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MATHEMATICS & APPLIED MATHEMATICS DEPARTMENT



MASTER THESIS

Periodic Lattices with Flat Bands

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To my parents Eva and Nikitas, and to my brothers Giorgos and Kostas.

Περίληψη

Σκοπός αυτής της εργασίας είναι να αναλύσουμε και να δείξουμε αριθμητικά τι είναι επίπεδη ζώνη ενός φωτονικού πλέγματος. Οι διαφορικές εξισώσεις που μελετάμε εξάγονται από την θεωρία συζευγμένων ρυθμών (Coupled Mode Theory) για οπτικούς πυρήνες και περιγράφουν την μεταβολή του πλάτους του πεδίου του πυρήνα (ή γενικότερα των κυματοδηγών) κατά την διεύθυνση διάδοσης του ηλεκτρομαγνητικού πεδίου. Το ηλεκτρομαγνητικό πεδίο κάθε πυρήνα επηρεάζεται από την ύπαρξη ηλεκτρομαγνητικού πεδίου στους γειτονικούς πυρήνες και αυτή η μεταβολή εκφράζεται μέσω του πλάτους.

Αρχικά εισάγουμε κάποιες βασικές έννοιες οδεύοντων κυμάτων όπως ταχύτητα φάσης, ταχύτητα ομάδας, σχέση διασποράς και τέλος την έννοια της επίπεδης ζώνης η οποία είναι μια επέκταση της σχέσης διασποράς. Σε επόμενο στάδιο βλέπουμε κάποια πλέγματα τα οποία δεν έχουν επίπεδες ζώνες και στα οποία μέσω της ταχύτητας ομάδας και άλλων μεγεθών ελέγχουμε την κατεύθυνση της δέσμης.

Αργότερα θεωρούμε τα πλέγματα Kagome και Lieb στα οποία εάν θεωρήσουμε αλληλεπίδραση μεταξύ των πρώτων γειτόνων των κυματοδηγών υπάρχουν επίπεδες ζώνες και οι οποίες παύουν να υφίστανται με την θεώρηση επόμενων (δεύτερων) γειτόνων. Βλέπουμε επίσης και διάφορα αριθμητικά αποτελέσματα για κάθε περίπτωση.

Κλείνοντας παρουσιάζουμε την ιδέα του πώς μπορεί κανείς να φτιάξει πλέγματα με επίπεδες ζώνες από την ομώνυμη δημοσίευση. Ξεκινώντας από ένα θεμελιώδες σύστημα κυματοδηγών βρίσκουμε τις λύσεις που υποστηρίζει. Έπειτα επεκτείνουμε με έναν συγκεκριμένο τρόπο το θεμελιώδες αυτό σύστημα και ψάχνουμε τι σχέσεις πρέπει να ισχύουν ούτως ώστε οι λύσεις που υποστηρίζονται από το θεμελιώδες σύστημα σε απομόνωση να συνεχίσουν να υπάρχουν και στα θεμελιώδη συστήματα που υπάρχουν στο επεκταμένο. Τέλος φτιάχνουμε τα περιοδικά πλέγματα, που προκύπτουν από τα επεκταμένα συστήματα, και βρίσκουμε τις ζώνες που εμφανίζονται σε αυτά.

Abstract

In this work our aim is to describe analytically and show numerically what is the flat band of a photonic lattice consisted of waveguides. The differential equations that we shall see later are derived from the Coupled Mode Equations which describe the variation of the amplitude of the electromagnetic field between fiber cores or "waveguides" as are called. The propagation of the electromagnetic fields are affected of the neighbouring fields of the waveguides so we consider various lattices of the waveguides and therefore various differential equations which describes them.

Initially we provide basic concepts of travelling waves such as phase velocity, group velocity and dispersion relation. Band and therefore the flat band, which is a specific type of band, is an extension of the dispersion relation for periodic systems. After that we consider some lattices 1d or 2d which have no flat bands and we observe various excitations of the fields for each of them.

Later we examine the Kagome and the Lieb lattice which possess flat bands and we see some excitations of them. We examine the excitations of these lattices considering interaction between only nearest neighbours at the first time and at the next stage we consider interaction between nearest and next nearest neighbours. At both of the lattices the flat bands vanishing with the consideration of the next nearest neighbours .

Finally we present an idea of how to construct flat band lattices based on the corresponding paper which is referred. We start from an initial (fundamental) system and we find the supported solution of this system. Next we are extend the fundamental system in a specific way and we look for relations in order to remain as solution to the extended system, the solution of the fundamental system in isolation. After that we construct the full periodic lattice which is constructed by extend infinitely many times the fundamental system as previous.

Acknowledgements

This work is the final stage for the acquisition of my Master Degree in "Applied and Computational Mathematics" with a specialization in "Scientific Computing" at University Of Crete. It was two years in which I improved a lot my knowledge and my experiences about the field. Simultaneously, it left me with the same enthusiasm for the field as before.

First and foremost I would like to express my gratitude and appreciation to my supervisor Assoc. Prof. Nikolaos Efremidis for his guidance, his patience and the knowledge he imparted to me through this work at which i have learned a lot about the subject. I would also like to thank the rest of my thesis committee, Prof. George Makrakis and Prof. Phoebus Rosakis for their time to read my thesis and for their questions at the presentation.

I couldn't have made this journey to knowledge if I didn't have my family to support me morally and materially, so i want to express my deepest gratitude and to say a big "thank you". Many thanks also to my friends and colleagues for the discussions about our thesis, for the sharing one's knowledge to another and the encouragement.

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Chapter 1

Introduction

In this chapter we will introduce some basic concepts of this work. At the beginning we will present the fundamental ideas of waves phenomena, the phase velocity which express the speed of a single "particle" of a wave when we have only one wave. Later we consider a wave which is consisted of superposition of waves (i.e sum of cosines) and we make clear what is the meaning of the group velocity which is the velocity of the "particles" moving as a group. After that the basic notion that governs this work is presented, the dispersion relation, from which the flatness of a band can be resulted. Finally we are going to speak about the equations describing the propagation of each of the field that we shall see later, the coupled mode equations.

1.1 Phase velocity, Group velocity and Dispersion Relation

Linearity

The vast majority of equations we will meet so far at this work are linear. This means that if u_1 and u_2 are solutions of a linear differential equation then any function of the form $c_1 u_1 + c_2 u_2$ (c_1, c_2 constants) is also a solution. Let us see an example.

Assume that we have the so-called wave equation in one space dimension :

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (1.1)$$

It is easily verified that, if u_1 and u_2 are solutions of the (1.1), then also will be the sum of them.

$$\begin{aligned} \frac{\partial^2 (c_1 u_1 + c_2 u_2)}{\partial t^2} &= c^2 \frac{\partial^2 (c_1 u_1 + c_2 u_2)}{\partial x^2} \\ c_1 \frac{\partial^2 u_1}{\partial t^2} + c_2 \frac{\partial^2 u_2}{\partial t^2} &= c^2 c_1 \frac{\partial^2 u_1}{\partial x^2} + c^2 c_2 \frac{\partial^2 u_2}{\partial x^2} \\ c_1 \left(\frac{\partial^2 u_1}{\partial t^2} - c^2 \frac{\partial^2 u_1}{\partial x^2} \right) &= c_2 \left(\frac{\partial^2 u_2}{\partial t^2} - c^2 \frac{\partial^2 u_2}{\partial x^2} \right) \end{aligned}$$

Hence if we have $u_k, k = 1, \dots, n$ solutions , then

$$u(x, t) = \sum_{k=1}^n c_k u_k(x, t)$$

is also a solution which is called a superposition of solutions $u_k, k = 1, \dots, n$.

Superposition of plane waves, dispersion relation

Suppose we have a wave with period T and a wavelength λ . Recall that the wavelength is the distance between two consecutively peaks of a wave, and T (period) is the time needed for a peak to displaced by λ . The **phase velocity** (v_p), is given from

$$v_p = \frac{\lambda}{T} \quad (1.2)$$

The last expression tells us the rate of a point of the wave that propagates along the propagation direction. Now if we introduce the wavenumber k as $k = \frac{2\pi}{\lambda}$ and the angular frequency $\omega = 2\pi f$, equivalently one can define as the phase velocity the below quantity

$$v_p = \frac{\omega}{k} \quad (1.3)$$

To understand this intuitively we have to observe the displacement of a single point of a wave. Let's suppose that we have the wave $\cos(k_1x - \omega_1t)$, at $t = 0$ the first peak is located at $k_1x = 0 \Rightarrow x = 0$, at the time $t = t'$ the point which previous were located at $x = 0$ is now located at $k_1x - \omega_1t' = 0 \Rightarrow x = \frac{\omega_1}{k_1}t'$. So the quantity $k_1x - \omega_1t$ must remains unaltered over time. In a more general way :

$$k_1x - \omega_1t = \text{con.} \Rightarrow \frac{d(k_1x - \omega_1t)}{dt} = 0 \Rightarrow k_1 \frac{dx}{dt} - \omega_1 = 0 \Rightarrow \frac{dx}{dt} = \frac{\omega_1}{k_1}$$

So it's reasonable to define the phase velocity as in eq. (1.3). Of course the idea of phase velocity is generalised to the complex wave function of the form :

$$A \exp(ikx - i\omega t) = A (\cos(kx - \omega t) + i \sin(kx - \omega t)) \quad (1.4)$$

where it is obvious that again the real and imaginary part are oscillating with the phase velocity we have defined at (1.3).

Group Velocity

To figure it out what is the group velocity we should see what is the result of two waves with equal amplitude superposed. Suppose we have two waves of the form of (1.4) with $k_2 = k_1 + \Delta_k$ and $\frac{\omega(k_2)}{dk} \approx \omega(k_1) + \frac{d\omega(k_1)}{dk} \Delta_k$,

$$\begin{aligned} & \exp(i(k_1x - \omega(k_1)t)) + \exp(i(k_2x - \omega(k_2)t)) = \\ & \exp\left(i\left(\frac{k_1+k_2}{2}x - \frac{\omega(k_1)+\omega(k_2)}{2}t\right)\right) \exp\left(i\left(\frac{k_1-k_2}{2}x - \frac{\omega(k_1)-\omega(k_2)}{2}t\right)\right) \\ & + \exp\left(i\left(\frac{k_1+k_2}{2}x - \frac{\omega(k_1)+\omega(k_2)}{2}t\right)\right) \exp\left(-i\left(\frac{k_1-k_2}{2}x - \frac{\omega(k_1)-\omega(k_2)}{2}t\right)\right) = \\ & = \exp\left(i\left(\frac{k_1+k_2}{2}x - \frac{\omega(k_1)+\omega(k_2)}{2}t\right)\right) (\exp(\dots) + \overline{\exp(\dots)}) = \\ & = \exp\left(i\left(\frac{k_1+k_2}{2}x - \frac{\omega(k_1)+\omega(k_2)}{2}t\right)\right) 2 \cos\left(\frac{k_1-k_2}{2}x - \frac{\omega(k_1)-\omega(k_2)}{2}t\right) \\ & \approx \exp\left(ik_1\left(x - \frac{\omega(k_1)}{k_1}t\right)\right) 2 \cos\left(\frac{-\Delta_k}{2}\left(x - \frac{d\omega(k_1)}{dk}t\right)\right) \end{aligned}$$

The first term oscillates with the phase velocity $v_p = \frac{\omega(k_1)}{k_1}$, and the second term, which forms the behaviour of every point of the wave, oscillates with speed

$$\frac{d\omega(k_1)}{dk} \quad (1.5)$$

Generally speaking the waves have an angular frequency ω that depends on the wavenumber k (and vice versa), in order to exist as a solution of differential equation that admits wave solutions. So we have used the notation $\omega = \omega(k)$. It would be explicated further and in more general way below why the last quantity is called **group velocity**.

Any wave of the form (1.4) can be written in the following form

$$\Psi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{\Psi}(k) \exp(i(kx - \omega t)) dk \quad (1.6)$$

where $\omega = \omega(k)$ as will be explained below (see dispersion relation). [2] Expanding the $\omega(k)$ as follows

$$\omega(k) \approx \omega(k_0) + \omega'(k_0)(k - k_0)$$

and replacing the last into eq. (1.6) we obtain

$$\Psi(x, t) \approx \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{\Psi}(k) \exp \left[i (k_0 x - \omega(k_0)t) + i ((k - k_0) x - \omega'(k_0) (k - k_0) t) \right] dk$$

or equivalently

$$\Psi(x, t) \approx \frac{\exp [i(k_0 x - \omega(k_0)t)]}{2\pi} \int_{-\infty}^{+\infty} \tilde{\Psi}(k) \exp [i(k - k_0) (x - \omega'(k_0)t)] dk \quad (1.7)$$

The last integral consists of a superposition of infinitely many waves that each of which are oscillating with velocity $\omega'(k_0)$ which is called **group velocity**. Of course the first term oscillates with the phase velocity $\frac{\omega(k_0)}{k_0}$. The term group velocity coined from the fact that we have infinitely many waves with the same speed and the superposition of all of these forms the behaviour of the one, which moves with the phase velocity.

Dispersion Relation

The relationship between the angular frequency ω and the wavenumber k (or reverse) is called **dispersion relation** and characterize every equation admits solution of the aforementioned wave form. Consider the following equation

$$u_t = i u_{xx}$$

and suppose there is a solution $u(x, t) = \exp(i(kx - \omega t))$ where k and ω are real. Replacing the last one into the equation, we get :

$$\begin{aligned} -i\omega \exp(i(kx - \omega t)) &= i^3 k^2 \exp(i(kx - \omega t)) \Rightarrow \\ \omega &= -k^2 \end{aligned}$$

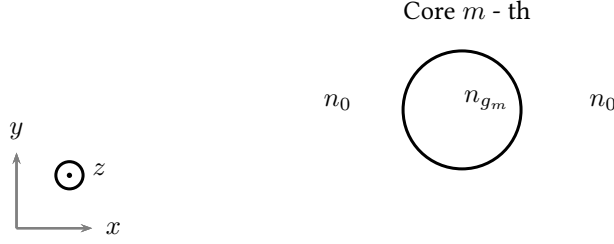
The last relation is the dispersion relation of the equation $u_t = i u_{xx}$. Of course there are many dispersion relations, for example $u_{tt} = u_{xx}$ where now $\omega^2 = k^2$ therefore $\omega = \pm k$ so we have two relations in this dispersion relation. The concept of "band" is the extension of the dispersion relation for periodic systems and we introduce it at the end of the next section.

1.2 Coupled Mode Equations and Waveguides

In this section we will derive the basic equations governing the majority of phenomena that exist in this work, the so-called Coupled Mode Equations. Before that, we have to say that the with the word "mode" we mean an electromagnetic field whose intensity remains constant in the direction of propagation (z axis). Moreover with the term "transverse field of the mode" we technically speak about the first eigenfunction of the 2d helmholtz equation (see eq. (1.10)) where we have a discrete spectrum of eigenvalues. The "transverse" is referred to the fact that we have oscillations of the electromagnetic field perpendicular to its propagation direction (z axis). In this derivation for simplicity we base the analysis on the scalar theory and we suppose that we have p cores, generally different, that run parallel into the direction of propagation (z axis). In addition we assume that the cores are oriented in the z -direction and are described by the refractive index $n_m(x, y)$, $m = 1, \dots, p$. So n_m is equal to n_0 everywhere, except the region occupied by the m -th core, where it is equal to $n_{g,m}$, as we can see at the Fig. 1.1 .

Suppose that the electric field is polarized in the y -direction, then the equation which describes the scalar electric field $E_m(x, y, z)$ when the m -th core is in isolation is given by :

$$\nabla^2 E_m(x, y, z) + n_m^2(x, y) k_0^2 E_m(x, y, z) = 0 \quad (1.8)$$

Figure 1.1: Refractive index $n_m(x, y)$

where n_m is the refractive index which we described above, $k_0 = \frac{2\pi}{\lambda_0}$ is the wavenumber and λ_0 is the vacuum wavelength. Notice that the product $n_m^2 k_0^2$ gives us the right wavenumber for every $(x, y) \in \mathbb{R}^2$ (inside or outside the fiber-core).

We seek solutions of the (1.8) of the form

$$E_m = F_m(x, y) \exp(i\beta_m z) \quad (1.9)$$

where F_m ($m = 1, \dots, p$) is the transverse field of the mode in the m -th guide and β_m is the mode propagation constant. Replacing eq. (1.9) in the eq. (1.8) gives us the following:

$$\begin{aligned} & \frac{\partial^2 E_m}{\partial x^2} + \frac{\partial^2 E_m}{\partial y^2} + \frac{\partial^2 E_m}{\partial z^2} + n_m^2 k_0^2 E_m = 0 \\ \Rightarrow & \frac{\partial^2 F_m}{\partial x^2} \exp(i\beta_m z) + \frac{\partial^2 F_m}{\partial y^2} \exp(i\beta_m z) - \beta_m^2 F_m \exp(i\beta_m z) \\ & + n_m^2 k_0^2 F_m \exp(i\beta_m z) = 0 \end{aligned}$$

or equivalently

$$\frac{\partial^2 F_m}{\partial x^2} + \frac{\partial^2 F_m}{\partial y^2} + [n_m^2 k_0^2 - \beta_m^2] F_m = 0 \quad (1.10)$$

The whole system consisted of p parallel cores is described by a refractive index $n(x, y)$ resulted from combining the refractive indices of each of the cores, i.e $n(x, y)$ is equal to n_{g_m} in the region occupied by the m -th core and equals to n_0 otherwise. The scalar wave equation for the whole system is :

$$\nabla^2 E(x, y, z) + n^2 k_0^2 E(x, y, z) = 0 \quad (1.11)$$

An essential assumption of the coupled mode theory is that the solution of (1.11) can be written as a superposition of modes of each of the cores, i.e sum of the form (1.12)

$$E(x, y, z) = \sum_{m=1}^p A_m(z) F_m(x, y) \exp(i\beta_m z) \quad (1.12)$$

where $A_m(z)$ is called "amplitude" and it tells us how the electromagnetic field travels from a core to another core. Thus replacing (1.12) into (1.11) we get

$$\begin{aligned} & \sum_{m=1}^p \nabla^2 [F_m A_m \exp(i\beta_m z)] + \sum_{m=1}^p [n^2(x, y) k_0^2 A_m F_m \exp(i\beta_m z)] = 0 \\ \Rightarrow & \sum_{m=1}^p \left[F_m \frac{d^2}{dz^2} [A_m \exp(i\beta_m z)] \right] + \sum_{m=1}^p \left[\left(\frac{\partial^2 F_m}{\partial x^2} + \frac{\partial^2 F_m}{\partial y^2} \right) A_m(z) \exp(i\beta_m z) \right] \\ & + \sum_{m=1}^p [n^2(x, y) k_0^2 A_m F_m \exp(i\beta_m z)] = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \sum_{m=1}^p & \left[\left(\frac{d^2 A_m}{dz^2} + 2i\beta_m \frac{dA_m}{dz} - \beta_m^2 A_m(z) \right) F_m \exp(i\beta_m z) \right] \\ & + \sum_{m=1}^p \left[\left(\frac{\partial^2 F_m}{\partial x^2} + \frac{\partial^2 F_m}{\partial y^2} \right) A_m \exp(i\beta_m z) \right] \\ & + \sum_{m=1}^p \left[n^2(x, y) k_0^2 A_m F_m \exp(i\beta_m z) \right] = 0 \end{aligned}$$

using eq. (1.10)

$$\begin{aligned} \Rightarrow \sum_{m=1}^p & \left[\left(\frac{d^2 A_m}{dz^2} + 2i\beta_m \frac{dA_m}{dz} - \beta_m^2 A_m \right) F_m \exp(i\beta_m z) \right] \\ & + \sum_{m=1}^p \left[\left(\beta_m^2 - n_m^2 k_0^2 \right) F_m A_m \exp(i\beta_m z) \right] \\ & + \sum_{m=1}^p \left[n^2(x, y) k_0^2 A_m F_m \exp(i\beta_m z) \right] = 0 \end{aligned}$$

or equivalently

$$\sum_{m=1}^p \left[\left(\frac{d^2 A_m}{dz^2} + 2i\beta_m \frac{dA_m}{dz} \right) F_m \exp(i\beta_m z) \right] + \sum_{m=1}^p \left[k_0^2 (n^2 - n_m^2) A_m F_m \exp(i\beta_m z) \right] = 0$$

