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# Turnaround density as a probe of the dark energy equation of state 

Master Thesis

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## Abstract

Turnaround density probes cosmology. In other words, the matter density within the turnaround radius of a large-scale structure depends on cosmology. So far, it has been shown, both through spherical collapse calculations and through N-body simulations, that measurements of the turnaround density can probe both $\Omega_{m}$, and $\Omega_{\Lambda}$. These calculations assumed that dark energy had the form of a cosmological constant. They showed the turnaround density is particularly sensitive to the halting of structure growth imparted by a cosmological constant or, alternatively, dark energy. In this thesis, we study how different equations of state of dark energy affect the turnaround density, and whether we can use observations of turnaround density in galaxy clusters to constrain the dark energy equation of state. We do so within the context of the spherical collapse model. We explore both clustering and non-clustering dark energy, and in particular we study the sensitivity of the evolution of the turnaround density on the value of $w$ which parameterizes the dark energy equation of state, in each of the two cases. We find that the behavior of the turnaround density with time is qualitatively similar in all the cases studied (converging for high redshift; decreasing at low redshift). However, they are different enough for the turnaround density to act as a meaningful probe of the dark energy equation of state. Interestingly, we also find no-go regimes for turnaround in the case of non-clustering dark energy, or even cases where structures disintegrate after turnaround, consistent with "big rip" scenarios.

## Table of Contents

Abstract ..... i
Table of Contents ..... ii
1 Introduction ..... 1
2 Formalism and Methodlogy ..... 3
2.1 Minimum turnaround density ..... 7
2.1.1 Clustering dark energy ..... 7
2.1.1.1 Way A ..... 7
2.1.1.2 Way B ..... 12
2.1.2 Non-clustering dark energy ..... 13
2.2 Solving for the turnaround density ..... 14
2.3 Numerical Methodology ..... 17
3 Results ..... 19
3.1 Inhomogeneous (Clustering) dark energy ..... 19
3.2 Homogeneous (Non-Clustering) dark energy ..... 22
4 Discussion ..... 27
References ..... 29
i. Books and Lecture Notes ..... 29
ii. Journal Articles ..... 29
Abbreviations ..... I

## Chapter 1

## Introduction

The study of large-scale structure is of fundamental importance in cosmology as it provides critical insights into the evolution of the Universe, placing crucial constraints on the cosmological parameters. Within the framework of the Spherical Collapse Model (SCM) (Gunn and Gott, 1972), one of the most prolific tools of the field, spherical shells around originally overdense regions, in an otherwise homogeneous and isotropic Universe consisting only of ordinary matter, dark matter and dark energy ( DE ), as is the case of our Universe since the beginning of the matterdomination era, initially expanding alongside the background, are predicted to slow down and eventually turnaround, leading to structure formation and the creation of large-scale structures: galaxies, galaxy groups, and galaxy clusters. In this context, the turnaround radius is defined as the distance from the center of a cosmic structure to the shell that is detaching from the Hubble flow at a given time and is the largest radius this structure can reach, if the material content it encloses remains conserved.

While the turnaround radius itself is not universal, the matter density within it (the turnaround density) is, varying only with redshift and the assumed cosmology. This means that, all structures turning around at the same redshift are characterized by the same turnaround density independently of their mass and assembly history. This has been shown to be true both in the context of the spherical collapse model, as well as in full N-body cosmological simulations (Pavlidou et al., 2020, Korkidis et al., 2020). It is exactly this universality, combined with the cosmological dependence, that makes turnaround density an attractive cosmological observable. For this reason, the turnaround density has been proposed to serve as a new, independent of all preceding (SNs, CMB, BAOs), cosmological probe. Its particular dependence on the cosmological parameters gives it the additional attractive feature that measurements of the turnaround density alone have the potential to lift the degeneracy in the parameter plane of $\Omega_{m}$ and $\Omega_{d e}$, the matter (dark and baryonic, combined) and dark energy density parameters, respectively, normalized to the critical density at present epoch, that renders the Universe Euclidian (see Fig. 1.1).

A wide range of techniques has been applied in determining the dependence of the turnaround density on the cosmic time and the cosmological parameters. Analytic and numerical calculations within the SCM framework, as well as full N-body cosmological simulations, allow for concrete predictions, that may be contrasted to observations and allow for model selection, or at least rejection of large parts of the parameter space of cosmological parameters that are incompatible with the data.

In most previous studies of the turnaround density (and in particular, of its evolution through cosmic time) as a cosmological observable, however, dark energy was introduced in terms of the cosmological constant, $\Lambda$. This corresponds to a perfect fluid of equal magnitude pressure and density that remain unaltered throughout the entire cosmic evolution. In contrast, in the


Fig. 1.1: While cosmological probes like the cosmic microwave background, baryon acoustic oscillations, and type Ia supernovae only determine a relation between matter and dark-energy density parameters the turnaround density can break the degeneracy and provide a measurement of both at the same time. Figure from Pavlidou et al., 2020.
present work, we relax this assumption, aiming to study the potential of the turnaround density to constrain the equation of state of dark energy.

In particular, we assume a perfect fluid equation of state (EoS) for dark energy, with proportional pressure, $p$, and energy density $\rho$, characterized by $w$, their dimensionless ratio, $w \equiv \frac{p}{\rho}$. We investigate a large range of $w$ values, from $-1 / 3$ to -2 (with $w=-1$, describing the EoS of the cosmological constant, $\Lambda$ ) to fully explore the regimes of the sensitivity of the turnaround density as a stand-alone cosmological observable.

We consider two distinct cases for the DE density profile and, in turn, the nature of its relation to the clustering matter. They constitute extreme choices as intermediate behaviour could also be fitting. In the first case, DE is assumed to be inhomogeneous, responding to the local deformations of spacetime, clustering synchronously with matter. This case describes scenarios within which the total energy content within a shell, as well as each component (matter, DE) individually, are conserved. In the second case, DE is assumed to be completely homogeneous, insensitive to the gravitational deformations in the vicinity of the overdense region, independently following the evolution of the background Universe.

## Chapter 2

## Formalism and Methodology

The Friedmann-Lemaitre-Robertson-Walker metric (FLRW) describes an always maximally symmetric (isotropic and homogeneous) universe, hence its time-dependent spatial (3d) part is the product of the time-independent metric that encodes the geometry of the maximally symmetric 3d space and a time-dependent scale factor, $a(t)$, common for all directions, encoding any change on the size of the universe with respect to time, dictated by the nature and dynamical evolution of its constituents.

If $d s$ and $d \Sigma$ are the line elements of the 4 d spacetime (FLRW) and its corresponding 3 d space, respectively, we have

$$
\begin{equation*}
d s^{2}=c^{2} d t^{2}-a^{2}(t) d \Sigma^{2} \tag{2.1}
\end{equation*}
$$

For maximally symmetric space the metric assumes the following form in reduced-circumference polar coordinates

$$
\begin{equation*}
d \Sigma^{2}=R_{0}^{2}\left(\frac{1}{1-k \xi^{2}} d \xi^{2}+\xi^{2} d \Omega^{2}\right), \quad \text { with } d \Omega^{2}=d \theta^{2}+\sin ^{2} \theta d \phi^{2} \tag{2.2}
\end{equation*}
$$

where $k$ the constant spatial curvature, which is positive, negative or equal to zero for closed, open or flat universes, respectively, and $R(t)=a(t) R_{0}$ the aerial radius of a sphere centered at the origin of the coordinate system, measured in units of its present epoch value, $R_{0}$, according to the definition of the scale factor, $a(t)=\frac{R}{R_{0}}=\frac{1}{1+z}$, which becomes equal to 1 at present day.

We solve the Friedmann equations (FE) in order to determine the scale factor of the universe as a function of time. They follow from the Einstein Field Equations (EFE),

$$
\begin{equation*}
G_{\mu \nu}=\frac{8 \pi G}{c^{4}} T_{\mu \nu} \tag{2.3}
\end{equation*}
$$

where $G_{\mu \nu}$ the Einstein tensor and $T_{\mu \nu}$ the stress-energy-momentum (SEM) tensor, when the form of the metric is assumed to be FLRW, for perfect fluids of the same symmetries (isotropic and homogeneous), with $\rho$ the spatially uniform density and $p$ the spatially uniform and isotropic pressure.

$$
\begin{gather*}
\left(\frac{\dot{a}}{a}\right)^{2}+\frac{k c^{2}}{a^{2}}-\frac{\Lambda c^{2}}{3}=\frac{8 \pi G}{3} \rho  \tag{2.4}\\
2 \frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^{2}+\frac{k c^{2}}{a^{2}}-\Lambda c^{2}=-\frac{8 \pi G}{c^{2}} p \tag{2.5}
\end{gather*}
$$

Allowing for dark energy with constant $w$, not necessarily equal to -1 , we promote the cosmological constant term to a perfect fluid contribution of the stress-energy tensor. Therefore, it enters
both the EFE and the resulting FE from the RHS as

$$
\begin{equation*}
\left(\frac{\dot{a}}{a}\right)^{2}=\frac{8 \pi G}{3}\left(\rho_{m}+\rho_{d e}\right)-\frac{k c^{2}}{R_{0}^{2} a^{2}} \tag{2.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\ddot{a}}{a}=-\frac{4 \pi G}{3}\left(\sum \rho_{i}+3 p_{i}\right) \tag{2.7}
\end{equation*}
$$

with $i$ referring to the matter and dark energy contributions. Since, $p_{i}=w_{i} \rho_{i}$ and $w_{m} \approx 0$, we denote $w_{d e}=w$ going forward and we have

$$
\begin{equation*}
\frac{\ddot{a}}{a}=-\frac{4 \pi G}{3}\left[\rho_{m}+(1+3 w) \rho_{d e}\right] \tag{2.8}
\end{equation*}
$$

## Uniform spherical overdensity

Moreover, the Friedmann equations also describe the evolution of a homogeneous and isotropic structure corresponding to an isolated overdensity of perfect fluid, in the context of the Spherical Collapse Model (SCM) (Gunn and Gott, 1972). We may therefore write for the evolution of any particular shell of the uniform overdensity

$$
\begin{equation*}
\frac{\ddot{R}_{s}}{R_{s}}=-\frac{4 \pi G}{3}\left[\rho_{m, s}+(1+3 w) \rho_{d e, s}\right] \tag{2.9}
\end{equation*}
$$

with $\rho_{m, s}, \rho_{d e, s}$, the densities of matter and dark energy it encloses at time t, respectively.
If we assume clustering dark energy, which translates to constant spatial curvature for this isolated spherical shell, we also have

$$
\begin{equation*}
\left(\frac{\dot{R}_{s}}{R_{s}}\right)^{2}=\frac{8 \pi G}{3}\left(\rho_{m, s}+\rho_{d e, s}\right)-\frac{k_{s} c^{2}}{R_{s}^{2}} \tag{2.10}
\end{equation*}
$$

where $k_{s}$ the constant 2d Gaussian curvature of the shell. On the other hand, if dark energy doesn't cluster, the spatial curvature cannot remain constant, since its evolution is associated to that of the background rather than being solely governed by local dynamics.

