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Conformal field theories and Analytic Bootstrap

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Abstract

This thesis is occupied with the collective methods known in the literature as the analytic bootstrap. The analytic bootstrap is one of the many existing techniques, which study conformal field theories (CFTs). It relies heavily on conformal invariance and the use of the crossing equation to reproduce universal results for generic CFTs and properties of particular models. Our first concern is to reproduce the theoretical foundation of a generic CFT, which will constitute the base of our upcoming analysis. We start our review on the analytic bootstrap by introducing the inversion formula, a relation analytic in spin, which connects the relevant physical information of a CFT with the double-discontinuities appearing on the four-point function. We also present the systematics of large spin perturbation theory (LSPT), which involve the more convenient expressions of twist conformal blocks (TCBs) and a relative frame for working on interacting CFTs in a perturbative setting. The combination of the inversion formula and LSPT provides us with the necessary formalism, so as to study the famous Wilson-Fischer model in $d = 4 - \epsilon$ dimensions and reproduce the means for obtaining several known results up to fourth-order in the expansion parameter. Among them lie the OPE coefficients and the anomalous dimensions of the intermediate operators appearing in the spectrum, the corrections of the external operator ϕ up to second order in the expansion parameter and the central charge of the theory up to fourth order in the expansion parameter. The last result is considered to be especially significant as it can be extracted with a relative ease compared to the other methods used for the same purpose.

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Contents

1 Introduction

Physical theories rely on mathematical descriptions that attempt to approximate the natural phenomena as accurately as possible. Their credibility is based upon the reproduction of known experimental results, how they organise and distinguish systematic behaviours of physical systems and foremost their potential to make predictions that can be verified from further experiments. This systematic way of studying our physical surroundings has met great success in the last few centuries, and among many triumphant results and theoretical constructions, lies the rich field of Quantum Field Theory (QFT).

In order to study more naturally the behaviour of quantum field theories in the varied spectrum of energy scales we often use the renormalisation group (RG), a frame first introduced by Wilson [47]. Moving along the energy scales one might stumble upon special points (fixed points), which are invariant under the RG flow and exhibit scale invariant behaviour. It has been shown that in four and two dimensions scaling invariance implies conformal invariance [37], hence the theories occupying these points are characterized as conformal field theories (CFTs). In general dimensions scale symmetry does not necessary result to conformal symmetry but under a few extra assumptions [17] these two are equivalent. In particular conformal field theories are generated by the usual Poincare group along with the scaling and special conformal transformations (translations of the infinity).

A very interesting point of view that depicts the important aspects of CFTs in field theory is the exclusive characterization of a QFT as a UV (high-energy) CFT that flows to an IR (low-energy) CFT under the scope of RG flow. This is the reason that CFTs have been called by many as the signposts in the space of QFTs, providing endpoints for the physically important theories. A physical system described by a

CFT exhibits scale invariance or equivalently its correlation length diverges, meaning that their is no characteristic scale suitable for describing the quantities of the theory. Therefore there should be only a few factors determining the precise behaviour of a system described by a CFT. These are typically the global symmetry of the theory and the number of relevant operators it contains. We say that to theories described by the same CFT belong to the same universality class and in fact there are several such examples available especially in the field of statistical mechanics. For instance the famous 3D Ising model, containing the $\mathbb{Z}_2$ global symmetry, belongs in the same universality class as boiling water and other liquids around at critical points. Also quite often one encounters other global group symmetries in CFTs describing second order phase transitions of different materials.

Now that the importance of CFTs as suitable theories of describing physical phenomena has become evident, the natural question of how to study these theories arises. This subject has been covered independently of high energy physics and the rich structure induced from conformal symmetry is known since the early 70's [19],[38]. In particular the additional constraints of conformal invariance fix completely the two- and three-point functions of the theory up to some overall constants called the CFT-data. These constants determine the theory completely, providing us with an alternative description independent of the Lagrangian formalism. The four-point functions though are not completely determined but decompose to theory independent functions, called the conformal blocks, as they require only a spin and a (scaling) dimension insertion. The main tools provided in the frame of a conformal field theory are the convergence of the operator product expansion (OPE) between the operators of two points at finite distance and the heavy constraints imposed by the crossing equation.

The OPE is the result of state operator correspondence, present in a generic CFT, claiming that for every operator of the theory there is a unique state produced from the action of an operator on the vacuum and conversely. By replacing two insertions in an n-point function we end up with a convergent expression of an $(n-1)$-point function. Therefore in principle CFT correlators can be reduced to three- or two-point functions. The crossing equation is the result of demanding the reproduction of the same physical information of the four-point function, regardless of the combination of which pairs we choose in the OPE expansion. Namely for the four point function

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle$$

it shouldn't matter if we take the OPE of x_1, x_2 and x_3, x_4 (s-channel) or equivalently of x_1, x_4 and x_3, x_4 (t-channel). These two decompositions ought to be the same and reproduce the same physical information. Equating these two expressions leads to the crossing equation that will be one of the major concerns in our analysis. The s and t channel decompositions are referred to as the direct and crossed channel respectively.

Specifically in two dimensions the situation is even more restraining because the conformal group contains an infinite number of generators, as holomorphic functions generate conformal transformations. This makes the crossing equation even more restrictive allowing to solve theories, in some cases completely, e.g. the complete classification of minimal models [7]. However there had not been major developments in higher dimensions up until 2008, when the seminal work of Rattazzi, Rychkov, Tonni and Vichi [[39]] was published. They introduced to the scientific community what later came to be known as (numerical) conformal bootstrap, paving the way for a new field with numerous applications. The bootstrap, apart from using the crossing equation, relies on the unitarity of the OPE coefficients appearing in the decomposition of the four-point function. By applying the reverse logic of excluding spectra from the constraints posed by crossing equation and unitarity, one can pin down the CFT-data of allowed theories[1]. However there have appeared a few publications, which also study numerical techniques in non-unitary theories e.g. [21]. One of the most remarkable results bootstrap has achieved is the precise measurement of the critical exponents of the 3D Ising [18] and $O(2)$ model [32], demonstrating the computational power of this new technology.

Since then there have been several breakthroughs, especially in the computational methods used in bootstrap itself. For instance the precise form of the conformal blocks in two and four dimensions may have been known [16] for quite a long time but in other dimensions there had not been found such a closed form. Therefore a variety of techniques have been employed to approximate conformal blocks numerically by using e.g. recursive relations upon "seed" conformal blocks [27], which are already known. Another computational breakthrough has been achieved by using mixed correlators, instead of four identical scalar fields in the correlation function [31]. In the mixed case the intermediate spectrum of the OPE is much more varied and allows for more detailed and precise measurements of the CFT data.

[1]This could be phrased more accurately as the "not disallowed theories".

1. Introduction

Despite numerical bootstrap meeting such a huge success, there have also been other methods of analyzing the crossing equation to deduce statements valid in any CFT. These collective methods have been dubbed in the literature as the *analytic bootstrap*. The analytic bootstrap relies heavily on the consistency of the crossing equation and its goal is to derive properties for generic CFTs or CFTs with additional properties, e.g. containing a global symmetry, and obtain universal results.

The starting point of analytic bootstrap is considered to be the proof [20],[30] of twist spectrum additivity, were we denote with twist τ of an operator $O_{\Delta,l}$, with scaling dimensions Δ ans spin l, the quantity $\tau = \Delta_l - l$. Provided the existence of two operators O_1 ans O_2 with twists τ_1 and τ_2 respectively, the presence of the identity operator in the crossed channel implied the accumulation of infinite operators with twists approaching $\tau = \tau_1 + \tau_2 + \gamma_l$, where $\gamma_l \to 0$ as $l \to \infty$. This last quantity can be considered as an anomalous dimension defined in a non-perturbative context. Among the many subsequent developments the proof that the OPE coefficients and the anomalous dimension of the intermediate operators can be expanded in inverse powers of the conformal spin $J^2 = (\Delta_l + l)(\Delta_l + l - 2)/4$, came to be a major point that allowed more detailed systematics to be created. These results were combined in the complementary publications [2],[1], where important methods were developed, that later came to be know as Large Spin Perturbation Theory (LSPT).

It turns out that in a perturbative setting one can decompose the four point function more efficiently in twist conformal blocks (TCBs), in order to capture the separate contributions of families of the twists described previously. These TCBs studied in Lorentzian kinematics can be determined completely in some cases but our main concern about them is the reproduction of their divergent behaviour to a certain limit of the CFT's invariant quantities, the cross ratios $z, \bar{z}$. This limit corresponds to the Lorentzian kinematics where $z \to 0, \bar{z} \to 1$ and provide interesting constraints on the conformal blocks, when applied to the crossing equation. The techniques focusing on these kinematics are known in the literature as the *lightcone bootstrap* and a lot of the results that we will reproduce fall in that category.

LSPT provided universal results for wide classes of CFTs but the schemes it employs relied on arbitrary values of spin l for the perturbative setting to be applied, as its descriptive name suggests. The analyticity of these results to lower values of spin was a main concern of the theory, that came to be resolved with the inversion formula discovered by Caron-Huot [10]. This formula apart from being analytic in spin encodes the CFT data in the double discontinuities of the conformal block expansion, hence does not require the full form of the four point function to examine its contribution to the crossed channel of the crossing equation. Another of its virtues, that it inherits from large spin perturbation theory, is that when applied in a perturbative setting, there are only a finite number of contributions [2] at each order in the expansion parameter. This renders the use of the inversion formula possible as long as we successfully determine the divergent behaviours contributing in the crossed channel of the crossing equation. If we finally demand these contributions to be reproduced in the direct channel one can obtain the OPE coefficients and the anomalous dimensions of the intermediate operators.

1.1 Outline

This thesis will start with a brief review of the conformal field theories in Chapter 2, covering the essential formalism and results required for the rest of our analysis. We will focus also on additional analysis in the context of analytic bootstrap, that we will need in subsequent chapters.

Chapter 3 is dedicated to studying the systematics around analytic bootstrap firstly presenting the inversion formula, along with the origins and a few details of its derivation. Afterwards we study the formulation and main results of large spin perturbation theory and we conclude with the combination of the two methods, by providing the frame of our upcoming applications and an overview of the combined tools that we have in our disposal.

In Chapter 4 our efforts come to fruition, as we apply our developed formalism in the well known Wilson-Fischer model and manage to obtain the OPE coefficients and anomalous dimensions up to fourth order in the expansion parameter. We will study the contributions appearing at each order in perturbation theory and resolve possible setbacks that appear as we move to higher orders.

[2] A contribution here denotes not a specific operator, but a family of operators fulfilling specific properties, e.g. belonging in the same twist family.

We conclude by reviewing our previous analysis and provide the latest results and developments in analytic bootstrap, along with the goals and further successes of the inversion formula.

Finally in Appendix A we attempt to highlight the main points of the derivation of the inversion formula in the context of [43] and in Appendix B we present and prove a few noteworthy results of analytic bootstrap.

2 Conformal Field Theory

This chapter is dedicated to the basics of conformal field theories with dimension $d \geq 3$, since the tools we will use in the main text do not apply for lower dimensions. The majority of the content we present here can be found in the informative reviews [41],[40], unless stated otherwise when we provide the additional reference.

2.1 Conformal invariance

We define a theory to be conformally invariant, when its metric obeys the transformation

$$g'_{\rho\sigma}(x') = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} g_{\mu\nu}(x) = c(x) g_{\mu\nu}(x), \tag{2.1}$$

where the function $c(x)$ is called the scale factor. In a sense conformal transformations can be thought to arise from scaling transformations[1], that leave the Hamlitonian H invariant, along with the existence of a traceless stress-energy tensor. This is expressed as

$$\Delta H = \int d^d x T_{\mu\nu} \delta g^{\mu\nu} = \int d^d x c(x) T^{\mu}_{\mu} = 0 \tag{2.2}$$

where we have considered by definition the metric deformation $\delta g^{\mu\nu} = c(x)\delta_{\mu\nu}$ in an arbitrary d-dimensional theory. This deformation is known in the literature as a Weyl transformation, which in its general definition alters the curvature of spacetime. However we can specify to a subclass of these transformations that leave spacetime flat and can expressed as the infinitesimal transformations

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x). \tag{2.3}$$

[1] Transformations that leave the Hamiltonian and the correlations functions invariant after appropriate rescaling.

The previous metric deformation can now be written

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = c(x)\delta_{\mu\nu} \tag{2.4}$$

and for $d \geq 3$ it has four classes of solutions.

By contracting both sides of (2.4) we derive that $c(x) = \frac{2}{d}\partial^\rho \epsilon_\rho$, which implies the relation

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial^\rho \epsilon_\rho)\delta_{\mu\nu}. \tag{2.5}$$

After a few trivial manipulations of the previous relation we derive the equations

$$(d-1)\Box(\partial \cdot \epsilon) = 0 \tag{2.6}$$

$$2\partial_\mu \partial_\nu = \frac{2}{d}(-\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu)(\partial \cdot \epsilon) \tag{2.7}$$

where we have used the usual definitions $\partial \cdot \epsilon = \partial_\mu \epsilon^\mu$ and $\Box \epsilon = \partial_\mu \partial^\mu \epsilon$. The first equation implies that $\partial_\mu \epsilon$ can be at most linear in x_μ, therefore ϵ_μ will be of the schematic form

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho. \tag{2.8}$$

It is straightforward now to use the relations (2.5), (2.7) and arrive to the previously mentioned solutions

$$\begin{aligned} &\epsilon^\mu = a^\mu, \;\; \epsilon^\mu = m^\mu_\nu x^\nu \\ &\epsilon^\mu = \alpha \cdot x^\mu, \;\; \epsilon^\mu = 2(b \cdot x)x^\mu - x^2 b^\mu \end{aligned}$$

corresponding to infinitesimal translations, rotations, scale transformations and special conformal transformations (SCT) respectively.

Their generators can be derived in a straightforward manner by using the convention for the generator G_a of a transformation action to be

$$\phi'(x) - \phi(x) \equiv -i\epsilon_a G_a \phi(x) \tag{2.9}$$

where

$$iG_a\phi = \frac{\delta x^\mu}{\delta \epsilon_a}\partial_\mu \phi - \frac{\delta F}{\delta \epsilon_a} \tag{2.10}$$

and $F(\phi(x)) = \phi'(x')$ relates the new field at x' with the old field at x. Since we are interested at the moment only on the transformation of the coordinates we may as well suppose that the fields remain unaffected from these transformations. Therefore the last part of the previous equation vanishes and we can use it to derive the explicit form of each generator. Namely for every transformation we have the generators

$$P_\mu = -i\partial_\mu \rightarrow \textbf{translations} \tag{2.11}$$

$$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \rightarrow \textbf{rotations} \tag{2.12}$$

$$D = ix^\mu\partial_\mu \rightarrow \textbf{dilations} \tag{2.13}$$

$$K_\mu = i(2x_\mu(x\cdot\partial) - x^2\partial_\mu) \rightarrow \textbf{SCTs}. \tag{2.14}$$

As you may have noticed we rederived the Poincare group, supplemented with two new elements the scale transformation and the SCT.

With these at hand we can obtain the corresponding conformal algebra or equivalently the commutation relations that these generators satisfy. These relations are of the form

$$[D, P_\mu] = -iP_\mu \tag{2.15}$$

$$[D, K_\mu] = iK_\mu \tag{2.16}$$

$$[K_\mu, P_\nu] = 2i(\delta_{\mu\nu}D - M_{\mu\nu}) \tag{2.17}$$

$$[M_{\mu\nu}, P_\rho] = -i(\delta_{\nu\rho}P_\mu - \delta_{\mu\rho}P_\nu) \tag{2.18}$$

$$[M_{\mu\nu}, K_\rho] = -i(\delta_{\nu\rho}K_\mu - \delta_{\mu\rho}K_\nu) \tag{2.19}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\delta_{\nu\rho}M_{\mu\sigma} - \delta_{\mu\rho}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\rho\mu} - \delta_{\mu\sigma}M_{\rho\nu}) \tag{2.20}$$

where we applied the definitions of the generators and simple algebraic manipulations. These generators correspond to one dilation, d translations, d SCTs and $\frac{d(d-1)}{2}$ rotations, in total $\frac{(d+2)(d+1)}{2}$ generators. This implies that the conformal algebra in d dimensions is isomorphic to $SO(d+1,1)$, the algebra of Lorentz transformations in $\mathbb{R}^{d+1,1}$. Therefore it would be natural to consider the conformal algebra in this context by defining appropriately a new set of generators to be

$$J_{\mu\nu} = M_{\mu\nu}$$

$$J_{\mu+} = P_\mu \tag{2.21}$$

$$J_{\mu-} = K_\mu \tag{2.22}$$

$$J_{+-} = D \tag{2.23}$$

$$\tag{2.24}$$

and under this definition we can check that they satisfy the commutation relation of the $SO(d+1,1)$ group

$$[J_{MN}, J_{RS}] = -i(\eta_{MR}J_{NS} \pm \text{permutations}). \tag{2.25}$$

The latin capital indices run over the $D+2$ coordinates, where we defined the last one to be the timelike direction and used the lightcone coordinates $x^+ = x^{d+2} + x^{d+1}$, $x^- = x^{d+2} - x^{d+1}$.

This identification of generators can be found in the literature as the embedding space formalism and makes use of the linearity of the conformal group, when considered in the lightcone of Lorentzian kinematics in $d+2$ dimensions. Several simplifications occur from this scheme as for example one can handle more conveniently the two- and three-point functions of operators with spin, and eventually fix them completely. At the same time we can project whichever result we derive in the null cone into our original d-dimensional space, and use there the projected relations.

Representations of the conformal group

Back to our previous formalism we are now in place to discuss the representations of the conformal group. It is instructive to consider first the Lorentz group-the subgroup of the Poincare group than leaves the point $x = 0$ invariant. By defining the infinitesimal generator on a field ϕ at this point, we can then use the Hausdorff formula

$$e^{-A}Be^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A]] \dots \tag{2.26}$$

and translate our results to an arbitrary point x. Then we will know the precise action of the generators on an arbitrary field $\phi(x)$. Having said that we can now define the infinitesimal transformation $M_{\mu\nu}\phi(0) = S_{\mu\nu}\phi(0)$, where $S_{\mu\nu}$ is a matrix representation. Using (2.26) we can translate $M_{\mu\nu}$ to non zero values of x, and derive that

$$e^{ix^\rho P_\rho} M_{\mu\nu} e^{-ix^\rho P_\rho} = M_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu.$$

Since this transformation contains a piece that transforms the fields and a separate piece that transforms the coordinates, we can safely determine the actions of the first two generators to be

$$P_\mu \phi(x) = -i\partial_\mu \phi(x) \tag{2.27}$$

$$M_{\mu\nu}\phi(x) = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi(x) + S_{\mu\nu}\phi(x). \tag{2.28}$$

Completely analogously we can write for the two other generators

$$D\phi(x) = -i(x^\mu \partial_\mu + \Delta)\phi(x) \tag{2.29}$$

$$K_\mu \phi(x) = (-2i\Delta x_\mu - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu \partial_\nu + ix^2 \partial_\mu)\phi(x) \tag{2.30}$$

where we define with Δ the eigenvalue of the dilation operator D acting upon $\phi(0)$,which is called scaling dimension of the field ϕ. Alternatively though it could also arise from its definition as the power of the factor appearing upon the action of a scaling transformation on a field ϕ according to

$$\phi(\lambda x) = \lambda^{-\Delta}\phi(x). \tag{2.31}$$

This is exactly the definition of a scaling transformation, which ultimately leads us to the same representation.

Equivalently we could think of these representations acting not on a field but on an operator $O(x)$, with a possible inner structure (spin indices). For such an operator evaluated at $x = 0$ we notice that

$$DK_\mu O(0) = ([D, K_\mu] + K_\mu D)O(0) = -i(\Delta - 1)K_\mu O(0) \tag{2.32}$$

where we made use of the relation (2.16). This renders K_μ an operator that lowers the scaling dimension by one and in a sensible theory his action needs to meet a certain bound. Therefore there will be operators that vanish under the action of K_μ and we define them as *primary* operators. In contrast the P_μ operator has the opposite effect of raising the scaling dimension of an operator by one and all the operators generated by its action on primaries are called *descendants*.

Finally by exponentiating the generators of the conformal group we derive that a field (or an operator) transforms under the conformal transformations through the relation

$$\phi^\alpha(x) = c(x')^{-\Delta} D\Big(M(x')\Big)^\alpha_\beta \phi^\beta(x') \tag{2.33}$$

where $c(x')$ is the factor in the definition of the conformal transformations and $D\Big(M(x')\Big)^\alpha_\beta$ is a matrix that contains the appropriate set of indices and corresponds to the representation that ϕ belongs to. So upon the action of $M^\mu_\nu \in SO(d)$ we could be in e.g. the scalar or the vector representation.

2.2 Conformal correlators

Let us now comment on how the conformal invariance constrains the two- and three-point functions. Scaling invariance by itself requires that the two point function satisfies the relation

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle \tag{2.34}$$

while Poincare invariance implies that $\langle \phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|)$. Therefore two-point function has to be of the form

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{d_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{2.35}$$

where d_{12} is a normalization constant. Finally transforming with the SCTs the two point function we arrive in the remarkable condition. It is non zero only when the scaling dimensions of the two fields are equal (hence they are the same). By choosing an appropriate normalization we can finally write

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{\delta_{12}}{|x_{12}|^{2\Delta_1}}. \tag{2.36}$$

where we used the notation $x_{12} = x_1 - x_2$.

The same reasoning applies to the three point function of scalar fields but this time the conformal invariance is less restraining. Using similar arguments with the two point function we arrive at the relation

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{123}}{x_{12}^{\Delta-2\Delta_3}x_{23}^{\Delta-2\Delta_1}x_{13}^{\Delta-2\Delta_2}} \tag{2.37}$$

where $\Delta = \sum_i \Delta_i$. This time though there is not an appropriate convention we can use, in order to normalize away the constants λ_{123}. In fact as we will see shortly these constants are essential data needed, which define completely an individual CFT.

With more appropriate arguments (e.g. the embedding space formalism) we can also constrain the correlators of fields containing spin indices. For example for a spin one field with scaling dimension Δ we can write

$$\langle \phi^\mu(x_1)\phi_\nu(x_2)\rangle = c_\phi \frac{I^\mu_\nu(x_{12})}{x_{12}^{2\Delta}} \tag{2.38}$$

where

$$I^\mu_\nu(x_{12}) = \delta^\mu_\nu - \frac{2x^\mu x_\nu}{x^2} \tag{2.39}$$

and c_ϕ an arbitrary constant. Also for the correlator of two scalars and a spin l field we have up to a single coefficient $f_{\phi_1\phi_2\phi}$ the relation

$$\langle\phi_1(x_1)\phi_2(x_2)\phi^{\mu_1\cdots\mu_l}(x_3)\rangle = f_{\phi_1\phi_2\phi}\frac{Z^{\mu_1}\cdots Z^{\mu_l} - \text{traces}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3+l}x_{23}^{\Delta_3+\Delta_2-\Delta_1+l}x_{13}^{\Delta_1+\Delta_3-\Delta_2+l}} \tag{2.40}$$

where we have defined the quantity $Z^\mu \equiv \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}$.