We can neglect the terms $\frac{d^2 A_m(z)}{dz^2}$ if we consider amplitudes that vary slowly along the propagation direction z . Due to the Kerr effect the refractive index $n^2 - n_m^2$ is affected by the intensity of the electric field and so we introduce a perturbation as follows

$$n^2 - n_m^2 \rightarrow n_c^2 - n_m^2 + \gamma \left| \sum_{m=1}^p A_m F_m \exp(i\beta_m z) \right|^2$$

where n_c is the refractive index resulted from combining the refractive indices n_m , $m = 1, \dots, p$, i.e n_c is equal to n_m in the region occupied by the m -th core otherwise is equal to n_0 . Replace the last formula into the equation we obtain

$$\begin{aligned} & \sum_{m=1}^p \left[\left(2i\beta_m \frac{dA_m}{dz} + k_0^2 (n_c^2 - n_m^2) A_m \right) F_m \exp(i\beta_m z) \right] \\ & + \gamma k_0^2 \sum_{m=1}^p \left[\left| \sum_{m'=1}^p [A_{m'} F_{m'} \exp(i\beta_{m'} z)] \right|^2 A_m F_m \exp(i\beta_m z) \right] = 0 \end{aligned} \quad (1.13)$$

Our aim is to find how a specific amplitude $A_l(z)$ of the core l , changes along the propagation direction z , i.e the term $\frac{dA_l(z)}{dz}$. Multiplying the last equation by F_l^* gives us

$$\begin{aligned} & F_l^* \sum_{m=1}^p \left[\left(2i\beta_m \frac{dA_m}{dz} + k_0^2 (n_c^2 - n_m^2) A_m \right) F_m \exp(i\beta_m z) \right] \\ & + k_0^2 \gamma F_l^* \sum_{m=1}^p \left[\left| \sum_{m'=1}^p [A_{m'} F_{m'} \exp(i\beta_{m'} z)] \right|^2 A_m F_m \exp(i\beta_m z) \right] = 0 \end{aligned}$$

If we assume that there is no overlap between non-adjacent cores we can neglect the terms of the form $F_l^* F_{l'}$ for $l' < l - 1$ and $l' > l + 1$ because F_l and therefore F_l^* is almost zero in the region where $F_{l'}$ is large (its absolute value) and vice versa. So the last equation becomes

$$F_l^* \sum_{m=l-1}^{l+1} \left[\left(2i\beta_m \frac{dA_m}{dz} + k_0^2 (n_c^2 - n_m^2) A_m(z) \right) F_m \exp(i\beta_m z) \right] \\ + k_0^2 \gamma F_l^* \sum_{m=l-1}^{l+1} \left[\left| \sum_{m'=l-1}^{l+1} [A_{m'} F_{m'} \exp(i\beta_{m'} z)] \right|^2 A_m F_m \exp(i\beta_m z) \right] = 0$$

or equivalently

$$F_l^* \sum_{m=l-1}^{l+1} \left[\left(2i\beta_m \frac{dA_m}{dz} + k_0^2 (n_c^2 - n_m^2) A_m \right) F_m \exp(i\beta_m z) \right] \\ + \gamma k_0^2 F_l^* \left(\left| \sum_{m'=l-1}^{l+1} [A_{m'} F_{m'} \exp(i\beta_{m'} z)] \right|^2 \right) \sum_{m=l-1}^{l+1} [A_m F_m \exp(i\beta_m z)] = 0$$

but if we expand the sum :

$$\left| \sum_{m'=l-1}^{l+1} [A_{m'} F_{m'} \exp(i\beta_{m'} z)] \right|^2 = |A_{l-1}|^2 |F_{l-1}|^2 + |A_l|^2 |F_l|^2 + |A_{l+1}|^2 |F_{l+1}|^2 \\ + A_{l-1} A_l^* F_{l-1} F_l^* \exp(-\Delta b_{l,-1}) + A_{l-1}^* A_l F_{l-1}^* F_l \exp(\Delta b_{l,-1}) \\ + A_l A_{l+1}^* F_l F_{l+1}^* \exp(-\Delta b_{l,+1}) + A_l^* A_{l+1} F_l^* F_{l+1} \exp(\Delta b_{l,+1})$$

where $\Delta b_{l,-1} = i(\beta_l - \beta_{l-1})z$ and $\Delta b_{l,+1} = i(\beta_{l+1} - \beta_l)z$. There are 2 terms which are neglected and is of the form $F_{l-1} F_{l+1}^*$ and $F_{l+1} F_{l-1}^*$ because they correspond to non adjacent cores. So now we can replace the last sum into the equation to obtain :

$$F_l^* \sum_{m=l-1}^{l+1} \left[\left(2i\beta_m \frac{dA_m}{dz} + k_0^2 (n_c^2 - n_m^2) A_m \right) F_m \exp(i\beta_m z) \right] \\ + \gamma k_0^2 F_l^* \left[|A_{l-1}|^2 |F_{l-1}|^2 + |A_l|^2 |F_l|^2 + |A_{l+1}|^2 |F_{l+1}|^2 + A_{l-1} A_l^* F_{l-1} F_l^* \exp(-\Delta b_{l,-1}) \right. \\ \left. + A_{l-1}^* A_l F_{l-1}^* F_l \exp(\Delta b_{l,-1}) + A_l A_{l+1}^* F_l F_{l+1}^* \exp(-\Delta b_{l,+1}) \right. \\ \left. + A_l^* A_{l+1} F_l^* F_{l+1} \exp(\Delta b_{l,+1}) \right] \sum_{m=l-1}^{l+1} [A_m F_m \exp(i\beta_m z)] = 0$$

So now if we represent the $\frac{dA_m}{dz} = A'_m$ and expand the sum, we take as a result

$$k_0^2 (n_c^2 - n_{l-1}^2) F_l^* F_{l-1} A_{l-1} \exp(i\beta_{l-1} z) + 2i\beta_l |F_l|^2 A'_l \exp(i\beta_l z) \\ + k_0^2 (n_c^2 - n_l^2) |F_l|^2 A_l \exp(i\beta_l z) \\ + k_0^2 (n_c^2 - n_{l+1}^2) F_l^* F_{l+1} A_{l+1} \exp(i\beta_{l+1} z) + \gamma k_0^2 |F_l|^4 A_l |A_l|^2 \exp(i\beta_l z) = 0$$

where we have neglected terms which are very small due to the cubic or higher order of power and in addition the corresponding cores have small overlapping modes, for example $\gamma F_{l-1} |F_{l-1}|^2 F_l^* |A_{l-1}|^2 A_{l-1}$

$\exp(i\beta_{l-1}z)$. Of course we've neglected terms of the form $F_m^* F_{m-1}$ due to the small overlapping of the modes of the corresponding cores. Now if we integrate over the entire xy plane we get the following

$$\begin{aligned} & k_0^2 \left(\int (n_c^2 - n_{l-1}^2) F_l^* F_{l-1} \right) A_{l-1} \exp(i\beta_{l-1}z) + 2i\beta_l \left(\int |F_l|^2 \right) A_l \exp(i\beta_l z) \\ & + k_0^2 \left(\int (n_c^2 - n_l^2) |F_l|^2 \right) A_l \exp(i\beta_l z) \\ & + k_0^2 \left(\int (n_c^2 - n_{l+1}^2) F_l^* F_{l+1} \right) A_{l+1} \exp(i\beta_{l+1}z) + \gamma k_0^2 \left(\int |F_l|^4 \right) A_l |A_l|^2 \exp(i\beta_l z) \\ & = 0 \end{aligned}$$

Multiplying both sides by $\frac{1}{2\beta_l \exp(i\beta_l z) \int |F_l|^2}$ we obtain

$$i \frac{dA_l}{dz} + \kappa_{l,l-1} A_{l-1} + \kappa_{l,l+1} A_{l+1} + \gamma_l A_l |A_l|^2 + \delta_l A_l = 0 \quad (1.14)$$

where

$$\begin{aligned} \kappa_{n,m} &= \frac{k_0^2}{2\beta_n} \frac{\int (n_c^2 - n_m^2) F_n^* F_m dx dy}{\int |F_n|^2 dx dy} \exp(i(\beta_m - \beta_n)z), \quad \gamma_l = \gamma \frac{k_0^2 \int |F_l|^4 dx dy}{2\beta_l \int |F_l|^2 dx dy}, \\ \delta_l &= \frac{k_0^2 \int (n_c^2 - n_l^2) |F_l|^2 dx dy}{2\beta_l \int |F_l|^2 dx dy}. \end{aligned}$$

The $\kappa_{n,m}$ are called coupling terms, γ is the Kerr effect constant and δ_l are called detuning terms. So the last equations constitute the **Coupled Mode Theory Equations** or **Discrete Nonlinear Schrödinger Equations** and are valid for $m = 2, \dots, p-1$. For $m = 1$ the coupling coefficient to the left waveguide does not exist, whereas for $m = p$ the coupling to the right waveguide is zero.

Waveguide in an abstract way in this work will be considered as something that can confine light as in the case above of the cores. Thus with the "waveguide array" or "waveguide lattice" or "periodic lattice" or simple a "lattice", is considered a number of waveguides ordered in a specific manner which is repeated again and again (a periodic lattice). With the orientation of the waveguide we mean the direction in which the electromagnetic field will be propagated. Furthermore with the term "unit cell of a lattice" we mean the minimum "box" which if repeat it infinitely many times i will end up with the periodic lattice.

Periodic lattices are governed by equations similar to the (1.14) and with the "similar" we mean that at some cases the nonlinear terms (e.g $\gamma(k_0^2/2\beta_l) (\int |F_l|^4 / \int |F_l|^2) A_l |A_l|^2$) are neglected and other cases we have to consider overlapping with non-adjacent waveguides. If we assume plane wave solution to the last equations we obtain the dispersion relation which is consisted of the bands (as previous the dispersion relation can have more than one relations) and if one band is constant then we say that we have a flat band.

Chapter 2

Non Flat Band Lattices

In this chapter we present various lattices, mono-atomic or diatomic and 1d or 2d. With the term "mono-atomic" is meant that we have one waveguide per unit cell and with "diatomic" we have two waveguides per unit cell. Of course these two terms are referred to a physical meaning which is not in the context of this work. With the term 1d lattice we referred to the dimension of the space the waveguides belong to, for example if the waveguides are extending along a straight line then we put them mathematically at the real line so the lattice is 1d. In similar way 2ds waveguides are extending at a plane.

2.1 1d Mono-atomic Waveguide Lattice

Let's suppose we have the following equation which describes the variation of the amplitude of the n -th waveguide in a periodic lattice [4].

$$i \frac{dE_n}{dz} + \kappa(E_{n+1} + E_{n-1}) + \gamma|E_n|^2 E_n = 0 \quad (2.1)$$

where E_n is the amplitude of the electromagnetic field of the n -th waveguide, κ is the coupling constant between the neighbor waveguides and finally γ is the Kerr effect constant.

Assume that we have a plane wave solution, which is of the following form

$$E_n = A e^{i(qn - kz)} \quad (2.2)$$

where A is a real constant, q is the phase difference between adjacent waveguides and of course $k = k(q)$ is the dispersion relation. Substituting eq. (2.2) into eq.(2.1) we get

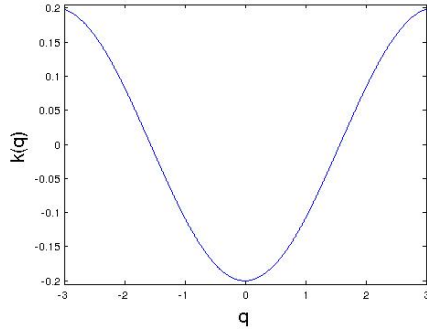
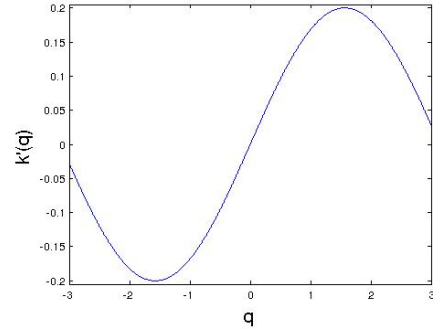
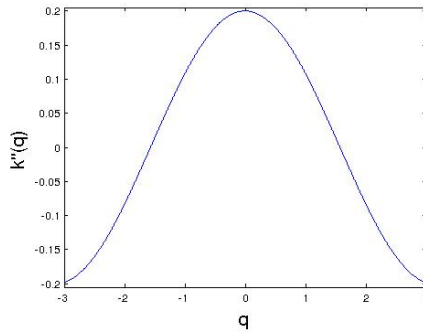
$$-i^2 k(q) E_n + E_n \kappa (e^{iq} + e^{-iq}) + \gamma A^2 E_n = 0$$

$$k(q) E_n + E_n (2\kappa \cos(q)) + \gamma A^2 E_n = 0$$

$$E_n (k(q) + 2\kappa \cos(q) + \gamma A^2) = 0$$

$$k(q) = -2\kappa \cos(q) - \gamma A^2 \quad (2.3)$$

The last relation is the band and is not constant, i.e k is not independent of q . For a given initial condition the wave propagation is completely controlled by the dispersion relation $k(q)$ [5]. At Fig. 2.1,2.2,2.3 we can see the dispersion, the group velocity and the second derivative of dispersion relation. As we see from the formula of the dispersion relation and the figures also, the dispersion relation is far from being flat.

Figure 2.1: Dispersion relation (k) for $\kappa = 0.1, \gamma = 0$ Figure 2.2: Group velocity ($\frac{dk}{dq}$) for $\kappa = 0.1, \gamma = 0$ Figure 2.3: $\frac{d^2k}{dq^2}$ for $\kappa = 0.1, \gamma = 0$

2.1.1 Numerical Results

We will excite some waveguides using a gaussian beam as initial condition. The "beam" has the form :

$$E_n(z=0) = e^{-\left(\frac{n}{n_0}\right)^2} e^{iq_0 n} \quad (2.4)$$

In fact every solution can be written in the form

$$E_n(z) = \int_{-\pi}^{\pi} A(q) e^{i(qn - kz)}$$

The $A(q)$ are the Fourier coefficients

$$A(q) = \sum_{n=-\infty}^{+\infty} E_n(0) e^{-iqn}$$

So in this case $A(q)$ can be written

$$A(q) = \sum_{n=-\infty}^{+\infty} e^{-\left(\frac{n}{n_0}\right)^2} e^{iq_0 n} e^{-iqn} = \sum_{n=-\infty}^{+\infty} e^{-\left(\frac{n}{n_0}\right)^2} e^{-i(q-q_0)n}$$

The $|A(q)|$ is a sharply peaked function around q_0 so we can write the solution $E_n(z)$ in the below form (as in the group velocity)

$$E_n(z) \approx \exp [i(q_0 n - k(q_0)z)] \int_{-\infty}^{+\infty} A(q) \exp [i(q - q_0) (n - k'(q_0)z)] dk$$

So, as previous the group velocity told us at which direction and with what speed will the beam propagates, the last form told us where the discrete wave propagates (which direction). Before we start to comment the figures we have to say that for every numerical result we will see from now on at this work h is the step at the z axis of the Runge-Kutta (explicit) method we have used to make all the numerical schemes. If we observe the Fig. 2.1-2.3 the positive maximum of the group velocity is at the $\pi/2$ and the negative minimum is at the $-\pi/2$ so at the Fig. 2.4 and 2.5 we see the "beam" propagates in the right and in the left direction respectively with almost no diffraction due to the fact that the second order diffraction is vanishing for $q_0 = -\pi/2, \pi/2$ (Fig. 2.3). At the Fig. 2.6 and 2.7 we see similar things, the "beam" propagates at the right direction and at the left respectively and in addition it presents a lot of diffraction since for $q_0 = -\pi/4, \pi/4$ the diffraction term is not vanishing as we see at the Fig. 2.3. At the Fig. 2.8 the "beam" propagates straight but it has a lot of diffraction since at $q_0 = 0$ the diffraction term have the positive maximum. Similar things holds for the Fig. 2.9-2.13 but because the transport and the diffraction are much smaller due to the smaller coupling term ($\kappa = 0.1$) we plot at the vertical axis the absolute value of the field and at the horizontal axis the number of waveguide. So we can see the transport and the small diffraction.

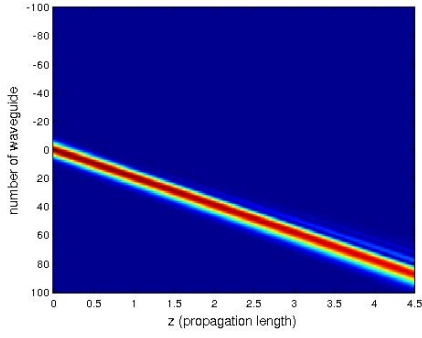


Figure 2.4: $q_0 = \pi/2, \kappa = 10, \gamma = 0, n_0 = 5, h = 10^{-2}, z = 4.5$

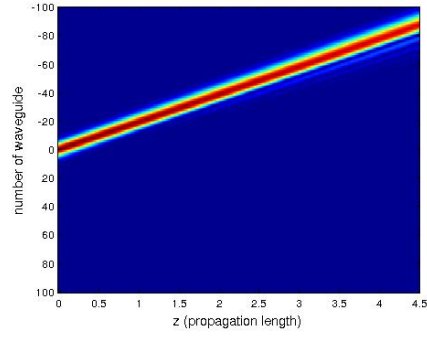


Figure 2.5: $q_0 = -\pi/2, \kappa = 10, \gamma = 0, n_0 = 5, h = 10^{-2}, z = 4.5$

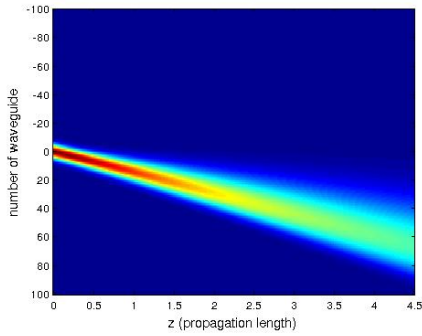


Figure 2.6: $q_0 = \pi/4, \kappa = 10, \gamma = 0, n_0 = 5, h = 10^{-2}, z = 4.5$

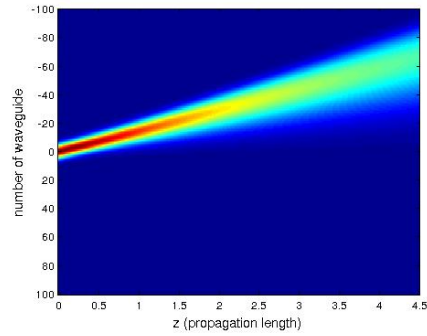


Figure 2.7: $q_0 = -\pi/4, \kappa = 10, \gamma = 0, n_0 = 5, h = 10^{-2}, z = 4.5$

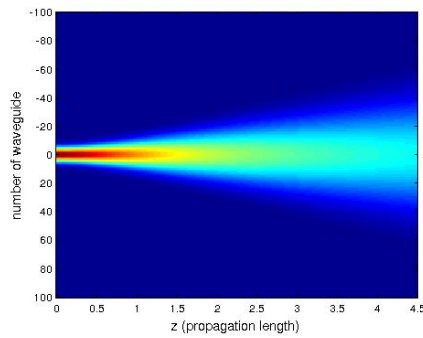


Figure 2.8: $q_0 = 0, \kappa = 10, \gamma = 0, n_0 = 5, h = 10^{-2}, z = 4.5$

Fig. 2.4-2.8 show the intensity of the field E_n (i.e. $|E_n|$) along the propagation length for every waveguide for initial condition of the form (2.4)

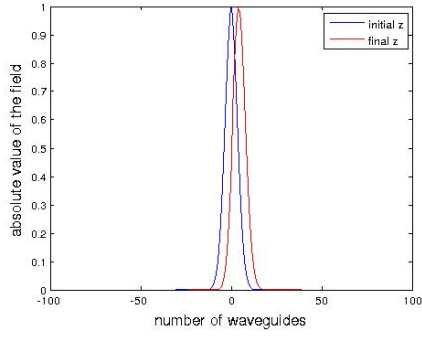


Figure 2.9: $q_0 = \pi/2, \kappa = 0.1, \gamma = 0, n_0 = 5, h = 10^{-2}, z = 4.5$

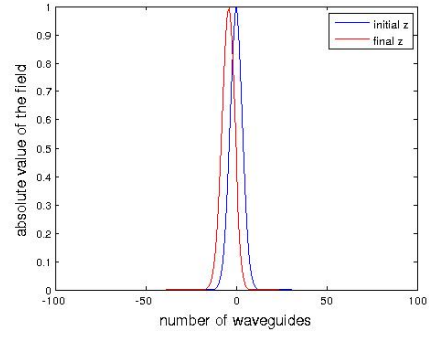


Figure 2.10: $q_0 = -\pi/2, \kappa = 0.1, \gamma = 0, n_0 = 5, h = 10^{-2}, z = 4.5$

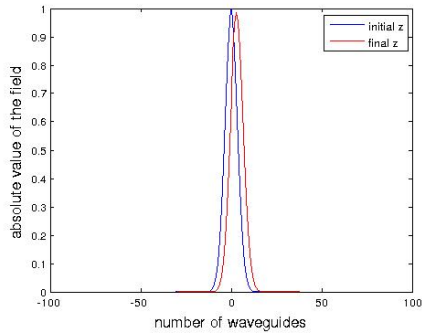


Figure 2.11: $q_0 = \pi/4, \kappa = 0.1, \gamma = 0, n_0 = 5, h = 10^{-2}, z = 4.5$

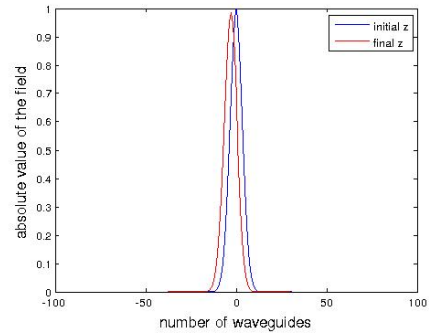


Figure 2.12: $q_0 = -\pi/4, \kappa = 0.1, \gamma = 0, n_0 = 5, h = 10^{-2}, z = 4.5$

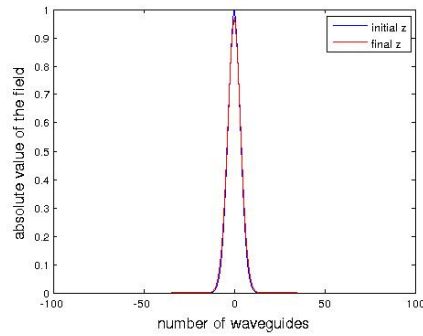


Figure 2.13: $q_0 = 0, \kappa = 0.1, \gamma = 0, n_0 = 5, h = 10^{-2}, z = 4.5$

Fig. 2.9-2.13 show the intensity of the field E_n in y axis (i.e. $|E_n|$) and the number of each waveguide in x axis for initial condition of the form (2.4)

2.2 1d Diatomic Waveguide Lattice

We have now two types of waveguides so we say that we have a diatomic lattice. We assign a detuning term to the waveguides, so waveguides of type u have a term $+\delta u_n$ and the waveguides of type v have a term $-\delta v_n$ as we can see at the eq. (2.5) and (2.6).

