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Undergraduate Thesis: Exotic Holographic Renormalization Group Flows

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Abstract: Recent developments in the study of holographic Renormalization Group (RG) flows have unveiled exotic local solutions which fall outside of the Fefferman Graham (FG) description of asymptotically anti-de Sitter space-times. These solutions lead to a lower dimensional boundary metric, posing a potential challenge to our understanding of the holographic principle if they can be globally extended. In this undergraduate thesis, we will review the framework of holography as well as the usual techniques used to solve such problems. Then, we will discuss possible generalizations to the FG methodology to incorporate these exotic solutions in the standard dictionary of holography and to be able to calculate the relevant vacuum expectational values (vevs).

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Intro

1. Fundamental Concepts of Quantum Field Theory (QFT)

We begin our introduction with fundamental concepts of QFT which will be necessary to understand the topics discussed later in this thesis. The first two subsections will introduce the concepts of renormalization and the renormalization group (RG). The book of Peskin and Schroeder "An introduction to Quantum Field Theory" is an excellent starting point to introduce these topics. [1]

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1.1 Renormalization

In QFT particles are not the most fundamental objects which describe a given system. Instead, we reduce the description to that of underlying quantum fields. The particles can then be thought of as excited states of these fields. There are some key advantages to that description, with the most important being arguably that we can study the creation and annihilation of particles. The physically relevant quantities in QFT are the correlation functions:

$$\langle 0|T(\varphi(x)\varphi(y))|0\rangle \quad , \quad T(\varphi(x)\varphi(y)) = \Theta(x^0 - y^0)\varphi(x)\varphi(y) + \Theta(y^0 - x^0)\varphi(y)\varphi(x) \tag{1.1}$$

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This quantity for example, which is known as the two point function, describes the probability amplitude of a field φ to propagate from the space-time event x to the space-time event y and it appears often in QFT calculations. In general, we can take correlation functions in any state, but the simples ones are the vacuum correlation functions, taken either at the free vacuum state $|0\rangle$, or the interacting vacuum state $|\Omega\rangle$.

The calculations become more complicated when one considers interacting QFTs. Delving into details would require multiple semesters worth of advanced undergraduate or graduate level physics to explain. For the purposes of understanding this thesis, it is sufficient to know that in most cases we are limited to perturbative techniques to study an interacting system. The usual approach is to expand the interacting part of the correlation function in powers of the coupling constants. These are the famous Feynman diagrams, which are a tool of expressing complicated integrals in a concise diagrammatic way. Each diagram comes with a set of rules that describe what the different lines and vertexes of a diagram correspond to. Of course, as we will discuss in later subsections, when the coupling constant is not sufficiently small to be considered a perturbation, as in the case of QCD, these methods fall apart.

To motivate the need of renormalization, consider as an example the so-called one-loop correction (the first non-trivial perturbative correction to the correlation function) for an interacting Lagrangian given by:

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_0^2\varphi^2 - \frac{\lambda_0}{4!}\varphi^4 \tag{1.2}$$

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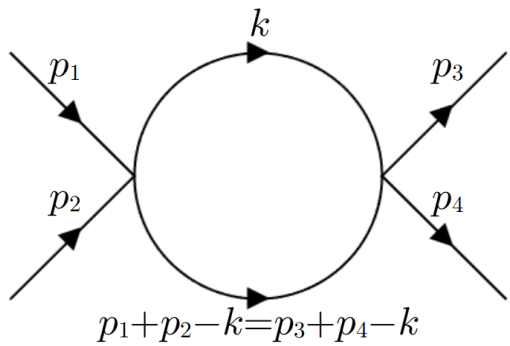
One of the Feynman diagrams we would have to evaluate to study a one-loop correction to the scattering $\varphi_1\varphi_2 \to \varphi_3\varphi_4$ would be the following:



The Feynman rules of the theory instruct us to integrate over all possible values of the internal momenta appearing in the diagram, which in our case is just k. These lines are interpreted as virtual particles. However, if we follow the rules for the free scalar propagators, i.e. the rules associated with the lines in the above diagram, we would find out that the integral diverges when k takes large values. This is a case of a UV divergence and as one might guess, these divergences appear a lot in QFT.

To make matters worse, even if all of the real particles' momenta $p_1 - p_4$ took small values, the virtual particles are still allowed to be created with arbitrarily large momenta. These divergences are problematic, because the emerging correlation functions would be infinite, which is contrary to our physical interpretation of them being associated with probability amplitudes. Renormalization is our way of dealing with these divergences and obtaining the physical information hidden behind the emerging infinities.

There are various techniques to make these infinities disappear, each with their advantages and disadvantages. In our case, the technique we will be using is the addition of counterterms in our Lagrangian. The physical argument for this method is that the coupling constants m_0 and λ_0 appearing in the equations are not necessarily the physical ones that we measure in our lab. The couplings appearing in the Lagrangian of (1.2) are called bare couplings and usually they are denoted with an index of 0. This choice of indexing is to distinguish them from the physical couplings, usually called dressed, which will appear without an index of 0.

To renormalize the Lagrangian via counterterms:

- We adopt a finite UV momentum cutoff Λ, ie we allow the momentum k to take values up to Λ. Now the integral over k is not divergent, or as it is usually called, it is regulated.

- We express the bare couplings $g_{0,i}$ that appear in our Lagrangian (in our case i takes 2 values, with the first being associated with m_0 and the second with λ_0) as:

$$g_{0,i} = g_i - \delta g_i \tag{1.3}$$ as03

The terms δg_i are called counterterms and the dressed couplings g_i are the physical observables that we measure in the lab.

- This is the most important step. Since the counterterms are arbitrary, we select them so that they cancel out the emerging infinities in the limit where $\Lambda \to \infty$. Now when we take that limit the regulated integral becomes finite.

- We evaluate the finite part of the integral, which has now become renormalized.

There is a perplexing conceptual observation which emerges upon describing the aforementioned process. The energy scale of the scattering we discussed is derived through the momenta of the external particles. We can think of the energy scale as an order of magnitude estimate for the energies we would expect to appear in the system. Renormalization would seem to imply that virtual particles whose momentum lies outside of the system's energy scale do not affect the physics of the system we are studying significantly. The renormalization group will help us understand this feature of QFT.

as2 **1.2 The Renormalization Group (RG)**

A useful quantity in QFT is the generating functional of vacuum correlation functions. In d-dimensional Minkowski space-time we have:

$$\mathcal{Z}[J] \equiv \int \mathcal{D}\varphi \;\; e^{i\int \mathcal{L}[\varphi;g_i]+J(x)\varphi(x)dx^d} \quad , \quad \mathcal{D}\varphi \equiv \prod_{k=0}^{\infty} d\varphi(k) \tag{1.4}$$ as20

Note that the momentum is in general a non integer. The product in (1.4) is a product over all possible momenta. The generating functionals offer a convenient way of studying correlation functions through functional differentiation. The momentum cutoff Λ we discussed in the previous section would translate to us integrating over momenta with $|k| < \Lambda$.

To study the effect of high energy virtual particles in our system we can explicitly integrate over momenta with values close to the cutoff and obtain an effective description of our system at lower energies. The comparison of the new generating functional with the old will uncover the effect of the high energy momenta. For simplicity we will also set:

$$J(x) \equiv 0, \tag{1.5}$$ as21

and we will study the specific example of the Lagrangian in the previous subsection. Of course it is straightforward, to generalize the following calculations to any renormalizable Lagrangian by expressing any arbitrary analytic potential $V(\varphi)$ as a power series in powers of φ.

We begin by bringing the generating functional to Euclidean form:

$$\mathcal{Z}_E[0] \equiv \int [\mathcal{D}\varphi]_\Lambda \;\; e^{-\int dx_E^d \left(\frac{1}{2}(\partial_E\varphi)^2 + \frac{1}{2}m_0^2\varphi^2 + \frac{\lambda_0}{4!}\varphi^4\right)} \quad , \quad [\mathcal{D}\varphi]_\Lambda \equiv \prod_{k=0}^{\Lambda} d\varphi(k) \tag{1.6}$$ as22

Henceforth we will drop the index E for convenience. We have to decompose the scalar field into high energy and low energy modes. To do that consider a real number b, with $0 < b < 1$ and define:

$$\chi(k) \equiv \begin{cases} \varphi(k) \ , \ b\Lambda \leq k < \Lambda, \\ 0 \ , \ \text{else} \end{cases} \tag{1.7}$$ as23

$$\phi(k) \equiv \begin{cases} \varphi(k) \ , \ k < b\Lambda, \\ 0 \ , \ \text{else} \end{cases} \tag{1.8}$$ as24

Using the definitions in (1.7) and (1.8) we may identify the high energy modes with χ and the low energy modes with ϕ. Now we can write:

$$\varphi = \phi + \chi \quad , \quad [\mathcal{D}\phi]_{b\Lambda} \equiv \prod_{k=0}^{b\Lambda} d\phi(k) \quad , \quad \mathcal{D}\chi \equiv \prod_{k=b\Lambda}^{\Lambda} d\chi(k) \tag{1.9}$$ as25

and we rewrite (1.6) as follows:

$$\mathcal{Z}[0] \equiv \int [\mathcal{D}\phi]_{b\Lambda} \;\; e^{-\int dx^d \mathcal{L}_{eff}[\phi;g_i]}, \tag{1.10}$$ as26

with:

$$e^{-\int \mathcal{L}_{eff}[\phi;g_i]dx^d} \equiv e^{-\int \mathcal{L}[\phi;g_i]} \int \mathcal{D}\chi \;\; e^{-\left(\int \mathcal{L}[\chi;g_i] + \mathcal{L}_I[\phi,\chi;g_i]\right)dx^d}, \tag{1.11}$$ as27

with the Lagrangian $\mathcal{L}_I$ containing all terms that include both the high and the low energy modes.

Upon integrating out the higher energy modes, we observe that the resulting effective Lagrangian takes the following form:

$$\mathcal{L}_{eff}[\phi; g_i] = \frac{1}{2}(1+\Delta\alpha)(\partial\phi)^2 + \frac{1}{2}(m^2 + \Delta m^2)\phi^2 + \frac{1}{4!}(\lambda + \Delta\lambda)\phi^4 + ... \tag{1.12}$$ as28

The integration of the high energy modes will generate corrections to all possible powers of ϕ, $\partial\phi$ and their products. Essentially, the effect of the higher energy modes has been absorbed into the coupling constants and now we have an effective

description of the system for lower energies.

Let us now compare the partition functions of (1.6) and (1.10). On one hand we have equation (1.6):

$$Z[0] = \int [\mathcal{D}\varphi]_\Lambda e^{-\int dx^d \mathcal{L}[\varphi;g_i]} \tag{1.13}$$ as29

On the other hand we have (1.10):

$$Z[0] = \int [\mathcal{D}\phi]_{b\Lambda} e^{-\int dx^d \mathcal{L}_{eff}[\phi;g_i]} \tag{1.14}$$ as210

By performing the rescaling:

$$k' = \frac{k}{b} \quad , \quad x' = xb \tag{1.15}$$ as211

in (1.14), we can write $\phi = \varphi$ and match the cutoffs. Equation (1.14) becomes:

$$Z[0] = \int [\mathcal{D}\varphi]_\Lambda e^{-\int dx^d b^{-d}(\frac{1}{2}(1+\Delta\alpha)b^2(\partial\varphi)^2+\frac{1}{2}(m^2+\Delta m^2)\varphi^2+\frac{1}{4!}(\lambda+\Delta\lambda)\varphi^4+...)} \tag{1.16}$$ as212

But now we can rescale the scalar and adopt new coupling constants in the effective Lagrangian, in order to obtain correlation functions of the same form as those that we would obtain from the original Lagrangian:

$$\varphi' = \sqrt{(1+\Delta\alpha)b^{2-d}}\varphi \quad , \quad m'^2 = \frac{m^2+\Delta m^2}{b^2(1+\Delta\alpha)} \quad ,$$

$$\lambda' = \frac{\lambda+\Delta\lambda}{(1+\Delta\alpha)^2 b^{4-d}} \quad , \quad ... \tag{1.17}$$ as213

To summarize, we saw that the process of integrating out the high energy momentum modes and rescaling momenta is analogous to a rescaling of the Lagrangian. In the limit $b \to 1$ this transformation is continuous. If we repeat this process an arbitrary number of times, we can think of its effects as a flow over all possible coupling constants. For historical reasons we call this transformation of the Lagrangian "The Renormalization Group", although it is not a group with its strict mathematical definition.

Let us return to the observation that motivated the study of RG flows, i.e., that the high energy modes of the virtual particles seem to be negligible in perturbative QFT corrections. We learned that the effect of these higher energy modes has been absorbed in the coupling constants that describe our system at the given energy scale. In a sense, we can think of the coupling constants as functions of the system's

energy scale. All the energy modes which lie far beyond the system's energy scale can be integrated through the Renormalization Group until we reach an effective description applicable to our system's energy scales.

Let us examine systematically how we can obtain the RG equations through which we can express a coupling constant as a function of the system's energy scale. Based on what we learned about the RG we saw that an infinitesimal variation of the cutoff given by:

$$\Lambda \to \Lambda + \delta\Lambda, \tag{1.18}$$ as214

corresponds to the following changes in the fields and the coupling constants:

$$\varphi \to (1+\delta\eta)\varphi \quad , \quad g_j \to g_j + \delta g_j \tag{1.19}$$ as215

The index j ranges over all possible coupling constants. An interacting n-point correlation function, i.e.:

$$G^{(n)}(\{x_i\}) \equiv \langle\Omega|T\left(\varphi(x_1)\varphi(x_2)...\varphi(x_n)\right)|\Omega\rangle \quad , \quad i = 1, 2, \cdots, n \tag{1.20}$$ as216

would transform (according to the first transformation in (1.19)) as:

$$G^{(n)}(\{x_i\}) \to (1+n\delta\eta)G^{(n)}(\{x_i\}) \tag{1.21}$$ as217

If we consider that G should be a function of the cutoff and the coupling constants we can write:

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial\Lambda}\delta\Lambda + \frac{\partial G^{(n)}}{\partial g_j}\delta g_j = n\delta\eta G^{(n)} \tag{1.22}$$ as218

Defining:

$$\beta_j(g_j) \equiv \frac{\Lambda}{\delta\Lambda}\delta g_j \quad , \quad \gamma(g_j) \equiv -\frac{\Lambda}{\delta\Lambda}\delta\eta \tag{1.23}$$ as219

we obtain the Callan-Symanzik equation:

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta_j\frac{\partial}{\partial g_j} + n\gamma\right] G^{(n)}(\{x_i\};\Lambda, g_j) = 0 \tag{1.24}$$ as220

The dimensionless quantities in (1.23) are called the β and γ functions. Since the n-point function is renormalized they are independent of the cutoff and they depend only upon the coupling constants.

Finally, let us consider what can happen when a Lagrangian transforms under the RG. Starting from an arbitrary point in the space of coupling constants, there are two possible outcomes:

- The process does not converge. This case is not interesting.

- The process converges to a fixed point of the flow, meaning a point that is invariant under an RG transformation.

As we can see from equation (1.15), for a fixed point to remain invariant under an RG transformation x must be either 0 or ∞. This means that our system has no characteristic length or energy scale. Moreover, the fixed points of a RG transformation are characterized by conformal symmetry, a symmetry in the scaling of the coordinates in a way that preserves angles.

Finally, let us return to (1.12) and consider what happens near a fixed point. The different powers of φ, $\partial\varphi$ and their products are often called operators. If we denote them with $\mathcal{O}(x)$, then the interacting part of the action can be expressed as:

$$S_I[\varphi] \propto \int dx^d g_i \mathcal{O}_i(x) \tag{1.25}$$ as221

Since the action remains invariant we observe that if the operators scale as:

$$\mathcal{O}_i(x) \to b^{-\Delta_{+,i}} \mathcal{O}_i(x) \tag{1.26}$$ as222

then the coupling constants must scale as:

$$g_i \to b^{-\Delta_{-,i}} g_i \quad , \quad \Delta_{-,i} \equiv d - \Delta_{+,i} \tag{1.27}$$ as223

The operators can then be classified according to the value of the corresponding Δ_+ as follows:

- If $\Delta_- > 0$ the operator is called relevant. According to (1.27) we see that this operator moves away from the fixed point as we move to the IR ($b \to 0$) and therefore it will end up describing different long distance physics.

- If $\Delta_- < 0$ the operator is called irrelevant. According to (1.27) we see that this operator moves towards the fixed point and therefore it will end up describing the same long distance physics.

- If $\Delta_- = 0$ the operator is called marginal. This case is a little more complicated, as the behavior of the operators depends on higher order quantum corrections. It could be marginally relevant or marginally irrelevant (i.e behaving like a relevant or irrelevant operator). In the special case where all quantum corrections vanish, it corresponds to our fixed point being part of a higher dimensional surface of fixed points. The RG will take us to another of these points.

holog

2. Holography

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2.1 A brief introduction to holography

The holographic principle is an axiom that was proposed by Gerard 't Hooft [2] and Leonard Susskind [3] in the 1990s. What started as an attempt to provide a quantum description of gravity and explain thermodynamical properties of black holes has evolved to a viable method of studying strongly coupled QFTs. As we discussed in previous subsections, the correlation functions of a QFT, are usually calculated using perturbative methods. However, this is impossible when the QFT is strongly coupled because the expansion parameter (the coupling constant) is not sufficiently small to be considered a perturbation.

The situation is further complicated when one considers what we learned from the study of the RG, ie, that in general, the coupling constant depends on the energy scale of the system. Although perturbative approaches may be applicable in certain energy scales, the same cannot be stated in the high energy (UV) limit or the low energy (IR) limit. This has led physicists to primarily use numerical approaches to study systems described by strongly coupled QFTs. One such example which has applications in QCD is to use a lattice, but even these numerical methods have their limitations. A great amount of computational strength is required and there are still cases where these methods fall apart [4].

In order to understand the holographic principle consider a d+1 dimensional space-time, described by a gravitational theory (bulk/gauge theory). The additional dimension, usually called the holographic dimension, ranges to infinity, where the space-time's d-dimensional boundary is located. Then, the holographic principle postulates that the dynamics of the bulk can be described by a QFT defined on that boundary (boundary theory). Essentially, it is as if the gravitational physics of the bulk space-time are encoded as a hologram on the boundary, hence the name holographic principle.

The duality becomes especially useful when one considers CFTs with a large number of colors, which corresponds to a $SU(N)$ gauge group in the large N limit for reasons that will soon become apparent. According to the holographic principle, the ground state of these CFTs corresponds to an AdS space-time [5]. The AdS/CFT correspondence is the most known and studied case of a holographic correspondence.