Conformal invariance is not capable of fixing completely the form of higher point functions. However in a CFT when we start with four points we can define the conformally invariant quantities

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2},\ v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \tag{2.41}$$

which are broadly known as the conformal cross-ratios. In addition the aggrement with the transformation properties of (2.33) requires that the four point function should be of the form

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}f(u,v) \tag{2.42}$$

which imposes a tremendous reduction of freedom in its structure, making it only depend on an arbitrary function f of the cross ratios.

We can also impose immediately a functional constrain on $f(u,v)$. Since the previous equations must be invariant under the exchange of indices $2 \leftrightarrow 4$ (essentially the exchange $u \leftrightarrow v$) we arrive at the following result

$$\frac{1}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}}f(u,v) = \frac{1}{|x_{14}|^{2\Delta}|x_{23}|^{2\Delta}}f(v,u) \tag{2.43}$$

which can be written equivalently

$$f(u,v) = \left(\frac{u}{v}\right)^{\Delta} f(v,u). \tag{2.44}$$

This equation will play an important role when we examine the crossing condition imposed on the four-point functions and is responsible for the vast majority of the results obtained in analytic and numerical bootstrap altogether.

2.3 Radial quantization

Since we are interested in scale-invariant theories it is sensible to quantize our theory in foliations of spacetime, with equal scaling dimension. This is similar to foliating our usual Poincare invariant theories with surfaces of equal time. Each surface corresponds to a Hilbert space and we move along Hilbert space's states without familiar unitary evolution operator. This time though our quantization will consist of concentral S^{d-1} spheres around the origin $x = 0$, each one associated with a scaling dimension Δ. A natural choice for the Hamiltonian, that will move us from one surface to the other, must involve the dilation operator D and will be of the form

$$U = e^{iD\Delta\tau} \tag{2.45}$$

where $\tau = \log r$. Each state "living" in a foliation of our spacetime will be characterized according to its scaling dimension $D|\Delta\rangle = i\Delta|\Delta\rangle$ and their $SO(D)$ spin l, satisfying the eigenvalue equation $M_{\mu\nu}|\Delta, l\rangle_s = (S_{\mu\nu})_s^\tau |\Delta, l\rangle_\tau$.

Among the various states in radial quantization the vacuum state $|0\rangle$ is a distinct one, since conformal invariance requires the action of any generator G on $|0\rangle$ to give zero. This is consistent with our defition of our familiar correlators as the vacuum averages of quantum mechanical operators, namely

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle = \langle 0|\phi_1(x_1) \dots \phi_n(x_n)|0\rangle. \tag{2.46}$$

Because of the previous claim it is also sensible to write the action of a generator G on a field ϕ as the commutator of the generator and the field, that is $[G, \phi]$. This implies that we can insert an operator O_Δ at the origin $x = 0$ and consequently generate a state $|\Delta\rangle = O_\Delta(0)|0\rangle$ with scaling dimension

$$D|\Delta\rangle = [D, O_\Delta(0)]|0\rangle = i\Delta|\Delta\rangle. \tag{2.47}$$

This relation also implies that our definition of primaries and descendants operators extends to the states of radial quantization. Primary states in fact correspond to the measurable quantities of our theory and all of the descendants states result by the repeated action of (2.15) on a primary state. Therefore they are used to classify CFTs and we define in total a primary state with its descendants as a *conformal multiplet.*

We must not fail though to exploit the peculiar traits of radial quantization. For example when we inserted primary operators at the origin they generated states with specific scaling dimension Δ, each unique with respect to the operator it corresponds. One may wonder whether we could apply the same logic in reverse. Provided that this is possible for a generic CFT there would be a state-operator correspondence, a statement which does not generally hold true in any quantum field theory. Truthfully for every primary state with a given scaling dimension Δ we can define an operator through its correlation function with the other operators of the theory. This is possible through the definition

$$\langle \phi(x_1)\phi(x_2)\dots O_\Delta(0)\rangle = \langle 0|\phi(x_1)\phi(x_2)\dots|\Delta\rangle \tag{2.48}$$

which satisfies all of the transformations imposed by conformal invariance.

Operator Product Expansion

The operator product expansion, a tool broadly used in quantum field theory, states that two local operators infinitesimally close to each other can be replaced by a series of operators at the midpoint. In the context of CFTs though we can arrive in a much more rigid result using the operator state correspondence. Two operators inserted inside a sphere in radial quantization will generate a state $\phi_1(x)\phi_2(0)|0\rangle = |\Psi\rangle$ on the surface of that sphere. Since that state will have a well defined scaling dimension Δ can be written as

$$|\Psi\rangle = O_\Delta(x)|0\rangle = e^{iPx}O_\Delta(0)e^{-iPx} = \sum_n \frac{1}{n!}(iPx)^n|\Delta\rangle = \sum_n c_n(x)|E_n\rangle \tag{2.49}$$

which is essentially a superposition of states. But now the state operator correspondence implies that these states will be in one-to-one correspondence with operators, primaries and descendants. This implies the relation

$$|\Psi\rangle = \sum_O C_O(x,\partial_y)O(y)\Big|_{y=0}|0\rangle \tag{2.50}$$

where the summation runs over primary operators O and $C_O(x,\partial_y)$ can be understood as a power series in ∂_y.

This result is precisely an OPE, which has the extraordinary property of being convergent not only when the operators are asymptotically close. The convergence of the OPE rellies heavily on the operator state correspondence and its detailed proof can be found in [36]. Its usefulness can be demonstrated immediately by applying it to the three-point function of three scalars. Their correlator can be written equivalently

$$\langle \phi_1(x)\phi_2(0)\Phi\rangle = \sum \lambda_O C_O(x,\partial_y)\langle O(y)|_{y=0}\Phi(z)\rangle \tag{2.51}$$

and by using the diagonal form of the two-point function we arrive at the result

$$\langle \phi_1(x)\phi_2(0)\Phi\rangle = \lambda_\Phi C_\Phi(x,\partial_y)\langle \Phi(y)|_{y=0}\Phi(z)\rangle \tag{2.52}$$

It is now possible by using the explicit form of the two- and three-point functions to fix the coefficients entering the function $C_\Phi(x,\partial_y)$ term by term. Besides the first few terms though this is quite a harsh computation and there are several other more efficient ways of deriving them. In fact one of these methods relies on the analytic bootstrap techniques that we will review in the chapters to come. We will see that our formalism is not only capable of deriving the OPE coefficients of a free theory but also collect their corrections, assuming that we can employ a perturbative expansion. The methods we will review extract these coefficients in a rather straightforward manner, in contrast to other methods used for the same purpose.

Unitarity bounds

Radial quantization is responsible for another famous result, the unitarity bounds. These are precisely positive bounds that determine the minimal allowed value for the scaling dimension that a primary operator can take. Unitarity of the theory implies that the norms of the states must be non-negative, a condition that in radial quantization can be exploited in the following way.

When we consider the K_μ and P_μ generators in radial quantization we can immediately observe that they are the hermitian conjugate of one another by writing their analytic expression in radial coordinates. Therefore we can write the relation

$$(P_\mu|O^a\rangle)^\dagger P_\nu|O^b\rangle = \langle O_a|K_\mu P_\nu|O^b\rangle = 2\Delta\delta_{\mu\nu}\delta_a^b - 2(M_{\mu\nu})_a^b \geq 0 \tag{2.53}$$

where the third line is the result of straightforward conformal algebra and the operator O^a transforms in a spin l representation, with a,b being the appropriate indices. Unitarity implies that the matrices entering the previous equation must be positive-definite. Therefore we derive that

$$\Delta \geq \lambda_{\max}((M_{\mu\nu})_a^b) \tag{2.54}$$

where we denote with $\lambda_{\max}$ the maximum eigenvalue of $(M_{\mu\nu})_a^b$.

We will employ a familiar manipulation from quantum mechanics, which involves the computation of the eigenvalues for the product of two amgular momenta. We recall that we can decompose the product

$$L^i \cdot S^i = \frac{1}{2}[(L+S)^2 - L^2 - S^2] \tag{2.55}$$

and through this diagonalized form the involved operators are Casimirs, so they have known eigenvalues. In a similar manner we can write the action of $M_{\mu\nu}$ as

$$-i(M_{\mu\nu})_a^b = -\frac{1}{2}(V^{kl})_{\mu\nu}(M_{kl})_a^b \tag{2.56}$$

where the generator of the vector representation $(V^{kl})_{\mu\nu}$ is

$$(V^{kl})_{\mu\nu} = i(\delta_\mu^k \delta_\nu^l - \delta_\nu^k \delta_\mu^l). \tag{2.57}$$

Then the decomposition we achieved consists of the spin l representation, the vector representation we "manually" inserted and their tensor product. By using the $SO(d)$ value of the Casimir $l(l+d-2)$, we derive for the maximum eigenvalue of the matrix $M_{\mu\nu}$ to be

$$\Delta \geq \lambda_{\max}((M_{\mu\nu})_a^b = \frac{1}{2}[\mathrm{Cas}(V_{l=1}) + \mathrm{Cas}(l) - \mathrm{minCas}(V \otimes l)] = l + d - 2 \tag{2.58}$$

where we used the spin $l-1$ representation appearing in the tensor product irreducible decomposition, which among these representations has the minimum eigenvalue.

By considering higher level of the descendants, e.g. level two with $l = 0$, we can examine the matrix element $\langle \Delta | K_{\mu'} K_{\nu'} P_\nu P_\mu | \Delta \rangle$ and the unitarity condition generates the boundary condition

$$\Delta \geq \frac{d-2}{2} \tag{2.59}$$

which bounds the scaling dimension of scalar operators. With these two constrains though we obtained all of the information that can be extracted for the unitarity bounds both for scalars and operators with spin, as for higher levels of descendants and other spin representations we do not obtain stronger bounds.

2.4 Conformal bootstrap

Having reviewed the most important aspects of conformal field theory, we are in place to start examining the general setup upon conformal bootstrap is formulated and the powerful tools that it has in its disposal.

As we saw in the previous section, the OPE in a conformal field theory can reduce effectively any n-point function to an $n-1$ point function, because of its finite range of convergence. This means in practice that we can reduce any correlation function to convergent sums of two point functions. Therefore when we think of the relevant information appearing on a CFT, that determines all of the measurable quantities, we should consider the OPE coefficients of the two-point function. We rewrite at this point the form of the OPE, as well as our final convention for the OPE coefficients to be

$$\phi_1(x)\phi_2(0) = \sum_{O \text{ primaries}} a_{12} C_O(x, \partial_y) O(y)|_{y=0} \tag{2.60}$$

These coefficients together with the spectrum of the operators appearing in the previous relation are called the *CFT-data*, and constitute all of the relevant information of a CFT. For this reason this data will be the main focus of our upcomming analysis.

Now that we've determined the main focus of conformal bootstrap, we can go back to our familiar four-point function. Since it consists of four point insertions x_1, x_2, x_3, x_4 we could organise the points in pairs of two, and consider the OPE of each pair. This reasoning reduces the four-point to a convergent sum of two point function that has the schematic form

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \sum_O a_{12O}a_{34O}[C_O(x_{12}, \partial_y)C_O(x_{34}, \partial_z)\langle O(y)O(z)\rangle]. \tag{2.61}$$

where we have consider the OPE channel (12)(34) (s-channel). The functions in the brackets are determined completely, since they are fixed by conformal invariance, and are known in the literature as Conformal Partial Waves (CPW). Each individual CPW is expected to transform as the four-point function itself, because it originates from the OPE, which is by definition conformally invariant. By stripping a convenient factor, which normalizes the contribution of the identity operator to one, we have for the four-point function of four identical scalar fields with scaling dimension Δ

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_O a_{12O}a_{34O}\frac{G_O(u,v)}{(x_{12})^{2\Delta}(x_{34})^{2\Delta}} = \frac{\mathcal{G}(u,v)}{(x_{12})^{2\Delta}(x_{34})^{2\Delta}} \tag{2.62}$$

where the quantities $G_O(u,v)$ are conformal invariant functionals, which depend only on the cross-ratios and are called conformal blocks (CB).

By employing once again conformal invariance we can obtain a different configuration for the invariant quantities determining a CFT, the cross-ratios. We can use conformal transformations to gauge fix each point entering the correlator, by setting $x_1 = 0$,$x_3 = (1, 0, \ldots, 0)$, $x_4 \to \infty$ and restricting x_2 at the complex x_1x_3 plane. Therefore the cross-ratios can be simplified to

$$u = |z|^2 = z\bar{z},\ \ v = |(1-z)|^2 = (1-z)(1-\bar{z}) \tag{2.63}$$

where $z = x + iy$ the complex coordinate of that plane and denote with the bar notation the complex conjugate of a complex number. This configuration will prove to be convenient at times and we might use it interchangeably with the u and v variables when we discuss the four-point function or the conformal blocks.

Another configuration in this plane, which is often encountered in the literature uses conformal invariance to set the location of these points in a more symmetrical way. It does so through the complex parameter $\rho = re^{i\alpha}$, $r < 1$ and by locating the points at $x_1 = re^{i\alpha}, x_2 = -re^{i\alpha}, x_3 = -1$ and $x_4 = 1$. Their relation with the z-variables can be found to be

$$z = \frac{4\rho}{(1+\rho)^2} \Leftrightarrow \rho = \frac{z}{(1+\sqrt{1-z})^2}. \tag{2.64}$$

Since the conformal blocks contain the decomposition of the four-point function in a functional form and sum the contribution of a primary operator O along with all of its descendants, they are the building blocks of a CFT and their computation has been a major goal from the start of the bootstrap program. There have been major advances in their computation, especially with methods involving numerical results, but their explicit form in two and form dimensions has been found [16]. This was rendered possible through the use of the Casimir differential equation satisfied by the generators of the conformal group. The differential equation satisfied by the CBs has the form of an eigenvalue equation and can be found to be

$$CG_O(u,v) = \Big(\Delta(\Delta-d) + l(l+d-2)\Big)G_O(u,v) \tag{2.65}$$

where the differential operator C is written in terms of the $z, \bar{z}$ variables as

$$C = D + \bar{D} + (d-2)\frac{z\bar{z}}{z-\bar{z}}((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}) \tag{2.66}$$

where

$$D = z^2(1-z)\partial_z^2 - z^2\partial_z \tag{2.67}$$

and $\bar{D}$ is of the same form with $z \leftrightarrow \bar{z}$. Through these differential equations we can write the well-known results

$$G^{(2d)}_{\Delta,l}(z,\bar{z}) = \frac{k_{\Delta-l}(z)k_{\Delta-l}(\bar{z}) + k_{\Delta-l}(\bar{z})k_{\Delta-l}(z)}{1+\delta_{l,0}} \tag{2.68}$$

$$G^{(4d)}_{\Delta,l}(z,\bar{z}) = \frac{z\bar{z}}{\bar{z}-z}[k_{\Delta-l-2}(z)k_{\Delta+l}(\bar{z}) - k_{\Delta-l-2}(\bar{z})k_{\Delta+l}(z)] \tag{2.69}$$

where we used the more convenient notation for the conformal blocks involving the scaling dimension Δ and spin l instead of the operator O and defined the functions

$$k_h(z) = z^h{}_2F_1(h,h;2h,z). \tag{2.70}$$

with ${}_2F_1(h,h;2h,z)$ our familiar hypergeometric functions.

Let us now comment on a final point regarding the decomposition of the four-point function in (2.62). It may have seemed arbitrary, and indeed it is, the choice of the pairings (12)(34) of the four point insertions and their subsequent OPE decomposition. Alternatively we could have select the pairings (14)(23), which ought to reproduce an equivalent decomposition of the four-point function. This condition though requires the equivalence of two functional equations, which in principal are not of the same form, and imposes infinite constrains on our theory. We can write this condition with the use of the relation (2.44) as

$$\mathcal{G}(u,v) = \left(\frac{u}{v}\right)^{\Delta}\mathcal{G}(v,u) \tag{2.71}$$

an overall result based on the crossing symmetry of the two pairings (s- and t-channel respectively), which is broadly known as the *crossing equation*. This relation will remain on the spotlight for the rest of our analysis.

Numerical Bootstrap

Even though the numerical techniques developed from the heavy constrains imposed by the crossing equation will not be in the scope of this analysis, it is reasonable for completeness purposes to present briefly the underlying ideas of these methods.

We will simplify our discussion by considering four identical external scalar operators ϕ with scaling dimension Δ and in particular this will also be the case for the majority of our upcoming analysis. Under this assumption we can write the crossing equation as

$$\sum_O a_O^2 [v^\Delta G_{\Delta,l}(u,v) - u^\Delta G_{\Delta,l}(v,u)] = \sum_O a_O^2 F^\Delta_{\Delta_O,l}(u,v) = 0 \tag{2.72}$$

where the function $F^\Delta_{\Delta_O,l}(u,v)$ is defined in the subspace

$$\mathcal{K} = \{(z,\bar{z}) \in \mathbb{C} : z \notin (-\infty, 0], \bar{z} \notin [1, +\infty]\} \tag{2.73}$$

in order for the conformal blocks to converge in both the s- and t-channel [36]. This condition though by itself does not make a proposition for values of the CFT data or even how to find them.

Instead the conformal bootstrap is based on the opposite reasoning. We will consider the vector space V of all the possible sums of the form of equation (2.72), that converge within the subspace $\mathcal{K}$. Our aim will be to use relation (2.72) and the restrictions it poses to the theory, to formulate a linear programming problem. For that we consider the space of functionals V^* and an appropriate base for that space would be the derivatives $\partial_z^m \partial_{\bar{z}}^n F^\Delta_{\Delta_O,l}(z,\bar{z})$, for sufficiently large values of m,n, possibly restricted by a cut-off Λ.

For a suitable functional (convergent in terms of the particular sum) $\alpha \in V^*$ we would have the relation

$$\sum_O a_O^2 \alpha(F^\Delta_{\Delta_O,l}) = 0 \tag{2.74}$$

which can be tested numerically in the formulation of the problem that we described. This formulation allows us to propose possible sets of CFT data encoded in the conformal blocks and search for functionals that disprove equation (2.72). Thus we can pin-down the CFT-data in the limits of our numerical precision.

An essential condition of this scheme is the unitarity of the theory, which demands the positiveness of the OPE coefficients. Therefore the numerical bootstrap has succeeded so far in unitary CFTs. Also the majority of the computations are performed around the values $z = \bar{z} = 1/2$, because this region treats symmetrically the direct and crossed channel. There have been attempts to compute numerical bounds for the CFT-data around other regions of these variables, but have not been efficient so far.

In conclusion this reverse logic allows us to constrain the CFT data of "allowed" theories or rather of theories not prohibited due to consistency reasons from the crossing equation. As noted previously, so far the results of modern numerical bootstrap have been extraordinary and this new field has provided an efficient way of testing known results obtained with other methods, along with generating many new results on its own.

2.5 Lorentzian kinematics

Even though numerical bootstrap focuses on the region where the variables $z, \bar{z}$ are equal and close to $\frac{1}{2}$, the analytic bootstrap exhibits its power in Lorentzian kinematics, where there are certain configurations of these variables that extract useful information from the crossing equation. We can reach Lorentzian kinematics by Wick rotating the complex coordinate in the configuration we described previously for z and $\bar{z}$. As a result these two variables will not be the complex conjugate of one another and all of our manipulations will be in Lorentzian signature. We set the x_2 coordinate to be $x_2 = (x_2^0, x_2^1, 0, \dots)$, where the time direction corresponds to the x_2^0 variable. Under this new scheme

$$z = x_2^1 - x_2^0, \ \bar{z} = x_2^1 + x_2^0. \tag{2.75}$$

For the convenience of our discussion we will organize the various limits of the kinematics in these two variables, which have been conveniently classified in [24]. The simplest one, known as the OPE limit, comes from $z, \bar{z} \to 0$ and it is usually referred together with its symmetric one, the crossed-channel OPE limit, which naturally comes from the region where $z, \bar{z} \to 1$. The OPE limit is useful in the sense that the leading contribution to the four point function comes from the operator with the smallest scaling dimension. Apart from that the limit itself will not be of our immediate concern.

By specifying though only in the z variable and the limit $z \to 0$, we reach the first region where the Lorentzian kinematics manifest. This becomes evident if we consider the quantity $|x_{12}|$, which approaches asymptotically 0 and does not have necessarily zero components in its coordinates. This limit is known as *collinear* and this time the contribution with the leading behaviour in the OPE, is the one with the lowest twist τ for each spin. We will discuss this in more detail in the next chapter, dedicated to analytic bootstrap, but there are several indications that the twist of an operator is a convenient quantity which we can use to classify the contributions in the CB expansion.

Finally we reach a completely Lorentzian regime by retaining the collinear limit and further specifying to $\bar{z} \to 1$. This region is known as the *double lightcone* limit as there is an additional potential null separation of the points x_2, x_3. The manipulations of the crossing equation in this limit are called the *lightcone* bootstrap and the techniques that we are going to develop here fall in this broad category. The most distinct characteristic of the double-lightcone limit is the domination in this region from operators with large values of spin. This trait precisely is responsible for the *large spin perturbation* theory, that we will review in the following chapter.

The domination of these operators has resulted in several analytic expressions for large values of spin. A big question that raised major concerns, was whether this results can be analytically continued for finite values of spin. This question was resolved with the introduction of the inversion formula [10] in the frame of lightcone bootstrap, which proved that this continuation is possible indeed. However the proof of this formula makes use of another Lorentzian limit, which is called Regge limit, and corresponds to the pairwise null separation of the insertions (x_1, x_4) and (x_2, x_3). Notice that the OPE is not convergent anymore as e.g. the points x_1, x_2 are not within a potential convergence radius. Despite that, the conformal blocks are proportional to inverse powers of their spin, and therefore we expect the four point function to be bounded. This will play an important role in Appendix A, where the proof of the inversion formula makes explicit use of that statement.

3
Analytic Bootstrap

3.1 Lorentzian Inversion formula

The author of [10] got the inspiration for the inversion formula mentioned previously from a brunch of physics known as Regge theory. It was first introduced by Regge in 1959 [11] and formulated by Gribov, Chew, Mandelstam and others. It flourished in the decades of 50's and 60's, as it had plenty of applications, provided new insight and tools in 2×2 scattering processes and posed a candidate for describing the strong interactions. In the years to come though Quantum Chromodynamics proved to be a more suitable theory to describe them, therefore the interest in Regge theory grew dull. Despite that its theoretical arguments still appear to be quite instructive. In fact its stringy description of hadrons prepared the ground for the string theories of gravity, where its techniques provided results in scattering amplitudes (for a relevant review [13]). A brief summary of Regge theory and its relevance for the inversion formula will be given in the section to come.