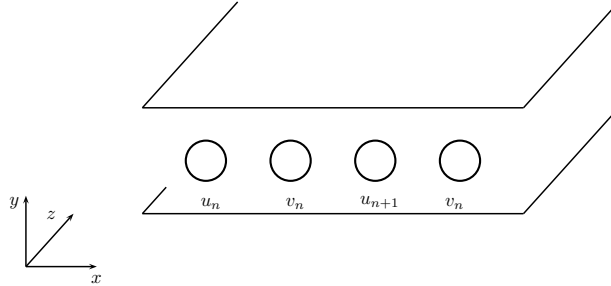


Figure 2.14: waveguide lattice

Below the equations describe the amplitudes of the electromagnetic fields of the waveguides:

$$i \frac{du_n}{dz} + \kappa(v_{n-1} + v_n) + \delta u_n = 0 \quad (2.5)$$

$$i \frac{dv_n}{dz} + \kappa(u_n + u_{n+1}) - \delta v_n = 0 \quad (2.6)$$

where δ it's called detuning term. We will find the dispersion relation for the system of eq.(2.5)-(2.6). Suppose the solution of the system is a planar wave of the form $u_n = ue^{iqn-ikz}$, $v_n = ve^{iqn-ikz}$,

$$ku_n + \kappa v_n(e^{-iq} + 1) + \delta u_n = 0$$

$$\kappa v_n + \kappa u_n(1 + e^{iq}) - \delta v_n = 0$$

writing last two equation in matrix form :

$$\begin{bmatrix} k + \delta & \kappa(e^{-iq} + 1) \\ \kappa(1 + e^{iq}) & k - \delta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2.7)$$

The only way to have non trivial solution is

$$\begin{vmatrix} k + \delta & \kappa(e^{-iq} + 1) \\ \kappa(1 + e^{iq}) & k - \delta \end{vmatrix} = 0 \Leftrightarrow k^2 - \delta^2 - \kappa^2(e^{-iq} + 1)(1 + e^{iq}) = 0$$

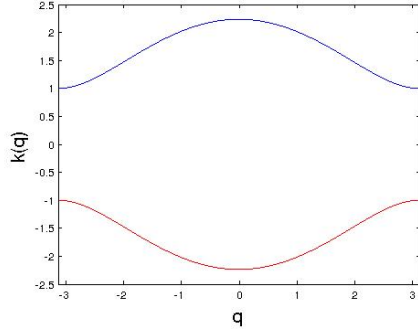
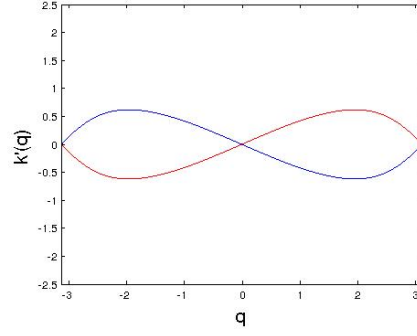
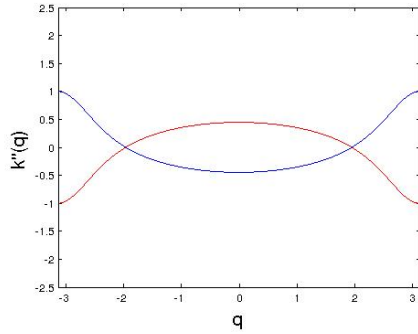
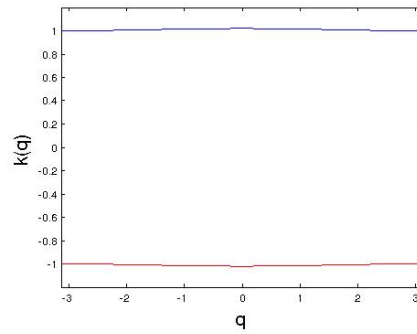
$$k^2 = \delta^2 + \kappa^2(\cos(q) - i \sin(q) + 1)(1 + \cos(q) + i \sin(q))$$

$$k^2 = \delta^2 + \kappa^2(\cos(q) + \cos^2(q) + \sin^2(q) + 1 + \cos(q))$$

$$k^2 = \delta^2 + 2\kappa^2(\cos(q) + 1)$$

$$k(q) = \pm \sqrt{\delta^2 + 2\kappa^2(1 + \cos(q))} \quad (2.8)$$

So now as we see at the (2.8) we have two non flat bands. There are the dispersion relation, group velocity and the second order diffraction at Fig. 2.15-2.17 for relatively big $\kappa = 1$ and at the next three Fig. (2.18 - 2.20) we see for good enough $\kappa = 0.1$ the same quantities.

Figure 2.15: Dispersion relation (k) for $\kappa = 1, \delta = 1$ Figure 2.16: Group velocity ($\frac{dk}{dq}$) for $\kappa = 1, \delta = 1$ Figure 2.17: ($\frac{d^2k}{dq^2}$) for $\kappa = 1, \delta = 1$ Figure 2.18: Dispersion relation (k) for $\kappa = 0.1, \delta = 1$

We are going to find the eigenvalues of the matrix (2.7)

$$\begin{vmatrix} k + \delta - \lambda & \kappa (e^{-iq} + 1) \\ \kappa (1 + e^{iq}) & k - \delta - \lambda \end{vmatrix} = 0 \Leftrightarrow k^2 - 2\lambda k + \lambda^2 - \delta^2 - 2\kappa^2 (1 + \cos(q)) = 0$$

But k is given from eq. (2.8) so the remaining terms are

$$\lambda^2 - 2\lambda k = 0$$

thus we have two eigenvalues

$$\lambda_1 = 0, \lambda_2 = 2k$$

The corresponding eigenvector of eigenvalue $\lambda_1 = 0$ is :

$$\begin{pmatrix} k + \delta & \kappa (e^{-iq} + 1) \\ \kappa (1 + e^{iq}) & k - \delta \end{pmatrix} \begin{matrix} \leftarrow -\frac{\kappa(1+e^{iq})}{k+\delta} \\ \leftarrow + \end{matrix}, k + \delta \neq 0 \\ \Rightarrow \begin{pmatrix} k + \delta & \kappa (e^{-iq} + 1) \\ 0 & \frac{-2\kappa^2(1+\cos(q))+k^2-\delta^2}{k^2-\delta^2} = 0 \end{pmatrix}$$

So the eigenvector corresponds to $\lambda_1 = 0$ is

$$\begin{bmatrix} -\frac{\kappa(e^{-iq}+1)}{k+\delta} \\ 1 \end{bmatrix}, \cos(q) \neq -1 \text{ or } k \neq -\sqrt{\delta^2 + 2\kappa^2(1 + \cos(q))}$$

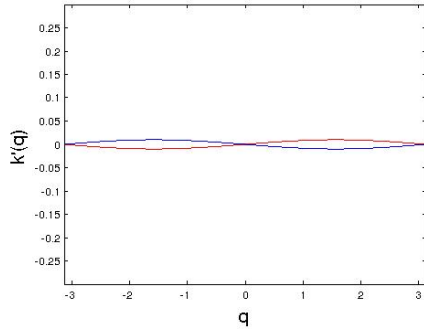


Figure 2.19: Group velocity $\left(\frac{dk}{dq}\right)$ for $\kappa = 0.1, \delta = 1$

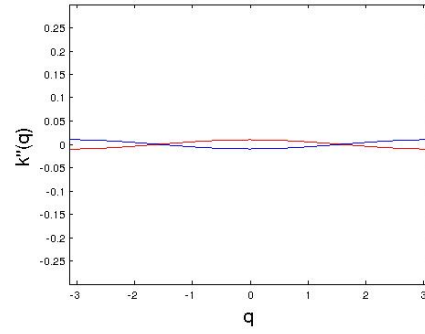


Figure 2.20: $\left(\frac{d^2k}{dq^2}\right)$ for $\kappa = 0.1, \delta = 1$

Similarly the second eigenvector is :

$$\begin{pmatrix} -k + \delta & \kappa(e^{-iq} + 1) \\ \kappa(1 + e^{iq}) & -k - \delta \end{pmatrix} \begin{matrix} \leftarrow - \\ \rightarrow + \end{matrix} \begin{matrix} -\frac{\kappa(1+e^{iq})}{-k+\delta} \\ \frac{\kappa(1+e^{-iq})}{-k-\delta} \end{matrix}, -k + \delta \neq 0$$

$$\Rightarrow \begin{pmatrix} -k + \delta & \kappa(e^{-iq} + 1) \\ 0 & \frac{-2\kappa^2(1+\cos(q)) - \delta^2 + k^2}{-k + \delta} = 0 \end{pmatrix}$$

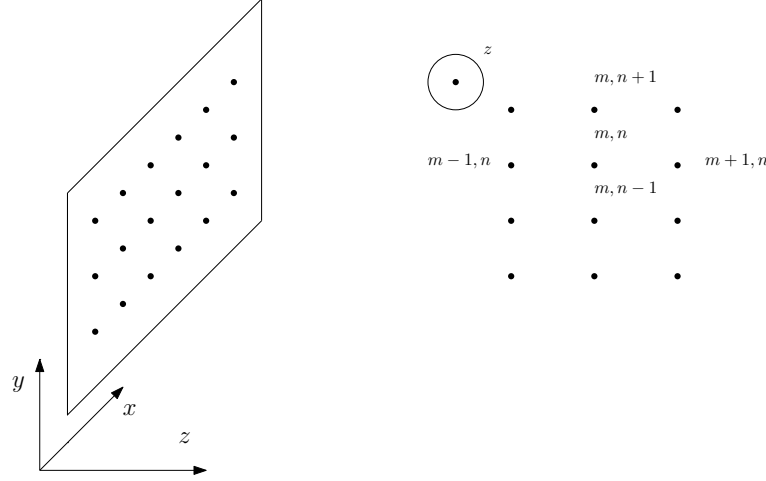
So the eigenvector corresponds to $\lambda_2 = 2k$ is

$$\begin{bmatrix} -\frac{\kappa(e^{-iq} + 1)}{-k + \delta} \\ 1 \end{bmatrix}, \cos(q) \neq -1 \text{ or } k \neq \sqrt{\delta^2 + 2\kappa^2(1 + \cos(q))}$$

Notice that last eigenvector doesn't correspond to a solution since we are looking to solve the homogeneous linear system as we see from the (2.7).

2.3 2d Mono-atomic Waveguide Lattice

At this time we have placed the waveguides at the plane so we have a 2d waveguide lattice as is depicted in the following illustration. Furthermore we have one type of waveguide so we have a mono-atomic lattice.



The lattice is consisted of repeated squares over the whole xy plane. Each waveguide has an enumeration two indices one for the vertical and one for the horizontal axis. The evolution of the field can be described by the following coupled mode equation :

$$i \frac{du_{m,n}}{dz} + \kappa (u_{m,n+1} + u_{m-1,n} + u_{m+1,n} + u_{m,n-1}) + \gamma |u_{m,n}|^2 u_{m,n} = 0 \quad (2.9)$$

If we assume that a solution of eq.(2.9) can be given by

$$u_{m,n} = A e^{i(q_x m + q_y n - k(\vec{q})z)} \quad (2.10)$$

where $\vec{q} = (q_x, q_y)$. Substituting eq.(2.10) into eq.(2.9) we get the following

$$\begin{aligned} -i^2 k(\vec{q}) u_{m,n} + \kappa u_{m,n} (e^{iq_x} + e^{-iq_x} + e^{iq_y} + e^{-iq_y}) + \gamma A^2 u_{m,n} &= 0 \\ k(\vec{q}) + \kappa (2 \cos(q_x) + 2 \cos(q_y)) + \gamma A^2 &= 0 \\ k(\vec{q}) &= -2\kappa (\cos(q_x) + \cos(q_y)) - \gamma A^2 \end{aligned} \quad (2.11)$$

As we can see the dispersion relation (2.11) is not flat, i.e it depends on the vector \vec{q} and is not constant. The group velocity is given by

$$v_g(\vec{q}) = 2\kappa (\sin(q_x) + \sin(q_y))$$

the second order diffraction is the following

$$k''(\vec{q}) = 2\kappa (\cos(q_x) + \cos(q_y))$$

2.3.1 Numerical Results

At the Fig. 2.21-2.23 are illustrated the dispersion relation, group velocity and the second derivative of dispersion relation. At the Fig. 2.24,2.25 we see the excitations for initial condition $u_{10,10}(z=0) = 1$, $u_{10,11}(z=0) = -1$, $u_{11,10}(z=0) = -1$, $u_{11,11}(z=0) = 1$ and $u_{4,5}(z=0) = 1$, $u_{5,4}(z=0) =$

$-1, u_{6,5}(z = 0) = 1, u_{5,6}(z = 0) = -1$ respectively. As we observe there is much diffraction in both cases. For every next numerical result, or "excitation" as we called it, we have the numbers M, N . The number M shows us how many times the unit cell, over the the direction which shows the first primitive cell, is repeated. Similarly the number N shows us how many times the unit cell, over the the direction which shows the second primitive cell, is repeated. The primitive vectors here in this simple lattice is $\vec{a}_1 = (1, 0)$ and $\vec{a}_2 = (0, 1)$ these forms the unit cell. The unit cell here is a simple square which contains only one waveguide.

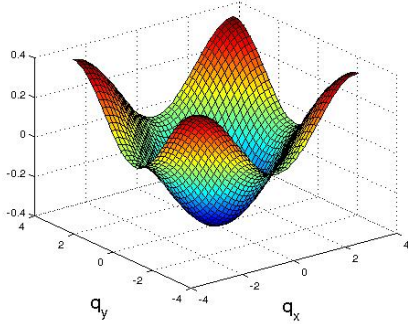


Figure 2.21: Dispersion relation ($k(\vec{q})$) for $\kappa = 0.1, \gamma = 0$

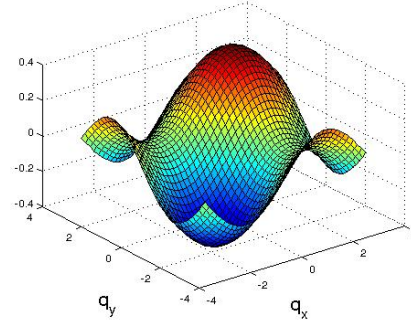


Figure 2.22: Group velocity for $\kappa = 0.1, \gamma = 0$

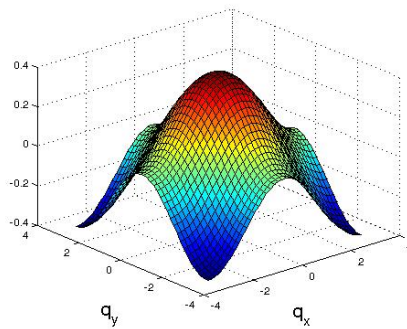


Figure 2.23: Second derivative of dispersion relation for $\kappa = 0.1, \gamma = 0$

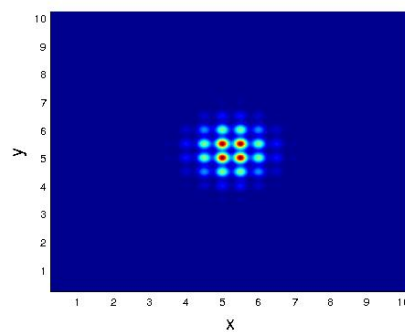
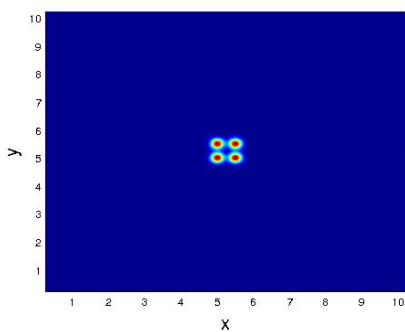


Figure 2.24: Excitation for $\kappa = 0.1, \gamma = 0, h = 10^{-2}, M = N = 20$, at the left initial condition for $z = 0$ and at the right $z = 5$

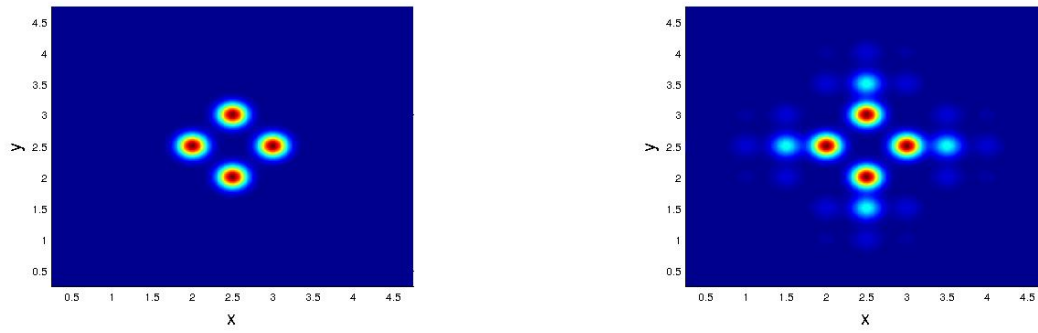


Figure 2.25: Excitation for $\kappa = 0.1, \gamma = 0, h = 10^{-2}, M = N = 9$, at the left initial condition for $z = 0$ and at the right $z = 4$

Chapter 3

Kagome Lattice

At this chapter we will consider the so-called Kagome lattice which is a 2d triatomic lattice possesses a flat band. Firstly we will see the first neighbour interactions between the waveguides where the flat band exists and after that we consider second neighbour interactions where the flat band is vanishing. Finally we introduce a small random displacement at the position of the waveguides to examine numerically the effect to the solutions corresponds to the flat band.

3.1 Coupled Mode Equations and Dispersion Relation

At the Fig. 3.1 we see the Kagome lattice and its unit cell repeats 3 times (black parallelograms), the hexagons which are formed is canonical hexagons.

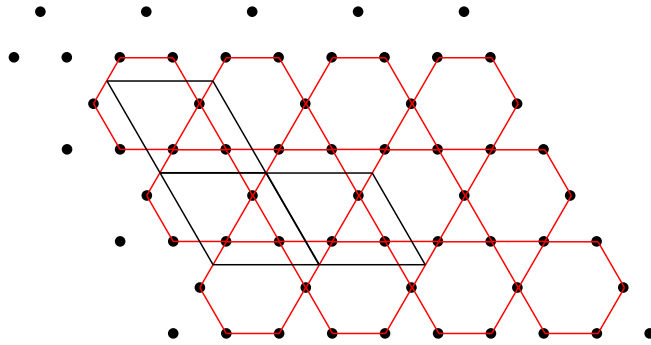


Figure 3.1: Kagome lattice and its unit cell

At the figure 3.2 below we see the geometry of the Kagome lattice from different point of views and in addition at the right part we see the enumeration of the waveguides. We have three waveguides (or atoms) in the unit cell so its a triatomic lattice.