AdS/CFT is an especially useful duality because the relation between the gravitational and the field theoretical description is a strong-weak one. To demonstrate

that, we begin from the one-loop β function [6]:

$$\frac{\partial g_{YM}}{\partial \log(\Lambda)} = -\frac{11}{3} N \frac{g_{YM}^3}{(4\pi)^2} + \mathcal{O}(g_{YM}^5) \tag{2.1}$$ a1

In order for the two sides of the equation to scale the same way to leading order in g_{YM}, we see that the following coupling must be kept constant:

$$\lambda = N g_{YM}^2 \tag{2.2}$$ a2

The coupling λ is called the t'Hooft coupling. A useful feature of the holographic principle is that the curvature radius of AdS scales as:

$$R = l_s \lambda^{\frac{1}{4}}, \tag{2.3}$$ a2a

where l_s is the natural length scale of string theory. We can perform perturbative gravitational calculations only when $R \gg l_s$, because then the space-time is asymptotically flat. But according to (2.3), this means that the t'Hooft coupling must be large, and therefore, the CFT must be strongly coupled. This confirms that the duality is a strong-weak one, which is the key advantage of AdS/CFT. In the large N-limit, computations that would be impossible in the QFT side of the duality become easy in the gravitational side and vice versa.

There is a fair amount of skepticism that should be present when one first learns of the holographic principle, since it is a conjecture. To ease these concerns, it should be mentioned that the theory does more than simply providing us with insight into the workings of strongly interacting systems via the study of toy models. There are plenty of examples across many different areas of physics that are in qualitative agreement with the holographic principle [7].

Perhaps, the most known is the entropy of a black hole. Bekenstein originally predicted and Hawking later confirmed that the entropy of a black hole is proportional to its area rather than its volume [8][9], which is in contrast to what would have been expected of a physical system. However, this perplexing relation is in total agreement with the holographic description, according to which the thermodynamical properties of black holes are encoded on their boundary, making the scaling of their entropy with their surface area appear natural.

Finally, since the holographic principle offers us with an inherently quantum mechanical description of gravity, it is natural to compare it with string theory. The key advantage of holography when compared to perturbative string theory, is that it allows us to create a non-perturbative theory of quantum gravity.

AdS

2.2 AdS space-time

The importance of AdS/CFT necessitates a brief overview of the most important features of AdS space-time. AdS is a maximally symmetric space-time with constant negative scalar curvature. To visualize it, consider a "d+2"-dimensional hyperboloid embedded in a "d+3"-dimensional space. This is described from the metric:

$$ds^2 = -dX_0^2 - dX_{d+2}^2 + \sum_{i=1}^{d} dX_i^2 + dX_{d+1}^2 \quad , \quad \begin{array}{c} X_j \in (-\infty, \infty), \\ j = 0, 1, \cdots, d+2 \end{array} \tag{2.4}$$

AdS0

along with the condition

$$X_0^2 + X_{d+2}^2 - \sum_{i=1}^{d} X_i^2 - X_{d+1}^2 = L^2 \tag{2.5}$$

AdS01

We can change coordinates via:

$$X_0 = \frac{r}{2}\left[1 + \frac{1}{r^2}\left(L^2 + \vec{x}^2 - t^2\right)\right] \quad , \quad X_i = \frac{L}{r} x_i$$

$$X_{d+1} = \frac{r}{2}\left[1 - \frac{1}{r^2}\left(L^2 - \vec{x}^2 + t^2\right)\right] \quad , \quad X_{d+2} = \frac{L}{r} t \tag{2.6}$$

AdS02

to obtain the metric of the "d+2"-dimensional AdS space-time, in what is usually referred to as Poincare coordinates:

$$ds^2 = \frac{L^2}{r^2}\left[dr^2 - dt^2 + d\vec{x}^2\right] \quad , \quad r \in (0, \infty) \tag{2.7}$$

AdS1

with $d\vec{x}$ being a d-dimensional vector of spatial coordinates. The Poincare coordinates cover only half of AdS, but the other half can be readily obtained by changing the sign of r. The Ricci tensor and the scalar curvature for a d-dimensional AdS space-time are given by:

$$R_{\mu\nu} = -\frac{d-1}{L^2} g_{\mu\nu} \quad , \quad R = -\frac{d(d-1)}{L^2} \tag{2.8}$$

AdS2

We can see two important features of AdS space-time from the metric in (2.7):

- The first feature is that for a given radial slice $r = const$, the metric becomes Minkowski. This is an important feature for holography, because the vevs defined on these submanifolds are well defined.

- The second feature is that this metric displays conformal symmetry. This partially motivates the AdS/CFT correspondence. We can see that as a symmetry of the metric in the scaling:

$$(r, t, x_i) \to (\alpha r, \alpha t, \alpha x_i) \quad , \quad \alpha > 0 \tag{2.9}$$ AdS3

Finally, another property of AdS worth discussing is its conformal boundary. We may parametrize the metric of AdS in a conformal way, such that the point at radial infinity becomes mapped in a compactified manifold. The boundary of the compactified manifold is called the conformal boundary of AdS. This is easier to see using an alternative form for the metric, where the coordinates are called the global coordinates of AdS:

$$ds^2 = L^2(-cosh^2\rho d\tau^2 + d\rho^2 + sinh^2\rho d\Omega_d^2) \quad , \quad \rho \in (0, \infty) \quad , \quad \tau \in [0, 2\pi) \tag{2.10}$$ AdS31

Under the transformation:

$$tan\theta = \sinh\rho, \tag{2.11}$$ AdS32

the metric takes the form:

$$ds^2 = \frac{L^2}{cos^2\theta}\left(-d\tau^2 + d\theta^2 + sin^2\theta d\Omega_d^2\right) \tag{2.12}$$ AdS4

The radial dimension has been compactified through θ, which now takes values in $\theta \in [0, \frac{\pi}{2})$. Now we can perform a conformal transformation to the metric by multiplying with $\frac{L^2}{cos^2\theta}$:

$$d\tilde{s}^2 = -d\tau^2 + d\theta^2 + sin^2\theta d\Omega_d^2 \tag{2.13}$$ AdS5

We have arrived at the metric of Einstein's static universe, however $\theta \in [0, \frac{\pi}{2})$ rather than $\theta in[0, \pi)$. The equator $\theta = \frac{\pi}{2}$ is a boundary of AdS. Through this comparison we can define Asymptotically AdS space-times, as the space-times which are conformal to space-times isomorphic to half of Einstein's static universe. In the case of the Poincare coordinate system, the boundary is at $r = 0$. This is where the boundary CFT is located in the AdS/CFT correspondence.

aas2

2.3 The Holographic Dictionary and Holographic RG Flows

We can now discuss specifically the relation between the two descriptions of the holographic duality. This relation is usually referred to as the Holographic Dictionary or the Gubser-Klebanov-Polyakov [10] and Witten [11] formula. Assume that we have an operator $\mathcal{O}$ of a QFT residing on the d-dimensional boundary of a space-time. The conformal dimension of $\mathcal{O}$ is Δ and the operator is coupled to a scalar source φ_0. Then, the Holographic Dictionary states that:

$$\mathcal{Z}_{QFT}[\varphi_0] = \mathcal{Z}_{string}\left[r^{\Delta-d}\varphi(x^\mu, r)|_{r=0}\right] \equiv \mathcal{Z}_{string}\left[\varphi_0\right], \tag{2.14}$$ Dict1

where $\mathcal{Z}_{QFT}$ and $\mathcal{Z}_{string}$ are the generating functionals of the QFT and the string/gravitational theory respectively. We see that the scalar source φ_0 acts as a boundary condition imposed on the space-time's boundary.

The generating functional of the QFT is:

$$\mathcal{Z}_{QFT}[\varphi_0] = \langle exp(i \int d^d x \varphi_0 \mathcal{O}) \rangle \tag{2.15}$$ Dict2

As we discussed in previous sections, the generating functional can be used to obtain correlation functions through functional differentiations. For example, the two-point function of $\mathcal{O}$ can be obtained by:

$$\langle \mathcal{O}(x)\mathcal{O}(y) \rangle = \frac{\delta^2 ln \mathcal{Z}_{QFT}[\varphi_0]}{\delta\varphi(x)\delta\varphi(y)}|_{\varphi_0=0} \tag{2.16}$$ Dict3

The Holographic Dictionary is useful because when the QFT is strongly coupled, the calculation of $\mathcal{Z}_{QFT}$ is not practical. Instead, we can use equation (2.14) and use the gravitational generating functional. In the case of AdS/CFT the gravitational generating functional is obtained through the classical gravity action:

$$\mathcal{Z}_{string} = exp(iS^{on-shell}[\varphi_0, g]) \tag{2.17}$$ Dict4

and since the gravitational theory is weakly curved for strongly coupled QFTs, the calculation of the on-shell action is easy. The only difficulty arises due to the IR divergences of the gravitational theory, which are associated with the infinite volyme of space-time. Therefore, the action must be renormalized. Through the holographic correspondence, the interpretation of these IR divergences is that they correspond to UV divergences of the QFT.

In the context of the AdS/CFT correspondence, the study of RG flows has an interesting translation. According to the holographic principle, the RG flows of the CFT correspond to the evolution of the bulk fields in the holographic dimension. The reason is that the Hamilton-Jacobi equations for the gravitational theory can be recast in the form of RG flow equations for the QFT, which asymptotically take the standard Callan-Symanzik form ([12]-[19]). This observation indicates that the radial/holographic dimension r is related to the energy scale of the CFT, with the UV corresponding to the limit where $r \to 0$ and the IR to the limit where $r \to \infty$.

There are many reasons one would want to study RG flows. Being inherently linked to renormalization, they are useful for the study of a QFT's UV and IR limits. RG flows can also be used to study phase transitions and critical phenomena.

Of course, if one accepts the holographic principle, they can be useful for studying properties of quantum gravity [20]. Finally, there have been recent developments in machine learning algorithms that can study RG flows [21].

as3

2.4 The dilaton action and the Equations of Motion (EsOM)

The standard action used to describe RG flows, from the ordinary solutions ([13]-[17]) to the exotic ones ([18],[19]), is the Einstein-dilaton theory in (3+1)-dimensions. It is the simplest action that can drive an RG flow, being a two-derivative action with a single scalar.

Specifically, the action we are using is[1]:

$$S[g,\varphi] = M^2 \int d^3x dr \sqrt{|g|} \left(R^{(g)} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) \right) + S_{GHY} \tag{2.18}$$ ga1

where S_{GHY} is the Gibbons-Hawking-York boundary term and can be written as:

$$S_{GHY} = 2M^2 \left[\int d^3x \sqrt{\gamma} K \right]_{UV} \tag{2.19}$$ ga2

We work in 4 dimensions, because the exotic RG flows were found during a study of squashed S^3. As we will see in section 4, γ_{ij} is the induced metric on the $r = 0$ boundary of AdS, whereas K is the trace of the extrinsic curvature. It is convenient to switch the action to Euclidean form by changing the signature of the metric:

$$S_E[g,\varphi] \equiv -S = -M^2 \int d^3x dr \sqrt{g} \left(R^{(g)} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) \right) - S_{GHY} \tag{2.20}$$ ga2a

Varying this action with respect to the metric $g_{\mu\nu}$ and the scalar φ we obtain the following equations of motion:

$$G_{\mu\nu} - \frac{1}{2}\left[\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\partial_\alpha\varphi\partial^\alpha\varphi \right] + \frac{g_{\mu\nu}}{2}V = 0 \tag{2.21}$$ d15

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\varphi\right) - \frac{\delta V}{\delta\phi} = 0, \tag{2.22}$$ d16

where G is the Einstein Tensor.

It is convenient to express the metric in the following form:

$$ds^2 = \frac{\ell^2}{r^2}dr^2 + \frac{\ell^2}{r^2}\hat{g}_{ij}(r,x^i)dx^i dx^j \tag{2.23}$$ d16a

[1]For the rest of this thesis unless specified we will adopt the usual notation of Greek indices taking the values 0-3 and Latin indices taking the values 1-3. The coordinate x^0 is the radial coordinate.

This coordinate system is known as the Fefferman-Graham system and it can describe any asymptotically AdS space-time. The boundary metric $\hat{g}_{ij}$ has a regular expansion if it is odd-dimensional:

$$\hat{g}_{ij}(r, x^i) = \sum_{m=0}^{\infty} \hat{g}_{ij}^{(m)}(x^i) r^m \tag{2.24}$$ d16b

More details abount this coordinate system are presented in Appendix (A.1).

The scalar equation (2.22) may be expanded in terms of the metric $\hat{g}_{\mu\nu}$:

$$\frac{1}{\sqrt{g}}\partial_r(\sqrt{g}g^{rr}\varphi') + \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j\varphi) - \frac{\delta V}{\delta\varphi} = 0 \quad \Rightarrow \tag{2.25}$$ d18

$$\frac{r^2}{\ell^2}\varphi'\partial_r \log(\sqrt{g}) + \partial_r(\frac{r^2}{\ell^2}\varphi') + \frac{r^4}{\ell^4\sqrt{\hat{g}}}\partial_i(\frac{\ell^2}{r^2}\sqrt{\hat{g}}\hat{g}^{ij}\partial_j\varphi) - \frac{\delta V}{\delta\varphi} = 0 \tag{2.26}$$ d19

The square root of the determinant is a scalar density with a vanishing covariant derivative. Therefore, using the Christoffel Symbols of Appendix (A) we have:

$$\partial_r \log(\sqrt{g}) = \Gamma_{\alpha r}{}^{\alpha} = -\frac{4}{r} + \frac{1}{2}Tr[\hat{g}^{-1}\hat{g}'], \tag{2.27}$$ d20

and thus, the scalar equation takes the following form:

$$r^2\varphi'' - 2r\varphi' + \frac{r^2}{2}Tr[\hat{g}^{-1}\hat{g}']\varphi' + r^2\Box_{\hat{g}}\varphi - \ell^2\frac{\delta V}{\delta\varphi} = 0 \tag{2.28}$$ d21

The trace-inverted Einstein equations are:

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi + \frac{V}{2}g_{\mu\nu} \tag{2.29}$$ d24

because

$$R = \frac{1}{2}(\partial\varphi)^2 + 2V \tag{2.30}$$ d25

In components, they read:

$$R_{rr} = \frac{1}{2}(\varphi')^2 + \frac{V}{2}\frac{\ell^2}{r^2} \tag{2.31}$$ d26

$$R_{ir} = \frac{1}{2}\partial_i\varphi\varphi' \tag{2.32}$$ d27

$$R_{ij} = \frac{1}{2}\partial_i\varphi\partial_j\varphi + \frac{V}{2}\frac{\ell^2}{r^2}\hat{g}_{ij} \tag{2.33}$$ d28

We parametrize:

$$V = -\frac{6}{L^2} + \delta V \tag{2.34}$$ d29

Using the results in Appendix (A), the equations of motion can be written as:

$$\frac{r^2}{2}(\varphi')^2 + \frac{\ell^2}{2}\delta V = -\frac{r^2}{2}Tr[\hat{g}^{-1}\hat{g}''] + \frac{r^2}{4}Tr[\hat{g}^{-1}\hat{g}'\hat{g}^{-1}\hat{g}'] + \frac{r}{2}Tr[\hat{g}^{-1}\hat{g}'] + 3\left(\frac{\ell^2}{L^2} - 1\right) \tag{2.35}$$ d30

$$r\hat{g}^{ik}\left[\hat{\nabla}_k\hat{g}'_{ij}-\hat{\nabla}_j\hat{g}'_{ik}\right]=r\varphi'\partial_j\varphi \tag{2.36}$$ d31

$$\frac{r^2}{2}\partial_i\varphi\partial_j\varphi+\frac{\ell^2\delta V}{2}\hat{g}_{ij}=r^2\hat{R}_{ij}-\frac{r^2}{2}\hat{g}''_{ij}+\frac{r^2}{2}(\hat{g}'\hat{g}^{-1}\hat{g}')_{ij}-$$
$$-\frac{r^2}{4}Tr[\hat{g}^{-1}\hat{g}']\hat{g}'_{ij}+\frac{r}{2}Tr[\hat{g}^{-1}\hat{g}']\hat{g}_{ij}+r\hat{g}'_{ij}+3\left(\frac{\ell^2}{L^2}-1\right)\hat{g}_{ij} \tag{2.37}$$ d33

In later parts, we will refer to the Einstein equations using the indexes of the Ricci scalar component from which they are stemming from.

Finally, using equation (2.30), the on-shell action may be rewritten as:

$$S_E^{on-shell}[g,\varphi]=-M^2\int d^3xdr\sqrt{g}V(\varphi)-2M^2\left[\int dx^3\sqrt{\gamma}K\right]_{r=\epsilon} \tag{2.38}$$ d34

s4

3. Perturbative solution near an extremum of the potential

To introduce the basic concepts and methodology which we will follow when we study the exotic RG flows, we will first study a specific potential and a 3-dimensional metric. The standard procedure was introduced by de Haro, Skenderis and Solodukhin in [22]. We will solve the equations "order by order" in the radial coordinate r. Then, we need to renormalize the action because, as we will see in section 4, the terms appearing in the action will be divergent. The renormalization can be performed by adding appropriate counterterms to the action, so that said divergences cancel each other out. Finally, we will calculate the vev of the induced Stress-Energy Tensor on the boundary.

