

# Magnetic skyrmions and their dependence on system parameters 

Master's Thesis<br>Programme on: Applied and Computational Mathematics-Modeling and Analysis in Sciences and Technology

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For my parents


#### Abstract

In the first chapters of this thesis we will start by introducing the experimental observations that motivated the idea that there is a relation between the magnetic moment and the angular momentum and by the end of the chapter 2, we will have introduced the equation of motion for one spin. In chapter 3, we will consider the structure of a spin chain and we will describe the different types of interactions between them. With these known, we will proceed to the discussion of the dynamics for a spin chain, and in the end we will have modelled the continuum form of the equation of motion and also the case when there are dissipative effects, another parameter that we should consider when we talk about spin chain dynamics.

In chapter 4 we mainly discuss the Landau-Lifshitz equation (later noted as LL Equation). We make some discussion about the basic ideas of this equation that origin on the Homotopy Theory. Later on this chapter we present the Landau-Lifshitz equation, the Laundau-Lifshitz-Gilbert Equation (later noted as LLG equation) and we solve them analytically in order to obtain the domain wall solution. With this chapter, the introductory part of the basic concepts and of the equations that will concern as on the next chapters, has ended.

So with everything that were discussed before, we introduce Skyrmions. We present the experiment where they were first observed and their configuration. In the first part of this thesis, we introduced the different types of interaction between spins, on a spin chain. Here, we are most interested in the Dzyaloshinskii-Moriya interaction (later referred as DMI) and in which way it affects the configuration of a magnetic skyrmion. Because, skyrmions are solutions of the LL equation, and the DMI is an important interaction for the formation of this structure, we discuss the existence of the DMI term on the LL equation and we present its domain wall solution.

Finally, we proceed with the micromagnetic simulations that were made for this thesis. We study the dependence of the skyrmion's diameter on different system parameters such as the DMI parameter, of the DMI energy, the anisotropy parameter K , of the anisotropy energy, the exchange parameter A , of the exchange energy, the external field parameter, of an external magnetic field that is being applied on the system. At last, we insert a dimensionless parameter on the problem, the parameter $\epsilon$ that combines the DMI, the A and the K parameters studied before and we observe how it affects the diameter of the skyrmion. All this observations are being done by presenting graphs which we will discuss further on.


## $\Pi \varepsilon \rho i \lambda \eta \psi \eta$





 тó to Einstein de Hass Effect to 1915. H x $\alpha \tau \alpha v o ́ n o \eta ~ \alpha u \tau n ́ s ~ t n s ~ \sigma \chi 气 ́ \sigma \eta s ~ \vartheta \alpha ~ \mu \alpha \varsigma ~$

































## 1 Introduction

In classical physics, a rotating object posseses a property known as angular momentum. Angular momentum is a form of inertia, reflecting the objects size, shape, mass
and rotational velocity. It is represented as a vector $\mathbf{L}$ pointing along the axis of rotation.

Electrons have a "built in" angular momentum, called spin. Spin manifests itself, for example, in how a charged particle, such as an electron interacts with a magnetic field. The concept of spin was introduced in 1925 by Ralph Kronig, and independently by George Uhlenbeck and Samuel Goudsmit.

The total angular momentum $\mathbf{J}$ is the sum of the spin and the orbital angular momentum, $\mathbf{J}=\mathbf{L}+\mathbf{S}$. Conservation of angular momentum applies in general to $\mathbf{L}$ and may not apply separately to $\mathbf{L}$ or $\mathbf{S}$. That means that the spin-orbit interaction allows angular momentum to transfer between $\mathbf{L}$ and $\mathbf{S}$, while $\mathbf{J}$ remains constant.


Figure 1: (Left) "Spin" angular momentum $\mathbf{S}$ is really orbital angular momentum of the object at every point. (Right) Extrinsic orbital angular momentum $\mathbf{L}$ about an axis. (Bottom) Momentum p and its radial position r from the axis. The total angular momentum (spin plus orbital) is $\mathbf{J}$. For a quantum particle the interpretations are different; particle spin does not have the above interpretation.

Atomic and subatomic particles posses a corresponding property known as spin or spin angular momentum. Protons, neutrons, whole nucleus and electrons all possess spin and are often represented as tiny spinning balls (Figure 1). Although inaccurate, this is not a bad way to think about spin, as long as we don't take it to far. Several key differences should be recognized.

- The particle is not actually spinning. The reason for that is that, in order for
the spin to equal $S_{z}=\frac{\hbar}{2}$, the velocity of the electron should be much bigger then the velocity of the light. Indeed,

$$
\begin{equation*}
S_{z}=I \omega \approx m \alpha^{2} \omega \approx m u=\frac{\hbar}{2} \approx \hbar \Rightarrow u \approx \frac{\hbar}{m \alpha} \approx 10^{17} \frac{\mathrm{~cm}}{\mathrm{sec}} \approx 10^{6} c \tag{1}
\end{equation*}
$$

where $\alpha \approx 10^{-17} \mathrm{~cm}$ the experimental upper bound of the radius of the electron.

- Spin, like mass, is a fundamental property of nature and does not arise from more basic mechanisms.
- Spin interacts with electromagnetic fields whereas classical angular momentum L interacts with gravitational fields.
- The magnitude of a spin is quantised, meaning that it can only take on a limited set of discrete values. Unlike macroscopic angular momentum, spin can only be measured discrete integer or half integer units ( $0, \frac{1}{2}, 1, \frac{3}{2} \ldots$ ). Protons, neutrons and electrons all have spin equal to $\frac{1}{2}$. Nuclear spin is traditionally denoted by the letter I, and electron denoted by the letter $\mathbf{S}$. Electron spin has only one value $\left(\frac{1}{2}\right)$, while proton spin has values ranging from $I=0$ to $I=8$. Particles with half-integer spin like, protons and electrons, are called fermions. Particles with integer spin are called bosons.

In the subatomic world governed by quantum mechanics, nuclei are better though as "fuzzy" "probability waves" rather than "solid objects". Because of the Heisenberg Uncertainty Principle, we cannot know the exact direction of a particles spin at any point in time. However, we can measure and know with certainty some limited properties about the spin, such as the component of its angular momentum along a single direction. When a quantum property is potentially observable or measurable, it is known as an eigenstate or spin state.

The number of eigenstates or spin states for a nuclei with spin $\mathbf{I}$ is given by

$$
\text { Number of nuclear spin states }=2 I+1
$$

Hence for the ${ }^{1} \mathrm{H}$ nucleus with only one electron and $I=\frac{1}{2}$, there are $2\left(\frac{1}{2}\right)+1=2$ possible spin states. These states are commonly denoted by $\left\lvert\,+\frac{1}{2}>\right.$ and $\left\lvert\,-\frac{1}{2}>\right.$ often reffered to "spin up" and "spin down", respectively. In the absence of the magnetic field $\mathbf{B}$, the two spin states for the hydrogen are not observable and said to be degenerate. If an external magnetic field $\mathbf{B}_{0}$ is applied, however, a quantum field interaction occurs, allowing the two separable states to be measured and, as a result, observed.

## 2 Equation of motion for a spin

### 2.1 The magnetic moment

Now that we have made an introduction about what is a spin, we shall proceed, in this section, by constructing the equation of motion and studying the dynamics of one spin.

A key property determining the motion of a spin in a magnetic field is its magnetic moment. Once this is known, the motion of the magnetic moment and energy of the moment can be calculated. Actually, the spin of a particle with a charge and a mass leads to a magnetic moment. An intuitive way to understand the magnetic moment is to imagine a current loop lying in a plane. If the loop has a current $I$ around an elementary or vanishingly small oriented loop of area dS, then the magnetic moment equals:

$$
\begin{equation*}
d \boldsymbol{\mu}=I d \boldsymbol{S} \tag{2}
\end{equation*}
$$

and the magnetic moment has the units of $\mathrm{A} m^{2}$. The length of the vector $d \boldsymbol{S}$ is equal to the area of the loop. The direction of the vector is normal to the loop and in a sense determined by the direction of current around the elementary loop.


Figure 2: an elementary magnetic moment $d \boldsymbol{\mu}=I d \boldsymbol{a}$, due to an elementary current loop

This object is also equivalent to a magnetic dipole, so called because it behaves analogously to an electric dipole (two electric charges, one positive and one negative, separated by a small distance). It is therefore possible to imagine a magnetic dipole as an object which consists of two magnetic monopoles of opposite magnetic charge separated by a small distance in the same direction as the vector dS. The magnetic moment $d \boldsymbol{\mu}$ points normal to the plane of the loop of the current and therefore can be
either associated with the charge which is going around the loop. For a loop of finite size, we can calculate a magnetic moment $\boldsymbol{\mu}$ by summing up the magnetic moments of lots of equal infinitesimal current loops distributed throughout the area of the loop.
(b)


Figure 3: a magnetic moment $\boldsymbol{\mu}=\int d \boldsymbol{\mu}=I \int d \boldsymbol{S}$ (now viewed from above the plane of the current loop) associated with a loop of current I can be considered by summing up the magnetic moments of the lots of infinitesimal current loops

All the currents from neighbouring infinitesimal loops cancel, leaving only a current running around the perimeter of the loop. Hence,

$$
\begin{gathered}
\boldsymbol{\mu}=\int d \boldsymbol{\mu}=I \int d \boldsymbol{S} \Rightarrow \\
\Rightarrow \boldsymbol{\mu}=I A \hat{n}
\end{gathered}
$$

A current loop occurs because of the motion of one or more electrical charges. All the charges that we will consider are associated with particles that have mass. Therefore there is also orbital motion of mass as well as charge in all the current loops and hence, the magnetic moment is always connected with the angular moment.

We have $\mu=I A$, where I is the current defined by $I=\frac{q}{t}$ and A is the area enclosed by the current, defined by $A=\pi r^{2}$ and q the charge. So $\mu=\frac{q}{t} \pi r^{2}$. If $t=\frac{2 \pi r}{u}$ is the time of one rotation and $m$ the mass of the particle, we obtain:

$$
\begin{equation*}
\mu=\frac{q u}{2 \pi r} \pi r^{2} \Rightarrow \mu=\frac{q u}{2 \pi r m} m \pi r^{2} \tag{3}
\end{equation*}
$$

But the angular momentum is $\mathbf{L}=\mathbf{r} \times \mathbf{p}$ or $\mathbf{L}=m \mathbf{u} \cdot \mathbf{r}$. So

$$
\mu=\frac{q}{2 m} L
$$

We set $\gamma=\frac{q}{2 m}$, the gyromagnetic ratio of the particle, and we have $\mu=\gamma L$.
Now the question is why this also holds true for the vectors. The more general case is that of a volume current $\mathbf{J}$ in some finite region of space. In this case, the general formula for the magnetic dipole moment of the configuration is

$$
\begin{equation*}
\boldsymbol{\mu}=\frac{1}{2} \int \mathbf{r} \times \mathbf{J} d^{3} r \tag{4}
\end{equation*}
$$

If we further assume that the current density is due to a number of particles with number density $n$, charge q , velocity $\mathbf{v}$ and mass m , then we have current density $\mathbf{J}=n q \mathbf{v}$. So, from (4) we obtain:

$$
\begin{gathered}
\boldsymbol{\mu}=\frac{1}{2} \int \mathbf{r} \times n q \mathbf{v} d^{3} r \Rightarrow \\
\Rightarrow \boldsymbol{\mu}=\frac{q}{2 m} \int \mathbf{r} \times n m \mathbf{v} d^{3} r
\end{gathered}
$$

But $\mathbf{v} \rho=n m \mathbf{v}$, where $\rho$ the mass density, thus the above integral can be written as:

$$
\begin{gathered}
\boldsymbol{\mu}=\frac{q}{2 m} \int \rho \mathbf{r} \times \mathbf{v} d^{3} r \Rightarrow \\
\Rightarrow \boldsymbol{\mu}=\frac{q}{2 m} \mathbf{L}
\end{gathered}
$$

### 2.2 Angular momentum and gyromagnetic ratio

In atoms, the magnetic moment $\boldsymbol{\mu}$ associated with an orbiting electron lies along the same direction as the angular momentum $\mathbf{L}$ of the electron and is proportional to it.

Thus we write

$$
\begin{equation*}
\boldsymbol{\mu}=\gamma \mathbf{L} \tag{5}
\end{equation*}
$$

where $\gamma$ is a constant known as gyromagnetic ratio. The gyromagnetic ratio of a particle or system is the ratio of its magnetic moment to its angular momentum. Its S.I. unit is the radian per second per tesla ( $\mathrm{rad} \cdot \mathrm{sec}^{-1} \cdot \mathrm{~T}^{-1}$ ). The gyromagnetic ratio can be negative or positive. The sign of the gyromagnetic ratio determines the sense of precession, which we'll discuss later on this chapter. In Figure 4, we observe that while the magnetic moment (green line) shown there, are oriented the same for both cases of $\gamma$,the spin angular momentum (black line) are in opposite directions. Spin and magnetic moment are oriented in the same direction for $\gamma>0$. This relation between the magnetic moment and the angular moment is demonstrated by Einstein de Hass effect, discovered in 1915, in which a ferromagnetic rod is suspended vertically, along its axis, by a thin fibre (Figure 5). It is initially in rest and unmagnetized, and is subsequently magnetized along its length by the application of a vertical magnetic field. This vertical magnetization is due to the alignment of the atomic magnetic moments and corresponds to net angular momentum. To conserve total angular momentum, the rod begins turning about its axis in the opposite sense. If the angular momentum of the rod is measured, the angular momentum associated with the atomic magnetic moments, and hence the gyromagnetic ratio, can be deduced. The Einstein de Hass effect is a rotation induced by magnetization, but there is also the reversed effect, known as Barnett Effect in which magnetization is induced by rotation. Both phenomena demonstrate the magnetic moments associated with angular momentum.


Figure 4: Gyromagnetic ratio and preccesion


Figure 5: The Einstein-de Haas effect. A ferromagnetic rod is suspended from a thin fibre. A coil is used to provide a magnetic field which magnetizes the ferromagnet and produces a rotation. The experiment can be done resonantly, by periodically reserving the current in the coil, and hence the magnetization in the ferromagnet, and observing the angular response as a function of frequency.


Figure 6: Hydrogen Atom

Before we proceed further, we shall perform a quick calculation to estimate the size of atomic magnetic moments and thus deduce the size of gyromagnetic ratio $\gamma$. Let us consider an electron performing a circular orbit around the nucleus of a hydrogen atom, as shown in the figure. The current $\mathbf{I}$ around the atom is $\mathbf{I}=-\frac{e}{\tau}$, where $\tau=\frac{2 \pi r}{v}$, is the orbital period, $v=|\mathbf{v}|$ is the speed and $\mathbf{r}$ is the radius of the circular orbit. The magnitude of the angular momentum $\boldsymbol{\mu}$ of the electron, $m_{e} v \mathbf{r}$ must equal $\hbar$ in the ground state so that the magnetic moment of the electron is

$$
\begin{equation*}
\boldsymbol{\mu}=\pi r^{2} \mathbf{I} \Rightarrow \mu=\frac{e \hbar}{2 m_{e}} \equiv-\mu_{B} \tag{6}
\end{equation*}
$$

where $\mu_{B}$ is the Bohr magneton, defined by

$$
\mu_{B}=\frac{e \hbar}{2 m_{e}}
$$

This is a convenient unit for describing the size of atomic magnetic moments and takes the value $9.274 \times 10^{24} \mathrm{Am}^{2}$. Note that sign of the magnetic moment in (4) is negative. Because of the negative charge of the electron, its magnetic moment is antiparallel to its angular momentum. The gyromagnetic ratio for the electron is $\gamma=-\frac{e}{2 m_{e}}$. The Larmor frequency is then $\omega_{L}=|\gamma| \mathbf{B}=\frac{e \mathbf{B}}{2 m_{e}}[2]$.

### 2.3 Equation of motion

Now we can relate the time derivative of the angular momentum $\mathbf{L}$ to the torque, in order to derive the equation of motion. In the presence of a magnetic field, the torque is the cross product of the magnetic moment $\boldsymbol{\mu}$ and the magnetic field that we apply, and we symbolize by $\mathbf{B}$. Because the angular momentum is proportional to the magnetic moment, we can write down a first order differential equation of the magnetic moment which describes the motion of one spin

$$
\begin{equation*}
\frac{d \boldsymbol{\mu}}{d t}=\gamma \boldsymbol{\mu} \times \mathbf{B} \tag{7}
\end{equation*}
$$

It's easy to see how we conclude this equation. We proved on the previous section that the magnetic moment $\boldsymbol{\mu}$ and the angular momentum $\mathbf{L}$ are propotional.

In this point, let us define the torque. For this we need to talk about the precession [2].

Again, we consider a magnetic moment $\boldsymbol{\mu}$ in a magnetic field $\mathbf{B}$ as shown in Figure 7. The energy of the magnetic moment is given by

$$
E=-\boldsymbol{\mu} \cdot \mathbf{B}
$$

so the energy is minimized when the magnetic moment is aligned with the magnetic field. We define a torque with magnitude

$$
\begin{equation*}
\boldsymbol{\tau}=\frac{\partial E}{\partial \psi}=\boldsymbol{\mu} \mathbf{B} \sin \psi=\boldsymbol{\mu} \times \mathbf{B} \tag{8}
\end{equation*}
$$

The direction of the torque should be perpendicular to both $\boldsymbol{\mu}$ (as its magnitude should remain constant) and $\mathbf{B}$ as the energy, and thus the angle $\psi$ between them, should be conserved. The sign of the energy is chosen so as the force derived from the energy be $F=-\nabla E$. Also, $\frac{d \mathbf{L}}{d t}=\sum \boldsymbol{\tau}$. So, we obtain $\frac{d \mathbf{L}}{d t}=\boldsymbol{\tau}=\boldsymbol{\mu} \times \mathbf{B}$.

Before we proceed further, we have to notice something about the magnetic moment $\boldsymbol{\mu}$. From (7) we understand that the change in $\boldsymbol{\mu}$ over time is perpendicular to both $\boldsymbol{\mu}$ and $\mathbf{B}$. Rather than turning $\boldsymbol{\mu}$ towards $\mathbf{B}$, the magnetic field causes the direction of $\boldsymbol{\mu}$ to precess around $\mathbf{B}$. This equation also implies that $|\boldsymbol{\mu}|$ is time-independent. Let us imagine a top spinning with its axis inclined to the vertical. The weight of the top, acting downwards, exerts a (horizontal) torque on the top. If it were not spinning it would just fall over. But because it is spinning, it has angular momentum parallel to its spinning axis, and the torque causes the axis of the spinning top to move parallel to the torque, in a horizontal plane. The spinning top precesses.

