Black Holes in String Theory: Entropy and Microstates

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Abstract

This thesis investigates black hole solutions involving branes, focusing on the entropy of 2-charge and 3-charge systems, the fuzzball proposal, and superstrata constructions. We explore how D-brane configurations account for black hole microstates, matching the Bekenstein-Hawking entropy. The fuzzball proposal is examined as a resolution to the black hole information paradox, suggesting that black holes are composed of horizonless, smooth microstates. Additionally, we present superstrata, which extend the fuzzball framework to provide a richer spectrum of microstate geometries for the 3-charge black hole. Our study aims to enhance the understanding of black hole microstates within the framework of string theory and especially supergravity.

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Contents

1 Introduction - Discussion

Black holes, one of the most intriguing predictions of Einstein's theory of general relativity, have long captured the interest of physicists and cosmologists. These enigmatic objects, defined by regions of spacetime exhibit strong gravitational effects that nothing, not even light can escape from them. Even more, they offer great insights into the nature of gravity, spacetime, and quantum mechanics. The simplest solution to Einstein's field equations is the Schwarzschild black hole, characterized only by its mass. However, more complex and realistic black holes have been theoretically discovered, verifying the field equations, that involve quantities like electric charge, angular momentum and more.

One of the most significant kinds of black holes is the Reissner-Nordström black hole. This object except of its mass it includes a non-zero total electric charge. It owns some very fascinating new properties like the existence of more than one event horizons. Another interesting phenomenon is the case where the charge is equal to the mass of the object. There, the solution obtains a high-symmetrical geometry that does not appear in the other cases.

The theoretical study of black holes was soon expanded in the framework of quantum mechanics. Great physicists like Hawking implemented the concepts of the quantum world and derived remarkable results for the dynamics of such objects. Perhaps the most notable are the Hawking evaporation process and the existence of an entropy proportional to the area from the horizon surface (Bekenstein-Hawking entropy). Having such an entropy implies a thermodynamical description of the system and especially the existence of different microstates. These results lead to two great problems involving the microstates of the system and the information paradox. The appearance of them challenge the fundamental concepts of gravity and quantum mechanics as none of them can give sufficient explanations.

In order to answer the problems, physicists turned to quantum gravity and especially string theory. To explore the microstates of black holes and the structure of spacetime, supersymmetric configurations using branes and strings have become a hot topic of research. Starting from simple one-type-brane configurations and combining different types we can make promising BPS systems of 2-charge and 3-charge black holes that give answers to our problems. Moreover, one prominent proposal that may resolve the information paradox is the fuzzball proposal. It suggests that black holes are complicated in general systems of branes and strings that are horizonless structures with regular solutions that make up all the microstates of the entropy. These systems are different up to the classical horizon scale.

In Section 2 we introduce the charged black hole solution of Reissner-Nordström using general relativity. We examine the various quantities included in the solution and analyse the geometries of such a black hole. From our analysis, it arises a special case of interest, the extremal solution.

Section 3 begins with an introduction upon the problems emerged from quantum mechanics and the great need for string theory. After a small description of the M-theory, we turn our concern in the type II string theory which will be used in our computations. We write the metrics and potentials for some supersymmetric configurations of Dp-branes and present a new type of black hole solutions. Most important are the D1-D5 and D1-D5-P systems and are extensively discussed. The importance of these systems is the relation with the 5-dimensional Reissner-Nordström black hole.

The last Section 4 is attributed to the calculation of the entropy for the 2-charge and 3-charge black holes and the construction of their microstates. In addition, a statistical analysis has been conducted for these systems. We state the fuzzball proposal and proceed to the microstate geometries. We find a family of such geometries for the 2-charge case and extend our results for the 3-charge case. The last structures are called superstrata and are an open subject in theoretical physics with a lot of possible applications but also several remaining issues.

2 Charged Black Holes In General Relativity

2.1 The Reissner-Nordström Black Hole

The first black hole ever discovered within the framework of general relativity is the Schwarzschild black hole solution and it is attributed to a spherically symmetric, non-rotating object with fixed mass. The metric describing such an object is as we know [1],[2]

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_2^2 \tag{2.1}$$

where

$$f(r) = 1 - \frac{2M}{r}, \quad d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2 \quad (c, G_N = 1) \tag{2.2}$$

We observe that the solution of (2.1) depends only on its mass M. But we know that an object can also be electromagnetically charged. So the next step is to search for a charged black hole solution. For simplicity we will only consider electrically charged solutions, the total magnetic charge will be zero.

To derive the equations of motion for the solution that we are searching, we need the Einstein-Maxwell action

$$S = \frac{1}{16\pi}\int dx^4\sqrt{-g}(R - F_{\mu\nu}F^{\mu\nu}) \tag{2.3}$$

with R to be the Ricci scalar and F the field strength of an 1-form Maxwell potential A

$$R = R^{\mu\nu}{}_{\mu\nu} \quad , \quad F = dA \tag{2.4}$$

The variations of the action (2.3) in respect to the metric g and the potential A will provide the equations of motion through the variation principle

$$\delta S = 0 \tag{2.5}$$

By varying the term of the Ricci scalar in respect to the metric ($\frac{\delta}{\delta g^{\mu\nu}}$) we obtain the Einstein tensor

$$\begin{aligned} \delta(\sqrt{-g}R) &= (R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\delta g^{\mu\nu}\sqrt{-g} \\ &= G_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu} \end{aligned} \tag{2.6}$$

This is the same result as for the Schwarzschild solution but now we will get an extra term from the same variation of the E/M term of the action

$$\begin{aligned} \delta(\sqrt{-g}F_{\mu\nu}F^{\mu\nu}) = \delta(\sqrt{-g}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}) = \\ \delta(\sqrt{-g})F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma} + 2\sqrt{-g}F_{\mu\nu}F_{\rho\sigma}g^{\nu\sigma}\delta g^{\mu\rho} = \\ \delta g^{\mu\nu}\sqrt{-g}(-\frac{1}{2}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + 2F_{\mu\rho}F_{\nu}^{\rho}) \end{aligned} \tag{2.7}$$

Combining them we have

$$G_{\mu\nu} = 8\pi(F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}) \tag{2.8}$$

Rewriting the field strength in respect of the potential as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.9}$$

and varying in respect of the potential A we get

$$\begin{aligned}
&\delta \int dx^4 \sqrt{-g} F_{\mu\nu} F^{\mu\nu} = 2 \int dx^4 \sqrt{-g} F^{\mu\nu} \delta(F_{\mu\nu}) = 4 \int dx^4 \sqrt{-g} F^{\mu\nu}\, \partial_\nu(\delta A_\mu) \\
&= -4 \int dx^4\, \delta A_\mu\, \partial_\nu(\sqrt{-g} F^{\mu\nu}) = -4 \int dx^4\, \delta A_\mu\, \partial_\nu(\sqrt{-g} F^{\mu\nu}) \\
&= -4 \int dx^4 \sqrt{-g}\, \delta A_\mu\, \nabla_\mu F^{\mu\nu}
\end{aligned} \tag{2.10}$$

Notice that for the above result we used the generalized version of Stokes' Theorem and the property that our ends of integration are at infinity where we demand the variations to vanish in order to have a stable and well defined solution. And from the (2.5) we must have

$$\nabla_\mu F^{\mu\nu} = 0 \tag{2.11}$$

Reissner and Nordström showed that there is a spherically symmetric solution that has the form of the (2.1) and satisfies (2.8) and (2.11). The radial function and the field strength of this solution are

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad F = \frac{Q}{r^2} dt \wedge dr \tag{2.12}$$

where Q is the total electric charge of the black hole. It is obvious that taking the limit of $Q = 0$ we obtain the Schwarzschild solution for a black hole of mass M. The addition of a magnetic charge P is also simple

$$Q^2 \to Q^2 + P^2 \tag{2.13}$$

2.2 Conserved Charges of Reissner-Nordström Black Hole

We know from the no-hair theorem that the parameters of mass, charge and angular momentum fully characterize stationary black hole solutions in asymptotically flat general relativity. In order to calculate these conserved charges from our metric we will use the Komar integrals. Specifically, as the solution for the RN black hole has no angular momentum, we will use the two following integrals [3]

$$m = \frac{1}{4\pi} \oint_{\partial\Sigma} \nabla^\mu k^\nu d\Sigma_{\nu\mu} \tag{2.14}$$

$$q = \frac{1}{4\pi} \oint_{\partial\Sigma} F^{\mu\nu} d\Sigma_{\nu\mu} \tag{2.15}$$

For both charges we must define a hypersurface Σ of constant t. Thus, we choose its boundary $\partial\Sigma$ to be a surface of constant radius r, in particular it is a two-sphere at spacial infinity. Next step is to find the unit normal vectors n and σ for these surfaces for the specific metric which are [3]

$$n = (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-\frac{1}{2}} \partial_t \quad , \quad \sigma = (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{\frac{1}{2}} \partial_r \tag{2.16}$$

Lets now start by calculating the mass. In the integral we have a factor named k, that is a timelike Killing vector field. Then from (2.14) we have

$$m = \frac{1}{4\pi} \oint_{\partial\Sigma} \nabla^\mu k^\nu n_\mu \sigma_\nu \sqrt{|\,\omega\,|} d^2\omega \tag{2.17}$$

where ω is the induced metric of the two-sphere and we also have that

$$\begin{aligned}
&n_\mu \sigma_\nu \nabla^\mu k^\nu = n^\mu \sigma^\nu g_{\nu\rho} \nabla_\mu k^\rho = n^t \sigma^r g_{rr} \nabla_t k^r = n^\mu \sigma^\nu g_{\nu\rho} \nabla_\mu k^\rho = n^t \sigma^r g_{rr} \Gamma^r_{t\nu} k^\nu = \\
&n^t \sigma^r g_{rr} \Gamma^r_{tt} = -\frac{1}{2} n^t \sigma^r g_{rr} g^{rr} \partial_r g_{tt} = \frac{1}{2} \frac{\partial}{\partial r} (1 - \frac{2M}{r} + \frac{Q^2}{r^2}) = \frac{M}{r^2} - \frac{Q^2}{r^3}
\end{aligned} \tag{2.18}$$

We are finally ready to calculate the integral for the mass

$$m = \lim_{r\to\infty} \frac{1}{4\pi} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \quad r^2 \sin\theta (\frac{M}{r^2} - \frac{Q^2}{r^3}) = \lim_{r\to\infty} (M - \frac{Q^2}{r}) = M \tag{2.19}$$

In a similar way, with the use of (2.12) and (2.15), we find the electric charge of the black hole as

$$q = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta r^2 \sin\theta \frac{Q}{r^2} = Q \tag{2.20}$$

To sum up, we showed that the parameters appeared in the metric of the black hole are indeed the total mass and electric charge of the object. Notice that our calculations are valid only for taking a sphere with radius bigger than the radius of the black hole in order to include the whole charges.

2.3 Singularities and Extremal Solution

Even when writing the Schwarzschild black hole we observed some particular anomalies when the $f(r)$ function becomes zero or infinite. Such anomalies are called singularities and they arise as a result from the breakdown of the theory's equations. There can be of two kinds: coordinate singularity and curvature singularity. The first indicates that our choice of coordinates is not good enough and does not include the full spacetime. On the contrast, a curvature singularity represents a point where the spacetime becomes infinite, leading to a point of infinite density.

While the solution for the Schwarzschild black hole seems to have two singularities at $r = 0$ and $r = 2M$, with the adjustment of Kruskal coordinates someone can show that it has only one curvature singularity in the center of the black hole $r = 0$. The surface of the $r = 2M$ is in fact an event horizon. Another way of showing this, is by finding the critical points of the Ricci squared invariant given by

$$R^2 = R^{\mu\nu} R_{\mu\nu} \tag{2.21}$$

Moving to the RN black hole we find the zeros of the radial function f from (2.12) to be

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2} \tag{2.22}$$

These two points with the $r = 0$ make up the singularities of our metric. For all these points the scalar curvature vanishes. But from (2.21) we see that

$$R^2 = 4\frac{Q^4}{r^8} \tag{2.23}$$

leaving only the $r = 0$ as a singularity from the geometry, while the other singularities of the metric are coordinate singularities and correspond just to horizons.