Any potential which is analytical near one of its extremums can be expanded in the general form:

$$V=V_0+V_2\varphi^2+\mathcal{O}(\varphi^3) \tag{3.1}$$ e1a

There are some convenient reparameterizations that we can perform for the coefficients V_0 and V_2. We may obtain V_0 from the scalar-free limit. By comparison with (2.34) we see that the scalar field limit corresponds to selecting:

$$V_0\equiv-\frac{6}{L^2}=2\Lambda_{AdS_4} \tag{3.2}$$ e1b

One can easily verify that with this selection for V_0 the Einstein-Hilbert term in (2.38) is the expected one for a 4-dimensional AdS space-time. To justify the reparameterization of V_2 we turn our attention to (2.22). By selecting :

$$V_2\equiv\frac{m^2}{2} \tag{3.3}$$ e1c

we see that equation (2.22) takes the familiar-looking form of the scalar equation:

$$\Box\varphi - m^2\varphi + \mathcal{O}(\varphi^2) = 0 \tag{3.4}$$ e1d

Thus, the potential can be written as:

$$V = -\frac{6}{L^2} + \frac{m^2}{2}\varphi^2 + \mathcal{O}(\varphi^3) \tag{3.5}$$ e1

To justify the perturbative expansion of the scalar, note that we are interested in studying the RG flow starting near the UV fixed point, which has been shifted at $\varphi = 0$. Since the UV regime of the QFT corresponds to the IR regime of AdS, we are close to the boundary $r \to 0$. Therefore, we will adopt expansions of the following form for the quantities that appear in equations (2.28), (2.35)-(2.37) and we will start to solve the equations "order by order" in r:

$$\varphi(r,x) = \sum_{\substack{n=1\\m=0}}^{\infty} \varphi_{n,m}(x) r^{n\Delta_+ + m} + \sum_{\substack{n=1\\m=0}}^{\infty} \bar{\varphi}_{n,m}(x) r^{n\Delta_- + m} + \sum_{m=0}^{\infty} \varphi_m(x) r^m \tag{3.6}$$ e3

$$\hat{g}_{ij}(r,x^\mu) = \sum_{\substack{n=1\\m=0}}^{\infty} g_{ij}^{(n,m)}(x) r^{n\Delta_+ + m} + \sum_{\substack{n=1\\m=0}}^{\infty} \bar{g}_{ij}^{(n,m)}(x) r^{n\Delta_- + m} + \sum_{n=0}^{\infty} g_{ij}^{(n)}(x^\mu) r^n \tag{3.7}$$ e4

$$\hat{g}^{ij}(r,x^\mu) = \sum_{\substack{n=1\\m=0}}^{\infty} g^{ij(n,m)}(x) r^{n\Delta_+ + m} + \sum_{\substack{n=1\\m=0}}^{\infty} \bar{g}^{ij(n,m)}(x) r^{n\Delta_- + m} + \sum_{n=0}^{\infty} g^{ij(n)}(x^\mu) r^n \tag{3.8}$$ e6

$$\hat{R}_{ij}(r,x^\mu) = \sum_{\substack{n=1\\m=0}}^{\infty} R_{ij}^{(n,m)}(x) r^{n\Delta_+ + m} + \sum_{\substack{n=1\\m=0}}^{\infty} \bar{R}_{ij}^{(n,m)}(x) r^{n\Delta_- + m} + \sum_{n=0}^{\infty} R_{ij}^{(n)}(x^\mu) r^n \tag{3.9}$$ e5

Since, the single index 0 appears the most in the equations to follow, for convenience, every coefficient that corresponds to the r^0 power will be denoted with a subscript 0. The aforementioned coefficients are entirely constructed from the metric $g_{ij}^{(0)}$.

s4s1

3.1 The solution of the scalar equation to order $\mathcal{O}(r^3)$.

We will begin from the scalar equation (2.28) up to order $\mathcal{O}(\varphi^2)$. In a later subsection, we will see what would have been different had we kept higher orders of the scalar in the potential. The indexes i and j will be omitted unless they are necessary. We have:

$$r^2\varphi'' - 2r\varphi' + \frac{r^2}{2}\phi' Tr[\hat{g}^{-1}\hat{g}'] + r^2\Box_{\hat{g}}\phi - m^2\ell^2\varphi + \mathcal{O}(\phi^2) = 0 \quad \Rightarrow \tag{3.10}$$ e7

$$r^2\varphi'' - 2r\varphi' + \frac{r^2}{2}\varphi' Tr[\hat{g}^{-1}\hat{g}'] + r^2\left(\frac{1}{2}Tr[\hat{g}^{-1}\partial_i\hat{g}]\hat{g}^{ij}\partial_j\varphi + \partial_i\hat{g}^{ij}\partial_j\varphi + \right. \tag{3.11}$$

$$\left. + \hat{g}^{ij}\partial_i\partial_j\varphi\right) - m^2\ell^2\varphi + \mathcal{O}(\varphi^2) = 0$$

In the last step, we performed the partial derivatives of the box operator with respect to x^i the same way we did in (2.26) using the logarithmic derivative of $\sqrt{g}$ as in (2.27). We will study separately the order $\mathcal{O}(r^4)$ and thus, we begin with the equations up to order $\mathcal{O}(r^3, r^{\Delta_\pm})$. Up to this order, we will not need the terms of the expansions (3.6)-(3.8) that contain two indexes n and m.

$$2r^2\varphi_2 - 2r(\varphi_1 + 2\varphi_2 r) + \frac{r^2}{2}\varphi_1 Tr[g_0^{-1}g^{(1)}] + r^2\Box_0\varphi_0 -$$

$$-m^2\ell^2(\varphi_0 + \varphi_1 r + \varphi_2 r^2) + \mathcal{O}(r^3, r^{\Delta_\pm}, \varphi^2) = 0 \tag{3.12}$$

We are led to the following equations by regrouping the terms of (3.12) according to their order in r:

$$-m^2\ell^2\varphi_0 = 0 \quad \Rightarrow \quad \varphi_0 = 0 \tag{3.13}$$

$$-2\varphi_1 - m^2\ell^2\varphi_1 = 0 \quad \Rightarrow \quad (m^2\ell^2 + 2)\varphi_1 = 0 \quad \Rightarrow \quad \varphi_1 = 0 \tag{3.14}$$

$$2\varphi_2 - 4\varphi_2 - m^2\ell^2\varphi_2 + \frac{1}{2}\varphi^1 Tr[g_0^{-1}g^{(1)}] + \Box_0\varphi_0 = 0 \quad \Rightarrow$$

$$\Rightarrow \quad (m^2\ell^2 + 2)\varphi_2 = 0 \quad \Rightarrow \quad \varphi_2 = 0 \tag{3.15}$$

We set φ_1 and φ_2 equal to zero because we are not interested in integer values for $\Delta_\pm$. We will see that this would have been the case when we revisit the scalar equation to order $\mathcal{O}(r^{\Delta_\pm+1})$.

3.2 The solution of the ij Einstein equation to order $\mathcal{O}(r^3)$.

Since the φ_m contribution starts at least to order $\mathcal{O}(r^4)$ we proceed with the solution of the ij equation (2.37), solving up to order $\mathcal{O}(r^4, r^{\Delta_\pm})$. Similar to the scalar equation, it is convenient to study the order $\mathcal{O}(r^4)$ separately. For our choice of potential, the ij equation becomes:

$$\frac{r^2}{2}\partial_i\varphi\partial_j\varphi + \frac{m^2\ell^2}{2}\varphi^2\hat{g}_{ij} = r^2\hat{R}_{ij} - \frac{r^2}{2}\hat{g}''_{ij} + \frac{r^2}{2}(\hat{g}'\hat{g}^{-1}\hat{g}')_{ij} -$$

$$-\frac{r^2}{4}Tr[\hat{g}^{-1}\hat{g}']\hat{g}'_{ij} + \frac{r}{2}Tr[\hat{g}^{-1}\hat{g}']\hat{g}_{ij} + r\hat{g}'_{ij} + 3\left(\frac{\ell^2}{L^2} - 1\right)\hat{g}_{ij} + \mathcal{O}(\varphi^3) \tag{3.16}$$

We substitute the relevant terms of the expansions (3.6)-(3.9) in (3.16) to obtain to order $\mathcal{O}(r^3, r^{\Delta_\pm}, \phi^3)$:

$$-r^2 R_{ij,0} = -r^2 g^{(2)}_{ij} + \frac{r^2}{2}g^{(1)}_{ik}g^{kl}_0 g^{(1)}_{lj} -$$

$$-\frac{r^2}{4}Tr\left[g_0^{-1}g^{(1)}\right]g_{ij}^{(1)}+r(g_{ij}^{(1)}+2g_{ij}^{(2)}r)+ \tag{3.17}$$ e16

$$+\frac{r}{2}Tr\left[\left(g_0^{-1}+(g^{-1})^{(1)}r\right)\left(g^{(1)}+2g^{(2)}r\right)\right](g_{ij,0}+g_{ij}^{(1)}r)+$$

$$+3\left(\frac{\ell^2}{L^2}-1\right)\left(\hat{g}_{0,ij}+rg_{ij}^{(1)}+r^2g_{ij}^{(2)}\right)+\mathcal{O}(r^3,r^{\Delta_\pm},\varphi^3)$$

The equation of order $\mathcal{O}(r^1)$ is trivially satisfied as long as we set:

$$\ell = L, \tag{3.18}$$ neq

therefore we immediately move to the equation of order $\mathcal{O}(r^2)$, which reads:

$$g_{ij}^{(1)}+\frac{1}{2}Tr\left[g_0^{-1}g^{(1)}\right]g_{ij,0}=0 \tag{3.19}$$ e19

By taking the trace of equation (3.19) with g_0^{ij} we have:

$$5Tr\left[g_0^{-1}g^{(1)}\right]=0 \quad\Rightarrow\quad g_{ij}^{(1)}=0 \tag{3.20}$$ e20

In the last step we used the tracelessness of $g_{ij}^{(1)}$ on (3.20). Since $g_{ij}^{(1)}$ vanishes so does $R_{ij}^{(1)}$.

The equation of order $\mathcal{O}(r^3)$ in (3.17) is solved by trace inverting:

$$-R_{ij,0}=g_{ij}^{(2)}+Tr\left[g_0^{-1}g^{(2)}\right]g_{ij,0} \quad\Rightarrow \tag{3.21}$$ e21

$$-R_0=4Tr\left[g_0^{-1}g^{(2)}\right] \quad\Rightarrow\quad Tr\left[g_0^{-1}g^{(2)}\right]=-\frac{R_0}{4} \tag{3.22}$$ e23

Therefore $g_{ij}^{(2)}$ takes the form:

$$g_{ij}^{(2)}=\frac{R_0}{4}g_{ij,0}-R_{ij,0} \tag{3.23}$$ e24

We will not solve the $\mathcal{O}(r^4)$ order equations yet, because there will be terms with two indexes that we have not calculated. It is better to study the terms that appear in equations (3.11) and (3.16) up to order $\mathcal{O}(r^{\Delta_++3})$. The equations for the $r^{n\Delta_-+m}$ powers are readily obtained from those of the $r^{n\Delta_++m}$ powers by switching Δ_+ to Δ_-, bared quantities to non-bared quantities and vice versa. Therefore we will not study them separately but we will instead use the mapping described earlier. These terms will be denoted by $+\,(+\leftrightarrow -)$.

s4s3 **3.3 The solution of the scalar equation to order $\mathcal{O}(r^{\Delta_\pm+3})$.**

We substitute the expansions (3.6) -(3.8) in (3.11) up to terms that will contribute to order $\mathcal{O}(r^{\Delta_\pm+3})$.

$$\begin{aligned}
&\Delta_+(\Delta_+-1)\varphi_{1,0}r^{\Delta_+}+(\Delta_++1)\Delta_+\varphi_{1,1}r^{\Delta_++1}+\\
&+(\Delta_++2)(\Delta_++1)\varphi_{1,2}r^{\Delta_++2}-2\Big(\Delta_+\varphi_{1,0}r^{\Delta_+}+(\Delta_++1)\varphi_{1,1}r^{\Delta_++1}+\\
&+(\Delta_++2)\varphi_{1,2}r^{\Delta_++2}\Big)+\frac{r}{2}\left(\Delta_+\varphi_{1,0}r^{\Delta_+}\right)Tr[g_0^{-1}2g^{(2)}r]+\\
&+r^2\Box_0(r^{\Delta_+}\varphi_{1,0})-m^2L^2\left(\varphi_{1,0}r^{\Delta_+}+\varphi_{1,1}r^{\Delta_++1}+\varphi_{1,2}r^{\Delta_++2}\right)\\
&+(+\leftrightarrow-)+\mathcal{O}(\varphi^2,r^{\Delta_\pm+3},r^{2\Delta_\pm})=0
\end{aligned} \tag{3.24}$$ e26

We are led to the following equations:

$$(\Delta_+^2-3\Delta_+-m^2L^2)\varphi_{1,0}=0 \quad\Rightarrow\quad \Delta_\pm=\frac{3\pm\sqrt{9+4m^2L^2}}{2} \tag{3.25}$$ e27

$$(\Delta_+^2-\Delta_+-2-m^2L^2)\varphi_{1,1}=0 \quad\Rightarrow\quad 2(\Delta_+-1)\varphi_{1,1}=0 \quad\Rightarrow$$

$$\varphi_{1,1}=\bar{\varphi}_{1,1}=0 \tag{3.26}$$ e28

$$(\Delta_+^2+\Delta_+-2-m^2L^2)\varphi_{1,2}+\Delta_+Tr[g_0^{-1}g^{(2)}]\varphi_{1,0}+\Box_0\varphi_{1,0}=0 \quad\Rightarrow \tag{3.27}$$ e29

$$\varphi_{1,2}=-\frac{\Delta_+Tr[g_0^{-1}g^{(2)}]\varphi_{1,0}+\Box_0\varphi_{1,0}}{2(2\Delta_+-1)} \tag{3.28}$$ e30

$$\bar{\varphi}_{1,2}=-\frac{\Delta_-Tr[g_0^{-1}g^{(2)}]\bar{\varphi}_{1,0}+\Box_0\bar{\varphi}_{1,0}}{2(2\Delta_--1)} \tag{3.29}$$ e31

Equation (3.25) provides us with some useful properties of $\Delta_\pm$:

$$\Delta_++\Delta_-=3 \tag{3.30}$$ ne1

$$\Delta_+^2-3\Delta_+=\Delta_-^2-3\Delta_-=-\Delta_+\Delta_-=m^2L^2 \tag{3.31}$$ ne2

Before proceeding with the solution of the ij equation (3.16) to higher orders we note that our justification for setting $\varphi_1=0$ is correct. Indeed, equation (3.25) has no dependency on any coefficients of the scalar expansion in (3.6) besides $\varphi_{1,0}$ and $\bar{\varphi}_{1,0}$. Therefore, equation (3.25) is independent of us setting $\varphi_1=0$. We see that, had we not selected $\varphi_1=0$ in equation (3.14) we would have integer values for $\Delta_\pm$, just as we had stated. This is also the reason why we selected $\varphi_{1,1}=\bar{\varphi}_{1,1}=0$. Finally, we did not select $\varphi_{1,0}=\bar{\varphi}_{1,0}=0$ in (3.25) because these are the boundary conditions through which we will express the rest of the scalar coefficients.

s4s4

3.4 The solution of the ij Einstein equation to order $\mathcal{O}(r^{\Delta_\pm+3})$.

We may now proceed to (3.16) and study it up to order $\mathcal{O}(r^{\Delta_\pm+3})$. We can use (3.31) to eliminate the mass, and thus, the ij equation reads:

$$\frac{r^2}{2}\partial_i\varphi\partial_j\varphi - \frac{\Delta_+\Delta_-}{2}\varphi^2\hat{g}_{ij} = r^2\hat{R}_{ij} - \frac{r^2}{2}\hat{g}''_{ij} + \frac{r^2}{2}(\hat{g}'\hat{g}^{-1}\hat{g}')_{ij} - \\ -\frac{r^2}{4}Tr[\hat{g}^{-1}\hat{g}']\hat{g}'_{ij} + \frac{r}{2}Tr[\hat{g}^{-1}\hat{g}']\hat{g}_{ij} + r\hat{g}'_{ij} + \mathcal{O}(\varphi^3) \tag{3.32}$$

e32

We substitute the terms of the expansions (3.6)-(3.9) that will contribute to order $\mathcal{O}(r^{\Delta_\pm+3})$:

$$\begin{gathered}
-r^2R^{(1,0)}_{ij}r^{\Delta_+} + \frac{1}{2}\Big(\Delta_+(\Delta_+-1)g^{(1,0)}_{ij}r^{\Delta_+} + \\
+(\Delta_++1)\Delta_+g^{(1,1)}_{ij}r^{\Delta_++1} + (\Delta_++2)(\Delta_++1)g^{(1,2)}_{ij}r^{\Delta_++2}\Big) - \\
-\frac{1}{2}\Big((2g^{(2)}_{ik}r^2 + \Delta_+g^{(1,0)}_{ik}r^{\Delta_+})g^{kl}_0(2g^{(2)}_{lj}r^2 + +\Delta_+g^{(1,0)}_{lj}r^{\Delta_+})\Big) + \\
+\frac{1}{4}Tr[(2g^{(2)}r^2 + \Delta_+g^{(1,0)}r^{\Delta_+})g_0^{-1}]\Big(2g^{(2)}_{ij}r^2 + \Delta_+g^{(1,0)}_{ij}r^{\Delta_+}\Big) - \\
-\Big(\Delta_+g^{(1,0)}_{ij}r^{\Delta_+} + (\Delta_++1)g^{(1,1)}_{ij}r^{\Delta_++1} + (\Delta_++2)g^{(1,2)}_{ij}r^{\Delta_++2}\Big) - \\
-\frac{1}{2}Tr[\big(g_0^{-1} + (g^{-1})^{(2)}r^2 + (g^{-1})^{(1,0)}r^{\Delta_+}\big)\big(2g^{(2)}r^2 + \Delta_+g^{(1,0)}r^{\Delta_+} + \\
+(\Delta_++1)g^{(1,1)}r^{\Delta_++1} + (\Delta_++2)g^{(1,2)}r^{\Delta_++2}\big)](g_{ij,0} + g^{(2)}_{ij}r^2 + \\
+g^{(1,0)}_{ij}r^{\Delta_+}) + (+\leftrightarrow -) + \mathcal{O}(\varphi^3, r^{\Delta_\pm+3}, r^{2\Delta_\pm}) = 0
\end{gathered} \tag{3.33}$$