## B



Figure 7: A magnetic moment $\boldsymbol{\mu}$ in a magnetic field $\mathbf{B}$ has energy equal to $-\boldsymbol{\mu} \cdot \mathbf{B}=-\mu B \cos \theta$

We will solve the equation of motion for the case $\mathbf{B}=(0,0, B)[2]$. We have the system of differential equations

$$
\begin{align*}
& \frac{d \mu_{x}}{d t}=\gamma \mu_{y} B \\
& \frac{d \mu_{y}}{d t}=-\gamma \mu_{x} B  \tag{9}\\
& \frac{d \mu_{z}}{d t}=0
\end{align*}
$$

From the first two equations of (9) we obtain

$$
\ddot{\mu}_{x}+\gamma^{2} B^{2} \mu_{x}=0
$$

with solution

$$
\mu_{x}=c_{1} \cos \gamma B t+c_{2} \sin \gamma B t
$$

for $c_{1}$ and $c_{2}$ arbitrary constant values. So from the second equation, we obtain the solution

$$
\mu_{y}=c_{3} \sin \gamma B t+c_{4} \cos \gamma B t
$$

for $c_{3}$ and $c_{4}$ arbitrary constant values. If we assume that $\mu_{x}(0)=0=\mu_{y}(0)$ we take

$$
\mu_{x}=c_{1} \cos \gamma B t
$$

and

$$
\mu_{y}=c_{3} \sin \gamma B t
$$

Now, a magnetic moment $\boldsymbol{\mu}$ in a magnetic field $\mathbf{B}$ precesses around the magnetic field at the Larmor precession frequency, $\gamma \mathbf{B}$. The magnetic field $\mathbf{B}$ lies among the
z-axis and the magnetic moment is initially in the xz-plane at an angle $\theta$ to $\mathbf{B}$. The magnetic moment precesses around a cone at semi-angle $\theta$. Therefore, we take

$$
\mu_{z}(t)=|\boldsymbol{\mu}| \cos \theta
$$

where $|\boldsymbol{\mu}|=\sqrt{\mu_{x}^{2}+\mu_{y}^{2}+\mu_{z}^{2}}$. Finally we have the solution for the equation of motion

$$
\begin{align*}
\mu_{x} & =|\boldsymbol{\mu}| \sin \theta \cos \gamma B t \\
\mu_{y} & =|\boldsymbol{\mu}| \sin \theta \sin \gamma B t  \tag{10}\\
\mu_{z} & =|\boldsymbol{\mu}| \cos \theta
\end{align*}
$$

Now if we assume an atom with a magnetic moment (due to the spin of an electron) $\boldsymbol{S}=\left(S_{1}, S_{2}, S_{3}\right)$ in an external field $\boldsymbol{h}=(0,0, h)$. The equation of motion is

$$
\begin{equation*}
\frac{d \boldsymbol{S}}{d t}=\boldsymbol{S} \times \boldsymbol{h} \tag{11}
\end{equation*}
$$

We will prove that $\boldsymbol{S} \cdot \boldsymbol{S}=S_{1}^{2}+S_{2}^{2}+S_{3}^{2}=$ const., that is, the equation preserves the length $|\boldsymbol{S}|$. Indeed

$$
\frac{d}{d t}(\boldsymbol{S} \cdot \boldsymbol{S})=\boldsymbol{S} \cdot \frac{d \boldsymbol{S}}{d t}+\frac{d \boldsymbol{S}}{d t} \cdot \boldsymbol{S}=\boldsymbol{S} \cdot(\boldsymbol{S} \times \boldsymbol{h})+(\boldsymbol{S} \times \boldsymbol{h}) \cdot \boldsymbol{S}=0
$$

## 3 A spin chain

In this section we will discuss the magnetic interaction which can be important in allowing the magnetic moments in a solid to communicate with each other and potentially to produce long range order. There are different types of magnetic interaction such as direct exchange, indirect exchange in ionic solids (superexchange), indirect exchange in metals, double exchange and anisotropic exchange interaction. In this section we will discuss the exchange and the anisotropy interaction.

### 3.1 Exchange Interaction

Exchange Interactions lie at the heart of the phenomenon of long range magnetic order. The exchange effect is subtle and not a little mysterious, since it seems surprising that one has to go to the bother of thinking about exchange operators and identical particles when all one is dealing with is a bar magnet and a pile of iron filings. Exchange interactions are nothing more than electrostatic interactions, arising because charges of the same sign cost energy when they are closer together and save energy when they are apart.

We consider a simple model with just two electrons which have spatial coordinates $r_{1}$ and $r_{2}$, respectively. The wave function for the joint state can be written as a product of single electron states, so that if the first electron is in state $\psi_{\alpha}\left(r_{1}\right)$ and the second electron is in state $\psi_{\beta}\left(r_{2}\right)$, then the joint wave function in $\psi_{\alpha}\left(r_{1}\right) \psi_{\beta}\left(r_{2}\right)$. However, this product state does not obey exchange symmetry, since if we exchange the two electrons, we get $\psi_{\alpha}\left(r_{2}\right) \psi_{\beta}\left(r_{1}\right)$, which is not a multiple of what we started with. Therefore, the only states which we are allowed to make are symmetrized or antisymmetrized product states which behave properly under the operation of particle exchange.

For electrons, the overall wave function must be antisymmetric so the spin part of the wave function must either be an antisymmetric singlet state $x_{S}(S=0)$, in the case of a symmetric spatial state and a symmetric triplet state $x_{T}(S=1)$, in the case of an antisymmetric spatial state. Therefore, we can write the wave function for the singlet case $\Psi_{S}$ and the triplet case $\Psi_{T}$ as

$$
\begin{align*}
\Psi_{S} & =\frac{1}{\sqrt{2}}\left[\psi_{\alpha}\left(r_{1}\right) \psi_{\beta}\left(r_{2}\right)+\psi_{\alpha}\left(r_{2}\right) \psi_{\beta}\left(r_{1}\right)\right] \\
\Psi_{T} & =\frac{1}{\sqrt{2}}\left[\psi_{\alpha}\left(r_{1}\right) \psi_{\beta}\left(r_{2}\right)-\psi_{\alpha}\left(r_{2}\right) \psi_{\beta}\left(r_{1}\right)\right] \tag{12}
\end{align*}
$$

The energies of the two possible states are

$$
\begin{aligned}
& E_{S}=\int \Psi_{S}^{*} H \hat{\Psi}_{S} d r_{1} d r_{2} \\
& E_{T}=\int \Psi_{T}^{*} H \hat{\Psi}_{T} d r_{1} d r_{2}
\end{aligned}
$$

with the assumption that the spin parts of the wave function $\Psi_{S}$ and $\Psi_{T}$ are normalized. The difference between the two energies is

$$
E_{S}-E_{T}=2 \int \Psi_{\alpha}^{*}\left(r_{1}\right) \hat{H} \Psi_{\alpha}\left(r_{2}\right) \Psi_{\beta}^{*}\left(r_{2}\right) \Psi_{\beta}\left(r_{1}\right) d r_{1} d r_{2}
$$

Hence the effective Hamiltonian describing the system is

$$
\hat{H}=\frac{1}{4}\left(E_{S}+3 E_{T}\right)-\left(E_{S}-E_{T}\right) S_{1} \cdot S_{2}
$$

The exchange constant or exchange integral $J$ is defined as

$$
J=\frac{E_{S}-E_{T}}{2}=\int \Psi_{\alpha}^{*}\left(r_{1}\right) \hat{H} \Psi_{\alpha}\left(r_{2}\right) \Psi_{\beta}^{*}\left(r_{2}\right) \Psi_{\beta}\left(r_{1}\right) d r_{1} d r_{2}
$$

and hence we may omit the constant term in the Hamiltonian and write the spindependent term as

$$
\hat{H}=-2 J \boldsymbol{S}_{1} \cdot \boldsymbol{S}_{2} .
$$

We consider the following cases.

- If $J>0$ then $E_{S}>E_{T}$ and the triplet state $S=1$ is favoured.
- If $J<0$ then $E_{S}<E_{T}$ and the singlet state $S=0$ is favoured.

It was recognized in the early days of quantum mechanics that interactions such as in the equations before, probably apply between all neighbouring atoms. This motivates the Hamiltonian of the Heisenberg model

$$
\begin{equation*}
\hat{H}=-\sum_{i j} J_{i j} \boldsymbol{S}_{i} \cdot \boldsymbol{S}_{j} \tag{13}
\end{equation*}
$$

where $J_{i j}$ is the constant between the $i^{\text {th }}$ and the $j^{t h}$ spins. The factor of 2 is omitted because the summation includes each pair of spins twice. Another way of writing the above equation is the following

$$
\hat{H}=2 \sum_{i>j} J_{i j} \boldsymbol{S}_{i} \cdot \boldsymbol{S}_{j}
$$

where the $i>j$ avoids the "double counting" and hence the factor of two returns. Often, it is a good approximation to take $J_{i j}$ to be equal to a constant $J$ for nearest neighbours and Eq. (13) can be written in the form

$$
\begin{equation*}
E_{e x}=-J \sum_{i} \boldsymbol{S}_{i} \cdot \boldsymbol{S}_{i+1} \tag{14}
\end{equation*}
$$

and to be zero otherwise. If the two electrons are on the same atom the exchange integral is usually positive. This stabilizes the triplet state, and ensures an antisymmetric spatial state which minimizes the Coulomb repulsion between the two electrons by keeping them apart. This is consistent with Hund's first rule.

When the electrons are on neighbouring atoms, the situation is very different. Any joint state will be a combination of a state centred on one atom and a state centred on the other atom. It is worth remembering that the energy of a particle in a one dimensional box of length L is proportional to $L^{-2}$. This is a kinetic energy and hence demonstrates that there is a large kinetic energy associated with being squized into a small box. The electrons therefore can save kinetic energy by forming bonds because this allows them to wander around both atoms rather than just one. The correct states to consider now are not atomic orbitals but molecular orbitals. These can be bonding or spatially symmetric, or andibonding-spatially antisymmetric, with the andibonding orbitals more energetically costly. This is because the antibomding orbital has a greater curvature and hence a larger kinetic energy. This favours singlet (antisymmetric) states and the exchange integral, therefore, is expected to be negative [2].

### 3.2 Anisotropic Exchange Interaction

Let us start by discussing an interaction between electrons, this is the Indirect Exchange in ionic Solids or otherwise Superexchange, or even Kramers-Anderson Superexchange.

Superexchange or Kramers-Anderson Superexchange is the strong antiferromagnetic coupling between two next to nearest neighbour cations through a non magnetic anion. It differs from direct exchange, where there is a coupling between nearest neighbour cations not involving an intermediary anion. Superexchange is the result of the electrons having come from the same donor atom and being coupled with the receiving ions' spins. If the two next to nearest neighbour positive ions are connected at 90 degrees to the bridging non magnetic anion, then the interaction can be a ferromagnetic interaction.

In the Anisotropic Exchange Interaction it is the spin orbit interaction that plays the role of the non magnetic anion, in the Superexchange interaction. Furthermore, in this type of interaction, the excited state is not connected with the non magnetic anion but it is produced by the spin orbit interaction in one of the magnetic ions. There is then an exchange interaction between the excited state of one anion and the ground state of the other ion. This is what we call as anisotropic exchange interaction or Dzyaloshinsky-Moriya interaction (DM).

When acting between two spins $S_{1}$ and $S_{2}$ it leads to a term in the Hamiltonian:

$$
\begin{equation*}
\hat{H}_{D M}=-\mathbf{D} \cdot\left(\boldsymbol{S}_{\mathbf{1}} \times \boldsymbol{S}_{\mathbf{2}}\right) \tag{15}
\end{equation*}
$$

The vector $\mathbf{D}$ vanishes when the crystal field has an inversion symmetry with respect to the center between the two magnetic ions. However, in general D may not vanish and then will lie parallel or perpendicular to the line connecting the two spins, depending on the symmetry. The form of the interaction is such that it tries to force $S_{1}$ and $S_{2}$ to be at right angles in a plane perpendicular to the vector $\mathbf{D}$ in such an orientation as to ensure that the energy is negative. Its effect is therefore very often to slightly rotate the spins by a small angle. It commonly occurs in antiferromagnets and then in a small ferromagnetic component of the moments which is produced perpendicular to the spin axis of the antiferromagnet. The effect is known as weak ferromagnetism.

It's interesting to compare the Hamiltonian of the Heisenberg model (13) with equation (15) that we derive above. Hence, let's consider the Hamiltonian of the symmetric Heisenberg exchange Interaction (13) which contains the dot product of the two spins $S_{1}$ and $S_{2}$. The energy is lowest if they have a collinear orientation (parallel for ferromagnetic case, antiparallel for the antiferromagnetic case with the sign of J reversed). Any deviation from the parallel (antiparallel, respectively) configuration is associated with an energy "penalty". So, if we suppose $S_{1}$ is fixed and $S_{2}$ has an
angle with $S_{1}$ we will see that the energy penalty remains the same, regardless if the deviation from collinearity is to this side or the other, hence, the interaction is symmetric. On the other hand, we see the DM interaction. The Hamiltonian (15) contains the cross product $S_{1} \times S_{1}$ which is a vector perpendicular to $S_{1}$ and $S_{2}$, times the DM vector D . This tells us that energy can be gained by introducing an angle between $S_{1}$ and $S_{2}$. However, this time, the deviation from collinearity has to be in the right direction while the opposite direction is energetically costly,thus one particular sense of spin rotation is favoured. So, as a result, DM interaction is an asymmetric interaction [2].

Now, we are ready to present the equation of motion for a spin chain.

### 3.3 Magnetocrystalline anisotropy

Due to the symmetries of the crystal structure of materials, the magnetic energy may depend on the orientation of the spin vectors in space. We model this effect by an additional energy term that is called the magnetocrystalline anisotropy (or simply, the anisotropy). For example,

$$
\begin{equation*}
E_{\text {an }}=K \sum_{i}\left[1-\left(S_{i, 3}\right)^{2}\right] \tag{16}
\end{equation*}
$$

where $S_{i, 3}$ is the third spin components at site $i$. This is called a single-ion anisotropy term because it depends only on the spin at each single site. It is an anisotropy term of the easy-axis type, that is, the third spin axis is energetically favoured [2].

### 3.4 Dynamics for a spin chain

The magnetic energy of the spin chain is the sum of the components that we show in the previous sections

$$
\begin{equation*}
E=E_{\mathrm{ex}}+E_{\mathrm{DM}}+E_{\mathrm{an}} . \tag{17}
\end{equation*}
$$

Let us assume a model with the exchange and anisotropy energy only. For the exchange energy, we obtain a minimum if all spins are aligned, $\boldsymbol{S}_{i}=\boldsymbol{S}_{0}$ where $\boldsymbol{S}_{0}=$ $\pm(0,0, s)$ is a constant vector. Indeed, if we assume that $\boldsymbol{S}_{\boldsymbol{i}} \| \boldsymbol{S}_{\boldsymbol{i + 1}}$ then, from the definition of the dot product we get: $\boldsymbol{S}_{\boldsymbol{i}} \cdot \boldsymbol{S}_{\boldsymbol{i + 1}}=\left|\boldsymbol{S}_{\boldsymbol{i}}\right| \cdot\left|\boldsymbol{S}_{\boldsymbol{i + 1}}\right| \cos \theta$, where $\theta$ is the angle between $\boldsymbol{S}_{\boldsymbol{i}}$ and $\boldsymbol{S}_{\boldsymbol{i + 1}}$, and we have that $\theta=0$. Therefore, we have that $\boldsymbol{S}_{\boldsymbol{i}} \cdot \boldsymbol{S}_{\boldsymbol{i + 1}}=\left|\boldsymbol{S}_{\boldsymbol{i}}\right| \cdot\left|\boldsymbol{S}_{\boldsymbol{i + 1}}\right|$ and if we take the sum, for all i, we have that: $J \sum_{i}\left|\boldsymbol{S}_{\boldsymbol{i}}\right| \cdot\left|\boldsymbol{S}_{\boldsymbol{i + 1}}\right|$ is the maximum and $-J \sum_{i}\left|\boldsymbol{S}_{\boldsymbol{i}}\right| \cdot\left|\boldsymbol{S}_{\boldsymbol{i}+\boldsymbol{1}}\right|$ is the minimum $E_{e x}$ that we can obtain. The anisotropy energy has a minimum when $\boldsymbol{S}_{0}= \pm(0,0, s)$. It is not difficult to prove why this is right. We want the vector to preserve the length, i.e $\left|S_{0}\right|=s$. Specifically, we want the $S_{i, z}$, which is the zth component of the vector $\boldsymbol{S}_{\mathbf{0}}$. A possible choice could
be ( $\mathrm{s}, 0,0$ ) , a second one could be the vector $(0, \mathrm{~s}, 0)$ and a final one ( $0,0, \mathrm{~s}$ ). For the first two choices, we would get that $E_{a n}=0$. But, could that be lower than zero? We see that from the final choice we get $E_{a n}=-\kappa s^{2}<0$. But we haven't excluded all the possibilities for the vector $(0,0, \mathrm{~s})$. This is because, a vector is, also, determined by its direction. As a result, we obtain two possible choices: $(0,0, s)$ and $(0,0,-\mathrm{s})$. Eventually, for the choice $(0,0,-s)$, we get a minimum value for the $E_{\text {ex }}$. We conclude that the total energy has a minimum for the two uniform configurations $\boldsymbol{S}_{i}= \pm(0,0, s)$.

The equations of motion for the spins have the following form

$$
\begin{equation*}
\dot{\boldsymbol{S}}_{k}(t)=\boldsymbol{S}_{k} \times \frac{\partial E}{\partial \boldsymbol{S}_{k}}, \quad k=1,2, \ldots, N-1 . \tag{18}
\end{equation*}
$$

For the energy in Eq. (17), we calculate

$$
\frac{\partial E_{\mathrm{ex}}}{\partial \overrightarrow{\boldsymbol{S}_{\boldsymbol{i}}}}=\frac{\partial}{\partial \overrightarrow{\boldsymbol{S}_{\boldsymbol{i}}}}\left(-J \sum_{i} \overrightarrow{\boldsymbol{S}_{\boldsymbol{i}}} \cdot \overrightarrow{\boldsymbol{S}_{\boldsymbol{i}+\boldsymbol{1}}}\right)=-J\left(S_{k+1}+S_{k-1}\right)
$$

and

$$
\frac{\partial E_{a n}}{\partial \overrightarrow{\boldsymbol{S}}_{\boldsymbol{i}}}=\frac{\partial}{\partial \overrightarrow{\boldsymbol{S}}_{\boldsymbol{i}}}\left(-K \sum_{i}\left(s_{i, z}\right)^{2}\right)=-K \sum_{i} 2\left(s_{i, z}\right) \cdot \frac{\partial s_{i, z}}{\partial \overrightarrow{\boldsymbol{S}_{\boldsymbol{i}}}}=-2 K s_{i, z} .
$$

Finally, we substitute the results in the equation of motion (18) and we obtain

$$
\frac{\partial E}{\partial \overrightarrow{\boldsymbol{S}}_{\boldsymbol{i}}}=-\left[J\left(S_{k+1}+S_{k_{1}}\right)+2 K s_{i, z}\right]
$$

and

$$
\begin{equation*}
\dot{\boldsymbol{S}}_{k}(t)=J \boldsymbol{S}_{k} \times\left(\boldsymbol{S}_{k+1}+\boldsymbol{S}_{k-1}\right), \quad k=1,2, \ldots, N-1 . \tag{19}
\end{equation*}
$$

Let us prove that the uniform configuration $\overrightarrow{\boldsymbol{S}}_{\boldsymbol{i}}= \pm(0,0, s)$ satisfies the equation of motion. If we substitute the vector $\pm(0,0, s)$ in $\frac{\partial E}{\partial \bar{S}_{i}}=-\left(J\left(S_{k+1}+S_{k_{1}}\right)+2 K s_{i, z}\right)$, we obtain $\frac{\partial E}{\partial S_{i}}=-(2 J s+2 K s)$. So if we compute the cross product $\overrightarrow{\boldsymbol{S}}_{\boldsymbol{i}} \times \frac{\partial E}{\partial \overrightarrow{\boldsymbol{S}}_{\boldsymbol{i}}}$ we end up with

$$
\frac{d \overrightarrow{\boldsymbol{S}}_{i}}{d t}=0
$$

The energy is conserved under the equation of motion

### 3.5 Continuum form of the equation of motion

We will write the continuum form of the equation of motion for a spin chain, if we have the approximation $\frac{\vec{S}_{i+1}-\vec{S}_{i-1}}{\alpha^{2}} \approx \frac{\partial^{2} \vec{S}}{\partial x^{2}}$, where $\alpha$ is defined as the distance between
neighbours spins, and we define as $x=i \alpha$, the position of the $i^{\text {th }}$ spin and with $\overrightarrow{\boldsymbol{S}}(\boldsymbol{x})$ we define $\overrightarrow{\boldsymbol{S}}_{\boldsymbol{i}}=\overrightarrow{\boldsymbol{S}}\left(\boldsymbol{x}_{\boldsymbol{i}}\right)$ the continuum magnetic field.

