[26], in these diagrams the double lines may cross each others, but others do not. The last of them are called **planar diagrams**. It turns out that in the 't Hooft limit the coefficients of these diagrams are proportional to

$$N^{V-E+F}\lambda^{E-V} = N^{\chi}\lambda^{E-V}$$

Where V are vertices, E are propagators and F are the loops of vacuum diagram. These are all related by the quantity $\chi = V - E + F$ known as Euler character. For example the first diagram in figure 9, which we identify it as planar in the double line notation $\chi = 2 - 3 + 3 = 2$. The second on the other hand has $\chi = 4 - 6 + 2 = 0$. Thus the planar diagram is proportional to N^2 and the other is proportional to N^0. From this we see that the leading contribution comes from planar diagrams, while other diagrams will be suppressed by powers of $1/N^2$

Actually this kind of approach holds true for any product of gauge invariant fields $\langle \prod_{j=1}^{n} G_j \rangle$, such that each G_j cannot be written as a product of two gauge-invariant fields[27]. We can study such correlation function by adding to the action $S \to S + N \sum g_j G_j$ and then if W is the sum of connected vacuum diagrams with the new action then

$$\langle \prod_{j=1}^{n} G_j \rangle = (iN)^{-n} [\frac{\partial^n W}{\prod_{j=1}^{n} \partial g_j}]_{g_j=0}$$

These diagrams follow the same analysis we had for vacuum diagrams above. Thus the leading contribution to n-point function in 't Hooft limit will come from planar diagrams with n additional will come, leading to

$$\langle \textstyle\prod_{j=1}^{n} G_j \rangle \propto N^{2-n} \qquad \text{(B.3)}$$

For example the two point functions comes out to be normalized, as for the three-point functions are proportional to $1/N$.

[26] Feynman diagrams with no external legs, which shift the vacuum of the free field theory.

[27] For example, G_j can be of the form $\frac{1}{N} Tr(\prod_i \Phi_i)$

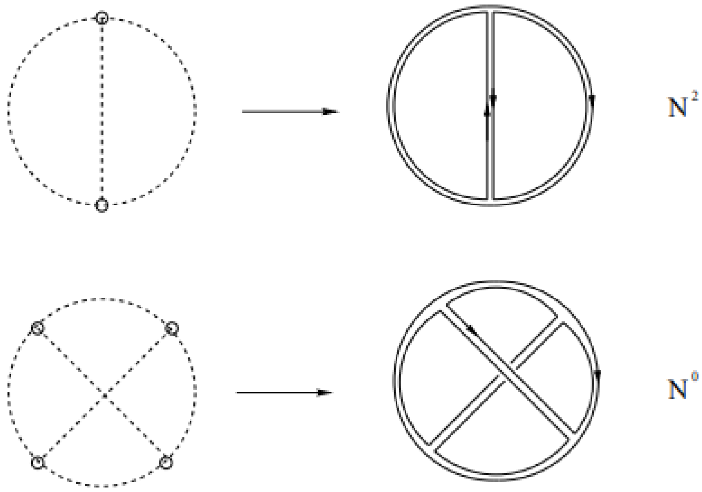

Figure 9: An example of some vacuum diagrams. Some diagrams in a field theory with adjoint fields in the standard representation (on the left) and in the double line representation (on the right). The dashed lines are propagators for the adjoint fields, the small circles represent interaction vertices

C Anomalous Dimensions

In a QFT when we want to find out the scaling dimensions we simply looking at the Langrangian of the theory and perform dimensional analysis. But, when interactions turned on something interesting happens. Due to quantum corrections the dimensions of the correlation functions are shifted from the dimensions of the free field theory and the method of dimensional analysis fails. This shifting of the dimensions is known as **anomalous dimensions**.

For example, consider a free field theory in 3+1 dimensions described by the action S. Now let's turn the interaction on, so that Langragian is now $S+g\int d^4x\Phi(x)$, with g being a dimensionless coupling constant and $\Phi(x)$ a scalar operator with scaling dimension 4. We want to examine the two point correlation function $\langle O_1(y)O_2(z)\rangle_g$. In this new interacting theory, with O_1 and O_2 having scaling dimensions Δ_1 and Δ_2 respectively. We can calculate it with perturbations for $g << 1$ and the first two terms are

$$\langle O_1(y)O_2(z)\rangle_g \approx \langle O_1(y)O_2(z)\rangle + ig\int dx^4 \langle O_1(y)O_2(z)\Phi(x)\rangle =$$

$$\frac{1}{|y-z|^{2\Delta_1}} + ig\int d^4x \frac{C_{123}}{|y-z|^{2\Delta_1-4}|z-x|^4|y-x|^4}$$

(C.1)

Let's focus on the integral

$$I = \frac{1}{[(y-z)^2]^{\Delta_1-2}}\int d^4x \frac{C_{123}}{[(x-z)^2]^2[(x-y)^2]^2} \qquad \text{(C.2)}$$

Let $x = x + z$, $w = y - z$ and (C.2) becomes

$$I = \frac{1}{[(w)^2]^{\Delta_1 - 2}} \int d^4x \frac{C_{123}}{[(x)^2]^2[(x-w)^2]^2} \qquad \text{(C.3)}$$

It is clear that the integral (C.3) diverges at 0 and w. We should investigate the behaviors of the integral near these values and disconnect them from the integral and the remain part should be well defined. Thus near 0 we have

$$I = \frac{1}{[(w)^2]^{\Delta_1 - 2}} \int d^4x \frac{C_{123}}{[(x)^2]^2[(w)^2]^2}$$

$$I = \frac{1}{|w|^{2\Delta_1}} \int d^4x \frac{C_{123}}{x^4} \qquad \text{(C.4)}$$

In four dimensional spherical coordinates $d^4x = r^3 dr d\Omega_3$ the integral (C.4) becomes

$$I = \frac{1}{|w|^{2\Delta_1}} C_{123} \int d\Omega_3 \int_{\Lambda} dr \frac{1}{r} \qquad \text{(C.5)}$$

The three dimensional integral with the solid angle is simple a finite number $A > 0$. The behavior of the integral at $\Lambda \to 0^+$ on the other hand is

$$I = -\frac{1}{|w|^{2\Delta_1}} C_{123} A ln(\Lambda) = \frac{1}{|y-z|^{2\Delta_1}} [-C \ln(\Lambda)]$$

The same kind of behavior can be obtained near w. Next, we can isolate this behavior from the total integral and write it as

$$I = \frac{1}{|y-z|^{2\Delta_1}} [-C \ln(\Lambda)] + \text{finite number} \qquad \text{(C.6)}$$

Now suppose that at order g the two point function in the interactive theory has its dimension $2\Delta_1$ shifted by $\delta\Delta << 1$

$$|y - z|^{-2\Delta_1 - \delta\Delta} = |y - z|^{-2\Delta_1} |y - z|^{-\delta\Delta} =$$

$$|y - z|^{-2\Delta_1} e^{-\delta\Delta \ln(|y-z|)} \approx |y - z|^{-2\Delta_1} [1 - \delta\Delta \ln(|y - z|)]$$

(C.7)

By comparing (C.6) and (C.7), it is clear that the values near 0 and w that diverge the integral are responsible for the dimensional shift of the two point function at order g in the interactive theory. Furthermore, with a more careful approach we can find the dimensional shift and this is the anomalous dimension at order g. In general we want to find the anomalous dimension at any order, but this process is rather difficult even for small orders.

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