Hence only Eq. 2.9) holds as is, but with the identification $\rho_{d e, s}=\rho_{d e}$. More explicitly, we have

$$
\begin{equation*}
\frac{\ddot{R}_{s}}{R_{s}}=-\frac{4 \pi G}{3}\left[\rho_{m, s}+(1+3 w) \rho_{d e}\right] \tag{2.11}
\end{equation*}
$$

In both cases alike, for the particular shell experiencing turnaround at a given time we have $\dot{R}_{s}\left(t=t_{t a}\right)=0$, with its radius and enclosed matter density promptly identified as the turnaround radius and turnaround density of the structure at that time. Therefore, defining the shell scale factor as $a_{s}=\frac{R_{s}(t)}{R_{t a}}$, with $R_{s}$ and $R_{t a}$ its radius at time $t$ and at turnaround, $t=t_{t a}$, respectively, we may trace its evolution up to $t_{t a}$, from $a_{s}=0$ to $a_{s}=1$, with the equations taking the form

$$
\begin{gather*}
\frac{\ddot{a}_{s}}{a_{s}}=-\frac{4 \pi G}{3}\left[\rho_{m, s}-|1+3 w| \rho_{d e, s}\right]  \tag{2.12}\\
\left(\frac{\dot{a}_{s}}{a_{s}}\right)^{2}=\frac{8 \pi G}{3}\left(\rho_{m, s}+\rho_{d e, s}\right)-\frac{k_{s} c^{2}}{R_{t a}^{2}} a_{s}^{-2} \tag{2.13}
\end{gather*}
$$

We also note that for the case of clustering dark energy, the constant shell curvature can be obtained by substituting the condition for shell turnaround, ie $\dot{a}_{s}=0$ and $a_{s}=1$, in the latter equation, which yields

$$
\begin{equation*}
k_{s} c^{2}=R_{t a}^{2} \frac{8 \pi G}{3}\left(\rho_{m, t a}^{s}+\rho_{d e, t a}^{s}\right) \tag{2.14}
\end{equation*}
$$

At this point, we use the continuity equation for each species individually, to derive the dependence of its density on the scale factor associated with its evolution. Starting with the matter density of the background, we have

$$
\begin{equation*}
\dot{\rho}_{m}^{u}+3 \frac{\dot{R}_{u}}{R_{u}} \rho_{m}^{u}=0 \quad \Rightarrow \quad \rho_{m}^{u}(t)=\rho_{m, 0}^{u}\left(\frac{R_{u}(t)}{R_{0}}\right)^{-3} \quad \Rightarrow \quad \rho_{m}^{u}(t)=\rho_{m, 0}^{u} a^{-3} \tag{2.15}
\end{equation*}
$$

where in the last equation, we used the definition of the universe scale factor for the FLRW metric, $R_{u}(t)=a(t) R_{0}$, with $a\left(t_{0}\right)=1$, $t_{0}$ standing for today. Likewise, by the continuity equation for dark energy in the universe, we derive

$$
\begin{equation*}
\dot{\rho}_{d e}^{u}+3(1+w) \frac{\dot{R}_{u}}{R_{u}} \rho_{d e}^{u}=0 \quad \Rightarrow \quad \rho_{d e}^{u}(t)=\rho_{d e, 0}^{u}\left(\frac{R_{u}(t)}{R_{0}}\right)^{-3(1+w)} \Rightarrow \rho_{d e}^{u}(t)=\rho_{d e, 0}^{u} a^{-3(1+w)} \tag{2.16}
\end{equation*}
$$

The continuity equation for matter within the spherical region bounded by the shell, yields

$$
\begin{equation*}
\dot{\rho}_{m}^{s}+3 \frac{\dot{R}_{s}}{R_{s}} \rho_{m}^{s}=0 \quad \Rightarrow \quad \rho_{m}^{s}(t)=\rho_{m, t a}^{s}\left(\frac{R_{s}(t)}{R_{t a}}\right)^{-3} \quad \Rightarrow \quad \rho_{m}^{s}(t)=\rho_{m, t a}^{s} a_{s}^{-3} \tag{2.17}
\end{equation*}
$$

Finally, regarding the continuity equation for dark energy in the shell, for the case of homogeneous (non-clustering) dark energy, 2.15) holds unaltered, while for clustering dark energy we have

$$
\begin{equation*}
\dot{\rho}_{d e}^{s}+3(1+w) \frac{\dot{R}_{s}}{R_{s}} \rho_{d e}^{s}=0 \Rightarrow \rho_{d e}^{s}(t)=\rho_{d e, t a}^{s}\left(\frac{R_{s}(t)}{R_{t a}}\right)^{-3(1+w)} \Rightarrow \rho_{d e}^{s}(t)=\rho_{d e, t a}^{s} a_{s}^{-3(1+w)} \tag{2.18}
\end{equation*}
$$

We proceed by introducing the critical density at present epoch, $\rho_{c, 0}=\frac{3 H_{0}^{2}}{8 \pi G}$, which provides a measure of the deviation from flat (Euclidean) geometry for the universe, when compared to the total density of its constituents. We shall use it in order to rewrite our equations in a more compact form, expressing all relevant densities as normalized to present epoch density parameters, $\Omega_{i, 0}=\frac{\rho_{i}}{\rho_{c, 0}}$, thus depending only on the scale factor and the present-day values of the cosmological and Hubble parameters, denoted from here on as $\Omega_{m, 0}$ for matter, $\Omega_{d e, 0}$ for dark energy, and $H_{0}$, respectively.

$$
\begin{equation*}
\rho_{m}=\frac{\rho_{m, 0} a^{-3}}{\rho_{c, 0}} \rho_{c, 0}=\Omega_{m, 0} a^{-3} \frac{3 H_{0}^{2}}{8 \pi G} \Rightarrow \frac{8 \pi G}{3} \rho_{m}=H_{0}^{2} \Omega_{m, 0} a^{-3} \tag{2.19}
\end{equation*}
$$

which, setting $a_{u}(t)=\frac{a(t)}{a_{t a}}$, becomes

$$
\begin{equation*}
\frac{8 \pi G}{3} \rho_{m}=H_{0}^{2} \Omega_{m, 0} a_{t a}^{-3} a_{u}^{-3} \tag{2.20}
\end{equation*}
$$

For dark energy

$$
\begin{equation*}
\rho_{d e}=\frac{\rho_{d e, 0} a^{-3(1+w)}}{\rho_{c, 0}} \rho_{c, 0}=\Omega_{d e, 0} a^{-3(1+w)} \frac{3 H_{0}^{2}}{8 \pi G} \quad \Rightarrow \quad \frac{8 \pi G}{3} \rho_{d e}=H_{0}^{2} \Omega_{d e, 0} a^{-3(1+w)} \tag{2.21}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{8 \pi G}{3} \rho_{d e}=H_{0}^{2} \Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{-3(1+w)} \tag{2.22}
\end{equation*}
$$

Similarly, for the shell we have

$$
\begin{equation*}
\rho_{m, s}=\frac{\rho_{m, t a}^{s} a_{s}^{-3}}{\rho_{c, 0}} \rho_{c, 0}=\Omega_{t a} a_{s}^{-3} \frac{3 H_{0}^{2}}{8 \pi G} \Rightarrow \frac{8 \pi G}{3} \rho_{m, s}=H_{0}^{2} \Omega_{t a} a_{s}^{-3} \tag{2.23}
\end{equation*}
$$

where we used the definition $\Omega_{t a}=\frac{\rho_{m, t a}^{s}}{\rho_{c, 0}}$ for the turnaround density, as the normalized to present epoch density parameter for the shell matter density, while for dark energy

$$
\begin{equation*}
\rho_{d e, s}=\frac{\rho_{d e, t a}^{s} a_{s}^{-3(1+w)}}{\rho_{c, 0}} \rho_{c, 0}=\Omega_{d e, t a}^{s} a_{s}^{-3(1+w)} \frac{3 H_{0}^{2}}{8 \pi G} \Rightarrow \frac{8 \pi G}{3} \rho_{d e, s}=H_{0}^{2} \Omega_{d e, t a}^{s} a_{s}^{-3(1+w)} \tag{2.24}
\end{equation*}
$$

Returning now to Eq. (2.6)

$$
\begin{equation*}
\left(\frac{\dot{a}}{a}\right)^{2}=\frac{8 \pi G}{3}\left(\rho_{m}+\rho_{d e}\right)-\frac{k c^{2}}{R_{0}^{2} a^{2}} \tag{2.25}
\end{equation*}
$$

we obtain the constant spatial curvature of the universe, $k$, by evaluating the expression at present epoch as

$$
\begin{equation*}
k c^{2}=-R_{0}^{2} q 0^{2} H_{0}^{2}\left(1-\Omega_{m, 0}-\Omega_{d e, 0}\right) \tag{2.26}
\end{equation*}
$$

Substitution of Eqs. (2.20) and (2.22) in the above relation yields

$$
\begin{equation*}
\left(\frac{\dot{a}}{a}\right)^{2}=H_{0}^{2}\left[\Omega_{m, 0} a^{-3}+\Omega_{d e, 0} a^{-3(1+w)}+\left(1-\Omega_{m, 0}-\Omega_{d e, 0}\right)\right] \tag{2.27}
\end{equation*}
$$

or, equivalently,

$$
\begin{align*}
\left(\frac{\dot{a}_{u}}{a_{u}}\right)^{2} & =H_{0}^{2}\left[\Omega_{m, 0} a_{t a}^{-3} a_{u}^{-3}+\Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{-3(1+w)}+\left(1-\Omega_{m, 0}-\Omega_{d e, 0}\right)\right]  \tag{2.28}\\
\frac{\dot{a}_{u}}{a_{u}} & = \pm H_{0}\left[\Omega_{m, 0} a_{t a}^{-3} a_{u}^{-3}+\Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{-3(1+w)}+\left(1-\Omega_{m, 0}-\Omega_{d e, 0}\right)\right]^{\frac{1}{2}}
\end{align*}
$$

with $\dot{a}_{u}>0$

$$
\begin{equation*}
\dot{a}_{u}=H_{0}\left[\Omega_{m, 0} a_{t a}^{-3} a_{u}^{-1}+\Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{-(3 w+1)}+\left(1-\Omega_{m, 0}-\Omega_{d e, 0}\right) a_{u}^{2}\right]^{\frac{1}{2}} \tag{2.29}
\end{equation*}
$$

or

$$
\begin{equation*}
\dot{a}_{u}=\frac{H_{0}}{a_{u}^{1 / 2}}\left[\Omega_{m, 0} a_{t a}^{-3}+\Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{|3 w+1|+1}+\left(1-\Omega_{m, 0}-\Omega_{d e, 0}\right) a_{u}^{3}\right]^{\frac{1}{2}} \tag{2.30}
\end{equation*}
$$