We know that $\frac{d \boldsymbol{S}_{\boldsymbol{i}}}{d t}=\boldsymbol{S}_{\boldsymbol{i}} \times J\left(\boldsymbol{S}_{\boldsymbol{i - 1}}+\boldsymbol{S}_{\boldsymbol{i}+\mathbf{1}}\right)$. If we assume that $\alpha$ is small, and we write the Taylor expansion, for $x=i \alpha$, we obtain, for $\boldsymbol{S}_{\boldsymbol{i}}=\boldsymbol{S}\left(\boldsymbol{x}_{\boldsymbol{i}}, \boldsymbol{t}\right)$.

$$
\boldsymbol{S}_{i+1} \overrightarrow{\boldsymbol{1}}(\boldsymbol{t})+\boldsymbol{S}_{\boldsymbol{i}-\mathbf{1}}(\boldsymbol{t})=2 \boldsymbol{S}(\overrightarrow{\boldsymbol{x}}, \boldsymbol{t})+\alpha^{2} \frac{\partial^{2} \boldsymbol{S}(\overrightarrow{\boldsymbol{x}}, \boldsymbol{t})}{\partial x^{2}}+\ldots
$$

where the rest of the terms are of higher order and they are neglected. We obtain

$$
\frac{\left.\boldsymbol{S}_{i+\boldsymbol{1}}(\boldsymbol{t})+\boldsymbol{S}_{\boldsymbol{i - 1}} \overrightarrow{\mathbf{1}} \boldsymbol{t}\right)}{2 \alpha^{2}}=\frac{\boldsymbol{S}(\overrightarrow{\boldsymbol{x}}, \boldsymbol{t})}{\alpha^{2}}+\frac{1}{2} \frac{\partial^{2} \boldsymbol{S}(\overrightarrow{\boldsymbol{x}}, \boldsymbol{t})}{\partial x^{2}}
$$

Taking the limit $\alpha \rightarrow 0$, we get

$$
\frac{d \overrightarrow{\boldsymbol{S}}}{d t}=\overrightarrow{\boldsymbol{S}} \times J \overrightarrow{\boldsymbol{S}}+\overrightarrow{\boldsymbol{S}} \times \frac{\partial^{2} \overrightarrow{\boldsymbol{S}}}{\partial x^{2}} \Rightarrow \frac{d \overrightarrow{\boldsymbol{S}}}{d t}=\overrightarrow{\boldsymbol{S}} \times \frac{\partial^{2} \overrightarrow{\boldsymbol{S}}}{\partial x^{2}}
$$

### 3.6 Dissipative effects

The equation

$$
\begin{equation*}
\dot{\boldsymbol{S}}_{k}=\boldsymbol{S}_{k} \times \boldsymbol{F}_{k}+\frac{\alpha}{s} \boldsymbol{S}_{k} \times \dot{\boldsymbol{S}}_{k}, \quad \boldsymbol{F}=\frac{\partial E}{\partial \boldsymbol{S}_{k}} \tag{20}
\end{equation*}
$$

is dissipative and $\alpha$ is a dissipation parameter.
Let us write explicitly the time derivative for every spin

$$
\begin{equation*}
\dot{\boldsymbol{S}}_{k}=\frac{\alpha}{s} \sum_{k} \boldsymbol{S}_{k} \times\left(\boldsymbol{F}_{k}+\dot{\boldsymbol{S}}_{k}\right) \tag{21}
\end{equation*}
$$

because from the properties of the cross product we have that:

$$
\mathbf{A} \times(\mathbf{B}+\mathbf{C})=\mathbf{A} \times \mathbf{B}+\mathbf{A} \times \mathbf{C}
$$

we can prove that the energy is decreasing under the equation of motion.

$$
\begin{gathered}
\frac{\partial E}{\partial t}=\frac{\partial E}{\partial \boldsymbol{S}_{k}} \frac{\partial \boldsymbol{S}_{k}}{\partial t}=-\boldsymbol{F}_{k}\left(\frac{\partial \boldsymbol{S}_{k}}{\partial t}\right)=-\boldsymbol{F}_{k}\left(\boldsymbol{S}_{k} \times \boldsymbol{F}_{k}+\frac{\alpha}{s} \boldsymbol{S}_{k} \times \dot{\boldsymbol{S}}_{k}\right)= \\
\quad=-\boldsymbol{F}_{k}\left(\boldsymbol{S}_{k} \times \boldsymbol{F}_{k}\right)-\boldsymbol{F}_{k}\left(\frac{\alpha}{s} \boldsymbol{S}_{k} \times \dot{\boldsymbol{S}}_{k}\right)=-\frac{\alpha}{s} \boldsymbol{F}_{k}\left(\boldsymbol{S}_{k} \times \dot{\boldsymbol{S}}_{k}\right) \leq 0
\end{gathered}
$$

## 4 The Landau-Lifshitz equation

### 4.1 Solitons: An introduction

Let us continue by introducing a new term that is going to be very useful, especially on the second part of the thesis. We will make a quick introduction in particles named Solitons [4].

In the 1960s and early 1970s a novel approach to quantum field theory developed and became popular. Physicists and mathematicians began to seriously study the classical field equations as candidates for particles of the theory. These particles had not been recognized before, they are different from the elementary particles such as electrons and protons that arise from the quantization of the wave-like excitations of the fields. Their properties are largely determines by the classical equations, althought a systematic treatment of quantum corrections is possible.

A characteristic feature of the new, particle-like solutions is their topological structure, which differs from the vacuum. If one supposes the quantum excitations about the vacuum are associated with smooth deformations of the field, then such excitations do not change the topology. So the usual elementary particles of quantum field theory, such as the photon, have no topological structure. The new particles owe their stability to their topological distinctivness. Although they are often of large energy, they can not simply decay into a number of elementary particles.

In many cases, the topological character of the field is captured by a single integer N , called the topological charge. This is usually a topological degree, or generalized winding charge. The topological charge N can be identified as the net number of the new type of particle, with the energy increasing as $|N|$ increases. The basic particle has $N=1$. The minimal energy field confihuration with $N=1$ is a classically stable solution, as it can not decay into a topologically trivial field. The energy density is smooth, and concentraded in some finite region of space. Such a field configuration is called a topological soliton or just Soliton. The ending "-on" indicates the particlelike nature of the solution. There is usually a reflection symmetry reversing the sign of N , and hence there is an antisoliton with $N=-1$. Soliton and anti-soliton pairs can annihilate or be pair-produced. Field configurations with $N>1$ are interpreted as multi-soliton state. Sometimes it is energetically favourable for these to decay into N well seperated charge 1 solitons. Alternatively the can relax to a classical bound state of N solitons.

The interesting part is that solitons are classified, in field theory. But in order to understan this classification, and espacially to understand their stability, it is important to remember some concepts from topology. There are two basic techniques for classifying solitons in theories with scalar fields. The first is homotopy theory, and the second is topological degree theory, which can sometimes be used to calculate a
homotopy class. Topological degree is a special case of homology ideas. In gauge theory, the Chern numbers classify solitons.

### 4.2 Homotopy Theory

Let X and Y be two manifolds without boundary, and consider the continuous maps between them, $\psi: X \mapsto Y$. Often it is helpful to identify base points $x_{0} \in X$ and $y_{0} \in Y$ and require $\Psi\left(x_{0}\right)=y_{0}$. Then $\Psi$ is a based map. A based map $\Psi_{0}: X \mapsto Y$ is said to be homotopic to another such map $\Psi_{1}$ if $\Psi_{0}$ can be continuously deformed into $\Psi_{1}$. Precisly, $\Psi_{0}$ is homotopic to $\Psi_{1}$ if there is a continuous map $\ominus: X \times[0,1] \mapsto Y$, with "time" $\tau$ parametrizing the interval $[0,1]$, such that $\left.\ominus\right|_{\tau=0}=\Psi_{0}$ and $\left.\ominus\right|_{\tau=1}=\Psi_{1}$ and $\ominus\left(\xi_{I}, \tau\right)=\Psi_{0}$ for all $\tau$.
"Homotopic" is an equivalence relation because:

- it is symmetric: $\Psi_{0}$ homotopic to $\Psi_{1}$ implies that $\Psi_{1}$ is homotopic to $\Psi_{0}$, because the time flow can be reversed.
- it is transitive: $\Psi_{0}$ homotopic to $\Psi_{1}$ and $\Psi_{1}$ homotopic to $\Psi_{2}$ implies that $\Psi_{0}$ homotopic to $\Psi_{2}$, because time intervals can be adjoined and then rescaled.
- it is reflexive: obviously, $\Psi_{0}$ is homotopic to itself.

Thus the maps Psi can be classified into homotopy classes. One class is the constant class, consisting of the maps homotopic to the constant map $\Psi$ for which $\Psi_{x}=y_{0}$ for all x.

One can say more about homotopy classes if X is a sphere. The n -sphere $\mathcal{S}^{n}$ is the set of points in $R^{n+1}$ at unit distance from the origin. We shall be especially interested in the cases $S^{1}$ the circle, which is also the manifold of the group $\mathrm{U}(1)$, $S^{2}$ the usual sphere, adn $S^{3}$ the unit sphere in four dimensions, which is also the manifold of the group $\mathrm{SU}(2)$.

The set of homotopy classes of based maps $\Psi: S^{n} \mapsto Y$ is denoted by $\pi^{n}(Y)$. We take as base points the North pole $\mathbf{p}$ in $S^{n}$, that is the point $(0,0, \ldots \ldots, 0,1) \in R^{n+1}$, and some chosedn point $y_{0} \in Y$. (In $R^{2}$ the usual choise of a base point is $(1,0)$ ). For $n \geq 1$, the set $\pi_{n}(Y)$ forms a group, the n-th homotopy group of Y. The constraction, for $\pi_{1}(Y)$, is schematically as in figure below.

A map $S^{1} \mapsto Y$, and also its image in Y , is called a loop. $\Psi_{0}$ and $\Psi_{1}$ are two based loops in Y. Their composition $\Psi_{0} \cdot \Psi_{1}$ is the loop $\Psi$ obtained by following $\Psi_{0}$ by $\Psi_{1}$. The composition is associative,

$$
\begin{equation*}
\Psi_{0} \cdot\left(\Psi_{1} \cdot \Psi_{2}\right)=\left(\Psi_{0} \cdot \Psi_{1}\right) \cdot \Psi_{2} \tag{22}
\end{equation*}
$$



Figure 8: Sketch illustrating the composition of two maps involved in the construction of $\pi_{1}(Y)$
and each of these is the loop $\Psi_{0}$ followed by $\Psi_{1}$ followed by $\Psi_{2}$. The class of the constant map $S^{1} \mapsto y_{0}$ is the identity element of the group $\pi_{1}(Y)$. When composed with another map $\Psi$, the class of $\Psi$ is unchanged. The inverse of $\Psi$ is $\Psi$ traversed in the opposite direction, which composes with $\Psi$ to give a loop in the constant class.Note that $\pi_{1}$ is generally non-avelian, since the composition of loops $\Psi_{0} \cdot \Psi_{1}$ is not necessarily homotopic to $\Psi_{1} \cdot \Psi_{0} . \pi_{1}$ is known as the fundamental group of Y.

There are a lot more that can be discussed about this type of group but we are not going to talk about everything here.

In general, the conclusion is that homotopy theory can be applied directly to a scalar field theory of the type governed by the Lagrangian $\mathcal{L}=\frac{1}{2} \int_{X}\left(\partial_{0} \phi_{l} \partial_{0} \phi_{m} H^{(l m}\right)-$ $\left.h^{i j} \partial_{i} \phi_{l} \partial_{j} \phi_{m} H^{l m}\right) \sqrt{\text { deth }} d^{d} x$ where $\phi: R \times X \mapsto Y$ and locally the field is represented by $\phi(t, x)=\left(\phi_{1}(t, x), \ldots, \phi_{n}(t, x)\right),\left(\phi_{1}, . ., \phi_{n}\right)$ are coordinates on Y , and $H_{l m}\left(\phi_{1}, . ., \phi_{n}\right)$ a metric to define the Lagrangian above.

If the field (strongly) satisfies the dynamical field equation then it is continuous in space and time. The homotopy class is a topological, conserved quantity. Homotopy theory can also be applied to field theories defined in $R^{d}$, but here the boundary conditions play a crucial role [4].

### 4.3 Topological number

Topological degree [4] is a more limited, but also more refined tool than homotopy theory, and it allows the calculation of the homotopy class of a map in certain circumstances. It is useful because it occurs in various ways in field theories with solitons. Often, the topological aspect of a soliton is entirely captured by the degree of a map directly related to the soliton field. However, the more general homotopy theory ideas are in the background, and can be brought into action where necessary.

The topological degree is defined for a map $\Psi$ between two closed manifolds of the same dimension, $\Psi: X \mapsto Y$. Let $\operatorname{dim} \mathrm{X}=\operatorname{dim} \mathrm{Y}=\mathrm{d}$. Both X and Y must be oriented, and the map should be differentiable everywhere, with continuous derivatives. To avoid trivial difficulties, we suppose X is connected. We may as well suppose that Y is also connected since the image of X will always lie in one of the connected components of Y.

We need next to suppose that a normalized volume form $\Omega$ is defined on Y . Locally, this maps an oriented frame of tangent vectors at each point of $Y$ to the reals, and preferably the positive reals. If Y is a Riemannian manifold, the Hodge dual of a positive function on Y is such a volume form. The normalization condition is

$$
\begin{equation*}
\int_{Y} \Omega=1 \tag{23}
\end{equation*}
$$

Now consider $\Psi^{*}(\Omega)$, the pull-back of $\Omega$ to X using the map $\Psi$.In terms of local coordinates, if $\Omega=\beta(y) d y^{1} \wedge d y^{2} \wedge \wedge d y^{d}$, and $\Psi$ is represented by functions of $y(x)$, then

$$
\begin{equation*}
\Psi^{*}(\Omega)=\beta(y(x)) \frac{\partial \psi^{1}}{\partial x^{j}} d x^{j} \wedge \frac{\partial \psi^{2}}{\partial x^{k}} d x^{k} \wedge \ldots . \wedge \frac{\partial \psi^{d}}{\partial x^{l}} d x^{l}=\beta(y(x)) \operatorname{det}\left(\frac{\partial \psi^{i}}{\partial x^{j}}\right) d x^{1} \wedge d x^{2} \wedge \ldots \wedge d x^{d} \tag{24}
\end{equation*}
$$

Now define

$$
\begin{equation*}
\operatorname{deg} \Psi=\int_{X} \Psi^{*}(\Omega) \tag{25}
\end{equation*}
$$

This integral occurs naturally in various field theories. $\operatorname{deg} \Psi$ is called the topological degree of the map $\Psi$, and is an integer. The topological degree is a homotopy invariant of $\Psi$, simply because an integer can not change under a continuous deformation. It is also independent of the choice of $\Omega$, because the difference of two normalized volume forms on Y is a d-form whose integral is zero, and hence an exact form. The pullback of the difference is therefore exact on X , and integrates to zero. For example, for a map $\Psi: S^{1} \mapsto S^{1}$ the degree is equal to the winding number. This is verified by choosing the volume form $\frac{1}{2 \pi} d \theta$ on $S^{1}$, and noting that for the map given by the function $f(\theta)(25)$ reduces to

$$
\begin{equation*}
\operatorname{deg} \Psi=\frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{d f}{d \theta} d \theta=\frac{1}{2 \pi}(f(2 \pi)-f(0))=k \tag{26}
\end{equation*}
$$

We will not work specifically on this, but it is important to know that these basic concepts that were discussed above and more can help as investigate whether a particular field theory has a required topological structure for topological solitons to exist.

### 4.4 The Landau-Lifshitz Equation

Now we are ready to make a breef introduction on the Landau-Lifshitz (LL) Equation [3] and discuss those properties of very simple quantum magnetic systems which are within the scope of the macroscopic theory of magnetic solitons.

The principal assumption of the macroscopic theory of ferromagnetism is that the state of a magnetic crystal is unambiguously describable by the magnetization vector M , and thus the dynamics kinetics of a ferromagnet is dictated by variations in its magnetization.

Static and dynamical properties of the magnetization $\boldsymbol{M}(x, t)$ are described by the Landau-Lifshitz equation

$$
\begin{equation*}
\frac{\partial \boldsymbol{M}}{\partial t}=-\gamma \boldsymbol{M} \times \boldsymbol{F} \tag{27}
\end{equation*}
$$

The effective magnetic field $\boldsymbol{F}$ is equal to the variational derivative of the magnetic crystal energy with respect to the vector $\boldsymbol{M}$,

$$
\begin{equation*}
\boldsymbol{F}=-\frac{\delta E}{\delta \boldsymbol{M}} \tag{28}
\end{equation*}
$$

In the equation (27) we noticed the term

$$
\begin{equation*}
\boldsymbol{M} \times \boldsymbol{F}=0 \tag{29}
\end{equation*}
$$

where $\boldsymbol{F}$ is the effective field, as we defined in (28). This field is made up from four parts

- The externally applied magnetic field.
- The magnetic field due to the magnetisation of the medium.
- A magnetic field which represents the effect of anisotropy, the anisotropy field, $\boldsymbol{B}_{\alpha}$.
- A magnetic field which represents the effect of exchange, the exchange field, $\boldsymbol{B}_{e x}$.

The magnetic crystal energy $E$ is assumed to be a functional of $\mathbf{M}$ and its spatial derivatives,

$$
\begin{equation*}
E=\int w\left(\boldsymbol{M}, \frac{\boldsymbol{M}}{\partial x_{\kappa}}\right) d^{3} x \tag{30}
\end{equation*}
$$

where $\mathrm{i}, \mathrm{k}$ are the coordinate indices $(i, k=1,2,3)$. Equation (30) gives the magnetic energy of a three-dimensional crystal.

The magnetization vector obeys the local constraint $\boldsymbol{M}^{2}=$ const. This statement is consistent with the assumption that the length of the vector $\boldsymbol{M}$ in a ferromagnet represents its equilibrium characteristic. In the ground state the value of $\boldsymbol{M}$ coincides with the so-called spontaneous magnetization $\boldsymbol{M}_{\mathbf{0}}=\frac{2 \mu_{0} s}{\alpha^{3}}$, where $s$ is the atomic spin.

The quantum ferromagnetic theory proceeds from the formulation of the spin Hamiltonian for the magnetic material under consideration. We shall use as the quantum model of the magnetic material a system of localized electronic spins with the exchange Heisenberg Hamiltonian, as we've seen in previous section

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2} \sum_{n \neq m} J_{n m} \boldsymbol{S}_{n} \cdot \boldsymbol{S}_{m} \tag{31}
\end{equation*}
$$

where n and m are the vectorial numbers of the crystal lattice sites. $\boldsymbol{S}_{n}$ is the spin operator of the n-th site, and $J_{n m}$ are the so called exchange integrals. Below we shall only allow for the exchange interaction between nearest neighbours in the lattice. Assuming that the exchange integrals are positive, we obtain a model of an isotropic ferromagnet. If we do not include the assumption that the exchange integral in expression (47) is isotropic, we obtain the so-called XYZ model with the Hamiltonian

$$
\begin{equation*}
\mathcal{H}=-\frac{1}{2} \sum_{n, n_{0}}\left(J_{1} S_{n}^{x} S_{n+n_{0}}^{x}+J_{2} S_{n}^{y} S_{n+n_{0}}^{y}+J_{3} S_{n}^{z} S_{n+n_{0}}^{z}\right) \tag{32}
\end{equation*}
$$

where $n_{0}$ labels the nearest neighbours of each lattice site.
There are various ways to proceed from Hamiltonian (31) or (32) to the macroscopic theory. The simplest way is to assume the the spin $S$ is a classical vector and to formulate for it equations of motion which are consistent with Hamiltonians (31) and (32). From (32) we find that the interaction of the spins is equivalent to the action of a certain effective magnetic field $\boldsymbol{F}$ given by

$$
\begin{equation*}
\mathcal{F}=\frac{2 \mu_{0}}{\hbar} \sum_{n} S_{n} \cdot \boldsymbol{F}, \frac{2 m_{0}}{\hbar} \boldsymbol{F}^{\kappa}=-\frac{1}{2} \sum_{n_{0}}\left(S_{n+n_{0}}^{\kappa}+S_{n-n_{0}}^{\kappa}\right)=\frac{\partial \mathcal{F}}{\partial \mathcal{S}_{n}^{\kappa}} \tag{33}
\end{equation*}
$$

$\kappa=1,2,3$. The dynamics of the vector $\mathcal{S}$ in a magnetic field is described by the equation of motion

$$
\begin{equation*}
\hbar \frac{d S_{n}}{d t}=-2 \mu_{0} S_{n} \times \boldsymbol{F} \tag{34}
\end{equation*}
$$

Equations (32),(33),(34) enable a smooth transition to a macroscopic description of the dynamics of a ferromagnet in the long-wevelength approximation. In order to introduce a continuous description of the magnetization, we define the magnetic moment of a unit volume, $\boldsymbol{M}$, in terms of the average spin of the lattice site,

$$
\begin{equation*}
\boldsymbol{M}_{n}=-\frac{2 \mu_{0}}{\alpha^{3}}<S_{n}>=-\frac{2 \mu_{0}}{\alpha^{3}} S\left(x_{n}\right), \quad S_{n} \mapsto-\frac{\alpha^{3}}{2 \mu_{0}} \boldsymbol{M}\left(x_{n}\right) \tag{35}
\end{equation*}
$$

where $\alpha$ is the interaction distance.
If the angle between adjacent spin vectors is small, then the vectors $S(\mathbf{x})$ and $\boldsymbol{M}(\mathbf{x})$ can be regarded as continuous functions of the x coordinate, and we can write the following expansion:

$$
\begin{equation*}
\boldsymbol{M}_{n+n_{0}}=\boldsymbol{M}\left(x_{n}\right)+x_{i}\left(n_{0}\right) \frac{\partial \boldsymbol{M}}{\partial x_{i}}+\frac{1}{2} x_{i}\left(n_{0}\right) x_{\kappa}\left(n_{0}\right) \frac{\partial^{2} \boldsymbol{M}}{\partial x_{i} \partial x_{\kappa}}+\ldots \tag{36}
\end{equation*}
$$

### 4.5 Magnetic domain wall

Ferromagnetodynamics is the study of the way in which the magnetisation of a ferromagnet can be changed in both space and time. Experimentally, the subject really began just after 1930, when the idea of ferromagnetic domains was very new and very little was known about the way in which the magnetisation could change with time.