As we can see the lattice is a 2d since the waveguides are extended in the xy plane and the field that we will describe below propagate along the z axis. The enumeration of the waveguides is based on the two primitive vectors considered here, \vec{a}_1 and \vec{a}_2 , and are displayed at the right part of the figure with blue color. Every waveguide of the lattice has its own position vector and it is different for the different types of atoms so the position vector of an arbitrary waveguide of type u say the waveguide $u_{m,n}$ is $\vec{R}_{m,n}$, for $v_{m,n}$ is the $\vec{R}_{m,n} + \vec{R}_v$ and similarly for $w_{m,n}$ is the $\vec{R}_{m,n} + \vec{R}_w$, where $\vec{R}_{m,n} = m\vec{a}_1 + n\vec{a}_2$. It is clear that the position vector describes the location of the respective waveguide if we start from the origin so for example if we set at the origin the waveguide $u_{0,0}$ then the position vector of the $v_{0,0}$ is $\vec{R}_{0,0} + \vec{R}_v = \vec{R}_v$. Notice that because of the symmetry of the lattice the triangle consisted of the vertices $u_{m,n}, v_{m,n}, w_{m,n}$ is equilateral thus the vectors \vec{R}_v and \vec{R}_w have equal magnitude and in addition $\vec{R}_v = \vec{a}_1/2$, $\vec{R}_w = (\vec{a}_1 + \vec{a}_2)/2$. At the

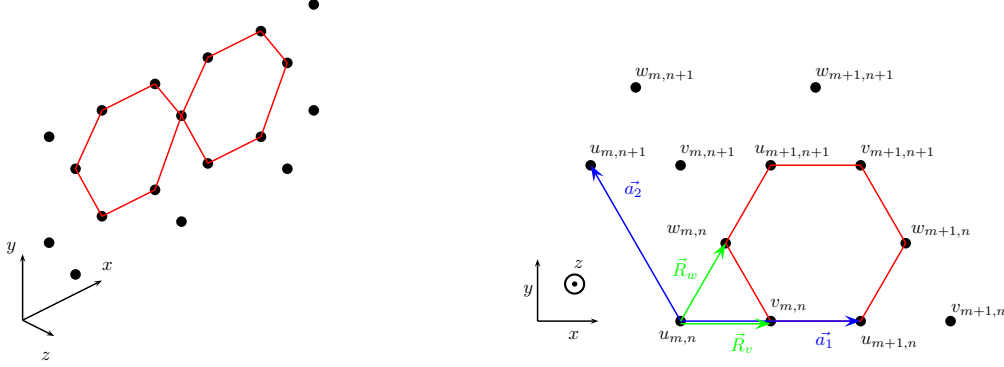


Figure 3.2: Kagome lattice

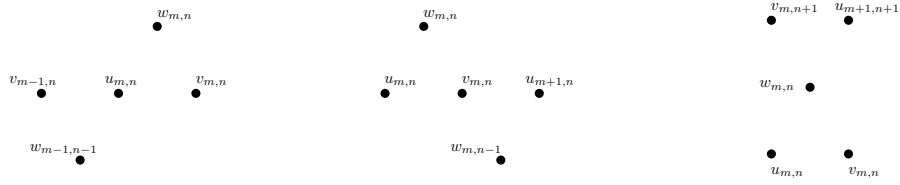


Figure 3.3: The first neighbours of each of the atoms

Fig. 3.3 are displayed the first neighbours of each of the atoms $u_{m,n}, v_{m,n}, w_{m,n}$, indicatively $u_{m,n}$ has first neighbours $w_{m,n}, v_{m,n}, v_{m-1,n}, w_{m-1,n-1}$ and all the distances from the central waveguide $u_{m,n}$ to these waveguides are equal.

Below we see the eq. (3.1)-(3.3) which have resulted from the coupled mode equations considering only the coupling term κ between the first neighbour waveguides. As we can see all the neighbours have equal coupling term because of the symmetry of the lattice, i.e all the distances from any waveguide of any atom to any first neighbour waveguide are equal.

$$i \frac{du_{m,n}}{dz} + \kappa (v_{m,n} + v_{m-1,n} + w_{m,n} + w_{m-1,n-1}) = 0 \quad (3.1)$$

$$i \frac{dv_{m,n}}{dz} + \kappa (u_{m,n} + u_{m+1,n} + w_{m,n} + w_{m,n-1}) = 0 \quad (3.2)$$

$$i \frac{dw_{m,n}}{dz} + \kappa (u_{m,n} + u_{m+1,n+1} + v_{m,n} + v_{m,n+1}) = 0 \quad (3.3)$$

Suppose we have the solution of the form :

$$u_{m,n} = ue^{-ikz+i\vec{R}_{m,n}\cdot\vec{q}} \quad (3.4)$$

$$v_{m,n} = ve^{-ikz+i(\vec{R}_{m,n}+\vec{R}_v)\cdot\vec{q}} \quad (3.5)$$

$$w_{m,n} = we^{-ikz+i(\vec{R}_{m,n}+\vec{R}_w)\cdot\vec{q}} \quad (3.6)$$

Substituting eq. (3.4),(3.5),(3.6) into eq. (3.1) we get the following :

$$ku_{m,n} + \kappa \left(v_{m,n} + ve^{-ikz+i(\vec{R}_{m,n}+\vec{R}_v)\cdot\vec{q}} + w_{m,n} + we^{-ikz+i(\vec{R}_{m-1,n-1}+\vec{R}_w)\cdot\vec{q}} \right) = 0$$

multiplying both sides by $e^{ikz}e^{-i\vec{R}_{m,n}\cdot\vec{q}}$

$$ku + \kappa \left(ve^{i\vec{R}_v\cdot\vec{q}} + ve^{i(-\vec{a}_1+\vec{R}_v)\cdot\vec{q}} + we^{i\vec{R}_w\cdot\vec{q}} + we^{i(-\vec{a}_1-\vec{a}_2+\vec{R}_w)\cdot\vec{q}} \right) = 0$$

or equivalently

$$\begin{aligned} ku + \kappa \left(ve^{i\vec{R}_v \cdot \vec{q}} + ve^{-i\vec{R}_v \cdot \vec{q}} + we^{i\vec{R}_w \cdot \vec{q}} + we^{-i\vec{R}_w \cdot \vec{q}} \right) &= 0 \\ ku + \kappa \left(2v \cos(\vec{R}_v \cdot \vec{q}) + 2w \cos(\vec{R}_w \cdot \vec{q}) \right) &= 0 \end{aligned} \quad (3.7)$$

Similarly from eq. (3.2) we get :

$$kv_{m,n} + \kappa \left(u_{m,n} + ue^{-ikz+i\vec{R}_{m+1,n} \cdot \vec{q}} + w_{m,n} + we^{-ikz+i\vec{R}_{m,n-1} \cdot \vec{q}} \right) = 0$$

multiplying by $e^{ikz+i(-\vec{R}_{m,n}-\vec{R}_v) \cdot \vec{q}}$

$$\begin{aligned} kv + \kappa \left(ue^{-i\vec{R}_v \cdot \vec{q}} + ue^{i\vec{R}_v \cdot \vec{q}} + we^{i\frac{a_2}{2} \cdot \vec{q}} + we^{-i\frac{a_2}{2} \cdot \vec{q}} \right) &= 0 \\ kv + \kappa \left(2u \cos(\vec{R}_v \cdot \vec{q}) + 2w \cos\left(\frac{a_2}{2} \cdot \vec{q}\right) \right) &= 0 \end{aligned} \quad (3.8)$$

And finally, from eq. (3.3), after multiplying with $e^{i(-\vec{R}_{m,n}-\vec{R}_w) \cdot \vec{q}}$, we obtain the following

$$\begin{aligned} kw + \kappa \left(ue^{-i\vec{R}_w \cdot \vec{q}} + ue^{i\vec{R}_w \cdot \vec{q}} + ve^{-i\frac{a_2}{2} \cdot \vec{q}} + ve^{i\frac{a_2}{2} \cdot \vec{q}} \right) &= 0 \\ kw + \kappa \left(2u \cos(\vec{R}_w \cdot \vec{q}) + 2v \cos\left(\frac{a_2}{2} \cdot \vec{q}\right) \right) &= 0 \end{aligned} \quad (3.9)$$

We can write as a linear system eq.(3.7),(3.8),(3.9)

$$2\kappa \begin{bmatrix} 0 & \cos(\vec{R}_v \cdot \vec{q}) & \cos(\vec{R}_w \cdot \vec{q}) \\ \cos(\vec{R}_v \cdot \vec{q}) & 0 & \cos\left(\frac{a_2}{2} \cdot \vec{q}\right) \\ \cos(\vec{R}_w \cdot \vec{q}) & \cos\left(\frac{a_2}{2} \cdot \vec{q}\right) & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = -k \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Substituting for convenience $\cos\left(\frac{a_2}{2} \cdot \vec{q}\right) = c_1$, $\cos(\vec{R}_v \cdot \vec{q}) = c_2$, $\cos(\vec{R}_w \cdot \vec{q}) = c_3$ and looking for eigenvectors :

$$\begin{aligned} & \begin{vmatrix} -\lambda & c_2 & c_3 \\ c_2 & -\lambda & c_1 \\ c_3 & c_1 & -\lambda \end{vmatrix} = 0 \\ \Leftrightarrow & -\lambda \begin{vmatrix} -\lambda & c_1 \\ c_1 & -\lambda \end{vmatrix} - c_2 \begin{vmatrix} c_2 & c_1 \\ c_3 & -\lambda \end{vmatrix} + c_3 \begin{vmatrix} c_2 & -\lambda \\ c_3 & c_1 \end{vmatrix} = 0 \\ \Leftrightarrow & -\lambda \left(\lambda^2 - c_1^2 \right) - c_2 \left(-\lambda c_2 - c_1 c_3 \right) + c_3 \left(c_1 c_2 + \lambda c_3 \right) = 0 \\ \Leftrightarrow & -\lambda^3 + \lambda \left(c_1^2 + c_2^2 + c_3^2 \right) + 2c_1 c_2 c_3 = 0 \Leftrightarrow \end{aligned}$$

using that $\cos^2(a) + \cos^2(b) + \cos^2(a+b) = 2 \cos(a) \cos(b) \cos(a+b) + 1$

$$\begin{aligned} \Leftrightarrow & -\lambda^3 + \lambda (2c_1 c_2 c_3 + 1) + 2c_1 c_2 c_3 = 0 \\ \Leftrightarrow & (\lambda + 1) (2c_1 c_2 c_3 + \lambda (1 - \lambda)) = 0 \\ \Leftrightarrow & (\lambda + 1) \left(-\lambda^2 + \lambda + 2c_1 c_2 c_3 \right) = 0 \end{aligned}$$

So we have 3 eigenvalues ($1 + 8c_1 c_2 c_3$ will certainly be non-negative, because the matrix is symmetric therefore it has real eigenvalues)

$$\lambda_1 = -1, \lambda_{2,3} = \frac{-1 \pm \sqrt{1 + 8c_1 c_2 c_3}}{-2} = \frac{1}{2} \pm \frac{\sqrt{1 + 8c_1 c_2 c_3}}{2}$$

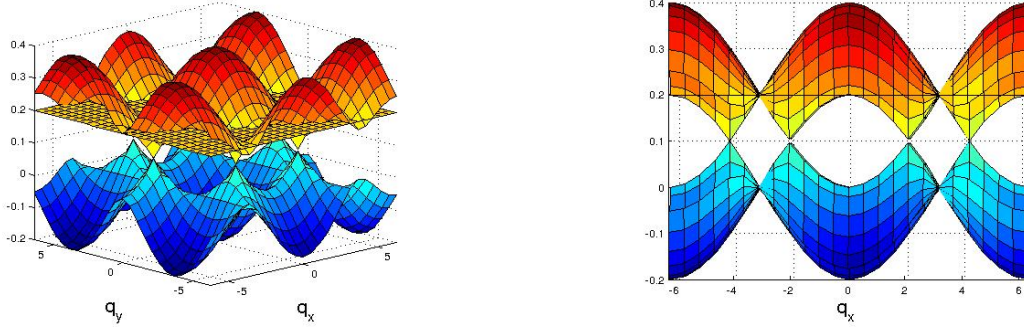


Figure 3.4: Bands of the Kagome lattice for $q_x, q_y \in [-2\pi, 2\pi]$, $\vec{a}_1 = (1, 0)$, $\vec{a}_2 = (-1/2, \sqrt{3}/2)$ and $\kappa = 0.1$, there are two different points of view

Thus there are two dispersive bands and one flat band

$$k_{1,2,3}(\vec{q}) = 2\kappa, \kappa(1 \pm \sqrt{1 + 8c_1c_2c_3})$$

Fig. 3.4 illustrates a 3D plot of the three bands $k_{1,2,3}(\vec{q}) = k_{1,2,3}(q_x, q_y)$. Finding the eigenvectors :

$$\begin{aligned} & \begin{pmatrix} -\lambda & c_2 & c_3 \\ c_2 & -\lambda & c_1 \\ c_3 & c_1 & -\lambda \end{pmatrix} \begin{array}{l} \leftarrow \frac{c_2}{\lambda} \\ + \\ \leftarrow \end{array} \lambda \neq 0 \Rightarrow \begin{pmatrix} -\lambda & c_2 & c_3 \\ 0 & \frac{-\lambda^2 + c_2^2}{\lambda} & \frac{\lambda c_1 + c_2 c_3}{\lambda} \\ c_3 & c_1 & -\lambda \end{pmatrix} \begin{array}{l} \leftarrow \frac{c_3}{\lambda} \\ + \\ \leftarrow \end{array} \\ \Rightarrow & \begin{pmatrix} -\lambda & c_2 & c_3 \\ 0 & \frac{-\lambda^2 + c_2^2}{\lambda} & \frac{\lambda c_1 + c_2 c_3}{\lambda} \\ 0 & \frac{\lambda c_1 + c_2 c_3}{\lambda} & \frac{-\lambda^2 + c_3^2}{\lambda} \end{pmatrix} \begin{array}{l} \leftarrow \frac{\lambda c_1 + c_2 c_3}{-\lambda^2 + c_2^2} \\ + \\ \leftarrow \end{array}, \lambda \neq \pm c_2 \\ \Rightarrow & \begin{pmatrix} -\lambda & c_2 & c_3 \\ 0 & \frac{-\lambda^2 + c_2^2}{\lambda} & \frac{\lambda c_1 + c_2 c_3}{\lambda} \\ 0 & 0 & \frac{(1+\lambda)[-2c_1c_2c_3 + \lambda^2 - \lambda]}{\lambda^2 - c_2^2} = 0 \end{pmatrix} \end{aligned}$$

Solving the linear system :

$$\frac{-\lambda^2 + c_2^2}{\lambda} v + \frac{\lambda c_1 + c_2 c_3}{\lambda} w = 0 \Leftrightarrow v = \frac{-\lambda c_1 - c_2 c_3}{-\lambda^2 + c_2^2} w$$

and

$$-\lambda u = \left(-c_2 \frac{-\lambda c_1 - c_2 c_3}{-\lambda^2 + c_2^2} - c_3 \right) w = \left(\frac{\lambda c_1 c_2 + c_3 \lambda^2}{-\lambda^2 + c_2^2} \right) w \Leftrightarrow u = \frac{-c_1 c_2 - c_3 \lambda}{-\lambda^2 + c_2^2} w$$

So, eigenvectors have the following form :

$$\begin{bmatrix} \frac{-c_1 c_2 - c_3 \lambda}{-\lambda^2 + c_2^2} \\ \frac{-\lambda c_1 - c_2 c_3}{-\lambda^2 + c_2^2} \\ 1 \end{bmatrix}, \lambda \neq 0, \pm c_2$$

where λ an eigenvalue .

Substituting $\lambda = -1$, which is the eigenvalue corresponding to the flat band, into the above eigenvector

gives us

$$\begin{aligned}
\begin{bmatrix} -c_1 c_2 + c_3 \\ c_2^2 - 1 \\ c_1 - c_2 c_3 \\ c_2^2 - 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} \frac{-\cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) \cos\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right) + \cos\left(\frac{\vec{a}_1 + \vec{a}_2}{2} \cdot \vec{q}\right)}{c_2^2 - 1} \\ \frac{\cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) - \cos\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right) \cos\left(\left(\frac{\vec{a}_2}{2} + \frac{\vec{a}_1}{2}\right) \cdot \vec{q}\right)}{c_2^2 - 1} \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-\cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) \cos\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right) + \cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) \cos\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right) - \sin\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) \sin\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right)}{c_2^2 - 1} \\ \frac{\cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) - \cos\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right) \left(\cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) \cos\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right) - \sin\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) \sin\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right)\right)}{c_2^2 - 1} \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-\sin\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) \sin\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right)}{-\sin^2\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right)} \\ \frac{\cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) - \cos^2\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right) \cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) + \cos\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right) \sin\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) \sin\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right)}{\cos^2\left(\frac{\vec{a}_1}{2}\right) - 1} \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sin\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right)}{\sin\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right)} \\ -\cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) + \frac{\cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) \sin\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) \sin\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right)}{-\sin^2\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right)} \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sin\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right)}{\sin\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right)} \\ -\cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) - \frac{\cos\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right) \sin\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right)}{\sin\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right)} \\ 1 \end{bmatrix}
\end{aligned}$$

Multiplying the last vector by $\sin\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right)$, the result remains an eigenvector so we have

$$\begin{bmatrix} \sin\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) \\ -\cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) \sin\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right) - \cos\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right) \sin\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) \\ \sin\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right) \end{bmatrix} = \begin{bmatrix} \sin\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) \\ \sin\left(\frac{\vec{a}_3}{2} \cdot \vec{q}\right) \\ \sin\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right) \end{bmatrix}$$

where $\vec{a}_3 = -\vec{a}_1 - \vec{a}_2$. A solution corresponds to the flat band is ($\vec{a}_3 = -\vec{a}_2 - \vec{a}_1$):

$$\begin{bmatrix} u_{m,n} \\ v_{m,n} \\ w_{m,n} \end{bmatrix} = \begin{bmatrix} \sin\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) e^{i(\vec{R}_{m,n} \cdot \vec{q} - 2\kappa z)} \\ \sin\left(\frac{\vec{a}_3}{2} \cdot \vec{q}\right) e^{i((\vec{R}_{m,n} + \vec{R}_v) \cdot \vec{q} - 2\kappa z)} \\ \sin\left(\frac{\vec{a}_1}{2} \cdot \vec{q}\right) e^{i((\vec{R}_{m,n} + \vec{R}_w) \cdot \vec{q} - 2\kappa z)} \end{bmatrix} = \vec{y}_{m,n}(\vec{q}) e^{-2i\kappa z} \quad (3.10)$$

The following superposition is also a solution corresponds to the flat band

$$\int \int A(\vec{q}) \vec{y}_{m,n}(\vec{q}) e^{-2i\kappa z} dq_x dq_y = e^{-2i\kappa z} \int \int A(\vec{q}) \vec{y}_{m,n}(\vec{q}) dq_x dq_y \quad (3.11)$$

It is easy to understand that the absolute value of the field of each of the waveguides remains unchanged along the direction z .

3.1.1 Numerical Results

At this section we will see some numerical results for the Kagome lattice for various initial conditions and as a result of the existence of a flat band we will see excitations that remain unchanged along the propagation direction. At the first excitation (Fig. 3.5) we see the initial condition : $v_{2,2}(z = 0) = -1, u_{3,2}(z = 0) = 1, w_{3,2}(z = 0) = -1, v_{3,3}(z = 0) = 1, u_{3,3}(z = 0) = -1, w_{2,2}(z = 0) = 1$ and the rest of the waveguides equal to zero. The second excitation (Fig. 3.6) is similarly with the first in the sense that we excite waveguides having in mind that the neighbouring waveguides must have opposite sign and equal absolute value, for $z = 0$. The third excitation (Fig. 3.7) is an excitation with diffraction, the initial conditions here are : $u_{2,2}(z = 0) = 1, v_{2,2}(z = 0) = -1, w_{2,2}(z = 0) = 1$ and the other waveguides equal to zero. The solutions corresponding to the first two cases can be written in the form (3.11), it is easy to see why the absolute value of the amplitude of the waveguides remain unchanged along the propagation direction. The solution corresponding to the third excitation can also be written in a superposition form but not only of the flat band. At the fourth excitation (Fig. 3.8) we see the excitation of the solution (3.1) where we have applied periodic boundary condition. This solution corresponds to an excitation of infinite system of waveguides but we can see it using periodic boundary conditions.