As for the shell, we start with Eq.(2.9) in the equivalent form

$$
\begin{equation*}
\frac{\ddot{a}_{s}}{a_{s}}=-\frac{1}{2} \frac{8 \pi G}{3}\left[\rho_{m, s}-|1+3 w| \rho_{d e, s}\right] \tag{2.31}
\end{equation*}
$$

where we emphasize the fact that for all the values of $w \in\left[-2,-\frac{1}{3}\right]$ that we investigate:

$$
\begin{equation*}
1+3 w<0 \Rightarrow 1+3 w=-|1+3 w| \tag{2.32}
\end{equation*}
$$

For the case of clustering dark energy, we substitute Eqs. 2.23 and (2.24), obtaining

$$
\begin{array}{r}
\frac{\ddot{a}_{s}}{a_{s}}=-\frac{H_{0}^{2}}{2}\left[\Omega_{t a} a_{s}^{-3}-|1+3 w| \Omega_{d e, t a}^{s} a_{s}^{-3(1+w)}\right] \\
\ddot{a}_{s}=-\frac{H_{0}^{2}}{2 a_{s}^{2}}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, t a}^{s} a_{s}^{-3 w}\right] \tag{2.33}
\end{array}
$$

while, for the case of clustering dark energy, with the constant shell curvature in (2.14) taking the form

$$
\begin{equation*}
k_{s} c^{2}=R_{t a}^{2} H_{0}^{2}\left(\Omega_{t a}+\Omega_{d e, t a}^{s}\right) \tag{2.34}
\end{equation*}
$$

we also have (2.13) reformed as

$$
\begin{align*}
\left(\frac{\dot{a}_{s}}{a_{s}}\right)^{2} & =H_{0}^{2}\left[\Omega_{t a} a_{s}^{-3}+\Omega_{d e, t a}^{s} a_{s}^{-3(1+w)}-\left(\Omega_{t a}+\Omega_{d e, t a}^{s}\right) a_{s}^{-2}\right] \\
& =\frac{H_{0}^{2}}{a_{s}^{2}}\left[\Omega_{t a} a_{s}^{-1}+\Omega_{d e, t a}^{s} a_{s}^{-(3 w+1)}-\left(\Omega_{t a}+\Omega_{d e, t a}^{s}\right)\right]  \tag{2.35}\\
\dot{a}_{s} & =H_{0}\left[\Omega_{t a}\left(a_{s}^{-1}-1\right)-\Omega_{d e, t a}^{s}\left(1-a_{s}^{|3 w+1|}\right)\right]^{\frac{1}{2}} \\
& =\frac{H_{0}}{a_{s}^{1 / 2}}\left[\Omega_{t a}\left(1-a_{s}\right)-\Omega_{d e, t a}^{s}\left(a_{s}-a_{s}^{|3 w|}\right)\right]^{\frac{1}{2}} \tag{2.36}
\end{align*}
$$

Similarly, for non-clustering dark energy we substitute Eqs.(2.23) and (2.22) in Eq.(2.12), obtaining

$$
\begin{align*}
& \frac{\ddot{a}_{s}}{a_{s}}=-\frac{H_{0}^{2}}{2}\left[\Omega_{t a} a_{s}^{-3}-|1+3 w| \Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{-3(1+w)}\right] \\
& \ddot{a}_{s}=-\frac{H_{0}^{2}}{2 a_{s}^{2}}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{-3(1+w)} a_{s}^{3}\right] \tag{2.37}
\end{align*}
$$

### 2.1. Minimum turnaround density

### 2.1.1. Clustering dark energy

### 2.1.1.1. Way A:

The argument of the square root in Eq. (2.36)

$$
\begin{equation*}
\dot{a}_{s}=\frac{H_{0}}{a_{s}^{1 / 2}}\left[\Omega_{t a}\left(1-a_{s}\right)-\Omega_{d e, t a}^{s}\left(a_{s}-a_{s}^{|3 w|}\right)\right]^{\frac{1}{2}} \tag{2.38}
\end{equation*}
$$

should be positive therefore

$$
\begin{equation*}
\Omega_{t a}>\Omega_{d e, t a}^{s} \frac{a_{s}-a_{s}^{|3 w|}}{1-a_{s}} \tag{2.39}
\end{equation*}
$$

and since $\Omega_{d e, t a}^{s}$ is a constant, we define

$$
\begin{equation*}
f\left(a_{s} ; w\right)=\frac{a_{s}-a_{s}^{|3 w|}}{1-a_{s}}, \quad a_{s} \in[0,1] \tag{2.40}
\end{equation*}
$$

Firstly, for $a_{s}=0$ we have

$$
\begin{equation*}
f(0 ; w)=\left.\frac{a_{s}-a_{s}^{|3 w|}}{1-a_{s}}\right|_{a_{s}=0}=0 \tag{2.41}
\end{equation*}
$$

and for $a_{s}=1$

$$
\begin{equation*}
f(1 ; w)=\lim _{a_{s} \rightarrow 1^{-}} \frac{a_{s}-a_{s}^{-3 w}}{1-a_{s}} \stackrel{\left(\frac{0}{0}\right)}{\text { D.L.H. }} \lim _{a_{s} \rightarrow 1^{-}} \frac{1+3 w a_{s}^{-3 w-1}}{-1}=-(1+3 w)=|3 w+1| \tag{2.42}
\end{equation*}
$$

We proceed with the computation of the derivative in order to investigate the behaviour of $f$ in the region of interest, setting for notational simplicity $a_{s}=u$

$$
\begin{align*}
\frac{\partial f}{\partial u} & =\left(\frac{u-u^{-3 w}}{1-u}\right)^{\prime}=\frac{\left(1+3 w u^{-(3 w+1)}\right)(1-u)+\left(u-u^{-3 w}\right)}{(1-u)^{2}} \\
& =\frac{1-\not x+3 w u^{-(3 w+1)}-3 w u^{-3 w}+\not x-u^{-3 w}}{(1-u)^{2}}  \tag{2.43}\\
& =\frac{u^{-3 w}\left[\frac{3 w}{u}-(3 w+1)\right]+1}{(1-u)^{2}} \\
& =\frac{u^{|3 w+1|}[|3 w+1| u-|3 w|]+1}{(1-u)^{2}}
\end{align*}
$$

The denominator is strictly positive for the open interval, $u \in(0,1)$. We further wish to check the sign of the numerator, so

$$
\begin{gather*}
A(u ; w)=u^{|3 w+1|}[|3 w+1| u-|3 w|]+1  \tag{2.44}\\
\frac{\partial A}{\partial u}=\frac{|3 w+1|}{u} u^{|3 w+1|}(\left\lvert\, \underbrace{|3 w+1|+1}_{=-3 w=|3 w|}-\frac{|3 w|}{u}\right.) \\
=-\underbrace{|3 w||3 w+1| \frac{u^{|3 w+1|}}{u}}_{>0} \underbrace{(1-u)}_{>0}<0  \tag{2.45}\\
\Rightarrow \frac{\partial A}{\partial u}<0, \text { for } u \in(0,1) \tag{2.46}
\end{gather*}
$$

Also, $\left.\frac{\partial A}{\partial u}\right|_{u=1}=0$, which combined with 2.46 means

$$
\begin{gather*}
A(u) \geq A(u=1)=0, \quad u \in(0,1) \\
A(u)>0, \quad u \in(0,1) \tag{2.47}
\end{gather*}
$$

Therefore, noting also that $A(u=0)=1$, we have:

$$
\begin{align*}
& \Rightarrow\left\{\begin{array}{l}
u \in[0,1): \quad \frac{\partial f(u ; w)}{\partial u}=\frac{A(u ; w)}{(1-u)^{2}}>0 \\
u=1: \quad\left(\frac{0}{0}\right) \Rightarrow \text { D.L.H.: }
\end{array}\right.  \tag{2.48}\\
& \begin{aligned}
\left.\frac{\partial f(u ; w)}{\partial u}\right|_{u=1} & =\lim _{u \rightarrow 1^{-}} \frac{\frac{\partial A(u ; w)}{\partial u}}{-2(1-u)}=\lim _{u \rightarrow 1^{-}} \frac{-(1-u)|3 w||3 w+1| \frac{u^{|3 w+1|}}{u}}{-2(1-\pi)} \\
& =\lim _{u \rightarrow 1^{-}} \frac{1}{2}|3 w||3 w+1| \frac{u^{|3 w+1|}}{u}=\frac{3 w(3 w+1)}{2}>0
\end{aligned}  \tag{2.49}\\
& \Rightarrow[f(u)]_{\max }=f(1) \quad \text { or } \quad\left[f\left(a_{s} ; w\right)\right]_{\max }=f\left(a_{s}=1 ; w\right)
\end{align*}
$$

which we have already calculated in (2.42) as $f\left(a_{s}=1 ; w\right)=|3 w+1|$, thus

$$
\begin{equation*}
\left[f\left(a_{s} ; w\right)\right]_{\max }=|3 w+1| \tag{2.52}
\end{equation*}
$$

Therefore, returning to Eq. $(2.39)$, we have

$$
\begin{gather*}
\Omega_{t a}>\Omega_{d e, t a}^{s} f\left(a_{s} ; w\right), \quad \forall a_{s} \in[0,1]  \tag{2.53}\\
\Rightarrow \Omega_{t a}>\Omega_{d e, t a}^{s}\left[f\left(a_{s} ; w\right)\right]_{\text {max }}  \tag{2.54}\\
 \tag{2.55}\\
\Rightarrow \Omega_{t a}>|3 w+1| \Omega_{d e, t a}^{s}  \tag{2.56}\\
\Omega_{t a, \text { min }}^{(c l)}=|3 w+1| \Omega_{d e, t a}^{s}\left(R_{t a}=R_{t a, \text { max }}\right) \\
\hline
\end{gather*}
$$

with $R_{t a, \max }$ denoting the maximum radius of turnaround.