I would like now to speak about a special case depending on the mass and the charge where $M = \mid Q \mid$. This case is called extremal RN black hole and will be of great use through the next sections. Notice that the two singularities in $r_{\pm} = Q$ coincide. So the two horizons become one. The radial function $f(r)$ for this case becomes

$$f(r) = \frac{(r - Q)^2}{r^2} \tag{2.24}$$

We observe that by changing coordinates as

$$\rho = r - Q \tag{2.25}$$

we can bring the metric to the form

$$ds^2 = -H^{-2}(\rho)dt^2 + H^2(\rho)[d\rho^2 + \rho^2 d\Omega_2^2] \quad , \quad H(\rho) = 1 + \frac{Q}{\rho} \tag{2.26}$$

In this coordinate system, the solution is regular at $\rho = 0$. The function $H(\rho)$ will be known to the observant reader as it is the harmonic Poisson function solving the Laplace's equation in a 3-dimensional space

$$\Delta_3 H = 0 \tag{2.27}$$

It is very important to examine the behavior of this metric in the near horizon limit $\rho \to 0$ in order to see the asymptotic structure of spacetime. In this limit we can neglect the one in the $H(\rho)$ function and write the metric as

$$ds^2 = -\frac{\rho^2}{Q^2}dt^2 + \frac{Q^2}{\rho^2}d\rho^2 + Q^2 d\Omega_2^2 \tag{2.28}$$

Making a last change in the radial coordinate as $\rho \to Q^2\rho$ (in the right units) we end up with

$$ds^2 = Q^2(-\rho^2 dt^2 + \frac{d\rho^2}{\rho^2} + d\Omega_2^2) \tag{2.29}$$

and this is now no other spacetime rather than the $AdS_2 \times S^2$ with radius of curvature $\mid Q \mid$. Analysing this geometry we start with a flat spacetime at infinity and as we move in the near-horizon area we fall inside an infinite throat with constant radius. The initial geometry of the general RN black hole had no obvious symmetries. But the current solution of the extreme case seems to be a lot more symmetric because it belongs to the family of the $AdS_n \times S^n$ geometries. Such geometries will appear in supergravity systems of branes later on. This is not an accident. The new solutions will be a sort of higher dimensional RN black holes with more exotic charges. Lastly, such geometries arise in holography and AdS/CFT correspondence and have been studied extensively.

3 String Theory And Black Hole Solutions

We have seen so far how to construct more general black hole solutions within the framework of general relativity. Starting with the Schwarzschild black hole, that depends only on its mass, we added an extra electric charge and got the Reissner-Nordström black hole. In this way, we could also add a magnetic charge, rotation (Kerr black hole) etc.

But so far we haven't spoke anything about quantum mechanics. From quantum mechanics and especially Quantum Field Theory (QFT) we know that energy can decay into a particle-antiparticle pair. With this knowledge, Hawking found that such pairs can be created in the near horizon area of the black hole and sometimes one of the two falls inside the black hole horizon while the other escapes to infinity. The black hole lower its mass and energy with the form of thermal radiation. Thus, the black hole behaves as a black body having a characteristic temperature T proportional to the gravitational field strength near the horizon.

But if it has a radiation temperature, by the universal laws of thermodynamics, must also have an entropy. In that spirit, Bekenstein and Hawking proved that the entropy of the black hole is proportional to the area of the horizon

$$S_{BH} = \frac{A}{4G_N} \tag{3.1}$$

where G_N is the Newton's constant related to Planck length.

The calculation of this entropy will give a very huge number even for a typical black hole. For instance, a black hole with one solar mass will have an entropy of about $10^{77} Joule/Kelvin$ [4]. Imagine what we will have for a supermassive black hole inside the center of a galaxy. Boltzmann demonstrated that the entropy is related to the number N of microstates in a dynamical system as follows

$$S \sim log(N) \tag{3.2}$$

This gives us an enormous number of microstates that correspond to the same macroscopic properties. Nonetheless, we cannot explain this number in the classical picture of general relativity, mostly due to two emerging problems.

The biggest mystery in the realm of classical GR is; where are these microstates? How do they look like? There are several propositions such us lying near the singularity of the black hole or extending even up to the horizon (fuzzball proposal). The other problem is called the information paradox. As the black hole looses mass through radiation it increases its temperature and thus the rate of emission. After a very long period of time, the black hole would be completely evaporated. So where did the initial information of the black hole go? These are the problems that we cannot answer using only general relativity.

By reason of the above problems, we will implement String Theory, which is our best theory of quantum gravity so far, in our computations in order to solve them. Subsequently, I will provide some basic elements of String theory like the type II string theory, D-branes etc. which will be used in extend in later problems. Our main goal is to compute the entropy of an analogous black hole in string theory and then compute and analyze the microstates emerging from each geometry of spacetime.

3.1 Type II String Theory and Fundamental Objects

String theory is a vast theoretical framework in physics consisting of lots of different physical ideas and mathematical constructs that try to interpret our world. It consists of different types of theories that only before some years physicists realized that all of them are connected by specific dualities. But in this project I will not provide a full review of these theories rather than I will stick only to the objects and features of these theories appeared in the problems of entropy and microstates inside a black hole as I mentioned above. Especially, we will consider the low-energy limits of type IIA, type IIB theory and a bit of M-theory. Also, for simplicity we are interested in the bosonic section and not the fermionic and supersymmetry extra terms.

We shall start by the most general case we will encounter that is not other than the 11-dimensional supergravity which is the low-energy limit of M-theory. From bibliography [5] we can go and write the action of this type

$$S_{11} = \frac{1}{2\kappa_{11}^2}[\int d^{11}x\sqrt{-g}(R - \frac{1}{2}F_4^2) - \frac{1}{6}\int \hat{C}_3 \wedge F_4 \wedge F_4 \quad], \quad F_4 = d\hat{C}_3 \tag{3.3}$$

where the κ_{11} is a constant of no interest for us, at least for now.

We observe an analogy with the Einstein-Maxwell action with the only difference that the 2-form field strength is transformed in a 4-form one and we have also an extra term depending on both the potential and the field strength. This extra term will change the equations of motions for the $\hat{C}_3$ potential. The fundamental objects of this theory, equivalent to the electron and magnetic monopole of Maxwell's theory of electromagnetism, will be the electric and magnetic objects $M2$ and $M5$ branes.

Now it is known that we can make a dimensional reduction and obtain a theory at ten dimensions [5],[6]. From this process the 11D metric will be replaced with a 10D and generate a gauge field C_1 and a scalar σ. The scalar σ has to be the dilaton field Φ, up to some field redefinition. The potential $\hat{C}_3$ reduces itself to two potentials C_3 and $B_2 = B_{\mu\nu} = \hat{C}_{\mu\nu 10}$. Moreover, choosing our compactification of the eleventh dimension on a circle with period $2\pi R$ we rescale our constants as $\kappa_{10}^2 = \frac{\kappa_{11}^2}{2\pi R}$.

After some changes in the fields we can write the new action as [5]

$$\begin{aligned} S_{IIA} =& \frac{1}{2\kappa_{10}^2}\int d^{10}x(\sqrt{-g})e^{-2\Phi}[(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}H_3^2) - \frac{1}{2}(\tilde{F}_4^2 + F_2^2)] \\ &- \frac{1}{4\kappa_{10}^2}\int B_2 \wedge F_4 \wedge F_4 \end{aligned} \tag{3.4}$$

where

$$F_2 = dC_1, \quad F_4 = dC_3, \quad H_3 = dB_2, \quad \tilde{F}_4 = dC_3 - C_1 \wedge H_3 \tag{3.5}$$

The C fields correspond to the R-R sector and the B, H to the NS-NS.

Similarly, we can write the metric for the type IIB theory [5]

$$\begin{aligned} S_{IIB} =& \frac{1}{2\kappa_{10}^2}\int d^{10}x(\sqrt{-g})e^{-2\Phi}[(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}H_3^2) - \frac{1}{2}(F_1^2 + \tilde{F}_3^2 + \tilde{F}_5^2)] \\ &- \frac{1}{4\kappa_{10}^2}\int C_4 \wedge H_3 \wedge F_3 \end{aligned} \tag{3.6}$$

where

$$\tilde{F}_3 = F_3 - C_0 \wedge H_3, \quad \tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \tag{3.7}$$

There is also a way to combine the actions of the two theories together in a single one [7] but I would not expand into that.

To sum up, starting from eleven dimensions and performing a dimensional reduction we obtain two 10-dimensional theories. Looking at the fields of each theory someone would think that they are not connected. The type IIA has potentials with odd indexes (C_1,C_3) while the type IIB has even (C_0,C_2,C_4). But in fact the two theories are connected through the T-duality explained in Appendix A.

To see what will be the objects of this theory we must be careful to the directions they extend. Remember that we have compactified the eleventh direction. So if we have a membrane in the directions 0,1,2 it will not change. But having a momentum wave in the directions of 0,10 will give just a particle after the dimensional reduction of the x_{10} dimension.

The fundamental electrical object of type II string theory is the simple quantized string F1, while the magnetically dual of this theory is the NS5-brane. Besides these objects, we have the so called p-branes. p-branes are extended objects with p spacial dimensions that are charged under a (p+1)-form potential. A special case of them are the Dp-branes which arise from the demand of Dirichlet boundary conditions on the edges of quantized strings. They can be regarded as large surfaces where the ends of strings move but one finds that they have a dynamics of their own.

Regarding the potentials under which are charged the different objects we saw that the fundamental F1 and NS5-brane are charged under the B field while a brane lets say the electric D1-brane is charged under the C_2 field potential and F_3 strength and its magnetic dual D5-brane under the Hodge dual $\star F_3 = F_7$. Keeping this in mind we can go and built a class of solutions for the D-branes.

3.2 D-brane Solutions

Let us present some D-brane supersymmetric solutions to enhance our understanding of these objects and the way they occupy the extra dimensions. We will restrict in the supergravity case which is the low-energy approximation of string theory.

3.2.1 D1-brane

We start with the simplest case of a supersymmetric D1-brane and write the naive metric (the metrics will be written in the string frame unless we say otherwise) of the form [4]

$$ds^2 = Z^{-1/2}(-dx_0^2 + dx_1^2) + Z^{1/2}(dx_2^2 + dx_3^2 + ... + dx_9^2) \tag{3.8}$$

which can be written in the more concrete way as

$$ds^2 = Z^{-1/2}(-dt^2 + dx_1^2) + Z^{1/2}(dr^2 + r^2 d\Omega_7^2) \tag{3.9}$$

and we also have for the 2-form potential

$$C_{01} = Z^{-1} \tag{3.10}$$

It is interesting that this metric resembles the extremal RN black hole and again has a harmonic function for a scaling factor.

So the brane extends in the time and one spacial direction and is pointlike in the remaining eight. The D1-brane behaves as a particle in the tranverse $\mathbb{R}^8$ space and the function Z admits the Maxwell potential in this space. It must obeys then the Laplace equation

$$\Delta_8 Z = \rho_{D1} \tag{3.11}$$

where ρ_{D1} is the density of branes in these coordinates. Taking the branes to lie at the origin of the coordinate system with a density of $\rho_{D1} = N_1\delta(\vec{x})$, thus we have spherical symmetry, we find

$$Z = c + \frac{N_1}{r^6} \tag{3.12}$$

where r is the radius of the transverse coordinates and c a constant that we can scale to 1.

As $r \to 0$ we approach the source and we have $Z \to \infty$. From the metric, we see that the $\mathbb{R}^{1,2}$ factor shrinks while the $\mathbb{R}^8$ blows up. This leads to the creation of a throat and a singularity at the $r = 0$.

To be explicit for the previous harmonic solution we recall the formula for the Laplacian operator in d dimensions [8]

$$\Delta_{\mathbb{R}^d} = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{d-1}} \tag{3.13}$$

and for a function that obeys the Laplace equation, we have

$$\Delta_d Z = 0 \implies Z = a + \frac{b}{r^{d-2}} \tag{3.14}$$

where a,b are constants.

3.2.2 Construction of Higher Brane Solutions

The D1-brane belongs to the type IIB theory. But as we mentioned previously, we can go from IIB to IIA and vice versa using the T-duality. To start this attempt we change the density of the D1-branes in the $\mathbb{R}^8$ space. We take a continuous distribution of such branes along a specific direction, lets say the direction x_7. This process is known as smearing of branes along one or many dimensions. The solution then of the transverse directions will not have any dependence of the dimension x_7.

To understand the effect of smearing, I will give an example of classical electromagnetism. Think that we have a charge distribution in a flat 2-dimensional space with coordinates x and y and want to find the potential at any point of space. If we distribute the total charge uniformly along the x axis from $-\infty$ to ∞ then the potential will depend only on the y coordinate.