3.1.1 An invitation for an inversion formula: Regge theory

Regge theory is essentially a theory of complex angular momenta. The spark that ignited the interest of many researchers, came from the finding that the poles of the amplitude in the complex l-plane were related with the scattering resonances/bound states of the non-relativistic scattering in quantum mechanics [23]. Regge further demonstrated the asymptotic behaviour of the scattering amplitude in the unphysical region $|t| \gg s$

$$A(s,t) \approx t^{l(s)} \tag{3.1}$$

where s, t are the so called Mandelstam variables and $l(s)$ is the position of the dominating pole of the partial wave f_l in the complex l-plane.

The reason though that Regge took this analytic approach in the first place, was to prove Mandelstam's conjecture stating that a 2×2 scattering amplitude can be written as a representation in double integrals, depending on the two variables s and t. Mandelstam's conjecture was part of a bigger scheme, the theory of the S-matrix, which attempted to provide a complete theory of the scattering in strong interactions based on three mathematical properties, each with a physical interpretation. These were unitarity of the scattering matrix, analyticity of the amplitude and crossing symmetry. The first one corresponds to conservation of probability in a scattering process and analyticity is tied to causality. The third property, crossing symmetry, states that if we exchange a particle in the initial and final state and replace them with their antiparticles, we will derive the same scattering amplitude, which of course corresponds to a scattering process in another channel. This last condition can be seen to apply generally in bootstrap schemes, as it poses great costraints for the theory at hand.

Following Regge's breakthrough result Gribov disproved, the standard then, black disc model, which considered hadrons to have a finite interaction radius and to be independent of the collision energy. His approach was to derive inconsistencies for the diffraction in the s-channel from the crossing equation, as t-channel violated unitarity conditions in the partial waves expansion. This is a very interesting result as it demonstrates how one can use crossing symmetry and analyticity in order deduce results for high energy processes from low energy regions. Gribov went a lot further than that, since he formulated the main properties of asymptotic scattering behaviour. These properties briefly summarise to a particular behaviour of hadron scattering in high energy. In fact hadrons become grey to interactions in the high energy limit, while they increase in size. Therefore the total cross section maintains a constant asymptotic behaviour, in contrast with the ever increasing cross section predicted from the black disc model.

Furthermore he introduced partial waves with complex angular momenta, that were used in a relativistic frame. This later became known as the Froissart-Gribov projection and led to analyticity of the partial waves in the complex l-plane, a result based on the unitarity of the crossing channel. This scheme allowed him to prove that in a 2×2 scattering the unitarity condition in the t-channel generates a moving pole singularity $a(t)$ in the complex l-plane. This singularity associates high energy (s-channel) scattering, with an exchange of a "particle" with spin $a(t)$ and t momentum transfer. This virtual particle is called reggeon and correlates the high energy behaviour, with resonances in low energy scattering. Finally this enabled Gribov to reduce through a diagrammatic technique the high energy scattering to a non-relativistic quantum field theory in 2+1 dimensions, which became known as Reggeon Field Theory (RFT).

The idea from the previous discussion, that we will extensively put to use, is the partial waves representation with complex angular momenta. In our case it will manifest as the principal series expansion derived in Appendix A and the role of angular momenta will be played by complex scaling dimensions. This expansion will lead to the Inversion formula an expression analytic in spin J, which presents a striking resemblance with Froissart-Gribov formula. This formula is a result that came along with various others, when Regge Theory was still flourishing and we will present its breef proof here.

We only need to consider a scattering amplitude $A(s,t)$ for a proccess involving four identical scalar fields with mass μ, a partial wave representation in terms of Legendre polynomials $P_n(z)$

$$f_n(t) = \frac{1}{2}\int_{-1}^{1} P_n(z)A(t,z)dz \tag{3.2}$$

and recall the Legendre function of the second kind for integer n

$$Q_n(z) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(z')}{z-z'}dz' \tag{3.3}$$

along with its complex valued form for $-1 < z < 1$

$$Q_n(z+i\epsilon) - Q_n(z-i\epsilon) = -i\pi P_n(z) \tag{3.4}$$

As discussed before our analysis will take part in the complex l-plane, where we will perform our contour integration. By replacing (3.4) in (3.2), the integral can be written as a contour integral in the complex l-plane. The contour (A) shown in Fig.(3.1) circles closely the $-1 < z < 1$ interval on the real axis and allows us to write

$$f_n(t) = \frac{1}{2\pi i}\oint_{(\mathrm{A}} Q_n(z)A(t,z)dz \tag{3.5}$$

The scattering amplitude $A(s,t)$ exhibits two branch points z_1, z_2 inherited from the s- and u-channel threshold $s = u = 4\mu^2$. We will therefore perform the contour deformation (B) around these branch cuts on the real z-axis, provided we do not encounter additional poles in the deformation. Note that in this contour deformation when z goes to infinity we do not get infinite contributions, since the Legendre functions of the second kind present the asymptotic behaviour

$$Q_n(z) \approx \frac{c}{z^{n+1}}, \quad |z| \to \infty \tag{3.6}$$

For sufficient large n we can write the following expression

$$f_n = \frac{1}{\pi}\int_{z_1}^{\infty} Q_n(z)A_1(z,t)dz + \frac{(-1)^n}{\pi}\int_{z_2}^{\infty} Q_n(z)A_2(-z,t)dz \tag{3.7}$$

where $A_{1,2}(z,t)$ are the discontinuities of the amplitude in each branch cut. We also made use of the relation $Q_n(-z) = (-1)^{n+1}Q_n(z)$. Here analytic continuation in the l-plane is explicit and an interesting feature of (3.7) is that there is no analytic function in relativistic scattering to involve both even and odd spin. To organize a bit better our previous expression we introduce the notation $f_l^{\pm}$ for even and odd spin in relation (3.7) respectively.

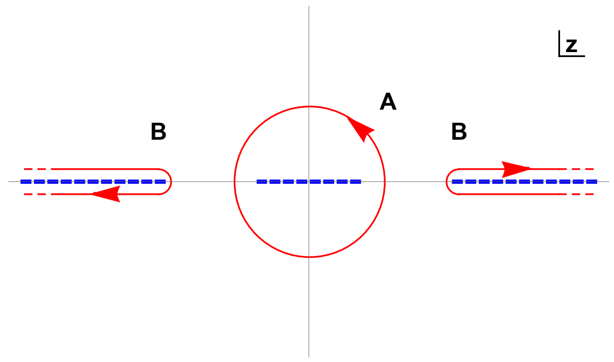

Figure 3.1: Contour deformations of Froisart-Gribov formula.

Now we are ready to represent the scattering amplitude as two sums of even and odd spin contributions $A^{\pm}$. Specifically we have

$$A = A^+ + A^- = \sum_{n=2r}(2n+1)P_n(z)f_n^+ + \sum_{n=2r+1}(2n+1)P_n(z)f_n^- \tag{3.8}$$

This formula was proved in detail not only for its significance in RFT but also because of the similarities it shares in its derivation with our inversion formula. Perhaps the most important one is that of the non-contributing asymptotic behaviour of the amplitude at infinities, also known as the Regge limit. The proper behaviour in the Regge limit, not only allows us to write an expression such as (3.7), but also constrains the overall behaviour of the coefficients f_n, as they cannot vary uncontrollably with respect to each other. On the contrary if we had to add contributions of the arcs in (3.8) we would not have analyticity in angular momenta. Also both formulas share the decomposition in two discontinuities of two different channels and have distinct contributions for even and odd values of spin. The detailed proof though for the inversion formula is much more complex and requires the use of more subtle arguments.

We've already mentioned previously that Regge theory paved the way for the extended use of string theories in attempts to describe the quantum nature of gravity. Since we know that conformal gauge theories are equivalent to Anti-de-Sitter (AdS) space string theories, it would be natural to extend the formalism of Regge theory to its conformal counterpart. As a matter of fact this reasoning was presented in [[13] and among other results the authors generalized Regge theory in Mellin amplitudes, which are the natural way of describing CFT scattering processes. In fact they also arrived in the principal series representation that we will use when we discuss the Inversion formula, but we will not settle with that derivation. Instead in Appendix A we will present a different way of obtaining and applying the principal series representation in a conformal field theory with an $SO(d+1,1)$ symmetry.

3.1.2 The Inversion formula

Following the previous discussion we are ready to present the Froissart-Gribov formula for the OPE data in conformal field theory. In order to preserve the compactness of the analysis we spare the technical details of the formula's derivation and discuss them in Appendix A along with the origins of the principal series expansion. This expansion is a representation consisting of the conformal partial waves we mentioned in Chapter 2. They form a complete base of single-valued functions and decompose as

$$\Psi^{\Delta_i}_{\Delta,l}(x_i) = K^{\Delta_3,\Delta_4}_{\tilde{\Delta},J} G^{\Delta_i}_{\Delta,l}(x_i) + K^{\Delta_1,\Delta_2}_{\Delta,l} G^{\Delta_i}_{\tilde{\Delta},l}(x_i) \tag{3.9}$$

where we have considered four scalar operators O_i with scaling dimensions Δ_i and denote with $\tilde{\Delta} = d - \Delta$ the scaling dimension of the "shadow" block. The coefficients in front of the blocks are given in by (A.10).

The four-point function of these operators can be written in terms of the principal series expansion as

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \sum_{J=0}^{\infty}\int_{d/2}^{d/2+i\infty}\frac{d\Delta}{2\pi i}\frac{I_{\Delta,l}}{n_{\Delta,l}}\Psi^{\Delta_i}_{\Delta,l}(x_i) \tag{3.10}$$

At this point we communicate with Caron-Huot's original formalism by defining the quantity

$$c(\Delta,l) = \frac{I_{\Delta,l}}{n_{\Delta,l}}K^{\Delta_3,\Delta_4}_{\tilde{\Delta},l} \tag{3.11}$$

and through the use of the relation (A.13) the four-point function admits the decomposition

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \sum_{l=0}^{\infty}\int_{d/2-i\infty}^{d/2+i\infty}\frac{d\Delta}{2\pi i}c(\Delta,l)G^{\Delta_i}_{\Delta,l}(x_i) \tag{3.12}$$

where the integration region in (3.12) was extended using a change of variables and shadow symmetry. For the rest of this discussion we stick to the formalism presented here, although in Appendix A we use the formalism and ideas of [43]. Nevertheless the two formalisms are completely equivalent but in the literature we usually come around the formalism of Caron-Huot.

Having deduced the relation (3.12) we can compare it with the usual form of the four-point function we discussed in Chapter 2. The two relations are equivalent and in fact by performing the contour integration in the principal series expansion, we should be able to arrive in our familiar four-point function. This though implies that the function $c(\Delta,l)$ has a very specific form in order for the contour integration to reproduce the appropriate result. In fact its poles in the Δ variable must be located at the scaling dimensions of physical operators in a given theory and the residues of these poles must be equal to their coefficients in the OPE. Thus this function should be of the schematic form

$$c(\Delta,l) \sim -\frac{a_l}{\Delta-(\tau+l)} \tag{3.13}$$

where we denote with a_l the squared OPE coefficients.

By performing the contour deformation depicted in Fig.(3.2) we pick up the poles along the real Δ-axis that correspond to the scaling dimensions of the physical operators. The arcs at infinity can be dropped as we perform this contour integration in Euclidean kinematics (2.64) where $|\rho| < 1$ and for large real Δ the conformal blocks vanish exponentially with a power law $|\rho|^{\Delta}$ [10]. This provides an appropriate Regge limit behaviour, just like in the Froissart-Gribov formula discussed previously.

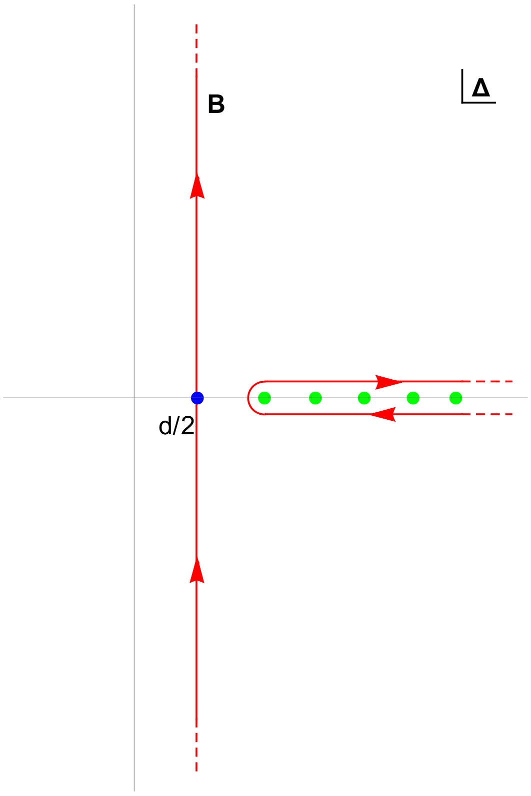

Figure 3.2: Contour deformation of the four-point function in the principal series expansion.

We now have a very firm motivation to invert (3.12). This is doable since the CPWs in the principle series expansion are single-valued and obey an orthogonality relation (A.14). Thus by integrating against them we have

$$c(\Delta, l) = N(\Delta, l) \int d^2 z \mu(z, \bar{z}) \Psi^{\Delta_i}_{\Delta, l} \mathcal{G}(z, \bar{z}) \tag{3.14}$$

where the integration runs over the entire complex plane ($\bar{z} = z^*$). The factor $N(\Delta, l)$ originates from the orthogonality relation we used and is proportional to (A.18), while the measure $\mu(z, \bar{z})$ can be derived from the self-adjointness of the Casimir differential operator (2.66) to be

$$\mu(z, \bar{z}) = \left| \frac{z - \bar{z}}{z\bar{z}} \right|^{d-2} \frac{\Big((1-z)(1-\bar{z})\Big)^{a+b}}{(z\bar{z})^2}, \; a = \frac{\Delta_2 - \Delta_1}{2}, b = \frac{\Delta_3 - \Delta_4}{2} \tag{3.15}$$

We also used the more convenient expression (2.62), where the previous four-point function is now expressed in terms of cross-ratios plus a multiplicative constant.

This inversion formula is not of much interest yet. The full expression of the four-point function is needed for this inversion to work and no new information can be obtained from this result. However if we restricted our analysis in Lorentzian kinematics, there is a possibility for a more interesting formula to arise, which involves other features that contain information regarding the CFT data. Fortunately this is true and by handling the above formula in Lorentzian signature we arrive in an extraordinary result.

We will now present, without dueling with the intermediate details yet, the final inversion formula for the s-channel OPE coefficients. This formula features analyticity in spin $l > 1$ and the CFT-data now are encoded in the double discontinuities of the function $\mathcal{G}(z, \bar{z})$. This celebrated result has the form

$$c(\Delta, l) = c^t(\Delta, l) + (-1)^l c^u(\Delta, l) \tag{3.16}$$

where the two other channel contributions are the integrals

$$c^t(\Delta, l) = \frac{\kappa_{\Delta+l}}{4} \int_0^1 dz d\bar{z} \mu(z, \bar{z}) G_{l+d-1, \Delta+1-d} \mathrm{dDisc}[\mathcal{G}(z, \bar{z})] \tag{3.17}$$

$$c^u(\Delta, l) = \frac{\kappa_{\Delta+l}}{4} \int_{-\infty}^0 dz d\bar{z} \mu(z, \bar{z}) G_{l+d-1, \Delta+1-d} \mathrm{dDisc}[\mathcal{G}(z, \bar{z})] \tag{3.18}$$

and defined the quantity

$$\kappa_\beta = \frac{\Gamma(\frac{\beta}{2} - a)\Gamma(\frac{\beta}{2} + a)\Gamma(\frac{\beta}{2} - b)\Gamma(\frac{\beta}{2}) + b}{2\pi^2 \Gamma(\beta)\Gamma(\beta - 1)} \tag{3.19}$$

Before we discuss this formula let us make a few comments. The use of Lorentzian signature led to the development of brunch cut singularities along the lightcone distances of the scalars in the correlator. Therefore the double discontinuities appeared, when we deformed the contours around these brunch cuts. The branch cuts on the plane of the contour deformations were on regions of t- and u-channels, hence the indices of these contributions in (3.16). They are pretty similar to the branch cuts of (3.8) on the positive and negative axis. Also for the t-channel we define

$$\begin{aligned} \mathrm{dDisc}[\mathcal{G}(z, \bar{z})] = 2\cos(\pi \frac{\Delta_2 - \Delta_1 + \Delta_3 - \Delta_4}{2}) \mathcal{G}(z, \bar{z}) - \frac{1}{2} e^{-i\pi \frac{\Delta_2 - \Delta_1 + \Delta_3 - \Delta_4}{2}} \mathcal{G}^{\circlearrowleft}(z, \bar{z}) \\ - \frac{1}{2} e^{i\pi \frac{\Delta_2 - \Delta_1 + \Delta_3 - \Delta_4}{2}} \mathcal{G}^{\circlearrowright}(z, \bar{z}) \end{aligned} \tag{3.20}$$

where $\mathcal{G}^{\circlearrowleft}$ or $\mathcal{G}^{\circlearrowright}$ imply a contour deformation around the $\bar{z} = 1$ branch point, leaving z fixed, in the direction indicated. For the u-channel contribution we have the similar definition

$$\begin{aligned}\mathrm{dDisc}[\mathcal{G}(z,\bar{z})] = 2\cos(\pi\frac{\Delta_2-\Delta_1+\Delta_3-\Delta_4}{2})\mathcal{G}(z,\bar{z}) - \frac{1}{2}e^{-i\pi\frac{\Delta_3-\Delta_4+\Delta_2-\Delta_1}{2}}\mathcal{G}^{\circlearrowleft}(z,\bar{z})\\ - \frac{1}{2}e^{i\pi\frac{\Delta_3-\Delta_4+\Delta_2-\Delta_1}{2}}\mathcal{G}^{\circlearrowright}(z,\bar{z}) \qquad (3.21)\end{aligned}$$

where this time $\bar{z}$ goes around $-\infty$ and z stays fixed. By comparison to (3.8) we can think of these double discontinuities as the analogue of a CFT "amplitude". Notice the interchanged roles of the arguments in the conformal blocks appearing in (3.17) and (3.18). This exchange is responsible for the analyticity in spin in our final result, but unfortunately for $l = 0, 1$ the formula may or may not behave properly. It can even give results differing from its Euclidean analogue.

We would like to bring (3.16) in a more convenient form, where the mechanism that generates the poles becomes evident. This is possible since the poles appear from the integration on the $z \to 0$ boundary, as we will see shortly. It is convenient to use the decomposition of the block into pure power solutions, a result arising through the Gegenbauer polynomials decomposition of the conformal block in the small z limit [10]. This decomposition is

$$G_{\Delta,l}(z,\bar{z}) = g^p_{\Delta,l}(z,\bar{z}) + \frac{\Gamma(l+d-2)\Gamma(-l-\frac{d-2}{2})}{\Gamma(l+\frac{d+2}{2})\Gamma(-l)}g^p_{\Delta,-l-d+2}(z,\bar{z}) \qquad (3.22)$$

where p stands for pure. The second block won't bother us, since it does not contribute to the poles at $\Delta > d/2$ of Fig.(3.2). This decomposition is useful, because for small values of z the power solutions behaves like

$$g^p_{\Delta,l} = z^{\frac{\Delta-l}{2}}k_{\Delta+l}(\bar{z}), \;\; k_\beta(\bar{z}) = \bar{z}^{\beta/2}\,{}_2F_1(\beta/2+\alpha, \beta/2+\beta, \beta, \bar{z}) \qquad (3.23)$$

We can therefore rewrite the inversion formula in a form that secludes the poles of $c(\Delta, l)$, hence the CFT-data information. By focusing only on the t-channel contribution this relation takes the form

$$c^t(\Delta,l)\Big|_{poles} = \int_0^1 \frac{dz}{2z}z^{\frac{l-\Delta}{2}}\int d\bar{z}\kappa_{\Delta+l}\mu(z,\bar{z})\frac{g^p_{l+d-1,\Delta+1-d}(z,\bar{z})}{z^{\frac{l-\Delta}{2}-1}}\mathrm{dDisc}[\mathcal{G}(z,\bar{z})]. \qquad (3.24)$$

The factor of 2 appearing in the denominator is a result of restricting to $\bar{z} > z$ in order to prevent double-counting. We did not perform all of the simplifications yet, because this relation depicts that the $\bar{z}$-integral generates a term $z^{\frac{\tau}{2}}$ corresponding to a pole $\frac{1}{\tau+l-\Delta}$. Thus if we focus now on the $z \to 0$ limit, we find that the $\bar{z}$ integral is independent of z and we can define the generating function of $SL_2(R)$ blocks

$$C^t(z,\beta) \equiv \int_z^1 \frac{d\bar{z}(1-\bar{z})^{\alpha+\beta}}{\bar{z}^2} \kappa_\beta k_\beta(\bar{z}) \mathrm{dDisc}[\mathcal{G}(z,\bar{z})] \tag{3.25}$$

The combination $\beta = \Delta + l$ is called conformal spin in the literature, since it is the Casimir invariant of the $SL_2(R)$ conformal symmetries in a null ray. This generating function projects $SL_2(R)$ primaries along the null direction $\bar{z}$ but fails to subtract the descendants along the z-directions. Most of the times though we will be interested in the leading twist behaviour and the relation (3.25) will suffice. We will discuss the subleading corrections in section 4.2.1, and there will be a few comments on the matter. We can also write a relation analogous to the even and odd spin decomposition of (3.8).

$$C^{(\pm)}(z,\beta) = \sum C_m^{(\pm)}(\beta) z^{\frac{1}{2}\tau_m^{(\pm)}(\beta)}, \;\; C^{(\pm)} \equiv C^t \pm C^u \tag{3.26}$$

In the examples to come, we will be studying cases with identical scalar operators appearing in the four-point function. Therefore since the u-channel contribution relates to the t-channel by the exchange of points x_1 and x_2, it is straightforward to change the integration limits in (3.18) and arrive in an identical result with the t-channel. Our formula then simplifies to the relation (3.24), with an additional factor of $1 + (-1)^l$ in front of it.