Around 1950, it became possible to discuss the experimental work from the point of view of existing theory. At the same time, the closely related fields of magnetic resonance and spin waves were undergoing rapid development. Magnetic resonance and spin waves involve changes in the magnetisation which are relatively small compared to the magnetisation itself and the time scales involved are usually shorter than $10^{-9} \mathrm{~s}$. Ferromagnetodynamics, on the other hand, is concerned with very large changes in the magnetisation, a complete reversal for example, occurring over distances which may be well under $10^{-6} \mathrm{~m}$ but the time scales involved are usually longer than $10^{-9} \mathrm{~s}$.

The concept of magnetic domains, which was introduced by Pierre Weiss in his famous paper on the molecular field theory of ferromagnetism (Weiss, 1907) was not taken up by experimentalists until Barkhausen published his work in 1919. This work showed that the magnetisation could change in a very discontinuous way, giving rise to the well known 'Barkhausen effect'. Two papers by van der Pol (1920), which can be found among his selected scientific papers (Bremmer and Bouwkamp, 1960) give considerable insight into the way in which the domain concept began to be introduced into ferromagnetism. The model was one in which the demagnetised state was a disordered array of very small regions, the domains, within the material and magnetisation due to an applied field involved these domains forming thread like chains, each domain in the thread having its magnetisation flipped around to point along the direction of the thread, which was the direction of the applied field.

Very strong experimental evidence for this model was given by de Waard (1927), who made calculations of the way in which the magnetisation of such ensembles of domains would vary with the applied field and then compared his models with experiment. The most surprising development of all was in 1931 when Francis Bitter published the results of his first 'Bitter pattern' observations (Erber and Fowler, 1969). The patterns were interpreted as further evidence for thread like domains.

Landau and Lifshitz (1935) introduced the idea that the magnetisation could change by a movement of the boundary between domains, that domains magnetised in the direction of the applied field would expand at the expense of domains magnetised against the applied field. We find this idea being taken up rapidly, first at a meeting in Göttingen in 1937 (Becker, 1938) and then by Kondorsky (1938) and by Brown (1939). However, as so often happens, and this is certainly the case in ferromagnetodynarnics today, the experimentalists were leading the theoreticians because it was not Landau and Lifshitz who were the common reference of the last three cited authors but Sixtus and Tonks and their experiments. These experiments mark the real beginning of ferromagnetodynamics.


Figure 9: The domain wall proposed by Landau and Lifshitz (1935). Between two domains in an infinite material having an easy direction along z. The elementary or atomic magnetic moments are of constant magnitude and rotate in $(x, z)$ as we go along y from one domain to the other

The paper by Landau and Lifshitz (1935), had not only introduced the idea of domain wall motion but had dealt with the atomic scale structure of a particularly simple domain wall. This is shown in figure 10 and represents the solution obtained by Landau and Lifshitz for the wall between two domains in a material which has only one preferred direction of magnetisation, the z-axis in figure 10. Landau and Lifshitz showed that one possible solution to the problem of the wall structure is that the atomic magnetic moments should rotate about the normal to the plane of the wall and that they should always lie in the plane of the wall as they rotate. The magnitude of the elementary magnetic moments is the same everywhere, the direction changes. The distance over which this rotation takes place depends, of course, upon the only material being considered but in the majority of materials the rotation is more or less
completed over about one hundred atomic spacings.
In order to understand the conclusions of Landau- Lifshitz, we should first study the experements that happened by Sixtus and Tonks.

These experiments are described in a series of five papers: Sixtus and Tonks (1931, 1932), Tonks and Sixtus (1933a, 1933b) and Sixtus (1935). Some further work was reported by Sixtus at the 1937 meeting in Göttingen (Becker,1938), which was referred to in the previous section.

The apparatus is shown in figure 11. This was centred around a nickel-iron alloy wire. The alloy was chosen so that when the wire was put in tension, the magnetostriction acted with the shape anisotropy of the wire to make the remnant state one single ferromagnetic domain with its magnetisation directed along the axis of the wire. This remnant state was achieved in the experiment by applying a large bias field, using the bias field coil shown. The bias field was then reduced, through zero, to be applied in the reverse direction but only increased to a small value so that it was below the value required to reverse the magnetisation of the wire by the spontaneous nucleation of a reverse domain.

Magnetic reversal of the wire was then achieved in a controlled way by passing a current through the nucleating field coil, shown in figure 11, so that the reverse bias field was increased in magnitude in that region of the wire. In later experiments this nucleation of a reverse, or seed, domain was done in a more definite manner by using a pulse of current through the nucleating field coil. The nucleated reverse domain then expanded, under the influence of the small applied bias field, and the propagation of one end of this expanding domain along the wire could be observed by means of the voltage pulses which were induced in the two pick-up coils shown in figure 11. By measuring the time difference between the two pulses, for various spacings and positions of the two coils, it was possible to work out the velocity of propagation of the domain boundary. The velocity was found to depend upon the applied bias field and the stress in the wire. It did not vary along the length of the wire. Some typical results are shown in figure 12. The striking point about the results shown in figure 12 is the very linear relationship between the velocity and the applied field. This does, however, only apply over a very small range of applied field because quite a large field, Bo, must be applied before any propagation can be observed. The maximum field that can be used is the one which would cause spontaneous reversal and this was found to be not very much greater than $B_{0}$. This linear relationship was expected by Sixtus and Tonks because they considered that the only resistance to the motion of the magnetic discontinuity would come from the coercivity of the material, which would explain $B_{0}$, and then from the eddy-current loss induced in the wire by the moving magnetic discontinuity which would give rise to a viscous damping term, linear in velocity.

In order to compare their experimental results with theory, Sixtus and Tonks


Figure 10: The apparatus used by Sixtus and Tonks (1931). The magnetisation of a nickel-iron wire reverses by the propagation of a domain boundary of the kind shown inset


Figure 11: Typical results copied from Sixtus and Tonks (1931) for a $14 / 86$, Ni/Fe, wire, $380 / \mathrm{Lm}$ in diameter, under tension. The tensile stress is shown
needed some model of the moving domain wall. This was inferred from the wave form of the voltage pulse induced in the pick-up coils as the domain wall passed through and the conclusion was that the boundary was a cone-like one, as shown inset in figure 11. We thus see that Sixtus and Tonks were proposing that a change in the magnetisation would take place by domain wall motion some years before the domain wall model had developed. In figure 11, the angle of the cone-like boundary has been exaggerated for clarity. The cone does, in fact, occupy a length of wire which is many times its diameter.

More detail may be seen if we consider a recent application of the technique developed by Sixtus and Tonks. O'Handley (1975) has used the technique to study domain wall kinetics in wire samples of the ferromagnetic glass $F_{76} P_{12} C_{7} C r_{4.5} B_{0.5}$ and his results are shown in figure 13. By observing the waveform of the voltage pulse induced in his pick-up coils, O'Handley was able to see that the length of the cone discontinuity was almost constant, independent of velocity and position, at 25 mm , over 200 times the diameter of the wire.

Figure 13 shows the same general features as figure 12. There is a very small positive intercept, Bo, when we extrapolate the linear velocity-applied field points back. The results can be expressed by the relationship

$$
\begin{equation*}
\nu_{n}=\mu_{w}\left(B-B_{0}\right) \tag{37}
\end{equation*}
$$

where $\nu_{n}$ is the velocity of the conical domain wall normal to its surface, B is the applied field, $B_{0}$ has been discussed above and $\mu_{w}$ is what we would now call a domain wall mobility. Equation 78 is going to come up again and again throughout this book because so many experimental results may be expressed in this way.


Figure 12: Results copied from O'Handley (1975) for a wire of the ferromagnetic glass Fe76P12C7Cr4.sBo.s, 115 ( Lm in diameter

The normal velocity of the wall, $\nu_{n}$, is, of course, much smaller than the velocity observed in the Sixtus and Tonks experiment because of the very small cone angle of the moving magnetic discontinuity. The value for $\mu_{w}$ in the case of O'Handley's experiments, shown in figure 13 is $2.7 \times 104(\mathrm{~m} / \mathrm{s})$ per T and he was able to obtain reasonable agreement with the theory of wall motion for insulating magnetic materials. Sixtus and Tonks were not able to get very good agreement with their own ideas for an eddy-current damping model but this was mainly because, while they had made the quite new and correct proposal that the magnetisation was changing by wall motion, they had not got any clear model for the structure of the wall itself [1].

### 4.6 Domain wall solution

Now we are ready to discuss the structure of the Landau-Lifshitz Wall [1]. Let's suppose that we have a picture frame sample, 1 mm thick and a few mm wide, and has a very narrow domain boundary between two domains which are both saturated along the $z$-axis, in opposite directions, this being the easy direction of magnetization.

In such a situation, there is no normal component of $\boldsymbol{M}$ at the surface of the sample, which we must imagine as either infinitely long or as a closed frame, except at the very small area where the domain wall meets in equation

$$
\begin{equation*}
\mathbf{H}=\left(\frac{1}{4 \pi}\right) \operatorname{grad}\left(\int_{\nu} \frac{\operatorname{div} \boldsymbol{M}}{r_{i \kappa}} d \nu+\int_{s} \frac{\boldsymbol{M} \cdot \nu}{r_{i \kappa}} d s\right) \tag{38}
\end{equation*}
$$



Figure 13: When the magnetization lies in-plane within the domains there is no surface divergence of $\boldsymbol{M}$ except over the small region where the domain wall cuts of surface.
from the equation of the magnetostatic field

$$
\begin{equation*}
\mathbf{B}=\mu_{0}(\mathbf{H}+\boldsymbol{M}) \tag{39}
\end{equation*}
$$

would be negligible.
If we know invoke the second condition imposed by Landau and Lifshitz, that the magnetization $\boldsymbol{M}$, rotates in $(x, z)$ with constant amplitude as we go along the $y$-axis from one domain into the other. The first integral in equation (38) then vanishes because $\operatorname{div} \boldsymbol{M}=0$.

Because $\boldsymbol{M}$ has only $x$ and $z$ components, so does the total field vector, $F$. These are given by

$$
\begin{equation*}
F_{x}=-\left(\frac{2 K_{u}^{2}}{M_{s}}\right) M_{x}+\left(\frac{2 A}{M_{s}^{2}}\right) \nabla^{2} M_{x}+\mu_{0} M_{x} \tag{40}
\end{equation*}
$$

and

$$
\begin{equation*}
F_{z}=\left(\frac{2 A}{M_{s}^{2}}\right) \nabla^{2} M_{z}+\mu_{0} M_{z} \tag{41}
\end{equation*}
$$

which follow from

$$
\begin{equation*}
\mathbf{B}_{e x}=\left(\frac{2 A^{2}}{\boldsymbol{M}_{s}}\right) \nabla^{2} \boldsymbol{M}, \quad \mathbf{B}=\mu_{0} \boldsymbol{M} \tag{42}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\mathbf{B}_{\alpha}\right)_{x}=-2 K_{u} \frac{M_{x}}{M_{s}^{2}}, \quad\left(\mathbf{B}_{\alpha}\right)_{y}=-2 K_{u} \frac{M_{y}}{M_{s}^{2}}, \quad\left(\mathbf{B}_{\alpha}\right)_{z}=0 \tag{43}
\end{equation*}
$$

We observe that equation (29) has on a $y$-component

$$
\begin{equation*}
\frac{A}{M_{s}^{2}}\left(M_{z} \frac{d^{2} M_{x}}{d y^{2}}-M_{x} \frac{d^{2} M_{z}}{d y^{2}}\right)-\frac{K_{u}}{M_{s}^{2}} M_{z} M_{x}=0 \tag{44}
\end{equation*}
$$

which is an ode because $\boldsymbol{M}$ is a function of y only. Because we assume that $\boldsymbol{M}$ has a constant magnitude, $M_{s}$ (which is called Saturation Magnetization and it is the maximum magnetic moment per unit volume of a magnetic field) and simply rotates in ( $\mathrm{x}, \mathrm{z}$ ) as we cross the wall, the components of $\boldsymbol{M}$ may be written

$$
\begin{align*}
& M_{x}=M_{s} \sin \theta  \tag{45}\\
& M_{z}=M_{s} \cos \theta \tag{46}
\end{align*}
$$

and substituted into equation (44) to give the very simple non-linear differential equation for $\theta$

$$
\begin{equation*}
\frac{d^{2} \theta}{d y^{2}}-\frac{K_{u}}{A} \sin \theta \cos \theta=0 \tag{47}
\end{equation*}
$$

In order to solve (47) we let $\frac{d \theta}{d y}=u$ and the variables separate in the equation to give

$$
\begin{equation*}
u d u=\left(\frac{K_{u}}{A}\right) \sin \theta \cos \theta \tag{48}
\end{equation*}
$$

Equation (48) integrates directly to give

$$
\begin{equation*}
u^{2}=\left(\frac{K_{u}}{A}\right) \sin ^{2} \theta+C_{1} \tag{49}
\end{equation*}
$$

and if we choose $y=0$ to be the central plane of the wall, the constant of integration, $C_{1}$ is zero. This follows because $\theta$ becomes constant an 0 or $\pi$, and $u=\frac{d \theta}{d y}$ tends to zero, as we move well into the magnetically satured domains which means that $y \mapsto \pm \infty$. Taking the square root of (49) to get $u=\frac{d \theta}{d y}$, the variables again separate to give

$$
\begin{equation*}
\frac{\left(\frac{A}{K_{u}}\right)^{2} d \theta}{\sin \theta}=d y \tag{50}
\end{equation*}
$$

and this equation integrates directly to give

$$
\begin{equation*}
y=\left(\frac{A}{K_{u}}\right)^{2} \ln \left( \pm \tan \frac{\theta}{2}\right)+C_{2} \tag{51}
\end{equation*}
$$

The choice of sign indicates that the domain wall may have either a clockwise or an anticlockwise screw-sense associates with the rotation of $\boldsymbol{M}$ in (x,z). That means
that $\theta$ may equal $\pm \frac{\pi}{2}$ at $y=0$. This latter boundary condition makes the constant of integration in equation (51) equal to zero and in the end the equation may be written

$$
\begin{equation*}
\pm \tan \left(\frac{\theta}{2}\right)=\exp \frac{y}{\Delta} \tag{52}
\end{equation*}
$$

where $\Delta=\left(\frac{A}{K_{u}}\right)^{\frac{1}{2}}$, the wall-width parameter. Finally, if we use the trigonometric relationships between $\cos \theta, \sin \theta$ and $\tan \theta$ we obtain

$$
\begin{equation*}
M_{z}=-M_{s} \tanh \frac{y}{\Delta}, \quad M_{x}= \pm M_{s} \operatorname{sech} \frac{y}{\Delta} \tag{53}
\end{equation*}
$$

to define the static structure of the Landau- Lifshitz domain wall. In the figure below we see the plot of the equation (53). We will descuss this again on a later section.


Figure 14: The variation of $\frac{M_{z}}{M_{s}}$ with $\frac{y}{\Delta}$ for the LL wall.
We observe that the LL wall occupies a width of about $2 \Delta$ and the rate of change of $M_{z}, \frac{d M_{z}}{d y}$, is $-\frac{M_{s}}{\delta}$ over quite a distance about the center of the wall.

In conclusion, we have given an analytical description of the LL domain wall and we found that indeed these kind of solutions could be expected in practice in the picture frame type sample shown in figure 14, where $\boldsymbol{M}$ lies in the plane of the sample.

### 4.7 Domain wall Solution - alternative method

We will solve, again analytically, the equation

$$
\begin{equation*}
\frac{\partial \boldsymbol{M}}{\partial t}=\boldsymbol{M} \times\left(\frac{\partial^{2} \boldsymbol{M}}{\partial x^{2}}+M_{z} \hat{\mathbf{e}}_{z}\right) \tag{54}
\end{equation*}
$$

applying a different method than before.

We will apply spherical coordinates,

$$
\begin{equation*}
M_{x}=\sin \Theta \cos \Phi \quad M_{y}=\sin \Theta \sin \Phi \quad M_{z}=\cos \Theta \tag{55}
\end{equation*}
$$

We obtain the following

$$
\begin{align*}
\frac{\partial M_{x}}{\partial t} & =\cos \Theta \cdot \Theta_{t} \cos \Phi-\sin \Theta \sin \Phi \cdot \Phi_{t} \\
\frac{\partial M_{y}}{\partial t} & =\cos \Theta \cdot \Theta_{t} \sin \Phi+\sin \Theta \cos \Phi \cdot \Phi_{t}  \tag{56}\\
\frac{\partial M_{x}}{\partial t} & =-\sin \Theta \cdot \Theta_{t}
\end{align*}
$$

Also
$\frac{\partial M_{x}}{\partial x}=\cos \Theta \cdot \Theta_{x} \cos \Phi-\sin \Theta \sin \Phi \cdot \Phi_{x} \frac{\partial M_{y}}{\partial x}=\cos \Theta \cdot \Theta_{x} \sin \Phi+\sin \Theta \cos \Phi \cdot \Phi_{x} \frac{\partial M_{z}}{\partial x}=-\sin \Theta \cdot \Theta_{x}$
and
$\frac{\partial^{2} M_{x}}{\partial x^{2}}=-\sin \Theta \cdot\left[\cos \Phi \cdot\left(\Theta_{x}^{2}+\Phi_{x}^{2}\right)+\sin \Phi \cdot \Phi_{x x}\right]+\cos \Theta \cdot\left[\Theta_{x x} \cdot \cos \Phi-2 \Theta_{x} \cdot \Phi_{x} \cdot \sin \Phi\right]$
$\frac{\partial^{2} M_{y}}{\partial x^{2}}=-\sin \Theta \cdot\left[-\sin \Phi \cdot\left(\Theta_{x}^{2}+\Phi_{x}^{2}\right)+\cos \Phi \cdot \Phi_{x x}\right]+\cos \Theta \cdot\left[\Theta_{x x} \cdot \sin \Phi+2 \Theta_{x} \cdot \Phi_{x} \cdot \cos \Phi\right]$
$\frac{\partial^{2} M_{z}}{\partial x^{2}}=-\cos \Theta \cdot \Theta_{x}^{2}-\sin \Theta \cdot \Theta_{x x}$
or

$$
\frac{\partial^{2} M_{z}}{\partial x^{2}}+S_{z} \cdot \hat{e_{z}}=-\cos \Theta \cdot\left[1-\Theta_{x}^{2}\right]-\sin \Theta \cdot \Theta_{x x}
$$

Substituting all the above equations in (54) we get

$$
\begin{equation*}
\Theta_{t}(x, t)=-\sin \Theta \cdot \Phi_{x x}-2 \cdot \cos \Theta \cdot \Theta_{x} \cdot \Phi_{x} \tag{60}
\end{equation*}
$$

and

$$
\begin{equation*}
\Phi_{t}(x, t)=\frac{1}{\sin \Theta} \cdot \Theta_{x x}-\cos \Theta \cdot\left[1+\Phi_{x}^{2}\right] \tag{61}
\end{equation*}
$$

Now let us assume $\Phi=0$. From Eq. (60), we get

$$
\begin{equation*}
\Theta_{t}(x, t)=0 \tag{62}
\end{equation*}
$$

and from (61), we get

$$
\begin{equation*}
\Theta_{x x}(x, t)=\cos \Theta \sin \Theta . \tag{63}
\end{equation*}
$$

Equation (63) is a second order nonlinear ODE of the form $\Theta^{\prime \prime}=f(\Theta)$. We will solve it by introducing the function

$$
\begin{equation*}
V(\Theta)=\Theta_{x}(x, t) \tag{64}
\end{equation*}
$$

such that

$$
\begin{equation*}
\Theta_{x x}(x, t)=\frac{d V}{d \Theta} \cdot V \tag{65}
\end{equation*}
$$

and (63) gives

$$
\begin{equation*}
V \cdot \frac{d V}{d \Theta}=\cos \Theta \cdot \sin \Theta \tag{66}
\end{equation*}
$$

This is a separable ODE with solution

$$
\begin{equation*}
\frac{V^{2}(\Theta)}{2}=\frac{\sin \Theta^{2}}{2}+C \Rightarrow V(\Theta)= \pm \sin \Theta+C \tag{67}
\end{equation*}
$$

Substituting (67) in (64), we get the ode

$$
\begin{equation*}
\Theta_{x}=\sin \Theta+C \tag{68}
\end{equation*}
$$

We are interested in localized solutions. These are obtained by requiring $C=0$. Then (68) is solved by separation of variables and we obtain

$$
\begin{equation*}
\pm \tan \frac{\Theta}{2}= \pm e^{k x} \tag{69}
\end{equation*}
$$

and if we keep the plus signs we observe that we get two cases:

- For $x \rightarrow-\infty$ we get $\Theta=0$, so the solution is localized on the north pole
- For $x \rightarrow \infty$ we get $\Theta=\pi$, so the solution is localized on the south pole For $\tan \frac{\Theta}{2}=-e^{k x}$, we will have the following cases:
- For $x \rightarrow-\infty$ we get $\Theta=0$, so the solution is localized on the north pole
- For $x \rightarrow+\infty$ we get $\Theta=-\pi$, so the solution is localized on the south pole

Also, for $\Theta=\frac{\pi}{2}$ we get $x=0$. So, we proved that there must be a magnetic wall that unites the points between $x=-\infty$ and $x=+\infty$.