We start from the equation of the terms which are of order $\mathcal{O}(r^{\Delta_\pm+1})$. Following the same process as in (3.19) and (3.20), we obtain:

$$\frac{1}{2}(\Delta_+^2 - 3\Delta_+)g^{(1,0)}_{ij} - \frac{\Delta_+}{2}Tr[g_0^{-1}g^{(1,0)}]g_{ij,0} = 0 \quad\Rightarrow\quad g^{(1,0)}_{ij} = \bar{g}^{(1,0)}_{ij} = 0 \tag{3.34}$$

e34

We reintroduce this result to (3.4) to simplify it. Note that $R^{(1,0)}_{ij}$ vanishes as well. Equation (3.4) takes the form:

$$\begin{gathered}
\frac{1}{2}\Big((\Delta_++1)\Delta_+g^{(1,1)}_{ij}r^{\Delta_++1} + (\Delta_++2)(\Delta_++1)g^{(1,2)}_{ij}r^{\Delta_++2}\Big) - \\
-\Big((\Delta_++1)g^{(1,1)}_{ij}r^{\Delta_++1} + (\Delta_++2)g^{(1,2)}_{ij}r^{\Delta_++2}\Big) - \\
-\frac{1}{2}Tr[g_0^{-1}\big((\Delta_++1)g^{(1,1)}r^{\Delta_++1} + (\Delta_++2)g^{(1,2)}r^{\Delta_++2}\big)]g_{ij,0} + \\
+(+\leftrightarrow -) + \mathcal{O}(\phi^3, r^{\Delta_\pm+3}, r^{2\Delta_\pm}) = 0
\end{gathered} \tag{3.35}$$

e35

We have the two following equations for the orders $\mathcal{O}(r^{\Delta_\pm+1})$ and $\mathcal{O}(r^{\Delta_\pm+2})$:

$$\frac{1}{2}(\Delta_+^2-\Delta_+-2)g_{ij}^{(1,1)}-\frac{\Delta_++1}{2}Tr[g_0^{-1}g^{(1,1)}]g_{ij,0}=0 \quad \Rightarrow$$

$$g_{ij}^{(1,1)}=\bar{g}_{ij}^{(1,1)}=0 \tag{3.36}$$ e36

$$\frac{1}{2}(\Delta_+^2+\Delta_++-2)g_{ij}^{(1,2)}-\frac{\Delta_++2}{2}Tr[g_0^{-1}g^{(1,2)}]g_{ij,0}=0 \quad \Rightarrow$$

$$g_{ij}^{(1,2)}=\bar{g}_{ij}^{(1,2)}=0, \tag{3.37}$$ e37

where we solved the equations by trace inverting them and demanding that $\Delta_\pm$ are not integers. Before continuing with the scalar equation (3.32) to order $\mathcal{O}(r^{2\Delta_\pm+1})$ to find at which order the scalar affects the metric for the first time, we will study the $\mathcal{O}(r^4)$ order of the ij Einstein equation (3.32) and the scalar equation (3.11).

s4s5

3.5 The $\mathcal{O}(r^4)$ order of the scalar and the ij Einstein equations.

We substitute the relevant terms of the expansion (3.6) in the scalar equation (3.11):

$$6\varphi_3r^3-6\varphi_3r^3+\Delta_+\Delta_-\varphi_3+\mathcal{O}(\varphi^2)=0 \quad \Rightarrow \quad \varphi_3=0 \tag{3.38}$$ e38

For the ij Einstein equation (3.32), the terms of the expansions (3.6)-(3.9) which are contributing to order $\mathcal{O}(r^4)$ are:

$$-\frac{\Delta_+\Delta_-}{4}(\varphi_{1,0}r^{\Delta_+}+\bar{\varphi}_{1,0}r^{\Delta_-})^2g_{ij,0}+3r^3g_{ij}^{(3)}-\frac{3r^3}{2}Tr[g_0^{-1}g^{(3)}]g_{ij,0}- \tag{3.39}$$ e39

$$-3r^3g_{ij}^{(3)}++\mathcal{O}(\varphi^3,r^4)=0 \quad \Rightarrow$$

$$\left(-\Delta_+\Delta_-\varphi_{1,0}\bar{\varphi}_{1,0}-3Tr[g_0^{-1}g^{(3)}]\right)g_{ij,0}+\mathcal{O}(\varphi^3,r^4)=0 \quad \Rightarrow \tag{3.40}$$ e40

$$Tr[g_0^{-1}g^{(3)}]=-\frac{\Delta_+\Delta_-}{3}\varphi_{1,0}\bar{\varphi}_{1,0} \tag{3.41}$$ e41

Equation (3.41) serves as a constraint between our four arbitrary boundary conditions.

s4s6

3.6 The solution of the ij Einstein equation to order $\mathcal{O}(r^{2\Delta_\pm+2})$.

By substituting the terms of the expansions (3.6) - (3.8) which will contribute up to order $\mathcal{O}(r^{2\Delta_\pm+2})$ in (3.32) we obtain:

$$-\frac{\Delta_+\Delta_-}{4}\varphi_{1,0}^2g_{ij,0}r^{2\Delta_+}+\frac{1}{2}\Big(2\Delta_+(2\Delta_+-1)g_{ij}^{(2,0)}r^{2\Delta_+}+$$

$$+(2\Delta_++1)2\Delta_+g_{ij}^{(2,1)}r^{2\Delta_++1}\Big)-\frac{1}{2}Tr\left[g_0^{-1}\left(2\Delta_+g^{(2,0)}r^{2\Delta_+}+\right.\right.$$

$$\left.\left.+(2\Delta_++1)g^{(2,1)}r^{2\Delta_++1}\right)\right]g_{ij,0}-(2\Delta_+g_{ij}^{(2,0)}r^{2\Delta_+}+ \tag{3.42}$$ e43

$$+(2\Delta_+ + 1)g_{ij}^{(2,1)} r^{2\Delta_+ + 1}) + (+ \leftrightarrow -) + \mathcal{O}(\phi^3, r^{2\Delta_+ + 3}, r^{3\Delta_+}) = 0$$

To order $\mathcal{O}(r^{2\Delta_\pm + 1})$ we have:

$$-\frac{\Delta_+\Delta_-}{4}\varphi_{1,0}^2 g_{ij,0} + (2\Delta_+^2 - 3\Delta_+)g_{ij}^{(2,0)} - \Delta_+ Tr[g_0^{-1} g^{(2,0)}]g_{0,ij} = 0 \tag{3.43}$$ e44

We trace the equation with g_0^{ij}:

$$-\frac{3\Delta_+\Delta_-}{4}\varphi_{1,0}^2 + (2\Delta_+^2 - 6\Delta_+)Tr[g_0^{-1} g^{(2,0)}] = 0 \quad \Rightarrow \tag{3.44}$$ e45

$$Tr[g_0^{-1} g^{(2,0)}] = \frac{3\Delta_+(3-\Delta_+)}{8\Delta_+(\Delta_+ - 3)}\varphi_{1,0}^2 = -\frac{3}{8}\varphi_{1,0}^2 \tag{3.45}$$ e46

$$Tr[g_0^{-1} \bar{g}^{(2,0)}] = -\frac{3}{8}\bar{\varphi}_{1,0}^2 \tag{3.46}$$ ne3

We substitute (3.45) in (3.43) to obtain:

$$-\frac{\Delta_+\Delta_-}{4}\varphi_{1,0}^2 g_{ij,0} + (2\Delta_+^2 - 3\Delta_+)g_{ij}^{(2,0)} + \frac{3}{8}\Delta_+\varphi_{1,0}^2 g_{ij,0} = 0 \quad \Rightarrow \tag{3.47}$$ e47

$$(2\Delta_+^2 - 3\Delta_+)g_{ij}^{(2,0)} = -\frac{2\Delta_+^2 - 3\Delta_+}{8}\varphi_{1,0}^2 g_{ij,0} \quad \Rightarrow \tag{3.48}$$ e48

$$g_{ij}^{(2,0)} = -\frac{1}{8}\varphi_{1,0}^2 g_{ij,0} \tag{3.49}$$ e49

$$\bar{g}_{ij}^{(2,0)} = -\frac{1}{8}\bar{\varphi}_{1,0}^2 g_{ij,0} \tag{3.50}$$ e50

The terms that contribute to order $\mathcal{O}(r^{2\Delta_+ + 2})$ are:

$$(\Delta_+ - 1)(2\Delta_+ + 1)g_{ij}^{(2,1)} - \frac{2\Delta_+ + 1}{2}Tr[g_0^{-1} g^{(2,1)}]g_{ij,0} = 0 \tag{3.51}$$ e51

By trace inverting and substituting the trace in equation (3.51) we have:

$$(2\Delta_+ + 1)(\Delta_+ - \frac{5}{2})Tr[g_0^{-1)} g^{(2,1)}] = 0 \quad \Rightarrow \quad Tr[g_0^{-1} g^{(2,1)}] = 0 \quad \Rightarrow \tag{3.52}$$ e52

$$g_{ij}^{(2,1)} = \bar{g}_{ij}^{(2,1)} = 0 \tag{3.53}$$ e53

We will limit ourselves to operators whose conformal dimension ranges between:

$$2 > \Delta_+ > 1.5 \quad \Rightarrow \quad 1.5 > \Delta_- > 1, \tag{3.54}$$ e53a

which is why we are not interested in the case $Tr[g_0^{-1} g^{(2,1)}] \neq 0$. As we will see in section (4), we do not need any additional orders to calculate the stress tensor as long as we limit ourselves to operators with the aforementioned dimensions. Nevertheless, we have to confirm that the other Einstein equations hold up to the order in r that we have solved.

s4s8 **3.7 The solution of the rr Einstein equation.**

We proceed to the solution of the rr Einstein equation. By substituting the expansions (3.6)-(3.8) we find out that most perturbation orders lead to identities of the form $0 = 0$. The rr Einstein equation reads:

$$\frac{r^2}{2}(\varphi')^2 - \frac{\Delta_+\Delta_-}{4}\varphi^2 + \frac{r^2}{2}Tr[\hat{g}^{-1}\hat{g}''] - \\ -\frac{r^2}{4}Tr[\hat{g}^{-1}\hat{g}'\hat{g}^{-1}\hat{g}'] - \frac{r}{2}Tr[\hat{g}^{-1}\hat{g}'] + \mathcal{O}(\varphi^3) = 0 \tag{3.55}$$ e64

We insert the relevant terms of the expansions (3.6)-(3.8):

$$\frac{1}{2}\left(\Delta_+\varphi_{1,0}r^{\Delta_+} + \Delta_-\bar{\varphi}_{1,0}r^{\Delta_-}\right)^2 - \frac{\Delta_+\Delta_-}{4}\left(\varphi_{1,0}r^{\Delta_+} + \bar{\varphi}_{1,0}r^{\Delta_-}\right)^2 + \\ + \left(\frac{1}{2}Tr\left[g_0^{-1}\left(2g^{(2)}r^2 + 6g^{(3)}r^3 + (2\Delta_+)(2\Delta_+ - 1)g^{(2,0)}r^{2\Delta_+}\right)\right] - \right. \\ \left. -\frac{1}{2}Tr[g_0^{-1}\left(2g^{(2)}r^2 + 3g^{(3)}r^3 + 2\Delta_+g^{(2,0)}r^{2\Delta_+}\right) + (+ \leftrightarrow -))\right] + \\ +\mathcal{O}(\varphi^3, r^4, r^{\Delta_\pm+3}, r^{2\Delta_\pm+2}, r^{3\Delta_\pm}) = 0 \quad \Rightarrow \tag{3.56}$$ e65

We regroup the terms according to their powers of r to obtain:

$$\left(Tr[g_0^{-1}g^{(2)}] - Tr[g_0^{-1}g^{(2)}]\right)r^2 + \left(\frac{\Delta_+\Delta_-}{2}\varphi_{1,0}\bar{\varphi}_{1,0} + \frac{3}{2}Tr[g_0^{-1}g^{(3)}]\right)r^3 + \\ +\left[\left(\frac{2\Delta_+^2 - \Delta_+\Delta_-}{4}\varphi_{1,0}^2 + 2\Delta_+(\Delta_+ - 1)Tr[g_0^{-1}g^{(2,0)}]\right)r^{2\Delta_+}\right. \\ \left. + (+ \leftrightarrow -)\right] + \mathcal{O}(\varphi^3, r^4, r^{\Delta_\pm+3}, r^{2\Delta_++2}, r^{3\Delta_+}) = 0 \tag{3.57}$$ e66

The $\mathcal{O}(r^3)$ order equation is satisfied as an identity. We may factor out the conformal dimension in the $\mathcal{O}(r^{2\Delta_\pm+1})$ order:

$$\frac{3}{2}\left(\frac{\Delta_+\Delta_-}{3}\varphi_{1,0}\bar{\varphi}_{1,0} + Tr[g_0^{-1}g^{(3)}]\right)r^3 + \\ +\left[(2\Delta_+^2 - 2\Delta_+)\left(\frac{3}{8}\varphi_{1,0}^2 + Tr[g_0^{-1}g^{(2,0)}]\right)r^{2\Delta_+} + (+ \leftrightarrow -)\right] + \\ +\mathcal{O}(\varphi^3, r^4, r^{\Delta_\pm+3}, r^{2\Delta_\pm+2}, r^{3\Delta_\pm}) = 0 \tag{3.58}$$ e67

By substituting in (3.58) equations (3.41), (3.45) and (3.46) we find out that the r^3 and $r^{2\Delta_\pm}$ order equations are also $0 = 0$ identities.

s4s9 **3.8 The solution of the rj Einstein equation.**

We are left with the rj equation for which we shall repeat the process we followed to solve the rr Einstein equation. However, we first have to express the covariant derivatives in a convenient way to use our expansions (3.6)-(3.8):

$$r\hat{g}^{ik}\left[\hat{\nabla}_k\hat{g}'_{ij}-\hat{\nabla}_j\hat{g}'_{ik}\right]=r\varphi'\partial_j\varphi \quad \Rightarrow \tag{3.59}$$ e68

$$r\hat{g}^{ik}\left[\partial_k\hat{g}'_{ij}-\hat{\Gamma}_{ik}^{\ \ l}\hat{g}'_{lj}-\partial_j\hat{g}'_{ik}+\hat{\Gamma}_{ij}^{\ \ l}\hat{g}'_{lk}\right]=r\varphi'\partial_j\varphi \quad \Rightarrow \tag{3.60}$$ e69

$$\begin{aligned} r\hat{g}^{ik}\Bigg[&\partial_k\hat{g}'_{ij}-\partial_j\hat{g}'_{ik}+\frac{1}{2}\hat{g}^{ls}\left(\partial_i\hat{g}_{sj}+\partial_j\hat{g}_{sj}-\partial_s\hat{g}_{ik}\right)\hat{g}'_{lk}-\\ &-\frac{1}{2}\hat{g}^{ls}\left(\partial_i\hat{g}_{ks}+\partial_k\hat{g}_{is}-\partial_s\hat{g}_{ki}\right)\hat{g}'_{lj}\Bigg]=r\varphi'\partial_j\varphi \end{aligned} \tag{3.61}$$ e70

We substitute the terms of the expansions (3.6)-(3.8) that will contribute up to order $\mathcal{O}(r^{2\Delta_\pm+2})$:

$$\begin{aligned} &\Big[g_0^{ik}\partial_k\left(2g_{ij}^{(2)}r^2+3g_{ij}^{(3)}r^3+2\Delta_+g_{ij}^{(2,0)}r^{2\Delta_+}\right)-\\ &-g_0^{ik}\partial_j\left(2g_{ik}^{(2)}r^2+3g_{ik}^{(3)}r^3+2\Delta_+g_{ik}^{(2,0)}r^{2\Delta_+}\right)+\\ &+\frac{1}{2}g_0^{ik}g_0^{ls}\Big[[s;ij]_0\left(2g_{lk}^{(2)}r^2+3g_{lk}^{(3)}r^3+2\Delta_+g_{lk}^{(2,0)}r^{2\Delta_+}\right)-\\ &-[s;ik]_0\left(2g_{lj}^{(2)}r^2+3g_{lj}^{(3)}r^3+2\Delta_+g_{lj}^{(2,0)}r^{2\Delta_+}\right)\Big]-\\ &-\Delta_+\varphi_{1,0}\partial_j\varphi_{1,0}r^{2\Delta_+}+(+\leftrightarrow-)\Big]-\Big(\Delta_+\varphi_{1,0}\partial_j\bar{\varphi}_{1,0}+\\ &+\Delta_-\bar{\varphi}_{1,0}\partial_j\varphi_{1,0}\Big)r^3+\mathcal{O}(\varphi^3,r^4,r^{\Delta_\pm+3},r^{2\Delta_\pm+2},r^{3\Delta_\pm})=0 \end{aligned} \tag{3.62}$$ e71

We regroup the coefficients that appear according to their powers of r.

$$\begin{aligned} &\left(2g_0^{ik}\partial_kg_{ij}^{(2)}-2g_0^{ik}\partial_jg_{ik}^{(2)}+2g_0^{ik}\Gamma_{0,ij}^{\ \ l}g_{lk}^{(2)}-2g_0^{ik}\Gamma_{0,ik}^{\ \ l}g_{lj}^{(2)}\right)r^2+\\ &\Big(3g_0^{ik}\partial_kg_{ij}^{(3)}-3g_0^{ik}\partial_jg_{ik}^{(3)}+3g_0^{ik}\Gamma_{0,ij}^{\ \ l}g_{lk}^{(3)}-3g_0^{ik}\Gamma_{0,ik}^{\ \ l}g_{lj}^{(3)}-\\ &-\Delta_-\bar{\varphi}_{1,0}\partial_j\varphi_{1,0}-\Delta_+\varphi_{1,0}\partial_j\bar{\varphi}_{1,0}\Big)r^3+\\ &+\Big[\Big(2\Delta_+g_0^{ik}\partial_kg_{ij}^{(2,0)}-2\Delta_+g_0^{ik}\partial_jg_{ik}^{(2,0)}+2\Delta_+g_0^{ik}\Gamma_{0,ij}^{\ \ l}g_{lk}^{(2,0)}-\\ &-2\Delta_+g_0^{ik}\Gamma_{0,ik}^{\ \ l}g_{lj}^{(2,0)}-\Delta_+\varphi_{1,0}\partial_j\varphi_{1,0}\Big)r^{2\Delta_+}+(+\leftrightarrow-)\Big]+\\ &+\mathcal{O}(\varphi^3,r^4,r^{\Delta_\pm+3},r^{2\Delta_\pm+2},r^{3\Delta_\pm})=0 \end{aligned} \tag{3.63}$$ e72

We will solve the equations "order by order" in r. The order r^2 equation is:

$$2g_0^{ik}\left(\partial_kg_{ij}^{(2)}-\Gamma_{0,ik}^{\ \ l}g_{lj}^{(2)}-\partial_jg_{ik}^{(2)}+\Gamma_{0,ij}^{\ \ l}g_{lk}^{(2)}\right)=0 \quad \Rightarrow \tag{3.64}$$ e73

$$g_0^{ik}\left(\nabla_{0,k}g_{ij}^{(2)}-\nabla_{0,j}g_{ik}^{(2)}\right)=0 \quad \Rightarrow \tag{3.65}$$ e74

$$\nabla_0^i g_{ij}^{(2)}=\nabla_0^i\left(g_{0,ij}Tr[g_0^{-1}g^{(2)}]\right) \tag{3.66}$$ e75

We substitute in the equation above the coefficient $g_{ij}^{(2)}$ as it appears in equations (3.22) and (3.23):

$$\nabla_0^i\left(\frac{R_0}{4}g_{0,ij}-R_{0,ij}\right)=-\nabla_0^i\left(\frac{R_0}{4}g_{0,ij}\right) \quad \Rightarrow \tag{3.67}$$ e76

$$\nabla_0^i\left(R_{0,ij}-\frac{R_0}{2}g_{0,ij}\right)=0 \tag{3.68}$$ e77

This is a Bianchi identity, therefore the 2nd order equation holds as an identity.