Finally if we are interested in an OPE coefficient at a fixed spin l, we would integrate the residue of $C^{(\pm)}(z,\beta)$ against a delta function

$$a_l = -\int dl \oint \frac{d\Delta}{2\pi i} c(\Delta,l)\delta(l'-l) = -\mathbf{Res}\, c(\Delta,l)\Big|_{\Delta=\Delta_l}. \tag{3.27}$$

Nevertheless the location of the pole depends on the quantity $l = \frac{1}{2}(\beta - \tau)$ with l fixed. Therefore when we integrate the delta function we should divide with the Jacobian factor

$$J = \frac{1}{2}\partial_\beta\Big(\beta - \tau_m^{(\pm)}(\beta)\Big) \tag{3.28}$$

and write for the squared OPE coefficients the result

$$a_l = J^{-1} C_m^{(\pm)}(\beta)\Big|_{l=\frac{\beta-\tau}{2}} \tag{3.29}$$

This formula has been deducted also as an arithmetic result in large-l expansions [3],[42]. However this new result is analytic almost in all values of the spin and this trait will prove to be very helpful in the sections to come.

3.2 Large spin perturbation theory

The Lorentzian inversion formula has a very interesting field of appliance, which is large spin perturbation theory in CFTs and falls in the broader category of theories which examine interacting CFTs. The methods developed for these kind theories made the introduction of twist conformal blocks (TCBs), as they incorporate more effectively the behaviour around degenerate points with infinite accumulation of operators of a certain twist. These schemes extract universal results in the spirit of analytic bootstrap, by exploiting the restrictive conditions posed by the crossing equation and the convenient properties of TCBs.

By studying the double-lightcone limit (equivalently the small u, v limit) of the TCBs the crossing equations transforms to an algebraic problem, because we relate the discontinuities of each channel and demand for them to be reproduced in both sides of the crossing equation. Note that for these values of the cross ratios we are in Lorentzian regions, which already makes the use of the inversion formula tempting. Finally these methods are especially powerful in a perturbative setting with a small parameter g, since the accumulation of operators becomes further organized as we will see shortly.

The first step towards understanding the large spin spectrum of scalar operators ϕ, is to understand the form of operators that the spectrum contains. Under the assumption of operator associativity, the additivity property of the large spin section has been proved for generic CFTs with $d > 2$ [20]. Lets review the implications of this property for the simple example of a scalar operator ϕ and its OPE in a generic CFT. Due to associativity of the operator algebra the OPE must contain a tower of operators of the form

$$\phi \times \phi = 1 + [\phi, \phi]_{n,l} + \dots \tag{3.30}$$

where $[\phi, \phi]_{n,l}$ is proportional to $\phi \Box^n \partial_{\mu_1 \dots \mu_l} \phi$ and for large values of spin its scaling dimension approaches

$$\Delta_{n,l} = 2\Delta_\phi + 2n + l + \frac{c}{l^{2m}} + \dots \tag{3.31}$$

This reasoning though implies the existence of additional towers because the OPE also contains the term $[\phi,[\phi,\phi]_{n',l'}]_{n,l}$. By iteration we deduce that the CFT spectrum contains infinite accumulation points in the spectrum of twists. We therefore point the fact that a CFT with a scalar operator ϕ must contain infinite sequencies of operators $O_{\tau,l}$ with twist $\tau = \Delta - l$, approaching $2\Delta_\phi + 2n$ for each integer n, as l becomes arbitrary large. The proof of this claim is a straightforward result presented in [20] and involves arguments which rely on consistency conditions of the crossing equation.

This seemingly uncontrollable behaviour can not be studied appropriately in the standard conformal block representation that we are used to. Instead of taking this approach the author of [1] introduced twist conformal blocks (TCBs)

$$H_\tau^{(0)}(u,v) = \sum_l a_{\tau,l}^{(0)} u^{\frac{\tau}{2}} g_{\tau,l}(u,v) \tag{3.32}$$

which allow us to write the four-point function (2.62) as

$$\mathcal{G}(u,v) = \sum_\tau H_\tau^{(0)}(u,v). \tag{3.33}$$

Here we made use of the notation $G_{\Delta,l}(u,v) = u^{\frac{\tau}{2}} g_{\tau,l}(u,v)$ and $a_{\tau,l}^{(0)}$, which will stand for the squared OPE coefficients when the expansion parameter g equals zero. These TCBs correspond to the contribution from a twist τ over all spins l and exhibit high levels of degeneracy.

An instructive start, in order to understand the large spin perturbation theory systematics, would be to consider a generic d-dimensional CFT with higher spin symmetry. Higher spin (HS) symmetry states that the existence of a single conserved current in a free theory implies the existence of an infinite number of higher spin conserved currents [44] saturating the unitarity bounds for the scaling dimensions of primary operators. Under this setup the authors of [2] proved that in a CFT with a scalar operator ϕ satisfying a conservation equation, there is a HS symmetric point where the four-point function of the theory reduces to the free field theory correlator. Our aim will be to break the HS symmetry with a perturbation parameter g and examine the behaviour of the spectrum under this new configuration. The crossing equation at zeroth order is of course our familiar

$$v^{\Delta_\phi^{(0)}} \mathcal{G}^{(0)}(u,v) = u^{\Delta_\phi^{(0)}} \mathcal{G}^{(0)}(v,u) \tag{3.34}$$

and the four-point function is written in terms of the TCB decomposition as

$$\mathcal{G}^{(0)}(u,v) = 1 + \sum_l a_l^{(0)} u^{\frac{d-2}{2}} g_{d-2,l}(u,v) \tag{3.35}$$

.

The free field correlator in d-dimensions can be found using Wick contractions to be equal to

$$\mathcal{G}^{(0)}(u,v) = 1 + u^{\frac{d-2}{2}} + \left(\frac{u}{v}\right)^{\frac{d-2}{2}} \tag{3.36}$$

and along with the CB expansion (3.35) suffice to compute the OPE coefficients $a_l^{(0)}$ of the free theory. These are found to be

$$a_l^{(0)} = (1 + (-1)^l) \frac{(\Delta_\phi)_l^2}{l!(l + 2\Delta_\phi - 1)_l}. \tag{3.37}$$

a result first predicted in [35], which agrees with the result obtained in [44]. This discussion at zeroth order also illustrates the possible behaviours of the decomposition in TCBs. There can be contributions with finite support in spin just like the identity operator in (3.36) and contributions resulting from infinite towers of operators for a given twist (in our case $\tau = d - 2$). Conclusively the general behaviour of the contributions to the four-point function, are either isolated operators with finite support in spin or tower of operators with unbounded spin. This becomes more evident as we introduce a perturbative parameter g, that organises further this accumulations of twists.

By breaking HS symmetry with a perturbation parameter g the correlator decomposes as

$$\mathcal{G}(u,v) = \mathcal{G}^{(0)}(u,v) + g\mathcal{G}^{(1)}(u,v) + \dots \tag{3.38}$$

since we expand the factor

$$u^{\tau/2} = u^{\Delta_\phi}(1 + \frac{1}{2}\gamma \log u + \frac{1}{8}\gamma^2 \log^2 u + ...) \tag{3.39}$$

contained in TCBs. The powers of logarithms in u are incorporated inside the conformal blocks for every order in perturbation theory. At first order the crossing equation (3.34) reads

$$\gamma_\phi^{(1)} \log v \mathcal{G}^{(0)}(u,v) + \mathcal{G}^{(1)}(u,v) = \gamma_\phi^{(1)} \log u \mathcal{G}^{(0)}(u,v) + \left(\frac{u}{v}\right)^{\frac{d-2}{2}} \mathcal{G}^{(1)}(u,v) \tag{3.40}$$

where we have expanded the extra powers of u, in terms of the anomalous dimensions appearing in the exponents. Also we expect the field ϕ and the intermediate operators appearing in the spectrum to acquire anomalous dimensions and their scaling dimensions to be of the form

$$\Delta_\phi = \frac{d-2}{2} + g\gamma_\phi^{(1)} + \dots \tag{3.41}$$

$$\Delta_l = d - 2 + g\gamma_l^{(1)} + \dots \tag{3.42}$$

The divergent behaviours of the terms in the crossing equation, that concern LSPT are known as enhanced singularities. These are defined as contributions contained in the CB expansion, which generate divergencies under finite actions of the Casimir operator upon them. For that to be possible they must arise from an infinite sum over spin, as individual CBs are eigenfunctions of the Casimir operators. We will see that the enhanced divergencies of LSPT can be identified as the double-discontinuities appearing in the inversion formula.

By studying the relation (3.40) at the double lightcone limit the authors of [2] deduced that except for the case of four dimensions, there are no non-trivial solution at first order in the expansion parameter. For the case of four dimensions the only operator acquiring an anomalous dimension at first order, is the bilinear operator with twist two and spin zero and the external operator ϕ has dimensional corrections only at order g^2. This analysis was rendered possible by the appliance of another famous property deduced in the context of analytic bootstrap [3], which is commonly referred as *reciprocity*. This property simply states that the anomalous dimensions and OPE coefficients of the theory can be written as expansions in inverse powers of the conformal spin. The Casimir operator which has the conformal spin as an eigenvalue is

$$\mathcal{C} = C + \frac{1}{4}\tau(2d - \tau - 2) \tag{3.43}$$

and from now on we will use the more usual definition

$$J^2_{\tau,l} = J^2 = \frac{1}{4}(2l+\tau)(2l+\tau-2). \tag{3.44}$$

Therefore we can write the relations

$$\gamma_l = 2\sum_m \frac{B_m}{J^{2m}}, \qquad a_l = \sum_m \frac{A_m}{J^{2m}} \tag{3.45}$$

which make sensible to introduce the functions

$$H_{d-2}^{(m)}(u,v) = \sum_l a_l^{(0)} \frac{u^{\frac{d-2}{2}}}{J^{2m}} g_{d-2,l}(u,v) \tag{3.46}$$

as they incorporate the anomalous dimensions appearing in the crossing equation at each order in a straightforward manner. Here m measures the departure of the TCB from the degenerate point and for $m = 0$ we have the TCB we introduced above. These functions complement naturally the previous analysis as they express the enhanced divergencies, which appear at each order in the crossing equation. They satisfy the recursion relation

$$\mathcal{C} H_{d-2}^{(m+1)}(u,v) = H_{d-2}^{(m)}(u,v) \tag{3.47}$$

and given the appropriate boundary conditions we can recover them for every $m \geq 0$. The very first TCBs needed for the analysis of [2] are reproduced in Appendix B.

The same line of thought may continue at second order in g and the LSPT systematics can be immediately applied in the well known Wilson-Fischer model in $d = 4 - \epsilon$ in order to reproduce its spectrum up to second order in the expansion parameter. Notice that these results follow without any reference in the Lagrangian description or equations of motion. Our essential assumptions come solely from conformal symmetry, along with invariance of the crossing equation under any global transformation. Using the same arguments we could reproduce the spectrum of the large N critical O(N) model or that of a weekly coupled gauge theory e.g. the $\mathcal{N} = 4SYM$. In both cases the previously known results in the literature are in perfect agreement with the results of LSPT [2].

This concludes our discussion in LSPT. The reproduction of the enhanced divergencies in both sides of the crossing equation in the double lightcone limit, posed sufficient conditions in the theory in order for the spectrum to be reproduced up to second order. We could continue with the same reasoning and obtain the corrections in the spectrum for higher order in perturbation theory, but the analysis would complicate and the sames schemes would not be efficient. This is the result of the ad hoc manner under which LSPT extracts the information from the crossing equation at each order. In fact the comparison of divergencies in each side is not guided by a principle, which could be used as a guide at higher orders. Fortunately we have a tool in our disposal, the inversion formula, that organises these divergencies in a more systematic way.

The inversion formula in principle obtains the CFT data that are encoded in its poles by extracting information from the double discontinuities of the conformal blocks. These double discontinuities though are exactly what we called enhanced divergencies in the language of LSPT. They can not be generated by a finite number of conformal blocks and express the asymptotic behaviour in the double lightcone limit we consider. Therefore instead of matching the divergencies of the crossing equation we could decompose the r.h.s of the crossing equation and demand the reproduction of the contributions in the inversion formula for its both sides.

The combination of these two schemes provides us with a new way of approaching the derivation of CFT data in a given theory. Using them we will attempt to find results up to fourth order in the same perturbative setting and create a toolset that has a varied appliance in many theories.

3.3 Combination of the two methods

Our first concern in order to combine the tools of the two previous sections, is to introduce a more appropriate way of handling the inversion formula in a perturbative setting. In fact when we introduce a small parameter g that brakes the higher spin symmetry all the other operators of the theory acquire small anomalous dimensions at a certain order. We can even consider the specific case of an infinite family of operators with twist $\tau_{O_l} = 2\Delta_\phi + \gamma_l$. Then as we saw previously the inversion formula contains the power $z^{\tau/2}$, which creates a pole of the form $\frac{1}{\tau+l-\Delta}$ and corresponds to the exchange of an operator with twist τ. This implies the schematic form

$$c(\Delta, l) \sim \sum_{O_l} \frac{-a_{O_l}}{\Delta - l - 2\Delta_\phi - \gamma_l} \cdots = \sum_{p=0}^{\infty} \sum_{O_l} \frac{-a_{O_l}(\gamma_l)^p}{(\Delta - l - \Delta_\phi)^{p+1}} \cdots \tag{3.48}$$

where in the second equality we expanded the denominator in terms of the small quantity γ_{O_l}. This relation though indicates a different scenario where the poles collide and come in higher orders, corresponding to different powers of the anomalous dimension. Therefore we need a more convenient way of expressing the residues arising from these poles.

First we recall that our analysis concentrates in the double lightcone limit where the cross ratios $z, \bar{z}$ approach zero and one respectively. We can therefore expand the correlator in the small z limit [15] as

$$\begin{aligned}\mathcal{G}(z,\bar{z}) &= \sum_l a_l z^{\tau/2} k_{\frac{\Delta_l+l}{2}}(\bar{z}) + O(z^{\Delta_\phi+1}) \\ &= z^{\Delta_\phi} \sum_p \frac{1}{2^p p!} \log^p(z) f_p(\bar{z}) + O(z^{\Delta_\phi+1}) \end{aligned} \tag{3.49}$$

where in the second line we expanded the power of z and defined the function

$$f_p(\bar{z}) := \sum_l a_l \gamma_J^p k_{\frac{\Delta_l+l}{2}}(\bar{z}). \tag{3.50}$$

We further define the quantities $\bar{h} = \bar{h}_b = \Delta_\phi + l$ and $\bar{h}_f = \frac{\Delta_l+l}{2}$ and a new normalization for the OPE coefficients: $\frac{\Gamma(\bar{h}_f)^2}{\Gamma(2\bar{h}_f-1)}\hat{a}_{O_l} = a_{O_l}$. Then together with the anomalous dimensions of the theory they can be recovered through the relation

$$\hat{a}_{O_l}\gamma^p_{O_l} = U^{(p)}_{\bar{h}} + \frac{1}{2}\partial_{\bar{h}} U^{(p+1)}_{\bar{h}} + \frac{1}{8}\partial^2_{\bar{h}} U^{(p+2)}_{\bar{h}} \dots \tag{3.51}$$

where we defined the function

$$U^{(p)}_{\bar{h}_f} \equiv \frac{\Gamma(\bar{h}_f)^2}{\Gamma(2\bar{h}_f-1)\pi^2} \int_0^1 d\bar{z} k_{\bar{h}_f} \mathrm{dDisc}[f_p(\bar{z})]. \tag{3.52}$$

Note the difference between the "bare" $\bar{h} = \bar{h}_f - \frac{1}{2}\gamma_l$ and "full" $\bar{h}_f$. The OPE coefficients obtained in (3.51) are expressed in terms of the "bare" variable, as it is a more convenient quantity to manipulate in perturbation theory. Note that from these definitions the conformal spin J is found to be equal to $\bar{h}(\bar{h}-1)$. The expression (3.51) was written in terms of $\bar{h}$ by changing variables and expanding the functions $f_p(\bar{z})$ in terms of the anomalous dimension γ_l^p. Also the presence of logarithmic powers is responsible for the generation of poles, which come in higher powers after integrating the z variable. These expressions for the CFT data were first introduced in [4].

From this standpoint it seems like we are ready to apply the inversion formula in the crossing equation. This would be possible by demanding the double discontinuities of the conformal blocks in the small z limit to appear in both sides of the crossing equation. For that purpose we will write the direct channel in terms of the relations (3.51) and the crossed channel will be expanded in the double lightcone limit. The demand that each side should reproduce the same double discontinuities will constrain completeley the CFT-data. One may wonder if this is a feasible attempt, since the operators appearing in the OPE may be arbitrary and increase the complexity of our computations significantly. This complication is solved naturally though, since we saw in LSPT that there are specific contributions of operators appearing in the spectrum at each order. This simple fact renders our analysis possible.

Despite having all of the main arguments ready, it will be useful to present a direct derivation of the inversion formula, which provides a connection with LSPT. These arguments were first presented in [5],[26] along with important applications of the inversion formula in analytic bootstrap for the Wilson-Fisher and critical $O(N)$ model. These publications will concern us extensively when we discuss the use of inversion formula in practice. Also the frame that we will describe here will be relevant for the analysis of the upcoming chapters.

The main model which will be of our concern is the famous Wilson-Fischer model around four dimensions. Our main assumptions are the same as in LSPT, since we expect our theory to contain a higher-spin symmetric point where it is essentially free and there the correlator to be given by (3.36). As noted before, the intermediate operators appearing in the spectrum are the bilinear operators of twist $2\Delta_\phi + 2n$ and the corrections of the operators appearing in the OPE will start at second order. The only exception is that of the bilinear operator with twist two and spin zero which will have

$$\Delta_0 = 2\Delta_\phi + g. \tag{3.53}$$

and the parameter g will be used as the expansion parameter of our scheme. The dimensionality of spacetime is defined traditionally to be $d = 4 - \epsilon$ and cannot differ arbitrary from d, but has to be of the same order as the expansion parameter. Therefore we conclude that $g \sim \epsilon$. In conclusion we write the expected corrections of the scaling dimensions and OPE coefficients to be of the form

$$\Delta_l = 2\Delta_\phi + l + \gamma_l^{(2)} g^2 + \dots \tag{3.54}$$

$$a_l = a_l^{(0)} + g^2 a_l^{(2)} + \dots \qquad\qquad l = 2, 4, \dots \tag{3.55}$$

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi^{(2)} g^2 \dots \tag{3.56}$$

We are now in place to set the question that will make the existence of the inversion formula emerge. For that we introduce the normalisation of $SL(2,\mathbb{R})$ conformal blocks

$$\begin{aligned} f_{\Delta,J}(\bar{z}) &= r_{\frac{\Delta+l}{2}} k_{\frac{\Delta+l}{2}}, \; r_h = \frac{\Gamma(h)^2}{\Gamma(2h-1)} \\ k_h(\bar{z}) &= \bar{z}^h \, {}_2F_1(h,h,2h,\bar{z}). \end{aligned} \tag{3.57}$$

The inversion problem we would like to solve, would be to find the coefficients $\hat{a}_l$ that satisfy the equation

$$\sum_{\substack{\Delta=2\Delta_\phi+l,\\ l=0,2,\ldots}} \hat{a}_l f_{\Delta,l}(\bar{z}) = G(\bar{z}) \tag{3.58}$$

with $G(\bar{z})$ containing enhanced singularities as $\bar{z} \to 1$. As we discussed before the equivalent of that statement would be to contain double-discontinuities defined as

$$\mathrm{dDisc}[G(\bar{z})] \equiv G(\bar{z}) - \frac{1}{2}G^{\circlearrowleft}(\bar{z})\frac{1}{2}G^{\circlearrowright}(\bar{z}). \tag{3.59}$$

The formulation of this problem in the limit $\bar{z} \to 1$, might as well remind us of the situation in LSPT, where our goal was to decompose the CBs in terms of their enhanced singularities. Here we specify the Casimir equation satisfied by the $SL(2,\mathbb{R})$ CBs in four dimensions to be

$$\bar{D} f_{\Delta,l}(\bar{z}) = J^2 f_{\Delta,l}(\bar{z}) \tag{3.60}$$

where $\bar{D} \equiv \bar{z}^2\bar{\partial}(1-\bar{z})\bar{\partial}$. As mentioned before we are justified to expand the OPE coefficients in inverse powers of the conformal spin J

$$\hat{a}_l \equiv \hat{a}(J) = \sum_m \frac{a_m}{J^{2m}} \tag{3.61}$$

and define in a similar manner as before the functions

$$h^{(m)}(\bar{z}) = \sum_{\substack{\Delta=2\Delta_\phi+l,\\ l=0,2,\ldots}} \frac{f_{\Delta,l}(\bar{z})}{J^{2m}}. \tag{3.62}$$

These can be obtained for arbitrary m through the relation

$$\bar{D} h^{m+1}(\bar{z}) = h^m(\bar{z}). \tag{3.63}$$

As we saw though in practice when we decomposed the CB in terms of these functions, m is expected to obey constrains involving the specific form of $G(\bar{z})$ and the order in perturbation theory we are considering. Therefore our computations will impose conditions regarding the order of m, that we will need to consider each time.

For this scheme to work one always has to find the explicit form of $h^{(m)}(\bar{z})$ for $m=0$. This though can be done explicitly by using the standard representation of the Hypergeometric functions

$${}_2F_1(h,h;2h,z) = \frac{\Gamma(2h)}{\Gamma(h)^2}\int_0^1 dt \frac{t^{h-1}(1-t)^{h-1}}{(1-tz)^h} \tag{3.64}$$

and exchanging the role of summation and integration in (3.62). By integrating the resulting sum we find the first function to be

$$h^{(0)}(\bar{z}) = \frac{1}{2}\frac{\bar{z}}{1-\bar{z}} + \text{regular terms} \tag{3.65}$$

where the regular terms do not exhibit double-discontinuities as $\bar{z} \to 1$. In fact the manipulation we used for this result can be also performed in the case of $m = 1$. We mention the final result to be

$$h^{(1)}(\bar{z}) = \frac{1}{4}\log^2(1-\bar{z}) + \mathrm{Li}_2(\bar{z}) + \log(1-\bar{z}) \tag{3.66}$$

which can be used as a consistency check for the solution we could obtain from the differential equation (3.63). Indeed the two results are in perfect agreement and reproduce correctly the divergent terms. In the previous relation the first term corresponds to the divergent part, while the two other functions are regular around $\bar{z} = 1$, in the sense we are describing these divergencies. Finally we used the definition of the polylogarithm function

$$\mathrm{Li}_n \bar{z} = \sum_{k=1}^{\infty} \frac{\bar{z}^k}{k^n}. \tag{3.67}$$

Instead of finding the OPE coefficients through these functions we would like to obtain them through an inversion formula. The inversion formula does not require the whole CB expansion in order to work but only the double discontinuities of the correlator. It seems sensible then to define the family of projectors

$$\int_0^1 d\bar{z} K^{(m)}(\bar{z}) \mathrm{dDisc}[h^{(m)}(\bar{z})] = \delta^{mn} \tag{3.68}$$

which allow us to write the relation for the coefficients

$$\hat{a}(J) = \int_0^1 d\bar{z} K(\bar{z}, J) \mathrm{dDisc}[G(\bar{z})] \tag{3.69}$$

where $K(\bar{z}, J) = \sum_m \frac{K^{(m)}(\bar{z})}{J^{2m}}$. The previous relation for the OPE coefficients can be inserted in (3.58) and together with the Casimir equation satisfied by the $SL(2,\mathbb{R})$ blocks provides us the equation

$$\bar{D}^\dagger K(\bar{z}, J) = J^2 K(\bar{z}, J). \tag{3.70}$$

where $\bar{D}^\dagger = \bar{\partial}(1-\bar{z})\bar{\partial}\bar{z}^2$.