In magnetism, a domain wall is an interface separating magnetic domains, as shown in the figure below. It is a transition between different magnetic moments and usually undergoes an angular displacement of $90^{\circ}$ or $180^{\circ}$. A domain wall is a gradual reorientation of individual moments across a finite distance. The domain wall


Figure 15: Domain wall (B) with gradual reorientation of the magnetic moments between two $180^{\circ}$ domains (A) and (C)
thickness depends on the anisotropy of the material, but on the average spans across around 100-150 atoms.

The energy of a domain wall is simply the difference between the magnetic moments before and after the domain wall was created and its value is expressed as energy per unit wall area.

The width of the domain wall varies due to the opposite energies that create it: the magnetocrystaline anisotropy energy and the exchange energy, both of which tend to be as low as possible so as to be in a more favorable energetic state. The anisotropy energy is lowest when the individual magnetic moments are aligned with the crystal lattice axes thus reducing the width of the domain wall. Conversely, the exchange energy is reduced when the magnetic moments are aligned parallel to each other and thus makes the wall thicker, due to the repulsion between them (where antiparallel alignment would bring them closer, working to reduce the wall thickness). In the end an equalibrium is reached between the two and the domain wall's width is set as such. An ideal domain wall would be fully independent of position, but the structures are not ideal and so get stuck on inclusion sites within the medium, also known as crystallographic defects. These include missing or different (foreign) atoms, oxides, insulators and even stresses within the crystal. This prevents the formation of the domain walls and also inhibits their propagation through the medium. Thus a greater applied magnetic field is required to overcome these sites.

As we have seen above, the magnetic domain walls are exact solutions to classical
nonlinear equations of magnets. We will discuss more on these type of equations on the following paragraphs.

### 4.8 The Landau-Lifshitz-Gilbert Equation

In section 2, we introduced the equation (5) because we wanted to relate the magnetic moment of a orbiting particle with his angular momentum, and the reason for that was that, in atoms, the magnetic moment $\boldsymbol{\mu}$ associated with an electron for example, lies along the same direction as the angular momentum $\mathbf{L}$ of the electron and is proportional to it. Later, we obtained the equation which described the motion of one spin $\frac{d \boldsymbol{\mu}}{d t}=\gamma \boldsymbol{\mu} \times \mathbf{B}$, where we remember that $\gamma$ is the gyromagnetic ratio, $\mathbf{B}$ is the magnetic field that we apply and $\boldsymbol{\mu}$ is the magnetic moment. Another way to see this equation is as $\frac{1}{\gamma} \frac{d \mu}{d t}=\boldsymbol{\mu} \times \mathbf{B}$. Let's try to read carefully this equation.

Equation of motion for one spin describes the uniform precession of the vector $\boldsymbol{M}$ about the magnetic field, as shown in figure 7, and does not represent the expected result that $\boldsymbol{M}$ and $\mathbf{B}$ should eventually become parallel to one another. For this reason, Landau Lifshitz proposed the idea of adding a second term to the right hand side of the equation $\frac{1}{\gamma} \frac{d \vec{\mu}}{d t}=\vec{\mu} \times \vec{B}$ which had the direction of $\mathbf{B}-\boldsymbol{M}$ and a magnitude which fell to zero when $\mathbf{B}$ and $\boldsymbol{M}$ became parallel. Such a vector may be formed as $\left(\mathbf{B}-(\mathbf{B} \cdot \boldsymbol{M}) \frac{M}{M_{s}^{2}}\right)$ and the complete Landau Lifshitz equation is written

$$
\begin{equation*}
\left(\frac{-1}{|\gamma|}\right) \frac{d \boldsymbol{M}}{d t}=(\boldsymbol{M} \times \mathbf{B})-\lambda\left[\mathbf{B}-(\mathbf{B} \cdot \boldsymbol{M}) \frac{\boldsymbol{M}}{M_{s}^{2}}\right] \tag{70}
\end{equation*}
$$

where $\lambda$ is a positive constant having the same dimensions as $\boldsymbol{M}$. Using $\gamma$ instead of $-|\gamma|$, we would have to make $\lambda$ negative when $\gamma$ positive. The final results would not depend upon the sign of $\gamma$, this simply determines the sense of the precession, and we shall avoid this complication.

Equation (70) represents $\boldsymbol{M}$ spiralling in toward $\mathbf{B}$ and eventually becoming parallel to $\mathbf{B}$, whereupon both terms on the right hand side of the equation are zero. The second term which has been added is thus a damping term upon the previous purely precessional motion. Another form of the LL equation, found at a parer of Galt (1952) is

$$
\begin{equation*}
\frac{d \boldsymbol{M}}{d t}=\gamma(\boldsymbol{M} \times \mathbf{B})-\frac{\lambda}{M_{s}^{2}}(\boldsymbol{M} \times \boldsymbol{M} \times \mathbf{B}) \tag{71}
\end{equation*}
$$

and this is the equation which is usually referred to as the Landau Lifshitz equation. This equation has the advantage of making the equation apply for both signs of $\gamma$ but the disadvantage of obscuring the condition $\lambda \ll M_{s}$, which Landau Lishitz underlined as essential to their original equation (70). The condition is $\lambda \ll M_{s}$ implied by the Landau and Lifshitz model in which the magnitude of $\boldsymbol{M}$ is constant
and the damping comes from a very weak relativistic interaction. If $\lambda$ no longer has the same dimensions as $\boldsymbol{M}$ a comparison of their magnitudes is obscure.

Later on, Gilbert and Kelly (1955) proposed a different form of equation (70)

$$
\begin{equation*}
\frac{-1}{\gamma} \frac{d \boldsymbol{M}}{d t}=\boldsymbol{M} \times\left(\mathbf{B}-\frac{\alpha}{|\gamma| M_{s}} \frac{d \boldsymbol{M}}{d t}\right) \tag{72}
\end{equation*}
$$

which follows from equation (71) if we use the triple product identity, and introduce the dimensionless damping constant $\alpha=\frac{\lambda}{M_{s}}$ and neglect terms in $\alpha^{2}$. This is justified because of the original assumption of Landau Lifshitz that $\lambda \ll M_{s}[1]$.

### 4.9 Domain wall propagation

Now let us see more carefully the application of the LL equation to domain wall motion.

Landau and Lifshitz applied their equation to the problem of domain wall motion for the particularly simple case of the wall structure they had proposed. This is the structure given here by equation (53). In order to avoid the complications of the magnetostatic field, they continued to consider the case of an infinite medium and also assumed that the wall structure remained almost identical to its previous static structure, when it was moving. In other words, they assumed a rigid wall structure.

There is a very important point here. If we assume that the wall moves as a rigid structure we have defined the space derivatives of $\mathbf{M}$. They are already given by the differential equation (47), which we solved to find this structure. If we now continue and assume that the wall is moving with a constant velocity, we have defined the time derivatives of $\mathbf{M}$ as well. We no longer have a differential equation to solve, the problem becomes purely algebraic, as we shall see, and can be solved by considering what is happening at any convenient point within the moving wall.

The other assumption which follows, once a rigid wall model is adopted, is that the total magnetic field, F, previously given by equations (40) and (41), remains antiparallel to $\mathbf{M}$ in the moving wall as it was in the stationary wall. We then only need to consider the applied field, $B_{z}$, in our equations of motion. The situation is illustrated in figure 16, where we again emphasise the kind of sample of material we are dealing with. The magnetisation must lie in-plane and the sample thickness must be very much greater than the wall width.

Let us now consider the sample, shown in figure 16, immersed in the applied field $B_{z}$. The domain on the left, in figure 16, is magnetised in the same direction as the applied field and consequently grows through the motion of the wall to the right with velocity $v_{y}$. Our diagram only shows M in the two domains and at the centre of the wall.


Figure 16: Coordinates used for the wall moving in an applied field $B_{z}$

We take the LL equation

$$
\begin{equation*}
\frac{-1}{|\gamma|} \frac{d \mathbf{M}}{d t}=(\mathbf{M} \times \mathbf{F})-\lambda\left[\mathbf{F}-(\mathbf{F} \cdot \mathbf{M}) \frac{\mathbf{M}}{M_{s}^{2}}\right] \tag{73}
\end{equation*}
$$

and, following the argument given above, we consider one convenient point within the wall[1]. We choose the centre where $M_{x}=M_{s}, M_{y}=0$. The total field, $\mathbf{F}$, is now made up from the field belonging to the previous static solution, $\mathbf{M} \times \mathbf{F}=0$, and therefore drops out of equation (73), and the applied field, $B_{z}$. Equation (73) thus has only two components

$$
\begin{equation*}
\frac{1}{|\gamma|} \frac{d M_{y}}{d t}=M_{s} B_{z} \tag{74}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{|\gamma|} \frac{d M_{z}}{d t}=\lambda B_{z} \tag{75}
\end{equation*}
$$

Now, because $\lambda \ll \mathbf{M}_{s}$ in the Landau-Lifshitz model we can concentrate on equation (74). This tells us that the proposal that the wall can move forward under the influence of the applied field $B_{z}$ must involve the vector $\mathbf{M}$ developing a component $M_{y}$, a component in the direction of motion. It is not possible to assume that the wall is really rigid and that it maintains exactly the same form which it has when it is stationary.

As shown in figure below, the magnetisation at the centre of the wall must tilt, to satisfy equation (73), by the angle $\phi$ shown. Let us go back to equation (72) again, with this modification, and see what happens.

Concentrating still on the centre of the wall, the vector $\mathbf{M}$ has components

$$
M_{x}=M_{s} \cos \phi, M_{y}=M_{s} \sin \phi, M_{z}=0
$$

For the vector $\mathbf{F}$, Landau and Lifshitz assumed that the parts of $\mathbf{F}$ due to exchange and anisotropy remained antiparallel to $\mathbf{M}$, as they were when $\phi$ was equal to zero. This assumption made it possible to proceed with the calculation bringing in only


Figure 17: The moving wall must develop a component of $\mathbf{M}$ which lies along the direction of motion
the magnetostatic field and the applied field. They also assumed that $M_{z}$ was still given by the function plotted in figure 14.

If the sum of the exchange and anisotropy fields remains antiparallel to $\mathbf{M}$, these do not enter into equation (73). The magnetostatic field is a particularly simple one to calculate in this case because the wall shown in figure 17 may be considered a thin layer with its magnetisation tilted out of the plane by the angle $\phi$. There will then be no magnetic field due to the component of magnetisation normal to the plane, $M_{y}$, because equation (38) would give $H_{y}=-M_{y}$ for a thin layer so that equation (39) would give $B_{y}=0$. In the plane of such a thin layer there is no such demagnetising effect and we have $B_{x}=\mu_{0} M_{s} \cos \phi$.

We conclude that the components of the total field $\mathbf{F}$ which should be substituted into equation (73) for the case shown in figure 17 are

$$
\begin{gather*}
F_{x}=\mu_{0} M_{s} \cos \phi  \tag{76}\\
F_{y}=0  \tag{77}\\
F_{z}=B_{z} \tag{78}
\end{gather*}
$$

The components of equation (73) are now

$$
\begin{align*}
& -\frac{1}{|\gamma|} \frac{d M_{x}}{d t}=B_{z} M_{s} \sin \phi-\mu_{0} \lambda M_{s} \cos \phi \sin ^{2} \phi  \tag{79}\\
& -\frac{1}{|\gamma|} \frac{d M_{y}}{d t}=B_{z} M_{s} \cos \phi+\mu_{0} \lambda M_{s} \cos ^{2} \phi \sin \phi \tag{80}
\end{align*}
$$

and

$$
\begin{equation*}
-\frac{1}{|\gamma|} \frac{d M_{z}}{d t}=-\mu_{0} M_{s}^{2} \cos \phi \sin \phi-\lambda B_{z} \tag{81}
\end{equation*}
$$

We now consider the wall moving forward at constant velocity. Concentrating upon the centre of the wall, as before, M lies in ( $\mathrm{x}, \mathrm{y}$ ) at the angle $\phi$. As the wall
moves forward, the vector $\mathbf{M}$ rotates about y and as it has constant magnitude, and $\phi$ is constant for constant velocity, equations (79) and (78) are both zero on the left hand side and both give

$$
\begin{equation*}
B_{z}=\mu_{0} \lambda \cos \phi \sin \phi \tag{82}
\end{equation*}
$$

The z -component of $\frac{d \mathbf{M}}{d t}$, on the other hand is given by

$$
\begin{equation*}
\frac{d M_{z}}{d t}=\frac{M_{s} v_{y}}{\Delta} \tag{83}
\end{equation*}
$$

because $\frac{\partial M_{z}}{\partial y}=-\frac{M_{s}}{\Delta}$ at the centre of the LL wall, as shown in figure 17. This may be formally deduced from equation (46). Substituting (83) in (81) we obtain

$$
\begin{equation*}
v_{y}=\left(\frac{|\gamma| \Delta}{M_{s}}\right)\left(\lambda B_{z}+\mu_{0} M_{s}^{2} \sin \phi \cos \phi\right) \tag{84}
\end{equation*}
$$

Equations (82) and (84) give as a picture of the domain wall moving forward at a constant velocity $v_{y}$, under the influence of a constant applied field $B_{z}$.Equation (82) shows that the effect of $B_{z}$ is to make $\mathbf{M}$ tilt out of the plane of the wall by an angle

$$
\begin{equation*}
\phi=\frac{1}{2} \arcsin \frac{2 B_{z}}{\mu_{0} \lambda} \tag{85}
\end{equation*}
$$

where it comes into equilibrium with the total field and the effective field of the second term in the Landau-Lifshitz equation, which we could call the damping field. If we now substitute the equilibrium condition, (82) into equation (84) we obtain the relationship between the velocity and the applied field as

$$
\begin{equation*}
v_{y}=\left(\frac{|\gamma| M_{s} \Delta}{\lambda}\right)\left(1+\frac{\lambda^{2}}{M_{s}^{2}} B_{z}\right) \tag{86}
\end{equation*}
$$

As expected, the problem has become purely algebraic since we assumed a given wall structure and constant velocity.

Because the formulation of LL equation demanded $\lambda \ll M_{s}$ we can rewrite equation (86) as

$$
\begin{equation*}
v_{y}=\frac{|\gamma| M_{s} \Delta}{\lambda} B_{z} \tag{87}
\end{equation*}
$$

which is often quoted result of Landau and Lifshitz, that the wall velocity is proportional to the applied field.If we set the damping constant $\alpha=\frac{\lambda}{M_{s}}$ we can also write

$$
\begin{equation*}
v_{y}=\left(\frac{|\gamma| \Delta}{\alpha}\right) B_{z} \tag{88}
\end{equation*}
$$

## 5 Skyrmions

In this section, we will discuss the basic theory about skyrmions. This will help us to proceed to the next section where we will talk about the micromagnetic simulations.

Topological solitons are solutions of non-linear differential equations which cannot be continuously deformed to a trivial solution. Examples of topological solitons can be found in exactly solvable models and in recently observed magnetic structures called skyrmions. Topological skyrmions always have received special interest from physicists and mathematicians because of their stability [6].

A soliton wave is not only an interesting and elegant observation but the stability of solitons attracts the attention of many. So as we observe, stability is a term that will think a lot about. And there is a reason for that.

The basic objects of Classical Mechanics are stable particles, charecterized by a non-zero mass, which live for ever. In contrast, at a fundamental level, elementary particles such as electrons and protons are described using quantum field theory, where they are thought of as wave-like excitations of an underlying field. It is however a non-trivial in field theory to make these wave-like excitations stable. They would generally dissipate similar to the disappearance of the waves in a pond once they are created. In the year 1962,physicist Tony Skyrme prososed the idea that the particles do not decay because they have a topological number, which can not be changed by a continuous deformation of the underlying field.In mathematical terms, there exists a topological integer that is unchanged. In topology, a donut, for example is topologically equivalent to a cup, as they both contain one hole, and one can be continuously deformed to the other. A sphere contains no hole and is topologically different, and thus can not be deformed into a donut by continuous deformation. Sometime back, in the year 1989, it was pointed out that topologically protected systems may be relevant in condensed matter physics. Recently, there has been a flurry of theoretical and experimental works showing that topological solitons can be stabilized in chiral magnets in the form of a swirling spin texture called "magnetic skyrmion". To be presice, this was first discovered in 2009.

A magnetic skyrmion is a topological soliton that is energetically stable. This means that a skyrmion state is energetically favored by the system. So, on the one hand skyrmions are fundamentally interesting because of their stability. On the other hand skyrmions are technologically interesting. By making use of the stability of the skyrmion configuration it might be possible to use skyrmions in a new kind of information carrying device.

We consider a ferromagnetic material [6]. This has two phases: the paramagnetic phase and the polarized phase. For temperatures above the Curie temperature the magnetization is randomly distributed in the magnet. However, below the Curie temperature the magnetization vectors are align in some direction. The two types
are shown on the figure below.


Figure 18: The magnetic spin structure of a ferromagnet for different values of T
The magnetization in this latter phase can be characterized by a magnetization profile, which is represented by $\mathbf{m}(\mathbf{x})$. This means that for every spatial variable $\mathbf{x}$ we have a vector $\mathbf{m}$ pointing in some direction. The polarized phase in a ferromagnet is a trivial phase where the magnetization is align throughout the system. Other types of magnetization profiles occur in, for example, chiral magnets. There, non-trivial magnetization profiles are formed that have a whirling structure. Spin textures, as long as a skyrmion may be one of the structures shown below.


Figure 19: (a) Hedgehog,(b) Neel-Type skyrmion, (c) Bloch-Type skyrmion, (d)antiskyrmion, (e)skyrmionium,(f) biskyrmion,(g) in-plane skyrmion,(h) skyrmion in helical background, (i) chiral bobber, (j) comped anti-hedgehog

These whirling types of magnetization in a chiral magnet are induced by the Dzyaloshinskii-Moriya interaction (DMI). The DM interaction is a microscopic characteristic of interacting spins that occurs in a system that lacks inversion symmetry and has a strong spin-orbit coupling. The DM interaction creates phases which have a winding magnetic configuration.