## Cosmological parameters

At this point, we wish to associate the density parameters of the shell with those of the universe, in search of a more informative direct relation between the lower limit $\Omega_{t a, \text { min }}$ and the cosmological parameters. In this course, we employ the definition of density contrasts, starting with the one for matter, which, making use of the matter density continuity (conservation) equations for the shell and the universe, yields

$$
\begin{equation*}
\zeta(t)=\frac{\rho_{m}^{s}(t)}{\rho_{m}^{u}(t)}=\frac{\rho_{m, t a}^{s} R_{t a}^{3} R_{s}^{-3}(t)}{\rho_{m, t a}^{u} a_{t a}^{3} a^{-3}(t)}=\underbrace{\left(\frac{\rho_{m, t a}^{s} R_{t a}^{3}}{\rho_{m, t a}^{u} a_{t a}^{3}}\right)}_{=\lambda_{1}}\left(\frac{R_{s}(t)}{a(t)}\right)^{-3} \tag{2.57}
\end{equation*}
$$

with $\lambda_{1}$ the constant

$$
\begin{align*}
& \lambda_{1}=\frac{\frac{M_{s} c^{2}}{4 \pi} B_{t a}^{3} B_{t a}^{3}}{\frac{M_{u} c^{2}}{\frac{4 \pi}{3} R_{0}^{3} y_{t a}^{3}} y_{t a}^{3 / 2}}=\frac{M_{s} c^{2}}{\frac{4 \pi}{3}\left(\frac{M_{u} c^{2}}{\frac{4 \pi}{3} R_{0}^{3}}\right)}=\frac{M_{s} c^{2}}{\frac{4 \pi}{3} \rho_{m, 0}^{u}}  \tag{2.58}\\
& \Rightarrow \quad \lambda_{1}=\frac{M_{s}}{M_{u}} R_{0}^{3}=R_{a}^{3} \text { such that } \rho_{m, 0}^{u}=\frac{M_{s} c^{2}}{\frac{4 \pi}{3} R_{a}^{3}} \tag{2.5}
\end{align*}
$$

which means that $R_{a}$ is the radius the shell would need to have in order for its matter density to equate that of the universe. Therefore, the ratio $\frac{R_{s}(t)}{R_{a}}$ expresses the relative size of the actual shell radius with respect to $R_{a}$. So

$$
\begin{equation*}
\zeta(t)=\lambda_{1}\left(\frac{R_{s}(t)}{a(t)}\right)^{-3}=\left(\frac{\frac{R_{s}(t)}{R_{a}}}{a(t)}\right)^{-3} \tag{2.60}
\end{equation*}
$$

and

$$
\begin{equation*}
\zeta_{t a}=\lambda_{1}\left(\frac{R_{t a}}{a_{t a}}\right)^{-3}=R_{a}^{3}\left(\frac{R_{t a}}{a_{t a}}\right)^{-3}=\left(\frac{\frac{R_{t a}}{R_{a}}}{a(t)}\right)^{-3} \tag{2.61}
\end{equation*}
$$

Accordingly, for dark energy

$$
\begin{equation*}
\zeta_{d e}(t)=\frac{\rho_{d e}^{s}(t)}{\rho_{d e}^{u}(t)}=\frac{\rho_{d e, t a}^{s} R_{t a}^{3(1+w)} R_{s}^{-3(1+w)}(t)}{\rho_{d e, t a}^{u} a_{t a}^{3(1+w)} a^{-3(1+w)}(t)}=\underbrace{\left(\frac{\rho_{d e, t a}^{s} R_{t a}^{3(1+w)}}{\rho_{d e, t a}^{u} a_{t a}^{3(1+w)}}\right)}_{=\lambda_{2}}\left(\frac{R_{s}(t)}{a(t)}\right)^{-3(1+w)} \tag{2.62}
\end{equation*}
$$

with $\lambda_{2}$ also constant, and

$$
\begin{align*}
& \lambda_{2}=\frac{\frac{Q_{s}}{\frac{4 \pi}{3} R_{t a}^{3(1)}} R_{t a}^{3(1+w)}}{\frac{Q_{u}}{\frac{4 \pi}{3} R_{0}^{3(1+w)} a_{t a}^{3(1+w)} a_{t a}^{3(1+\sigma)}}=\frac{Q_{s}}{\frac{4 \pi}{3}\left(\frac{Q_{u}}{\left.\frac{4 \pi}{3} R_{0}^{3(1+w)}\right)}=\frac{Q_{s}}{\frac{4 \pi}{3} \rho_{d e, 0}^{u}}\right.}} \begin{array}{ll}
\Rightarrow \quad & \lambda_{2}=\frac{Q_{s}}{Q_{u}} R_{0}^{3(1+w)}=R_{b}^{3(1+w)}
\end{array} \text { such that } \rho_{d e, 0}^{u}=\frac{Q_{s}}{\frac{4 \pi}{3} R_{b}^{3(1+w)}}
\end{align*}
$$

where $Q_{s}$ and $Q_{u}$ are the integrated dark energy content of the spherical region and the universe, respectively, and $R_{b}$ is the radius the shell would need to have in order for its dark energy density to equate that of the universe, with

$$
\begin{equation*}
\zeta_{d e}(t)=\lambda_{2}\left(\frac{R_{s}(t)}{a(t)}\right)^{-3(1+w)}=\left(\frac{\frac{R_{s}(t)}{R_{b}}}{a(t)}\right)^{-3(1+w)} \tag{2.64}
\end{equation*}
$$

and

$$
\begin{equation*}
\zeta_{d e, t a}=\lambda_{2}\left(\frac{R_{t a}}{a_{t a}}\right)^{-3(1+w)}=R_{b}^{3(1+w)}\left(\frac{R_{t a}}{a_{t a}}\right)^{-3(1+w)}=\left(\frac{\frac{R_{t a}}{R_{b}}}{a(t)}\right)^{-3(1+w)} \tag{2.65}
\end{equation*}
$$

while

$$
\begin{equation*}
\frac{\lambda_{1}^{1+w}}{\lambda_{2}}=\left(\frac{R_{a}}{R_{b}}\right)^{3(1+w)}=\frac{\left.\left(\frac{\rho_{m, t a}^{s}}{\rho_{m, t a}^{s}}\right)^{1+w}\right)^{\frac{R_{t a t}^{3(1+w)}}{a_{t a}^{3(1+w)}}}}{\frac{\rho_{d e, t a}^{s+w}}{\rho_{d e, t a}^{3}} \frac{R_{t a}^{3(1+w)}}{g_{t a}^{3(1+w)}}} \Rightarrow \zeta_{d e, t a}=\left(\frac{R_{b}}{R_{a}}\right)^{3(1+w)} \zeta_{t a}^{1+w} \tag{2.66}
\end{equation*}
$$

Now, we may proceed with simplifying the notation. Since the time evolution of the scale factor for the background universe, governed by Eq. (2.30), depends on the densities rather than the absolute quantities of enclosed material, we may as well consider a second notional universe of the same densities, which would result in an identical evolution of the scale factor, only this time with content equal to the absolute quantity of material enclosed in the overdense region of our original system, matter and dark energy alike, ie $M_{u}=M_{s}$ and $Q_{u}=Q_{s}$. This choice allows for a direct translation of density contrasts to radius contrasts, which, by Eqs. (2.59) and (2.63), means

$$
\left.\begin{array}{l}
R_{a}=R_{0, \star}  \tag{2.67}\\
R_{b}=R_{0, \star}
\end{array}\right\} \Rightarrow R_{a}=R_{b}=R_{0, \star} \quad \text { and } \quad \lambda_{1}=\lambda_{2}^{\frac{1}{1+w}}=\lambda
$$

where $\lambda=R_{0, \star}^{3}$. We should note the star is used to indicate that the aerial radius of the auxiliary universe is different than the original's, $R_{0}$, since for the same density but different mass the radius is correspondingly adjusted. Substituting in Eqs. (2.60) and (2.64), promptly gives

$$
\begin{gather*}
\zeta(t)=\left(\frac{\frac{R_{s}(t)}{R_{0, *}}}{a(t)}\right)^{-3}, \quad \zeta_{d e}(t)=\left(\frac{\frac{R_{s}(t)}{R_{0}}}{a(t)}\right)^{-3(1+w)}  \tag{2.68}\\
\zeta_{t a}=\lambda\left(\frac{R_{t a}}{a_{t a}}\right)^{-3}, \quad \zeta_{d e, t a}=\lambda^{1+w}\left(\frac{R_{t a}}{a_{t a}}\right)^{-3(1+w)} \text { and } \zeta_{d e, t a}=\zeta_{t a}^{1+w} \tag{2.69}
\end{gather*}
$$

Returning to our original endeavour of relating the shell and universe density parameters, we may thus write

$$
\begin{gather*}
\Omega_{t a}=\frac{\rho_{m, t a}^{s}}{\rho_{c, 0}}=\frac{\rho_{m, t a}^{s}}{\rho_{m, t a}^{u}} \frac{\rho_{m, t a}^{u}}{\rho_{c, 0}}=\frac{\rho_{m, 0}^{u} a_{t a}^{-3}}{\rho_{c, 0}} \zeta_{t a}=\Omega_{m, 0} a_{t a}^{-Z} \lambda\left(\frac{R_{t a}}{\ell_{t a}}\right)^{-3} \\
 \tag{2.70}\\
\Rightarrow \Omega_{t a}=\Omega_{m, 0} \lambda R_{t a}^{-3} \\
\Omega_{d e, t a}^{s}=\frac{\rho_{d e, t a}^{s}}{\rho_{c, 0}}=\frac{\rho_{d e, t a}^{s}}{\rho_{d e, t a}^{u}} \frac{\rho_{d e, t a}^{u}}{\rho_{c, 0}^{u}}=\frac{\rho_{d e, 0}^{u} a_{t a}^{-3(1+w)}}{\rho_{c, 0}} \zeta_{d e, t a}=\Omega_{d e, 0} a_{t a}^{-3(1+m)} \lambda^{1+w}\left(\frac{R_{t a}}{a_{t a}}\right)^{-3(1+w)}  \tag{2.71}\\
\Rightarrow \Omega_{d e, t a}^{s}=\Omega_{d e, 0} \lambda^{1+w} R_{t a}^{-3(1+w)}
\end{gather*}
$$

with the latter, more explicitly written as

$$
\begin{array}{rlr}
\Omega_{d e, t a}^{s} & =\Omega_{d e, 0} \lambda^{1+w} R_{t a}^{|3 w+1|-2} \\
& =\left\{\begin{array}{rr}
\Omega_{d e, 0} \lambda^{1+w} \frac{1}{\mid R_{t a}^{|3 w+1|-2 \mid}}, & w \in\left(-1,-\frac{1}{3}\right] \\
\Omega_{d e, 0} \lambda^{1+w}=\Omega_{d e, 0}, & w=-1 \\
\Omega_{d e, 0} \lambda^{1+w} R_{t a}^{||3 w+1|-2|}, & w \in[-2,-1)
\end{array}\right. \tag{2.72}
\end{array}
$$