Since we did not specify the form of these projectors they can be arbitrary as long as they satisfy (3.70). Then by writing the conformal spin J in terms of the bare variable $\bar{h}$ as $J = \bar{h}(\bar{h}-1)$ and using the representation (3.64) we arrive at the result

$$\hat{a}(\bar{h}) = \frac{1}{\pi^2}\int_0^1 dtdz \frac{\bar{z}^{\bar{h}-2}(t(1-t))^{\bar{h}-1}}{(1-t\bar{z})^{\bar{h}}} \mathrm{dDisc}[G(\bar{z})]. \tag{3.71}$$

where we choose the normalization of this solution to be $\hat{a}(J) = 1$ for $G(\bar{z}) = h^{(0)}(\bar{z})$. Note that integrating over t brings us back to the inversion formula (3.16).

This formula is the main tool that we will use in the following chapter. The non-zero contributions of the correlator to the inversion formula, involve logarithms diverging in the limit $\bar{z} \to 1$ or inverse powers of $(1-\bar{z})$, since only these types of functions produce double-discontinuities. Throughout our analysis we will use the inversions for this types of functions summarized in Appendix A of [5], with several of those tested in the context of this thesis.

Finally we note that the expression (3.71) converges when $\bar{h} > 1$, or equivalently when $l > 0$. Indeed the inversion integral produces the correct results for these conditions, but the case of $l = 0$ is not irrelevant and in fact will constitute a key argument for our analysis.

4
Inversion of the W.F. model

The model introduced by Wilson and Fischer [46] for a scalar field ϕ with a quartic term ϕ^4, relied on a Lagrangian description and concerned an interacting theory at $d = 4-\epsilon$ dimensions, where ϵ played the role of an expansion parameter. Their purpose was to approach perturbatively the Ising model in three and four dimensions by inducing an RG flow in the parameter space.

In this chapter we will apply the tools of analytic bootstrap developed in the previous chapters in order to study the CFT data of the W.F. model around four spacetime dimensions, an attempt first presented in [5]. We will follow closely their original derivation. Our overall setup was described previously and we can immediately state the inversion problem at hand. In the spirit of our previous manipulations the crossing equation admits the following decomposition

$$\sum_{\substack{\Delta=\tau_l+l,\\ l=0,2,\ldots}} \hat{a}_l z^{\tau_l/2} f_{\Delta,l}(\bar{z}) = z^{\Delta_\phi}\Big(\frac{\bar{z}}{1-\bar{z}}\Big)^{\Delta_\phi} \mathcal{G}(1-\bar{z},1-z)\Big|_{z\to 0} \tag{4.1}$$

where we wrote the sum over the leading twist operators $O_{\Delta,l}$ appearing in the spectrum at first order and concentrated on the small z limit on the l.h.s. through the relation (3.49). The leading twist operators have twist $\tau_l = 2\Delta_\phi + g^2\gamma_l^{(2)}\ldots$ and the hatted OPE coefficients defined in the previous chapter acquire a correction $a_l^{(2)}$ at order g^2, as was made clear in section 3.2. Note that the sum over l runs over even indices as the W.F. model is invariant under the $\mathbf{Z}_2$ symmetry, therefore it constrains the operators appearing in the OPE.

4.1 Contributions to order ϵ^3

The contributions appearing up to cubic order in the r.h.s. of the crossing equation are those of the identity operator and the bilinear operator with spin zero. This can be understood in the sense that only the bilinear operator with zero spin has an anomalous dimension at first order and the identity operator is always present in the crossing equation. Therefore our inversion problem simplifies to

$$\sum_{\substack{\Delta=\tau_l+l,\\ l=0,2,\ldots}} \hat{a}_l z^{\tau_l/2} f_{\Delta,l}(\bar{z}) = z^{\Delta_\phi}\Big(\frac{\bar{z}}{1-\bar{z}}\Big)^{\Delta_\phi}\Big(1 + a_0(1-\bar{z})^{\Delta_0/2} g_{\Delta_0,0}(1-\bar{z},1-z) + \text{regular}\Big) \tag{4.2}$$

where the regular terms do not contribute to the inversion formula at this order. The OPE coefficient a_0 is also expected to acquire corrections in our perturbative setting. Nevertheless our frame does not suffice completely for determining these corrections and we refer to [9] and [25], which have computed these corrections to be

$$a_0 = 2(1 - g - g^2 + \dots) \tag{4.3}$$

We will use the d-dimensional conformal block for an exchange between two identical scalar operators, which was given in [16]. It is of the form

$$\begin{aligned} g_{\Delta,0}(1-\bar{z},1-z) &= \sum_{m,n=0} \frac{(\Delta/2)_m^2(\Delta/2)_{m+n}^2}{m!n!(\Delta+1-d/2)_m(\Delta)_{2m+n}}(1-z)^m(1-\bar{z})^m(1-\bar{z}z)^n \\ &= \sum_{m=0} \frac{(\Delta/2)_m^2}{(\Delta+1-d/2(\Delta/2)_{2m})}(1-z)^m(1-\bar{z})^m \times \\ &\quad {}_2F_1(m+\Delta/2, m+\Delta/2, 2m+\Delta, 1-z\bar{z}). \end{aligned} \tag{4.4}$$

In the second line we performed the sum over the n-index so as to separate the z-dependence. For the double-discontinuities to arise we must expand the powers of g and calculate the infinite sum on the r.h.s. This calculation can be done symbolically and we find the following results in perfect agreement with those of [5] to be:

$$\begin{aligned} &\sum_{\substack{\Delta=\tau_l+l,\\ l=0,2,\ldots}} \hat{a}_l z^{\tau_l/2} f_{\Delta,l}(\bar{z}) = z^{\Delta_\phi}\Big(\frac{\bar{z}}{1-\bar{z}}\Big)^{\Delta_\phi} + \\ &z^{\Delta_\phi}\bar{z}^{\Delta_\phi} a_0\Big(\frac{g^2}{8}\log^2(1-\bar{z})(1+\epsilon\partial_\epsilon + g\partial_\Delta) + \frac{g^3}{48}\log^3(1-\bar{z})\Big) g^{4d}_{2,0}(1-\bar{z},1-z) \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} g^{4d}_{2,0}(1-\bar{z},1-z) &= \frac{\log\bar{z}-\log z}{\bar{z}} \\ \partial_\epsilon g^{4d}_{2,0}(1-\bar{z},1-z) &= \frac{(\log\bar{z}-\log z)(\log\bar{z}-2)+2\zeta_2}{2\bar{z}} \\ \partial_\Delta g^{4d}_{2,0}(1-\bar{z},1-z) &= \frac{\mathrm{Li}_2(1-\bar{z})+\log\bar{z}-\log z-\zeta_2}{\bar{z}}. \end{aligned} \tag{4.6}$$

Note that in the r.h.s. we expanded the overall power of $(1-\bar{z})^g$ and didn't take into account the $\log(1-\bar{z})$ term at first order, since its double discontinuity is equal to zero.

After this decomposition we further expand the z power on the l.h.s of (4.5) and the inversion problem takes the more familiar form in the order we are considering

$$\sum_{\substack{\Delta=\tau_l+l,\\ l=0,2,\ldots}} \left(U^{(0)}_{\bar{h}}+\frac{1}{2}\log z U^{(1)}_{\bar{h}}\right) f_{\Delta,l}(\bar{z}) = (\textbf{Contributions up to third order}). \tag{4.7}$$

where we took into account the relations of section 3.3

$$\begin{aligned} \hat{a}_l &= U^{(0)}_{\bar{h}}+\frac{1}{2}\partial_{\bar{h}}U^{(1)}_{\bar{h}} \\ \hat{a}_l\gamma_l &= U^{(1)}_{\bar{h}} \end{aligned} \tag{4.8}$$

Now it is straightforward to use the inversion integrals mentioned previously and obtain

$$\begin{aligned} U^{(0)}_{\bar{h}} &= \hat{a}^{(0)}_l - \frac{g^2}{(\bar{h}-1)^2\bar{h}^2} + \frac{\zeta_2(\bar{h}-1)\bar{h}+1}{(\bar{h}-1)^2\bar{h}^2}g^2\epsilon - \frac{\zeta_2(\bar{h}-1)\bar{h}-S_1}{\zeta_2(\bar{h}-1)\bar{h}}g^3+,\ldots \\ U^{(1)}_{\bar{h}} &= -\frac{2g^2}{(\bar{h}-1)\bar{h}} + \frac{2g^2\epsilon}{(\bar{h}-1)\bar{h}} + \frac{2S_1g^3}{(\bar{h}-1)\bar{h}}+\ldots \end{aligned} \tag{4.9}$$

where S_1 stands for the harmonic number with argument $\bar{h}-1$. The contribution of the identity operator corresponds to the zeroth order result

$$\hat{a}^{(0)}_l = \frac{2\Gamma(\bar{h}+\Delta_\phi-1)}{\Gamma(\Delta_\phi)^2\Gamma(\bar{h}-\Delta_\phi+1)}, \tag{4.10}$$

which can be related to the OPE coefficients of the free-field theory if we take into account the definition of $\hat{a}_l$. As a first test of this method we reproduce correctly the free field theory OPE coefficients found in (3.37). The rest of the CFT-data at leading order can be obtained from these expressions, through the relations (4.8).

The CFT-data at leading order admits another decomposition, which is more appropriate for our upcoming discussion. They can be expressed in terms of the "full"

$\bar{h}_f = l + \Delta_\phi + \frac{1}{2}\gamma_l$, which contains in its definition the anomalous dimension γ_l. The prescription we follow is to replace the "bare" variable in (4.9) and expand the result in terms of the anomalous dimensions, in order to arrive in an expression for $\hat{a}_l$. We can solve this expression in terms of γ_l and by further expanding in the perturbative parameter g we acquire the leading order corrections for the scaling dimensions of bilinear operators. These are found to be

$$\Delta_l = 2\Delta_\phi + l - \frac{g^2}{(\bar{h}_f - 1)\bar{h}_f} + \frac{g^2\epsilon + (g^3 - g^2\epsilon)S_1}{(\bar{h}_f - 1)\bar{h}_f} + \dots \tag{4.11}$$

This last relation may not be the final result of this section, but it highlights the virtues of the inversion formula, which we used in combination with crossing symmetry. It involved only a few secluded contributions of specific conformal blocks, whose exchange was dictated from the perturbation theory assumptions. One might feel curious though if we could use sensible constrains that will allow us to find the relation between g and ϵ at leading order. This would determine the corrections in the spectrum of the external operator ϕ and of the intermediate operators $O_{\Delta,l}$. This is precisely what we will examine in the following section and as we will see our reasoning involves the extrapolation of the results obtained from the inversion formula down to spin zero.

4.1.1 Analytic continuation at spin zero

We will impose two sensible conditions upon the small values of spin

$$\Delta_2 = d \tag{4.12}$$

$$\Delta_0 = 2\Delta_\phi + g. \tag{4.13}$$

The first one is nothing more than the conservation of the stress-energy tensor of our theory. Notice that this condition cancels the distinction between the two variables $\bar{h}, \bar{h}_f$ as the anomalous dimension for that case is zero. We can solve in terms of Δ_ϕ in (4.11) and by expanding in the ϵ variable we arrive at the result

$$\Delta_\phi = 1 - \frac{1}{2}\epsilon + \frac{1}{12}g^2 - \frac{1}{8}g^3 + \frac{11}{144}g^2\epsilon + \dots \tag{4.14}$$

The second condition stems from our definition of the expansion parameter g and we will apply it by assuming the standard continuation of (4.11) in $\bar{h}_f$ across its pole at $\bar{h}_f = 1$. We might rephrase that as the analytic continuation of our results down to spin zero. The limit around $\bar{h}_f = 1$ can be considered somewhat stable, since for $l = 0$ the approximate relation $\bar{h}_f - 1 \sim \epsilon$ holds true. Therefore the second condition gives as the relation between g and ϵ to be

$$-g\epsilon + 3g^2 = 0. \tag{4.15}$$

where we matched the terms of the same order in the relation that we obtained from the second condition. The trivial solution $g = 0$ corresponds to the free theory, while the non-trivial solution is

$$g = \frac{1}{3}\epsilon + \dots \tag{4.16}$$

and allows us to write the final result for the scaling dimension of the external operator

$$\Delta_\phi = 1 - \frac{1}{2}\epsilon + \frac{1}{108}\epsilon^2 \dots \tag{4.17}$$

in perfect agreement with the standard values of the Wilson-Fisher model found in the literature.

The extrapolation of our results down to spin zero is a crucial assumption that allowed us to obtain precise results up to third order in the expansive parameter. As an assumption it is interesting on its own, because the inversion formula used for the production of the overall results, fails to make predictions for $l = 0$. Therefore it remains an open question to define the cases when this extrapolation is possible.

4.2 Contributions to order ϵ^4

We will now extend our formalism at second order in perturbation theory, which will allow us to extract results up to fourth order in the expansion parameter. At this order though there are a few complications arising, as we expect new operators to appear in the crossing equation. These are our familiar bilinear operators of twist $\tau = 2\Delta_\phi + 2n$ and spin l. Because for the rest of this analysis we will work for the g^2 contributions we are free to replace the scaling dimension of the external scalar operator with its precise value in four dimensions, which is $\Delta_\phi = 1$.

Our problem is of the same schematic form we previously encountered

$$\sum_{\substack{\Delta=\tau_l+l,\\ l=0,2,\dots}} \left(U^{(0)}_{\bar{h}} + \frac{1}{2}\log z U^{(1)}_{\bar{h}} + \frac{1}{8}\log^2 z U^{(2)}_{\bar{h}}\right) f_{\Delta,l}(\bar{z}) = (\textbf{Contributions up to fourth order}). \tag{4.18}$$

We will start by analysing the new contributions that enter the crossing equation and attempt to determine them completely. We expect the contributions appearing in the previous order to remain in the crossing equation and in fact the bilinear operator of spin zero contributes with terms proportional to g^4. We will denote this contribution as I_{ϕ^2}, and it is obtained by simply taking into account another order in the expansion (4.5). By also using the explicit relation (4.16) for g and ϵ and corrections of the OPE coefficients (4.3) we acquire the contribution of I_{ϕ^2} at fourth order, which is given by the relation (4.42) at the end of this section.

The non-trivial operators entering the spectrum are those with twist two an four. This can be made evident by considering the Wick contractions in a three-point function of bilinear operators with twist greater than four. We immediately see that they contain higher loop computations than the operators of lower twist and therefore they appear at higher order in perturbation theory.

The emergence of the twist two operators can be understood from the order at which they acquire an anomalous dimension. We have already mention that their anomalous dimensions start at order ϵ^2, thus the sum of their squares give a contribution to the double-discontinuity at order ϵ^4. We denote this operators as I_2 and these contributions will be of the form

$$I_2 = \frac{1}{8}\log^2(1-\bar{z}) \sum_{l=0,2,\ldots} \hat{a}_l(\gamma_l)^2 G^{4d}_{2+l,l}(1-\bar{z},1-z)\Big|_{z\to 0} + \mathcal{O}(\epsilon^5). \tag{4.19}$$

The perturbation expansions we have used up until now and the explicit results of the functions appearing in the decomposition of the TCBs, suggest that the functions appearing in the correlator organise themselves in pure transcendental functions. The degree of their transcendality increases with the order of perturbation theory we are considering, which is a familiar setup encountered in other perturbative schemes. These functions are also expected to present a pole around 0 or 1, because of the kinematics we are interested. Having said that we will create an ansatz for the functions, which possibly appear in the conformal block expansion. It contains the set of functions

$$\{1, \log\bar{z}, \log^2\bar{z}, \log^3\bar{z}, \mathrm{Li}_2(1-\bar{z}), \log\bar{z}\mathrm{Li}_2(1-\bar{z}), \mathrm{Li}_3(1-\bar{z}), \mathrm{Li}_3(\frac{\bar{z}-1}{\bar{z}})\}. \tag{4.20}$$

We will impose the condition that the decomposition of the contributions in the limit $\bar{z}\to 1$, can be written as a combination of these functions. This requirement can be expressed equivalently as, the decomposition of the contributions and the decomposition of the functions in the ansatz must have the same Taylor expansions around $\bar{z} = 1$. Now we are ready to put this idea into practice.

For the I_2 contribution apart from this idea, we used an implicit condition to restrict the functions appearing in its decomposition through the TCBs scheme. The OPE coefficients appearing in (4.19) are of zeroth order, since the overall power of the corrections we are considering is four. Therefore the explicit use of the decomposition (2.69) of the CBs in four dimensions and the behaviour $\gamma_l \sim 1/J^2$ to leading order, relate the $\bar{z}$ dependence of (4.19) with a sum of the form

$$h^{(2)}_{\tau,l}(\bar{z}) \equiv \sum_{l=0,2,\ldots} \frac{\hat{a}^{(0)}_l}{J^4} k_{l+1}(1-\bar{z}). \tag{4.21}$$

This function can be obtained from the Casimir recursive relation (3.63) and makes a suggestion on the functions of the ansatz, appearing in the decomposition of I_2. Specifically the functions we should restrict to are

$$\{\log \bar{z}, \log^3 \bar{z}, \mathrm{Li}_3(1-\bar{z}), \mathrm{Li}_3(\frac{\bar{z}-1}{\bar{z}})\}. \tag{4.22}$$

This further restriction allows us to determine the $\bar{z}$ contribution explicitly, by demanding the agreement of the Taylor expansions between the ansatz and (4.21). The z contribution though can be determined completely from the TCBs formalism due to its simpler behaviour around the small-z limit. We refer to the explicit result in (4.40).

The contribution from operators with twist four, denoted as I_4, can be traced in the Lagrangian description of the model which contains a quartic coupling. This time not only the anomalous dimension produce this contribution at fourth order, but also their OPE coefficients contain second order corrections. Their computation relies again on the ansatz scheme but it contains an additional setback. Twist two operators were not degenerate in the spectrum of operators, as the equation of motion canceled the combinations which could potentially cause mixing. However this is not the case for twist four operators, as their schematic form $\phi^2\partial_{\mu_1}\ldots\partial_{\mu_l}\phi^2$ makes evident the degeneracy that they may exhibit. This degeneracy does not allow us to know their explicit form completely but instead we write their decomposition to be

$$I_4 = \frac{1}{8}\log^2(1-\bar{z}) \sum_{l=0,2,\ldots} \langle\langle \hat{a}_{4,l}(\gamma_{4,l})^2\rangle\rangle G^{(4d)}_{4,l}(1-z, 1-\bar{z}) + \mathcal{O}(\epsilon^5) \tag{4.23}$$

where we denoted with the double brackets the weighted average

$$\langle\langle \hat{a}_{4,l}(\gamma_{4,l})^2\rangle\rangle \equiv \sum_i \hat{a}_{4,l,i}(\gamma_{4,l,i})^2. \tag{4.24}$$

Since we cannot access the information of this decomposition due to twist degeneracy, we cannot restrict the functions appearing in the ansatz in an immediate way. Nevertheless we can start by considering the small-z limit of (4.23).

The explicit form of CBs with twist four in four dimensions decomposes in the small-z limit as

$$I_4 = (\log z g(\bar{z}) - \log \bar{z} g(z)) \log^2(1-\bar{z}) \tag{4.25}$$

where we defined the function

$$g(\bar{z}) = \frac{1}{8}\sum_l \langle\langle \hat{a}_{4,l}(\gamma_{4,l})^2\rangle\rangle k_{2+l}(\bar{z}). \tag{4.26}$$

The contributions (4.40), (4.42) do not contain a $\log^2 z$ term, therefore we expect that it should originate from I_4. This contribution should be of the schematic form $-a\log\bar{z}\log^2(1-\bar{z})$ and match with the terms on the l.h.s. of the inversion problem (4.18) proportional to $\log^2 z$. This obviously is $U^{(2)}_{\bar{h}}$ and through the inversion integrals can be found to be

$$U^{(2)}_{\bar{h}} = 8a\frac{4}{\bar{h}^2(1-\bar{h})^2}g^4 \tag{4.27}$$

But this term has been already computed form out previous computations, as $U^{(2)}_{\bar{h}}$ is been defined to be $\hat{a}_l(\gamma_l)^2$ and at fourth order this expression involves the values of lower order corrections. Indeed we can read from the relations (4.10) and (4.11) that the coefficient of the previous expression must be $a = \frac{1}{16\cdot 81}$[1].

Having said that we can start setting the conditions involved in the expansion of the functions in (4.20). The l.h.s. of (4.18) does not contain a $\log^3 z$ term, therefore we expect that the terms of the ansatz should cancel altogether the term $\log^3 z$ up to fifth order. This generates our first condition. The rest of the conditions can be expressed in a closed system which requires the expression of $g(\bar{z})$ to agree with the expansion of the ansatz up to fifth order. This system has the following solution up to a single undetermined coefficient

$$g(\bar{z}) = \frac{1}{16}\log^2\bar{z} + \beta\Big(-\frac{1}{6}\log^3\bar{z} - \frac{2}{3}\log\bar{z}\mathrm{Li}_2(1-\bar{z}) + \mathrm{Li}_3(1-\bar{z}) + \mathrm{Li}_3\Big(\frac{\bar{z}-1}{\bar{z}}\Big)\Big). \tag{4.28}$$

[1] This computation of ours differs from the one found in [5] but is in agreement with the collective results for the $O(N)$ model found in [26].

For that last coefficient to be determined we employ a different strategy. The twist four operators with spin zero and two are known in the literature [29] to be non-degenerate. The computation of their OPE coefficients and anomalous dimensions can be performed in the scheme we present here, but for the moment we simply quote their known values to be

$$
\begin{aligned}
a_{4,0} &= g^2/6 + \dots, \ \gamma_{4,0} = 3g + \dots \\
a_{4,2} &= g^2/160 + \dots, \ \gamma_{4,2} = 4/3g + \dots
\end{aligned}
\tag{4.29}
$$

The spin zero corrections when inserted in the system of equations we described previously define the undetermined coefficient to be $\beta = -3/2$. The corrections for spin two also confirm the validity of the expression for $g(\bar{z})$ up to fifth order.