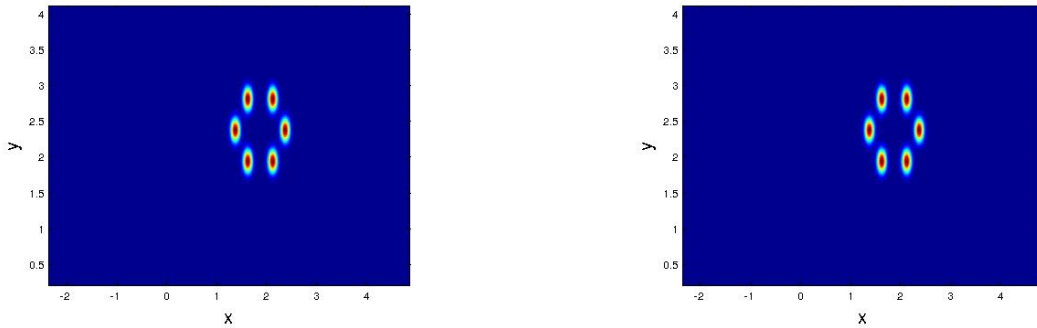


Figure 3.5: Excitation for $\kappa = 0.1, h = 10^{-2}, M = 5, N = 5$, at the left initial condition for $z = 0$ and at the right $z = 10$

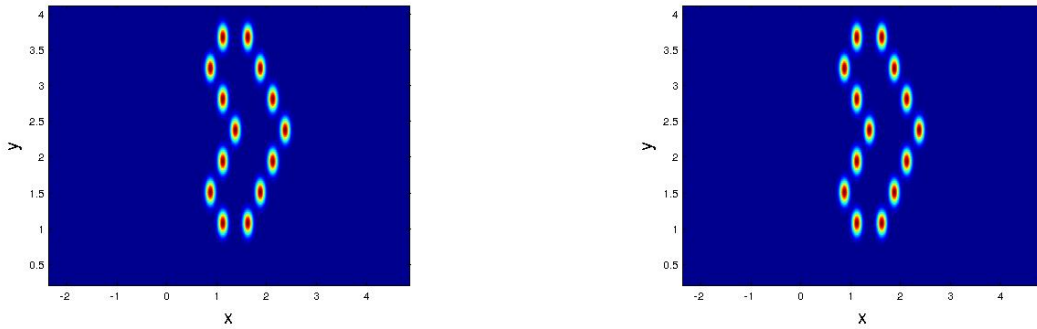


Figure 3.6: Excitation for $\kappa = 0.1, h = 10^{-2}, M = 5, N = 5$, at the left initial condition for $z = 0$ and at the right $z = 10$

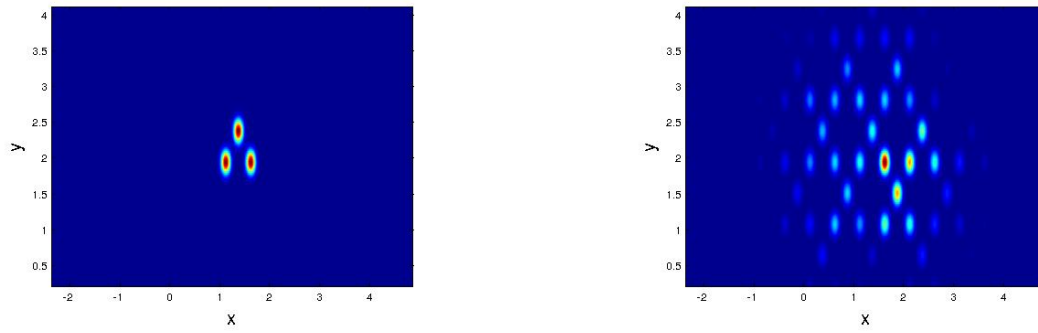


Figure 3.7: Excitation for $\kappa = 0.1$, $h = 10^{-2}$, $M = 5$, $N = 5$, at the left initial condition for $z = 0$ and at the right $z = 10$

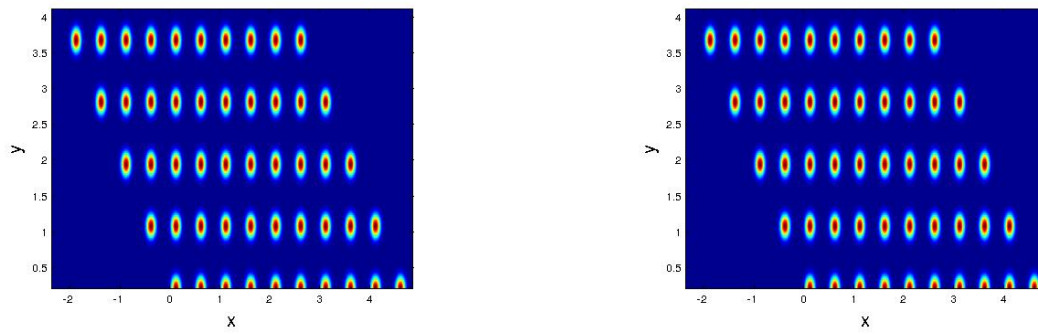


Figure 3.8: Excitation for $\kappa = 0.1$, $h = 10^{-2}$, $M = 5$, $N = 5$, at the left initial condition for $z = 0$ and at the right $z = 10$

3.2 Next Nearest Neighbor

At the below figure is depicted the nearest and next nearest neighbours for each type of atom say u, v, w . From left to right we have the neighbours of $u_{m,n}, v_{m,n}, w_{m,n}$ respectively, at the center of each image we have the aforementioned waveguides and we can see also the red lines which connect the central waveguide with the next nearest neighbours. There is no line connects the central waveguide of each image with the nearest neighbors.

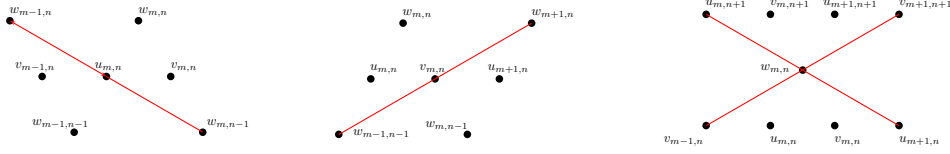


Figure 3.9: Nearest and next nearest neighbours

The equations, with the consideration of the next nearest neighbours, become now

$$i \frac{du_{m,n}}{dz} + \kappa (v_{m,n} + v_{m-1,n} + w_{m,n} + w_{m-1,n-1}) + \kappa_1 (w_{m,n-1} + w_{m-1,n}) = 0 \quad (3.12)$$

$$i \frac{dv_{m,n}}{dz} + \kappa (u_{m,n} + u_{m+1,n} + w_{m,n} + w_{m,n-1}) + \kappa_1 (w_{m+1,n} + w_{m-1,n-1}) = 0 \quad (3.13)$$

$$i \frac{dw_{m,n}}{dz} + \kappa (u_{m,n} + u_{m+1,n+1} + v_{m,n} + v_{m,n+1}) + \kappa_1 (u_{m,n+1} + u_{m+1,n} + v_{m-1,n} + v_{m+1,n+1}) = 0 \quad (3.14)$$

Suppose we have the solution of the form :

$$u_{m,n} = u e^{-ikz + i\vec{R}_{m,n} \cdot \vec{q}} \quad (3.15)$$

$$v_{m,n} = v e^{-ikz + i(\vec{R}_{m,n} + \vec{R}_v) \cdot \vec{q}} \quad (3.16)$$

$$w_{m,n} = w e^{-ikz + i(\vec{R}_{m,n} + \vec{R}_w) \cdot \vec{q}} \quad (3.17)$$

Replacing the last three equations into equations (3.12),(3.13),(3.14) respectively :

$$ku + \kappa w (e^{i\vec{R}_w \cdot \vec{q}} + e^{i(-\vec{a}_1 - \vec{a}_2 + \vec{R}_w) \cdot \vec{q}}) + \kappa v (e^{i(-\vec{a}_1 + \vec{R}_v) \cdot \vec{q}} + e^{i\vec{R}_v \cdot \vec{q}}) + \kappa_1 w (e^{i(-\vec{a}_2 + \vec{R}_w) \cdot \vec{q}} + e^{i(-\vec{a}_1 + \vec{R}_w) \cdot \vec{q}}) = 0$$

$$kv + \kappa w (e^{i(\vec{R}_w - \vec{R}_v) \cdot \vec{q}} + e^{i(-\vec{a}_2 + \vec{R}_w - \vec{R}_v) \cdot \vec{q}}) + \kappa u (e^{i(\vec{a}_1 - \vec{R}_v) \cdot \vec{q}} + e^{i(-\vec{R}_v) \cdot \vec{q}}) + \kappa_1 w (e^{i(\vec{a}_1 + \vec{R}_w - \vec{R}_v) \cdot \vec{q}} + e^{i(-\vec{a}_1 - \vec{a}_2 + \vec{R}_w - \vec{R}_v) \cdot \vec{q}}) = 0$$

$$kw + \kappa u (e^{i(-\vec{R}_w) \cdot \vec{q}} + e^{i(\vec{a}_1 + \vec{a}_2 - \vec{R}_w) \cdot \vec{q}}) + \kappa v (e^{i(\vec{R}_v - \vec{R}_w) \cdot \vec{q}} + e^{i(\vec{a}_2 + \vec{R}_v - \vec{R}_w) \cdot \vec{q}}) + \kappa_1 u (e^{i(\vec{a}_2 - \vec{R}_w) \cdot \vec{q}} e^{i(\vec{a}_1 - \vec{R}_w) \cdot \vec{q}}) + \kappa_1 v (e^{i(-\vec{a}_1 + \vec{R}_v - \vec{R}_w) \cdot \vec{q}} + e^{i(\vec{a}_1 + \vec{a}_2 + \vec{R}_v - \vec{R}_w) \cdot \vec{q}}) = 0$$

Recall that $\vec{R}_v = \vec{a}_1/2$, $\vec{R}_w = (\vec{a}_1 + \vec{a}_2)/2$ we can write the last three equations in a more simple form :

$$ku + 2\kappa \cos(\vec{R}_w \cdot \vec{q}) w + 2\kappa \cos(\vec{R}_v \cdot \vec{q}) v + 2\kappa_1 \cos\left(\left(\frac{\vec{a}_1 - \vec{a}_2}{2}\right) \cdot \vec{q}\right) w = 0 \quad (3.18)$$

$$\begin{aligned}
& kv + 2\kappa \cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) w + 2\kappa \cos\left(\vec{R}_v \cdot \vec{q}\right) u \\
& + 2\kappa_1 \cos\left(\left(\vec{a}_1 + \frac{\vec{a}_2}{2}\right) \cdot \vec{q}\right) w = 0
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
& kw + 2\kappa \cos\left(\vec{R}_w \cdot \vec{q}\right) u + 2\kappa \cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) v + 2\kappa_1 \cos\left(\frac{\vec{a}_1 - \vec{a}_2}{2} \cdot \vec{q}\right) u \\
& + 2\kappa_1 \cos\left(\left(\vec{a}_1 + \frac{\vec{a}_2}{2}\right) \cdot \vec{q}\right) v = 0
\end{aligned} \tag{3.20}$$

If we write again the linear system resulted from (3.18),(3.19),(3.20)

$$\begin{bmatrix} 0 & c_2 & c_3 \\ c_2 & 0 & c_1 \\ c_3 & c_1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = -k \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

where $c_2 = 2\kappa \cos\left(\vec{R}_v \cdot \vec{q}\right)$, $c_3 = 2\kappa \cos\left(\vec{R}_w \cdot \vec{q}\right) + 2\kappa_1 \cos\left(\left(\frac{\vec{a}_1 - \vec{a}_2}{2}\right) \cdot \vec{q}\right)$, and $c_1 = 2\kappa \cos\left(\frac{\vec{a}_2}{2} \cdot \vec{q}\right) + 2\kappa_1 \cos\left(\left(\vec{a}_1 + \frac{\vec{a}_2}{2}\right) \cdot \vec{q}\right)$.

It is not an easy task to find the analytical form of the eigenvalues, so we found it numerically. As we can

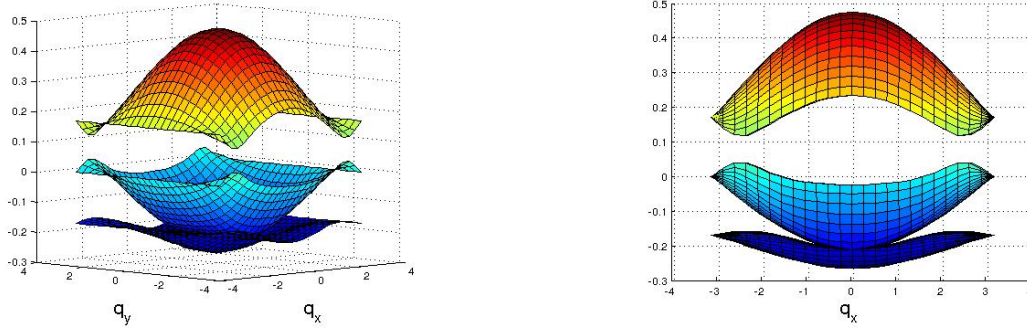


Figure 3.10: Eigenvalues for $q_x, q_y \in [-\pi, \pi]$ for $\kappa = 0.1$, $\kappa_1 = 0.02$ and $\vec{a}_1 = (1, 0)$, $\vec{a}_2 = (-1/2, \sqrt{3}/2)$, two different points of view.

see in Fig. 3.10 there is much variation of the eigenvalues, so there is no flat band.

3.2.1 Numerical Results

Now we will show some excitations. The first excitation, at the Fig. 3.11, is for initial conditions for the waveguides $v_{2,2}, u_{3,2}, w_{3,2}, v_{3,3}, u_{3,3}, w_{2,2}$ equal to $-1, 1, -1, 1, -1, 1$ respectively and the other waveguides have 0, this excitation without the consideration of the next nearest neighbours is a zero diffraction excitation but as we can see here, there is a diffraction due to the fact that the Kagome is not a flat band lattice if we consider next nearest neighbours. At the second excitation (Fig. 3.12) we have as initial condition $v_{1,1}(z=0) = -1, u_{2,1}(z=0) = 1, w_{2,1}(z=0) = -1, u_{3,2}(z=0) = 1, w_{3,2}(z=0) = -1, v_{3,3}(z=0) = 1, w_{3,3}(z=0) = -1, v_{3,4}(z=0) = 1, u_{3,4}(z=0) = -1, w_{2,3}(z=0) = 1, v_{2,3}(z=0) = -1, w_{2,2}(z=0) = 1, u_{2,2}(z=0) = -1, w_{1,1}(z=0) = 1$.

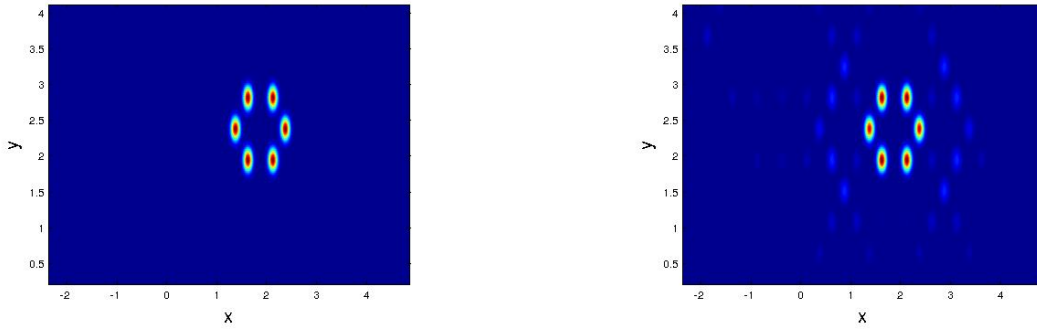


Figure 3.11: Excitation for $\kappa = 0.1$, $\kappa_1 = 0.02$, $z = 0$, $h = 10^{-2}$, $M = 5$, $N = 5$, at the left initial condition for $z = 0$ and at the right $z = 10$

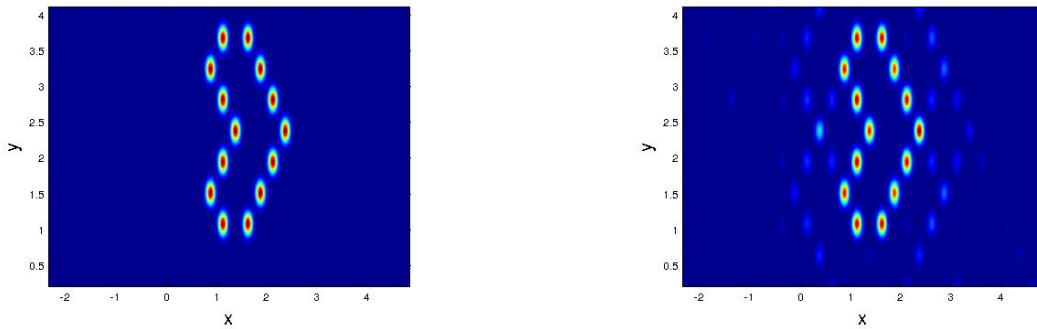


Figure 3.12: Excitation for $\kappa = 0.1$, $\kappa_1 = 0.02$, $z = 0$, $h = 10^{-2}$, $M = 5$, $N = 5$, at the left initial condition for $z = 0$ and at the right $z = 10$

3.3 Disorder

At this section we consider the usual Kagome lattice but we introduce a small random displacement around the standard position of the waveguides at the Kagome lattice. Each waveguide moves within a square of side ϵ around its fixed position, as the Fig. 3.13 illustrates, and the probability of the waveguide to be found at a certain point in the square is distributed uniformly. The random displacement of each of the

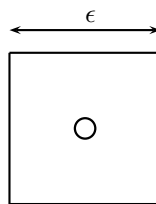


Figure 3.13: A waveguide and the area in which it can be displaced randomly

waveguides holds for every z , hence we have a permanent displacement. We show some excitations at the below figures where for each point on the xy plane we see a color, the more the color is close to blue the smaller the absolute value at that point. The coupling term is not constant here and the reason for that is that the distance between the waveguides is not constant, therefore we choose the coupling term to decrease exponentially with the distance. Notice that only the **nearest neighbours** are considered here. At the first couple of figures (Fig. 3.14) we see the excitation corresponds to the flat band, we excite the waveguides $v_{0,0}, u_{1,0}, w_{1,0}, v_{1,1}, u_{1,1}, w_{0,0}$ with initial conditions $-1, +1, -1, +1, -1, +1$ (left) respectively as we see due to the small perturbation of the positions of the waveguides there is a difference between the coupling

terms of the neighbours for each waveguide thus there is a small diffraction (right).

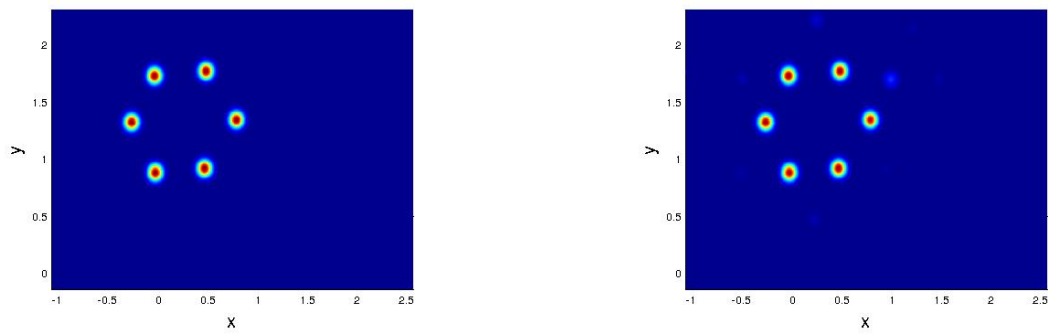


Figure 3.14: Excitation for $\epsilon = 0.1$, $h = 10^{-2}$, $M = 2$, $N = 2$, at the left initial condition for $z = 0$ and at the right $z = 10$

Chapter 4

Lieb Lattice

At this chapter we examine the Lieb lattice, it is a lattice which is consisted of three simple lattices as we will see. It possesses one flat band at the first neighbour interaction consideration, which is vanishing again with the second neighbours consideration. Finally we introduce a small random displacement of the position of the waveguides to examine the effect to the flat band excitations.

4.1 Coupled Mode Equations and Dispersion Relation

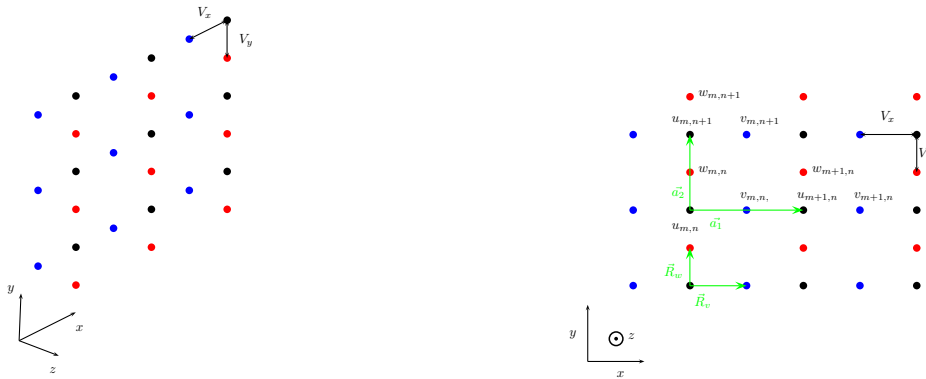


Figure 4.1: Lieb lattice

At the Fig. 4.1 we see at the left the Lieb lattice in the \mathbb{R}^3 , the dots represent the waveguides which are on the xy plane and the direction of propagation is the z axis. The lattice is triatomic and this is the reason we chose three different colors for the dots which represents the waveguides at the two images (right and left). At the right image there are the lattice enumeration and the primitive vectors, which are used for the enumeration of the lattice, in addition we also see the vectors \vec{R}_v and \vec{R}_w . The enumeration of the waveguides following the same rule as the Kagome lattice i.e the waveguide $u_{m,n}$ has position vector $\vec{R}_{m,n}$ the $v_{m,n}$ has position vector $\vec{R}_{m,n} + \vec{R}_v$ and the $w_{m,n}$ has position vector $\vec{R}_{m,n} + \vec{R}_w$ where $\vec{R}_{m,n} = m\vec{a}_1 + n\vec{a}_2$ and $\vec{R}_v = \vec{a}_1/2$ and $\vec{R}_w = \vec{a}_2/2$. At the Fig. 4.2 we have from left to right the nearest neighbours of each of the atoms so every waveguide of u has 4 neighbours and every waveguide of v and w has 2, this is because we consider nearest neighbouring between waveguides which have distance between them V_x or V_y . Thus we are not considering as nearest neighbours for example $v_{m,n}$ and $w_{m,n}$ which are distant by $\sqrt{V_x^2 + V_y^2}$.