The r^3 order equation is:

$$3g_0^{ik}\left(\partial_k g_{ij}^{(3)}-\Gamma_{0,ik}{}^l g_{lj}^{(3)}-\Gamma_{0,jk}{}^l g_{li}^{(3)}-\partial_j g_{ik}^{(3)}+\Gamma_{0,ij}{}^l g_{lk}^{(3)}+\right. \tag{3.69}$$ e78

$$\left.+\Gamma_{0,jk}{}^l g_{li}^{(3)}\right)=\Delta_-\bar{\varphi}_{1,0}\partial_j\varphi_{1,0}+\Delta_+\varphi_{1,0}\partial_j\bar{\varphi}_{1,0} \quad \Rightarrow$$

$$\nabla_0^i g_{ij}^{(3)}=\nabla_0^i\left(g_{0,ij}Tr[g_0^{-1}g^{(3)}]\right)+\frac{\Delta_-\bar{\varphi}_{1,0}\partial_j\varphi_{1,0}+\Delta_+\varphi_{1,0}\partial_j\bar{\varphi}_{1,0}}{3} \tag{3.70}$$ e79

Therefore, by substituting the trace of $g_{ij}^{(3)}$ written in (3.41) we obtain:

$$\nabla_0^i g_{ij}^{(3)}=-\frac{\Delta_+\Delta_-}{3}\nabla_0^i\left(g_{0,ij}\varphi_{1,0}\bar{\varphi}_{1,0}\right)+\frac{\Delta_-\bar{\varphi}_{1,0}\partial_j\varphi_{1,0}+\Delta_+\varphi_{1,0}\partial_j\bar{\varphi}_{1,0}}{3} \quad \Rightarrow \tag{3.71}$$ e80

$$\nabla_0^i g_{ij}^{(3)}=-\frac{\Delta_+\Delta_--\Delta_-}{3}\nabla_0^i\left(g_{0,ij}\varphi_{1,0}\bar{\varphi}_{1,0}\right)+\frac{\Delta_+-\Delta_-}{3}\varphi_{1,0}\partial_j\bar{\varphi}_{1,0} \quad \Rightarrow \tag{3.72}$$ e81

$$\hat{\nabla}_0^i\left[3g_{ij}^{(3)}-(\Delta_-^2-2\Delta_-)g_{0,ij}\bar{\varphi}_{1,0}\varphi_{1,0}\right]=(2\Delta_+-3)\varphi_{1,0}\partial_j\bar{\varphi}_{1,0} \quad \Rightarrow \tag{3.73}$$ e81b

$$\hat{\nabla}_0^i\left[3g_{ij}^{(3)}-(\Delta_-^2-2\Delta_-)g_{0,ij}\bar{\varphi}_{1,0}\varphi_{1,0}\right]=\langle\mathcal{O}\rangle\partial_j\bar{\varphi}_{1,0} \tag{3.74}$$ e81a

We can already see which must be the form of the stress-tensor by demanding that the Ward identity is satisfied, however, it is better to follow the usual approach of deriving the Stress-Tensor from the action and to use (3.70) as a verification check.

Finally, the $r^{2\Delta_+}$ order equation is:

$$2\Delta_+ g_0^{ik}\left(\partial_k g_{ij}^{(2,0)}-\Gamma_{0,ik}{}^l g_{lj}^{(2,0)}-\Gamma_{0,jk}{}^l g_{li}^{(2,0)}-\partial_j g_{ik}^{(2,0)}+\Gamma_{0,ij}{}^l g_{lk}^{(2,0)}+\right. \tag{3.75}$$ e82

$$\left.+\Gamma_{0,jk}{}^l g_{li}^{(2,0)}\right)=\Delta_+\varphi_{1,0}\partial_j\varphi_{1,0} \quad \Rightarrow$$

$$\nabla_0^i g_{ij}^{(2,0)}=\nabla_0^i\left(g_{ij}^{(0)}Tr[g_0^{-1}g^{(2,0)}]\right)+\frac{\varphi_{1,0}\partial_j\varphi_{1,0}}{2} \tag{3.76}$$ e83

We substitute the metric coefficient $g_{ij}^{(2,0)}$ as it appears in equations (3.45) and (3.49) to obtain:

$$-\frac{1}{8}\nabla_0^i\left(\varphi_{1,0}^2 g_{0,ij}\right) = -\frac{3}{8}\nabla_0^i\left(g_{0,ij}\varphi_{1,0}^2\right) + \frac{\varphi_{1,0}\partial_j\varphi_{1,0}}{2} \quad \Rightarrow \tag{3.77}$$ e84

$$\frac{1}{4}\partial_j\left(\varphi_{1,0}^2\right) = \frac{\varphi_{1,0}\partial_j\varphi_{1,0}}{2} \quad \Rightarrow \tag{3.78}$$ e85

$$\frac{\varphi_{1,0}\partial_j\varphi_{1,0}}{2} = \frac{\varphi_{1,0}\partial_j\varphi_{1,0}}{2} \tag{3.79}$$ e86

We see that equation (3.79) is satisfied as an identity and similarly, the equation for $\bar{g}_{ij}^{(2,0)}$ holds as well:

$$\nabla_0^i\bar{g}_{ij}^{(2,0)} = \nabla_0^i\left(g_{0,ij}Tr[g_0^{-1}\bar{g}^{(2,0)}]\right) + \frac{\bar{\varphi}_{1,0}\partial_j\bar{\varphi}_{1,0}}{2} \quad \Rightarrow \quad 0 = 0 \tag{3.80}$$ e87

s4s10

3.9 A potential of cubic order in φ.

Before proceeding to the calculation of the stress energy tensor it is a good idea to consider what would have happened had we kept cubic terms of the scalar in our potential. Assume that:

$$V = -\frac{6}{L^2} + \frac{m^2}{2}\varphi^2 + \frac{V_3}{3L^2}\varphi^3 + \mathcal{O}(\varphi^4) \tag{3.81}$$ e88

Equation (2.28) shows us that in the scalar equation, there would be an extra term:

$$-\ell^2\frac{\delta V}{\delta\varphi} \to -\frac{\ell^2}{L^2}V_3\varphi^2 + \mathcal{O}(\varphi^3) \tag{3.82}$$ e89

We expand the scalar according to (3.6):

$$\begin{aligned}
-\ell^2\frac{\delta V}{\delta\varphi} \to -\frac{\ell^2}{L^2}V_3\left(\varphi_0^2 + 2\varphi_0\varphi_1 r + \left(2\varphi_0\varphi_2 + \varphi_1^2\right)r^2\right) - \\
-\frac{\ell^2}{L^2}V_3\left(2\varphi_0\varphi_3 + 2\varphi_1\varphi_2 + 2\varphi_{1,0}\bar{\varphi}_{1,0}\right)r^3 - \frac{\ell^2}{L^2}V_3\left[2\varphi_{1,0}\varphi_0 r^{\Delta_+} + \right. \\
+(2\varphi_0\varphi_{1,1} + 2\varphi_1\varphi_{1,0})r^{\Delta_+ + 1} + \left(2\varphi_0\varphi_{1,2} + 2\varphi_1\varphi_{1,1} + 2\varphi_2\varphi_{1,0}\varphi\right)r^{\Delta_+ + 2} \\
\left. +(2\varphi_0\varphi_{1,3} + 2\varphi_1\varphi_{1,2} + 2\varphi_2 + 2\varphi_3\varphi_{1,0} + 2\varphi_{2,0}\bar{\varphi}_{1,0})r^{\Delta_+ + 3} + (+ \leftrightarrow -)\right] + \\
+\mathcal{O}(\varphi^3, r^4, r^{\Delta_\pm + 4}, r^{2\Delta_\pm})
\end{aligned} \tag{3.83}$$ e89a

Now equation (3.13) becomes:

$$-m^2L^2\varphi_0 - V_3\varphi_0^2 = 0 \quad \Rightarrow \tag{3.84}$$ e90

$$\varphi_0\left(m^2L^2 + V_3\varphi_0\right) = 0 \tag{3.85}$$ e91

It looks like φ_0 could take the non-trivial value:

$$\varphi_0 = -\frac{m^2 L^2}{V_3} \tag{3.86}$$ e92

However, this corresponds to a different fixed point from the one we started and to be consistent with our assumptions we need to take $\varphi_0 = 0$. As we can see from (3.83), the corrections in most of the orders of r will vanish with a few exceptions. The non-trivial corrections are:

$$-\ell^2 \frac{\delta V}{\delta \varphi} \to -2\frac{\ell^2}{L^2} V_3 \left(\varphi_{1,0}\bar{\varphi}_{1,0} r^3 + \left[(\varphi_3 \varphi_{1,0} + \varphi_{2,0}\bar{\varphi}_{1,0}) r^{\Delta_+ + 3} + \right.\right. \\ \left.\left. +(+ \leftrightarrow -)\right]\right) + \mathcal{O}(\varphi^3, r^4, r^{\Delta_\pm + 4}, r^{2\Delta_\pm}) \tag{3.87}$$ e93

We see that the first terms that would be corrected in the scalar expansion would be φ_3 and $\varphi_{1,3}$. However, the terms containing a potential in the on-shell action are quadratic or cubic in the scalar for a cubic potential. We see that any combination of the existing scalar terms or the corrected ones will be of greater order than the terms we have omitted, which are of order $\mathcal{O}(\varphi^4, r^4, r^{\Delta_\pm + 3}, r^{2\Delta_\pm + 2}, r^{3\Delta_\pm})$. Therefore, from the scalar's point of view, the Stress-Tensor is unaffected by potentials with greater powers of the scalar than φ^2. However, before concluding that a potential with a cubic scalar term would leave the Stress-Tensor unaffected, we have to see the corrections in the ij equation, because through that equation we calculated the metric coefficients.

The correction in the ij equation comes in the form of two extra terms:

$$\frac{\delta V}{2} \ell^2 \hat{g}_{ij} \to \left(\frac{m^2 \ell^2}{4}(\varphi^2 - \varphi_{old}^2) + \frac{V_3 \ell^2}{6L^2}\varphi^3\right) \hat{g}_{ij} + \mathcal{O}(\varphi^4), \tag{3.88}$$ e99

where the scalar φ_{old} contains the coefficients we calculated ignoring the cubic term in the potential. By inserting the new and old terms of the scalar, we obtain:

$$\frac{\delta V}{2} \ell^2 \hat{g}_{ij} \to \left(\frac{m^2 \ell^2}{4}\left[\varphi_3^2 r^6 + 2\varphi_3 \varphi_{1,0} r^{\Delta_+ + 3} + (+ \leftrightarrow -)\right] + \right. \\ +\mathcal{O}(\varphi^4, r^7, r^{\Delta_\pm + 4}, r^{2\Delta_\pm + 3}, r^{3\Delta_\pm}) + \\ \left. + \frac{V_3 \ell^2}{6L^2}\left[6\varphi_3 \varphi_{1,0}\bar{\varphi}_{1,0} r^6 + \left(3\varphi_{1,0}^2 \bar{\varphi}_{1,0} r^{\Delta_+ + 3} + (+ \leftrightarrow -)\right)\right]\right) \hat{g}_{ij} \tag{3.89}$$ e100

We see that the first corrections to the metric coefficients will be at $g_{ij}^{(6)}$, $g_{ij}^{(1,3)}$ and $\bar{g}_{ij}^{(1,3)}$. However, as we can see from their order in r, all corrections lie after the terms required for the Stress-Energy tensor. Therefore, a cubic potential would not alter the results of this analysis in a physically meaningful way.

s5 4. The On shell action.

We now move to the on-shell action, which we had already written in (2.38), but we will reproduce it for convenience after we substitute the potential in (3.5) to see explicitly the scalar:

$$S_E^{on-shell}[g,\varphi] = M^2 \int d^3x dr \sqrt{g} \left(\frac{6}{L^2} - \frac{m^2}{2}\varphi^2 + \mathcal{O}(\varphi^3) \right) - S_{GHY} \tag{4.1}$$ f3

The induced metric (at the boundary $r = \epsilon$) is:

$$\gamma_{ij} = \frac{L^2}{\epsilon^2} \hat{g}_{ij}(\epsilon, x^i) \tag{4.2}$$ f4

Therefore the determinant γ is:

$$\gamma = \left(\frac{L^2}{\epsilon^2}\right)^3 \hat{g} \quad \Rightarrow \quad \sqrt{\gamma} = \left(\frac{L}{\epsilon}\right)^3 \sqrt{\hat{g}} \tag{4.3}$$ f5

The vector which is orthogonal to the hypersurface $r = \epsilon$ has the form:

$$n^\rho = (A, 0, 0, 0) \tag{4.4}$$ f6

We normalize it using the metric $g_{\mu\nu}$ in (A.1):

$$n^\mu g_{\mu\nu} n^\nu = 1 \quad \Rightarrow \quad A = \pm \frac{r}{L} \tag{4.5}$$ f7

The sign is selected to point toward the direction where n^ρ increases and thus we take it to be positive.

The extrinsic curvature in terms of $\hat{g}_{ij}$ is given by:

$$K_{ij} = -\frac{1}{2} n^\rho \partial_\rho g_{ij} = -\left(\frac{r}{2L}\partial_r \gamma_{ij}\right)_{r=\epsilon} = -\frac{\epsilon}{2L}\left[\partial_r \left(\frac{L^2}{r^2}\hat{g}_{ij}\right)\right]_{r=\epsilon} \quad \Rightarrow \tag{4.6}$$ new1

$$K_{ij} = \frac{L}{\epsilon^2}\left[\hat{g}_{ij} - \frac{\epsilon}{2}\hat{g}'_{ij}(\epsilon, x^i)\right] \tag{4.7}$$ new2

The extrinsic curvature's trace is taken with γ^{ij} :

$$K = \gamma^{ij} K_{ij} = \frac{1}{L}\left[3 - \frac{\epsilon}{2} Tr[\hat{g}^{-1}\hat{g}'(\epsilon, x^i)]\right] \tag{4.8}$$ f10

Therefore, the Euclidean action takes the form:

$$S_E^{on-shell}[\hat{g},\varphi] = M^2 L^2 \int d^3x dr \frac{1}{r^4}\sqrt{\hat{g}} \left(6 - \frac{m^2 L^2}{2}\varphi^2 + \mathcal{O}(\varphi^3)\right) - \tag{4.9}$$ f12

$$-2M^2L^2 \left[\int d^3x \frac{1}{r^3}\sqrt{\hat{g}}(3 - \frac{r}{2}Tr[\hat{g}^{-1}\hat{g}'])\right]_{r=\epsilon} \quad \Rightarrow$$

$$S_E^{on-shell}[\hat{g},\varphi] = M^2L^2\int d^3xdr\frac{\sqrt{\hat{g}}}{r^4}\left(6+\frac{\Delta_+\Delta_-}{2}\varphi^2+\mathcal{O}(\varphi^3)\right)- \tag{4.10}$$ f13

$$-M^2L^2\left[\int d^3x\frac{\sqrt{\hat{g}}}{r^3}(6-rTr[\hat{g}^{-1}\hat{g}'])\right]_{r=\epsilon}$$

We expand the determinant of the metric in powers of r variationally to identify the counterterms that need to be added for the renormalization of the action. We define:

$$\hat{g}_{ij} = g_{0,ij}+\delta\hat{g}_{ij} = g_{0,ik}(\delta^k{}_j + g_0^{kl}\delta\hat{g}_{lj}) \tag{4.11}$$ f14

and:

$$\sqrt{\hat{g}} = \sqrt{g_0}\sqrt{I+g_0^{-1}\delta\hat{g}} \tag{4.12}$$ f15

The term $\delta\hat{g}$ contains all the powers of r. Therefore we rewrite (4.12) as follows:

$$\sqrt{\hat{g}} = \sqrt{g_0}\left(1+\frac{1}{2}Tr[g_0^{-1}\delta\hat{g}]+\mathcal{O}\left((g_0^{-1}\delta\hat{g})^2\right)\right) \tag{4.13}$$ f17

There is no reason to go beyond the first order in δg since higher powers would produce terms that vanish at $r\to 0$. Therefore we obtain:

$$\sqrt{\hat{g}} = \sqrt{g_0}\Bigg(1+\frac{1}{2}Tr[g_0^{-1}g^{(2)}]r^2+\frac{1}{2}Tr[g_0^{-1}g^{(3)}]r^3++ \tag{4.14}$$ f18

$$+\frac{1}{2}Tr[g_0^{-1}\bar{g}^{(2,0)}]r^{2\Delta_-}\Bigg)+\mathcal{O}(\varphi^3,r^4,r^{\Delta_\pm+3},r^{2\Delta_+},r^{2\Delta_-+2},r^{3\Delta_\pm})$$

We see that as long as we limit ourselves to dimensions $2>\Delta_+>1.5$ and $1.5>\Delta_->1$, all of the neglected terms will not contribute to the counterterms. For the scalar field, the expansion is:

$$\varphi^2 = \left(\varphi_{1,0}r^{\Delta_+}+(+\leftrightarrow -)+\mathcal{O}(\varphi^3,r^4,r^{\Delta_\pm+2},r^{2\Delta_\pm},r^{3\Delta_\pm})\right)^2 \quad\Rightarrow \tag{4.15}$$ f19

$$\varphi^2 = 2\varphi_{1,0}\bar{\varphi}_{1,0}r^3+\bar{\varphi}_{1,0}^2r^{2\Delta_-}+\mathcal{O}(\varphi^3,r^4,r^{\Delta_\pm+3},r^{2\Delta_+},r^{2\Delta_-+2},r^{3\Delta_\pm}) \tag{4.16}$$ f20