The symmetric contribution $g(z)$ can be now found immediately by replacing variables and considering the small z limit. The order of the anomalous dimension we consider suggests the following expansion

$$
g(z) = \frac{1}{16}\log^2 z + \alpha_1 \log z + \alpha_2 \tag{4.30}
$$

which in the small z limit restrains these two constants to be

$$
\alpha_1 = -\frac{1}{3}\zeta_2, \ \alpha_2 = -\frac{3}{2}\zeta_3 \tag{4.31}
$$

With that we have all the ingredients that constitute the I_4 contribution. Therefore we are now in place to compute the quantities $U_{\bar{h}}^{(0)}$ and $U_{\bar{h}}^{(1)}$ up to fourth order and together with $U_{\bar{h}}^{(2)}$ they reproduce the CFT-data we are interested in. This is again possible by decoupling the set of equations (3.51), as we did in the previous order. The order of corrections for the OPE coeffients appearing in [5] is the highest that has appeared in the literature up till now and exhibits the strength of analytic bootstrap compared with traditional diagrammatic techniques.

As in the previous section we might as well extrapolate our results down to spin zero and obtain the corrections on the scalar external operator, through a more precise relation between g and ϵ. Working in the same spirit as before we obtain

$$
\Delta_\phi = 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{108} + \frac{109\epsilon^3}{11664} + \dots \tag{4.32}
$$

$$
g = \frac{\epsilon}{3} + \frac{8}{81}\epsilon^2 + \dots \tag{4.33}
$$

The most prolific application for the OPE coefficients is to use the $l = 2$ corrections, in order to extract the central charge of the theory. The central charge relates to these coefficients through the expression [25]

$$\hat{a}_2 = \frac{d^2 \Delta_\phi}{4(d-1)^2 C_T}. \tag{4.34}$$

As previously the hatted OPE coefficients are found from the explicit expression

$$\hat{a}_l = U_{\bar{h}}^{(0)} + \frac{1}{2}\partial_{\bar{h}} U_{\bar{h}}^{(1)} + \frac{1}{8}\partial_{\bar{h}}^2 U_{\bar{h}}^{(2)} \tag{4.35}$$

and they can be related to the usual OPE coefficients with the relation

$$\hat{a}_l = \frac{\Gamma(\Delta_l + l)}{\Gamma(\frac{\Delta_l + l}{2})^2} a_l \tag{4.36}$$

The corrections to these coefficients are obtained by expanding them in the expansion parameter ϵ. For this to be possible we need to determine the corrections in the spectrum of the bilinear operators. For example the anomalous dimension at order ϵ^2 is found to be

$$\gamma_l = -\frac{1}{9l(l+1)} \tag{4.37}$$

and the OPE corrections at the same order are

$$a_l = a_l^{(0)} + \frac{\epsilon^2}{9l(l+1)}\Big(S_1(2l) - S_1(l) + \frac{1}{l+1}\Big)\frac{2\Gamma(l+1)^2}{\Gamma(2l+1)} + \mathcal{O}(\epsilon^3) \tag{4.38}$$

where we denote again with $a_l^{(0)}$ the free field theory coefficients, but this time expanded to order ϵ^2 due to the dimensional corrections of the external operator ϕ. For these computations the identities, which relate the Harmonic numbers in our expressions with the Polygamma functions, proved to be very useful.

For completeness we also mention the fourth order expression for the central charge of the theory

$$\frac{C_T}{C_{\text{free}}} = 1 - \frac{5}{324}\epsilon^2 - \frac{233}{8748}\epsilon^3 - \Big(\frac{100651}{3779136} - \frac{55}{2916}\zeta_3\Big)\epsilon^4 + \dots \tag{4.39}$$

The fourth order correction is a result that appeared in the literature for the first time as an analytic expression and by setting $\epsilon = 1$ we approach even more the high precision numerical value for the 3d Ising model found in [33].

Finally as promised we quote the explicit results for the I_2 and I_{ϕ^2} contributions to be

$$I_2 = \frac{1}{8}\log^2(1-\bar{z})\Big(\log z(\zeta_2 - 2) + 2\log\bar{z} + \frac{1}{6}\log^3\bar{z} + \mathrm{Li}_3(1-\bar{z}) - \mathrm{Li}_3\Big(\frac{\bar{z}-1}{\bar{z}}\Big)\Big) \tag{4.40}$$

$$\begin{aligned} I_{\phi^2} =& \log^4(1-\bar{z})\frac{1}{192}(\log\bar{z} - \log z) + \log^3(1-\bar{z})\frac{1}{24}(\mathrm{Li}_2(1-\bar{z}) + 3\log z - 3\log\bar{z} + 2\zeta_2) \\ &+ \log^2(1-\bar{z})\Big(- \log z\frac{46 + \mathrm{Li}_2(1-\bar{z}) + \log\bar{z} + 12\zeta_2}{48} + \frac{5}{8}\mathrm{Li}_3(1-\bar{z} + \frac{1}{2}\mathrm{Li}_3\Big(\frac{\bar{z}-1}{\bar{z}}\Big) \qquad (4.41) \\ &+ \frac{2(23+6\zeta_2)\log\bar{z} - \mathrm{Li}_2(1-\bar{z})(21\log\bar{z}+34) - 106\zeta_2 - 4\log^3\bar{z} + \log^2\bar{z}24\zeta_3}{48}\Big) \qquad (4.42) \end{aligned}$$

Note that the derivation of these contributions, especially when there was mixing involved, required several manipulations and maneuvers in order to arrive at the final result. In fact despite the great deal of simplification imposed in analytic bootstrap from the introduction of the inversion formula, the calculation of the distinct contributions at each order is not systematically organised but rather depends on the case at hand.

4.2.1 The $O(N)$ model

The authors of [5] generalised the previous discussion in the case of the $O(N)$ model in $d = 4 - \epsilon$ dimensions [26]. The essential difference between the two publications concerns the additional tensor structure present in the $O(N)$ model, under which the transformation properties of the operators organise in different representations. These are found to be the singlet (S), traceless symmetric (T) and anti-symmetric (A) representations. The derivation of the CFT-data is similar to the manipulations we described previously and organises in categories depending on the representation that the intermediate operators belong. Their corrections also reach the fourth order in the expansion parameter and confirm that the analytic bootstrap rediscovers the critical $O(N)$ model obtained from the diagrammatic ϵ-expansions.

They are of particular interest the arguments they present regarding the subleading contributions of higher twist operators to lower orders and the information we can extract from them. The subleading corrections in z may be traced in the crossing equation through the use of the projection

$$\frac{1}{2\pi i}\oint \frac{dz}{z^2} k_\alpha(z) k_{1-\beta}(z) = \delta_{\alpha\beta} \tag{4.43}$$

which relates them with contributions from operators of higher spin. This relation is sensible for our purpose, since the subleading corrections involve the explicit expression (2.69) for the CBs that contains the functions $k_h(z)$. Make no mistake here: the subleading contributions arise from leading twist operators that their decomposition matches the attributes of higher twist operators and have non-vanishing projections considered in terms of them. The only higher twist operators that we must take into account are the quadrilinear operators of τ=4. For the respective crossing equation of the $O(N)$ model in the singlet representation[1] at order ϵ^2, the only terms in the l.h.s. of that equation that have non-vanishing projections on quadrillinear operators are the subleading contributions of the identity operator and the leading twist operators. The identity contribution corresponds to the free field theory coefficients at order ϵ^2, while the leading-twist operators have a non-zero projection into quadrilinear operators through the sum

$$\sum_l a^{(0)}_{S,l} \gamma_{S,l} \partial_\tau G^{4d}_{\tau,l}(z,\bar{z})\Big|_{\tau=2}$$

Ultimately though these subleading contributions at ϵ^2 ought to be zero. Therefore the double-discontinuities that they produce must cancel out and through this ascertainment we can obtain an expression that relates the contributions of the identity and the leading twist operator. This allows us to determine completely the coefficients of twist four operators in the singlet irrep at order ϵ^2 to be

$$a_{S,4,l} = \frac{\Gamma(l+2)^2}{\Gamma(2l+3)} \frac{l^2+3l+8}{12(l+1)(l+2)} g_S^2. \tag{4.44}$$

This result though reproduces exactly the coefficients we used for the non-degenerate operators of twist four and spin 0 and 1 respectively. With a similar manipulation we can extract the anomalous dimensions for these operators and therefore the analysis we perform on the previous section is self-consistent. At the same time this argument provides another justification for the order that the quadrilinear operators appear in the crossing equation.

[1]Notice that a field in the singlet irrep transforms as a field in our familiar Wilson-Fischer model.

5
Conclusions and Discussion

In this master thesis we attempt to present the collective methods known in the literature as analytic bootstrap. For that purpose we describe in the introduction the emergent importance of conformal field theories in physics and the evolution of techniques used to study those theories. In the heart of those techniques lies the analytic bootstrap, a promising field which aims to produce universal statements for generic CFTs with $d > 2$. Chapter 2 is dedicated on an overview of conformal field theories in general, along with a more focused analysis on the necessary formalism used in the upcoming chapters. Having obtained a complete toolkit we start reviewing the basics of analytic bootstrap in Chapter 3, by considering the much celebrated inversion formula that complemented many of the large spin perturbation theory (LSPT) arguments and techniques. We also present the frame upon which LSPT was developed, along with the core results that raised the interest in these methods. The chapter concludes with a few more technical details, which concern the combination of LSPT and the inversion formula. We provide the additional formalism for the inversion formula to work in the perturbative setting and define the framework upon which the examples of the next chapter are developed. In Chapter 4 we use the tools of analytic bootstrap in the well known Wilson-Fischer model and present the necessary results in order to reproduce the OPE coefficients and anomalous dimensions of the intermediate operators appearing in the spectrum, up to fourth order in the expansion parameter. Furthermore it follows a brief discussion regarding the generalisation of our scheme to the CFT with an $O(N)$ global symmetry and how the previous analysis constrains the CFT at hand to agree with that of the critical $O(N)$ model found with the diagrammatic ϵ expansion. We want to stress that up to this point all of our results are independent of the Lagrangian description of these theories and obey the spirit of analytic bootstrap that extracts universal results for CFTs and their spectrum, by imposing restrictions on the crossing equation. Appendix A contains the basic details for the proof of the inversion formula, while Appendix B discusses technical aspects on the computation of twist conformal blocks.

At this point we would like to complement the previous analysis with a discussion upon details appearing on this thesis that present further interest. Our first comment relies on the fact that the inversion formula itself contains all of the theory specific information of the CFT under consideration. This demonstrates a broader idea present in the literature that the CFT-data can be expressed only in terms of a few parameters. In our case this involves the double-discontinuities of the conformal blocks, a quantity much less detailed than the conformal blocks themselves. This trait can be traced back in the Regge limit behaviour of the four-point function as it has been indicated in [10]. To that limit, the correct behaviour does not only fix the value of its individual OPE coefficient, but imposes constraints on the uniform behaviour of these coefficients. This is a clear manifestation of the analyticity in spin, present in the formula.

The gravitas of the inversion formula raises the immediate question, whether it is possible to generalise it into non-scalar correlators and arrive to expressions analytic in spin. A method for that purpose has been discussed in [27] but there are technical details that prevent an immediate practical result. There is also a promising direction, as noted in [24] that involves the so-called *light-ray operators* [34], which provide a firmer formulation of the problem. These operators are Lorentzian by definition and take arbitrary, not necessary integer, values of spin J and the inversion formula can be viewed as a combination of them. Additionally the light-ray operators could potentially answer the question of analyticity all the way down to spin zero. When evaluated for integer spin they are equivalent to the integration of a local operator along a null direction, namely a *light transform.* There is a possibility that a suitable combination of light and shadow (Appendix A) transformations could explain the analytic continuation for all values of spin.

At the same time there have been quite recent developments and extensions of the previous methods in the context of analytic bootstrap. These were presented in a series of papers [5],[26],[6],[25] which include the majority of the context presented here. As noted previously the scheme applied on the Wilson-Fishcer model can be used immediately in its generalisation, a CFT with a global $O(N)$ symmetry. Then the analysis of analytic bootstrap determines the only possible CFT equipped with such a global symmetry to coincide with the critical $O(N)$ model, at the order of perturbation considered. We would like to stress here that the methods used for the W.F. model and its generalisation, the $O(N)$ model, treat the OPE coefficients and the anomalous dimensions in the same manner, therefore their extraction requires the same amount of difficulty. This is a major advantage of analytic bootstrap, as for example it made possible to determine the correction of the central charge of the W.F. model up to forth order in the expansion parameter in a straightforward manner. In comparison this result was added in the literature up to order three only recently [22] and diagrammatic techniques have not succeeded to achieve that precision.

Finally there is one more significant benefit of analytic bootstrap compared to other methods. It is based on the fact that the crossing equation and the contributions it contains at a given order of the ϵ-expansion, do not immediately require information regarding the global symmetry of the theory we are concerned with. Therefore the distinction of each theory equipped with a global symmetry results from the tensor structure, which multiplies the conformal blocks of the model. This was exactly the case in [25], where the authors generalise the analytic bootstrap methods in $d = 4 - \epsilon$ dimensions to an arbitrary global symmetry and the CFT-data of each model are obtained after decomposing the conformal blocks into irreducible representations. This allows one to examine the fixed points of each theory in a uniform way and considerably simplifies the analysis in specific global symmetries where there is a great amount of complexity in order for the CFT-data to be obtained.

Appendix A

In this chapter we will highlight the non-trivial points in the derivation of the Lorentzian inversion formula. We will not follow Caron-Huot's original line of thought but instead present a different approach, which uses spacetime arguments rather than a manipulation of the cross-ratios and contour deformations like in the original paper. Also in this chapter we use the notation $\chi, \bar{\chi}$ in place of the variables $z, \bar{z}$ used in the main text, in order to be in agreement with the notation of [43], which contains the proof that we will present here.

Principal series representation

Our starting point will be writing the four-point function in terms of, the so called, principal series representation. This is no more than an expansion of the correlator in terms of a set of functions with convenient properties. Specifically these functions are single-valued, form an almost complete set of Casimir equation's solutions and they satisfy an orthogonality relation. They can be thought just as the original representation of operators in a CFT but with unphysical complex dimensions $\Delta = \frac{d}{2} + ir$, where $d > 1$ and r is a positive real number in a traceless symmetric representation ρ of integer spin J. Their derivation is possible through the use of harmonic analysis on the conformal group $G = SO(d+1, 1)$, the analogue of Fourier analysis on a topological group. This subject has been studied with great care in the 70's [14] and a detailed overview on the subject has been provided in [45]. We also refer to [28] for modern analysis on harmonic analysis in CFTs.

We will denote with G' the space of unitary irreducible representations π, also known as the unitary dual of G. G' is composed by the discrete series of representations and the principal series representation $\Pi_{\Delta,\rho}$. Only the later will be of our concern, since only they contribute on the Plancherel formula of scalar operators, which we will define and use below. This space G' contains a Plancherel measure defined [28]

$$\mu(\Delta,\rho) = \frac{\dim\pi}{\mathrm{vol}G} = \frac{Tr_\pi(\mathbf{1})}{\mathrm{vol}G} \tag{A.1}$$

where for a $g \in G$ we defined the functional

$$Tr_\pi(f) = \int dg f(g)\pi(g)$$

In this definition $f(g)$ is an infinite differentiable function with compact support on G and dg the Haar measure of G, which acts like our familiar differential, but this time in subsets of locally compact topological groups [12]. By abuse of notation we also noted with $\pi(g)$ the representation map of the linear transformation to the associated vector space of the irreducible representation π.

The existence of Haar measure is based on the fact that G is locally compact, which justifies our upcoming analysis of (globally) non-compact groups. For compact spaces the Plancherel measure is a finite quantity for each irreducible representation π, while when we deal with non-compact cases, $\dim\pi$ and $\mathrm{vol}G$ may be infinite quantities. In our non-compact space G the Plancherel measure can be redefined as

$$\frac{\mu(\Delta,\rho)}{\mathrm{vol}SO(1,1)} = \frac{Tr_{\Pi_{\Delta,\rho}}(\mathbf{1})}{\mathrm{vol}G}$$

with the whole fraction being a finite quantity, playing the role of a regularized quotient. From this point we will also denote the Plancherel measure with $d\pi$, as it can be understood as a "measure" in the space of representations.

We now consider the function

$$f(g) = \langle g \cdot O(0)O(\infty)O(x_3)O(x_4)\rangle$$

which acts on the first two operators and produces all the physical correlators of G. Now all conditions are met to apply Plancherel theorem, which simply states that

$$f(g) = \int_{G'} d\pi Tr\left[\pi(g)\hat{f}(\pi)\right] \tag{A.2}$$

where $\hat{f} = \int_G dg f(g)\pi(g)^{-1}$. This equation is the analogue of a Fourier transform in a topological group G and tells us precisely that the representations appearing in the Plancherel formula provide us with a basis for any function on the group. By introducing a discrete set of states $\{|\psi\rangle\}$ the above formula can be rewritten as

$$f(g) = \int_{G'} d\pi \sum_{\psi,\psi'} \langle\psi'|\pi(g)|\psi\rangle \cdot \langle\psi|\hat{f}(g)|\psi'\rangle \tag{A.3}$$

with the first term corresponding to the matrix elements of representations, appearing in the integral of the Plancherel measure and the second term to coefficients closely related to those of a Fourier transform.

By using the above results, we can specify to the principal series representation $\Pi_{\Delta,\rho}$ as a set of states and write the four-point function in the form

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle = \sum_{J=0}^{\infty} \int_{\frac{d}{2}}^{\frac{d}{2}+\infty} \frac{d\Delta}{2\pi i}\mu(\Delta, J)\times \tag{A.4}$$

$$\int d^dx \langle O(x_1)O(x_2)O_{\mu_1,\ldots,\mu_J}(x)\rangle_S P^{\mu_1,\ldots,\mu_J}_{\Delta,J}(x, x_3, x_4) \tag{A.5}$$

where we have also used the completeness relation of this base

$$\mathbf{1} = \int d^dx |O_{\mu_1\ldots\mu_J}\rangle\langle\tilde{O}^{\mu_1\ldots\mu_J}| \tag{A.6}$$

and introduced the notation $\tilde{O}$ for an operator with scaling dimension $\tilde{\Delta} = d - \Delta$. We denote with $\langle\phi(x_1)\phi(x_2)O_{\mu_1,\ldots,\mu_J}(x)\rangle_S$ the conformal invariant structure fixed from conformal symmetry and the normalisation of the theory, not a physical correlator. Also the quantity $P^{\mu_1,\ldots,\mu_J}_{\Delta,J}(x, x_3, x_4)$ that encodes the coefficients of the transformation in the space of representations, transforms itself like a conformally invariant three point function. Therefore it is not necessary to compute it explicitly but simply write it as a product of the invariant three-point structure times a constant $I(\Delta, J)$. Then the integrand of (A.5) can be written as

$$\Psi^{\Delta_O}_{\Delta,J}(x_1, x_2, x_3, x_4) = \int d^dx \langle O(x_1)O(x_2)O_{\mu_1,\ldots,\mu_J}(x)\rangle_S \langle\tilde{O}_{\mu_1,\ldots,\mu_J}(x)O(x_3)O(x_4)\rangle_S \tag{A.7}$$

which as a decomposition is essentially the four point function in radial quantization with the additional insertion of the identity operator. By comparing this expression with the decomposition of the four-point function (2.62) we deduce that the previous relation is our familiar expression of the conformal partial waves (CPW). Thus the decomposition of the four point function takes finally the form

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle = \sum_{J=0}^{\infty} \int_{\frac{d}{2}}^{\frac{d}{2}+\infty} \frac{d\Delta}{2\pi i}\mu(\Delta, J)I(\Delta, J)\Psi^{\Delta_O}_{\Delta,J}(x_i) \tag{A.8}$$

The symmetry of the previous expression under the exchange $\Delta \leftrightarrow d-\Delta$ indicates that a CPW can be written as a linear combination of a conformal block and the corresponding *shadow block* with scaling dimension $d-\Delta$. We can therefore write the conformal partial wave as

$$\Psi^{\Delta_i}_{\Delta,J}(x_i) = K^{\Delta_3,\Delta_4}_{\tilde{\Delta},J} G^{\Delta_i}_{\Delta,J}(x_i) + K^{\Delta_1,\Delta_2}_{\Delta,J} G^{\Delta_i}_{\tilde{\Delta},J}(x_i) \tag{A.9}$$

By limiting our interest in the small x_{12}-region, where the integrand inside of (A.8) factorizes nicely, we can derive the coefficients of the conformal block in the previous expression. This manipulation will lead us to a conformally covariant integral, which has been computed in various publications and produces the result for the first coefficient to be

$$K^{\Delta_3,\Delta_4}_{\tilde{\Delta},J} = (-\frac{1}{2})^J \frac{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})\Gamma(\Delta+J-1)\Gamma(\frac{\tilde{\Delta}+\Delta_1-\Delta_2+J}{2})\Gamma(\frac{\tilde{\Delta}+\Delta_2-\Delta_1+J}{2})}{\Gamma(\Delta-1)\Gamma(d-\Delta+J)\Gamma(\frac{\Delta+\Delta_1-\Delta_2+J}{2})\Gamma(\frac{\Delta+\Delta_2-\Delta_1+J}{2})}. \tag{A.10}$$

The additional factor $(-\frac{1}{2})^J$ is the result of the conformal blocks normalization

$$G^{\Delta_i}_{\Delta,J}(0,x,e,\infty) \sim (-2)^J |x|^{\Delta-\Delta_1-\Delta_2-J} \langle O_{\nu_1\ldots\nu_J}(0) O_3(e) O_4(\infty)\rangle + \ldots \tag{A.11}$$

$$= |x|^{\Delta-\Delta_1-\Delta_2} 2^J \hat{C}_J\Big(\frac{x\cdot e}{|x|}\Big) + \ldots \tag{A.12}$$

where e is a unit vector and $\hat{C}_J$ is a d-dimensional Gegenbauer polynomial. Performing the same analysis in the limit $x_{3,4}\to 0$, we derive the coefficient of the shadow block. Finally we write the useful expression

$$\Psi^{\Delta_i}_{\tilde{\Delta},J} = \frac{K^{\Delta_3,\Delta_4}_{\Delta,J}}{K^{\Delta_1,\Delta_2}_{\Delta,J}} \Psi^{\Delta_i}_{\Delta,J} \tag{A.13}$$

an immediate result after performing a few trivial substitutions.