Now we are going to write down the equations which describe the variation of the amplitude of the waveguides in the lattice. Waveguides which are spaced by V_x and V_y are associated with the coupling term κ_x and κ_y respectively.

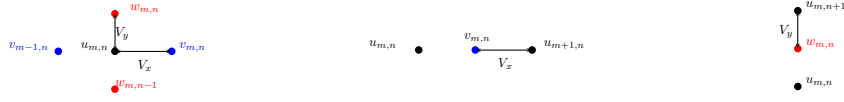


Figure 4.2: Next nearest neighbours of each of the atoms

Below the eq. (4.1)-(4.3) describe the propagation of the field along the propagation direction.

$$i \frac{du_{m,n}}{dz} + \kappa_x (v_{m,n} + v_{m-1,n}) + \kappa_y (w_{m,n} + w_{m,n-1}) = 0 \quad (4.1)$$

$$i \frac{dv_{m,n}}{dz} + \kappa_x (u_{m,n} + u_{m+1,n}) = 0 \quad (4.2)$$

$$i \frac{dw_{m,n}}{dz} + \kappa_y (u_{m,n} + u_{m,n+1}) = 0 \quad (4.3)$$

Suppose we have the solution of the form :

$$u_{m,n} = u e^{-ikz + i\vec{R}_{m,n} \cdot \vec{q}} \quad (4.4)$$

$$v_{m,n} = v e^{-ikz + i(\vec{R}_{m,n} + \vec{R}_v) \cdot \vec{q}} \quad (4.5)$$

$$w_{m,n} = w e^{-ikz + i(\vec{R}_{m,n} + \vec{R}_w) \cdot \vec{q}} \quad (4.6)$$

Substituting eq. (4.4)-(4.6) into the equation (4.1) gives us :

$$ku_{m,n} + \kappa_x (v_{m,n} + v_{m,n} e^{-i\vec{a}_1 \cdot \vec{q}}) + \kappa_y (w_{m,n} + w_{m,n} e^{-i\vec{a}_2 \cdot \vec{q}}) = 0$$

multiplying both sides by $e^{ikz} e^{-i\vec{R}_{m,n} \cdot \vec{q}}$

$$ku + \kappa_x (v e^{i\vec{R}_v \cdot \vec{q}} + v e^{i(-\vec{a}_1 + \vec{R}_v) \cdot \vec{q}}) + \kappa_y (w e^{i\vec{R}_w \cdot \vec{q}} + w e^{i(-\vec{a}_2 + \vec{R}_w) \cdot \vec{q}}) = 0$$

or equivalently

$$ku + \kappa_x (v e^{i\vec{R}_v \cdot \vec{q}} + v e^{-i\vec{R}_v \cdot \vec{q}}) + \kappa_y (w e^{i\vec{R}_w \cdot \vec{q}} + w e^{-i\vec{R}_w \cdot \vec{q}}) = 0$$

$$ku + 2\kappa_x \cos(\vec{R}_v \cdot \vec{q})v + 2\kappa_y \cos(\vec{R}_w \cdot \vec{q})w = 0 \quad (4.7)$$

Similarly from eq. (4.2) we get :

$$kv_{m,n} + \kappa_x (u_{m,n} + u_{m,n} e^{i\vec{a}_1 \cdot \vec{q}}) = 0$$

multiplying by $e^{ikz + i(-\vec{R}_{m,n} - \vec{R}_v) \cdot \vec{q}}$

$$kv + \kappa_x (u e^{-i\vec{R}_v \cdot \vec{q}} + u e^{i\vec{R}_v \cdot \vec{q}}) = 0$$

$$kv + 2\kappa_x \cos(\vec{R}_v \cdot \vec{q})u = 0 \quad (4.8)$$

And finally, from eq. (4.3) after multiplying with $e^{i(-\vec{R}_{m,n} - \vec{R}_w) \cdot \vec{q}}$, we get :

$$kw + \kappa_y (u e^{-i\vec{R}_w \cdot \vec{q}} + u e^{i\vec{R}_w \cdot \vec{q}}) = 0$$

$$kw + 2\kappa_y \cos(\vec{R}_w \cdot \vec{q})u = 0 \quad (4.9)$$

We can write as a linear system eq.(4.7),(4.8),(4.9)

$$2 \begin{bmatrix} 0 & \kappa_x \cos(\vec{R}_v \cdot \vec{q}) & \kappa_y \cos(\vec{R}_w \cdot \vec{q}) \\ \kappa_x \cos(\vec{R}_v \cdot \vec{q}) & 0 & 0 \\ \kappa_y \cos(\vec{R}_w \cdot \vec{q}) & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = -k \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Replace $\cos(\vec{R}_v \cdot \vec{q}) = c_2'$ and $\cos(\vec{R}_w \cdot \vec{q}) = c_3'$

$$\begin{aligned} & \begin{vmatrix} -\lambda & \kappa_x c_2' & \kappa_y c_3' \\ \kappa_x c_2' & -\lambda & 0 \\ \kappa_y c_3' & 0 & -\lambda \end{vmatrix} = 0 \\ \Leftrightarrow & -\lambda \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} - \kappa_x c_2' \begin{vmatrix} \kappa_x c_2' & 0 \\ \kappa_y c_3' & -\lambda \end{vmatrix} + \kappa_y c_3' \begin{vmatrix} \kappa_x c_2' & -\lambda \\ \kappa_y c_3' & 0 \end{vmatrix} = 0 \\ \Leftrightarrow & -\lambda^3 + \lambda \kappa_x^2 c_2'^2 + \lambda \kappa_y^2 c_3'^2 = 0 \\ \Leftrightarrow & \lambda(-\lambda^2 + \kappa_x^2 c_2'^2 + \kappa_y^2 c_3'^2) = 0 \\ \Leftrightarrow & \lambda = 0 \text{ or } \lambda = \pm \sqrt{\kappa_x^2 \cos^2(\vec{R}_v \cdot \vec{q}) + \kappa_y^2 \cos^2(\vec{R}_w \cdot \vec{q})} \end{aligned}$$

So the bands of the system of equations are :

$$k_{1,2,3}(\vec{q}) = 0, \pm 2 \sqrt{\kappa_x^2 \cos^2(\vec{R}_v \cdot \vec{q}) + \kappa_y^2 \cos^2(\vec{R}_w \cdot \vec{q})}$$

At the Fig. 4.3 we see the bands for different \vec{q} .

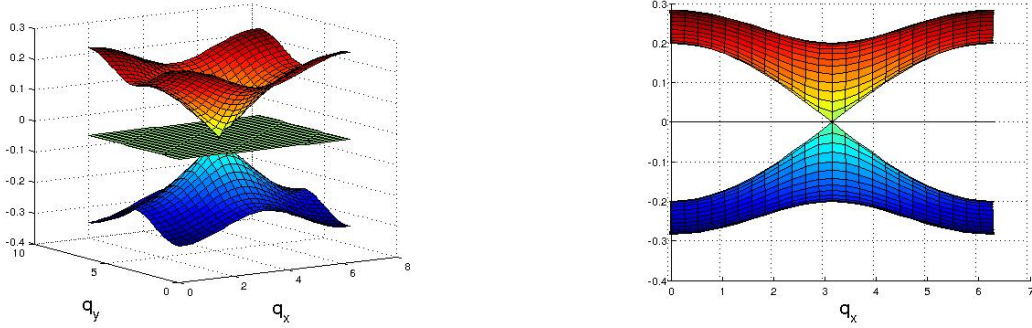


Figure 4.3: Bands of the Lieb lattice for $q_x, q_y \in [0, 2\pi]$, $\vec{a}_1 = (1, 0)$, $\vec{a}_2 = (0, 1)$ and $\kappa_x = \kappa_y = 0.1$ ($V_x = V_y$), there are two different points of view.

Eigenvectors for $\lambda \neq 0$ are :

$$\begin{aligned} & \begin{pmatrix} -\lambda & \kappa_x c_2' & \kappa_y c_3' \\ \kappa_x c_2' & -\lambda & 0 \\ \kappa_y c_3' & 0 & -\lambda \end{pmatrix} \begin{matrix} \leftarrow + \\ \leftarrow + \\ \leftarrow + \end{matrix} \frac{\kappa_x c_2'}{\lambda} \lambda \neq 0 \Rightarrow \begin{pmatrix} -\lambda & \kappa_x c_2' & \kappa_y c_3' \\ 0 & \frac{\kappa_x^2 c_2'^2 - \lambda^2}{\lambda} & \frac{\kappa_x \kappa_y c_2' c_3'}{\lambda} \\ \kappa_y c_3' & 0 & -\lambda \end{pmatrix} \begin{matrix} \leftarrow + \\ \leftarrow + \\ \leftarrow + \end{matrix} \frac{\kappa_y c_3'}{\lambda} \\ & \Rightarrow \begin{pmatrix} -\lambda & \kappa_x c_2' & \kappa_y c_3' \\ 0 & \frac{\kappa_x^2 c_2'^2 - \lambda^2}{\lambda} & \frac{\kappa_x \kappa_y c_2' c_3'}{\lambda} \\ 0 & \frac{\kappa_x \kappa_y c_2' c_3'}{\lambda} & \frac{\kappa_y^2 c_3'^2 - \lambda^2}{\lambda} \end{pmatrix} \begin{matrix} \leftarrow + \\ \leftarrow + \\ \leftarrow + \end{matrix} -\frac{\kappa_x \kappa_y c_2' c_3'}{\kappa_x^2 c_2'^2 - \lambda^2}, c_3' \neq 0 \\ & \Rightarrow \begin{pmatrix} -\lambda & \kappa_x c_2' & \kappa_y c_3' \\ 0 & \frac{\kappa_x^2 c_2'^2 - \lambda^2}{\lambda} & \frac{\kappa_x \kappa_y c_2' c_3'}{\lambda} \\ 0 & 0 & \frac{-\kappa_x^2 \kappa_y^2 c_2'^2 c_3'^2 + (\kappa_y^2 c_3'^2 - \lambda^2)(\kappa_x^2 c_2'^2 - \lambda^2)}{(\kappa_x^2 c_2'^2 - \lambda^2)\lambda} = 0 \end{pmatrix} \end{aligned}$$

Hence we have

$$\begin{aligned} & \frac{\kappa_x^2 c_2'^2 - \lambda^2}{\lambda} v + \frac{\kappa_x \kappa_y c_2' c_3'}{\lambda} w = 0, \quad (\lambda^2 = \kappa_x^2 c_2'^2 + \kappa_y^2 c_3'^2) \\ & \Leftrightarrow -\kappa_y^2 c_3'^2 v + \kappa_x \kappa_y c_2' c_3' w = 0 \\ & \Leftrightarrow v = \frac{\kappa_x c_2'}{\kappa_y c_3'} w, \quad c_3' \neq 0 \\ & -\lambda u + \frac{\kappa_x^2 c_2'^2 + \kappa_y^2 c_3'^2}{\kappa_y c_3'} w = 0 \Leftrightarrow u = \pm \frac{\sqrt{\kappa_x^2 c_2'^2 + \kappa_y^2 c_3'^2}}{\kappa_y c_3'}, \quad c_3' \neq 0 \end{aligned}$$

As a result eigenvectors for $\lambda \neq 0$ and $c_3' \neq 0$ have the following form

$$\begin{bmatrix} \pm \frac{\sqrt{\kappa_x^2 c_2'^2 + \kappa_y^2 c_3'^2}}{\kappa_y c_3'} \\ \frac{\kappa_x c_2'}{\kappa_y c_3'} w \\ 1 \end{bmatrix}, \lambda = \pm \sqrt{\kappa_x^2 c_2'^2 + \kappa_y^2 c_3'^2}, c_3' \neq 0$$

if $\lambda \neq 0$ and $c_3' = 0$

$$\begin{bmatrix} \pm \text{sign}(c_2') \\ 1 \\ 0 \end{bmatrix}, \lambda = \pm \sqrt{\kappa_x^2 c_2'^2}, c_3' \neq 0$$

where $\text{sign}(c_2') = 1$ if $c_2' \geq 0$ otherwise $\text{sign}(c_2') = -1$.

In the case that $\lambda = 0$

$$\begin{pmatrix} 0 & \kappa_x c_2' & \kappa_y c_3' \\ \kappa_x c_2' & 0 & 0 \\ \kappa_y c_3' & 0 & 0 \end{pmatrix}$$

So if $c_2' \neq 0$ and $c_3' \in \mathbb{R}$

$$\begin{bmatrix} 0 \\ -\frac{\kappa_y c_3'}{\kappa_x c_2'} \\ 1 \end{bmatrix}$$

$c_2' = 0$ and $c_3' \neq 0$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The solution which corresponds to the flat band is

$$\begin{bmatrix} u_{m,n} \\ v_{m,n} \\ w_{m,n} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\kappa_y c_3'}{\kappa_x c_2'} e^{i(\vec{R}_{m,n} + \vec{R}_v)\vec{q}} \\ e^{i(\vec{R}_{m,n} + \vec{R}_w)\vec{q}} \end{bmatrix} = \vec{y}_{m,n}(\vec{q}) \text{ for } c_2' \neq 0 \quad (4.10)$$

$$\text{or } \begin{bmatrix} u_{m,n} \\ v_{m,n} \\ w_{m,n} \end{bmatrix} = \begin{bmatrix} 0 \\ e^{i(\vec{R}_{m,n} + \vec{R}_v)\vec{q}} \\ 0 \end{bmatrix} = \vec{y}'_{m,n}(\vec{q}) \text{ for } c_2' = 0 \quad (4.11)$$

Again the superposition of solutions correspond to the flat band, is also a solution that corresponds to the flat band. This superposition is the following

$$\int \int A(\vec{q}) \vec{y}_{m,n}(\vec{q}) dq_x dq_y \text{ or } \int \int A(\vec{q}) \vec{y}'_{m,n}(\vec{q}) dq_x dq_y \quad (4.12)$$

4.1.1 Numerical Results

At the first excitation we have as initial condition $v_{2,2}(z=0) = 1, v_{2,3}(z=0) = 1, w_{2,2}(z=0) = -1, w_{3,2}(z=0) = -1$. This excitation correspond to the flat band as we see at the Fig. 4.4. At the second excitation (Fig. 4.5) we again excite the flat band using as initial condition $v_{2,2}(z=0) = \kappa_y/\kappa_x, v_{2,3}(z=0) = \kappa_y/\kappa_x, w_{2,2}(z=0) = -1, w_{3,2}(z=0) = -1$ as the eigenvector correspond to the flat band implies. The first two excitation can be written as a linear combination of the two forms we see at the (4.12). At the third excitation (Fig. 4.6) the initial condition is $v_{3,2}(z=0) = 1, v_{2,3}(z=0) = 1, w_{2,2}(z=0) = -1$ and there is a lot of diffraction.

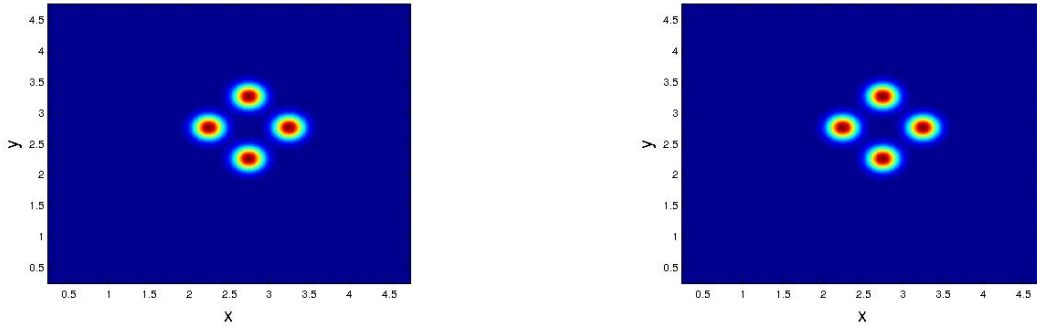


Figure 4.4: Excitation for $\kappa_x = \kappa_y = 0.1$ ($V_x = V_y = 0.5$), $h = 10^{-2}$, $M = N = 5$, at the left initial condition for $z = 0$ and at the right $z = 10$

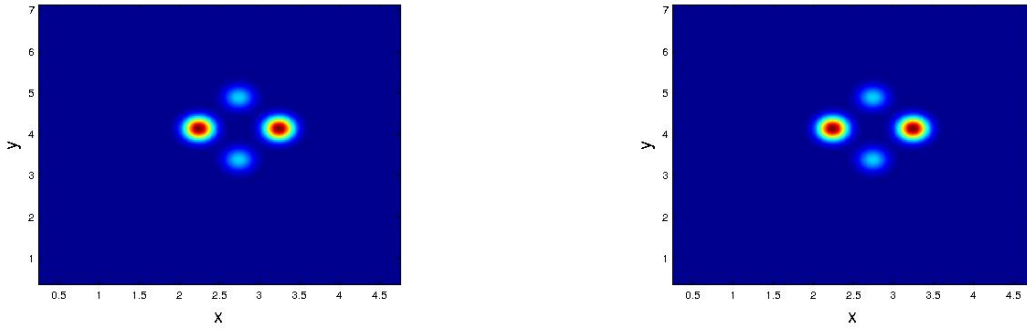


Figure 4.5: Excitation for $\kappa_x = 0.1, \kappa_y = 0.03$, $(V_x = 0.5, V_y = 0.7)$, $h = 10^{-2}$, $M = N = 5$, at the left initial condition for $z = 0$ and at the right $z = 10$

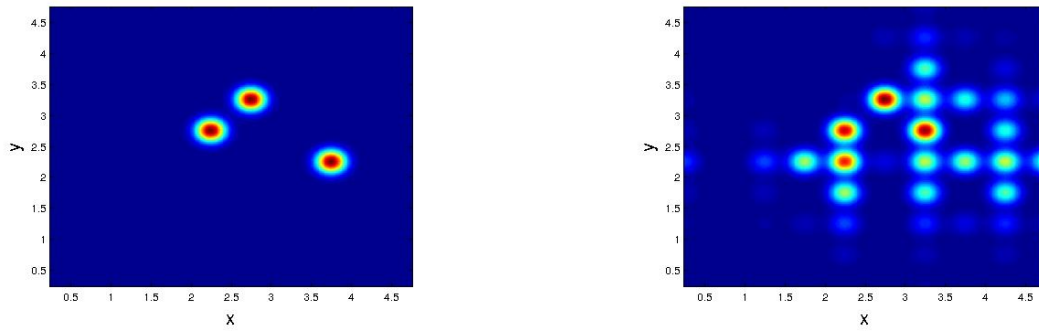


Figure 4.6: Excitation for $\kappa_x = \kappa_y = 0.1$ ($V_x = V_y = 0.5$), $h = 10^{-2}$, $M = N = 5$, at the left initial condition for $z = 0$ and at the right $z = 10$

4.2 Next Nearest Neighbor

The Fig. 4.7 shows us the nearest and next nearest waveguides of each type of waveguide, as is seen for the waveguides of type u there is no difference in neighbouring, with or without considering next nearest neighbours. There are 4 extra neighbours at the other two types of waveguides with the consideration of next nearest neighbours. This occur because waveguides of v, w have neighbour waveguides that are spaced by $\sqrt{V_x^2 + V_y^2}$ but u has not such a neighbours. If we again associate as previous the distances

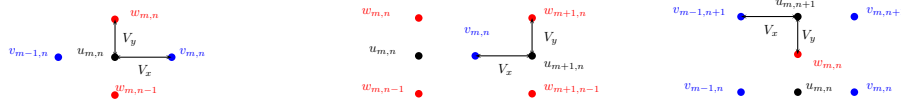


Figure 4.7: Nearest and next nearest neighbours

V_x, V_y and $\sqrt{V_x^2 + V_y^2}$ with coupling terms $\kappa_x, \kappa_y, \kappa_{xy}$ respectively then we can conclude to the following coupled mode equations.