### 5.1 First observation of Skyrmions

Skyrmion spin structures were first observed in 2009, using neutron scattering [6]. This was the first time the A-phase, as it was called, in a chiral magnet was identified as skyrmion phase. The A-phase is shown in figure 20 among two other chiral phases. In the first experiment people used neutron scattering to observe the spontaneous formation of a two-dimensional lattice of skyrmions in MnSi . The first observation of a magnetic skyrmion structure has been made in the chiral ferromagnetic MnSi which lacks of inversion symmetry. Due to this property it allows for non-inversion symmetric magnetic structures to appear. A skyrmion is such a structure. In the


Figure 20: Different magnatic phases of MnSi
phase diagram of a ferromagnet there are two different phases. The spins are ordered below the Curie temperature and above this temperature point the spins change direction and become disordered. We expect to see the same two phases in the phase diagram of MnSi , since this too is a ferromagnet.

We see five different phases when we look at the phase diagram of MnSi in figure 20. The most familiar phases are the polarized phase, which occurs for large magnetic fields and the paramagnetic phase, which appears above the Curie temperature. Between these two phases, there are three additional phases which are not present in a basic ferromagnet. These three phases are chiral phases. These chiral phases occur due to the DMI, as explained in the previous section. There is a conical phase, a helical phase and an A-phase. There is a strong exchange energy present in this magnet which favors uniform magnetization. The weaker energy scale comes from the DMI. This interaction favors twisted spin structures, such as the three chiral phases we see in the figure. There are other interactions at play here, but they are negligible.

The helical phase appears below the critical Curie temperature at small (or zero) magnetic field. The magnetization in the helical phase precesses around an axis which is perpendicular to the external magnetic field. The helical magnetization is shown in figure 21. If the temperature is below the Curie temperature, and the external magnetic field is increased a crossover to the conical phase happens.


Figure 21: The helical phase
The crossover between the helical phase and the conical phase occurs at $B_{c 1}>0.1 T$ , when $T<T_{c}$.In the conical phase the magnetization obtains a component parallel to the magnetic field. The angle of the cone continuously decreases to zero when the magnetic field increases to a value of $B=0.55 T$, at which all spins allign. Below is the magnetization profile of a conical phase.


Figure 22: The canonical phase, where the magnetization precesses around the propagation vector, and has a component parallel to the external magnetic field.

It is important to see that at a magnetic field value of $B>0.55 \mathrm{~T}$ the effect of the DMI is very weak compared to the exchange energy scale. The ferromagnetic energy scale dominates the magnetization and the DMI is negligible. So for a stronger magnetic field we have a field polarized state as ground state.

The A-phase occurs in a small region of the phase-diagram. The region for temperatures just below $T_{c}$ and for a magnetic field value around $B=0.2 T$. In this small area a two dimensional hexagonal lattice of (anti-) skyrmions is the stable ground state of the system. The magnetic field is perpendicular to the skyrmion lattice.

The individual skyrmion appearing in the lattice is a structure we have already seen: recall figure19 (b). The magnetic structure is translationally invariant along the direction of the magnetic field, which is perpendicular to the skyrmion structure. The color in the picture indicates whether the spins are parallel (blue) or anti-parallel (red) to the external magnetic field. Also, the skyrmion is rotationally symmetric if we rotate around the axes parallel to the external magnetic field. We can imagine skyrmion tubes could form if we translate the skyrmion structure along the magnetic field.

Before the discovery of the skyrmion structure in 2009, the consensus was that the A-phase was some kind of helix with a wave vector aligned perpendicular to the applied field. By using neutron scattering this hypothesis is disproven, since there it shows that the structure emerging in the A-phase is a hexagonal lattice. However, using neutron scattering alone, it is not possible to deduce the magnetic structure of the A-phase. Measurements of the topological Hall effect of the A-phase are necessary to prove that the A-phase corresponds to a skyrmion structure. The topological Hall effect is induced by the magnetic field of the skyrmions on conduction electrons. The movement of a skyrmion leads to a change in the magnetic field produced by the skyrmion and thus changes the electromagnetic induction. Then the induced electric field gives an additional contribution to the Hall effect when skyrmions move.

### 5.2 Magnetic skyrmion configuration

A magnetic skyrmion is a topological object consisting of a skyrmion core, an outer domain, and $a$ domain wall that separates the skyrmion core from the outer domain [12].

A skyrmion is mainly characterized by three numbers: the skyrmion number $Q_{s}$, the vorticity number $Q_{\nu}$, which is defined by the winding number of the spin configurations projected into the x-y plane, and the helicity number $Q_{h}$. The magnetization profile of a skyrmion can be described with:

$$
\begin{equation*}
\boldsymbol{m}(r)=\left[\sin \theta(r) \cos Q_{\nu} \phi+Q_{h}, \sin \theta(r) \sin \left(Q_{\nu} \phi+Q_{h}\right), \cos \theta(r)\right] \tag{89}
\end{equation*}
$$

where $\theta(r)$ is the radial function hat determines the z-component of $\boldsymbol{m}(r)$ [16].
The skyrmion configuration is physically stable [6]. This means that the spin configuration of a skyrmion could minimize the energy of the system and therefore the system favours this configuration. Moreover, small deformations of the system cannot transform the spin structure to some trivial structure. This topological property is
made explicit with a topological charge,or the skyrmion number $Q_{s}$ (also defined sometimes by the symbol m , as used in the figure 25). The skyrmion number $Q_{s}$ can be calculated with

$$
\begin{equation*}
Q_{s}=-\frac{1}{4 \pi} \iint\left[\boldsymbol{m}(r) \cdot\left(\partial_{x} \boldsymbol{m}(r) \times \partial_{y} \boldsymbol{m}(r)\right)\right] d x d y=\frac{1}{4 \pi} \int_{\pi}^{0} \sin \theta d \theta \int_{\pi}^{0} d \phi=\frac{Q_{\nu}}{2}\left[\lim _{r \rightarrow \infty} \cos \theta(r)-\cos \theta(0)\right] \tag{90}
\end{equation*}
$$

which is determined by the product of the vorticity number and the difference between the spin direction of the core and the tail of the skyrmion. It is notable that the helicity number does not contribute to the topological number, which is uniquely determined by the type of the DMI.

Figure 24 shows how the topological charge works. It shows a skyrmion magnetization from the side. If we start wrapping this structure around a unit sphere we go round once. This gives the skyrmion structure a winding number of 1 . Depending on whether the center of the skyrmion is parallel or anti-parallel with the external magnetic field the skyrmion number either is 1 or -1 .

In fact, chiral magnetic skyrmions are not limited to the case of topological charges $|Q|=1$, and could be of any topological charge. For example, the skyrmionium can be regarded as a topological combination of a skyrmion with $Q=+1$ and a skyrmion with $Q=-1$, which carries a net topological charge of $Q=0$. The skyrmionnium structure was first studied in a theoritical work by Bogdanov and Huber in 1999. It is also referred to as the target skyrmion. The topological charge difference between the skyrmionium with $Q=0$ and the skyrmion with $Q=+1$ originates form their out-of-plane spin textures. As other examples, the biskyrmion has a topological charge of $Q=-2$, which can be formed in some materials such as chiral bulk or frustrated magnets when two skyrmions with the same topological number ( $Q=-1$ in this case) are approaching to each other [19].

In order for someone to be able to completely characterize the spin texture, we must calculate the vorticity number $Q_{\nu}$ and the helicity number $Q_{h}$. These two numbers are given by the following expressions

$$
Q_{\nu}=\frac{1}{2 \pi} \oint_{C} d \phi=\frac{1}{2 \pi}[\phi]_{\phi=0}^{\phi=2 \pi}
$$

and

$$
\phi=Q_{\nu} \phi+Q_{h}
$$

In figure 23, we see different skyrmions' configurations for different values of skyrmion, vorticity and helicity numbers.


Figure 23: Illustrations of 2D magnetic skyrmions with differnet topological charge, vorticity number and helicity number $\left(Q_{s}, Q_{\nu}, Q_{h}\right)$. The arrow denotes the spin direction and the out-of-plane spin component $\left(m_{z}\right)$ is represented by the color: red is out of plane, white is in-plane and blue is into the plane

It should be noted for later that $Q_{s}$ is calculated over the whole magnetic element considering in the simulations we will describe.It has been reported that the presence of iDMI certifies the nonexistence of a complete uniform state. Micromagetically iDMI imposes boundary conditions which lead to the tilt of the magnetization states at the edges. For the arbitraty orientation of normal to edge ( $\mathbf{n}$ ), the boundary condition is given by

$$
\begin{equation*}
\frac{d \mathbf{m}}{d n}=\frac{d}{2 A}(\hat{\mathbf{z}} \times \mathbf{n}) \times \mathbf{m} \tag{91}
\end{equation*}
$$

This condition tilts the magnetization states near the edges of the sample in a plane normal to the surface. We will have the opportunity to discuss more on the magnetic simulations later on this part.

If we would deform the skyrmion state to some trivial state we would end up with the configuration shown in figure 27. Based on energy considerations this state is not allowed (it costs a lot of energy). If we were to take a different magnetization, a domain wall for example, and deform the state, we would be able to obtain a state where all spins are aligned.

Side view a skyrmion magnetization:


Figure 24: Skyrmion wrapping around a sphere


Figure 25: A state which is not allowed due to energy considerations-hence the cross

People categorize different skyrmion structures by their helicity and value of their skyrmion number, depicted in figure 26. The helicity of a skyrmion depends on the DMI that is induced and this depends on the direction in which inversion symmetry is broken in the compound. The skyrmion structure we saw in MnSi corresponds to the top right configuration in figure $26, m=1$ and $\gamma=\frac{\pi}{2}$.


Figure 26: Several skyrmion structures with different skyrmion numbers $m$ and helicity $\gamma$


Figure 27: Two typed of magnetic skyrmion texture

We are more interested in to types of skyrmions:Neel type and Bloch type skyrmions, which correspond to different symmetries of the interaction between spins (this can be due to the underlying crystal lattice or to the presence of an interface), resulting to different directions of rotation. As we see on the figure 27, in a Bloch type skyrmion, the spins rotate in the tangential planes, that is, perpendicular to the radial directions, when moving from the core to the periphery ,while in the Neel type skyrmion, the spins rotate in the radial planes from the core to the periphery. A skyrmion with helicity number of $\frac{\pi}{2}$ or $\frac{3 \pi}{2}$ corresponds to the Bloch type skyrmion while a skyrmion with helicity number of 0 or $\pi$ corresponds to a Neel type skyrmion, when the vorticity number of the skyrmion equals one [16].

### 5.3 Skyrmion State

The Skyrmion state is defined such that $\theta(r)$ changes from $\theta=\pi$ at the center to $\theta=0$ at the boundary of the Skyrmion. Single-valuedness of the spin orientation demands that the azimuthal angle is of the form $\phi=m \alpha+\gamma$, where $m$ is the winding number and $\gamma$ the helicity parameter [7].

A magnetic Skyrmion is a swirling magnetic structure of spins. It is usually a two-dimensional object, existing at interfaces between two materials or in magnetic thin films. The topological properties depend on the geometry of the structure. The two key characteristic geometrical properties are the vorticity and the helicity of the structure, which are a characteristic of how the spin orientation $\vec{n}(\vec{r})$ changes over space. Note that for the lattice, the position $\vec{r}$ takes discrete values, while in continuum models it is a continuous variable. The spin orientation at each point $\vec{r} \equiv(r, \alpha)$ on the 2D plane is described in the spherical coordinates by specifying the polar and azimuthal angles, $\theta(\vec{r})$ and $\phi(\vec{r})$, respectively. For the Skyrmion state, $\theta(\vec{r})$ is a function of the radial distance r only and $\phi(\vec{r})$ is a function of the polar angle $\alpha$ only, so that in the cartesian coordinates, the local magnetization vector is $\vec{n}(\vec{r})=$ $(\sin \theta(r) \cos \phi(\alpha), \sin \theta(r) \sin \phi(\alpha), \cos \theta(r))$, where m is a non-zero integer called the winding number. Skyrmions and anti-Skyrmions are defined as those for which the winding number is positive or negative, respectively. The helicity parameter $\gamma$ takes specific values for helical states. If $\gamma= \pm \frac{\pi}{2}$, then the helicity $h= \pm 1$ (the two signs indicate left or right handedness), while for $\gamma=0$ or $\gamma=\pi$, we have a radial spin structure, as we have seen from Figure 26.

### 5.4 Dzyaloshinskii-Moriya interaction in the LL equation

In this section we will introduce the Dzyaloshinskii-Moriya term in the LL equation [1]. For that, let us consider a ferromagnet described by the magnetization vector as a function of space and time $\mathbf{M}=\mathbf{M}(x, t)$, with constant magnitude $|\mathbf{M}|=M_{s}$, where $M_{s}$ is called the saturation magnetization. Statics and dynamics of the magnetization are governed by the dimensionless Laundau-Lifshitz equation

$$
\begin{equation*}
\frac{\partial \mathbf{m}}{\partial t}=-\mathbf{m} \times \mathbf{f} \tag{92}
\end{equation*}
$$

where $\mathbf{m}=\frac{\mathbf{M}}{M_{s}}$ is the normalized magnetization, in component form $\mathbf{m}=\left(m_{1}, m_{2}, m_{3}\right)$. The variable t is the dimensionless time that is measured in units of $t_{0}=\frac{1}{\gamma_{0} \mu_{0} M_{s}}$, where $\gamma_{0}$ is the gyromagnetostatic ratio as we have stated on the previous part, and $\mu_{0}$ is the permeability of vacuum. The effective field $\mathbf{f}$ contains the interactions in the material. We will assume a ferromagnet with exchange, an easy-axis anisotropy, and a DMI interaction. We will consider configurations where the magnetization is varying in only one space direction, that is, we assume $\mathbf{m}=\mathbf{m}(x, t)$. The energy of this system is

$$
\begin{equation*}
\left.E(\mathbf{m})=\int\left[\frac{\left(\partial_{x} \mathbf{m}\right)^{2}}{2}\right]+\frac{\kappa^{2}}{2}\left(1-m_{3}^{2}\right)+\lambda\left(\mathbf{m} \times \partial_{x} \mathbf{m}\right) \cdot \hat{\mathbf{e}}_{1}\right] d x \tag{93}
\end{equation*}
$$

where $\hat{e}_{1}$ is the unit vector for the magnetization in the x direction. We measure distance in units of exchange length $l_{e x}=\sqrt{\frac{2 A}{\mu_{0} M_{s}^{2}}}$ where A is the exchange constant. There are two length scales in the model, $l_{w}=\sqrt{\frac{2 A}{K}}$, where K is the anisotropy constant, and $l_{D}=\frac{2 A}{|D|}$, where D is the DMI constant. The dimensionless parameters appearing in the energy above are $\kappa^{2}=\frac{2 K}{\mu_{0} \mathrm{M}_{s}^{2}}=\left(\frac{l_{e x}}{l_{w}}\right)^{2}$, and $\lambda=\frac{l_{e x}}{l_{D}}$. We will consider $\lambda>0$. The case $\lambda<0$ corresponds to the transformation $x \mapsto-x$. The general form of the DM term is given in terms of Lifshitz invariants

$$
\begin{equation*}
\mathcal{L}_{j k}=\left(\mathbf{m} \times \partial_{j} \mathbf{m}\right)_{k} \tag{94}
\end{equation*}
$$

In the energy (93) we have only kept the Lifshitz invariant $\mathcal{L}_{11}$ in the $\hat{\mathbf{e}}_{1}$ direction corresponding to cubic DMI given by $\mathbf{m} \cdot(\nabla \times \mathbf{m})$. Replacing $\mathcal{L}_{11}$ by $\mathcal{L}_{12}$ (interfacial DMI) or a linear combination of both, yields a model that is mathematically equivalent except a rigid rotation around the $\hat{\mathbf{e}}_{3}$ axis.

The effective field in (92) is obtained by varying the energy

$$
\begin{equation*}
\mathbf{f}=-\frac{\delta E}{\delta \mathbf{m}}=\partial_{x}^{2} \mathbf{m}+\kappa^{2} m_{3} \hat{\mathbf{e}}_{3}-2 \lambda \hat{\mathbf{e}}_{1} \times \partial_{x} \mathbf{m} \tag{95}
\end{equation*}
$$

The uniform (ferromagnetic) states $\mathbf{m}=(0,0, \pm 1)$ are the simplest time-independent (static) solutions of the LL equation. For large anisotropy such that,

$$
\begin{equation*}
\kappa>\kappa_{c} \equiv \frac{\pi}{2} \lambda \tag{96}
\end{equation*}
$$

the ferromagnetic configuration in the ground state of the system, while for $\kappa<\kappa_{c}$ a spiral configuration becomes the ground state. The period of the spiral increases for increasing anisotropy and goes to infinity as $\kappa \mapsto \kappa_{c}$.

Let us assume a material with $\kappa>\kappa_{c}$ and we are looking for domain wall solutions as excitations of the ferromagnetic ground state. A standard Bloch wall

$$
\begin{equation*}
m_{1}=0, m_{2}= \pm \operatorname{sech}(k x), m_{3}= \pm \tanh k x \tag{97}
\end{equation*}
$$

for any combination of the signs, is a solution of (92) also in the presence of DMI $(\lambda \neq 0)$ as the contribution of the DM term on the right-hand-side of equation (92) vanishes identically for these configurations. As the DMI is chiral, the walls with the same signs for $m_{2}$ and $m_{3}$ in equation (97), have lower energy, for $\lambda>0$. The walls with opposite signs for $m_{2}$ and $m_{3}$ are energy maxima.

One can easily prove that traveling domain walls are not discribed by equation (92), when the DMI is not included in the effective field (95). To see that let us consider the total magnetization in the direction perpendicular to the film

$$
\begin{equation*}
\mathcal{M}=\int_{-\infty}^{\infty} m_{3} d x \tag{98}
\end{equation*}
$$

in the sense of the Cauchy principle value and calculate its time derivative using equation (92)

$$
\begin{equation*}
\frac{d \mathcal{M}}{d t}=-2 \lambda \int_{-\infty}^{\infty} m_{1} \partial_{x} m_{3} d x \tag{99}
\end{equation*}
$$

This result tells us that the exchange and anisotropy interactions are invariant with respect to rotations around the third axis of the magnetization, and therefore the total magnetization $\mathcal{M}$ is conserved in the absence of DMI, where $\lambda=0$. Since a propagating domain wall configuration is equivalent to expanding one domain in favour of the other, for example, it favours the "up" domain instead the "down" domain, thus changing $\mathcal{M}$, domain wall propagation is not possible in a model where the total magnetization $\mathcal{M}$ is preserved.

In a model with effective field described by equation (92), it is entirely due to DMI that the symmetry is broken and the associated conservation law is not valid, thus allowing for possibility of propagating domain walls. If we assume a rigid wall connecting the south pole $\left(m_{3}=-1\right)$ at $x \mapsto-\infty$ to the north pole $\left(m_{3}=1\right)$ at
$x \mapsto+\infty$ and travelling with velocity v then we obtain that $v=-\frac{1}{2} \frac{d \mathcal{M}}{d t}$ and from (99) we obtain

$$
\begin{equation*}
v=\lambda \int_{-\infty}^{\infty} m_{1} \partial_{x} m_{3} d x \tag{100}
\end{equation*}
$$

This gives an upper bound for the speed

$$
\begin{equation*}
|c| \leq 2 \lambda \tag{101}
\end{equation*}
$$

More generally, a Lifshitz invariant $\mathcal{L}_{1 \kappa}$ gives rise to an integrand $m_{\kappa} \partial_{x} m_{3}$ in equation (94). In particular, no non-trivial traveling domain wall solution is possible in the case $\kappa=3$ corresponding to a wire along $\hat{\mathbf{e}}_{3}$ with a cubic DMI and a stray-field induced anisotropy.

Lastly, considering the conditions of the equations (100) and (101) we have, for positive c , the ordering

$$
\begin{equation*}
0<\frac{v}{2}<\lambda \leq \frac{2}{\pi} \kappa \tag{102}
\end{equation*}
$$

### 5.5 Dzyaloshinskii-Moriya Interaction (DMI) in the configuration of skyrmions

In the previous section, we introduced the DMI term on the LL equation. Also, in the first part, we shared some information about what is this type of interaction. As we will see in this section, the existence of DMI is crucial for the formation of a skyrmion, along with some other mechanisms, sometimes working together.

Chirality is a form of assymetry of a system.If the atomic structure of a magnet lacks inversion symmetry we call them chiral magnets. The chirality expresses itself through the phase diagram which shows additional chiral phases. In this phases the magnetization is whirled in some way, for example helical. What mechanism is responsible for these additional phases?