Turning back to our problem, the harmonic function will depend only on seven and not eight spacial coordinates and thus

$$Z \sim 1 + \frac{N_1}{r^5} \tag{3.15}$$

We can compact the x_7 on a circle and apply the T-duality. Furthermore, the application of the T-duality along x_7 changes the size of the circle as $g_{77} \to (g_{77})^{-1}$(Appendix A). Hence, the metric becomes

$$ds^2 = Z^{-1/2}(-dt^2 + dx_1^2 + dx_7^2) + Z^{1/2}(dr^2 + r^2 d\Omega_6^2) \tag{3.16}$$

and the 3-form potential gets an additional index

$$C_{017} = Z^{-1} \tag{3.17}$$

This solution corresponds to a D2-brane. With the reverse process we can go back to a D1-brane. Then, someone can create a solution for each brane using the first solution and the T-duality. I will not provide the solution for each brane except two of them. The first one is the D5-brane which will use a lot later on. The metric and the harmonic function for this brane is

$$\begin{aligned} ds^2 &= Z^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2) + Z^{1/2}(dr^2 + r^2 d\Omega_3^2) \\ Z &= 1 + \frac{N_5}{r^2} \end{aligned} \tag{3.18}$$

3.2.3 D3-brane

The second is the D3-brane. For this one we have

$$\begin{aligned} ds^2 &= Z^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + Z^{1/2}(dr^2 + r^2 d\Omega_5^2) \\ Z &= 1 + \frac{N_3}{r^4} \end{aligned} \tag{3.19}$$

This solution has a very unique behaviour in the near horizon limit. As we take the limit of $r \to 0$ we can neglect the one in the harmonic function and write it as $Z \sim N_3/r^4$. Replacing in the metric

$$ds^2 = \frac{r^2}{\sqrt{N_3}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{\sqrt{N_3}}{r^2}dr^2 + \sqrt{N_3}d\Omega_5^2 \tag{3.20}$$

This geometry can be identified as the $AdS_5 \times S^5$. For large radius, away from the source the solution approaches the flat Minkowskian spacetime. But as we travel near the center of our coordinates the spacetime creates a throat. The interesting fact is that this throat does not closes but acquire a finite size fixed with radius of $N_3^{1/4}$. In addition, the S^5 sphere maintains the same size. We have then a throat around the $r = 0$ going to infinity. This feature of the D3-brane is what makes it so special for holography and the reason for this is that the D3 is both electrically and magnetically charged under the C_4 due to its self-duality, $F_5 = \star F_5$.

3.3 New Black Hole Solutions

In the previous sections we saw how to create black hole solutions within the framework of general relativity and we introduced some brane solutions from type II string theory. We will try now to combine them and create black hole solutions consisting of Dp-branes or a system of them.

In Section 2 we saw two examples of black holes in the 4-dimensional spacetime, the Schwarzschild and the Reissner-Nordström black hole. We can generalize in more dimensions and find simple solutions (without adding any string theory) of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2 \tag{3.21}$$

For example, there exists a black hole solution for a simple static black hole, like the Schwarzschild black hole, in a d-dimensional spacetime. It is called the Schwarzschild-Tangherlini black hole [9] and is given by the metric

$$\begin{aligned} ds^2 &= -(1 - \frac{\mu}{r^{d-3}})dt^2 + \frac{dr^2}{(1 - \frac{\mu}{r^{d-3}})} + r^2 d\Omega_{d-2}^2 \\ \mu &= \frac{16\pi G_d M}{(d-2)\Omega_{d-2}}, \quad \Omega_{d-2} = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} \end{aligned} \tag{3.22}$$

Again this is a black hole characterized only by its mass, which we can find with the appropriate Komar integral as we did in 2.2. It has a singularity at $r = 0$ and a horizon at $r = \mu$.

3.3.1 Non-extremal Case

We can go now back to the solution of the D3-brane (3.19) and deform it in order to resemble the (3.22). Under this consideration we can take a solution like

$$ds^2 = Z^{-1/2}(-f(r)dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + Z^{1/2}(\frac{dr^2}{f(r)} + r^2 d\Omega_5^2) \tag{3.23}$$

When the function takes the value $f(r) = 1$, we take the simple supersymmetric solution of a D3-brane we saw in the previous subsection. Otherwise, we have obtained a solution of a non-extremal black hole from D3-branes. The function $f(r)$ is also harmonic in the transverse plain, obeying the Laplace equation

$$\Delta_6 f = 0 \tag{3.24}$$

and someone can consider the solution

$$f(r) = 1 - \frac{m}{r^4} \tag{3.25}$$

The "electric" charge Q of the D3 can be given by the flux of the field strength

$$Q = \int F_5 \tag{3.26}$$

which will be proportional to the number N_3 of branes up to some constants as a multiplication factor.

The ADM mass then, coming from the g_{tt} term, will be

$$M = Q + m \tag{3.27}$$

Examining the result, we observe that when $m < 0$ the metric will describe a singular solution with a naked singularity. So for physical reasons we consider only the case where $m > 0$. In that case, we cannot obtain a stable system of such branes for this metric. If we put two or more of these D3 black branes together the branes will collapse to a single body. This is because the gravitational attraction is larger than the electrical repulsion.

A special case is when the mass is equal to the charge $M = Q$. The solution is called extremal as we mentioned in the RN black hole case and is identical with the supersymmetric solutions we gave earlier. The gravitational attraction will be compensated from the electrical repulsion and the system will come to an equilibrium. These black objects will have zero Hawking temperature and do not radiate but they have non zero mass and entropy. These solutions may sound extremely theoretical but they are very important due to their simplicity in construction. From now on we will be interested only in such extremal cases of black branes and also correspond to BPS states of string theory. These states are useful as they preserve a huge number of symmetries and have a significant impact in the computation of Hawking-Bekenstein entropy that will concern us.

3.3.2 Example of Extremal D2-D2 Black Brane System

To complete the discussion of such black branes solutions we will go and write such a system of two branes. For simplicity, we will take two supersymmetric D2-branes in the following directions

$$\begin{aligned} D2_1 &\to t, x_1, x_2 \\ D2_2 &\to t, x_3, x_4 \end{aligned} \tag{3.28}$$

Then we smear each brane in the directions occupied by the other. For instance the $D2_1$ is smeared in the directions of x_3, x_4. The metric will become

$$ds^2 = -(Z_1 Z_2)^{-1/2} dt^2 + (\frac{Z_1}{Z_2})^{1/2}(dx_1^2 + dx_2^2) + (\frac{Z_2}{Z_1})^{1/2}(dx_3^2 + dx_4^2) + (Z_1 Z_2)^{1/2} ds^2(\mathbb{R}^5) \tag{3.29}$$

where the functions Z_1 and Z_2 are harmonic in the $\mathbb{R}^5$ transverse space, so

$$Z_i = 1 + \frac{Q_i}{r^3}, \quad i = 1, 2 \tag{3.30}$$

where r is the radial coordinate of $\mathbb{R}^5$ and Q_i are the charges of each pack of branes. As we smeared the branes in the specific directions, the radial functions do not depend upon them. The directions x_1 to x_4 have been compactified and create a T^4 manifold.

Lets make some remarks for this solution. First, if any of the H_i functions becomes one, we obtain the solution for a simple supersymmetric D2-brane configuration like in (3.16). This can be thought as the limit of taking the charge of a pack of D2, for instance Q_1, and eliminate it. We would expect then to have the main contribution from the other stack of branes with the larger total charge.

In the near horizon limit where $r \to 0$ we neglect the ones in the H functions and obtain the metric

$$ds^2 = -\frac{r^3}{\sqrt{Q_1 Q_2}}dt^2 + \sqrt{\frac{Q_1}{Q_2}}(dx_1^2 + dx_2^2) + \sqrt{\frac{Q_2}{Q_1}}(dx_3^2 + dx_4^2) + \frac{\sqrt{Q_1 Q_2}}{r^3}ds^2(\mathbb{R}^5) \tag{3.31}$$

Taking the two charges to be of the same value $Q_1 = Q_2 = Q$, the metric simplifies to

$$ds^2 = -\frac{r^3}{Q}dt^2 + ds^2(T^4) + \frac{Q}{r^3}ds^2(\mathbb{R}^5) \tag{3.32}$$

and rewrite it as

$$ds^2 = -\frac{r^3}{Q}dt^2 + Q(\frac{dr^2}{r^3} + \frac{d\Omega_4^2}{r}) + ds^2(T^4) \tag{3.33}$$

which looks like the AdS geometry but with a different dependence over the radial coordinate. It would be very nice if we could indeed have a dependence over r^2 and obtain the AdS spacetime which has been studied extensively for many years and is very useful to holography. This can be done by the compactification or smearing of the branes of extra dimensions from the $\mathbb{R}^5$ or using higher dimensional branes. A system of such black branes that has an asymptotic AdS behaviour in the near horizon limit is the D1-D5 system.

3.4 The D1-D5 Supergravity Solution

In the past years, theorists have discovered a lot of different solutions from combinations of branes that have special properties and applications to modern problems. One of these problems was the explanation of the entropy from a black hole and the calculation of the microstates that are related to it. The first and most simple solution of a black hole, in the regime of string theory, that gave a glimpse to the approach for the problem was the three charge D1-D5-P BPS black hole. For the purpose of extending our discussion to the analysis of black hole entropy, it is crucial to examine such solutions, starting from the simpler two-charge system of D1-D5, coming from the type IIB string theory. This system is also mapped in the D2-D2 under some T-dualities.

Starting with the naive approach that we discussed in the previous subsection we write the metric of the system

$$ds^2 = \frac{1}{\sqrt{Z_1 Z_5}}(-dt^2 + dy^2) + \sqrt{Z_1 Z_5}ds^2(\mathbb{R}^4) + \sqrt{\frac{Z_1}{Z_5}}ds^2(T^4) \tag{3.34}$$

where we have compactified the four dimensions of the branes in a T^4 torus and the remaining y direction on a circle S^1 with periodicity

$$y \to y + 2\pi R_y \tag{3.35}$$

The functions Z_i, that are harmonical in the transverse dimensions, and the dilaton field are given from the expressions

$$\begin{aligned} Z_i &= 1 + \frac{Q_i}{r^2}, \quad i = 1,5 \\ e^{-\Phi/2} &= (\frac{Z_5}{Z_1})^{1/4} \end{aligned} \tag{3.36}$$

The charges are connected with the number of branes as

$$Q_1 = \frac{a'^3}{V} N_1, \quad Q_5 = a' N_5 \tag{3.37}$$

where we set $g = 1$ (the string coupling), $a' = l_s^2$ and the volume of the torus T^4 is given as $v = (2\pi)^4 V$.

It is important to fully understand the structure of the branes in (3.34). The metric describes D5-branes wrapped around the T^4 manifold and the S^1 circle. They intersect with the D1-branes that are enclosed in the S^1 circle. Remember that the D1-branes are smeared in the directions of the torus. Lastly, the functions Z_i have a dependence of r^2 as they are harmonic in the transverse 4-dimensional space of $\mathbb{R}^4$.

The dimensions on S^1 and T^4 are compact and can be reduced (Appendix B) leaving a five dimensional metric

$$ds_5^2 = -k^{-2/3} dt^2 + k^{1/3}(dr^2 + r^2 d\Omega_3^2) \tag{3.38}$$

where

$$k = Z_1 Z_5 = (1 + \frac{Q_1}{r^2})(1 + \frac{Q_5}{r^2}) \tag{3.39}$$

At large distances this spacetime is asymptotically flat. The only point of concern is the $r = 0$ that creates a singularity. To clarify the type of singularity, we compute the Ricci scalar

$$R = -\frac{1}{6}(\frac{2\partial_r^2 k}{k^{4/3}} + \frac{6\partial_r k}{r k^{4/3}} - \frac{(\partial_r k)^2}{k^{7/3}}) \tag{3.40}$$

In the near $r = 0$ region the function k behaves as $k \sim r^{-4}$ and thus the scalar

$$R \sim r^{-2/3} \tag{3.41}$$

The Ricci scalar diverges in the $r = 0$. The solution of the 2-charge D1-D5 system then will have a naked singularity without a macroscopic horizon. We can change that by the alternation of the function of k. Specifically, the addition of an extra charge with a Z function similar to the others will provide a dependence of

$$k \sim r^{-6} \implies R \sim r^0 \sim c \tag{3.42}$$

where c a constant. The curvature now is finite and the solution regular.