The orthogonality relation of satisfied by the CPWs should be of the form

$$\begin{aligned}\Big(\Psi^{\Delta_i}_{\Delta,J}, \Psi^{\tilde{\Delta}_i}_{\tilde{\Delta}',J'}\Big) &\equiv \int \frac{d^dx_1\ldots d^dx_4}{vol(SO(d+1,1))} \Psi^{\Delta_i}_{\Delta,J}(x_i)\Psi^{\tilde{\Delta}_i}_{\tilde{\Delta}',J'}(x_i) \\ &= n_{\Delta,J} 2\pi\delta(r-r')\delta_{J,J'}\end{aligned} \tag{A.14}$$

where the volume of $SO(d{+}1,1)$ acts like a normalization quantity, because this integral is a conformally invariant. Our convention for defining this volume is so that when we gauge fix three points to 0,1,∞ we have a Faddeev-Popov determinant equal to 1. Our previous equation therefore reads

$$\left(\Psi^{\Delta_i}_{\Delta,J},\Psi^{\tilde{\Delta}_i}_{\tilde{\Delta}',J'}\right)=\int\frac{d^dx}{vol(S0(d-1))}\Psi^{\Delta_i}_{\Delta,J}(0,x,e,\infty)\Psi^{\tilde{\Delta}_i}_{\tilde{\Delta}',J'}(0,x,e,\infty) \tag{A.15}$$

At this point we also specify the convention we use regarding the measurement of vol$SO(d)$ to be

$$\text{vol}SO(d)=\text{vol}S^{d-1}\text{vol}SO(d-1)=\frac{2\pi^{d/2}}{\Gamma(d/2)}\text{vol}SO(d-1) \tag{A.16}$$

where vol$SO(1)=1$ and S^d is the manifold of the sphere in d-dimensions [8].

The computation of the normalization factor $n_{\Delta,J}$ can be found in [10],[43] but we will provide a few details justifying the relation (A.14). The normalization of the conformal block which we introduced previously involved Gegenbauer polynomials, which satisfy the orthogonality relation

$$\int_{-1}^{1}C^{(\alpha)}_J(x)C^{(\alpha)}_{J'}(x)(1-x^2)^{\alpha-\frac{1}{2}}dx=\frac{\pi 2^{1-2\alpha}\Gamma(J+2\alpha)}{J!(J+\alpha)\Gamma(\alpha)^2}\delta_{JJ'}. \tag{A.17}$$

The form of (A.14) also suggests that we should focus on singular terms around $x=0$ in order to produce a term proportional to $\delta(r-r')$. By restricting again only on the conformal block's coefficient we can exploit the spherical symmetry in $(d-1)$-dimensions and obtain a relation proportional to (A.17). In fact by arranging the remaining coefficients we arrive in a result independent of the scaling dimensions of the external operators. Therefore our analysis does not need to carry on the shadow-block contribution. Finally we quote the final result for $n_{\Delta,J}$ to be

$$2\frac{K^{\Delta_3,\Delta_4}_{\tilde{\Delta},J}K^{\tilde{\Delta}_3,\tilde{\Delta}_4}_{\Delta',J'}vol(S^{d-2})}{vol(SO(d-1))}\frac{(2J+d-2)\pi\Gamma(J+1)\Gamma(J+d-2)}{2^{d-2}\Gamma(J+d/2)^2}\pi\delta(r-r')\delta_{JJ'} \tag{A.18}$$

To conclude this section we will first write the final form of the decomposition of the four-point function in terms of the principal series expansion.

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle=\sum_{J=0}^{\infty}\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+\infty}\frac{d\Delta}{2\pi i}\mu(\Delta,J)I_{\Delta,J}K^{\Delta_3,\Delta_4}_{\tilde{\Delta},J}G^{\Delta_O}_{\Delta,J}(x_i) \tag{A.19}$$

The new domain of integration can be understood, from the invariance of the integral under the exchange of the scaling dimension with its shadow counterpart. Therefore the shadow contribution could be incorporated in the original integral by changing variables and combining together the two integrals.

We would like to stress that, since there is a completeness and orthogonality relation for the principal series representation, we can invert equation (A.19) and obtain the expression

$$I_{\Delta,J} = \frac{n(\Delta,J)}{\mu(\Delta,J)} \int \frac{d^d x_1 \dots d^d x_4}{vol(SO(d+1,1))} \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \Psi^{\tilde{\Delta}_i}_{\tilde{\Delta},J} \tag{A.20}$$

where the volume in the denominator normalizes the divergency produced by the integral due to conformal invariance. Also the factors appearing in front of the previous expression are finite and can be omitted, since they do not meddle with our upcoming manipulations. The quantity $I_{\Delta,J}$ communicates with the one presented in [10], through the relation

$$c(J,\Delta) = \frac{I_{\Delta,J}}{n_{\Delta,J}} K^{\Delta_3,\Delta_4}_{\tilde{\Delta},J} \tag{A.21}$$

and as remarked in the main text it contains all of the theory specific information, namely the CFT-data, in its poles and residues. The important details of its derivation will be spelled below.

Inversion formula

Before we start our main discussion, regarding the analytic decomposition in spin J of the inversion formula, we have to clarify a few points regarding the validity and details in derivation of equation (A.20).

Let's note first that when we wrote this expression we purposely left out the non-normalizable contributions to the principal series expansion, which may arise from e.g. the identity operator. We expect these ill-behaved contributions to arise near $\chi = 0$, $\chi, \bar{\chi} = 1$ and $\chi, \bar{\chi} \to \infty$, where we expressed the regions of interest in term of the conformal cross ratios (2.41). However each type of these unwanted contributions can be handled separately in a way that does not spoil the behaviour of our inversion integral (A.20). More details on the subject can be found in Appendix B of [43].

Finally a remark regarding a type of spurious poles arising from the contour deformation over Δ. There are two ways from additional poles to appear, aside from those of $I_{\Delta,J}$. From the conformal block $G^{\Delta_i}_{\Delta,J}$ and the factor $\Gamma(d-\Delta+J-1)$ inside of $K^{\Delta_3,\Delta_4}_{d-\Delta,J}$. The conformal block contains a series of poles when $\Delta = J + d - 1 - k$ for

$k = 1, \ldots, J$, while the poles of the gamma function when $\Delta = J+k+d-1$. Fortunately these contributions to the residues of $I_{\Delta,J}$ cancel out, after we use the appropriate relations between conformal partial waves[refs]. Therefore there aren't any unexpected contributions when we try to recover the OPE decomposition of the four-point function.

Two dimensions

Our derivation of the Lorentzian inversion formula will start for the two-dimensional case, where our arguments take a preliminary and simpler form. We will simplify the analysis in this section even more by considering a four-point function of identical scalar operators, with equal external dimensions Δ. The general case will be studied in the d-dimensional case.

The conformal group in two dimensions has two independent quadratic Casimirs, whose eigenfunctions are labeled by a pair of left and right weights $(h, \bar{h})$. In terms of the shadow representation, these eigenfunctions are

$$\Psi^{\Delta}_{h,\bar{h}}(z_i, \bar{z}_i) = \frac{1}{|z_{12}|^{2\Delta}|z_{34}|^{2\Delta}} \Psi_{h,\bar{h}}(z_i, \bar{z}_i) \tag{A.22}$$

$$\Psi_{h,\bar{h}}(z_i, \bar{z}_i) = \int d^2 z_5 \Big(\frac{z_{12}}{z_{15}z_{25}}\Big)^h \Big(\frac{\bar{z}_{12}}{\bar{z}_{15}\bar{z}_{25}}\Big)^{\bar{h}} \Big(\frac{z_{34}}{z_{35}z_{45}}\Big)^{1-h} \Big(\frac{\bar{z}_{34}}{\bar{z}_{35}\bar{z}_{45}}\Big)^{1-\bar{h}}. \tag{A.23}$$

Let us note here that these expressions are valid only when J is an integer, since the term $z_{34}^h \bar{z}_{34}^{\bar{h}}$ and the function $\Psi_{1-h,1-\bar{h}}$ are not single valued otherwise. If we further define $h = \frac{1+l+ir}{2}$ and $\bar{h} = \frac{1-l+ir}{2}$ the four-point function admits the expansion

$$\langle O_1(z_1)O_2(z_2)O_3(z_3)O_4(z_4)\rangle = \sum_{l=-\infty}^{\infty} \int_0^{\infty} \frac{dr}{2\pi} \frac{I_{h,\bar{h}}}{n_{h,\bar{h}}} \Psi^{\Delta}_{h,\bar{h}}(z_i, \bar{z}_i) \tag{A.24}$$

In the same spirit we discussed in the previous section we perform the inversion

$$I_{h,\bar{h}} = \int \frac{d^2z_1 d^2z_2 d^2z_3 d^2z_4}{vol(SO(3,1))} \langle O_1(z_1)O_2(z_2)O_3(z_3)O_4(z_4)\rangle \Psi^{\tilde{\Delta}}_{1-h,1-\bar{h}}(z_i, \bar{z}_i) \tag{A.25}$$

By setting $z_1 = 0$, $z_2 = \chi$, $z_3 = 1$ and $z_4 = \infty$ we would go to a gauge, where our integral becomes the usual Euclidean integral in term of the cross-ratios $\chi, \bar{\chi}$. We are not interested in this configuration though, since we would like to Wick rotate and perform calculations in Lorentzian signature. By using the explicit expression (A.23) we have

$$\int \frac{d^2z_1 d^2z_2 d^2z_3 d^2z_4 d^2z_5}{vol(SO(3,1))} \frac{\langle O_1O_2O_3O_4\rangle}{|z_{12}|^{4-2\Delta}|z_{34}|^{4-2\Delta}} \Big(\frac{z_{12}}{z_{15}z_{25}}\Big)^{1-h} \Big(\frac{\bar{z}_{12}}{\bar{z}_{15}\bar{z}_{25}}\Big)^{1-\bar{h}} \Big(\frac{z_{34}}{z_{35}z_{45}}\Big)^{h} \Big(\frac{\bar{z}_{34}}{\bar{z}_{35}\bar{z}_{45}}\Big)^{\bar{h}}.$$

If we further choose $z_1 = 1, z_2 = 0, z_5 = \infty$ the previous quantity results to

$$I_{h,\bar{h}} = \int \frac{d^2 z_3 d^2 z_4}{|z_{34}|^{4-2\Delta}} \langle O_1 O_2 O_3 O_4 \rangle z_{34}^h \bar{z}_{34}^{\bar{h}}. \tag{A.26}$$

and we are ready to Wick rotate and work under Lorentzian signature.

Specifically by setting $t = i\tau$ we have $|z|^2 = x^2 + t^2 = x^2 - \tau^2 + i\epsilon$, where in the second equality we used the standard Feynman prescription. If we further set $u = x - \tau$ and $\upsilon = x + \tau$ we can rewrite (A.26) as

$$I_{h,\bar{h}} = -\frac{1}{4} \int \frac{du_3 d_\upsilon du_4 d\upsilon_4}{|u_{34}\upsilon_{34}|^{2-\Delta}} \langle O_1 O_2 O_3 O_4 \rangle u_{34}^h \upsilon_{34}^{\bar{h}} \tag{A.27}$$

The fraction in front of this expression comes from the differential, while the rest factors result from trivial substitutions. Due to conformal symmetry we are free to fix $u_1, \upsilon_1 = 1, 1$ and $u_2, \upsilon_2 = 0, 0$. In this format singularities occur when two operators become light-like separated, therefore the possible singularities obtained from the OPE are of the form

$$u_{i,j}\upsilon_{i,j} + i\epsilon = 0, \; i, j = 1, \ldots 4 \tag{A.28}$$

Since we will perform contour integration we need to sort out the behaviour around branch cuts and the asymptotic behaviour at infinities.

We would like to have a decaying asymptotic behaviour when a contour approaches infinity. This is possible through the choice of either u_{34}^h or $\upsilon_{34}^{\bar{h}}$, when J is negative or positive respectively. For our case we choose the factor $\upsilon_{34}^{\bar{h}}$ that has the correct asymptotic behaviour and we will integrate over the υ_3 and υ_4 variables. The factors u_{41}, u_{42}, u_{43} vary independently and their sign determines the location of the poles in (A.28). Also from that same relation we deduce that there are three brunch points for the υ variables. Their respective brunch cuts lie just below or above of the real axis depending on the sign of the integration variables.

Lets examine the possible behaviour of the integrand for the various values of its variables. We would like to have two of the branch points below or above of the real axis, since otherwise an appropriate contour deformation around the real axis gives a vanishing result. If we recall the placement of u_1 and u_2 this condition is satisfied, when $0 < u_3, u_4 < 1$. In the case of $u_3 < u_4$ we deform the contours, as indicated in Fig. (A.1) in order not to encounter additional singularities during that deformation. As the branch cut approaches the real axis and υ moves just above the singularity

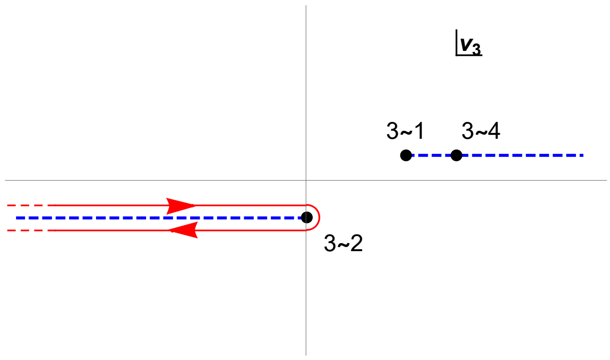

Figure A.1: Contour deformation for the υ variable.

$\upsilon_{32} = -i\epsilon/u_{32}$, it picks up the discontinuities (hence the contributions) of the OPE of the operators O_3O_2. When it does the same from below it will pick up the contributions of $-O_2O_3$, where the minus sign is a result of the opposite direction of the contour. This though produces the commutator $[O_3, O_2]$ and following the same line of thought for the υ_4 integral we get the commutator $[O_1, O_4]$. The arguments for the case $u_4 < u_3$ are identical to the ones presented above and we can write the relation

$$I_{h,\bar{h}} = -\frac{(-1)^J}{4}\int_{R_1}\frac{du_3 d_\upsilon du_4 d\upsilon_4}{|u_{34}\upsilon_{34}|^{2-\Delta}}\langle[O_3, O_2][O_1, O_4]\rangle u_{43}^h \upsilon_{43}^{\bar{h}} \\ -\frac{1}{4}\int_{R_2}\frac{du_3 d_\upsilon du_4 d\upsilon_4}{|u_{34}\upsilon_{34}|^{2-\Delta}}\langle[O_4, O_2][O_1, O_3]\rangle u_{34}^h \upsilon_{34}^{\bar{h}} \tag{A.29}$$

where

$$R_1 : \upsilon_3 < 0, \;\; \upsilon_4 > 1, \;\; 0 < u_3 < u_4 < 1 \\ R_2 : \upsilon_3 > 1, \;\; \upsilon_4 < 0, \;\; 0 < u_4 < u_3 < 1 \tag{A.30}$$

the appropriate integration regions, acquired from (A.28).

Our work here is almost done, since we only have to write an expression in terms of the cross-ratios, through the relations

$$\chi = \frac{u_{34}}{(u_3 - 1)u_4}, \;\; \bar{\chi} = \frac{\upsilon_{34}}{(\upsilon_3 - 1)\upsilon_4} \tag{A.31}$$

In two dimensions converting the integral to cross ratios is immediate.In higher dimensions though a few non-trivial manipulations are required as we will see shortly. Performing the integrations over u_4, υ_4 the previous expression admits a decomposition of $SL(2, \mathbb{R})$ conformal blocks

$$k_{2h}(\chi) \equiv \chi^h\, {}_2F_1(h, h, 2h, \chi), \;\; \hat{k}_{2h}(\chi) \equiv (-\chi)^h\, {}_2F_1(h, h, 2h, \chi) \tag{A.32}$$

where we have used the integral representation of the hypergeometric functions

$$ {}_2F_1(a,b,c,\chi) = \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-t\chi)^a} \tag{A.33}$$

After a few substitutions arrive in the explicit result

$$ I_{h,\bar{h}} = -\frac{1}{4}\frac{\Gamma(h)^2\Gamma(1-\bar{h})^2}{\Gamma(2h)\Gamma(2-2\bar{h})}\times \tag{A.34}$$

$$ \Big[(-1)^J \int_0^1\int_0^1 \frac{d\chi d\bar{\chi}}{(\chi\bar{\chi})^{2-\Delta_O}}\langle [O_3,O_2][O_1,O_4]\rangle k_{2h}(\chi)k_{2(1-\bar{h}}(\bar{\chi}) \tag{A.35}$$

$$ + \int_{-\infty}^0\int_{-\infty}^0 \frac{d\chi d\bar{\chi}}{(\chi\bar{\chi})^{2-\Delta_O}}\langle [O_4,O_2][O_1,O_3]\rangle \hat{k}_{2h}(\chi)\hat{k}_{2(1-\bar{h})}(\bar{\chi})\Big] \tag{A.36}$$

Higher dimensions

We are ready to proceed to the generalization of the inversion formula in higher dimensions by extending the arguments of the previous section, while adding a few more details in our derivation.

Along with the conversion of the integral into an integral of cross ratios, there is another problem arising in higher dimensions. As we discussed previously we did not handle the full expression of the partial wave, rather its "chiral half". The derivation for the other "chiral half" $\Psi_{\bar{h},h}$ is essentially the same but with the roles of u and v exchanged, since now the term $u_{34}^{\bar{h}}$ decays at the infinity. In higher dimensions though since the operators themselves transform as a symmetric traceless tensors (STTs), there are infinite null directions contributing at infinity. The solution to this problem will be to isolate only one direction, which determines the behaviour of the integral at hand and behaves appropriately at infinity.

Using the expression for the conformal partial waves we can write immediately for the Euclidean inversion formula

$$ I_{\Delta,J} = \int \frac{d^2z_1d^2z_2d^2z_3d^2z_4d^2z_5}{\mathrm{vol}(SO(d+1,1))}\langle O_1O_2O_3O_4\rangle\langle \tilde{O}_1\tilde{O}_2\tilde{O}_5^{\mu_1\ldots\mu_J}\rangle\langle O_{5,\mu_1\ldots\mu_J}\tilde{O}_3\tilde{O}_4\rangle \tag{A.37}$$

Due to its construction this integral is conformally-invariant. We are therefore free to gauge fix $x_5=\infty$, $x_1=(1,0,\ldots,0)$ and $x_2=(0,\ldots,0)$ and arrive to a more convenient expression. At the same time the product of the three-point functions is proportional to a Gegenbauer polynomial, which transforms as an STT and has null contributions in every direction at infinities. By choosing a convenient representation for it we can isolate only one null direction, that has the appropriate behaviour at the infinity. Then we immediately arrive at the result

$$I_{\Delta,J} = \hat{C}_J(1) \int \frac{d^d x_3 d^d x_4}{\mathrm{vol}(SO(d-1))} \frac{\langle O_1 O_2 O_3 O_4 \rangle}{|x_{34}|^{J+2d-\Delta_3-\Delta_4-\Delta}} (x_{34}^0 + i x_{34}^1)^J \tag{A.38}$$

which is of the same form as (A.26) and can be handled in a similar manner.

Like before we Wick rotate the second coordinate $x^1 = it$ and define $u = x^0 - t$. Thus the previous equation results to

$$I_{\Delta,J} = -\hat{C}_J(1) \int \frac{d^d x_3 d^d x_4}{\mathrm{vol}(SO(d-1))} \frac{\langle O_1 O_2 O_3 O_4 \rangle}{(x_{34}^2)^{\frac{J+2d-\Delta_3-\Delta_4-\Delta}{2}}} (u_{34})^J \tag{A.39}$$

where the minus sign comes form two consecutive Wick rotations and the change of variables in the differentials. Our argument here is the same as in two dimensions, since the extra spatial coordinates simply change the locations of the singularities of the correlator but do not affect their behaviour and we can restrict our analysis in the υ-planes. Due to the use of Lorentzian signature the regions of integration now are the future lightcones of points 1 and 2 and one can get insight in that configuration from Fig.(A.2).

Let us note here that the integration in the complex values of υ, if not treated properly, can be invalidated by singularities of the four-point function in the contour deformation. We treat this properly by writing

$$O(\upsilon) = O(\upsilon_R + i\upsilon_I) = e^{\upsilon_I P_\upsilon} O(\upsilon_R) e^{-\upsilon_I P_\upsilon} \tag{A.40}$$

where $P_\upsilon \leq 0$ is the non-positive operator generating translations in the υ direction. This quantity is well defined for the various values of υ_I and analytically continued in the upper or lower half plane regarding the time ordering of the operators. Their time ordering in our scheme depends on the sign of the u variable, which as of its definition specifies the spacetime separation of the x variables. It is straightforward to see that the contour deformations we used in the plane of the variables υ_3, υ_4 are in agreement with the analytic continuation of the complex variable υ, therefore do not produce extra singularities. For example the υ_4 integration in the region where $u_4 < u_1$ and u_4 is greater than u_2, u_3 is analytically continued in the upper half plane, when x_4 is spacelike separated from x_1 or in its past lightcone. This happens precisely by putting the O_4 operator in the rightmost of the correlation function, in order to produce a unit when $e^{-\upsilon_I P_\upsilon}$ acts on the vacuum. On the other hand when x_4 is in the future of x_1 this is no longer possible and the O_1 operator goes first on the correlator. This though describes precisely the commutator that appears in equation (A.29), justifying the validity of our arguments for the contour deformations in higher dimensions.

By further introducing the null vector $m^\mu = (1, 1, 0, \ldots, 0)$ we arrive at

$$\begin{aligned} I_{\Delta,J} = -\hat{C}_J(1)\times \\ &\Big[(-1)^J \int_{4>1,2>3} \frac{d^d x_3 d^d x_4}{\text{vol}(SO(d-1))} \times \frac{\langle [O_3, O_2][O_1, O_4]\rangle}{|x_{34}|^{J+2d-\Delta_3-\Delta_4-\Delta}} (-m\cdot x_{34})^J \theta(-m\cdot x_{34}) \\ &+ \int_{3>1,2>4} \frac{d^d x_3 d^d x_4}{\text{vol}(SO(d-1))} \times \frac{\langle [O_4, O_2][O_1, O_3]\rangle}{|x_{34}|^{J+2d-\Delta_3-\Delta_4-\Delta}} (m\cdot x_{34})^J \theta(m\cdot x_{34})\Big] \end{aligned} \tag{A.41}$$

where $i > j$ denotes that x_i is in the future lightcone of x_j. All other x_{ij} intervals are spacelike, therefore we can replace them with their absolute values. Having deduced (A.41) we would like to express the integral in terms of cross ratios. This will become possible after we average over the null direction we've conveniently chosen, go to a new gauge and integrate out the non-cross ratio variables.