$$i \frac{du_{m,n}}{dz} + \kappa_x (v_{m,n} + v_{m-1,n}) + \kappa_y (w_{m,n} + w_{m,n-1}) = 0 \quad (4.13)$$

$$i \frac{dv_{m,n}}{dz} + \kappa_x (u_{m,n} + u_{m+1,n}) + \kappa_{xy} (w_{m,n} + w_{m+1,n} + w_{m,n-1} + w_{m+1,n-1}) = 0 \quad (4.14)$$

$$i \frac{dw_{m,n}}{dz} + \kappa_y (u_{m,n} + u_{m,n+1}) + \kappa_{xy} (v_{m,n} + v_{m,n+1} + v_{m-1,n+1} + v_{m-1,n}) = 0 \quad (4.15)$$

In order to find the bands of this system of equations we have to suppose wave solution of the following form :

$$u_{m,n} = u e^{-ikz + i\vec{R}_{m,n} \cdot \vec{q}} \quad (4.16)$$

$$v_{m,n} = v e^{-ikz + i(\vec{R}_{m,n} + \vec{R}_v) \cdot \vec{q}} \quad (4.17)$$

$$w_{m,n} = w e^{-ikz + i(\vec{R}_{m,n} + \vec{R}_w) \cdot \vec{q}} \quad (4.18)$$

Replacing the last three equations into eq. (4.13),(4.14),(4.15) respectively and factoring out mutual terms we obtain the following:

$$\begin{aligned} & ku + 2\kappa_x \cos(\vec{R}_v \cdot \vec{q}) v + 2\kappa_y \cos(\vec{R}_w \cdot \vec{q}) w = 0 \\ & kv + 2\kappa_x \cos(\vec{R}_v \cdot \vec{q}) u \\ & + 2\kappa_{xy} \left[\cos\left(\left(\vec{R}_v - \vec{R}_w\right) \cdot \vec{q}\right) + \cos\left(\left(\vec{R}_v + \vec{R}_w\right) \cdot \vec{q}\right) \right] w = 0 \\ & kw + 2\kappa_y \cos(\vec{R}_w \cdot \vec{q}) u \\ & + 2\kappa_{xy} \left[\cos\left(\left(\vec{R}_v - \vec{R}_w\right) \cdot \vec{q}\right) + \cos\left(\left(\vec{R}_v + \vec{R}_w\right) \cdot \vec{q}\right) \right] v = 0 \end{aligned}$$

Recall that $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$ we can write the last three equation in a more simple form :

$$ku + 2\kappa_x \cos(\vec{R}_v \cdot \vec{q})v + 2\kappa_y \cos(\vec{R}_w \cdot \vec{q})w = 0 \quad (4.19)$$

$$kv + 2\kappa_x \cos(\vec{R}_v \cdot \vec{q})u + 4\kappa_{xy} \left(\cos(\vec{R}_v \cdot \vec{q}) \cos(\vec{R}_w \cdot \vec{q}) \right) w = 0 \quad (4.20)$$

$$kw + 2\kappa_y \cos(\vec{R}_w \cdot \vec{q})u + 4\kappa_{xy} \left(\cos(\vec{R}_v \cdot \vec{q}) \cos(\vec{R}_w \cdot \vec{q}) \right) v = 0 \quad (4.21)$$

If we write again (4.19),(4.20),(4.21) as a linear system

$$\begin{bmatrix} 0 & c_2 & c_3 \\ c_2 & 0 & c_1 \\ c_3 & c_1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = -k \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

where $c_2 = 2\kappa_x \cos(\vec{R}_v \cdot \vec{q})$, $c_3 = 2\kappa_y \cos(\vec{R}_w \cdot \vec{q})$, and $c_1 = 4\kappa_{xy} \cos(\vec{R}_v \cdot \vec{q}) \cdot \cos(\vec{R}_w \cdot \vec{q})$. The roots of the determinant of the above matrix, as we have already seen, given from :

$$-\lambda^3 + \lambda(c_1^2 + c_2^2 + c_3^2) + 2c_1c_2c_3 = 0 \quad (4.22)$$

The analytical form of these bands is not an easy task to be found, so we found it numerically. The Fig. 4.8

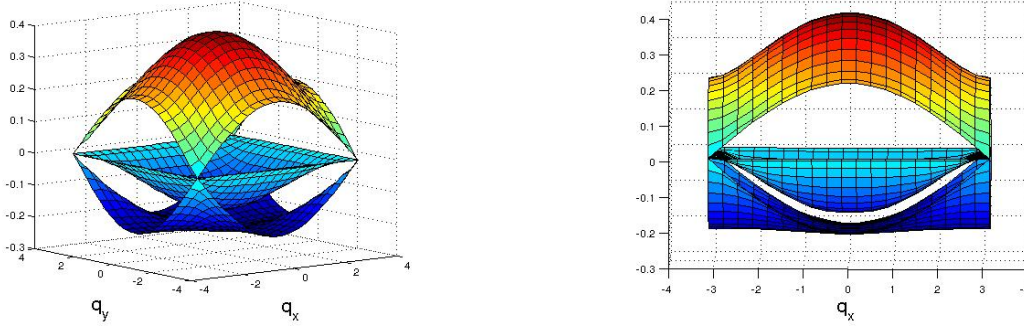


Figure 4.8: Eigenvalues for $q_x, q_y \in [-\pi, \pi]$, $\kappa_x = 0.1$, $\kappa_y = 0.1$ ($V_x = V_y = 0.5$), $\kappa_{xy} = 0.04$ and $\vec{a}_1 = (1, 0)$, $\vec{a}_2 = (0, 1)$

illustrates for different $\vec{q} = \{q_x, q_y\}$ the different bands that are created. We see 3 non flat bands, 2 with extremely much curvature and 1 with much less curvature. The conclusion is that there is no flat band with the consideration of interaction, between the waveguides, of the next nearest neighbours.

4.2.1 Numerical Results

The first excitation we see at the Fig. 4.9 corresponded at the flat band without the next nearest neighbour consideration. We have for initial condition : $v_{2,2}(z=0) = 1, v_{2,3}(z=0) = 1, w_{2,2}(z=0) = -1, w_{3,2}(z=0) = -1$. At the second excitation (Fig. 4.10) we see an excitation which corresponded again to the flat band if we don't consider next nearest neighbouring, the initial condition is $v_{2,2}(z=0) = \kappa_y/\kappa_x, v_{2,3}(z=0) = \kappa_y/\kappa_x, w_{2,2}(z=0) = -1, w_{3,2}(z=0) = -1$. As we see there is a small diffraction due to the presence of the next nearest neighbour interaction. At the third excitation (Fig. 4.11) we see the excitation with initial conditions $v_{3,2}(z=0) = 1, v_{2,3}(z=0) = 1, w_{2,2} = -1$, there is a lot of diffraction here.

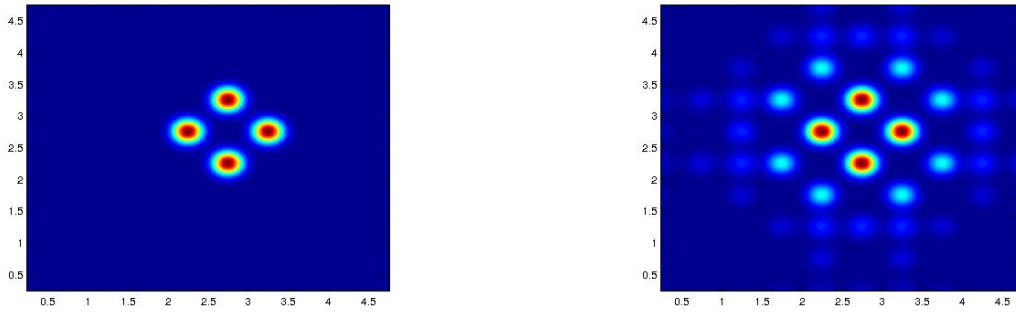


Figure 4.9: Excitation for $\kappa_x = \kappa_y = 0.1$ ($V_x = V_y = 0.5$), $\kappa_{xy} = 0.04$, $h = 10^{-2}$, $M = N = 5$, at the left initial condition for $z = 0$ and at the right $z = 10$

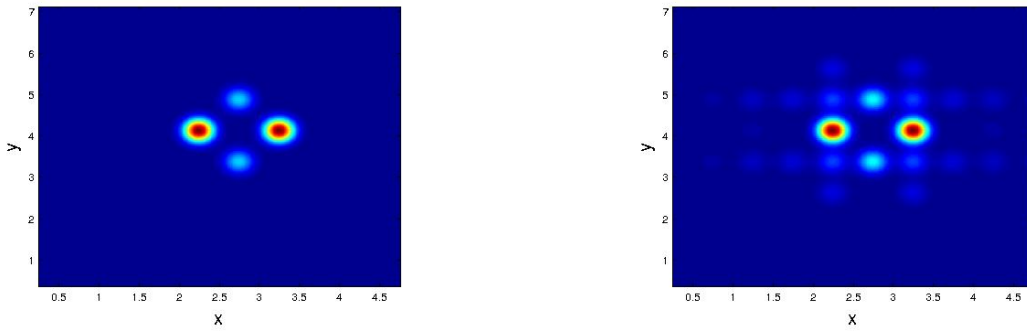


Figure 4.10: Excitation for $\kappa_x = 0.1$, $\kappa_y = 0.03$, $\kappa_{xy} = 0.01$ ($V_x = 0.5$, $V_y = 0.7$), $h = 10^{-2}$, $M = N = 5$, at the left initial condition for $z = 0$ and at the right $z = 10$

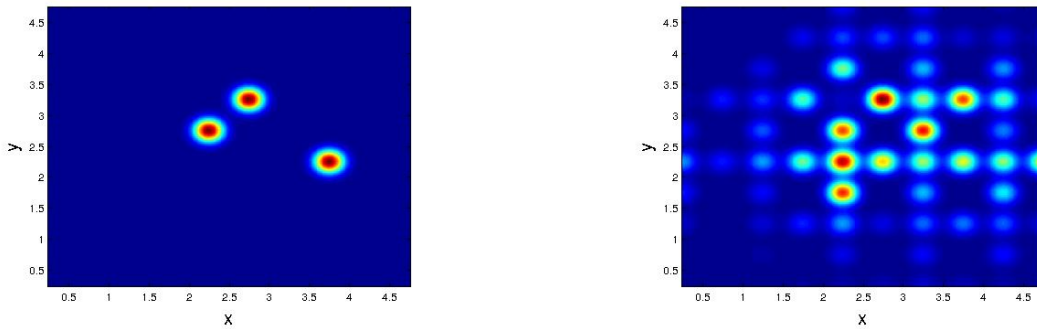


Figure 4.11: Excitation for $\kappa_x = \kappa_y = 0.1$ ($V_x = V_y = 0.5$), $\kappa_{xy} = 0.04$, $h = 10^{-2}$, $M = N = 5$, at the left initial condition for $z = 0$ and at the right $z = 10$

4.3 Disorder

We consider the Lieb lattice but instead of the fixed positions of the waveguides we assume a random displacement within a square of side ϵ around the fixed positions as previous to the Kagome lattice. The probability of the waveguide to be found in a certain point is distributed uniformly. Fig. 4.12 show us the excitation corresponds to the flat band as we previous saw at the previous section, notice that there exist a small diffraction due to the unequal coupling terms.

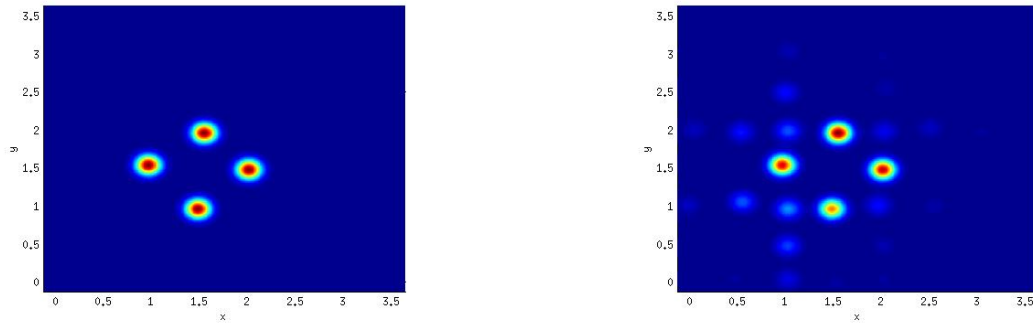


Figure 4.12: Excitation for $\epsilon = 0.1, h = 10^{-2}, M = 4, N = 4, V_x = 0.5 = V_y$, at the left initial condition for $z = 0$ and at the right $z = 10$

Chapter 5

How to Construct Flat Band Lattices

At this chapter we examine in a more abstract way the coupled mode equations we have considered all over this work. After that we present the basic idea of how to construct flat band lattices and we apply it to construct 1d and 2d lattices.

5.1 General Model

A general equation which we've used till now, describes the variation of the amplitude, as have been described at the derivation of Coupled Mode Equations is the following[9]

$$-i \frac{d\psi_{\vec{n}}}{dz} = \sum_{\vec{m} \neq \vec{n}} V_{nm} \psi_{\vec{m}} \quad (5.1)$$

where \vec{n}, \vec{m} describe the position vectors of the corresponding waveguides, z is the direction in which the light propagates and V_{nm} is the coupling term between the waveguide at the position \vec{n} and the waveguide at the position \vec{m} , of course $V_{nm} = V_{mn}$ due to the dependence of the coupling term from the distance between the waveguides.

We are going to follow the same procedure as previous at every lattice we have studied in this work. Assume that the solution of the eq. (5.1) is of the form

$$\psi_{\vec{n}} = A_{\vec{n}} e^{i\vec{q} \cdot \vec{n} - ikz} \quad (5.2)$$

where k is as previous the spatial frequency, \vec{q} is the lattice wave vector and $A_{\vec{n}}$ is the amplitude. Replacing the eq. (5.2) into eq. (5.1) one gets the following :

$$\begin{aligned} -i \frac{d\psi_{\vec{n}}}{dz} &= \sum_{\vec{m} \neq \vec{n}} V_{nm} \psi_{\vec{m}} \Rightarrow k\psi_{\vec{n}} = \sum_{\vec{m} \neq \vec{n}} V_{nm} A_{\vec{m}} e^{i\vec{q} \cdot \vec{m}} \\ &\Rightarrow -k A_{\vec{n}} e^{i\vec{q} \cdot \vec{n} - ikz} = \sum_{\vec{m} \neq \vec{n}} V_{nm} A_{\vec{m}} e^{i\vec{q} \cdot \vec{m} - ikz} \end{aligned}$$

multiplying both sides by $e^{-i\vec{q} \cdot \vec{n} + ikz}$ gives us

$$-k(\vec{q}) A_{\vec{n}} = \sum_{\vec{m} \neq \vec{n}} V_{nm} A_{\vec{m}} e^{i\vec{q} \cdot (\vec{m} - \vec{n})} \quad (5.3)$$

The last equation gives us the linear system such as these we've met at the Kagome and Lieb lattice. The amplitude of each of the waveguides are given by $A_{\vec{n}}$ and $k(\vec{q})$ gives us the bands of the system. Notice that the multitude of the distinct amplitudes $A_{\vec{n}}$ are equal to the number of the kinds of waveguides, i.e if we speak about a triatomic lattice then we have three kinds of waveguides thus we have three distinct $A_{\vec{n}}$.

Now we've set-up all the necessary ingredients which are necessary to describe as general as possible the creation of a flat band lattice or FB lattice.

5.1.1 From Dimer to Rhombic lattice

If we suppose that there are two waveguides which are spaced by a certain distance and there is an interaction between them which is associated with the coupling term V then we have in some sense a trivial waveguide array named **Coupler** or **Dimer**. The Dimer shown at the Fig. 5.1 below

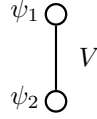


Figure 5.1: A Dimer, two waveguides associated with coupling term V

The equations describes this simple system are the following

$$i \frac{d\psi_1}{dz} + V\psi_2 = 0$$

$$i \frac{d\psi_2}{dz} + V\psi_1 = 0$$

If we suppose that we have solutions of the form

$$\psi_1 = A_1 e^{iq-ikz}$$

$$\psi_2 = A_2 e^{iq-ikz}$$

So we can conclude easily that the eigenvalue problem for this system is the below

$$\begin{bmatrix} 0 & V \\ V & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = -k \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

the above matrix has eigenvalues $\lambda_{1,2} = \pm V$, thus the dispersion relation is $k_{1,2} = \mp V$. Eigenvectors corresponds to the eigenvalues

$$\lambda_1 \leftrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_2 \leftrightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So the modes of the Dimer are of the form $[+a, +a], [-a, +a]$ and for simplicity from now on we will write $[+, +], [-, +]$. We have to declare that from now on with the term mode we speak about the eigenvectors of the system (for example the Dimer we saw). From now on in this chapter we will call the components of the modes as amplitudes for our convenience. We often consider as a component of the mode the real constant which multiply the sinusoidal at the eigenvectors. If we add a waveguide between two Dimers as the Fig. 5.2 shows, we compose a more complicated system .

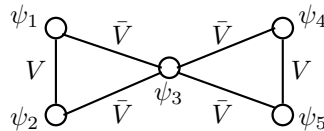


Figure 5.2: A system consists of two Dimer connected with a central waveguide and the coupling terms

It is easy to verify that the mode $[-, +]$ of the Dimer continue to exist as a mode in the extended system if we have zero amplitude at the central waveguide. Therefore the system has the mode $[-, +, 0, -, +]$ where the ordering of the mode corresponds to the ordering of the waveguides, i.e $[\psi_1, \psi_2, \psi_3, \psi_4, \psi_5]$. Often the waveguides are called **sites** and the central waveguide of the last case called **connector site** and it is

notated as ψ_c . We can see that the $[-, +, 0, -, +]$ is a mode from eq. (5.3) and if we suppose that there is no phase difference between the waveguides, i.e the term $e^{i\bar{q}(\bar{m}-\bar{n})}$ is neglected.

$$\begin{aligned}\psi_3 : -k \cdot 0 &= \bar{V}(a - a + a - a) = 0 \\ \psi_1 : -k(-a) &= V(+a) + 0 \Rightarrow k = V \\ \psi_2 : -k(+a) &= V(-a) + 0 \Rightarrow k = V\end{aligned}$$

the equations for ψ_3, ψ_4 are similar with the cases ψ_1, ψ_2 . Notice that the mode $[+, +]$ of the Dimer is not a mode of the effenxed system because it will change the 0 (zero) amplitude of the central waveguide (ψ_3). Now we can extend the system of the Fig. 5.2 to the full lattice which is known as **Rhombic** or **Diamond** lattice. The equations which describes the Rhombic lattice are

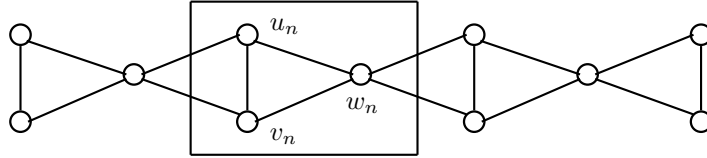


Figure 5.3: The Rhombic or Diamond lattice with its unit cell

$$i \frac{du_n}{dz} + Vv_n + \bar{V}(w_{n-1} + w_n) = 0 \quad (5.4)$$

$$i \frac{dv_n}{dz} + Vu_n + \bar{V}(w_{n-1} + w_n) = 0 \quad (5.5)$$

$$i \frac{dw_n}{dz} + \bar{V}(u_n + u_{n+1} + v_n + v_{n+1}) = 0 \quad (5.6)$$

and if we suppose solutions of the form $u_n = ue^{iq_x n - ikz}$, $v_n = ve^{iq_x n - ikz}$, $w_n = we^{iq_x(n+1/2) - ikz}$ where q_x is the wave number (it isn't a vector because the lattice is repeated over one direction). Replacing the three wave solutions into the eq. (5.4) - (5.6) gives us the following :

$$\begin{aligned}ku + Vv + 2\bar{V} \cos(q_x/2) w &= 0 \\ kv + Vu + 2\bar{V} \cos(q_x/2) w &= 0 \\ kw + 2\bar{V} \cos(q_x/2) u + 2\bar{V} \cos(q_x/2) v &= 0\end{aligned}$$

or equivalently

$$\begin{bmatrix} 0 & V & 2\bar{V} \cos(q_x/2) \\ V & 0 & 2\bar{V} \cos(q_x/2) \\ 2\bar{V} \cos(q_x/2) & 2\bar{V} \cos(q_x/2) & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = -k \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

The bands and the modes are associated with the eigenvalues and the eigenvectors of the above linear system.

$$\begin{vmatrix} -\lambda & V & 2\bar{V} \cos(q_x/2) \\ V & -\lambda & 2\bar{V} \cos(q_x/2) \\ 2\bar{V} \cos(q_x/2) & 2\bar{V} \cos(q_x/2) & -\lambda \end{vmatrix} = 0$$

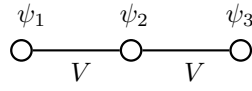


Figure 5.4: A Trimer, three waveguides and the associated coupling terms

$$i \frac{d\psi_2}{dz} + V(\psi_1 + \psi_3) = 0 \quad (5.8)$$

$$i \frac{d\psi_3}{dz} + V\psi_2 = 0 \quad (5.9)$$

We suppose that the solution is of the form $\psi_1 = A_1 e^{-ikz}$, $\psi_2 = A_2 e^{iqx - ikz}$, $\psi_3 = A_3 e^{2iqx - ikz}$. So the eq. (5.7)-(5.9) gives us

$$(5.7) \Rightarrow kA_1 + VA_2 e^{iqx} = 0$$

$$(5.8) \Rightarrow kA_2 + V(A_1 e^{-iqx} + A_3 e^{iqx}) = 0$$

$$(5.9) \Rightarrow kA_3 + VA_2 e^{-iqx} = 0$$

or equivalently

$$\begin{bmatrix} 0 & Ve^{iqx} & 0 \\ Ve^{-iqx} & 0 & Ve^{iqx} \\ 0 & Ve^{-iqx} & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = -k \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

We are going to find the bands and the modes.

$$\begin{vmatrix} -\lambda & c_1 & 0 \\ \bar{c}_1 & -\lambda & c_1 \\ 0 & \bar{c}_1 & -\lambda \end{vmatrix} = 0$$

where $c_1 = Ve^{iqx}$ and \bar{c}_1 is the complex conjugate of c_1 .

$$\Leftrightarrow -\lambda \begin{vmatrix} -\lambda & c_1 \\ \bar{c}_1 & -\lambda \end{vmatrix} - c_1 \begin{vmatrix} \bar{c}_1 & c_1 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow -\lambda(\lambda^2 - c_1 \bar{c}_1) - c_1(-\lambda \bar{c}_1) = 0$$

$$\Leftrightarrow -\lambda^3 + 2\lambda c_1 \bar{c}_1 = 0$$

So the bands are $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{2}V$

The procedure for the modes follows.