Therefore, for $R_{t a}=R_{t a, \max }$, we may write

$$
\Omega_{d e, t a}^{s}\left(R_{t a}=R_{t a, \max }\right)=\left\{\begin{array}{lr}
\Omega_{d e, t a, \text { min }}^{s}, & w \in\left(-1,-\frac{1}{3}\right]  \tag{2.73}\\
\Omega_{d e, 0}, & w=-1 \\
\Omega_{d e, t a, \text { max }}^{s}, & w \in[-2,-1)
\end{array}\right.
$$

Therefore, Eq. 2.56 ), that provides the lower limit of turnaround density in the case of clustering dark energy, takes the final form

$$
\Omega_{t a, \text { min }}^{(c l)}=|3 w+1|\left\{\begin{array}{lr}
\Omega_{d e, t a, \text { min }}^{s}, & w \in\left(-1,-\frac{1}{3}\right]  \tag{2.74}\\
\Omega_{d e, 0}, & w=-1 \\
\Omega_{d e, t a, \text { max }}^{s}, & w \in[-2,-1)
\end{array}\right.
$$

## Final result

Finally, substituting our results (2.70) and (2.71) in Eq.(2.56), we obtain

$$
\begin{align*}
\left(\Omega_{m, 0} \lambda_{1} R_{t a}^{-3}\right)_{\min } & =\left.|3 w+1|\left(\Omega_{d e, 0} \lambda_{2} R_{t a}^{-3(1+w)}\right)\right|_{R_{t a}=R_{t a, \text { max }}} \\
\Rightarrow \quad \Omega_{m, 0} R_{t a, \text { max }}^{-3} & =|3 w+1| \frac{\lambda_{2} \Omega_{d e, 0}}{\lambda_{1}} R_{t a, \max }^{-3} R_{t a, \text { max }}^{-3 w} \\
\Rightarrow \quad R_{t a, \text { max }}^{3 w} & =|3 w+1| \frac{\lambda_{2} \Omega_{d e, 0}}{\lambda_{1} \Omega_{m, 0}} \\
\Rightarrow \quad R_{t a, \text { max }}^{-3} & =\left[|3 w+1| \frac{\lambda_{2} \Omega_{d e, 0}}{\lambda_{1} \Omega_{m, 0}}\right]^{-\frac{1}{w}}  \tag{2.75}\\
\Rightarrow \quad \Omega_{m, 0} \lambda_{1} R_{t a, \text { max }}^{-3} & =\Omega_{m, 0} \lambda_{1}\left[|3 w+1| \frac{\lambda_{2} \Omega_{d e, 0}}{\lambda_{1} \Omega_{m, 0}}\right]^{-\frac{1}{w}} \\
\Rightarrow \quad \Omega_{t a, \text { min }} & =\left[\lambda_{1} \Omega_{m, 0}\right]^{\frac{w+1}{w}}\left[|3 w+1| \lambda_{2} \Omega_{d e, 0}\right]^{-\frac{1}{w}} \\
& =\left[\frac{\lambda_{1}^{1+w}}{\lambda_{2}}\right]^{\frac{1}{w}}\left[\Omega_{m, 0}\right]^{\frac{w+1}{w}}\left[|3 w+1| \Omega_{d e, 0}\right]^{-\frac{1}{w}}
\end{align*}
$$

which, since $\lambda_{1}^{1+w}=\lambda_{2}$, becomes

$$
\begin{equation*}
\Rightarrow \quad \Omega_{t a, \text { min }}^{(c l)}=|3 w+1| \Omega_{d e, 0}\left[|3 w+1| \frac{\Omega_{d e, 0}}{\Omega_{m, 0}}\right]^{-\frac{w+1}{w}} \tag{2.76}
\end{equation*}
$$

### 2.1.1.2. Way B:

In this case we use Eq. (2.33)

$$
\begin{equation*}
\ddot{a}_{s}=-\frac{H_{0}^{2}}{2 a_{s}^{2}}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, t a}^{s} a_{s}^{-3 w}\right] \tag{2.77}
\end{equation*}
$$

demanding that $\ddot{a}_{s}=0$, alongside $\dot{a}_{s}=0$, in order to identify the outermost shell that could eventually turnaround. Since this equation characterizes the radial geodesics in the vicinity of the spherical overdense region, it describes the radial acceleration of test particles. Hence, the shell for which this acceleration vanishes marks the boundary between two distinct regions. The outer one, where all test particles experience outward acceleration and cannot escape from joining the Hubble flow, and the inner, where deceleration dominates and, if the test particles don't have excess outward initial velocity, it will lead them to eventually stop, turnaround and fall back towards the center of the sphere until virialization takes hold. We obtain, from (2.33)

$$
\begin{align*}
& \left.\begin{array}{l}
\ddot{a}_{s}=0 \\
\dot{a}_{s}=0
\end{array}\right\} \Rightarrow \begin{array}{l}
\Omega_{t a, \text { min }}=|3 w+1| \Omega_{d e, t a}^{s} a_{s}^{-3 w} \\
a_{s}=1
\end{array}  \tag{2.78}\\
& \Rightarrow \Omega_{t a, \text { min }}^{(c l)}=|3 w+1| \Omega_{d e, t a}^{s}\left(R_{t a}=R_{t a, \text { max }}\right) \tag{2.79}
\end{align*}
$$

Comparing with the previous method, we note its equivalence with this one, since a differentiation was involved there, that effectively increased the order of the derivative, we needed to take into account in reaching this result. Thus, although in the first method we started from a first order derivative of $a_{s}(t)$ (essentially, $\left.R_{s}(t)\right)$ that could be associated with the velocities of test particles,
ultimately, we ended up using the second derivatives, associated with their acceleration, that truly governs the evolution and decides the outcome, in both cases. As we have already obtained the expression relating $\Omega_{d e, t a}^{s}\left(R_{t a}=R_{t a, \max }\right)$ with the cosmological parameters, we promptly restate our final result for $\Omega_{t a, \text { min }}$ here.

$$
\begin{equation*}
\Omega_{t a, \text { min }}^{(c l)}=|3 w+1| \Omega_{d e, 0}\left[|3 w+1| \frac{\Omega_{d e, 0}}{\Omega_{m, 0}}\right]^{-\frac{w+1}{w}} \tag{2.80}
\end{equation*}
$$

### 2.1.2. Non-clustering dark energy

For non-clustering dark energy, we may proceed in the same manner as in the previous section, in search of a limiting value of $\Omega_{t a, \text { min }}$. We repeat Eq. (2.37) here

$$
\begin{equation*}
\ddot{a}_{s}=-\frac{H_{0}^{2}}{2 a_{s}^{2}}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{-3(1+w)} a_{s}^{3}\right] \tag{2.81}
\end{equation*}
$$

The demand is once again the same, namely that $\ddot{a}_{s}=0$, alongside $\dot{a}_{s}=0$, in order to identify the largest shell that could eventually turnaround, for the exact same reasons described in the previous situation. Only, this time we derive

$$
\left.\begin{array}{rl}
\ddot{a}_{s}=0 \\
\dot{a}_{s}=0
\end{array}\right\} \Rightarrow \begin{aligned}
& \Omega_{t a, \text { min }}=|3 w+1| \Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{-3(1+w)} a_{s}^{3}  \tag{2.83}\\
& a_{s}=1
\end{aligned} \quad \Rightarrow \quad \Omega_{t a, \text { min }}=|3 w+1| \Omega_{d e, 0}[a(t)]^{-3(1+w)} \text {. }
$$

and, since $-3 w-3=-3 w-1-2=|3 w+1|-2$, we have

$$
\begin{equation*}
\Omega_{t a, \text { min }}=|3 w+1| \Omega_{d e, 0} \frac{[a(t)]^{|3 w+1|}}{[a(t)]^{2}} \tag{2.84}
\end{equation*}
$$

which means that

$$
\Omega_{t a, \text { min }}^{(h o m)}=\left\{\begin{array}{lr}
\Omega_{t a, \text { min }}=0, & w \in\left(-1,-\frac{1}{3}\right]  \tag{2.85}\\
\Omega_{t a, \text { min }}=|3 w+1| \Omega_{d e, 0}=2 \Omega_{d e, 0}, & w=-1 \\
\Omega_{t a, \text { min }} \rightarrow \infty, & w \in[-2,-1)
\end{array}\right.
$$

The last case, for phantom dark energy, which means that beyond a certain evolutionary point, turnaround will be impossible, is in fine accordance with the associated, towards Big Rip, evolution of the background universe.

Moreover, though, we should note that turnaround is only possible if $\ddot{a}_{s}$ is strictly negative prior to the turnaround time, otherwise any associated solution for the evolution of the spherical region is non physical. This means for each $w<-1$, in the case of non-clustering DE, there exists an exact upper limit on the cosmic time, $a_{t a}^{(\max )}(w)$, beyond which turnaround is impossible, which is decreasing as the absolute value of $w$ increases. Larger $|w|$ correspond to smaller $a_{t a}^{(\text {max })}(w)$ limits (earlier $t_{t a}^{(\max )}(w)$ times). Or, equivalently, the smallest overdensities capable of halting the expansion of their neighborhood, are larger for larger $|w|$.

### 2.2. Solving for the turnaround density

We repeat here compactly the equations we need to solve numerically in order to determine the dependence of the turnaround density, $\Omega_{t a}$, on the dark energy equation of state parameter, $w$, and the scale factors of the universe and the spherical region that surrounds the initial overdensity, for any specific choice of cosmological parameters $\Omega_{m, 0}, \Omega_{d e, 0}$ and $H_{0}$. In the present work, we use the 2018 Planck cosmological parameters (Planck Collaboration VI, 2020): $\Omega_{m, 0}=0.315$, $\Omega_{d e, 0}=\Omega_{\Lambda, 0}=1-\Omega_{m, 0}=0.685$ and $H_{0}=67.4 \mathrm{~km} \mathrm{~s}^{-1} \mathrm{Mpc}^{-1}$, in order to ensure consistency with current estimates of $\Lambda \mathrm{CDM}$ values. This choice imparts no qualitative effect on the results.