Let's review this process step by step. Our first concern would be to reintroduce inside the integral a solution to the Gegenbauer differential equation, which expresses all of the averaged null directions. We average over a transformation g of $SO(d-2,1)$ that acts on vector m, which leaves the x^0 component invariant and as a Lorentz transformation on the remaining $(d-1)$-components. Therefore we can insert in (A.41) the expression

$$\frac{1}{\text{vol}(SO(d-2,1))} \int_{SO(d-2,1)} dg (gm\cdot x_{34})^J \theta(gm\cdot x_{34}) \tag{A.42}$$

which has been normalized properly by dividing with the volume in the denominator, due to the invariance of the conformal integral.

However because of this invariance we are free to set $x_{34} = x = (x^0, x^1, 0, \ldots, 0)$ and simplify the previous expression in an one dimensional integral. In order to do this we take into account the factors emerging from rotational invariance and the possible boost directions in the $(d-3)$-dimensional plane. We namely have

$$\begin{aligned}
&\int_{SO(d-2,1)} dg(gm\cdot x_{34})^J\theta(gm\cdot x_{34})\\
&=\mathrm{vol}(SO(d-2))\mathrm{vol}(S^{d-3})\int_0^{\mathrm{arccosh}\frac{x^0}{x^1}} d\beta(\mathrm{sinh}\beta)^{d-3}(x^0-x^1\mathrm{cosh}\beta)^J\\
&=vol(SO(d-2))|x|^J B_J\Big(\frac{x^0}{|x|}\Big)
\end{aligned}\tag{A.43}$$

a valid expression, since $\mathrm{arccosh}\frac{x^0}{x^1}$ is well defined for $x^1<x^0$, when x is spacelike. In the third line we have defined the function

$$B_J(y)\equiv\frac{\pi^{d/2-1}\Gamma(J+1)}{2^J\Gamma(J+d/2)}(1+y)^{2-d-J}\,{}_2F_1(J+\frac{d-1}{2},J+d-2,2J+d-1,\frac{2}{1+y})\tag{A.44}$$

which is the result of the integration in the second line after we perform the change of variables $z=\mathrm{cosh}\beta$ and a few standard hypergeommetric function identities. One can easily check that it has the asymptotic expansion

$$B_J(y)\approx\frac{\pi^{d/2-1}\Gamma(J+1)}{2^J\Gamma(J+d/2)}y^{2-d-J},\quad |y|\gg 1.\tag{A.45}$$

Even though we introduced the explicit form of this function, only its asymptotic behaviour will be of use in our derivation.

Having said that equation (A.41) admits the decomposition

$$\begin{aligned}
I_{\Delta,J}=&-\frac{\hat{C}_J(1)}{\mathrm{vol}(S^{d-2})}\Big[(-1)^J\int_{4>1,2>3}\frac{d^dx_3d^dx_4}{\mathrm{vol}(SO(d-2,1))}\times\frac{\langle[O_3,O_2][O_1,O_4]\rangle}{|x_{34}|^{2d-\Delta_3-\Delta_4-\Delta}}B_J(-\eta)\\
&+\int_{3>1,2>4}\frac{d^dx_3d^dx_4}{\mathrm{vol}(SO(d-2,1))}\times\frac{\langle[O_4,O_2][O_1,O_3]\rangle}{|x_{34}|^{2d-\Delta_3-\Delta_4-\Delta}}B_J(\eta)\Big]
\end{aligned}\tag{A.46}$$

and at this point we are in place to un-gauge fix the previous expression and express it in terms of cross ratios. We will focus in detail on the contribution of the first block, since the contribution of its shadow block is obtained in a similar manner. This contribution takes the form

$$\begin{aligned}
I_{\Delta,J}\supset&-\frac{\hat{C}_J(1)}{\mathrm{vol}(S^{d-2})}(-1)^J\times\\
&\int_{4>1,2>3}\frac{d^dx_1^dx_5}{vol(SO(d,2))}\frac{\langle[O_3,O_2][O_1,O_4]\rangle}{|x_{12}|^{\tilde{\Delta}_1+\tilde{\Delta}_2-\tilde{\Delta}}|x_{15}|^{\tilde{\Delta}_1+\tilde{\Delta}-\tilde{\Delta}_2}|x_{25}|^{\tilde{\Delta}_2+\tilde{\Delta}-\tilde{\Delta}_1}}\times\\
&\frac{B_J(-\eta)}{|x_{34}|^{\tilde{\Delta}_3+\tilde{\Delta}_4-\Delta}|x_{35}|^{\tilde{\Delta}_3+\Delta-\tilde{\Delta}_4}|x_{45}|^{\tilde{\Delta}_4+\Delta-\tilde{\Delta}_3}}
\end{aligned}\tag{A.47}$$

where the Lorentzian signature is now implicit. By introducing the standard change of variables

$$\chi = \frac{4\rho}{(1+\rho)^2}, \quad \bar{\chi} = \frac{4\bar{\rho}}{(1+\bar{\rho})^2} \tag{A.48}$$

and integrating out the variable x_5, gauge fixed to be in all regions that are spacelike separated from the other four points, we will arrive to an expression in terms of the cross ratios. This configuration can be seen in Fig.(A.2) below.

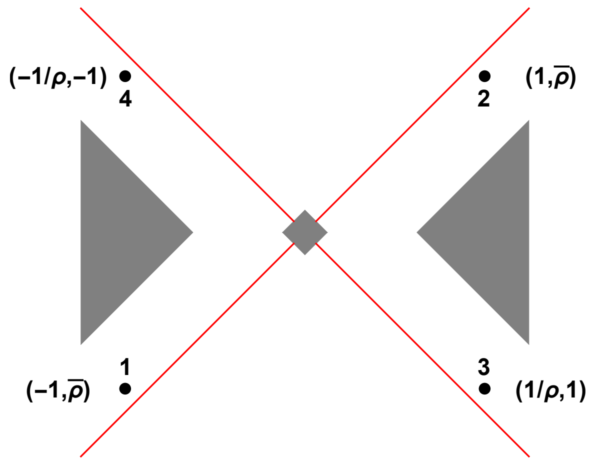

Figure A.2: Selected point configuration in the uv-plane.

For that purpose we seclude the x_5 integral and define the quantity

$$H_{\Delta,J}(x_i) \equiv \int_{\text{spacelike}} d^d x_5 \frac{1}{|x_{12}|^{\tilde{\Delta}_1+\tilde{\Delta}_2-\tilde{\Delta}}|x_{15}|^{\tilde{\Delta}_1+\tilde{\Delta}-\tilde{\Delta}_2}|x_{25}|^{\tilde{\Delta}_2+\tilde{\Delta}-\tilde{\Delta}_1}} \times \frac{B_J(-\eta)}{|x_{34}|^{\tilde{\Delta}_3+\tilde{\Delta}_4-\Delta}|x_{35}|^{\tilde{\Delta}_3+\Delta-\tilde{\Delta}_4}|x_{45}|^{\tilde{\Delta}_4+\Delta-\tilde{\Delta}_3}} \tag{A.49}$$

It is reasonable to rewrite the previous expression in terms of cross ratios by adding a few extra factors in front of it. We recall that

$$\chi = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \quad \bar{\chi} = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} \tag{A.50}$$

and by using simple manipulations we derive

$$H_{\Delta,J}(x_i) = \frac{1}{|x_{12}|^{2d}|x_{34}|^{2d}} \frac{1}{T^{\Delta_i}(x_i)} H_{\Delta,J}(\chi,\bar{\chi}) \tag{A.51}$$

where

$$T^{\Delta_i}(x_i) \equiv \frac{1}{|x_{12}|^{\Delta_1+\Delta_2}|x_{34}|^{\Delta_3+\Delta_4}}\Big(\frac{|x_{14}|}{|x_{24}|}\Big)^{\Delta_2-\Delta_1}\Big(\frac{|x_{14}|}{|x_{13}|}\Big)^{\Delta_3-\Delta_4} \tag{A.52}$$

Note here that the left and right-most expressions of (A.51) are distinct since they have different arguments.

The factors in the denominators of (A.49) denote that the integrand is an eigenfunction of the quadratic and quartic Casimirs of the pairs 1+2 and 3+4. Therefore $H_{\Delta,J}(\chi,\bar{\chi})$ will have the same property and as a solution of these Casimir equations will be fully determined through its asymptotic behaviour for small values of the cross ratios, acting as boundary conditions. In order for the (u,v) coordinates to maintain their spacetime correlation we will examine the solutions behaviour in the limit $\rho \ll 1$, $\bar{\rho} \gg 1$. There the conformal cross ratios have the approximate form

$$\chi \approx 4\rho,\ \bar{\chi} \approx \frac{4}{\bar{\rho}},\ \rho \sim \epsilon,\ \bar{\rho} \sim \epsilon^{-1} \tag{A.53}$$

where we introduced a small parameter ϵ, that proves to be helpful for the rest of the analysis. Our only restriction on this scheme is for the product $\rho\bar{\rho}$ to be fixed.

Under this setup the integral can be evaluated indeed. The range of integration will consist of the plane where the cross ratio variables lay, along with the transverse directions on that plane. We will denote the radius in the transverse directions with r and the integration over these variables becomes essential, when $r \sim 1/\sqrt{\epsilon}$ and $u, v \sim 1$. We can approximate now the order of the conformal invariant η, which appears in the function $B_J(-\eta)$ discussed previously. It is more convenient to first compute the quantity

$$4\eta^2 \approx \frac{\bar{\rho}}{\rho}\frac{(1/\rho + r^2)^2}{(\frac{1-v}{\rho}+r^2)(\frac{1+v}{\rho}+r^2))}\frac{(\bar{\rho}+r^2)^2}{(\bar{\rho}(1-u)+r^2)(\bar{\rho}(1+u)+r^2)} \tag{A.54}$$

and approximate its value in terms of ϵ. Note the "Cartesian" inner product in the u, v plane and the manipulation in the radial component in the transverse directions. This approximation can be found to be of order $1/\epsilon^2$, implying the relation $\eta \sim 1/\epsilon$. In that case η is rather large and we can use the approximate form (A.45) for $B_J(y)$.

The rest of the integral can be simplified similarly by considering only the terms of order $1/\sqrt{\epsilon}$ in the distances $|x_{ij}|$. For example we may write

$$|x_{15}| = |(u+1, v+\bar{\rho}, r)| = ((u+1)(v+\bar{\rho}) + r^2)^{1/2} \approx ((u+1)(\bar{\rho}) + r^2)^{1/2} \tag{A.55}$$

and arrive in a result with factorized u,v dependence and region of integration. Computing these two integral leaves as with the integration on the radial coordinate to be

$$(\rho\bar{\rho})^{(\tilde{\Delta}-1)/2}\int_0^\infty dyy^{(d-4)/2}(\rho\bar{\rho}+y)^{1-\tilde{\Delta}}(1+y)^{1-\Delta} = \frac{\Gamma(d/2-1)^2}{\Gamma(d-2)}\,{}_2F_1\Big(\tilde{\Delta}-1,\Delta-1,\frac{d-1}{2},\frac{1-x}{2}\Big) \tag{A.56}$$

where we've defined $y=\rho r^2$ and $x=\frac{1}{2}(\sqrt{\rho\bar{\rho}})+\frac{1}{\sqrt{\rho\bar{\rho}}}$.

By collecting the multiplicative factors from the previous integrations and translating to cross ratios, the previous expression admits the decomposition

$$H_{\Delta,J}\approx \alpha_{\Delta,J}(\chi\bar{\chi})^{\frac{J+d-1}{2}}\Big(\frac{\chi}{\bar{\chi}}\Big)^{-\frac{\Delta-d+1}{2}} \tag{A.57}$$

under the scope of the kinematics $\chi \ll \bar{\chi} \ll 1$. This though corresponds precisely the asymptotic behaviour of a conformal block in these kinematics with "dimension" equal to $J+d-1$ and "spin" equal to $\Delta-d+1$. Therefore our function is fully determined to be

$$H_{\Delta,J}\approx \alpha_{\Delta,J}G^{\tilde{\Delta}}_{J+d-1,\Delta-d+1}(\chi,\bar{\chi}) \tag{A.58}$$

where the constant $\alpha_{\Delta,J}$ is an expression regular in the limit under consideration. This exchange of the arguments of the CB is where the explicit manifestation of spin lies in the inversion formula.

The only thing left is to gauge fix our equation appropriately and obtain the final result. After a few modifications we will have exactly the same expression derived in [[10]. The gauge we choose relies in the variables defined in (A.48) and results to a Faddeev-Popov determinant. Our modifications will involve the expression

$$I\equiv\int\frac{d^dx_i}{\mathrm{vol}(SO(d,2))}\frac{1}{|x_{12}|^{2d}\cdot|x_{34}|^{2d}}=\int\frac{d^dx_i}{\mathrm{vol}(SO(d,2))}f(x_i) \tag{A.59}$$

In a d-dimensional space we have to consider $4d$ variables in total but our gauge fixing will leave as with only two variables, ρ and $\bar{\rho}$. On the other hand the generators of the coset $\frac{SO(d+2)}{SO(d-2)}$ are

$$\frac{(d+2)(d+1)}{2}-\frac{(d-2)(d-3)}{2}=4d-2 \tag{A.60}$$

therefore after gauge fixing we are restricted on the transformations of $SO(d,2)$ modulo those of $SO(d-2)$. By inserting the identity

$$1=\int_{\Lambda\in\frac{SO(d+2)}{SO(d-2)}}d\Lambda\delta\Big(G_a(\Lambda(x_i))\Big)\Big|\det\Big(\frac{\delta G_a}{\delta\Lambda}\Big)\Big| \tag{A.61}$$

inside the expression (A.59) the $4d-2$ delta function integration will leave only two integrations left, as we have argued already. The gauge fixing conditions are precisely those of Fig.(A.2). The determinant over the Λ transformations produces additional factors that translated in terms of the cross ratios, reduce the expression (A.59) to be

$$I = \frac{1}{2\mathrm{vol}(SO(d-2))}\int_0^1\int_0^1 \frac{1}{2}\frac{d\chi d\bar{\chi}}{(\chi\bar{\chi})^d}|\chi-\bar{\chi}|^{d-2} \tag{A.62}$$

Then our final result for $I_{\Delta,J}$ is

$$\begin{aligned} I_{\Delta,J} = \alpha_{\Delta,J}\Bigg[(-1)^J\int_0^1\int_0^1 \frac{d\chi d\bar{\chi}}{(\chi\bar{\chi})^d}|\chi-\bar{\chi}|^{d-2} G^{\tilde{\Delta}}_{J+d-1,\Delta-d+1}(\chi,\bar{\chi})\frac{\langle[O_3,O_2][O_1,O_4]\rangle}{T^{\Delta_i}} \\ + \int_{-\infty}^0\int_{-\infty}^0 \frac{d\chi d\bar{\chi}}{(\chi\bar{\chi})^d}|\chi-\bar{\chi}|^{d-2}\hat{G}^{\tilde{\Delta}}_{J+d-1,\Delta-d+1}(\chi,\bar{\chi})\frac{\langle[O_4,O_2][O_1,O_3]\rangle}{T^{\Delta_i}} \end{aligned} \tag{A.63}$$

where $\hat{G}_{\Delta,J}$ is defined as the conformal block,which behaves like $(-\chi)^{\frac{\Delta-J}{2}}(-\bar{\chi})^{\frac{\Delta+J}{2}}$, for negative cross ratios satisfying $|\chi| \ll |\bar{\chi}| \ll 1$. The constants appearing in this expression are found to be exactly equal with those of the formula derived in [10]. The only notable difference between the two results is the un-tilded external dimension contained in $I(\Delta, J)$. It is straightforward to get around this by using the identity

$$G^{\tilde{\Delta}}_{J+d-1,\Delta-d+1}(\chi,\bar{\chi}) = ((1-\chi)(1-\bar{\chi}))^{\frac{\Delta_{21}+\Delta_{34}}{2}} G^{\tilde{\Delta}}_{J+d-1,\Delta-d+1}(\chi,\bar{\chi}). \tag{A.64}$$

Our last concern is to relate the double commutators of our expression with the double discontinuity encountered in (3.16). This is possible by applying the $i\epsilon$ prescription in the configuration of the u and v variables. For example lets examine an ordering of the correlator to be $\langle O_4O_1O_2O_3\rangle$. We would therefore want the O_2 operator to be in the lightcone of O_3 and to O_1 operator in the past of O_4. The operators O_4 and O_3 are spacelike separated and commute under this configuration. This implies for the time argument of the other two operators to acquire complex phases of the form

$$t_2 \to t_2 - i\epsilon,\ t_1 \to t_1 + i\epsilon. \tag{A.65}$$

Since these deformations happen in the uv-plane we are free to consider that initially our insertions are in the same time slice. In order for these operators to become null separated in the format we described, we only need to move them in the v-direction, where the v-variable will transform like

$$v_2 \to v_2 - i\epsilon,\ v_1 \to v_1 + i\epsilon. \tag{A.66}$$

The v-dependence concerns only the $\bar{\chi}$ variables, which have the decomposition

$$\bar{\chi} = \frac{v_{21}v_{34}}{v_{31}v_{24}}, \ 1 - \bar{\chi} = \frac{v_{32}v_{14}}{v_{31}v_{24}}. \tag{A.67}$$

The second equation involves the time deformations discussed previously, therefore is the only one that will change essentially. When we move the insertion of the operator O_3 in the future of O_2 the difference v_{32} acquires a phase $e^{i\pi}$. The same holds true for v_{14} resulting for the quantity $1-\bar{\chi}$ to acquire a total phase $e^{2i\pi}$. This is precisely the analytic continuation around 1 that appears in equation (3.17). The other prefactors found in double discontinuities of (3.16), result from the quantities in front of the stripped correlator, which we defined as $T^{\Delta_i}(x_i)$ in equation (A.52). Performing the rest of the analytic continuations in a similar manner we arrive in the expression corresponding to the double discontinuity defined in the main text.

This concludes the proof of the inversion formula in dimensions greater than one. An analogous result in one dimension can be obtained as in higher dimensions but we will not stress with it here, since it is not relevant with the analysis made on this thesis.

B

Appendix B

Twist conformal blocks

In the main text we used several arguments involving twist conformal blocks, when we tried to compare and reproduce the divergencies in both sides of the crossing equation. Here we will explain in more detail the factorization properties of the twist conformal blocks and how one can derive the functions $H^{(m)}_{\tau}(u,v)$ for any positive m, through the recursive relation that these functions obey.

Our starting point will be an asymptotic expansion satisfied from the TCBs around $v=0$ and the Casimir equation that the satisfy [1]. This expansion is of the schematic form

$$H^{(m)}_{\tau}(u,v) = \frac{1}{v^{\Delta_\phi}}\Big(h^{(0)}_{\tau}(u) + h^{(1)}_{\tau}(u)v + \dots\Big) \tag{B.1}$$

and we can use it to solve that Casimir equation order by order. The (physical) solution of this differential equation in first order is

$$h^{(0)}_{\tau}(u) = c_0(1-u)^{1-d/2+\Delta_\phi}u^{\tau/2}F_{\frac{2+\tau-d}{2}}(u) \tag{B.2}$$

where $F_a(u) = {}_2F_1(a,a,2a;u)$ and c_0 a constant that will not matter in the following discussion. For the computation of $(H^{(m)}_{\tau}(u,v))$ we noticed that the only difference between the equation for $h^{(0)}_{\tau}$ and $h^{(1)}_{\tau}$ is the overall power of v, therefore the solutions at each order will differ by a few, hopefully, relative factors. We are also free to specify to any dimension preferable and focus on the form of the TCBs at the limit $\bar{z} \to 1$.

Since we are interested in four space-time dimensions we will examine this case in particular but a similar argument can be carried in two dimensions, where the TCBs factorize nicely too. It is not necessary though to determine the value of the twist τ

and instead we will present the general case. By substituting the solution of (B.1) at every order we arrive in the expression

$$\begin{aligned}
H_\tau^{(m)}(u,v) &= \frac{1}{v}(z\bar{z})^{\tau/2}F_{\frac{\tau-2}{2}}(u)\Big(1+\frac{v}{1-u}+\Big(\frac{v}{1-u}\Big)^2\dots\Big)\\
&= \frac{1}{v}(z\bar{z})^{\tau/2}F_{\frac{\tau-2}{2}}(u)\frac{1}{1-\frac{v}{1-u}}\\
&= \frac{1}{(1-z)(1-\bar{z})}(z\bar{z})^{\tau/2}F_{\frac{\tau-2}{2}}(z\bar{z})\frac{1-z\bar{z}}{\bar{z}+z(1-2\bar{z})}\\
&= \frac{1}{1-\bar{z}}z^{\tau/2}F_{\frac{\tau-2}{2}}(z)\frac{1}{\bar{z}-z}
\end{aligned} \tag{B.3}$$

where in the last line we restricted our analysis to $\bar{z}\to 1$, which allows us to make the appropriate simplifications. The resulting equation is of the factorized form

$$H_\tau^{(m)}(u,v) = \frac{1}{\bar{z}-z}z^{\tau/2}F_{\frac{\tau-2}{2}}(z)\bar{H}_\tau^{(0)}(\bar{z}) \tag{B.4}$$

where $\bar{H}_\tau^{(0)}(\bar{z}) = \frac{1}{1-\bar{z}}$.

In the relative case of two dimensions the Casimir recursion relation (**??**) has only a $\bar{z}$ dependence and the rest of the factorization involves only the z variable. Therefore the recursive differential equation is one dimensional. A similar result can be derived in four dimensions, as the recursive relation can be brought to the form

$$\bar{D}_{4d}\bar{H}_\tau^{(m+1)}(\bar{z}) = \bar{H}_\tau^{(m)}(\bar{z}),\ \ \bar{D}_{4d} = \bar{z}\bar{D}\bar{z}^{-1} \tag{B.5}$$

where as noted before $D = (1-\bar{z})\bar{z}^2\bar{\partial}^2 - \bar{z}^2\bar{\partial}$. It is now straightforward to derive for $\tau = 2$ and $m = 1, 2$ the TCBs

$$\bar{H}_2^{(1)}(\bar{z}) = \frac{\bar{z}}{2}\log^2(1-\bar{z}),\ \ \bar{H}_2^{(1)}(\bar{z}) = -\frac{\bar{z}}{2}\log\bar{z}\log^2(1-\bar{z}) \tag{B.6}$$

which were used in the main text. These solutions obey the boundary condition of having the correct asymptotic behaviour around $\bar{z}\to 1$ and express the leading asymptotic behaviour in that limit. In practice this means the complete solutions contained also other parts that were discarded for not obeying these conditions. In principle we could continue and derive the functions $H_\tau^{(m)}(u,v)$ for any m we would like, but the process is straightforward.

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