In 1960 Dzyaloshinskii constructed a model to describe weak ferromagnetism. Based on symmetries he introduced an assymmetrical term which later on was dubbed the DMI. Moriya connected his name to this term when he found the mechanism behind this interaction is partly based on spin-orbit coupling. So, DMI is induced by a lack of inversion symmetry of the compound and strong spin-orbit coupling. Let us see an example. A compound that lacks inversion symmetry is MnSi (magnanese silicide). In the figure below, we see that inversion symmetry is broken in a unit cell. Aside from lack of inversion symmetry, MnSi has a strong spin-orbit coupling.


Figure 28: The atomic structure of MnSi . The circles represent atoms and the dashed lines depict the boundaries of a unit cell. This structure lacks inversion symmetry on a unit cell

Inversion symmetry can be broken in different directions leading to a different induced DMI. In practice, this means that the magnetization is different. We consider an example of just two spins here, must for the sake of simplicity. The DMI for two spins has the following form:

$$
\begin{equation*}
\mathcal{H}_{D M}=-\mathbf{D}_{12} \cdot\left(\mathbf{S}_{1} \times \mathbf{S}_{2}\right) \tag{103}
\end{equation*}
$$

where $\mathbf{S}_{1}$ and $\mathbf{S}_{2}$ are the atomic spins.
In the next figure, there is a DMI emerging from the interplay of two atomic spins with the neightboring atom having spin-orbit coupling in a thin film. The resulting DMI points outwards from the plan of the atoms. The same mechanism is responsible for the Interfacial DMI between a ferromagnetic thin layer and a non-magnetic leyer with a large spin-orbit coupling. Here, at the interface between the two layers, the tringle mechanism produces a DMI for the interfacial spins $\mathbf{S}_{1}$ and $\mathbf{S}_{2}$. The DMI vector, $\mathbf{D}_{12}$, is perpendicular to the triangle.


Figure 29: iDMI for two spins
Starting with a ferromagnetic state where all spin are alligned: $S 1 \| S 2$, we then assume a strong spin-orbit coupling present that induces a DMI. The resulting magnetic structure depends on the direction of the D-vector, which in turn depends on the way which the symmetry in the compound is broken. Different helicities are obtained for different DMI.

We call the vector joining the sites of the spins $\mathbf{S}_{1}, \mathbf{S}_{2}$ and $\mathbf{R}_{12}$. The energy of the system is minimized if either $\mathbf{R}_{12} \perp \mathbf{D}_{12}$ or $\mathbf{R}_{12} \| \mathbf{D}_{12}$. If $\mathbf{R}_{12} \perp \mathbf{D}_{12}$, the DMI tilts $\mathbf{S}_{1}$ around $\mathbf{D}_{12}$, with respect to $\mathbf{S}_{2}$ 6].

For the interfacial DMI (iDMI) $\mathbf{D}_{12}$ can be written as $\mathbf{D}_{12} \cdot\left(\mathbf{z} \times \mathbf{u}_{12}\right)$, where the $\mathbf{z}$ and $\mathbf{u}_{12}$ ) are unit vectors, respectively perpendicular to the interface in the direction of a amgnetic layer and pointing from site 1 to site 2. For $\mathbf{D}_{12}>0$ the DMI favours anticlockwise rotations from $\mathbf{S}_{1}$ to $\mathbf{S}_{2}$ while $\mathbf{D}_{12}<0$ correspondes to the lower energy for clockwise mangetization rotation. In coclusion, the DMI is a chiral interaction that lowers or increases the energy of the spins depending on whether the rotation from $\mathbf{S}_{1}$ to $\mathbf{S}_{2}$ around $\mathbf{D}_{12}$ in the clockwise or in the anticlockwise sense.If $\mathbf{S}_{1}$ and $\mathbf{S}_{2}$ are initially parallel, the effect of a strong DMI is to introduce a relative tilt around $\mathbf{D}_{12}$. For a purely interfacial DMI, D is inversely proportional to the thickness of the film. It is positive for anticlockwise rotations.


Figure 30: (a)Bulk DMI vector (white arrow) originate in a non-centrosymmetric crystal because of the interaction of the ferromagnetic atoms.(b)Interfacial DMI vector (whitre arrow) in a ferromagnet/heavy metal bilayer

So that was a simple example for when we have two spins. For the case that we study a Neel or a Bloch type skyrmion, the definition of the DMI is a little different. There are two different types of DMI responsible for the stability of skyrmions. The corresponding Hamiltonian term ( $H_{D M I}$ ) for the bulk DMI, which supports the Blochtype materials can be expressed in the form of:

$$
\begin{equation*}
H_{D M I}=D \boldsymbol{m}(\boldsymbol{r}) \cdot[\nabla \times \boldsymbol{m}(\boldsymbol{r})] \tag{104}
\end{equation*}
$$

where $\boldsymbol{m}(\boldsymbol{r})$ represents the local magnetic moment orientation with $|\boldsymbol{m}(\boldsymbol{r})=1|$, and D as always the DMI contant.

The Neel type skyrmions are supported by the interfacial DMI (iDMI), as described above, have attracted great attention, owing to the increased thermal stability and structural simplicity from the perspective of electronic applications. The corresponding Hamiltonian term has the form of

$$
\begin{equation*}
H_{D M I}=D\left[m_{z}(\boldsymbol{r}) \nabla \cdot \boldsymbol{m}(\boldsymbol{r})-(\boldsymbol{m}(\boldsymbol{r}) \cdot \nabla) m_{z}(\boldsymbol{r})\right] \tag{105}
\end{equation*}
$$

where $m_{z}(\boldsymbol{r})$ is the z component of $\boldsymbol{m}(\boldsymbol{r})$.

### 5.6 Domain wall solution

We will compute the analytical solution of the equation, with DMI term

$$
\begin{equation*}
\frac{\partial \boldsymbol{M}}{\partial t}=\boldsymbol{M} \times\left[\frac{\partial^{2} \boldsymbol{M}}{\partial x^{2}}+M_{z} \hat{\mathbf{e}}_{z}+\lambda \hat{\mathbf{e}}_{y} \times \partial_{x} \boldsymbol{M}\right] \tag{106}
\end{equation*}
$$

We remember two cross product properties:

$$
\vec{A} \times(\vec{B}+\vec{C})=\vec{A} \times \vec{B}+\vec{A} \times \vec{C}
$$

and

$$
\vec{A} \times \lambda e_{y} \times \partial_{x} \vec{A}=-A_{y} \partial_{x} \vec{A}
$$

So, besed on these properties, equation (106) takes the form

$$
\begin{equation*}
\frac{\partial \boldsymbol{M}}{\partial t}=\boldsymbol{M} \times\left(\frac{\partial^{2} \boldsymbol{M}}{\partial x^{2}}+M_{z} \hat{e}_{z}\right)-M_{y} \partial_{x} \boldsymbol{M} \tag{107}
\end{equation*}
$$

We substitute spherical coordinates

$$
M_{x}=\sin \Theta \cos \Phi, M_{y}=\sin \Theta \sin \Phi, M_{z}=\cos \Theta
$$

We have:

$$
\begin{align*}
\frac{\partial M_{x}}{\partial t} & =\cos \Theta \cdot \Theta_{t} \cdot \cos \Phi-\sin \Theta \cdot \sin \Phi \cdot \Phi_{t} \\
\frac{\partial M_{y}}{\partial t} & =\cos \Theta \cdot \Theta_{t} \cdot \sin \Phi+\sin \Theta \cdot \cos \Phi \cdot \Phi_{t}  \tag{108}\\
\frac{\partial M_{z}}{\partial t} & =-\sin \Theta \cdot \Theta_{t}
\end{align*}
$$

and

$$
\begin{align*}
& \frac{\partial M_{x}}{\partial x}=\cos \Theta \cdot \Theta_{x} \cdot \cos \Phi-\sin \Theta \cdot \sin \Phi \cdot \Phi_{x} \\
& \frac{\partial M_{y}}{\partial x}=\cos \Theta \cdot \Theta_{x} \cdot \sin \Phi+\sin \Theta \cdot \cos \Phi \cdot \Phi_{x}  \tag{109}\\
& \frac{\partial M_{z}}{\partial x}=-\sin \Theta \cdot \Theta_{x}
\end{align*}
$$

We have

$$
\begin{align*}
\frac{\partial^{2} M_{x}}{\partial x^{2}} & =-\sin \Theta\left[\cos \Phi\left(\Theta_{x}^{2}+\Phi_{x}^{2}\right)+\sin \Phi \cdot \Phi_{x x}\right]+\cos \Theta\left(\Theta_{x x} \cos \Phi-2 \Theta_{x} \Phi_{x} \sin \Phi\right) \\
\frac{\partial^{2} M_{y}}{\partial x^{2}} & =-\sin \Theta\left[-\sin \Phi\left(\Theta_{x}^{2}+\Phi_{x}^{2}\right)+\cos \Phi \cdot \Phi_{x}\right]+\cos \Theta\left(\Theta_{x x} \sin \Phi+2 \Theta_{x} \Phi_{x} \cos \Phi\right) \\
\frac{\partial^{2} M_{z}}{\partial x^{2}} & =-\cos \Theta \cdot \Theta_{x}^{2}-\sin \Theta \cdot \Theta_{x x} \tag{110}
\end{align*}
$$

So, from (110) we obtain

$$
\begin{aligned}
\frac{\partial^{2} \boldsymbol{M}}{\partial x^{2}}+M_{z} \hat{\mathbf{e}}_{z} & =\left[-\sin \Theta\left[\cos \Phi\left(\Theta_{x}^{2}+\Phi_{x}^{2}\right)+\sin \Phi \cdot \Phi_{x x}\right]+\cos \Theta\left(\Theta_{x x} \cos \Phi-2 \Theta_{x} \Phi_{x} \sin \Phi\right)\right] \hat{\mathbf{e}}_{x} \\
& +\left[\sin \Theta\left[-\sin \Phi\left(\Theta_{x}^{2}+\Phi_{x}^{2}\right)+\cos \Phi \cdot \Phi_{x}\right]+\cos \Theta\left[\Theta_{x x} \sin \Phi+2 \cos \Phi \cdot \Theta_{x} \cdot \Phi_{x}\right]\right] \hat{\mathbf{e}}_{y} \\
& +\left[\cos \Theta\left(1-\Theta_{x}^{2}\right)-\sin \Theta \cdot \Theta_{x x}\right] \hat{\mathbf{e}}_{z}
\end{aligned}
$$

and

$$
\begin{align*}
\boldsymbol{M} \times\left(\frac{\partial^{2} \boldsymbol{M}}{\partial x^{2}}+M_{z} \hat{e}_{z}\right) & =\left[\sin \Theta \sin \Phi \cos \Theta-\sin \Phi \cdot \Theta_{x x}+\cos \Theta \sin \Theta \sin \Phi \cdot \Phi_{x}^{2}\right. \\
& \left.-\cos \Theta \sin \Theta \cos \Phi \cdot \Phi_{x}-2 \cdot \cos ^{2} \Theta \cos \Phi \cdot \Phi_{x} \Theta_{x}\right] \hat{\mathbf{e}}_{x} \\
& -\left[\sin \Theta \cos \Phi \cos \Theta-\sin \Theta \cos \Phi \cos \Theta \cdot \Theta_{x}^{2}-\cos \Phi \cdot \Theta_{x x}\right. \\
& +\cos \Theta \sin \Theta \cos \Phi \cdot\left(\Theta_{x}^{2}+\Phi_{x}^{2}\right)+\cos \Theta \sin \Theta \sin \Phi \cdot \Phi_{x x} \\
& \left.+2 \cdot \cos ^{2} \Theta \cdot \Theta \cdot \Phi_{x} \cdot \sin \Phi\right] \hat{\mathbf{e}}_{y} \\
& +\left[-\sin ^{2} \Theta \cos \Phi \sin \Phi\left(\Theta_{x}^{2}+\Phi_{x}^{2}\right)+\sin ^{2} \Theta \cos ^{2} \Phi \cdot \Phi_{x}\right. \\
& \left.+\sin \Theta \cos \Phi \cos \Theta \sin \Phi \cdot \Theta_{x x}+2 \sin \Theta \cos \Theta \cos \Phi \cdot \Phi_{x} \cdot \Theta_{x}\right] \hat{\mathbf{e}}_{z} \tag{111}
\end{align*}
$$

Also we have

$$
\begin{align*}
M_{y} \partial_{x} \boldsymbol{M} & =\left[\sin \Theta \sin \Phi \cos \Theta \cos \Phi \cdot \Theta_{x}-\sin ^{2} \Theta \sin ^{2} \Phi \cdot \Phi_{x}\right] \hat{\mathbf{e}}_{x} \\
& +\left[\sin \Theta \sin ^{2} \Phi \cos \Theta \Theta_{x}+\sin ^{2} \Theta \sin \Phi \cos \Phi \cdot \Phi_{x}\right] \hat{\mathbf{e}}_{y}  \tag{112}\\
& +\left[-\sin ^{2} \Theta \sin \Phi \Theta_{x}\right] \hat{\mathbf{e}}_{z}
\end{align*}
$$

Finally, substituting the equations (111), (112) and (108) in (107), we obtain $\cos \Theta \cos \Phi \cdot \Theta_{t}-\sin \Theta \sin \Phi \cdot \Phi_{t}=\sin \Theta \sin \Phi \cos \Theta\left(1-\Phi_{x}^{2}\right)-\sin \Phi \cdot \Theta_{x x}-\cos \Theta \sin \Theta \cos \Phi \cdot \Phi_{x}$

$$
\begin{aligned}
& -2 \cdot \cos ^{2} \Theta \cos \Phi \cdot \Phi_{x} \cot \Theta_{x}-\sin \Theta \sin \Phi \cos \Theta \cos \Phi \cdot \Theta_{x} \\
& +\sin ^{2} \Theta \sin ^{2} \Phi \cdot \Phi_{x}
\end{aligned}
$$

$\cos \Theta \sin \Phi \cdot \Theta_{t}+\sin \Theta \cos \Phi \cdot \Phi_{t}=-\sin \Theta \cos \Phi \cos \Theta\left(1+\Phi_{x}^{2}+\Phi_{x x}\right)$

$$
+\cos \Phi\left(\Theta_{x x}+\lambda \sin \Theta \cdot \Phi_{x} \Phi_{x}\right)+\sin \Phi \cos \Theta \cdot \Theta_{x}\left(\lambda-2 \cos \Theta \Phi_{x}\right)
$$

$$
-\sin \Theta \cdot \Theta_{t}=\sin ^{2} \Theta\left(\cos ^{2} \Phi \cdot \Phi_{x}\right.
$$

$$
\begin{equation*}
\left.+\sin ^{2} \Phi \cdot \Phi_{x x}\right)+\sin \Theta \cdot \Theta_{x}\left(2 \cos \Theta \cdot \Phi_{x}-\lambda\right) \tag{113}
\end{equation*}
$$

From the last equation of (113) we have

$$
\begin{align*}
\Theta_{t} & =\sin \Theta \cos \Phi \sin \Phi\left(\Theta_{x}^{2}+\Phi_{x}^{2}\right)-\sin \Theta \cos \Phi \cdot \Phi_{x}-\cos \Phi \cos \Theta \sin \Phi \Theta_{x x}-2 \cos \Theta \cos \Phi \cdot \Phi_{x} \cdot \Theta_{x} \\
& -\sin \Theta \sin \Phi \cdot \Theta_{x} \tag{114}
\end{align*}
$$

Now if we multiply the first equation of (113) with $-\sin \Phi$ and the second equation with $\cos \Phi$ and subtract them we get

$$
\begin{align*}
\Phi_{t} & =-\cos \Theta+\sin ^{2} \Phi \cos \Theta \cdot \Phi_{x}^{2}+\cos \Theta \cos \Phi \sin ^{2} \Phi \cdot \Phi_{x} \\
& +\sin \Theta \sin ^{3} \Phi \cdot \Phi_{x}+\cos ^{2} \Phi \Theta_{x x}-\cos ^{2} \Theta \cos \Phi \cdot \Phi_{x}^{2}+\cos \Phi \sin ^{2} \Phi \cos \Theta+\cos ^{2} \Phi \sin \Theta \sin \Phi \cdot \Phi_{x} \tag{115}
\end{align*}
$$

Let us look for solutions with $\Phi=0$. Then we obtain

$$
\begin{equation*}
\Theta_{t}=0 \tag{116}
\end{equation*}
$$

This tells us that $\Theta$ is constant with respect to time $t$. and equation (115) gives

$$
\begin{equation*}
\Theta_{x x}=\cos \Theta \sin \theta \tag{117}
\end{equation*}
$$

which as we have seen in the LL equation before it is solved by considering the function

$$
V(\Theta)=\Theta_{x}(x, t)
$$

and by substituting it on (149) we finally obtain

$$
\begin{equation*}
\pm \tan \frac{\Theta}{2}= \pm c \exp x \tag{118}
\end{equation*}
$$

Now that we have also obtained the domain wall solution for the LL equation with the DMI term we can notice that in this case we had to set $\Phi=0$ in order to get the (118). While in the case of the domain wall solution for the LL equation, this solution coulsd be obtained by setting $\Phi=C$ where $C$ a random constant value and not specifically 0 .

### 5.7 Axially symmetric Skyrmions

In this section, we will give the equations that describe the different regions of a skyrmion with a big diameter [8].

We consider a two-dimensional ferromagnet on the xy-plane with exchange, DMI, and anisotropy of the easy-axis type perpendicular to the plane. The micromagnetic structure is described via the magnetization vector $\mathbf{m}=\mathbf{m}(x, y)$ with a fixed magnitude normalized to unity, $\mathbf{m}^{2}=1$. The normalized form of the micromagnetic energy is

$$
\begin{equation*}
E_{\epsilon}(m)=\int\left[\frac{1}{2} \partial_{\mu} \mathbf{m} \cdot \partial_{\mu} \mathbf{m}+\frac{1}{2}\left(1-m_{3}^{2}\right)+\epsilon e_{D M}\right] d x \tag{119}
\end{equation*}
$$

A summation over repeated indices $\mu=1,2$ is assumed.
Also, $e_{D M}=\hat{e}_{\mu} \cdot\left(\partial_{\mu} \mathbf{m} \times \mathbf{m}\right)$ models the bulk DM interaction and the interfacial DM interaction form $e_{D M}=\epsilon_{\mu \nu} \hat{e}_{\mu} \cdot\left(\partial_{\mu} \mathbf{m} \times \mathbf{m}\right)$, where $\epsilon_{\mu \nu}$ is the totally antisymmetric two-dimensional tensor. Here $\hat{e}_{1}, \hat{e}_{2}, \hat{e}_{3}$ are the unit vectors for the magnetization in the respective directions. Static magnetization configurations satisfy the LL equation

$$
\begin{equation*}
\mathbf{m} \times\left(\partial_{\mu} \partial_{\mu} \mathbf{m}+m_{3} \hat{e}_{3}-2 \epsilon \mathbf{h}_{D M}\right)=0 \tag{120}
\end{equation*}
$$

where the last term is the DM field with $\mathbf{h}_{D M}=\hat{e}_{3} \times \partial_{\mu} \mathbf{m}$ in case of bulk interaction or $\mathbf{h}_{D M}=\epsilon_{\mu \nu} \hat{e}_{3} \times \partial_{\nu} \mathbf{m}$ in case of interfacial DM. In equations (119) and (120), lengths are measured in units of the domain wall width $l_{w}=\sqrt{\frac{A}{K}}$, where A is the exchange and K the anisotropy constant. The equation contains a single parameter

$$
\begin{equation*}
\epsilon=\frac{l_{s}}{l_{w}}=\frac{D}{2 \sqrt{A K}} \tag{121}
\end{equation*}
$$

defined via an additional length scale of this model $l_{s}=\frac{D}{2 K}$, where D is the DM parameter. We will refer to $\epsilon$ as the dimensionless DM parameter, which can also be controlled by changing the anisotropy or the exchange parameter. The lowest energy (ground) state is the spiral for $\epsilon>\frac{2}{\pi}$ and the ferromagnet state, which we are interested in, for $\epsilon<\frac{2}{\pi}$.

Let us consider the angles $(\Theta, \Phi)$ for the spherical parametrization of the magnetization vector, and the polar coordinates $(r, \phi)$ for the film plane. We assume an axially symmetric skyrmion with $\Phi=\phi+\phi_{0}$ and $\Theta=\Theta(r)$. For a bulk DM term the energy is minimized for $\phi_{0}=\frac{\pi}{2}$ (Bloch skyrmion) and for interfacial DM interaction we choose $\phi_{0}=0$ (Neel skyrmion). A value $0<\phi_{0}<\frac{\pi}{2}$ should be chosen if the DM term is a combination of the bulk and interfacial terms.