To preserve a better understanding of the geometry, we consider the decoupling limit in the region near the singularity for the full metric of (3.34). It is frequent to use for this limit

$$a' \to 0, \quad \rho = \frac{r}{a'} fixed, \quad R_y, \frac{v}{a'^2} fixed \tag{3.43}$$

We can neglect the ones of the Z functions in this limit and the metric takes the form (in a' units)

$$ds^2 = \frac{\rho^2}{\sqrt{Q_1 Q_5}}(-dt^2 + dy^2) + \frac{\sqrt{Q_1 Q_5}}{\rho^2}(d\rho^2 + \rho^2 d\Omega_3^2) + \sqrt{\frac{Q_1}{Q_5}} ds^2(T^4) \tag{3.44}$$

and with the proper coordinate transformation

$$\rho \to \frac{R_y}{\sqrt{Q_1 Q_5}}\rho, \quad t \to \frac{t}{R_y}, \quad y \to \frac{y}{R_y} \tag{3.45}$$

we have the final form of

$$ds^2 = \sqrt{Q_1 Q_5}\left(\rho^2(-dt^2 + dy^2) + \frac{d\rho^2}{\rho^2} + d\Omega_3^2\right) + \sqrt{\frac{Q_1}{Q_5}}ds^2(T^4) \tag{3.46}$$

This metric obviously corresponds to a $AdS_3 \times S^3 \times T^4$ spacetime geometry. Both the AdS_3 and S^3 parts acquire the same radius in the throat region with value $(Q_1 Q_5)^{1/4}$.

To sum up, we looked at a 2-charge case of black holes in type IIB supergravity. Starting from the simple construction of black branes from a previous subsection we wrote down the metric for a system of D1 and D5 branes and proceeded to the study of this solution at the critical region around the $r = 0$, i.e the center of the transverse space where the branes appear like point particles. After the reduction of the compact dimensions we observed an anomaly of the solution. The metric has a singularity at $r = 0$ and an absence of a macroscopic horizon, making the singularity to be naked. This fact, in combination with the metric of (3.46) suggests that the solution matches with the massless limit of the extremal BTZ black hole. In this limit the position of the singularity coincides with the horizon and the naked singularity appears. So the 3-charge solution will correspond to the massive extremal BTZ [10].

Although the 2-charge solution lacks of a macroscopic horizon it has an entropy of the form

$$S \sim \sqrt{N_1 N_5} \tag{3.47}$$

as we will discuss more extensively in Section 4. But the appearance of such an entropy should imply the existence of a horizon due to the relation (3.1). The simplest explanations to this are that either this 2-charge system is not a well-defined black hole solution or the metric (3.34) is not the proper one. In a way both of them are true. As we see later, this metric is just a mathematical tool for understanding black brane solutions and is not valid for real problems. Also, the D1-D5 system serves as a simple test for our theory. If anyone wants to deal with a solution that can be addressed to real problems and provide sufficient results, he should include the 3-charged D1-D5-P system, where P stands for a momentum wave. Someone can create it by adding momentum perturbations along the S^1 circle. This system is special because we can obtain from it the 5-dimensional RN black hole.

3.5 Type IIA Solutions

The above solutions consist only of branes from type IIB theory. These systems excite a lot of potentials like C_2, C_4... etc, making sometimes the derivation and formulation of the systems' equations tough. But we can simplify our computations by using simpler branes combinations which are dual to our system through T and S dualities. In this subsection I will record these dual systems of different number of conserved charges. The objects that we will work with are the $F1$ and $NS5$, thus the fundamental electric and magnetic objects of type IIA theory that are charged under the B field.

3.5.1 1-charge Solution

The first case is the 1-charge solution that is simple the fundamental string in ten dimensions.[11]

$$ds^2 = Z^{-1}(-dt^2 + dy^2) + ds^2(\mathbb{R}^8) \tag{3.48}$$

where the harmonic function and the dilaton are

$$Z = 1 + \frac{Q_1}{r^6}, \quad e^{2\Phi} = Z^{-1} \tag{3.49}$$

As before the y coordinate will be compactified in a circle S^1 with radius R_y. Notice that the radial function Z is the same with the D1-brane from (3.9) as we were expecting. This solution has also a zero area horizon at $r = 0$ and thus its entropy from (3.1) will be zero.

3.5.2 2-charge Solution

We move to the 2-charge solution by placing extra $NS5$ branes. To do this we need some compact dimensions so we compactify other four spacial dimensions on a T^4 torus and we also smear the strings on those directions. The metric for this F1-NS5 system is [11]

$$ds^2 = Z_1^{-1}(-dt^2 + dy^2) + Z_5 ds^2(\mathbb{R}^4) + ds^2(T^4) \tag{3.50}$$

where

$$Z_i = 1 + \frac{Q_i}{r^2}, \quad i = 1, 5. \qquad e^{2\Phi} = \frac{Z_5}{Z_1} \tag{3.51}$$

We can transform this solution in the D1-D5 one using the S-duality. The two systems are analogous and provide the same results for the entropy up to perhaps some constants and numerical factors. But this is not obvious from their metrics that can deceive us. The intuitive problem can be solved if we write both metrics (3.34),(3.50) in the Einstein frame.

To go from the string to the Einstein frame we use the dilaton field as

$$g^E_{\mu\nu} = e^{-\Phi/2} g^S_{\mu\nu} \tag{3.52}$$

starting from the (3.34) and using the (3.38) we find the metric of the D1-D5 in Einstein frame

$$ds^2_E = (Z_1^3 Z_5)^{-1/4}(-dt^2 + dy^2) + (Z_1^3 Z_5)^{1/4} ds^2(\mathbb{R}^4) + (\frac{Z_1}{Z_5})^{1/4} ds^2(T^4) \tag{3.53}$$

We take the same result if we use (3.52),(3.53) and (3.54). We are then convinced that the two systems are in a way different manifestations of the same effects. The same common properties like the zero value of the horizon at $r = 0$ which worried us in the D1-D5 description appear again. We proceed in our last case of a 3-charge system analogous to D1-D5-P by adding again momentum along the S^1 circle.

3.5.3 3-charge Solution

The metric for the 3-charged solution of F1-NS5-P is (we continue now to write our metrics in the string frame)[11]

$$ds^2 = Z_1^{-1}(-dt^2 + dy^2 + K(dt + dy)^2) + Z_5 ds^2(\mathbb{R}^4) + ds^2(T^4) \tag{3.54}$$

with K to be the harmonic function of the momentum wave

$$K = \frac{Q_p}{r^2} \tag{3.55}$$

The solution has the same value for the dilaton field with the 2-charge case before the addition of momentum. Also, this solution has a non-zero horizon area at $r = 0$ thanks to the momentum that stabilizes the circle as we shrink it.

We observe that as we added a charge and obtained a higher charged solution, we can go all the way around and lower our solution to a 2-charge one. This can be done by just eliminating the number of NS5 branes. Suddenly our system features N_1 fundamental strings or one string winded N_1 times around the circle and N_p momentum waves in the same circle on the y direction. It is clear that the metric is

$$ds^2 = Z_1^{-1}(-dt^2 + dy^2 + K(dt + dy)^2) + ds^2(\mathbb{R}^4) + ds^2(T^4) \tag{3.56}$$

We then have the simplest system possible from our theory, the F1-P, which can be mapped in the D1-D5 geometry with the correct dualities.

3.6 Mapping F1-P to D1-D5

In the next section we will try to construct the microstate geometries for the 2-charge F1-P and then translate them to the D1-D5 black hole. The results will depend on the charges, the volume of the torus and the radius of the compact circle. These quantities will obviously change under the dualities. So here I will very briefly mention the dualities for mapping the two systems and discuss about the conservation of the charges.

The direction of our circle S^1 will be the $y \equiv x_5$ where the F1 is wrapped and the momentum travels. The dimensions of x_i, $i = 6, 7, 8, 9$ are the directions of the T^4 torus. The procedure can be routed as

$$\begin{pmatrix} F1(5) \\ P(5) \end{pmatrix} \overset{S}{\rightarrow} \begin{pmatrix} D1(5) \\ P(5) \end{pmatrix} \overset{T6789}{\rightarrow} \begin{pmatrix} D5(56789) \\ P(5) \end{pmatrix} \overset{S}{\rightarrow} \begin{pmatrix} NS5(56789) \\ P(5) \end{pmatrix} \overset{T5}{\rightarrow} \tag{3.57}$$

$$\begin{pmatrix} NS5(56789) \\ F1(5) \end{pmatrix} \overset{S}{\rightarrow} \begin{pmatrix} D5(56789) \\ D1(5) \end{pmatrix} \tag{3.58}$$

To understand each step of this process one can look at Appendix A where I provide some basic elements of T and S duality.

Someone could rush now and state that the new charges will be

$$Q_1' = Q_1, \quad Q_5' = Q_p \tag{3.59}$$

which is completely wrong. But if we keep track of the transformation of branes from the dualities we will find that

$$Q_1' = \mu^2 Q_p, \quad Q_5' = \mu^2 Q_1 \tag{3.60}$$

where the scaling factor μ came from the dualities [11].

Having shown the relation between F1-P and D1-D5 we can find and write a solution for the fundamental string that carries momentum and then transform it to a solution for the D1-D5 case by just rescaling it.

4 Entropy And Microstates Of Black Holes

The previous section has provided us with all the necessary tools for studying black holes within the framework of type II string theory and especially of supergravity. We found new solutions of black holes using D-branes, called black branes, and even combinations of them preserving the supersymmetry. These systems, particularly the D1-D5 and the D1-D5-P systems, are some of the most promising structures in string theory that may correspond to physical macroscopic BPS black holes.

The first thing that we do in this section is to use these systems and calculate the Bekenstein-Hawking entropy we introduced in (3.1). But as we stated there, this entropy oblige a thermodynamical/statistical description. In this description, the entropy is related to some kind of microstates of the system. Unfortunately, this cannot be observed macroscopically resulting to the information paradox.

The second part of this section will cover the research of the microstates. We start from the 2-charge black hole and find a class of microstate geometries that belong to the fuzzball proposal. Then, we use the previous solutions to expand in the 3-charge black hole and obtain a similar family of microstate geometries, the called superstrata.

4.1 Entropy of the 3-charge Black Hole

The 3-charge solution of the previous chapter has a horizon at $r = 0$ and thus we can calculate its Bekenstein-Hawking entropy. We start from the metric (3.54) and take the limit of $r \to 0$

$$ds^2 = \frac{r^2}{Q_1}(-dt^2 + dy^2 + K(dt + dy)^2) + Q_5(\frac{dr^2}{r^2} + d\Omega_3^2) + ds^2(T^4) \tag{4.1}$$

And now we must find the area of the horizon. We must be carefully to convert the area to the right frame. While our metric (3.54) is written in the string frame, the (3.1) is referring to the Einstein frame. We will find the area using our metric and it is given by

$$A^S = A_{S^3} \cdot L_y \cdot V_{T^4} \tag{4.2}$$

where A_{S^3} is the surface of the 3-sphere, L_y the length of the S^1 compact circle at the horizon limit and V_{T^4} the volume of the torus.