Similar to the metric, we see that the neglected terms will not contribute to the counterterms. We substitute the expansions of (4.14) and (4.16) in the integrals of (4.10) to obtain:

$$S_E^{on-shell}[\hat{g},\varphi] = M^2L^2\int d^3xdr\frac{\sqrt{g_0}}{r^4}\left[\left(1+\frac{1}{2}Tr[g_0^{-1}g^{(2)}]r^2+\right.\right.$$

$$+\frac{1}{2}Tr[g_0^{-1}g^{(3)}]r^3+\ \frac{1}{2}Tr[g_0^{-1}\bar{g}^{(2,0)}]r^{2\Delta_-}\Big)\Big(6+\frac{\Delta_+\Delta_-}{2}(2\varphi_{1,0}\bar{\varphi}_{1,0}r^3+$$

$$+\bar{\varphi}_{1,0}^2r^{2\Delta_-})\Big)+\mathcal{O}(\varphi^3,r^4,r^{\Delta_\pm+3},r^{2\Delta_+},r^{2\Delta_-+2},r^{3\Delta_\pm})\Big]- \tag{4.17}$$ f21

$$-M^2L^2\Big[\int d^3x\frac{\sqrt{g_0}}{r^3}\Big(\Big(1+\frac{1}{2}Tr[g_0^{-1}g^{(2)}]r^2+\frac{1}{2}Tr[g_0^{-1}g^{(3)}]r^3+$$

$$+\frac{1}{2}Tr[g_0^{-1}\bar{g}^{(2,0)}]r^{2\Delta_-}\Big)\Big(6-2Tr[g_0^{-1}g^{(2)}]r^2-3Tr[g_0^{-1}g^{(3)}]r^3-$$

$$-2\Delta_-Tr[g_0^{-1}\bar{g}^{(2,0)}]r^{2\Delta_-}\Big)\Big)+\mathcal{O}(\varphi^3,r^4,r^{\Delta_\pm+3},r^{2\Delta_+},r^{2\Delta_-+2},r^{3\Delta_\pm})\Big]_{r=\epsilon}$$

We regroup terms and perform the appropriate multiplications to obtain:

$$S_E^{on-shell}[\hat{g},\varphi]=M^2L^2\int d^3xdr\frac{\sqrt{g_0}}{r^4}\Big[6+3Tr[g_0^{-1}g^{(2)}]r^2+$$

$$+\left(3Tr[g_0^{-1}g^{(3)}]+\Delta_+\Delta_-\varphi_{1,0}\bar{\varphi}_{1,0}\right)r^3+\Big(3Tr[g_0^{-1}\bar{g}^{(2,0)}]+ \tag{4.18}$$ f22

$$+\frac{\Delta_+\Delta_-}{2}\bar{\varphi}_{1,0}^2\Big)r^{2\Delta_-}+\mathcal{O}(\varphi^3,r^4,r^{\Delta_\pm+3},r^{2\Delta_+},r^{2\Delta_-+2},r^{3\Delta_\pm})\Big]-$$

$$-M^2L^2\Big[\int d^3x\frac{\sqrt{g_0}}{r^3}\Big(6+Tr[g_0^{-1}g^{(2)}]r^2+(3-2\Delta_-)Tr[g_0^{-1}\bar{g}^{(2,0)}]r^{2\Delta_-}+$$

$$+\mathcal{O}(\varphi^3,r^4,r^{\Delta_\pm+3},r^{2\Delta_+},r^{2\Delta_-+2},r^{3\Delta_\pm})\Big)\Big]_{r=\epsilon}$$

And finally we distribute the factors of r and use (3.41) and (3.46) to simplify the integrals:

$$S_E^{on-shell}[\hat{g},\varphi]=M^2L^2\int d^3x\int_\epsilon^{r_{IR}}dr\sqrt{g_0}\left[6r^{-4}+3Tr[g_0^{-1}g^{(2)}]r^{-2}+\right.$$

$$+\left(3-\frac{4\Delta_+\Delta_-}{3}\right)Tr[g_0^{-1}\bar{g}^{(2,0)}]r^{2\Delta_--4}+ \tag{4.19}$$

$$\left.+\mathcal{O}(\varphi^3,r^0,r^{\Delta_\pm-1},r^{2\Delta_+-4},r^{2\Delta_--2},r^{3\Delta_\pm-4})\right]-$$ f23

$$-M^2L^2\int d^3x\sqrt{g_0}\left(6\epsilon^{-3}+Tr[g_0^{-1}g^{(2)}]\epsilon^{-1}+(3-2\Delta_+)Tr[g_0^{-1}g^{(2,0)}]\epsilon^{2\Delta_+-3}+\right.$$

$$\left.+(3-2\Delta_-)Tr[g_0^{-1}\bar{g}^{(2,0)}]\epsilon^{2\Delta_--3}+\mathcal{O}(\varphi^3,\epsilon^1,\epsilon^{\Delta_\pm},\epsilon^{2\Delta_+-3},\epsilon^{2\Delta_--1},\epsilon^{3\Delta_\pm-3})\right)$$

We integrate over r and separate the infrared from the ultraviolet part of the action. S_{IR} will denote the action that contains all powers of the infrared endpoint r_{IR}:

$$S_E^{on-shell}[\hat{g},\varphi]=S_{IR}+M^2L^2\int d^3x\sqrt{g_0}\left[2\epsilon^{-3}+3Tr[g_0^{-1}g^{(2)}]\epsilon^{-1}-\right.$$

$$-\left(\frac{4\Delta_-^2-12\Delta_-+9}{3(2\Delta_--3)}\right)Tr[g_0^{-1}\bar{g}^{(2,0)}]\epsilon^{2\Delta_--3}+ \tag{4.20}$$
$$+\mathcal{O}(\varphi^3,\epsilon,\epsilon^{\Delta_\pm},\epsilon^{2\Delta_+-3},\epsilon^{2\Delta_--1},\epsilon^{3\Delta_\pm-3})\Big]-\boxed{\texttt{f24}}$$
$$-M^2L^2\int d^3x\sqrt{g_0}\left(6\epsilon^{-3}+Tr[g_0^{-1}g^{(2)}]\epsilon^{-1}+(3-2\Delta_-)Tr[g_0^{-1}\bar{g}^{(2,0)}]\epsilon^{2\Delta_--3}+\right.$$
$$\left.+\mathcal{O}(\varphi^3,\epsilon,\epsilon^{\Delta_\pm},\epsilon^{2\Delta_+-3},\epsilon^{2\Delta_--1},\epsilon^{3\Delta_\pm-3})\right)$$

We combine the integrals and regroup the terms to obtain:

$$S_E^{on-shell}[\hat{g},\varphi]=S_{IR}+M^2L^2\int d^3x\sqrt{g_0}\left[-4\epsilon^{-3}+2Tr[g_0^{-1}g^{(2)}]\epsilon^{-1}+\right.$$
$$\left.+\frac{2}{3}(2\Delta_--3)Tr[g_0^{-1}\bar{g}^{(2,0)}]\epsilon^{2\Delta_--3}+\mathcal{O}(\varphi^3,\epsilon,\epsilon^{\Delta_\pm},\epsilon^{2\Delta_+-3},\epsilon^{2\Delta_--1},\epsilon^{3\Delta_\pm-3})\right] \tag{4.21}$$ $\boxed{\texttt{f25}}$

$\boxed{\texttt{s5s1}}$ 4.1 The renormalization of the on-shell action

To obtain the renormalized action we must add counterterms for the divergences and take the limit $\epsilon\to 0$. However, the counterterms must be expressed in terms of the induced metric. We will do that order by order in ϵ up to terms that vanish in the limit $\epsilon\to 0$. Using equation (3.9) we see that the induced metric's determinant up to finite terms in the limit $\epsilon\to 0$ is:

$$\sqrt{\gamma}=\left(\frac{L^3}{r^3}\right)\sqrt{\hat{g}}=L^3\sqrt{g_0}(r^{-3}+\frac{1}{2}Tr[g_0^{-1}g^{(2)}]r^{-1}+\frac{1}{2}Tr[g_0^{-1}g^{(3)}]+$$
$$\left.\frac{1}{2}Tr[g_0^{-1}\bar{g}^{(2,0)}]r^{2\Delta_--3}\right)+\mathcal{O}(\varphi^3,r,r^{\Delta_\pm+1},r^{2\Delta_+-3},r^{2\Delta_--1},r^{3\Delta_\pm-3})\quad\Rightarrow \tag{4.22}$$ $\boxed{\texttt{f26}}$

$$-4M^2L^2\sqrt{g_0}\epsilon^{-3}=-\frac{4M^2}{L}\sqrt{\gamma}+2M^2L^2\sqrt{g_0}Tr[g_0^{-1}g^{(2)}]\epsilon^{-1}+$$
$$+2M^2L^2\sqrt{g_0}Tr[g_0^{-1}g^{(3)}]+2M^2L^2\sqrt{g_0}Tr[g_0^{-1}\bar{g}^{(2,0)}]\epsilon^{2\Delta_--3}+$$
$$+\mathcal{O}(\varphi^3,\epsilon,\epsilon^{\Delta_\pm},\epsilon^{2\Delta_+-3},\epsilon^{2\Delta_--1},\epsilon^{3\Delta_\pm-3}) \tag{4.23}$$ $\boxed{\texttt{nf1}}$

We can now express the leading singularity in (4.21) in terms of γ. The action in the limit $\epsilon\to 0$ becomes:

$$\lim_{\epsilon\to 0}\left(S_E^{on-shell}[\hat{g},\varphi]\right)=S_{IR}+\lim_{\epsilon\to 0}\left(M^2L^2\int d^3x\left(-\frac{4}{L^3}\sqrt{\gamma}\right)+\right.$$
$$\left.+\sqrt{g_0}\left[4Tr[g_0^{-1}g^{(2)}]\epsilon^{-1}+2Tr[g_0^{-1}g^{(3)}]+\frac{4\Delta_-}{3}Tr[g_0^{-1}\bar{g}^{(2,0)}]\epsilon^{2\Delta_--3}\right]\right) \tag{4.24}$$ $\boxed{\texttt{f28}}$

And using (3.22) we have:

$$\lim_{\epsilon\to 0}\left(S_E^{on-shell}[\hat{g},\varphi]\right)=S_{IR}+\lim_{\epsilon\to 0}\left(M^2L^2\int d^3x\left(-\frac{4}{L^3}\sqrt{\gamma}\right)+\sqrt{g_0}\left[-R_0\epsilon^{-1}+\right.\right.$$

$$+2Tr[g_0^{-1}g^{(3)}] + \frac{4\Delta_-}{3}Tr[g_0^{-1}\bar{g}^{(2,0)}]\epsilon^{2\Delta_- -3}\Bigg]\Bigg) \tag{4.25}$$ f29

Since the Ricci scalar and Ricci tensor of the metric $\hat{g}_{ij}$ contain no derivatives with respect to r, expressing them in terms of the induced metric is a matter of scaling. Each term in the definition of the Ricci tensor is a product of one inverse metric and one metric, therefore the scaling is:

$$\hat{R}_{ij} = R_{ij}[\gamma] \quad \Rightarrow \quad R[\gamma] = \frac{r^2}{L^2}\hat{R} \tag{4.26}$$ f30

We will express the next singularity in terms of γ_{ij}:

$$\sqrt{\gamma}R[\gamma] = L\sqrt{g_0}\Bigg(r^{-3} + \frac{1}{2}Tr[g_0^{-1}g^{(2)}]r^{-1} + \frac{1}{2}Tr[g_0^{-1}g^{(3)}] +$$

$$+\frac{1}{2}Tr[g_0^{-1}\bar{g}^{(2,0)}]r^{2\Delta_- -3}\Bigg)r^2\hat{R} + \mathcal{O}(\varphi^3, r, r^{\Delta_\pm}, r^{2\Delta_+ -3}, r^{2\Delta_- -1}, r^{3\Delta_\pm -3}) \quad \Rightarrow \tag{4.27}$$ f31

$$\sqrt{\gamma}R[\gamma] = \frac{L}{r}\sqrt{g_0}R_0 + \mathcal{O}(\varphi^3, r, r^{\Delta_\pm}, r^{2\Delta_+ -3}, r^{2\Delta_- -1}, r^{3\Delta_\pm -3}) \quad \Rightarrow \tag{4.28}$$ f32

$$-M^2L\sqrt{\gamma}R[\gamma] = -M^2L^2\frac{1}{\epsilon}\sqrt{g_0}R[g_0] + \mathcal{O}(\varphi^3, \epsilon, \epsilon^{\Delta_\pm +1}, \epsilon^{2\Delta_+ -3}, \epsilon^{2\Delta_- -1}, \epsilon^{3\Delta_\pm -3}) \tag{4.29}$$ nf2

Using (4.29), the action takes the form:

$$\lim_{\epsilon\to 0}\left(S_E^{on-shell}[\hat{g},\varphi]\right) = S_{IR} + \lim_{\epsilon\to 0}\Bigg(M^2L^2\int d^3x\left(-\frac{4}{L^3}\sqrt{\gamma} - \frac{1}{L}\sqrt{\gamma}R[\gamma]\right) +$$

$$\sqrt{g_0}\left[2Tr[g_0^{-1}g^{(3)}] + \frac{4\Delta_-}{3}Tr[g_0^{-1}\bar{g}^{(2,0)}]\epsilon^{2\Delta_- -3}\right]\Bigg) \tag{4.30}$$ f34

To express the final divergence in terms of the induced metric at the boundary, we use equation and (3.46):

$$\lim_{\epsilon\to 0}\left(S_E^{on-shell}[\hat{g},\varphi]\right) = S_{IR} + \lim_{\epsilon\to 0}\Bigg(M^2L^2\int d^3x\left(-\frac{4}{L^3}\sqrt{\gamma} - \frac{1}{L}\sqrt{\gamma}R[\gamma]\right) +$$

$$\sqrt{g_0}\left[2Tr[g_0^{-1}g^{(3)}] - \frac{\Delta_-}{2}\bar{\varphi}_{1,0}^2\epsilon^{2\Delta_- -3}\right]\Bigg) \tag{4.31}$$ f36

We have to study the following product in the limit $\epsilon \to 0$. Using (4.16) and (4.22) we find:

$$\sqrt{\gamma}\varphi^2 = L^3\sqrt{g_0}\Bigg(r^{-3} + \frac{1}{2}Tr[g_0^{-1}g^{(2)}]r^{-1} + \frac{1}{2}Tr[g_0^{-1}g^{(3)}] +$$

$$+\frac{1}{2}Tr[g_0^{-1}g^{(2,0)}]r^{2\Delta_- -3}\Big)\left(2\varphi_{1,0}\bar{\varphi}_{1,0}r^3+\bar{\varphi}_{1,0}^2 r^{2\Delta_-}\right)+ \tag{4.32}$$ f37

$$+\mathcal{O}(\varphi^3, r, r^{\Delta_\pm}, r^{2\Delta_+ -3}, r^{2\Delta_- -1}, r^{3\Delta_\pm -3})$$

By performing the multiplication and dropping the terms that vanish when $\epsilon \to 0$ we have:

$$\sqrt{\gamma}\varphi^2 = L^3\sqrt{g_0}\left(2\varphi_{1,0}\bar{\varphi}_{1,0}+\bar{\varphi}_{1,0}^2 r^{2\Delta_- -3}\right)+ \tag{4.33}$$ f38

$$+\mathcal{O}(\varphi^3, r, r^{\Delta_\pm}, r^{2\Delta_+ -3}, r^{2\Delta_- -1}, r^{3\Delta_\pm -3}) \quad \Rightarrow$$

$$-\frac{\Delta_-}{2}M^2L^2\sqrt{g_0}\bar{\varphi}_{1,0}^2\epsilon^{2\Delta_- -3} = -\frac{\Delta_-}{2}\frac{M^2}{L}\sqrt{\gamma}\varphi^2+\Delta_- M^2L^2\sqrt{g_0}\varphi_{1,0}\bar{\varphi}_{1,0} \tag{4.34}$$ f39

$$+\mathcal{O}(\varphi^3, \epsilon, \epsilon^{\Delta_\pm}, \epsilon^{2\Delta_+ -3}, \epsilon^{2\Delta_- -1}, \epsilon^{3\Delta_\pm -3})$$

Therefore, by defining:

$$S_{reg}[\hat{g},\varphi] \equiv \lim_{\epsilon\to 0}\left(S_E^{on-shell}[\hat{g},\varphi]\right) \tag{4.35}$$ f39a

the regulated action takes the form:

$$S_{reg}[\hat{g},\varphi] = \lim_{\epsilon\to 0}\Bigg(M^2L^2\int d^3x\sqrt{\gamma}\left(-\frac{4}{L^3}-\frac{1}{L}R[\gamma]-\frac{\Delta_-}{2L^3}\varphi^2\right)+ \tag{4.36}$$ f40

$$+\sqrt{g_0}\left[2Tr[g_0^{-1}g^{(3)}]+\Delta_-\varphi_{1,0}\bar{\varphi}_{1,0}\right]\Bigg)+S_{IR}$$

We can identify, the counterterm action from (4.36):

$$S_{ct} = M^2L^2\int dx^3\sqrt{\gamma}\left(\frac{4}{L^3}+\frac{R[\gamma]}{L}+\frac{\Delta_-}{2L^3}\varphi^2\right) \tag{4.37}$$ f41

And we have completed the renormalization of the action:

$$S_{ren} = \lim_{\epsilon\to 0}\left(S_{ct}+S_{reg}\right) = S_{IR}+M^2L^2\int d^3x\sqrt{g_0}\left(2Tr[g_0^{-1}g^{(3)}]+\Delta_-\varphi_{1,0}\bar{\varphi}_{1,0}\right) \tag{4.38}$$ f42

s5s2

4.2 The Stress-Energy Tensor

The expected value of the Stress-Energy Tensor is given by:

$$\langle T_{ij}\rangle = \frac{2}{\sqrt{g_0}}\frac{\delta S_{ren}}{\delta g_0^{ij}} = \lim_{\epsilon\to 0}\frac{2}{\sqrt{\hat{g}}}\frac{\delta S_{ren}}{\delta \hat{g}^{ij}} = \lim_{\epsilon\to 0}\left(\frac{L}{\epsilon}\frac{2}{\sqrt{\gamma}}\frac{\delta S_{ren}}{\delta\gamma^{ij}}\right) \tag{4.39}$$ f53

We may freely move from one expression to the next because the renormalized action contains no divergences. Furthermore, we shall return to the original expression of the supergravity action in (4.1), to avoid implicit dependencies between the remaining parameters.