For $\lambda = 0$

$$\begin{pmatrix} 0 & c_1 & 0 \\ \bar{c}_1 & 0 & c_1 \\ 0 & \bar{c}_1 & 0 \end{pmatrix}, \text{ exchange between the 1st row and the 2nd} \\ \Rightarrow \begin{pmatrix} \bar{c}_1 & 0 & c_1 \\ 0 & c_1 & 0 \\ 0 & \bar{c}_1 & 0 \end{pmatrix} \begin{matrix} \leftarrow -\frac{\bar{c}_1}{c_1} \\ \leftarrow + \end{matrix} \Rightarrow \begin{pmatrix} \bar{c}_1 & 0 & c_1 \\ 0 & c_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore

$$c_1 A_2 = 0 \Leftrightarrow (c_1 \neq 0) A_2 = 0$$

$$\bar{c}_1 A_1 + c_1 A_3 = 0 \Leftrightarrow A_1 = -\frac{c_1}{\bar{c}_1} A_3$$

So the mode corresponds to $\lambda = 0$ is $[-, 0, +]$.

For $\lambda = \sqrt{2}V$.

$$\begin{aligned} & \begin{pmatrix} -\sqrt{2}V & c_1 & 0 \\ \bar{c}_1 & -\sqrt{2}V & c_1 \\ 0 & \bar{c}_1 & -\sqrt{2}V \end{pmatrix} \begin{array}{l} \leftarrow \frac{c_1}{\sqrt{2}V} \\ \leftarrow + \end{array} \Rightarrow \begin{pmatrix} -\sqrt{2}V & c_1 & 0 \\ 0 & -V/\sqrt{2} & c_1 \\ 0 & \bar{c}_1 & -\sqrt{2}V \end{pmatrix} \begin{array}{l} \leftarrow \frac{\sqrt{2}c_1}{V} \\ \leftarrow + \end{array} \\ & \Rightarrow \begin{pmatrix} -\sqrt{2}V & c_1 & 0 \\ 0 & -V/\sqrt{2} & c_1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Thus we have

$$\begin{aligned} A_2 &= \sqrt{2}e^{iq_x} A_3 \\ A_1 &= \frac{c_1}{\sqrt{2}V} A_2 = e^{2iq_x} A_3 \end{aligned}$$

Thus the mode corresponds to $\lambda = \sqrt{2}V$ is $[+, +\sqrt{2}, +]$.

Finally for $\lambda = -\sqrt{2}V$

$$\begin{aligned} & \begin{pmatrix} \sqrt{2}V & c_1 & 0 \\ \bar{c}_1 & \sqrt{2}V & c_1 \\ 0 & \bar{c}_1 & \sqrt{2}V \end{pmatrix} \begin{array}{l} \leftarrow \frac{-c_1}{\sqrt{2}V} \\ \leftarrow + \end{array} \Rightarrow \begin{pmatrix} \sqrt{2}V & c_1 & 0 \\ 0 & V/\sqrt{2} & c_1 \\ 0 & \bar{c}_1 & \sqrt{2}V \end{pmatrix} \begin{array}{l} \leftarrow \frac{-\sqrt{2}c_1}{V} \\ \leftarrow + \end{array} \\ & \Rightarrow \begin{pmatrix} \sqrt{2}V & c_1 & 0 \\ 0 & V/\sqrt{2} & c_1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

So

$$\begin{aligned} A_2 &= -\sqrt{2}e^{iq_x} A_3 \\ A_1 &= e^{2iq_x} A_3 \end{aligned}$$

Therefore the mode corresponds to $\lambda = -\sqrt{2}V$ is $[+, -\sqrt{2}, +]$.

Now we will look one by one the modes we have found. The mode $[+, +\sqrt{2}, +]$ can't be the starting point where we can connect a connector waveguide in order to create a more complicated system of waveguides. The reason for this is that there is no change of sign between its components, consequently if one connects a connector waveguide at the Trimer which is exciting the last mode and the connector waveguide has zero amplitude then it is obvious from eq. (5.1), if we again neglect the terms $e^{i\vec{q}(\vec{m}-\vec{n})}$, that the amplitude of the connector waveguide will not remain unchanged :

$$-i \frac{d\psi_c}{dz} \Big|_{z=0} = V_{c1}(+1) + V_{c2}(+\sqrt{2}) + V_{c3}(+1)$$

where the ψ_c describes the amplitude of the connector waveguide, $V_{ci}, i = 1, 2, 3$ describes the coupling terms between the connector waveguide and the rest waveguides, which forms the Trimer.

Suitable modes which presents alteration of sign are the other two modes, i.e $[-, 0, +]$ and $[+, -\sqrt{2}, +]$. We consider firstly the first of the two modes and then the second, we are going to create two different lattices.

5.1.2.1 Cross Lattice

The construction begins from the fact that we have two Trimer and between them we place a connector waveguide as the Fig. 5.5 shows. This system must have as mode the mode which is consisted of the

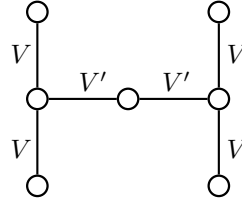


Figure 5.5: Two Trimers connected with a central (connector) waveguide with one connection from each of the Trimers to the central waveguide

$[-, 0, +]$ at the two Trimers and zero to the connector waveguide. At the Fig. 5.6 we see the possible modes¹ to which we have resulted without deep investigation but using the fact that we seek for cancellation of the amplitudes at every waveguide and mainly at the connector waveguide.

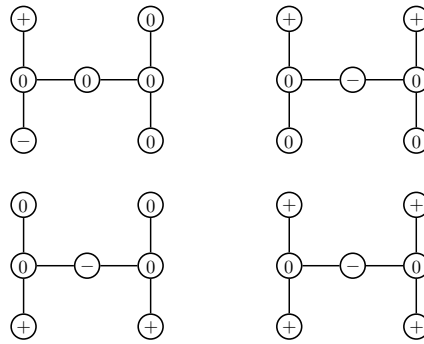


Figure 5.6: The sign of the possible modes of the composed system

In order to find all the possible flat bands, one must construct the full lattice which is shown below at the Fig. 5.7 and is named **Cross** lattice.

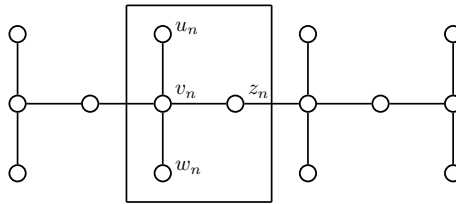


Figure 5.7: The Cross lattice and its unit cell

As we can see there is 4 kind of waveguides and the lattice which extends in one direction. The basic equations governing the propagation of the light along the waveguides are the following

$$i \frac{du_n}{dz} + Vv_n = 0 \tag{5.10}$$

$$i \frac{dv_n}{dz} + V(u_n + w_n) + V'(z_{n-1} + z_n) = 0 \tag{5.11}$$

¹The meaning of the mode here is not the same with previous, we mean only the sign of each of the components of the composed system. For example + at the figure might mean $+\sqrt{2}$ or $+2$ or something different. As long as the constant is unknown we called it possible mode.

$$i \frac{dw_n}{dz} + V v_n = 0 \quad (5.12)$$

$$i \frac{dz_n}{dz} + V' (v_n + v_{n+1}) = 0 \quad (5.13)$$

and the form of the solution is assumed to be $u_n = u e^{iq_x n - ikz}$, $v_n = v e^{iq_x n - ikz}$, $w_n = w e^{iq_x n - ikz}$, $z_n = z e^{iq_x (n+1/2) - ikz}$. Therefore we get the following

$$\begin{aligned} ku + Vv &= 0 \\ kv + V(u + w) + 2V' \cos(q_x/2) z &= 0 \\ kw + Vv &= 0 \\ kz + 2V' \cos(q_x/2) v &= 0 \end{aligned}$$

or equivalently

$$\begin{bmatrix} 0 & V & 0 & 0 \\ V & 0 & V & 2V' \cos(q_x/2) \\ 0 & V & 0 & 0 \\ 0 & 2V' \cos(q_x/2) & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = -k \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix}$$

The eigenvalue are given

$$\begin{aligned} & \begin{vmatrix} -\lambda & V & 0 & 0 \\ V & -\lambda & V & 2V' \cos(q_x/2) \\ 0 & V & -\lambda & 0 \\ 0 & 2V' \cos(q_x/2) & 0 & -\lambda \end{vmatrix} = 0 \\ \Leftrightarrow & -\lambda \left(-\lambda \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} - V \begin{vmatrix} V & 0 \\ 2V' \cos(q_x/2) & -\lambda \end{vmatrix} + 2V' \cos(q_x/2) \begin{vmatrix} V & -\lambda \\ 2V' \cos(q_x/2) & 0 \end{vmatrix} \right) \\ & -V \left(V \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} \right) = 0 \\ \Leftrightarrow & \lambda^2 (\lambda^2 - 2V^2 - 4V'^2 \cos^2(q_x/2)) = 0 \end{aligned}$$

So the bands are

$$\lambda_{1,2} = 0, \quad \lambda_{2,3} = \pm \sqrt{2V^2 + 4V'^2 \cos^2(q_x/2)}$$

Modes for $\lambda = 0$ are given from the following

$$\begin{aligned} & \begin{pmatrix} 0 & V & 0 & 0 \\ V & 0 & V & 2V' \cos(q_x/2) \\ 0 & V & 0 & 0 \\ 0 & 2V' \cos(q_x/2) & 0 & 0 \end{pmatrix}, \text{ exchange between the 1st row and the 2nd} \\ \Rightarrow & \begin{pmatrix} V & 0 & V & 2V' \cos(q_x/2) \\ 0 & V & 0 & 0 \\ 0 & V & 0 & 0 \\ 0 & 2V' \cos(q_x/2) & 0 & 0 \end{pmatrix} \begin{matrix} \leftarrow - \\ \leftarrow + \end{matrix} \\ \Rightarrow & \begin{pmatrix} V & 0 & V & 2V' \cos(q_x/2) \\ 0 & V & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2V' \cos(q_x/2) & 0 & 0 \end{pmatrix} \begin{matrix} \leftarrow \frac{-2V' \cos(q_x/2)}{V} \\ \leftarrow + \end{matrix} \end{aligned}$$

$$\Rightarrow v = 0$$

$$u = -w - 2 \frac{V'}{V} \cos(q_x/2) z$$

So the modes are $[-, 0, +, 0]$, $[-\frac{V'}{V}, 0, 0, +]$ for $\cos(q_x/2) = 1/2$, the first mode is that we are looking for because this verify our hypothesis that the mode of Trimer $[-, 0, +]$ with 0 to the connector waveguide i.e the mode $[-, 0, +, 0]$ exist as a mode of the periodic lattice (i.e Cross lattice).

5.1.2.2 Sawtooth Lattice

Returning back to the modes of Trimer this time we will use the third one which is $[-, \sqrt{2}, -]$. At this time we connect two waveguides of each of the Trimers to the central waveguide, of course it doesn't matter which one because as we can see the mode we mentioned is the same at the first and third component. Therefore we do the connection as usual, taking two Trimers and connected with the central waveguide as the Fig. 5.8 shows. Let's see what must be held so that the mode of the Trimer exist as a mode of the

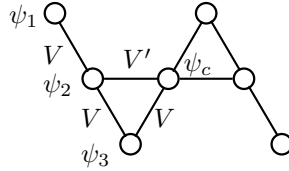


Figure 5.8: Two Trimers connected with a central waveguide with two waveguides of each of the Trimers connected to central waveguide

new system. If we excite the connector waveguide with zero amplitude and the left Trimer as the mode $[-, \sqrt{2}, -]$ shows us, i.e $[\psi_1, \psi_2, \psi_3, \psi_c] = [-a, +\sqrt{2}a, -a, 0]$, then from the relation (5.3) (neglect the phase difference) we can export the below relation

$$k\psi_c = V'(\sqrt{2}a) - Va + V(0) + V'(0)$$

$$\Leftrightarrow 0 = a(V'\sqrt{2} - V)$$

$$\Leftrightarrow \frac{V}{V'} = \sqrt{2}$$

Notice that the amplitudes of the second Trimer are considered as zero.

Now we have to show that for the full **Sawtooth** lattice there is a flat band for the ratio of the coupling terms we told above. There is the Sawtooth lattice at the Fig. 5.9 which depicts the unit cell with continuous line and a hypothetical with dashed line. Due to the symmetry of the distances and therefore of the coupling terms we can reduce the system to just two waveguides, although the hypothetical unit cell² is not exactly a unit cell but it shows us how to make more easier the calculations.

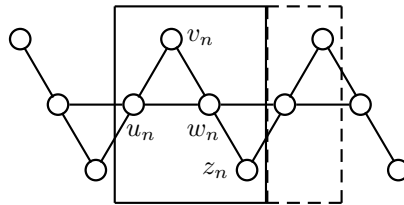


Figure 5.9: Sawtooth lattice and its unit cell

² The hypothetical unit cell is not that which if i repeated over the x axis i construct the full lattice.

The equations describe the propagation of the light along the waveguides are the following.

$$i \frac{du_n}{dz} + V(v_n + v_{n-1}) + V'(u_{n+1} + u_{n-1}) = 0 \quad (5.14)$$

$$i \frac{dv_n}{dz} + V(u_n + u_{n+1}) = 0 \quad (5.15)$$

We consider plane wave solutions, i.e. solutions of the form $u_n = ue^{iq_x n - ikz}$, $v_n = ve^{iq_x(n+1/2) - ikz}$. Consequently the eq. (5.14),(5.15) become :

$$\begin{aligned} ku + 2V \cos(q_x/2)v + 2V' \cos(q_x)u &= 0 \\ kv + 2V \cos(q_x/2)u &= 0 \\ \Leftrightarrow \begin{bmatrix} 2V' \cos(q_x) & 2V \cos(q_x/2) \\ 2V \cos(q_x/2) & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} &= -k \begin{bmatrix} u \\ v \end{bmatrix} \end{aligned}$$

The eigenvalues are given from the following.

$$\begin{aligned} \begin{vmatrix} 2V' \cos(q_x) - \lambda & 2V \cos(q_x/2) \\ 2V \cos(q_x/2) & -\lambda \end{vmatrix} &= 0 \\ \Leftrightarrow \lambda^2 - 2\lambda V' \cos(q_x) - 4V^2 \cos^2(q_x/2) &= 0 \\ \lambda_{1,2} = V' \cos(q_x) \pm \sqrt{V'^2 \cos^2(q_x) + 4V^2 \cos^2(q_x/2)} \\ = V' \left(\cos(q_x) \pm \sqrt{1 + 4 \left(\frac{V^2}{V'^2} - 1 \right) \cos^2(q_x/2) + 4 \cos^4(q_x/2)} \right) \end{aligned}$$

where we have concluded using that the $\cos(q_x) = 2 \cos^2(q_x/2) - 1$. Suppose that the $\frac{V}{V'} = \sqrt{2}$ then we get

$$\begin{aligned} \lambda_{1,2} &= V' \left(\cos(q_x) \pm \sqrt{1 + 4 \cos^2(q_x/2) + 4 \cos^4(q_x/2)} \right) \\ &= V' \left(\cos(q_x) \pm (\cos(q_x) + 2) \right) \end{aligned}$$

So the flat band is for $\lambda_2 = -2V'$ and $\frac{V}{V'} = \sqrt{2}$. The relation between the amplitudes of the mode is

$$\begin{aligned} u &= -\frac{\sqrt{2} \cos(q_x/2)}{\cos(q_x) + 1} v \text{ for } \cos(q_x) \neq -1 \\ u = 0, v = 0 &\text{ for } \cos(q_x) + 1 = 0 \text{ and } \cos(q_x/2) \neq 0 \\ v = 0 &\text{ for } \cos(q_x) + 1 = 0 \text{ and } \cos(q_x/2) = 0 \end{aligned}$$

5.2 Lieb Lattice

Let's recall the Lieb lattice which is a FB lattice. At this section we will study how one can conclude to this lattice starting from a smaller system in similar way with the things we have already seen at this chapter. Suppose we have an array of waveguides as the illustration in Fig. 5.10 (left), it is understood that this array is a part of the Lieb lattice. At the right we see an excitation which is exactly the excitation we saw at the chapter of the Lieb lattice where we excited the waveguides of type v and w . The point here is that at the corners of the waveguides excitation (right) we have zero amplitudes thus these waveguides can be used as connector waveguides. This means that we can reconstruct the Lieb lattice from the mini array presented here if we connect the mini array in the way that the Fig. 5.11 shows.

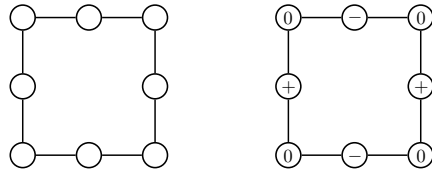


Figure 5.10: A mini array (left) and one of its modes (right)

We can apply the same idea, i.e using the zero amplitude waveguides of a mode of the mini array as connector waveguides to produce other lattices. Another mode of the mini array is that depicted at the Fig. 5.12 (left) which can be the starting mini array from which we construct a new lattice, via connecting the zero amplitude waveguides of the mode with a similar way as previous. Hence if we take two mini arrays connected with a central waveguide as the Fig. 5.12 shows, then we have constructed the **Lieb 2** lattice. Notice that it is hard to find the bands of the new lattice because it has 10 waveguides into its unit cell in contrary with the Lieb lattice has only 3, hence we can approximate the bands with numerical methods.

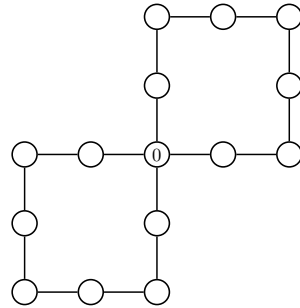


Figure 5.11: A miniarray (left) and one of its modes (right)

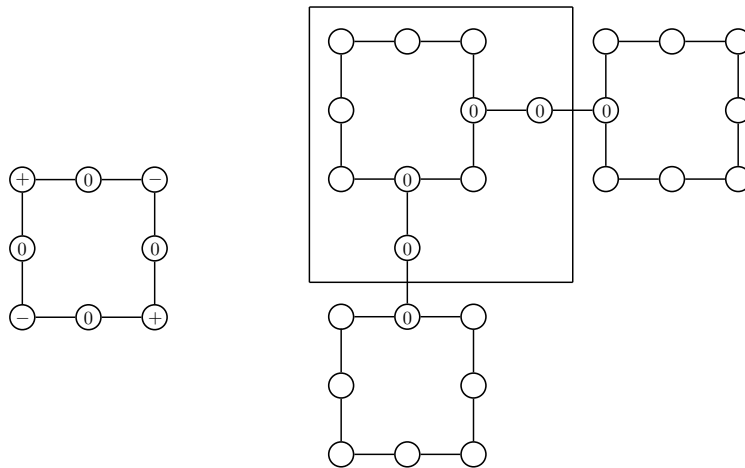


Figure 5.12: A mode of the mini array (left) and the construction of Lieb 2 lattice and its unit cell (right)

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