The 3-sphere has a radius of $\sqrt{Q_5}$. The surface of a n-dimensional sphere is given by

$$A_{S^n} = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} R^n \tag{4.3}$$

So we have

$$A_{S^3} = 2\pi^2 Q_5^{3/2} \tag{4.4}$$

We saw the volume of the torus when we wrote the D1-D5 solution

$$V_{T^4} = (2\pi)^4 V \tag{4.5}$$

The only left is the length of the circle as we approach r to zero. For that we must look the behaviour of the dy^2 term in our metric. This term has a factor of

$$\frac{r^2}{Q_1} + \frac{Q_p}{Q_1} \overset{r\to 0}{\to} \frac{Q_p}{Q_1} \tag{4.6}$$

The lack of the $\frac{Q_p}{Q_1}$ term will give a zero-area horizon. As a consequence of that, the entropy of the 2-charge black hole would be zero as the radius and thus the perimeter of the circle would be eliminated. Perform the transformation

$$y \to \sqrt{\frac{Q_p}{Q_1}} y \implies R_y \to \sqrt{\frac{Q_p}{Q_1}} R_y \tag{4.7}$$

and the length of the circle is

$$L_y = 2\pi R_y \sqrt{\frac{Q_p}{Q_1}} \tag{4.8}$$

Combining all of them together we have

$$A^S = 2^6 \pi^7 R_y V Q_1^{-1/2} Q_5^{3/2} Q_p^{1/2} \tag{4.9}$$

We remember the expression for the dilaton field from (3.51) and the way of changing frames from (3.52). We then get in the limit of the horizon

$$g^E_{\mu\nu} = (\frac{Q_1}{Q_5})^{1/4} g^S_{\mu\nu} \tag{4.10}$$

The two areas then are connected by the form

$$A^E = (\frac{g^E}{g^S})^4 A^S \implies A^E = 2^6 \pi^7 R_y V \sqrt{Q_1 Q_5 Q_p} \tag{4.11}$$

With such simple steps we have shown that the entropy is proportional to the square root of the charges from the branes and the momentum modes

$$S_{BH} \sim \sqrt{Q_1 Q_5 Q_p} \tag{4.12}$$

Moreover, we know that these charges are proportional to the number of each one

$$Q_1 \sim N_1, \quad Q_5 \sim N_5, \quad Q_p \sim N_p \tag{4.13}$$

up to numerical factors related to units and string theory. Someone can find these factors and the value of G_{10} which is the Newtons' constant of gravity in ten dimensions and substitute to the (3.1) and get

$$S_{BH} = 2\pi \sqrt{N_1 N_5 N_p} \tag{4.14}$$

Note that it depends only on the number of our elements that consist our system. This number does not change under the dualities. Thus, this entropy is also valid for the D1-D5-P black hole. And if we were about to take the case of making a physical black hole we expect this number to be really high. But does this result agree with the micro-statistical description of such a system of branes?

4.2 Microscopic Entropy

4.2.1 F1-P System

We study again the 2-charge system of F1-P (3.56) where we have only one elementary string that winds the S^1 circle N_1 times. Upon that string, we add an extra charge of momentum. In order to be bounded this momentum will travel as a transverse wave with the speed of light. There is a variety of possible ways for that momentum to travel along the string. We approach the calculation of the degeneracy for this system using statistical analysis.

There are a lot of different paths that someone can follow for counting these states. For our approach we can simulate the momentum waves as a 1-dimensional massless gas. This gas lies inside a 1-dimensional box of the string's length $L = 2\pi R_y N_1$, where R_y is the radius of the compact circle. We can also take our waves to be only left-moving. So all the elements of the gas travels in the same direction. Then the total energy for the gas is the same with the total momentum of the waves given by

$$E = P = \frac{N_p}{R_y} = \frac{2\pi N_1 N_p}{L} \tag{4.15}$$

The vibrations of the string can be in eight dimensions. A vibration in each dimension can be thought as an oscillator. Then we have in total eight bosonic and eight fermionic degrees of freedom for our gas. The two types of degrees of freedom (bosonic, fermionic) are of the same number that results from supersymmetry. Keeping that in mind we write the partition function for the system. The partition function is given as

$$Z = \sum_{states} e^{-\beta E_{state}} \tag{4.16}$$

where the constant β is the inverse of the "temperature" if we take $k_B = 1$ (Boltzmann's constant).

For each direction we can translate the vibrations into Fourrier modes. The energy of each mode is $e_k = \frac{2\pi k}{L}$. For the bosonic and fermionic parts of each mode we have

$$\begin{aligned} Z_k^B &= \sum_{m=0}^{\infty} e^{-\beta m e_k} = \frac{1}{1 - e^{-\beta e_k}} \\ Z_k^F &= \sum_{m=0}^{1} e^{-\beta m e_k} = 1 + e^{-\beta e_k} \end{aligned} \tag{4.17}$$

The m here stands as a counter of the times each mode is appeared. We call it occupation number. We see that for a fermion can be zero or one. This is an implication from the principle that states of two fermions of the same kind cannot occupy a state with the same energy and the same quantum numbers. But bosons can and their counter can take all values from zero to infinity. Writing the natural logarithm of the full partition function for all modes we can replace the sum with an integral if we have a high number of modes.

$$Z = (Z^B Z^F)^8 \implies lnZ = 8(lnZ^B + lnZ^F) \tag{4.18}$$

For the bosonic part

$$\begin{aligned} Z^B &= \sum_k Z_k^B \to \int Z^B(k) dk \to \int Z^B(e_k) \frac{L}{2\pi} de_k \implies \\ lnZ^B &= -\frac{L}{2\pi} \int_0^{\infty} de_k \quad ln(1 - e^{-\beta e_k}) = \frac{L\pi}{12\beta} \end{aligned} \tag{4.19}$$

In a similar way for the fermionic part

$$lnZ^F = \frac{L}{2\pi} \int_0^{\infty} de_k \quad ln(1 + e^{-\beta e_k}) = \frac{L\pi}{24\beta} \tag{4.20}$$

Combining the two parts

$$lnZ = \frac{L\pi}{\beta} = (8+4)\frac{L\pi}{12\beta} = (f_B + \frac{1}{2} f_F)\frac{L\pi}{12\beta} \equiv c(\frac{L\pi}{12\beta}) \tag{4.21}$$

where the f_B and f_F are the bosonic and fermionic degrees of freedom.

We can determine the β and the entropy using the known statistical relations for the thermodynamical quantities.

$$E = -\partial_\beta lnZ = \frac{cL\pi}{12\beta^2} \implies \beta = \sqrt{\frac{cL\pi}{12E}} \tag{4.22}$$

and for the entropy we have that

$$S = lnZ + \beta E = \frac{c\pi L}{6\beta} = \frac{c\pi L}{6}\sqrt{\frac{12E}{c\pi L}} = 2\sqrt{\pi LE} \tag{4.23}$$

where from the equation (4.15)

$$S_{micro} = 2\sqrt{2}\pi\sqrt{N_1 N_p} \tag{4.24}$$

To our surprise we found a non-zero entropy for the 2-charge system for which we have no horizon and thus $S_{BH} = 0$. One could say that the previous computation of the S_{micro} is wrong or it has no correlation with the Bekenstein-Hawking entropy. As we need to be assured that the above procedure is correct we compute this entropy for the 3-charge solution.

4.2.2 F1-NS5-P System

In addition to the above system, we have some NS5 living in another four transverse directions to the F1 and intersecting with them in the y direction of the circle. Due to the occupation of extra dimensions the number of the degrees of freedom will become $c = 6$ consisting of four bosonic and four fermionic. They relate to the four compact dimensions of the NS5 (T^4). The string cannot oscillate to the other four dimensions ($\mathbb{R}^4$) because the string is stuck in the NS5.

If we had this number c in the previous calculations we would find

$$S = 2\pi\sqrt{N_1 N_p} \tag{4.25}$$

which is the entropy if we had only one NS5, $N_5 = 1$. Moving towards the general case of $N_5 > 1$ we remember that the systems F1-P and NS5-F1 are connected through dualities shown in the (3.57). We will use a slightly different way for counting the current microstates.

Having the F1-NS5 system from the duality

$$F1(N_1) \quad P(N_p) \rightarrow NS5(N_1) \quad F1(N_p) \tag{4.26}$$

where the number of our objects did not changed, we take the simple case of $N_5' = N_1 = 1$. We have only one NS5 and we want to place an amount of elementary strings upon it. But these strings can be placed with different ways. The strings are winded in the circle with coordinate y. Immediately there are two possible microstates. There can be N_p strings wounded only one time or one string winded N_p times. Beyond them we can have any possible combination of strings and winding numbers. But their sum must always be equal to N_p.

$$\sum_i m_i k_i = N_p \tag{4.27}$$

where m_i is the number of strings and k_i the winding number for each one. If now are more than one NS5 we have $N_1 N_p$ ways of placing the strings

$$\sum_i m_i k_i = N_1 N_p \tag{4.28}$$

Statistics implies that the total number of the microstates will come from the number of partitions of the $N_1 N_p$ as

$$N_{micro} = e^{2\sqrt{2}\pi\sqrt{N_1 N_p}} \tag{4.29}$$

Notice that from here we obtain (4.24).

Rename $N_1 \to N_5$ and $N_p \to N_1$ for the 3-charge case. We take this bound state and we add momentum in the form of waves. The N_p units of momentum can be distributed along the distinct strings. Following the previous technique and taking into consideration that now we have $c = 6$ for the degrees of freedom we have

$$S_{micro} = ln[N_{micro}] = 2\pi\sqrt{N_1 N_5 N_p} \tag{4.30}$$

This is the microscopic entropy of a 3-charge BPS system with left-moving momentum. To our relief, this entropy agrees with the one we calculated in the 4.1 subsection.

$$S_{micro} = S_{BH} \tag{4.31}$$

for the 3-charge F1-NS5-P system, dual to the D1-D5-P. Our microscopic approach then is valid. This implies a flaw in the 2-charge F1-P system. For this system, we found a microscopic entropy but we know that it cannot have a S_{BH} due to the absence of horizon. This obliges us to go back to the 2-charge system and look closely the gravitational solution. We have to deal with the absence of horizon that leads to a naked singularity, making this system a nonphysical black hole.

4.3 Fuzzball Proposal

As far we know, gravity is an attractive force for all bodies. The magnitude of this force can be measured and it depends on a constant called Newton's gravitational constant G_N. This in turn depends on our spacetime. It is different if we have four or ten dimensions. Moreover, in string theory is related to the string length (l_s or a') and the string coupling (g). For the purpose of our discuss we are interested in the relation with the coupling. We have that the gravitation constant is proportional to the square of the string coupling constant

$$G_N \sim g^2 \tag{4.32}$$

Looking back to the gravitational analysis, we see that for an increase in the value of G_N each object becomes smaller except the horizon radius. As we increase G_N the radius becomes larger. Take for example a star. If we increase the gravity, the radius of the star will start becoming smaller and the horizon bigger. This will happen until a point where they will coincide and the star will collapse into a black hole. Under this idea we can create black holes by taking configurations of branes at a strong limit (increasing string coupling). The branes and strings of our system lies now behind the horizon close to the singularity. But still we have the problem of the information paradox. As the branes sit behind the horizon we cannot have any information about the initial state of the system. A modification of this idea should be provided in order to address this problem.

The proposal now is that as we increase the coupling we do not form a horizon. The components of the system grow at size and they acquire the same size of the analogous horizon. Starting from each microstate we will obtain the same object with the exact horizon size. Hawking radiation contains now the information of the branes due to the lack of a horizon. These solutions look like the black hole asymptotically, up to the horizon scale. In order then to observe a well behaved black hole without a naked singularity, our geometries are regular in their center. It will be shown below, when we construct one such type of geometries, how we eliminate the singularity and obtain a regular solution for every point in spacetime.

Even though we are not sure if this proposal is true, it is a very promising candidate for explaining the mysterious form of black holes. Next we will construct a type of fuzzballs for the 2-charge and 3-charge case. These microstate geometries will be horizon-less and non-singular. Furthermore, the area in the calculation of the Bekenstein-Hawking entropy corresponds to the boundary area of the fuzzball.

4.4 F1-P Revisited

We remember that the metric of this system is given by the (3.56) and is dual to the D1-D5 2-charge system. To find a family of fuzzballs for the 2-charge black hole we shall be able to use the simplest system we have that is not other than the F1-P. But we should change some of the features for this solution and modify it. If we find the type of solution that we seek, we can straightforward expand our research for the 3-charge black hole by applying small perturbations (momentum waves).