The renormalized action contains two contributions, one from the supergravity action and one from the counterterm action. The bulk action does not contribute because by definition its variation with respect to the metric produces the equations of motion and since the on-shell action is evaluated for the solutions to these equations, the bulk term vanishes. We begin by examining the contribution of the GHY term. Using:

$$\frac{\delta K}{\delta \gamma^{ij}} = \frac{1}{2} K_{ij} \tag{4.40}$$ f53a

we obtain:

$$T_{GHY,ij} = \lim_{\epsilon \to 0} \left(\frac{L}{\epsilon} \frac{2}{\sqrt{\gamma}} \frac{\delta S_{GHY}}{\delta \gamma^{ij}} \right) =$$

$$= -4M^2 L \lim_{\epsilon \to 0} \left(\frac{1}{\epsilon} \left(-\frac{1}{2} K \gamma_{ij} + K_{ij} - \frac{1}{2} K_{ij} \right) \right) \quad \Rightarrow \tag{4.41}$$ f58

$$T_{GHY,ij} = -2M^2 L^2 \lim_{\epsilon \to 0} \left(\frac{1}{L\epsilon} \left(K_{ij} - K \gamma_{ij} \right) \right) \tag{4.42}$$ f59

Using (4.2), (4.7) and (4.8) we express T_{GHY} in terms of the metric $\hat{g}_{ij}$:

$$T_{GHY,ij} = -2M^2 L^2 \lim_{\epsilon \to 0} \left(\frac{1}{L\epsilon} \left(\frac{\epsilon}{2L} \left[2 \frac{L^2}{\epsilon^3} \hat{g}_{ij} - \frac{L^2}{\epsilon^2} \hat{g}'_{ij} \right] - \right.\right.$$

$$\left.\left. -\frac{1}{L} \left[3 - \frac{\epsilon}{2} Tr[\hat{g}^{-1} \hat{g}'] \right] \frac{L^2}{\epsilon^2} g_{ij} \right) \right) \tag{4.43}$$ f60

We rearrange the terms to obtain:

$$T_{GHY,ij} = M^2 L^2 \lim_{\epsilon \to 0} \left(\frac{4}{\epsilon^3} \hat{g}_{ij} + \frac{1}{\epsilon^3} \epsilon \hat{g}'_{ij} - \frac{1}{\epsilon^3} Tr[\hat{g}^{-1} \epsilon \hat{g}'] \hat{g}_{ij} \right) \tag{4.44}$$ f61

The contribution of the counterterm action comes from the action in (4.37):

$$T_{ct,ij} = \lim_{\epsilon \to 0} \left(\frac{L}{\epsilon} \frac{2}{\sqrt{\gamma}} \frac{\delta S_{ct}}{\delta \gamma^{ij}} \right) =$$

$$= 2M^2 L^2 \lim_{\epsilon \to 0} \left(\frac{L}{\epsilon} \left(-\frac{1}{2} \gamma_{ij} \left(\frac{4}{L^3} + \frac{R[\gamma]}{L} + \frac{\Delta_-}{2L^3} \varphi^2 \right) + \frac{R_{ij}[\gamma]}{L} \right) \right) \tag{4.45}$$ f62

We use (4.2) and (4.26) to express the counterterm contribution in terms of the metric $\hat{g}_{ij}$:

$$T_{ct,ij} = 2M^2 L^2 \lim_{\epsilon \to 0} \left(\frac{L}{\epsilon} \left(-\frac{L^2}{2\epsilon^2} \hat{g}_{ij} \left(\frac{4}{L^3} + \frac{\epsilon^2 \hat{R}}{L^3} + \frac{\Delta_-}{2L^3} \varphi^2 \right) + \frac{\hat{R}_{ij}}{L} \right) \right) \tag{4.46}$$ f63

We then rearrange the resulting terms to obtain:

$$T_{ct,ij} = M^2 L^2 \lim_{\epsilon \to 0} \left(\frac{2}{\epsilon} \left(\hat{R}_{ij} - \frac{1}{2} \hat{R} \hat{g}_{ij} \right) - \frac{4}{\epsilon^3} \hat{g}_{ij} - \frac{\Delta_-}{2\epsilon^3} \varphi^2 \hat{g}_{ij} \right) \tag{4.47}$$ f64

We can now obtain the expected value of the Stress-Energy tensor by adding the two contributions:

$$\langle T_{ij}\rangle = T_{ct,ij} + T_{GHY,ij} = M^2L^2 \lim_{\epsilon\to 0}\left(\frac{2}{\epsilon}\left(\hat{R}_{ij} - \frac{1}{2}\hat{R}\hat{g}_{ij}\right) - \frac{\Delta_-}{2\epsilon^3}\varphi^2\hat{g}_{ij} + \right. \tag{4.48}$$

$$\left. + \frac{1}{\epsilon^3}\epsilon\hat{g}'_{ij} - \frac{1}{\epsilon^3}Tr[\hat{g}^{-1}\epsilon\hat{g}']\hat{g}_{ij}\right)$$

We use the expansions (3.6)- (3.9) up to finite terms. If the renormalization was correct we will see the divergent terms cancel each other out. For this reason, we will not write explicitly the limit of ϵ going to zero:

$$\langle T_{ij}\rangle = M^2L^2\left(2\epsilon^{-1}\left(R_{0,ij} - \frac{1}{2}R_0 g_{0,ij}\right) - \frac{\Delta_-}{2}\bar{\varphi}^2_{1,0}g_{0,ij}\epsilon^{2\Delta_- -3} - \right. \tag{4.49}$$

$$-\Delta_-\varphi_{1,0}\bar{\varphi}_{1,0}g_{0,ij} + 2\epsilon^{-1}g^{(2)}_{ij} + 3g^{(3)}_{ij} + 2\Delta_-\epsilon^{2\Delta_- -3}\bar{g}^{(2,0)}_{ij} -$$

$$\left. -2\epsilon^{-1}Tr[g_0^{-1}g^{(2)}]g_{0,ij} - 3Tr[g_0^{-1}g^{(3)}]g_{0,ij} - 2\Delta_-\epsilon^{2\Delta_- -3}Tr[g_0^{-1}\bar{g}^{(2,0)}]g_{0,ij}\right)$$

We regroup the divergences and use (3.22), (3.41) and (3.46):

$$\langle T_{ij}\rangle = M^2L^2\left(2\epsilon^{-1}\left(g^{(2)}_{ij} - \left(\frac{R_0}{4}g_{0,ij} - R_{0,ij}\right)\right) + \right. \tag{4.50}$$

$$+3g^{(3)}_{ij} - (\Delta_-^2 - 2\Delta_-)\varphi_{1,0}\bar{\varphi}_{1,0}g_{0,ij}$$

$$\left. +2\Delta_-\epsilon^{2\Delta_- -3}\left(\bar{g}^{(2,0)}_{ij} - \left(-\frac{1}{8}\bar{\varphi}^2_{1,0}g_{0,ij}\right)\right)\right)$$

Finally, we use (3.23) and (3.50) to eliminate the divergences. The expected value of the Stress-Tensor is:

$$\langle T_{ij}\rangle = M^2L^2\left(3g^{(3)}_{ij} - (\Delta_-^2 - 2\Delta_-)\varphi_{1,0}\bar{\varphi}_{1,0}g_{0,ij}\right) \tag{4.51}$$

Indeed, its trace can be expressed through (3.41) :

$$\langle T^i_i\rangle = M^2L^2\left(3\Delta_- - 2\Delta_-^2\right)\varphi_{1,0}\bar{\varphi}_{1,0} \quad \Rightarrow \tag{4.52}$$

$$\langle T^i_i\rangle = M^2L^2\Delta_-(2\Delta_+ - 3)\varphi_{1,0}\bar{\varphi}_{1,0} = \Delta_-\bar{\varphi}_{1,0}\langle\mathcal{O}\rangle \tag{4.53}$$

And it satisfies the Ward identity as we can see from (3.74).

Conclusion

5. Discussion of the Exotic RG flows

sc1

5.1 The properties of the exotic solutions

We are ready to begin discussing exotic RG flows. One such example comes in the from of the Bergman metric discussed in [23]. We can express it in FG coordinates as:

$$ds^2 = \frac{du^2}{\ell^2 u^2} + \frac{1}{2\ell^2 u^{2\sqrt{2}}}\left((d\psi + cos\theta d\phi)^2 + u^{\sqrt{2}}(d\theta^2 + \sin^2\theta d\phi^2)\right)\left(1 + \mathcal{O}(u^{\sqrt{2}})\right) \quad (5.1)$$ C0a

Using the coordinate transformation:

$$r = u^{\sqrt{2}} \quad (5.2)$$ Ctra

we can bring the metric to the familiar FG form with integer powers:

$$ds^2 = \frac{\tilde{\ell}^2}{r^2}dr^2 + \frac{\tilde{\ell}^2}{r^2}\left((d\psi + cos\theta d\phi)^2 + r(d\theta^2 + \sin^2\theta d\phi^2)\right)(1 + \mathcal{O}(r)) \quad , \quad \tilde{\ell} = \frac{1}{\sqrt{2}\ell} \quad (5.3)$$ C0b

We see that the metric is exotic in the sense that the boundary metric becomes degenerate when $r \to 0$. The boundary metric is one-dimensional, meaning that it has only one non-zero eigenvalues and two zero eigenvalues.

Of course this makes the calculation of vevs non trivial, because the boundary inverse metric is ill-defined and the determinant cannot be expanded variationally using the method described in section 4. To understand these properties we would need to somehow modify the FG expansion so that it can consistently describe metrics with degenerate boundaries. We would also want to find the dual description of these degeneracies in the context of the boundary QFT.

Another example of exotic RG flows was studied in [19]. A summary of the process followed to find these exotic solutions is presented in Appendix (B). We will reproduce for convenience the asymptotic form of the metrics in (B.36) and (B.48), which govern the exotic RG flows:

$$ds^2 = \frac{\tilde{\ell}^2 dr^2}{r^2} + \frac{\tilde{\ell}^2}{r^2}\left(\frac{16}{\tilde{\ell}^4 T_{(2)0}^2}(d\psi + \cos\theta d\phi)^2 + \frac{4}{\tilde{\ell}^2 T_{(2)0}} r d\Omega^2\right)(1 + \mathcal{O}(r)) \quad , \quad \tilde{\ell} = \frac{\ell}{\sqrt{2}} \quad (5.4)$$ C1

$$ds^2 = \frac{\tilde{\ell}^2}{r^2}dr^2 + \frac{\tilde{\ell}^2}{r^2}\left(\frac{\tilde{\ell}^2 \tilde{T}_{(1)0}}{4}(d\psi + \cos\theta d\phi)^2 - r^2 d\Omega^2\right)(1 + \mathcal{O}(r)) \quad , \quad \tilde{\ell} = \frac{\ell}{\sqrt{3}} \quad (5.5)$$ C2

The expansions of the corresponding scalar fields are respectively:

$$\varphi = \varphi_0 + r^{\frac{1}{\nu_\pm}}(1 + \ldots) \quad , \quad \nu_\pm = \frac{1}{1 \pm \sqrt{\frac{(\Delta_+ - 1)(\Delta_+ - 2)}{2}}} \quad (5.6)$$ C1a

$$\varphi = \varphi_0 + r^{\frac{2}{\tilde{\alpha}}}(1+\ldots) \quad , \quad \tilde{\alpha} = -\frac{3+\sqrt{9-12\Delta_+\Delta_-}}{\Delta_+\Delta_-} \ , \ \Delta_+ > 3, \tag{5.7}$$ C2a

Again, we see that the boundary metric is one-dimensional and the rest of the dimensions appear at higher perturbative orders. This does not allow us to have a regular expansion for the inverse metric, thus hindering their analysis and the calculation of the corresponding Stress-Energy tensor. Furthermore, in the case of the metric in (5.4), the missing dimensions appear at the first perturbative order in r, which was not present when studying a regular metric.

A more general form of the FG expansion was discussed in [23], with the generalization being that the g_0 boundary metric depends on r as well. This approach however has some problems, as the author pointed out. As we discussed previously, since the determinant of g_0 in general could vanish in the limit $r \to 0$ (in fact, in both of our cases it does), the inverse of the boundary metric is ill-defined. Moreover, the resulting action cannot be reformulated in a covariant way, meaning that it is not renormalisable.

Although such an approach does not seem to be able to solve our problem, it was useful since it highlighted some key features that must be present in the approach that will eventually solve the problem. One is that we need a way to be able to somehow have a non-zero contribution to first order in r. This feature could be incorporated using an approach where the metric g_0 has an explicit r dependence.

However, there should be caution when using the inverse metric in the equations, because it is divergent in the $r \to 0$ limit. The same caution should be present when using perturbative expansions of the metric since the determinant of g_0 vanishes in the $r \to 0$ limit. Finally, the method should permit the covariant reformulation of the action, because otherwise, renormalization is impossible.

sc2

5.2 A possible solution

A possible solution to this problem is to use a Kaluza/Klein-like ansatz for the metric of the following form[2]:

$$ds^2 = \frac{\tilde{\ell}^2}{r^2}\Big[dr^2 + r^n \hat{B}_{\alpha\beta}(r,x^M)dx^\alpha dx^\beta +$$

$$+\hat{C}_{ij}(r,x^\mu)(dx^j + \hat{A}^j{}_\beta(r,x^M)dx^\beta)(dx^i + \hat{A}_\alpha{}^i(r,x^M)dx^\alpha)\Big], \tag{5.8}$$ C4

[2]For this subsection we change our index notation, because we will need to refer to separate blocks of the metric. The convention we are adopting is as follows: Upper case Greek indexes take the values 1, 2, $\cdots$, d. The Latin indexes take the values 1, 2, $\cdots$, d_1. Finally, lower case Greek indexes $\alpha\beta$ take the values d_1+1, d_1+2, $\cdots$, d.

where the metrics $\hat{B}_{\alpha\beta}$, $\hat{C}_{ij}$ and the "vectors" $\hat{A}^j{}_\beta$, $\hat{A}_\alpha{}^i$ have now regular FG expansions. Usually, the factor r^n multiplying $\hat{B}_{\mu\nu}$ would be omitted, however, the case of the exotic RG flows demands that the two parts of the metric scale at different rates with respect to r.

In the case of the exotic solutions of the Squashed S^3 there exists a single fiber, therefore the metric $\hat{C}_{ij}$ is one-dimensional and n takes the values 1 for the $\nu_\pm$ solutions and 2 for B_3. However, we will study the most general case in the sequel, where $\hat{C}_{ij}$ is d_1-dimensional and $\hat{B}_{ij}$ is d_2-dimensional, with the dimensions satisfying $d_1 + d_2 = d$. One important simplification comes in the form of the cylinder condition, which is usually implemented in such systems. According to this condition, the metric does not depend on the fibered coordinates of the $d_1 \times d_1$ block [24]. Through the cylinder condition (which holds at least for the lowest order of the exotic solutions) partial derivatives with respect to the fibered coordinates vanish.

In matrix representation, we have:

$$\hat{g}_{MN} = \begin{pmatrix} \hat{C}_{ij} & \hat{C}_{ij}\hat{A}^j{}_\beta \\ \hat{A}_\alpha{}^i\hat{C}_{ij} & r^n\hat{B}_{\alpha\beta} + \hat{A}_\alpha{}^i\hat{C}_{ij}\hat{A}^j{}_\beta \end{pmatrix} \tag{5.9}$$ ex3

The inverse metric according to the ansatz is:

$$\hat{g}^{MN} = \begin{pmatrix} \hat{C}^{ij} + \hat{A}^i{}_\alpha\hat{B}^{\alpha\beta}\hat{A}_\beta{}^j r^{-n} & -\hat{A}^i{}_\alpha\hat{B}^{\alpha\beta}r^{-n} \\ -\hat{B}^{\alpha\beta}\hat{A}_\beta{}^j r^{-n} & r^{-n}\hat{B}^{\alpha\beta} \end{pmatrix} \tag{5.10}$$ ex4

This ansatz has many promising attributes. For one, it is the most general form of many fibered metrics, for example, the metrics discussed in [25] and [26]. The second strength it offers is that it allows us to isolate the divergent parts of the metric and study them separately. Therefore, we do not need to worry about using g_0^{-1}, because the divergences have been absorbed in the equations. Instead, we will use the inverses of the matrices C and B, which are well-defined.

Another strength is that by construction we have a contribution in the r^1 order from the term $r^n\hat{B}_{\alpha\beta}$. In short, the proposed generalization to the FG expansion is to use independent expansions for each of the blocks of the metric. The determinant of g seems initially difficult to handle, however, this is not the case. The problem can be solved by implementing a math theorem presented in [27], which allows us to express the determinant in terms of the determinants of B and C. The only possible issue is the covariant reformulation of the action. This is an issue that will be dealt with once the equations of motion have been solved.

To summarize how one would study such a metric, the standard procedure demonstrated in section (3) can be repeated for each block independently, allowing us to impose boundary conditions on each of them. Instead of a system of 4 equations, we would have 8. One scalar equation, one equation for the trace of g, corresponding to the rr equation, 2 for the conservation of each square matrix, corresponding to the rj equations and 4 for the coefficients of each block, corresponding to the ij equations. Using symmetry arguments, the equations for the coefficients of the iα and αi blocks would not be independent, leaving us with a system of 7 independent equations.

The challenge comes in the form of finding the Ricci tensor elements for this metric and expressing them in terms of the quantities appearing in the equations. This endeavor can be slightly simplified by noting that we have already calculated the Ricci tensor elements for a metric in FG form in Appendix (A). One must simply take all combinations of indexes belonging to each block and be extra careful when using the inverse metric in products and traces.

Finally, one good correspondence check when one finishes the calculations would be to set $d_1 = n = 0$, $d_2 = d$ and set all matrices besides B to zero. The results should correspond to those of a regular FG expansion. The same is not true for letting only C be non-zero, because of the cylinder condition. One would have to calculate the full equations of motion, without using the cylinder condition if they want to use C as a correspondence check. This problem will be continued and solved in the future.

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Appendix

FG A. The Fefferman-Graham Coordinate System

An asymptotically AdS 3+1 dimensional metric $g_{\mu\nu}$ expressed in Fefferman -Graham coordinates has the following form:

$$ds^2 = \frac{\ell^2}{r^2}dr^2 + \frac{\ell^2}{r^2}\hat{g}_{ij}(r, x^i)dx^i dx^j \tag{A.1}$$ FG1

where the boundary metric has a regular expansion if it is odd-dimensional:

$$\hat{g}_{ij}(r, x^i) = \sum_{m=0}^{\infty} \hat{g}_{ij}^{(m)}(x^i) r^m \tag{A.2}$$ FG2

Usually, an additional logarithmic term is required for even-dimensional metrics, but this is beyond the scope of this thesis since our metric is 3+1 dimensional. The expansion may have corrections to account for scalar sources coupled to gravity. If one is interested in such cases, the expansion of $\hat{g}_{ij}$ becomes:

$$\hat{g}_{ij}(r, x^i) = \sum_{\substack{n=1\\m=0}}^{\infty} g_{ij}^{(n,m)}(x^i) r^{n\Delta_+ + m} + \sum_{\substack{n=1\\m=0}}^{\infty} \bar{g}_{ij}^{(n,m)}(x^i) r^{n\Delta_- + m} + \sum_{n=0}^{\infty} g_{ij}^{(n)}(x^i) r^n, \tag{A.3}$$ FG3

where Δ_+ is the dimension of the dual operator and Δ_- is defined as $\Delta_- = d - \Delta_+$.