## Inhomogeneous (clustering) dark energy

We have the first Friedmann equation for the background (2.30)

$$
\begin{equation*}
\dot{a}_{u}=\frac{H_{0}}{a_{u}^{1 / 2}}\left[\Omega_{m, 0} a_{t a}^{-3}+\Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{|3 w+1|+1}+\left(1-\Omega_{m, 0}-\Omega_{d e, 0}\right) a_{u}^{3}\right]^{\frac{1}{2}} \tag{2.86}
\end{equation*}
$$

and for the shell 2.36 )

$$
\begin{equation*}
\dot{a}_{s}=\frac{H_{0}}{a_{s}^{1 / 2}}\left[\Omega_{t a}\left(1-a_{s}\right)-\Omega_{d e, t a}^{s}\left(a_{s}-a_{s}^{|3 w|}\right)\right]^{\frac{1}{2}} \tag{2.87}
\end{equation*}
$$

while, for the shell we may prefer to use (2.33)

$$
\begin{equation*}
\ddot{a}_{s}=-\frac{H_{0}^{2}}{2 a_{s}^{2}}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, t a}^{s} a_{s}^{-3 w}\right] \tag{2.88}
\end{equation*}
$$

which, by combining the relations (2.70) and 2.71), as

$$
\begin{equation*}
\lambda R_{t a}^{-3}=\frac{\Omega_{t a}}{\Omega_{m, 0}} \Rightarrow \Omega_{d e, t a}^{s}=\Omega_{d e, 0}\left(\frac{\Omega_{t a}}{\Omega_{m, 0}}\right)^{1+w} \tag{2.89}
\end{equation*}
$$

in order to use the explicit dependence of $\Omega_{d e, t a}^{s}$ on the cosmological parameters, takes the equivalent form

$$
\begin{equation*}
\ddot{a}_{s}=-\frac{H_{0}^{2}}{2 a_{s}^{2}}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, 0} \Omega_{m, 0}^{-(1+w)} \Omega_{t a}^{1+w} a_{s}^{-3 w}\right] \tag{2.90}
\end{equation*}
$$

which now contains only one unknown variable, $\Omega_{t a}\left(t_{t a} ; w\right)$, the one we intend to evaluate for any specific choice of DE EoS $w$ parameter and cosmic time of turnaround. The corresponding boundary conditions are

$$
\begin{align*}
& a_{u}(t=0)=0  \tag{2.91}\\
& a_{u}\left(t=t_{t a}\right)=1
\end{aligned} \quad \begin{aligned}
& a_{s}(t=0)=0 \\
& a_{s}\left(t=t_{t a}\right)=1 \\
& \dot{a}_{s}\left(t=t_{t a}\right)=0
\end{align*}
$$

Notice how the first two are overdetermining the system, since the equation governing the evolution of $a_{u}$ is first order, although this poses no problem since the two are not incompatible. One of them may be reserved simply to check the validity of the returned solution. Moreover, one may also choose to use the second Friedmann equation for the background as well, in which case both conditions would be necessary. At the same time, the presence of three (3) boundary conditions for the shell, will allow us to compute the value of the unknown variable, since two of them will serve in solving for the evolution of the shell's scale factor and the third will be used solely in
solving for $\Omega_{t a}$. As a final remark, we wish to underline that obtaining $\Omega_{t a}$ does not immediately yield a specific $R_{t a}$, since its value depends on the actual $M_{s}$ and $Q_{s}$ of the overdensity through (2.60), (2.64) and (2.70), details not necessary if one deals directly with densities. Exactly here, in its scale universality, lies the unique value of the turnaround density as a cosmological probe, even more so when contrasted to the more restrictive measure that is the turnaround radius.

## Homogeneous (non-clustering) dark energy

In this case, we also have the first Friedmann equation for the background 2.30

$$
\begin{equation*}
\dot{a}_{u}=\frac{H_{0}}{a_{u}^{1 / 2}}\left[\Omega_{m, 0} a_{t a}^{-3}+\Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{|3 w+1|+1}+\left(1-\Omega_{m, 0}-\Omega_{d e, 0}\right) a_{u}^{3}\right]^{\frac{1}{2}} \tag{2.92}
\end{equation*}
$$

as for the shell (2.37)

$$
\begin{equation*}
\ddot{a}_{s}=-\frac{H_{0}^{2}}{2 a_{s}^{2}}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{-3(1+w)} a_{s}^{3}\right] \tag{2.93}
\end{equation*}
$$

with the same conditions as in the case of clustering dark energy. Contrary to the previous case, the system of ODEs is now coupled.

## Singularity circumvention

As one may verify with a quick visual inspection of our equations, there is a singularity at $t=0 \Leftrightarrow a_{u}=0$ and $a_{s}=0$, yielding

$$
\begin{equation*}
\lim _{t \rightarrow 0^{+}} \dot{a}_{u}=\infty, \quad \lim _{t \rightarrow 0^{+}} \dot{a}_{s}=\infty \quad \text { and } \quad \lim _{t \rightarrow 0^{+}} \ddot{a}_{s}=\infty \tag{2.94}
\end{equation*}
$$

In order to avoid treating infinity in our numerical calculation, we use the following transformations

$$
\begin{equation*}
a_{u} \rightarrow u_{1}=a_{u}^{3 / 2} \quad \text { and } \quad a_{s} \rightarrow u_{2}=a_{s}^{3 / 2} \tag{2.95}
\end{equation*}
$$

that will allow us to rewrite our equations in a form that is nowhere singular. We have

$$
\begin{gather*}
\dot{u}_{1}=\frac{3}{2} a_{u}^{1 / 2} \dot{a}_{u} \quad \Rightarrow \quad \frac{\dot{a}_{u}}{a_{u}}=\frac{2}{3} \frac{\dot{u}_{1}}{u_{1}}  \tag{2.96}\\
\Rightarrow \quad \ddot{u}_{1}=\frac{3}{2} \frac{1}{2} a_{u}^{-1 / 2} \dot{a}_{u}^{2}+\frac{3}{2} a_{u}^{1 / 2} \ddot{a}_{u} \\
\Rightarrow \quad u_{1} \ddot{u}_{1}=\frac{1}{3} \dot{u}_{1}^{2}+\frac{3}{2} a_{u}^{2} \ddot{a}_{u} \quad \text { or } \quad a_{u}^{2} \ddot{a}_{u}=\frac{2}{3} u_{1} \ddot{u}_{1}-\frac{2}{9} \dot{u}_{1}^{2} \tag{2.97}
\end{gather*}
$$

and, likewise,

$$
\begin{gather*}
\dot{u}_{2}=\frac{3}{2} a_{s}^{1 / 2} \dot{a}_{s} \quad \Rightarrow \quad \frac{\dot{a}_{s}}{a_{s}}=\frac{2}{3} \frac{\dot{u}_{2}}{u_{2}}  \tag{2.98}\\
\Rightarrow \quad u_{2} \ddot{u}_{2}=\frac{1}{3} \dot{u}_{2}^{2}+\frac{3}{2} a_{s}^{2} \ddot{a}_{s} \quad \text { or } \quad a_{s}^{2} \ddot{a}_{s}=\frac{2}{3} u_{2} \ddot{u}_{2}-\frac{2}{9} \dot{u}_{2}^{2} \tag{2.99}
\end{gather*}
$$

Therefore, for the evolution of the background we have

$$
\begin{gather*}
\dot{a}_{u}=\frac{H_{0}}{a_{u}^{1 / 2}}\left[\Omega_{m, 0} a_{t a}^{-3}+\Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{|3 w+1|+1}+\left(1-\Omega_{m, 0}-\Omega_{d e, 0}\right) a_{u}^{3}\right]^{\frac{1}{2}} \\
a_{u}^{1 / 2} \dot{a}_{u}=H_{0}\left[\Omega_{m, 0} a_{t a}^{-3}+\Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{-3 w}+\left(1-\Omega_{m, 0}-\Omega_{d e, 0}\right) a_{u}^{3}\right]^{\frac{1}{2}} \\
\dot{u}_{1}=\frac{3}{2} H_{0}\left[\Omega_{m, 0} a_{t a}^{-3}+\Omega_{d e, 0} a_{t a}^{-3(1+w)} u_{1}^{|2 w|}+\left(1-\Omega_{m, 0}-\Omega_{d e, 0}\right) u_{1}^{2}\right]^{\frac{1}{2}} \tag{2.100}
\end{gather*}
$$

and for the shell, in the clustering dark energy case,

$$
\begin{gather*}
\ddot{a}_{s}=-\frac{H_{0}^{2}}{2 a_{s}^{2}}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, 0} \Omega_{m, 0}^{-(1+w)} \Omega_{t a}^{1+w} a_{s}^{-3 w}\right] \\
\Rightarrow \quad a_{s}^{2} \ddot{a}_{s}=-\frac{H_{0}^{2}}{2}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, 0} \Omega_{m, 0}^{-(1+w)} \Omega_{t a}^{1+w} a_{s}^{-3 w}\right] \\
\Rightarrow \quad \\
u_{2} \ddot{u}_{2}=\frac{1}{3} \dot{u}_{2}^{2}-\frac{3 H_{0}^{2}}{4}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, 0} \Omega_{m, 0}^{-(1+w)} \Omega_{t a}^{1+w} u_{2}^{-2 w}\right]  \tag{2.101}\\
\Rightarrow \quad \\
\ddot{u}_{2}=\left\{\frac{1}{3} \dot{u}_{2}^{2}-\frac{3 H_{0}^{2}}{4}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, 0} \Omega_{m, 0}^{-(1+w)} \Omega_{t a}^{1+w} u_{2}^{2|w|}\right]\right\} \frac{1}{u_{2}}
\end{gather*}
$$

and, finally, in the case of non-clustering dark energy

$$
\begin{gather*}
\ddot{u}_{s}=-\frac{H_{0}^{2}}{2 a_{s}^{2}}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, 0} a_{t a}^{-3(1+w)} a_{u}^{-3(1+w)} a_{s}^{3}\right] \\
u_{2} \ddot{u}_{2}=\frac{1}{3} \dot{u}_{2}^{2}-\frac{3 H_{0}^{2}}{4}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, 0} a_{t a}^{-3(1+w)} u_{1}^{-2(1+w)} u_{2}^{2}\right] \\
\ddot{u}_{2}=\left\{\frac{1}{3} \dot{u}_{2}^{2}-\frac{3 H_{0}^{2}}{4}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, 0} a_{t a}^{-3(1+w)} u_{1}^{-2(1+w)} u_{2}^{2}\right]\right\} \frac{1}{u_{2}} \tag{2.102}
\end{gather*}
$$

It is immediately apparent from Eq. 2.100 that $\dot{u}_{1}(0)=\frac{3}{2} H_{0} \sqrt{\Omega_{m, 0} a_{t a}^{-3}}$, therefore it is not singular for $t=0\left(u_{1}=0, u_{2}=0\right)$. The same holds for $\dot{u}_{2}(0)$ and $\ddot{u}_{2}(0)$, as can be shown with application of the Mean Value Theorem on Eqs. (2.101) and (2.102), that also yields the approximation

$$
\begin{gather*}
\tilde{\dot{u}}_{2}(t)=\dot{u}_{2}(0)  \tag{2.103}\\
\tilde{u}_{2}(t)=\dot{u}_{2}(0) t \tag{2.104}
\end{gather*}
$$

appropriately close to the time of the singularity $(t \rightarrow 0)$.

## Final expressions

Compactly, the final form of our equation systems, applied for a flat background universe, where $k=0 \Leftrightarrow\left(1-\Omega_{m, 0}-\Omega_{d e, 0}\right)=0$ is as follows.