First we introduce light-cone coordinates

$$u = \frac{t-y}{\sqrt{2}}, \quad v = \frac{t+y}{\sqrt{2}} \tag{4.33}$$

and the metric (3.56) transforms to

$$\begin{aligned} ds^2 &= 2Z^{-1}[-dudv + Kdv^2] + \sum_{i=1}^{4} dx_i^2 + \sum_{a=1}^{4} dz_a^2 \\ Z &= 1 + \frac{Q_1}{r^2}, \quad K = \frac{Q_p}{r^2} \end{aligned} \tag{4.34}$$

where x_i are the non-compact dimensions and z_a the compact ones. The dilaton and the NS field are

$$e^{2\Phi} = Z^{-1}, \quad B_{uv} = -\frac{1}{2}[Z^{-1} - 1] \tag{4.35}$$

This is the naive metric for the F1-P BPS bound system. The most important point in our calculations is the fact that the momentum is carried only on the transverse directions, longitudinal vibrations do not occur. As the momentum travels through these directions the string should bend away from the point of $r = 0$. So the string will not be a point particle in these dimensions anymore. The movements of the string can be then parametrized by a vibration profile $\vec{g}(v')$. The metric after the intersection with the non-compact dimensions becomes [11]

$$\begin{aligned} ds^2 &= \frac{2}{Z}[-dudv + Kdv^2 + \sqrt{2}A_i dx^i dv] + \sum_{i=1}^{4} dx_i^2 + \sum_{a=1}^{4} dz_a^2 \\ B_{uv} &= -\frac{1}{2}[Z^{-1} - 1], \quad B_{ui} = Z^{-1}A_i \\ Z(\vec{x}, t, y) &= 1 + \frac{Q_1}{|\vec{x} - \vec{g}|^2}, \quad K(\vec{x}, t, y) = \frac{Q_1|\dot{\vec{g}}|^2}{|\vec{x} - \vec{g}|^2} \\ A_i(\vec{x}, t, y) &= -\frac{Q_1|\dot{g}_i|^2}{|\vec{x} - \vec{g}|^2}, \quad i = 1, ..., 4 \end{aligned} \tag{4.36}$$

Each winding of the string can also carry different types of vibration profiles. Yet, given that every strand of the string is BPS, we can superpose the functions for each strand and write the general solution which has the same metric and fields with (4.36) but with different functions as

$$Z \to Z(\vec{x}, t, y) = 1 + \sum_s \frac{Q_1^{(s)}}{|\vec{x} - \vec{g}^{(s)}|^2} \tag{4.37}$$

and in a similar way for the K and A_i. Finding an exact solution to this problem is really difficult and most times impossible as we have N_1 strands. The difficulty lies in the existence of the sum over the different vibration profiles. Nevertheless, there is a way to get rid of this sum. By getting the so called black hole limit we practically increase the number of the winding and momentum waves

$$N_1, N_p \to \infty \tag{4.38}$$

In this limit we can change the sum into an integral. We followed a similar way in the computation of the partition function of the microscopic entropy for the 2-charge system.

$$\sum_{s=1}^{N_1} \to \int_{s=0}^{N_1} ds = \int_0^{2\pi R_y N_1} \frac{ds}{dy} dy = \int_0^L \frac{dy}{2\pi R_y} \to \int_0^L dv \tag{4.39}$$

up to a numerical factor. Using the description with the integrals and following the dualities from 3.57 we write the metric and the functions for the D1-D5 (in the form of Mathur's review [11])

$$\begin{aligned}
ds^2 &= \frac{1}{\sqrt{Z(1+K)}}[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2] + \sqrt{Z(1+K)} \sum_{i=1}^{4} dx_i^2 + \sqrt{\frac{1+K}{Z}} \sum_{a=1}^{4} dz_a^2 \\
Z &= 1 + \frac{\mu Q_1}{L} \int_0^{\mu L} \frac{dv'}{|\vec{x} - \mu \vec{g}(v')|^2}, \quad K = \frac{\mu Q_1}{L} \int_0^{\mu L} \frac{dv' (\mu^2 \dot{g}(v'))^2}{|\vec{x} - \mu \vec{g}(v')|^2} \\
A_i &= -\frac{\mu Q_1}{L} \int_0^{\mu L} \frac{dv' (\mu \dot{g}_i(v'))}{|\vec{x} - \mu \vec{g}(v')|^2}
\end{aligned} \tag{4.40}$$

where we saw the μ factor in the 3.6, the B_i is relating to the Hodge dual of A_i

$$dB = - \star_4 dA \tag{4.41}$$

and Q_1 is the charge of the F1 string. The above metric simplifies to the naive metric (3.34) in the correct large limit with the match

$$\begin{aligned}
Z &\to Z_5 \\
1 + K &\to Z_1
\end{aligned} \tag{4.42}$$

We will rewrite the above metric and functions in a more concrete and general way.

4.5 General Metric

From [12] we can write the most general metric for the D1-D5-P system which can also be reduced to the D1-D5 metric. We will follow the notation from [13]. This metric must be invariant under the rotations on T^4 and preserve all the necessary supercharges. Of course someone can easily replace T^4 with a $K3$ manifold but within the context of this project we will use only the torus. The metrics are [13]:

$$\begin{aligned}
ds_{10}^2 &= \frac{1}{\sqrt{\alpha}} ds_6^2 + \sqrt{\frac{Z_1}{Z_2}} d\hat{s}_4^2 \\
ds_6^2 &= -\frac{2}{\sqrt{\mathcal{P}}}(dv + \beta)[du + \omega + \frac{\mathcal{F}}{2}(dv + \beta)] + \sqrt{\mathcal{P}} ds_4^2
\end{aligned} \tag{4.43}$$

where the $d\hat{s}_4^2$ is the metric for the torus T^4 and the ds_4^2 is a non-trivial, v-dependent Euclidean metric in the four non compact dimensions of the spatial base, which is denoted as $\mathcal{B}$. The volume form of the torus is $\widehat{vol_4}$ and u,v are the light-cone coordinates we introduced earlier.

The potentials are [13]:

$$
\begin{aligned}
&e^{2\Phi} = \frac{Z_1^2}{\mathcal{P}}\\
&B = -\frac{Z_4}{\mathcal{P}}(du+\omega)\wedge(dv+\beta) + a_4\wedge(dv+\beta) + \delta_2\\
&C_0 = \frac{Z_4}{Z_1}\\
&C_2 = -\frac{Z_2}{\mathcal{P}}(du+\omega)\wedge(dv+\beta) + a_1\wedge(dv+\beta) + \gamma_2\\
&C_4 = \frac{Z_4}{Z_2}\widehat{vol}_4 - \frac{Z_4}{\mathcal{P}}\gamma_2\wedge(du+\omega)\wedge(dv+\beta) + x_3\wedge(dv+\beta) + \mathcal{C}\\
&C_6 = \widehat{vol}_4\wedge[-\frac{Z_1}{\mathcal{P}}(du+\omega)\wedge(dv+\beta) + a_2\wedge(dv+\beta) + \gamma_1]\\
&-\frac{Z_4}{\mathcal{P}}\mathcal{C}\wedge(du+\omega)\wedge(dv+\beta)
\end{aligned}
\tag{4.44}
$$

with the functions

$$
\alpha \equiv \frac{Z_1 Z_2}{\mathcal{P}}, \quad \mathcal{P} \equiv Z_1 Z_2 - Z_4^2 \tag{4.45}
$$

Lets explain the unknown quantities that appear. First the $Z_1, Z_2, Z_4, \mathcal{F}$ are scalar functions. We have the 1-forms: $\beta, \omega, a_1, a_2, a_4$, the 2-forms: $\gamma_1, \gamma_2, \delta_2$ and the 3-form x_3 upon the base $\mathcal{B}$. We also have the 4-form $\mathcal{C}$ but it can be set equal to zero using an appropriate gauge. They all depend in general on the coordinates of $\mathcal{B}$ and on v. From now on we will choose the base space to be the $\mathbb{R}^4$. To understand where all these functions and forms came from someone should use Generalised Geometry and find the restricted spinorial structure for the D1-D5-P. But this is beyond our interest for the moment, so we will not address it any further. When is needed I will just provide the set of equations related to the analogous quantities.

If we look closely to the (4.43) we see a resemblance with the (4.40) when we have $\mathcal{F} = 0$. For $\frac{\mathcal{F}}{2} = -\frac{Q_p}{r^2}$ we can get the naive metric for the D1-D5-P. It is true that from the general metric of (4.43) we can obtain the D1-D5 general metric and find the functions and forms appeared in regard to the vibration profiles we introduced. In the next subsection we will produce a microstate geometry for the D1-D5 black hole which we will use as a seed to a more general family of fuzzballs for the 3-charge case, the superstrata. To do this we express the scalar functions and the potentials in respect of the vibrations profile [13]:

$$
\begin{aligned}
&Z_2 = 1 + \frac{Q_5}{L}\int_0^L \frac{dv'}{|\vec{x}-\vec{g}(v')|^2}, \quad Z_4 = -\frac{Q_5}{L}\int_0^L \frac{\dot{g}_5(v')}{|\vec{x}-\vec{g}(v')|^2}dv'\\
&Z_1 = 1 + \frac{Q_5}{L}\int_0^L \frac{|\dot{\vec{g}}|^2}{|\vec{x}-\vec{g}(v')|^2}dv', \quad d\gamma_2 = \star_4 dZ_2, \quad d\delta_2 = \star_4 dZ_4\\
&A_i = -\frac{Q_5}{L}\int_0^L \frac{\dot{g}_i}{|\vec{x}-\vec{g}(v')|^2}dv', \quad dB = -\star_4 dA\\
&\beta = \frac{-A+B}{\sqrt{2}}, \quad \omega = \frac{-A-B}{\sqrt{2}}, \quad Q_1 = \frac{Q_5}{L}\int_0^L |\dot{\vec{g}}|^2 dv'
\end{aligned}
\tag{4.46}
$$

Now the vibration profile has five directions, we will explain that in a moment. With all the above we are fully equipped to go and find a class of microstate geometries (fuzzballs) for the 2-charge case.

4.6 Microstate Geometries from Circular Profile

We will use the so called circular vibration profile where the non-vaniching components are

$$g_1(v') = a\cos(\frac{2\pi}{L}v'), \quad g_2(v') = a\sin(\frac{2\pi}{L}v'), \quad a > 0 \tag{4.47}$$

where $L = \frac{2\pi Q_5}{R_y}$. The radius R_y is in the denominator now because of the T-duality. The other components of the profile are equal to zero ($g_3, g_4, g_5 = 0$). As you can see there is an extra component than before. The first four are the vibrations that correspond to the movement of the string in the $\mathbb{R}^4$. These do not break the rotational symmetry of the torus. Yet, is has been verified that if we use also a dimension from the torus we still have the rotational invariant solution. Thus, the g_5 is the element of the profile emerging from the parametrization of the one direction of the torus.

A good description of the $\mathbb{R}^4$ is in spherodial coordinates

$$\begin{aligned} &x_1 = \sqrt{r^2+a^2}\sin\theta\cos\phi, \quad x_2 = \sqrt{r^2+a^2}\sin\theta\sin\phi \\ &x_3 = r\cos\theta\cos\psi, \quad x_4 = r\cos\theta\sin\psi \end{aligned} \tag{4.48}$$

and the metric is

$$ds_4^2 = (r^2 + a^2\cos^2\theta)(\frac{dr^2}{r^2+a^2} + d\theta^2) + (r^2+a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2 \tag{4.49}$$

In these coordinates the $r = 0$ point describes a disk with radius a. The specific profile produces a "tube" or more exactly from literature a supertube geometry that lies in the perimeter of the disk ($r = 0, \theta = \pi/2$).

From the profile we can immediately deduce that

$$Z_4 = 0 \implies \delta_2 = 0 \tag{4.50}$$

For the forthcoming computations we will name

$$\Sigma = r^2 + a^2\cos^2\theta \tag{4.51}$$

So from (4.46) we have

$$\begin{aligned} &Z_2 = 1 + \frac{Q_5}{L}\int_0^L \frac{dv'}{x_3^2 + x_4^2 + (x_1 - a\cos(\frac{2\pi}{L}v'))^2 + (x_2 - a\sin(\frac{2\pi}{L}v'))^2} \xrightarrow{k=\frac{2\pi}{L}v'} \\ &Z_2 = 1 + \frac{Q_5}{2\pi}\int_0^{2\pi} \frac{dk}{a' + b'\cos k + c'\sin k}, \\ &a' = r^2 + a^2(1+\sin^2\theta), \quad b' = -2ax_1, \quad c' = -2ax_2 \end{aligned} \tag{4.52}$$

This integral has a solution for the condition

$$a'^2 > b'^2 + c'^2 \tag{4.53}$$

You can easily see that is true in our case. So,

$$Z_2 = 1 + \frac{Q_5}{2\pi}\frac{2\pi}{\sqrt{a'^2 - b'^2 - c'^2}} \implies Z_2 = 1 + \frac{Q_5}{\Sigma} \tag{4.54}$$

In a similar way we find for the other scalar function

$$Z_1 = 1 + \frac{Q_5 a^2 4\pi^2}{\Sigma L^2} \tag{4.55}$$

But if we compute the Q_1 from (4.46) we find

$$Q_1 = \frac{Q_5 a^2 4\pi^2}{L^2} \tag{4.56}$$

and the Z_1 can be written as

$$Z_1 = 1 + \frac{Q_1}{\Sigma} \tag{4.57}$$

Next we have to find the fields A, B. Starting from A, it has only two components for $i = 1, 2$.