The skyrmion profile arises as a local minimizer of the energy

$$
\begin{equation*}
E_{\epsilon}(\mathbf{m})=2 \pi \int_{0}^{\infty}\left[\frac{1}{2}\left(\frac{d \Theta}{d r}\right)^{2}+\frac{1}{2}\left(1+\frac{1}{r^{2}}\right) \sin ^{2} \Theta+\epsilon\left(\frac{d \Theta}{d r}+\frac{1}{2 r} \sin 2 \Theta\right)\right] r d r \tag{122}
\end{equation*}
$$

of
$\mathbf{m}(r, \phi)=\left(\sin \Theta \cos \left(\phi+\phi_{0}\right), \sin \Theta \sin \left(\phi+\phi_{0}\right), \cos \Theta\right)$ where $\Theta=\Theta(r)$ satisfies the equation

$$
\begin{equation*}
\Theta^{\prime \prime}+\frac{\Theta^{\prime}}{r}-\frac{\sin (2 \Theta)}{2 r^{2}}-\frac{\sin (2 \Theta)}{2}+2 \epsilon \frac{\sin ^{2}(\Theta)}{r}=0 \tag{123}
\end{equation*}
$$

with boundary conditions $\Theta(0)=\pi$ and $\lim _{r \mapsto \infty} \Theta(r)=0$. The same equation applies to all types of skyrmions, Bloch or Néel for the respective DM terms.

### 5.8 Skyrmion profile

Let us study skyrmions with large radius $R$, defined by the equation

$$
\begin{equation*}
\Theta(R)=\frac{\pi}{2} \tag{124}
\end{equation*}
$$

. The skyrmion profile exhibits three spatial regions [8]. The skyrmion core is the region where the value of $\Theta$ is close to $\pi$. That means that the magnetization pointing close to the south pole. The outer region (or far field) is where $\Theta$ is exponentially close to zero and the magnetization pointing close to the north pole. The skyrmion domain wall is the thin region that connects the core and the outer region.Using assyptotic analysis we obtain the following results. Close to the skyrmion center, the deviation of the skyrmion profile from $\pi$ is linear with an exponentially small factor

$$
\begin{equation*}
\Theta \approx \pi-\exp ^{-R} \sqrt{2 \pi R r}, r \ll 1 \tag{125}
\end{equation*}
$$

As r increases, the deviation attains exponential growth. This is held in check by the small factor throughout the skyrmion core, up ti approach to the domain wall

$$
\begin{equation*}
\Theta \approx \pi-2 \sqrt{\frac{R}{r}} \exp ^{r-R}, 1 \ll r \ll R \tag{126}
\end{equation*}
$$

The leading approximation of the skyrmion domain wall profile is independent of the radius when the radius is large. Past the domain wall, in the far field, the behaviour is similar to the one of skyrmions of small redius. We have

$$
\begin{equation*}
\Theta \approx 2 \sqrt{\frac{R}{r}} \exp ^{-(r-R)} \tag{127}
\end{equation*}
$$

## 6 Micromagnetic Simulations

Now that we have discussed the basics of the concept of skyrmions, we are ready to study how their diameter increases or reduces and as a result their size, in relation with the DMI, anisotropy $\left(E_{\text {ani }}\right)$ and exchange $\left(E_{e x}\right)$ energy.In the simulations that we discuss below, we are interested in the stydy of Neel skyrmions. Therefore, we study how the size of the skyrmion is affected by the variation of the iDMI (interfacial DMI), because as we have discussed in previous sections, this is the interaction that leads to the formation of a Neel skyrmion with radial chirality.

Micromagnetic simulations solve the LLG equation

$$
\begin{equation*}
\frac{d \mathbf{m}}{d t}=-|\gamma| \mathbf{m} \times \boldsymbol{F}+\alpha\left(\mathbf{m} \times \frac{d \mathbf{m}}{d t}\right) \tag{128}
\end{equation*}
$$

to obtain the stable magnetization configuration in a ferromagnetic material, where $\gamma$ is the gyromagnetic ratio and $\alpha$ is the Gilbert damping constant. $\mathbf{F}$ is the effective magnetic field of the system which is given by:

$$
\begin{equation*}
\boldsymbol{F}=\frac{-1}{\mu_{0} M_{s}} \frac{\delta E_{\text {total }}}{\delta \mathbf{m}} \tag{129}
\end{equation*}
$$

where $\mu_{0}$ is the permeability in vacuum and $E_{\text {total }}$ is the magnetic energy of the system which consists of various energy terms such as exchange ( $E_{e x}$ ), anisotropy ( $E_{\text {ani }}$ ) and DMI $\left(E_{D M I}\right)$. Therefore, $E_{\text {total }}$ is

$$
\begin{equation*}
E_{\text {total }}=E_{\text {demag }}+E_{\text {ani }}+E_{e x}+E_{D M I} \tag{130}
\end{equation*}
$$

where $E_{\text {demag }}$ is the demagnetization energy, $E_{e x}=-J_{i, j}\left(\hat{\mathbf{s}_{i}} \cdot \hat{\mathbf{s}_{j}}\right), E_{\text {ani }}=K\left(\hat{\mathbf{s}}_{i} \cdot \hat{z}\right)^{2}$ and $E_{D M I}=\vec{d}_{i j} \cdot\left(\hat{s}_{i} \times \hat{s}_{j}\right)$ respectively.

### 6.1 The dependence of skyrmion diameter on the DM parameter

We remind once again that such whirling structures as skyrmions are induced by the DMI, which is a macroscopic characteristic of interacting spins that lacks inversion symmetry and that has a strong spin-orbit coupling.

The magnetic simulations were performed on thick multilayer structures with area of 150 nmx 150 nm , with saturation magnetization $M_{s}=10^{6} \frac{A}{m}$, and exchange constant $A=15.0 \times 10^{-12} \frac{\mathrm{~J}}{\mathrm{~m}}$. The figure below shows formation of a typical Neel skyrmion from a bubble domain state at the center of a ferromagnetic element ( 150 nmx 150 nmx 1 nm ). The bubble domain state has two distinct spin configurations with the red and blue areas representing the +z and -z spin directions, respectively. The somewhat white


Figure 31: Skyrmion
area in the middle of the configuration has spins in the xy plane which represents the width of the domain wall. As the simulation begins, the influence of the iDMI starts realigning the spins into skyrmionic state where the change in spin direction from $+z$ to $-z$ occurs continuously over a large area. The simulation ends when the spin configuration of the skyrmion becomes stable.

We observe, from the figure below, that the increase of the diameter of the skyrmion is analogous to the DMI increase. This is more clear if we recall how we defined the DMI energy. Indeed, $E_{D M I}=\boldsymbol{d}_{i j} \cdot\left(\boldsymbol{s}_{i} \times \boldsymbol{s}_{j}\right)$. Let us think when is this energy maximized. We obtain the maximum $E_{D M I}$ when the spins are aligned perpendicular to each other $\left(\theta=\frac{\pi}{2}\right)$, because of the cross product on the definition of the DMI energy. That results an increasement on the diameter of the skyrmion. So, as the DMI increases (becomes stronger), the spins are being perpendicular to each other and the diameter of the skyrmion increases. For DMI values between $1.0 \times 10^{-3} \frac{\mathrm{~mJ}}{\mathrm{~m}^{2}}$ and $1.4 \times 10^{-3} \frac{\mathrm{~mJ}}{\mathrm{~m}^{2}}$, there is no skyrmion observed by the simulations.


Figure 32: The skyrmion diameter as a function of the DM parameter $D$ obtain by numerical simulations.

### 6.2 Dependence of skyrmion diameter on the anisotropy parameter

The magnetic anisotropy describes how an object's magnetic properties can be different depending on the direction. For most magnetically anisotropic materials, which are $180^{\circ}$ rotation apart. The line parallel to these direction is called the easy axis. In other words, the easy axis is an energetically favourable direction of spontaneous magnetization.

For the magnetic simulations the exchange constant $A=15.0 \times 10^{-12} \frac{\mathrm{~J}}{\mathrm{~m}}$, the DMI constant is $D=2.0 \times 10^{-3} \frac{m J}{m^{-2}}, M_{s}=10^{6} \frac{\mathrm{~A}}{\mathrm{~m}}$. From the figure down below we observe that the skyrmion diameter is not analogous to the magnetic anisotropy but decreases as the second one increases. Again, we have defined the anisotropy energy as $E_{\text {ani }}=K\left(\boldsymbol{s}_{i} \cdot \hat{z}\right)^{2}$ and we observe that as this energy increases, the spins are being aligned along the easy axis. This has as a result the reducing of the skyrmion diameter.


Figure 33: Skyrmion diameter for various values of the anisotropy parameter $K$ obtained by numerical simulations.

An interesting part of this figure is for anisotropy between $0.92 \times 10^{6}$ and $0.94 \times$ $10^{6} \frac{\mathrm{~J}}{\mathrm{~m}^{3}}$, where we observe that the skyrmion diameter reduces so, in the end there is no skyrmion to be found by the simulation. In general, when the simulations do not find


Figure 34: Size of the skyrmion with 0.942 anisotropy
a skyrmion structure, we need to control whether this is a numerical error or indeed there is no skyrmion to be found. This can happen with many ways. The changing on the boundary conditions of the initial problem is one of them. But this would change the problem drastically and this is something we don't want. Instead, at first we proceed in the simulation by checking what is happening for a much smaller step for values of anisotropy above $0.94 \times 10^{6}\left(0.942 \times 10^{6}, 0.944 \times 10^{6}, 0.946 \times 10^{6}, 0.948 \times\right.$ $\left.10^{6}, 0.949 \times 10^{6}, 0.9492 \times 10^{6}\right)$ and when there is no skyrmion to be found for values of anisotropy above $0.9492 \times 10^{6}$ we choose to change the number of cells.For the later case, we remember that we started our simulations with an area of 150 nmx 150 nm . So the idea is that if we increase the number of cells, for instance, 300 nmx 300 nm , while at the same time reducing the discretization from $4.0 \times 10^{-9}$ to $2.0 \times 10^{-9}$, the simulation will be done on a bigger sample. We observe that the skyrmion size still reduces for values above $0.94 \times 10^{6} \frac{\mathrm{~J}}{\mathrm{~m}^{3}}$, while later for values above $0.9492 \times 10^{6}$, working on a bigger sample, there is no skyrmion. So this leads us to the initial thought of ours, which was that there must be an error from the numerical method that had been held. This error could be that, above the limit of $0.9492 \times 10^{6}$ anisotropy, the diameter of the skyrmion is very small, so the simulation is not able to detect whether there is a skyrmion.

### 6.3 Dependence of skyrmion diameter on an external field

In the figure below, we observe what is the change of the skyrmion size if we apply an external magnetic field.


Figure 35: Size of the skyrmion in relation to the applied external magnetic field
We notice that as the external magnetic field becomes stronger, the size of the skyrmion reduces. The explanation here is very simple, and it has to do with the external field being applied. Moreover, if we apply an external magnetic field the spins tend to align with it, on the direction that we are applying it. We need to be a little careful here, because the application of the external magnetic field plays a crucial role on the variation of the size of the skyrmion. In this specific case that we study here and that is shown on the figure below, we apply an external field right above the film. As a result this makes all spins being aligned on the direction of the applied field, which is right above, on the z-axis, so the size of the skyrmion reduces. Instead, if we had applied an external field left or right to the film, that would have resulted the increase of the size of the skyrmion. But we didn't study this case.

### 6.4 Dependence of skyrmion's diameter on the exchange parameter



Figure 36: Diameter of the skyrmion in relation to the exchange parameter A
The exchange energy is defined as $E_{e x}=-A_{i j}\left(\hat{s}_{i} \cdot \hat{s}_{j}\right)$. This is maximized when the spins are aligned parallel $(\theta=\pi)$ to each other. As we increase the exchange parameter $A$, the spins begin to align parallel to each other causing a reducing of the size of the skyrmion as shown on the figure above. For the simulations, we worked with DMI parameter $D=2.0 \times 10^{-3} \frac{\mathrm{~mJ}}{\mathrm{~m}^{-2}}, M_{s}=10^{6} \frac{\mathrm{~A}}{\mathrm{~m}}$, and anisotropy parameter $K=0.80 x 10^{6} \frac{\mathrm{~J}}{\mathrm{~m}^{3}}$.

### 6.5 Dependence of the Skyrmion Radius on dimensionless parameter $\epsilon$



Figure 37: The radius of the skyrmion in relation to the dimensionless parameter $\epsilon$
We assume a ferromagnetic material as a two-dimensional system lying on the xy-plane. The micromagnetic structure is described via the magnetization vector $\boldsymbol{m}=\boldsymbol{m}(x, y)$ with a fixed magnitude normalized to unity, $\boldsymbol{m}^{2}=1$. The normalized magnetic energy is

$$
\begin{equation*}
E(\boldsymbol{m})=\int\left(\frac{\partial_{\mu} \boldsymbol{m} \cdot \partial_{\mu} \boldsymbol{m}}{2}+\frac{1-m_{3}^{2}}{2}+\epsilon \hat{\mathbf{e}}_{\mu} \cdot\left(\partial_{\mu} \boldsymbol{m} \times \boldsymbol{m}\right)\right) d x \tag{131}
\end{equation*}
$$

where summation over repeated indices $\mu=1,2$ is implied and $\hat{e}_{1}, \hat{e}_{2}, \hat{e}_{3}$ are the unit vectors for the magnetization in the respective directions. Static magnetization fields are local minimizers of the E satisfying the normalized LL equation

$$
\boldsymbol{m} \times \boldsymbol{h}=0
$$

where the effective field is

$$
\boldsymbol{h}=\partial_{\mu} \partial_{\mu} \boldsymbol{m}+m_{3} \hat{e}_{3}-2 \epsilon \hat{e}_{\mu} \times \partial_{\mu} \boldsymbol{m}
$$

is minus the variational gradient of $E=E(\boldsymbol{m})$. We measure legths in units of the domain wall width $l_{w}=\sqrt{\frac{A}{K}}$ and the equation contains a single parameter

$$
\epsilon=\frac{l_{s}}{l_{w}}=\frac{D}{2 \sqrt{A K}}
$$

where $l_{s}=\frac{D}{2 K}$. The ferromagnet state is for $\epsilon<\frac{2}{\pi}$.
We consider angles $(\Theta, \Phi)$ for the spherical parameterization of the magnetization vector, and the polar coordinates $(r, \phi)$ for the film plane. We assume an axially symmetric skyrmion with $\Phi=\phi+\frac{\pi}{2}$ and $\Theta=\Theta(\boldsymbol{r})$, called Bloch skyrmion. Of course, we could consider other types of skyrmions, for example Neel skyrmion where $\Phi=\phi$ with iDMI. The equation for the profile $\Theta=\Theta(r)$ is

$$
\begin{equation*}
\Theta^{\prime \prime}+\frac{\Theta^{\prime}}{r}-\frac{\sin 2 \Theta}{2 r^{2}}-\frac{\sin 2 \Theta}{2}+2 \epsilon \frac{\sin ^{2} \Theta}{r}=0 \tag{132}
\end{equation*}
$$

with boundary conditions $\Theta(0)=\pi$ and $\lim _{r \mapsto \infty} \Theta(r)=0$ is the same for all typed of skyrmions, therefore the following calculations apply equally to all of them.

Now we can find the relation between the radius of the skyrmion with the parameter $\epsilon$.

We have

$$
\begin{equation*}
\epsilon(R)=-R \ln \left(\frac{R}{\alpha}\right) \tag{133}
\end{equation*}
$$

where $\alpha=0.2065$ and $\epsilon \ll 1$.
We will present two equations for the radius R in relation to the parameter $\epsilon$. The first equation describes the a small radius of a skyrmion in relation to $\epsilon$ and it is

$$
\begin{equation*}
R(\epsilon)=\frac{-\epsilon}{\ln \left(\frac{\epsilon}{\beta}\right)} \tag{134}
\end{equation*}
$$

where $\beta=0.72, R \ll 1$ and in the graph is represented by the blue line. On the other hand, the equation that describes large radius of skyrmions in relation to $\epsilon$ is

$$
\begin{equation*}
R(\epsilon)=\sqrt{\frac{\delta}{\frac{2}{\pi}-\epsilon}} \tag{135}
\end{equation*}
$$

where $\delta=0.3057$, and in the graph is represented by the green line.
For the simulations we set the parameters: $A=15.0 \times 10^{-12} \frac{\mathrm{~J}}{\mathrm{~m}}, K=0.92 \times$ $10^{6} \frac{\mathrm{~J}}{\mathrm{~m}^{3}}, M_{s}=10^{6} \frac{\mathrm{~A}}{\mathrm{~m}}$. We check how the radius change for different values of the DMI parameter, starting from the value $1.48 \times 10^{-3} \frac{J}{m^{2}}$ to $2.18 \times 10^{-3} \frac{J}{m^{2}}$ while making sure to change the number of cells and the discretization value. Specifically, for
small radius we change the discretization value for the specific values of the DMI that give us small radius (from $1.48 \times 10^{-3}$ until $1.90 \times 10^{3}$ ). For the discretization we set $2.0 \times 10^{-9}, 2.5 \times 10^{-9}$ and $4.0 \times 10^{-9}$. On the other hand, for the values of the DMI that we get skyrmions with large radius, we changed the number of cells ( $180 \mathrm{nmx} 180 \mathrm{~nm}, 200 \mathrm{nmx} 200 \mathrm{~nm}, 300 \mathrm{nmx} 300 \mathrm{~nm}$ ). The numerical results are noted by the red line and dots on the graph. In conclusion we see that the numerical results (red line and dots) are matching the analytical results (green and blue line). So indeed, the equations (134) and (135) are representing the small and the large radius of the skyrmions,respectively.

### 6.6 Concluding remarks

Based on the observations above, we conclude that the dependence of the skyrmion's size is different from the DMI parameter than it is from the others. At this point we should note that, by radius we define the distance between the end of -z spinorientation (blue region of figure 30) to the center of $z+$ spin orientation (red region of figure 30). We noticed so far that as we increase the DMI parameter of the system, the size of the skyrmion tends to increase as well, and above the value $2.20 \times 10^{-3}$, the stable state is not a skyrmion, but stripe domains as shown on the figure 37 below. On the other hand,the dependence of the skyrmion's diameter on anisotropy,


Figure 38: Skyrmion Diameter for $D=2.4 \times 10^{-3}$
exchange and the external field parameters, are similar. Obviously, from the figures above, we conclude that as we increase the values of these three parameters the size of the skyrmion reduces, until there is no skyrmion to be found by the simulations. These observations come directly by the simulation that have been made and their results are shown up above.

At last, we make some observations for the parameter $\epsilon$. We start with equation (134). In order for the radius to be well defined we set the restriction $\epsilon>0.72$. We notice that as $\epsilon \rightarrow 0$ then $R \rightarrow 0$ while on the other hand, as $\epsilon \rightarrow \infty$ then R diverges to $-\infty$. This behaviour was expected, as is shown from the figure 36. The blue line that represents the small radius of skyrmions, tends to 0 for small values of $\epsilon$ while for big values of $\epsilon$, it gives as no valid description for what happens there, because, as we defined, this equation shall describes only small radius. Now, from equation (135) at first we take the restriction $\epsilon<\frac{2}{\pi}$, because the radius must be positive and real. We notice that as $\epsilon \rightarrow 0$ the radius tends to a constant value, $\sqrt{\frac{\delta \pi}{2}} \approx 0.6929$. On the other hand, we notice that when $\epsilon \rightarrow \frac{2}{\pi}$, the radius R diverges to infinity. Again, these two observations agree with the green line that is shown on figure 36, and describes the behaviour of big radius. We can conclude some more information from the radius of the skyrmion in relation to the $\epsilon$ parameter. If we substitute the (133) on equation (134) we obtain that when $K \rightarrow+\infty$ the radius R goes to zero. This agrees with the results that we discussed for the skyrmion's diameter in relation to the anisotropy parameter. Also, if we let $D \rightarrow 0$ in equation (134) we obtain that $R \rightarrow 0$, which again we observe from figure 31 . So, in conclusion, for very big values of the anisotropy parameter K or for very small values of the parameter D , the skyrmion's size tends to zero.

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