## Inhomogeneous (clustering) dark energy

$$
\begin{aligned}
& \dot{u}_{1}=\frac{3}{2} H_{0}\left[\Omega_{m, 0} a_{t a}^{-3}+\Omega_{d e, 0} a_{t a}^{-3(1+w)} u_{1}^{|2 w|}\right]^{\frac{1}{2}} \\
& \ddot{u}_{2}=\left\{\frac{1}{3} \dot{u}_{2}^{2}-\frac{1}{3}\left(\frac{3}{2} H_{0}\right)^{2}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, 0} \Omega_{m, 0}^{-(1+w)} \Omega_{t a}^{1+w} u_{2}^{|2 w|}\right]\right\} \frac{1}{u_{2}}, \quad u_{2} \neq 0
\end{aligned}
$$

Homogeneous (non-clustering) dark energy

$$
\begin{aligned}
& \dot{u}_{1}=\frac{3}{2} H_{0}\left[\Omega_{m, 0} a_{t a}^{-3}+\Omega_{d e, 0} a_{t a}^{-3(1+w)} u_{1}^{|2 w|}\right]^{\frac{1}{2}} \\
& \ddot{u}_{2}=\left\{\frac{1}{3} \dot{u}_{2}^{2}-\frac{1}{3}\left(\frac{3}{2} H_{0}\right)^{2}\left[\Omega_{t a}-|1+3 w| \Omega_{d e, 0} a_{t a}^{-3(1+w)} u_{1}^{-2(1+w)} u_{2}^{2}\right]\right\} \frac{1}{u_{2}}, \quad u_{2} \neq 0
\end{aligned}
$$

In both cases, for $t=0$

$$
\begin{equation*}
\dot{u}_{1}(0)=\frac{3}{2} H_{0} \sqrt{\Omega_{m, 0} a_{t a}^{-3}} \quad \text { and } \quad \dot{u}_{2}(0)=\frac{3}{2} H_{0} \sqrt{\Omega_{t a}} \tag{2.105}
\end{equation*}
$$

and in the vicinity of $t \rightarrow 0$ the following approximations hold. For the background universe

$$
\begin{array}{|}
\tilde{\tilde{u}}_{1}(t)=\dot{u}_{1}(0) \\
\tilde{u}_{1}(t)=\dot{u}_{1}(0) t  \tag{2.107}\\
\hline
\end{array}
$$

and for the shell

$$
\begin{array}{r}
\tilde{\tilde{u}}_{2}(t)=\dot{u}_{2}(0)  \tag{2.108}\\
\tilde{u}_{2}(t)=\dot{u}_{2}(0) t
\end{array}
$$

while the boundary conditions for the transformed system, following from (2.91), are

$$
\begin{align*}
& u_{1}(t=0)=0  \tag{2.110}\\
& u_{1}\left(t=t_{t a}\right)=1
\end{aligned} \quad \begin{aligned}
& u_{2}(t=0)=0 \\
& u_{2}\left(t=t_{t a}\right)=1 \\
& u_{2}\left(t=t_{t a}\right)=0
\end{align*}
$$

### 2.3. Numerical Methodology

In order to evaluate numerically the turnaround density, $\Omega_{t a}\left(t_{t a} ; w\right)$, for varying dark energy equation of state $(w)$ and epochs $\left(a_{t a}\right)$, we need to solve the following systems of coupled ODEs, for the specified boundary conditions.

## Inhomogeneous (clustering) dark energy

$$
\begin{align*}
& \dot{a}_{u}=f_{1}\left(t, a_{u}, a_{s}, \dot{a}_{s}\right)=f_{1}\left(a_{u}\right)  \tag{2.111}\\
& \dot{a}_{s}=f_{2}\left(t, a_{u}, a_{s}, \dot{a}_{s}\right)=f_{2}\left(\dot{a}_{s}\right)=\dot{a}_{s} \\
& \ddot{a}_{s}=f_{3}\left(t, a_{u}, a_{s}, \dot{a}_{s}\right)=f_{3}\left(a_{s}, \dot{a}_{s}\right) \\
& \hline
\end{align*}
$$

or, equivalently,

$$
\begin{array}{|l|}
\hline \dot{u}_{1}=f_{1}\left(t, u_{1}, u_{2}, \dot{u}_{2}\right)=f_{1}\left(u_{1}\right) \\
\dot{u}_{2}=f_{2}\left(t, u_{1}, u_{2}, \dot{u}_{2}\right)=f_{2}\left(\dot{u}_{2}\right)=\dot{u}_{2}  \tag{2.112}\\
\ddot{u}_{2}=f_{3}\left(t, u_{1}, u_{2}, \dot{u}_{2}\right)=f_{3}\left(u_{2}, \dot{u}_{2}\right) \\
\hline
\end{array}
$$

## Homogeneous (non-clustering) dark energy

$$
\begin{align*}
& \dot{a}_{u}=f_{1}\left(t, a_{u}, a_{s}, \dot{a}_{s}\right)=f_{1}\left(a_{u}\right) \\
& \dot{a}_{s}=f_{2}\left(t, a_{u}, a_{s}, \dot{a}_{s}\right)=f_{2}\left(\dot{a}_{s}\right)=\dot{a}_{s}  \tag{2.113}\\
& \ddot{a}_{s}=f_{3}\left(t, a_{u}, a_{s}, \dot{a}_{s}\right)=f_{3}\left(\boldsymbol{a}_{\boldsymbol{u}}, a_{s}, \dot{a}_{s}\right)
\end{align*}
$$

or, equivalently,

$$
\begin{align*}
& \dot{u}_{1}=f_{1}\left(t, u_{1}, u_{2}, \dot{u}_{2}\right)=f_{1}\left(u_{1}\right) \\
& \dot{u}_{2}=f_{2}\left(t, u_{1}, u_{2}, \dot{u}_{2}\right)=f_{2}\left(\dot{u}_{2}\right)=\dot{u}_{2}  \tag{2.114}\\
& \ddot{u}_{2}=f_{3}\left(t, u_{1}, u_{2}, \dot{u}_{2}\right)=f_{3}\left(\boldsymbol{u}_{1}, u_{2}, \dot{u}_{2}\right) \\
& \hline
\end{align*}
$$

We acquired the solution of our system in all cases, using the Newton-Raphson (NR) collocation method, a generalization of the well known iterative root-finding algorithm characterized by guaranteed local convergence of order 2, whenever an appropriate initial step seed is provided. Specifically, we made use of a python routine called solve_bvp, included in the integrate package of the scipy library for specialized scientific applications (Virtanen et al., 2020). It comprises an optimized realization of the NR algorithm, designed for (non-linear) first order systems of ODEs, coupled or otherwise.

In the case of clustering dark energy the system is not coupled. This allowed us to crosstest the correctness and accuracy of our results for this case, using a particularly fast method of numerical quadrature (tanh-sinh (Mori, 2005), provided by both the scipy and mpmath python libraries), at arbitrarily high precision, ensured by use of mpmath (mpmath Development Team, 2023). Analytical solutions were also procured for the special cases that admit them, further reaffirming the validity of our associated results.

## Chapter 3

## Results

In this section we present the results of our work. We have integrated the Friedmann equations describing the evolution of a spherical overdensity in an otherwise homogeneous and isotropic universe, together with the ones governing the background's evolution, exploring the behavior of the turnaround density as a function of cosmic time and the equation of state of DE parameterized by $w$. Their solution allows us to evaluate $\Omega_{t a}\left(a_{t a} ; w\right)$, as described in the previous section.

### 3.1. Inhomogeneous (Clustering) dark energy

In Fig.(3.1) we plot the turnaround density as a function of redshift (or the scale factor) for varying dark energy equations of state. From the range of values we have investigated, $w \in[-2,-1 / 3]$, we have selected to display as representative choices $(-2,-1.5,-1)$. In all frames of Fig. (3.1) we show how $\Omega_{t a}$ varies with the cosmic time, encoded in the first row by the scale factor of the background Universe and in the second row by redshift. The first column covers the scale factor range $a \in\left[10^{-3}, 10^{2}\right]$ and its equivalent redshift range. In the second column, we focus in the vicinity of the present cosmic epoch, with the scale factor in the region $a \in[0.25,1]$ (redshift: $z \in[0,3])$, starting with the Universe in a quarter of its present size leading up to today. The final column zooms in even further, with $a \in[0.75,1]$ and $z \in\left[0, \frac{1}{3}\right]$.

The most prominent characteristic, across all epochs and for all cases of DE equation of state, is the decrease of the turnaround density with time, falling from over $10^{9}$ for $a_{t a} \sim 0.001$ (high redshift) consistently towards a few hundreds, for a Universe half-sized compared to today, to clearly discernible values of the order of one for the present day. For high redshifts, all the lines are closing in together, apparently rendering the dependence of the turnaround density on the cosmology, via the $w$ parameter, increasingly negligible. Therefore, we would naively not expect observations at increasingly high redshift to add significant constraining power. However, assuming the observing accuracy for the turnaround density at higher redshifts does not substantially deteriorate, we should note that the absolute difference between models is in fact important, so high-z measurements could in principle be also very constraining. Of course, it is by no means obvious that the turnaround density is measurable with similar accuracies, or even at all, at high redshifts, so the relative importance of measurements at different redshifts will have to be studied in detail taking into account specific measurement methods, their uncertainties, and how these depend on $z$.

To estimate the constraining power of turnaround density for $w$, we assume $1 \%$ uncertainty in its measurement and $3.5 \%$ uncertainty in the measurement of its derivative, in accordance with the estimation suggested by Pavlidou et al., 2020, which we also adopt here for consistency. Then, the


Fig. 3.1: Three sets (vertical) of time ranges for the evolution of turnaround density, $\Omega_{t a}$, as a function of the background scale factor (Upper panels) and redshift (Lower panels), for the same three values of $w$, in the case of inhomogeneous (clustering) dark energy.
predicted variation of $\Omega_{t a}$ with $w$ for low-redshift structures, shown in the final column of Fig.(3.1), translates to considerable constraining power on $w$ alone, assuming other cosmological parameters, e.g. $\Omega_{m, 0}$, are obtained independently. However, if the remaining cosmological parameters are not obtained independently, a more detailed study is in order, to determine the constraining power of the turnaround density as a stand-alone cosmological observable.

In Fig.(3.2), we plot the turnaround density as a function of $w$ for $z=0$ (current epoch). We are interested in comparing the behaviour of $\Omega_{t a}$ for the same redshift, here chosen to be the present cosmic epoch, for the full range of $w$ rather than a representative sample. We observe a continuous but not monotonic dependence of $\Omega_{t a}$ with $w$, starting from the lowest value for the lowest w , reaching a peak for $w \sim-0.6$, and then decreasing for even larger (less negative) values of $w$. This behavior is not surprising: Eq. 2.80 informs us of the behaviour of $\Omega_{t a, \text { min }}^{(c l)}(w)$, the minimum turnaround density predicted for each $w$ in the clustering DE case, which indeed has a $w$-dependence similar to the one observed for the current epoch.