$$A_1 = \frac{Q_5 a}{L} \int_0^{2\pi} \frac{\sin k}{a' + b' \cos k + c' \sin k} dk = R_y a^2 \frac{\sin\phi \sin\theta}{\Sigma\sqrt{r^2 + a^2}} \tag{4.58}$$

and in the same way

$$A_2 = -R_y a^2 \frac{\cos\phi \sin\theta}{\Sigma\sqrt{r^2 + a^2}}, \quad A_3 = A_4 = 0 \tag{4.59}$$

The A can be written using only one direction

$$A_\phi = A_1 \frac{\partial x_1}{\partial\phi} + A_2 \frac{\partial x_2}{\partial\phi} \implies A_\phi = -\frac{R_y a^2}{\Sigma} \sin^2\theta \tag{4.60}$$

where

$$R_y = \frac{\sqrt{Q_1 Q_5}}{a} \tag{4.61}$$

As the B can be evaluated from A from (4.46)

$$B_\psi = -\frac{R_y a^2}{\Sigma} \cos^2\theta \tag{4.62}$$

we have all we need in order to write the 1-forms and then the metric for our solution.

$$\beta = \frac{R_y a^2}{\sqrt{2}\Sigma}(\sin^2\theta d\phi^2 - \cos^2\theta d\psi^2), \quad \omega = \frac{R_y a^2}{\sqrt{2}\Sigma}(\sin^2\theta d\phi^2 + \cos^2\theta d\psi^2) \tag{4.63}$$

and the metric in the original coordinates is

$$\begin{aligned} ds_{10}^2 =& \frac{1}{\sqrt{\mathcal{P}}}(-dt^2 + dy^2) - \frac{2R_y a^2}{\sqrt{\mathcal{P}}\Sigma}(\sin^2\theta dt d\phi + \cos^2\theta d\psi dy) + \sqrt{\mathcal{P}}\Sigma(\frac{dr^2}{r^2 + a^2} + d\theta^2) \\ & + [\sqrt{\mathcal{P}}(r^2 + a^2) - \frac{R_y^2 a^4}{\sqrt{\mathcal{P}}\Sigma^2} \sin^2\theta] \sin^2\theta d\phi^2 + [\sqrt{\mathcal{P}} r^2 + \frac{R_y^2 a^4}{\sqrt{\mathcal{P}}\Sigma^2} \cos^2\theta] \cos^2\theta d\psi^2 \\ & + \sqrt{\frac{1 + \frac{Q_1}{\Sigma}}{1 + \frac{Q_5}{\Sigma}}} d\hat{s}_4^2 \end{aligned} \tag{4.64}$$

where

$$\mathcal{P} = Z_1 Z_2 = (1 + \frac{Q_1}{\Sigma})(1 + \frac{Q_5}{\Sigma}) \tag{4.65}$$

For very large r this metric goes over to flat space. The mixed terms of the metric signify an existence of angular momentum. This momentum stops the system from collapse from the gravitational attraction between the D1 and D5 branes when they are in the supertube geometry. Remember that this geometry is like a hyper-cylinder.

Furthermore, we study the decoupling limit in the area when

$$r^2 + a^2 \to 0 \implies r, a \to 0 \tag{4.66}$$

Then we have

$$\sqrt{\mathcal{P}} \sim \frac{\sqrt{Q_1 Q_5}}{\Sigma} \tag{4.67}$$

Also, we change the angles

$$\phi \to \phi + \frac{t}{R_y}, \quad \psi \to \psi + \frac{y}{R_y} \tag{4.68}$$

After substitution we find that the mixed terms cancel out and we are left with the metric

$$\begin{aligned} ds^2 = \sqrt{Q_1 Q_5}(-\frac{r^2 + a^2}{a^2 R_y^2} dt^2 + \frac{r^2}{a^2 R_y^2} dy^2 + \frac{dr^2}{r^2 + a^2}) \\ + \sqrt{Q_1 Q_5}(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2) + \sqrt{\frac{Q_1}{Q_5}} d\hat{s}_4^2 \end{aligned} \tag{4.69}$$

The first three terms are the AdS_3 spacetime in huper-polar coordinates. The full geometry is of $AdS_3 \times S^3 \times T^4$ with radius after the reduction upon the torus of $\mathcal{R} = (Q_1 Q_5)^{1/4}$. It is the same with the (3.46) obtained from the decoupling limit of the naive metric of (3.34) but it has no singularity at $r = 0$. Now instead of an infinite throat at the singularity at $r = 0$ we end up in a smooth cap at the end of the throat [14],[15]. We have created a type of geometries which are non-singular, horizonless and preserve the appropriate supercharges. Thus, they belong to the family of fuzzballs. In specific, the geometry created from the exact circular profile is called a Kaluza-Klein supertube monopole [16], [17]. We state here that we can create different geometries depending on the profile. Different profiles will produce different types of caps.

4.7 Size of Fuzzball and Entropy

We said that the previous metric is a more realistic interpretation of the 2-charge black hole. Also, it becomes the flat metric at long radius r as the naive metric (3.34) does. So it is reasonable to say that there must exist an area where the metric (4.40) becomes the naive metric (3.34) with the proper limit. To find it we look again at the dual system of F1-P.

The typical wavelength of the vibration from a strand of the string is [11]

$$\lambda \sim R_y \sqrt{\frac{N_1}{N_p}} \tag{4.70}$$

The transverse coordinates shift by a small change of

$$\Delta x = |\dot{\vec{g}}| \Delta y \sim |\dot{\vec{g}}| \lambda \tag{4.71}$$

From (4.46)

$$Q_p \sim Q_1 |\dot{\vec{g}}|^2 \tag{4.72}$$

$$\implies \Delta x \sim \sqrt{\frac{Q_p}{Q_1}} R \sqrt{\frac{N_1}{N_p}} \sim \sqrt{\alpha'} = l_s \tag{4.73}$$

For $r > \sqrt{\alpha'}$ the metric settles down to the naive metric of the 2-charge case. This surface is the boundary of our fuzzball. The computation of this area will provide a very interesting result.

At $r = \sqrt{\alpha'}$ in the naive metric of (3.56) we roughly have:

$$\begin{aligned} A_{S^3} \sim \alpha'^{3/2}, \quad V_{T^4} \sim V \\ L_y \sim R_y \sqrt{\frac{K}{Z_1}}|_{r=\sqrt{\alpha'}} \sim R_y \sqrt{\frac{Q_p}{Q_1}} \end{aligned} \tag{4.74}$$

where we did the approximation that

$$Z_1, K = 1 + \frac{Q_i}{l_s} \sim \frac{Q_i}{l_s} \tag{4.75}$$

Then we have after the multiplication with the dilaton as we did in the 4.1 (we must compute the area at the Einstein frame)

$$A^E \sim \alpha'^4 \sqrt{N_1 N_p} \quad (e^{-2\Phi} \sim \frac{Q_1}{\alpha'}, g = 1) \tag{4.76}$$

Finally, if we divide with the gravitational constant G_N for the 10-dimensional spacetime

$$\frac{A^E}{G_N} \sim \sqrt{N_1 N_p} \sim S_{micro} \tag{4.77}$$

This is a quite fascinating result. The area of the boundary surface of our fuzzball produces a Bekenstein-Hawking type entropy that agrees with the microscopic entropy computed from the same system. The fuzzball proposal is valid within the energy limits we have discussed and gives the correct results. The boundary area is called now 'horizon' of the fuzzball.

4.8 Superstrata

The two previous subsections showed the methodology to construct and analyze a 2-charge type of microstate geometries for a specific vibration profile. Especially, we were able to fabricate such geometries, that have all the expected properties of the fuzzball proposal, using a circular profile. There is now the question if we can create a similar family of microstate geometries for the 3-charge black hole that will also constitutes a class of fuzzballs. It was first shown in [18] that such geometries really exist and can be generated from the 2-charge supertube solution of (4.64).

The starting point is the supertube solution of the $\frac{1}{4}$-BPS D1-D5 system that has eight supercharges and are parametrized by functions of only one variable. Adding momentum units as deformations of the originally maximally rotating supertube will break the symmetry into $\frac{1}{8}$-BPS and make the functions that parametrize it to be continuous and rely on (at least) two variables. The functions are obtained by solving the BPS conditions that we can gather in three groups ("layers"). These conditions emerge from the enforcement of the preservation of supersymmetry.

4.8.1 Zeroth Layer

This layer fixes on the base space and provides the equations for the β 1-form. The condition is the u-independence. The base space must be in general an almost hyper-Kähler space. For the purposes of our interest we have chosen our base space $\mathcal{B}$ to be the $\mathbb{R}^4$. The choice of the Euclidean flat space makes us define a covariant derivative of the form

$$\mathcal{D} \equiv d - \beta \wedge \frac{\partial}{\partial_u} \tag{4.78}$$

with d to be the exterior derivative on the spatial space. Moreover, we choose the β to be v-independent which implies

$$d\beta = \star_4 d\beta \tag{4.79}$$

4.8.2 First Layer

This layer of equations are responsible for the characterization of the wrap functions Z_i functions and the 2-forms Θ_i fields sourced by the branes. These fields can be written in regard to the known forms [12]:

$$\Theta_1 \equiv \mathcal{D}a_1 + \dot{\gamma}_2, \quad \Theta_2 \equiv \mathcal{D}a_2 + \dot{\gamma}_1, \quad \Theta_4 \equiv \mathcal{D}a_4 + \dot{\delta}_2 \tag{4.80}$$

Then the equations are [13]:

$$\begin{aligned} \star_4 \mathcal{D}\dot{Z}_1 = \mathcal{D}\Theta_2, \quad \mathcal{D}\star_4 \mathcal{D}Z_1 = -\Theta_2 \wedge d\beta, \quad \Theta_2 = \star_4\Theta_2 \\ \star_4 \mathcal{D}\dot{Z}_2 = \mathcal{D}\Theta_1, \quad \mathcal{D}\star_4 \mathcal{D}Z_2 = -\Theta_1 \wedge d\beta, \quad \Theta_1 = \star_4\Theta_1 \\ \star_4 \mathcal{D}\dot{Z}_4 = \mathcal{D}\Theta_4, \quad \mathcal{D}\star_4 \mathcal{D}Z_4 = -\Theta_4 \wedge d\beta, \quad \Theta_4 = \star_4\Theta_4 \end{aligned} \tag{4.81}$$

4.8.3 Second Layer

The second and final layer determines the $\mathcal{F}, \omega$ forms that are connected with the momentum and the angular momentum respectively. They are [13]:

$$\begin{aligned} &\mathcal{D}\omega \star_4 \mathcal{D}\omega + \mathcal{F}d\beta = Z_1\Theta_1 + Z_2\Theta_2 - 2Z_4\Theta_4 \\ &\star_4 \mathcal{D}\star_4 (\dot{\omega} - \frac{1}{2}\mathcal{D}\mathcal{F}) = \partial_v^2(Z_1Z_2 - Z_4^2) - (\dot{Z}_1\dot{Z}_2 - (\dot{Z}_4)^2 - \frac{1}{2}\star_4 (\Theta_1\wedge\Theta_2 - \Theta_4\wedge\Theta_4) \end{aligned} \tag{4.82}$$

4.8.4 Adding the Extra Charge

We shall find a way to add the third charge in the form of momentum units. It can be achieved if we add another component to the g profile. Indeed this works, but not for any component. It must be the component in the direction of the torus. Then we have the circular profile with the extra component of

$$g_5(v') = -\frac{b}{k}\sin(\frac{2\pi k}{L}v') \tag{4.83}$$

where k is a positive integer and b an arbitrary parameter like a.

The 1-forms β,ω and the Z_2 remain unchanged. Also, it still is $\mathcal{F} = 0$. On the other hand we have a non-zero Z_4 and an extra term in the Z_1 function.

$$\begin{aligned} Z_1 &= 1 + \frac{R_y^2}{2Q_5}[\frac{2a^2+b^2}{\Sigma} + b^2a^{2k}\frac{\sin^{2k}\theta\cos(2k\phi)}{(r^2+a^2)^k\Sigma}] \\ Z_4 &= R_y b a^k \frac{\sin^k\theta\cos(k\phi)}{(r^2+a^2)^{k/2}\Sigma} \end{aligned} \tag{4.84}$$

The relation now for the parameters a,b with the radius and the charges is

$$R_y = \sqrt{\frac{Q_1Q_5}{a^2 + \frac{b^2}{2}}} \tag{4.85}$$

We observe that for a fixed radius and charges we have a family of 2-charge solutions with a free parameter b/a. If we set $b = 0$ we obtain the supertube solution (4.64).

Applying the solution generating technique used in [19] in order to produce the 3-charge solution we get

$$Z_4^{(k,m)} = R_y\frac{\Delta_{k,m}}{\Sigma}\cos(m\frac{\sqrt{2}v}{R_y} + (k-m)\phi - m\psi) \tag{4.86}$$

where

$$\Delta_{k,m} \equiv (\frac{a}{\sqrt{r^2+a^2}})^k \sin^{k-m}\theta\cos^m\theta \tag{4.87}$$

Z_4 is a linear superposition of modes of the two integers k, m, $m \geq 0$.

4.8.5 General Solution

We can check that each mode solves the first layer of equations. Thus, we can write the general solution as a linear combination of all the modes (Fourier expansion) with arbitrary coefficients $b_{k,m}$:

$$Z_4 = R_y \sum_{(k,m)} b_{k,m} \frac{\Delta_{k,m}}{\Sigma} \cos \hat{v}_{k,m} \tag{4.88}$$

$$\Theta_4 = -\sqrt{2} \sum_{(k,m)} b_{k,m} m \Delta_{k,m} (r \sin(\theta) \Omega^{(1)} \sin \hat{v}_{k,m} + \Omega^{(2)} \cos \hat{v}_{k,m}) \tag{4.89}$$

where the $\Omega^{(i)}, i = 1, 2, 3$ are a basis of self-dual 2-forms on $\mathbb{R}^4$ and

$$\hat{v}_{k,m} \equiv m \frac{\sqrt{2}v}{R_y} + (k - m)\phi - m\psi + \eta_{km} \tag{4.90}$$

The modes of k, m are related to the ϕ, ψ coordinates so the Fourier coefficients will be functions of two variables. The terms with the dependence upon ϕ, ψ denote the deformations upon the supertube. The η_{km} are some phase constants, non-zero in general. As we were able to write the solution in a basis expansion parametrized by the $b_{k,m}$ coefficients we can expand in a similar way the other two pairs $(Z_i, \Theta_i), i = 1, 2$. In addition, the constant b is a combination of the coefficients.

We found a set of solutions that solve the first layer. Next step is the imposition of the second layer and the regularity of the total geometry.

Because the source terms in the second-layer conditions are quadratic in the first-order fields, the expansion will be of the form

$$\sim \sum_{(k,m),(k',m')} b_{k,m} b_{k',m'} (\cdots) \tag{4.91}$$

Therefore, someone should solve only the second-layer equations for each pair $(k, m), (k', m')$. Once done that, we can write the $\mathcal{F}, \omega$ as superpositions of the pairs' modes

$$\mathcal{F} \sim \sum_{(k,m),(k',m')} b_{k,m} b_{k',m'} \mathcal{F}_{k,m} \mathcal{F}_{k',m'} \tag{4.92}$$

and similar for ω.

This is the furthest we will go into technical stuff. Just to mention that physicists have not found a solution for general pairs but only some special cases. This is because of the complexity of the equations from the sourced terms of the second-layer.

4.8.6 Some Remarks

Starting from a 2-charge geometry with non-trivial Z_4 we ended up in a 3-charge one with both angular momentum and momentum along S^1. This momentum is encoded in the $\mathcal{F}$ function while the angular momentum is described by the extra term appearing in the ω field. Our system is a rotating D1-D5-P configuration.

Rotating black holes of this kind exist when

$$N_1 N_5 N_p - j^2 > 0 \tag{4.93}$$

where j is the angular momentum. This imposes for our parameters the condition [18]

$$\frac{b^2}{a^2} > \frac{k}{n + \sqrt{(m+n)(k-m+n)}} \tag{4.94}$$

From this bound, [18], where the authors added an extra mode n in front of the v term ($m \to m + n, \quad n \geq 0$) in the (4.90) which is set to zero in [13] for simplicity, exhibited that this regime of parameters belongs to smooth, horizonless microstate geometries. Moreover, for a specific choice of modes we can obtain the non-rotating black hole of Strominger-Vafa type. The momentum units were produced from the deformations upon our geometry of a supertube. They are carried upon the compact circle and make the circle to shrink smoothly. This results in a cap similar to the 2-charge case with the analogous properties. So we reached where we wanted. Founded a possible way to compute the microstates for the general 3-charge black hole.

For further and deeper reading someone can start from [19] and familiarize himself with the technique of generating 3-charge solutions from 2-charge seeds. Also [12] provides a useful insight in the construction of the general metric used in the 2-charge and 3-charge geometries. The first paper of introducing the importance and capabilities of superstrata is the [18] but it lacks a lot of explanations and explicit computations. A more concrete construction of superstrata is carried in [13],[20]. In addition to these, [21] provides a sufficient coverage of the superstratum material. Lastly, [22] gives some explicit examples of specific-mode solutions of superstrata.

Unfortunately, superstrata cannot yet produce all the possible microstates of a black hole. Even though, they have a huge contribution in the understanding of these geometries, the fuzzball proposal and they offer significant insights into the microscopic physics of black holes.

A Dualities

Dualities provide a way of mapping the various types of theories appeared in string theory. We are interested only in two types of dualities, the S and T duality.

A.1 S-duality

S-duality or strong-weak duality refers to a relation between two theories, one with a strong coupling constant and another with a weak one (as the name suggests). In string theory this can be viewed by the substitution

$$g \to \frac{1}{g} \tag{A.1}$$

where g is the string coupling constant.

With this duality we can organise the distinct branes into pairs like

$$F1 \overset{S}{\Longleftrightarrow} D1, \quad NS5 \overset{S}{\Longleftrightarrow} D5, \quad ... \tag{A.2}$$

The conversion from weak to strong coupling is more obvious in the first example. The elementary string is equivalent, under this duality, to the stronger coupled D1-brane. Imagine the string like a classic rope, then the D1-brane is like a metal rope of still one dimension. You can bend the metal rope just like the normal one but you need more strength (energy).

One last comment in this duality is that does not act upon a momentum wave.

A.2 T-duality

We stated that this duality is used to switch between type IIA and IIB theory. It is also used in the writing of the metric for suppersymetric higher Dp-branes. To understand it intuitively we start from a string theory compactified on a circle with radius R and a Dp-brane wrapped upon it. After the T-duality the circle will have a new radius of

$$R \overset{T}{\Longleftrightarrow} R' = \frac{l_s^2}{R} \tag{A.3}$$

For $R >> l_S$ the new circle is approximately a line and so our theory is now compact in one dimension instead of two. The Dp-brane lost one spatial dimension so it has also to map into a D(p-1)-brane.

Lowering or increasing the spatial dimensions of the branes we can go from IIA to IIB and vice versa. On top of that, it is obvious that the type IIA theory comes from the M-theory after one T-dulaity in the x_{10} direction. The exact changes of this duality, if we act on the direction of z, upon the background fields is given in [23]:

$$\begin{aligned}
&\tilde{g}_{zz} = \frac{1}{g_{zz}}, \quad e^{2\tilde{\Phi}} = \frac{e^{2\Phi}}{g_{zz}}, \quad \tilde{g}_{\mu z} = \frac{B_{\mu z}}{g_{zz}}, \quad \tilde{B}_{\mu z} = \frac{g_{\mu z}}{g_{zz}} \\
&\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{g_{\mu z} g_{\nu z} - B_{\mu z} B_{\nu y}}{g_{zz}} \\
&\tilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu z} g_{\nu z} - g_{\mu z} B_{\nu y}}{g_{zz}}
\end{aligned} \tag{A.4}$$

To give an example we take a D2-brane which is wrapped in the (t, x_1, x_2) directions and pointlike in the others. We can perform a T-duality to an occupied direction and obtain a lower brane or to a transverse dimension and obtain a higher brane.

$$\begin{aligned}
&D2(012) \xrightarrow{T(1)} D1(02) \\
&D2(012) \xrightarrow{T(3)} D3(0123)
\end{aligned} \tag{A.5}$$

Likewise in a system of branes like a D2-D2 (3.3.2):

$$\begin{pmatrix} D2(012) \\ D2(034) \end{pmatrix} \overset{T(5)}{\to} \begin{pmatrix} D3(0125) \\ D3(0345) \end{pmatrix} \overset{T(2)}{\to} \begin{pmatrix} D2(015) \\ D4(02345) \end{pmatrix} \overset{T(1)}{\to} \begin{pmatrix} D1(05) \\ D5(012345) \end{pmatrix} \tag{A.6}$$

The only exception is when it acts upon the elementary objects F1, NS5. Acting in the transverse directions leaves the objects unchanged. But it has a very unique property when it acts upon the worldvolume. Say we have an elementary string wrapped N_1 times around a circle and we also have some momentum units N_p. If we take a T-duality in the circle we cannot eliminate that direction because

$$2\pi R N_1 = l_s \implies R \sim l_s, \quad R' \sim l_s \tag{A.7}$$

It is still of the same magnitude. Instead of changing the dimensions of the worldvolume it will exchange the winding number with the momentum units

$$F1(N_1) \quad P(N_p) \overset{T}{\to} F1(N_p) \quad P(N_1) \tag{A.8}$$

B Dimensional Reduction

In 3.4 we perform a dimensional reduction for the compact dimensions. We are left with a 5-dimensional metric which is easier to analyze. To perform such a reduction someone should study the Kaluza-Klein theory. A sufficient coverage of the theory is provided in chapter 44 of [6] and an explicit example in [24]. We will not provide all the technical stuff behind this theory. Rather we use these ideas in the specific system of D1-D5 to extract the 5D reduced metric.

The exact metric in 10D was

$$ds_{10}^2 = \frac{1}{\sqrt{Z_1 Z_5}}(-dt^2 + dy^2) + \sqrt{Z_1 Z_5} ds^2(\mathbb{R}^4) + \sqrt{\frac{Z_1}{Z_5}} ds^2(T^4) \tag{B.1}$$

The torus is easily reduced in the absence of a Z_4 function as seen in (4.43). The reason in that is that the scalar field coming from the reduction is the same with the dilaton.

$$\begin{aligned} ds_{10}^2 &= ds_6^2 + \sqrt{\frac{Z_1}{Z_5}} ds^2(T^4) \\ ds_6^2 &= \frac{1}{\sqrt{Z_1 Z_5}}(-dt^2 + dy^2) + \sqrt{Z_1 Z_5} ds^2(\mathbb{R}^4) \end{aligned} \tag{B.2}$$

Only thing left is the reduction upon the circle S^1 of the y direction. Let's rewrite the metric in a more convenient way

$$ds_6^2 = -\frac{1}{\sqrt{Z_1 Z_5}} dt^2 + \sqrt{Z_1 Z_5} ds^2(\mathbb{R}^4) + \frac{1}{\sqrt{Z_1 Z_5}} dy^2 \tag{B.3}$$

and with $k = Z_1 Z_5$

$$ds_6^2 = [-\frac{1}{\sqrt{k}} dt^2 + \sqrt{k} ds^2(\mathbb{R}^4)] + \frac{1}{\sqrt{k}} dy^2 \tag{B.4}$$

We can separate the metric into two parts. One that depends on y and one independent. The factor in front of the dy^2 will become the scalar field in the reduced lower-dimensional theory. We write the metric in realtion to the scalar field

$$ds_6^2 = e^{2\alpha\phi'} ds_5^2 + e^{2\beta\phi'} (dy + A_\mu dx^\mu)^2 \tag{B.5}$$

The A_μ is a gauge field that will be produced additionally to the scalar field. But in our case is zero.

If we compare the two forms of the 6D metric we observe

$$e^{2\beta\phi'} = \frac{1}{\sqrt{k}} \tag{B.6}$$

The values of β, α are found to be connected from the theory of reduction as $\beta = -(D-2)\alpha$. With $D = 5$ the field is

$$e^{2\alpha\phi'} = k^{1/6} \tag{B.7}$$

$$\begin{aligned} ds_5^2 &= e^{-2\alpha\phi}[-\frac{1}{\sqrt{k}}dt^2 + \sqrt{k}ds^2(\mathbb{R}^4)] = k^{-1/6}[-\frac{1}{\sqrt{k}}dt^2 + \sqrt{k}ds^2(\mathbb{R}^4)] \\ \implies ds_5^2 &= -k^{-2/3}dt^2 + k^{1/3}ds^2(\mathbb{R}^4) \end{aligned} \tag{B.8}$$

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