Furthermore, at $w=-1$ and $w=-1 / 3$ the Friedmann equations we are solving for become identical to their non-clustering case counterparts, since in the case of the cosmological constant or the complete absense of DE, respectively, DE behaves effectively homogeneously. This serves as a verification for the values we expect our curve to cross, when the analytical solutions are also known.

We notice that the value of $\Omega_{t a}^{(z=0)}(w)$, although it has a degeneracy in $w$ for the region that cover its $\Omega_{t a}^{(z=0)}(w=-1 / 3)$ value and higher, once again varies enough to be discernible by observations of the estimated accuracy.


Fig. 3.2: The predicted value of current turnaround density, $\Omega_{t a}^{(t o d a y)}$, for varying $w$, in the case of inhomogeneous (clustering) dark energy.

In the following figure, Fig. 3.3), we show how the ratio of the turnaround density at present epoch to that of redshift $z=1$, when the Universe was half its current size, $\frac{\Omega_{t a}^{(z=0)}}{\Omega_{t a}^{z=1)}}$, varies with the $w$ parameter, assuming all other cosmological parameters remain fixed. This serves as an estimate of the change in the time dependence of $\Omega_{t a}\left(a_{t a} ; w\right)$ with changing $w$. There is clear distinction
of this parameter (at the level of a $\sim 35 \%$ range of values) for the range of $w$ we investigated, although its behavior is, again, not monotonic.


Fig. 3.3: The ratio of current turnaround density to the turnaround density @ $z=1$, for varying $w$, in the case of inhomogeneous (clustering) dark energy.

### 3.2. Homogeneous (Non-Clustering) dark energy

As in the case of clustering dark energy, in Fig.(3.4) we plot the turnaround density as a function of cosmic time, parametrized by the scale factor (top panels) or the redshift (bottom panels), for varying $\operatorname{DE} \operatorname{EoS}$ ( $w$ parameter). We have, once again, selected to display ( $-2,-1.5,-1$ ) as representative choices from the range of $w$ we investigated, $[-2,-1 / 3]$, only this time we further add $w=-0.5,-1 / 3$ as representative choices of $w>-1$, in order to illustrate the distinct difference between the regions separated by $w=-1$ (cosmological constant), i.e. quintessence $(w \in[-1,-1 / 3])$ and phantom energy $(w<-1)$. As in the previous section, the first column covers the scale factor range $a \in\left[10^{-3}, 10^{2}\right]$ and the equivalent redshift range. The second column, centers in the vicinity of the present cosmic epoch, starting with the Universe in a quarter of its present size (scale factor: $a \in[0.25,1]$, redshift: $z \in[0,3]$ ), leading up to present day. The final column zooms in even further, with $a \in[0.75,1]\left(z \in\left[0, \frac{1}{3}\right]\right)$.

We once again observe the decrease of the turnaround density with cosmic time (increase with redshift), across all epochs and for all cases of DE equation of state, starting, with a convergence of all the lines, from over $10^{9}$ for $a_{t a} \sim 0.001$ (high redshift) and also characterized by similar absolute differences between models, as in the previous case (clustering DE). This means, as discussed in the previous section, that distinction between $w$ cosmological models could be, in principle achieved, if measurements of enough accuracy could be acquired for high- $z$ structures in the future, although it should be noted, that further distinction between clustering and nonclustering DE is not attainable by high- $z$ observations alone. This is due to the aggregate nature of the effect that an ever-increasing DE scalar field, around an original overdensity, has on the ability of the overdensity to impact the detachment of its neighborhood from the Hubble flow.

For high redshifts this effect hasn't had enough time to fully play out, therefore rendering the two cases effectively indistinguishable, the further back in time we look.

Importantly, the most prominent qualitative difference between this scenario (non-clustering DE ) and the one discussed in the previous section (clustering DE) becomes apparent close the present day and can be seen in the first column, which also includes future values of the turnaround density. For $w<-1$ (phantom energy) there exists an exact upper limit on the cosmic time, $a_{t a}^{(\max )}(w)$, beyond which turnaround is impossible, as we have already noted in section 2.1.2, when discussing the minimum turnaround density for the case of non-clustering DE. This limit decreases for increasing absolute value of $w$. For larger $|w|$, the rate at which the relative contribution of DE to the energy density budget of the Universe increases is higher. As a result, overdense regions that haven't yet managed to induce turnaround on their surroundings become unable to do so faster. This means that the smallest overdensities capable of halting the expansion of their neighborhood, $\Omega_{t a, \text { min }}^{(h o m)}(w)$, are larger for larger $|w|$. For a given $w$, any overdensities below this limit will be incapable of ever inducing turnaround, a result in accordance with the Big Rip fate of all Universes with $w<-1$, that predicts the eventual disintegration of all existing structures.

On the other hand, assuming $1 \%$ uncertainty in the measurement of the turnaround density and $3.5 \%$ uncertainty in the measurement of its derivative, as per the estimation suggested by Pavlidou et al., 2020, the predicted variation of $\Omega_{t a}$ with $w$ (for $w \geq-1$ ) for low- $z$ structures, is again enough for achieving considerable constraining power on $w$, provided the remaining cosmological parameters are independently obtained with high enough accuracy.


Fig. 3.4: Three sets (vertical) of time ranges for the evolution of turnaround density, $\Omega_{t a}$, as a function of the background scale factor (Upper panels) and redshift (Lower panels), for the same five values of $w$, in the case of homogeneous (non-clustering) dark energy.


Fig. 3.5: The predicted value of current turnaround density, $\Omega_{t a}^{(t o d a y)}$, varying $w$, in the case of homogeneous (non-clustering) dark energy.

As in the case of non-clustering dark energy, in Fig.(3.5), we plot the turnaround density as a function of $w$ for $z=0$, in order to examine the behaviour of $\Omega_{t a}(z=0 ; w)$ (present cosmic epoch) for the full range of $w$. We observe, once again, continuous dependence of $\Omega_{t a}$ with $w$, although in this case it is also monotonic, increasing from lowest to highest $w$. We confirm once again that for $w=-1,-1 / 3$ we retrieve the same value of the turnaround density as in the clustering case, as we should.

In the homogeneous DE case, the value of $\Omega_{t a}^{(z=0)}(w)$, does not exert any degeneracy with $w$ and it varies enough to be discernible by observations of the estimated accuracy, as in the case of clustering DE.


Fig. 3.6: The ratio of current turnaround density to the turnaround density @ $z=1$, for varying $w$, in the case of homogeneous (non-clustering) dark energy.

In the following figure, Fig. (3.6), we present as in the case of clustering dark energy the dependence of the ratio of the turnaround density at present epoch to that of redshift $z=1$, when the Universe was half its current size, $\frac{\Omega_{t a}^{(z-0)}}{\Omega_{t a}^{(z-1)}}$, with the $w$ parameter (DEEOS), assuming all other cosmological parameters remain fixed, since it serves as an estimate of the change in the time dependence of $\Omega_{t a}\left(a_{t a} ; w\right)$ with changing $w$, as previously discussed. There is strictly monotonic decrease with increasing $w$ and the distinction of this parameter (at the level of again a $\sim 40 \%$ range of values) for the range of $w$ we investigated, is observationally attainable, within estimated accuracy.

## Chapter 4

## Discussion

In this thesis, we have examined the behavior of $\Omega_{t a}$ as a function of the equation of state of dark energy in the following two cases: inhomogeneous (clustering) dark energy and homogeneous (non-clustering) dark energy. We have done so in the context of the spherical collapse model of structure formation, by integrating the Friedmann equations that describe the evolution of a spherical overdensity. In the first case, the overdensity evolves effectively isolated by the otherwise homogeneous and isotropic background universe that contains it. In the second case, the overdensity's growth is more directly affected by the background's evolution. The evolution of the background, also governed by Friedmann equations, is in both cases independent of that of the overdensity as, for the scales we are considering, the effect of the perturbation in altering the expansion of the Universe as a whole is negligible. Combining this system of equations with the appropriate boundary conditions that ensure the eventual turnaround of the spherical region allows us to determine the value of $\Omega_{t a}\left(a_{t a} ; w\right)$ for each specific evolutionary scenario.

Since $\Omega_{t a}$ has been suggested as a cosmological observable, we have sought to identify for which cases it is indeed possible to draw direct conclusions for the cosmological parameters based on its observed value.

In the context of the spherical collapse model, or, equivalently, if all effects that make the true value deviate from the spherical collapse prediction are below statistical uncertainty, we have reached the following conclusions.

First, we have seen that for both investigated cases (clustering/non-clustering DE) the highredshift behaviour of the turnaround density for all $w$ cosmological models converges. However, the absolute differences of the turnaround density between models remain significant at high redshift, so depending on the achievable accuracy of high-redshift measurements of the turnaround density, important constraints could in principle be derived from the early Universe.

Second, in the low-redshift regime, if we were to adopt the estimated accuracy of Pavlidou et al., 2020 for measuring the turnaround density, we would conclude that indeed different values of $w$ produce turnaround densities different enough so that the turnaround density can be considered as a useful observable for the dark energy equation of state.

Third, for the non-clustering DE case the behaviour of the turnaround density as a function of cosmic time for $w<-1$ (phantom energy) vastly deviates from the one seen in the case of clustering DE , since for later times the ever-increasing contribution of DE around overdensities that have not yet managed to stop the expansion of their surroundings, render them incapable of doing so in the future, which is reasonable, especially considering the Big Rip fate of their background Universe.

In summary, we qualitatively conclude that indeed the turnaround density could yield important constraints for the dark energy equation of state. Immediate next steps for the investigation
presented in this thesis include a follow up on these results in order to produce quantitative constraints on the values of the cosmological parameters $\Omega_{m, 0}, \Omega_{d e, 0}, w$ that we can obtain, assuming we can achieve a measurement of $\Omega_{t a}$ with a given accuracy in several redshift bins. We will additionally quantify the constraints that can be placed on $w$ alone, assuming that the values of $\Omega_{m, 0}$, $\Omega_{d e, 0}$ and $H_{0}$ are obtained from independent cosmological datasets, at accuracies comparable with those quoted by current cosmological studies.

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## Abbreviations

| Acronym | Definition |
| :--- | :--- |
| GR | General Relativity |
| EFE | Einstein Field Equations |
| FE | Friedmann Equations |
| EoS | Equation of State |
| SCM | Spherical Collapse Model |
| FLRW (metric/model) | Friedmanni-Lemaitre-Robertson-Walker (metric/model) |
| $\Lambda$ CDM (cosmology/model) | Lambda Cold Darm Matter (cosmology/model) |
| NR | Newton-Raphson (method) |