For the rest of this appendix, a hat denotes a quantity constructed from the metric $\hat{g}_{ij}$ and a prime denotes a derivative with respect to r. The non-trivial Christoffel symbols of $g_{\mu\nu}$ are:

$$\Gamma_{rr}{}^{r} = -\frac{1}{r} \quad , \quad \Gamma_{ij}{}^{r} = -\frac{1}{2}\left(\hat{g}'_{ij} - \frac{2}{r}\hat{g}_{ij}\right) \quad , \tag{A.4}$$ FG4

$$\Gamma_{rj}{}^{i} = \frac{1}{2}\left((\hat{g}^{-1}\hat{g}')^{j}{}_{i} - \frac{2}{r}\delta^{j}{}_{i}\right) \quad , \quad \Gamma_{ij}{}^{k} = \hat{\Gamma}_{ij}^{\ k}$$

The non-trivial independent components of the Riemann tensor are

$$R^{r}{}_{ijr} = \frac{1}{2}\left[\hat{g}''_{ij} - \frac{1}{2}(\hat{g}'\hat{g}^{-1}\hat{g}')_{ij} - \frac{\hat{g}'_{ij}}{r} + \frac{2}{r^2}\hat{g}_{ij}\right] \tag{A.5}$$ FG5

$$R^{r}{}_{ijk} = \frac{1}{2}\left[\hat{\nabla}_k \hat{g}'_{ij} - \hat{\nabla}_j \hat{g}'_{ik}\right] \tag{A.6}$$ FG6

$$R^{l}{}_{ijk} = \hat{R}^{l}_{ijk} + \frac{1}{4}\left[\left(\hat{g}'_{ij} - \frac{2}{r}\hat{g}_{ij}\right)\left((\hat{g}^{-1}\hat{g}')^{l}{}_{k} - \frac{2}{r}\delta^{l}{}_{k}\right) - \right. \tag{A.7}$$ FG7

$$\left. - \left(\hat{g}'_{ik} - \frac{2}{r}\hat{g}_{ik}\right)\left((\hat{g}^{-1}\hat{g}')^{l}{}_{j} - \frac{2}{r}\delta^{l}{}_{j}\right)\right]$$

The non-trivial components of the Ricci tensor are:

$$R_{ij} = \hat{R}_{ij} + \frac{1}{4}\Bigg(2(\hat{g}'\hat{g}^{-1}\hat{g}')_{ij} - 2\hat{g}''_{ij} - Tr[\hat{g}^{-1}\hat{g}']\hat{g}'_{ij} +$$

$$+\frac{2}{r}Tr[\hat{g}^{-1}\hat{g}']\hat{g}_{ij} + \frac{2(d-1)}{r}\hat{g}'_{ij} - \frac{4d}{r^2}\hat{g}_{ij}\Bigg) \tag{A.8}$$ FG8

$$R_{rj} = \frac{1}{2}\hat{g}^{ik}\left(\hat{\nabla}_k\hat{g}'_{ij} - \hat{\nabla}_j\hat{g}'_{ik}\right) \tag{A.9}$$ FG9

$$R_{rr} = -\frac{1}{2}\left(Tr[\hat{g}^{-1}\hat{g}''] - \frac{1}{2}Tr[\hat{g}^{-1}\hat{g}'\hat{g}^{-1}\hat{g}'] - \frac{1}{r}Tr[\hat{g}^{-1}\hat{g}'] + \frac{2d}{r^2}\right) \tag{A.10}$$ FG10

The Ricci curvature becomes:

$$R = \frac{r^2}{\ell^2}\hat{R} + \frac{1}{4}\frac{r^2}{\ell^2}\Bigg(\frac{4d}{r}Tr[\hat{g}^{-1}\hat{g}'] - 4Tr[\hat{g}^{-1}\hat{g}''] +$$

$$+ 3Tr[\hat{g}^{-1}\hat{g}'\hat{g}^{-1}\hat{g}'] - (Tr[\hat{g}^{-1}\hat{g}'])^2 - \frac{4d(d+1)}{r^2}\Bigg) \tag{A.11}$$ FG11

We may combine our results for the Ricci curvature and Ricci tensor, to obtain the non-trivial elements of the Einstein tensor:

$$G_{rr} = -\frac{1}{2}\hat{R} - \frac{1}{8}\left(Tr[\hat{g}^{-1}\hat{g}'\hat{g}^{-1}\hat{g}'] - (Tr[\hat{g}^{-1}\hat{g}'])^2 +\right.$$

$$\left.+\frac{4(d-1)}{r}Tr[\hat{g}^{-1}\hat{g}'] - \frac{4d(d-1)}{r^2}\right) \tag{A.12}$$ FG12

$$G_{rj} = R_{rj} \tag{A.13}$$ FG13

$$G_{ij} = \hat{G}_{ij} + \frac{1}{8}\Bigg(4(\hat{g}'\hat{g}^{-1}\hat{g}')_{ij} - 4\hat{g}''_{ij} - 2Tr[\hat{g}^{-1}\hat{g}']\hat{g}'_{ij} - \frac{4(d-1)}{r}Tr[\hat{g}^{-1}\hat{g}']\hat{g}_{ij} + \tag{A.14}$$ FG14

$$+\frac{4(d-1)}{r}\hat{g}'_{ij} + \Bigg(4Tr[\hat{g}^{-1}\hat{g}''] - 3Tr[\hat{g}^{-1}\hat{g}'\hat{g}^{-1}\hat{g}'] +$$

$$+ (Tr[\hat{g}^{-1}\hat{g}'])^2 + \frac{4d(d-1)}{r^2}\Bigg)\hat{g}_{ij}\Bigg)$$

RGF B. A review of the Exotic RG Flows

In this appendix, we have collected the expansions of the exotic solutions we are asked to describe up to the orders which will be relevant for the calculation of the Stress-Energy Tensor. For more details, read [19]. For a potential of the form:

$$V(\varphi) = -\frac{6}{\ell^2} - \frac{\Delta_+\Delta_-}{2\ell^2}(\varphi-\varphi_0)^2 + \mathcal{O}\left((\varphi-\varphi_0)^3\right), \tag{B.1}$$ RGF1

and the following ansatz for the bulk metric:

$$ds^2 = du^2 + L^2\left(e^{2A_1(u)}(d\psi + cos\theta d\phi)^2 + e^{2A_2(u)}d\Omega_2^2\right), \tag{B.2}$$ RGF2

the resulting equations of motion obtained variationally from the action in (2.20) are (dots denote derivatives with respect to u and primes denote derivatives with respect to r):

$$\ddot{\varphi} + (\dot{A}_1 + 2\dot{A}_2)\dot{\varphi} + V'(\varphi) = 0 \tag{B.3}$$ RGF3

$$4\dot{A}_2^2 + 8\dot{A}_1\dot{A}_2 + \frac{1}{L^2}e^{2A_1-4A_2} - \frac{4}{L^2}e^{-2A_2} - \dot{\varphi}^2 + 2V(\varphi) = 0 \tag{B.4}$$ RGF4

$$4\ddot{A}_2 + 4\dot{A}_2(\dot{A}_2 - \dot{A}_1) + \frac{1}{L^2}e^{2A_1-4A_2} + \dot{\varphi}^2 = 0 \tag{B.5}$$ RGF5

$$\ddot{A}_1 - \ddot{A}_2 + (\dot{A}_1 - \dot{A}_2)(\dot{A}_1 + 2\dot{A}_2) + \frac{1}{L^2}e^{-2A_2} - \frac{1}{L^2}e^{2A_1-4A_2} = 0 \tag{B.6}$$ RGF6

The conformal dimension of the operator takes the values $\frac{3}{2} \leq \Delta_+ \leq 3$ when we are at a maximum and $\Delta_+ > 3$ when we are at a minimum of the potential.

The equations can be expressed in terms of the following superpotentials:

$$W_1(\varphi) \equiv -4\dot{A}_1 \tag{B.7}$$ RGF7

$$W_2(\varphi) \equiv -4\dot{A}_2 \tag{B.8}$$ RGF8

$$T_1(\varphi) \equiv \frac{4}{L^2}e^{2A_1-4A_2} \tag{B.9}$$ RGF9

$$T_2(\varphi) \equiv \frac{4}{L^2}e^{-2A_2} \tag{B.10}$$ RGF10

$$S \equiv \dot{\varphi} \tag{B.11}$$ RGF11

It is possible to show that the equation inside the parenthesis is not independent of the rest, however for clarity's sake we shall leave it here:

$$SS' - \frac{1}{4}S\left(W_1 + 2W_2\right) - V' = 0, \tag{B.12}$$ RGF12

$$W_2^2 + 2W_1W_2 + T_1 - 4T_2 - 4S^2 + 8V = 0, \tag{B.13}$$ RGF13

$$\left(-4SW_2' + W_2^2 - W_1W_2 + T_1 + 4S^2 = 0\right) \tag{B.14}$$ RGF14

$$4(W_2' - W_1')S + (W_1 - W_2)(W_1 + 2W_2) + 4(T_2 - T_1) = 0, \tag{B.15}$$ RGF15

From the definitions of the T superpotentials, we may also obtain:

$$ST_1' = -\frac{1}{2}(W_1 - 2W_2)T_1 \quad , \quad ST_2' = \frac{1}{2}W_2T_2. \tag{B.16}$$ RGF16

Besides regular power series expansions, the superpotential may also contain deformations corresponding to non-analytic expansions, as long as they are subleading. To solve the equations we define:

$$S = S_L + \delta S \ , \ W_1 = W_{1L} + \delta W_1 \ , \ W_2 = W_{2L} + \delta W_2, \tag{B.17}$$ F12

$$T_1 = T_{1L} + \delta T_1 \ , \ T_2 = T_{2L} + \delta T_2, \tag{B.18}$$ RGF17

and introduce these redefinitions in the EsOM. The functions with an L index, are assumed to satisfy the non-linear equations.

$$(S_L\delta S)' - \frac{1}{4}S_L\left(\delta W_1 + 2\delta W_2\right) - \frac{1}{4}\delta S\left(W_{1L} + 2W_{2L}\right) = 0, \tag{B.19}$$ RGF18

$$\left(W_{2L} + W_{1L}\right)\delta W_2 + W_{2L}\delta W_1 + \frac{1}{2}\delta T_1 - 2\delta T_2 - 4S_L\delta S = 0 \tag{B.20}$$ RGF19

$$\left(W_{2L} - W_{1L}\right)'\delta S + \left(\delta W_2 - \delta W_1\right)' S_L + \frac{1}{4}\left(W_{1L} - W_{2L}\right)\left(\delta W_1 + 2\delta W_2\right),$$
$$+\frac{1}{4}\left(\delta W_1 - \delta W_2\right)\left(W_{1L} + 2W_{2L}\right) + \delta T_2 - \delta T_1 = 0, \tag{B.21}$$ RGF20

$$S_L\delta T_1' + \delta S T_{1L}' + \frac{1}{2}\left((W_{1L} - 2W_{2L})\delta T_1 + (\delta W_1 - 2\delta W_2)T_{1L}\right) = 0. \tag{B.22}$$ RGF21

$$S_L\delta T_2' + \delta S T_{2L}' = \frac{1}{2}\left(W_{2L}\delta T_2 + \delta W_2 T_{2L}\right), \tag{B.23}$$ RGF22

We can solve first equations (B.19)-(B.21) in order to obtain $\delta W_1, \delta T_1, \delta T_2$. After that, we shall substitute the resulting expressions into equations (B.22), (B.23). We will finally solve two differential equations for the functions $\delta W_2, \delta S$.

The $\nu_\pm$ solutions

Defining:

$$x \equiv \varphi - \varphi_0, \tag{B.24}$$ RGF24

and:

$$\nu_\pm = \frac{1}{1 \pm \sqrt{\frac{(\Delta_+ - 1)(\Delta_+ - 2)}{2}}} \tag{B.25}$$ RGF24a

the expansions for the $\nu_\pm$ solutions are:

$$S_{\nu_\pm} = \frac{\sqrt{2}}{\ell \nu_\pm} x + \frac{3 V_3 \ell \nu_\pm}{\sqrt{2}(3 - 2\nu_\pm)} x^2 + \tag{B.26}$$ RGF25

$$+ \frac{\frac{4(2\nu_\pm^3 + 27\nu_\pm^2 + 66\nu_\pm - 72)}{\ell \nu_\pm (12\nu_\pm^2 + 33\nu_\pm - 32)(12 + \nu_\pm)} + 4 V_4 \ell \nu_\pm - \frac{9 V_3^2 \ell^3 \nu_\pm^3}{(3 - 2\nu_\pm)^2}}{2\sqrt{2}(2 - \nu_\pm)} x^3 + \mathcal{O}(x^4) +$$

$$+ \frac{\ell}{8\sqrt{2}(2 - \nu_\pm)} T_{(2)0} x^{1+\nu_\pm} (1 + \mathcal{O}(x)) + \frac{\mathcal{C}}{8(2 + \nu_\pm)} x^{1+3\nu_\pm} (1 + \mathcal{O}(x)) +$$

$$+ \left(-\frac{3\ell \nu_\pm \mathcal{D}}{8\sqrt{2}} x^{2\nu_\pm - 1}\right) (1 + \mathcal{O}(x)),$$

$$W_{1,\nu_\pm} = \frac{4\sqrt{2}}{\ell} + \frac{8\sqrt{2}(\nu_\pm^2 + \nu_\pm - 2)}{(12\nu_\pm^2 + 33\nu_\pm - 32)\ell \nu_\pm} x^2 + \mathcal{O}(x^3) + \mathcal{C} x^{3\nu_\pm} (1 + \mathcal{O}(x)), \tag{B.27}$$ RGF26

$$W_{2,\nu_\pm} = \frac{2\sqrt{2}}{\ell} + \frac{4\sqrt{2}(\nu_\pm^3 + 14\nu_\pm^2 + 56\nu_\pm - 48)}{(12 + \nu_\pm)(12\nu_\pm^2 + 33\nu_\pm - 32)\ell \nu_\pm} x^2 + \mathcal{O}(x^3) + \tag{B.28}$$ RGF27

$$+ \frac{\ell}{4\sqrt{2}} T_{(2)0} x^{\nu_\pm} (1 + \mathcal{O}(x)) - \frac{\mathcal{C}}{4} x^{3\nu_\pm} (1 + \mathcal{O}(x)),$$

$$T_{1,\nu_\pm} = \frac{8}{\ell^2} + \frac{16(2\nu_\pm - 1)}{(12\nu_\pm^2 + 33\nu_\pm - 32)\ell^2} x^2 + \mathcal{O}(x^3) + \tag{B.29}$$ RGF28

$$+ T_{(2)0} x^{\nu_\pm} (1 + \mathcal{O}(x)) - \frac{\sqrt{2}}{\ell} \mathcal{C} x^{3\nu_\pm} (1 + \mathcal{O}(x)) + (\mathcal{D} x^{2\nu_\pm}) (1 + \mathcal{O}(x)),$$

$$T_{2,\nu_\pm} = T_{(2)0} x^{\nu_\pm} (1 + \mathcal{O}(x)) - \frac{\sqrt{2}}{3\nu_\pm \ell (2 + \nu_\pm)} \mathcal{C} x^{2+3\nu_\pm} (1 + \mathcal{O}(x)) + \tag{B.30}$$ RGF29

$$+ (\mathcal{D} x^{2\nu_\pm}) (1 + \mathcal{O}(x)).$$

In the above expansions, $\mathcal{C}$, $\mathcal{D}$ and $T_{(2)0}$ are integration constants. The terms proportional to $\mathcal{D}$ exist only when $\nu_\pm \geq 1$, whereas the rest all exist simultaneously when $\nu_\pm > 0$. The terms proportional to $\mathcal{D}$ are inside parentheses to emphasize this fact. Specifically, we see from (B.25) that they exist for the ν_- solution but not for the ν_+ solution. Now we may find the expansion of the scalar from (B.11). In both cases, it is:

$$\varphi = \varphi_0 + e^{\frac{\sqrt{2}}{\ell \nu_\pm}(u - u_0)} (1 + \ldots) \tag{B.31}$$ RGF30

We may then use relations (B.9) and (B.10) to directly compute the exponentials appearing in the metric ansatz. To leading order, we obtain:

$$e^{2A_2} = \frac{4}{L^2 T_{(2)0}} e^{-\sqrt{2}\frac{u-u_0}{\ell}} \left(1 + \mathcal{O}\left(e^{\sqrt{2}\frac{u-u_0}{\ell}}\right)\right), \tag{B.32}$$ RGF31

$$e^{2A_1} = \frac{32}{\ell^2 L^2 T_{(2)0}^2} e^{-2\sqrt{2}\frac{u-u_0}{\ell}} \left(1 + \mathcal{O}\left(e^{\sqrt{2}\frac{u-u_0}{\ell}}\right)\right). \tag{B.33}$$ RGF32

Which give the following form for the metric near φ_0 (we define $\tilde{\ell} = \frac{\ell}{\sqrt{2}}$):

$$ds^2 \approx du^2 + \frac{32 e^{-2\sqrt{2}\frac{u-u_0}{\ell}}}{\ell^2 T_{(2)0}^2} \left((d\psi + \cos\theta d\phi)^2 + \frac{\ell^2 T_{(2)0}}{8} e^{\sqrt{2}\frac{u-u_0}{\ell}} d\Omega^2 \right) \quad \Rightarrow \tag{B.34}$$ RGF33

$$ds^2 \approx \frac{\ell^2 dz^2}{z^2} + \frac{32 z_0^{2\sqrt{2}}}{\ell^2 T_{(2)0}^2 z^{2\sqrt{2}}} \left((d\psi + \cos\theta d\phi)^2 + \frac{\ell^2 T_{(2)0}}{8} \left(\frac{z}{z_0}\right)^{\sqrt{2}} d\Omega^2 \right) \quad \Rightarrow \tag{B.35}$$ RGF34

$$ds^2 \approx \frac{\tilde{\ell}^2 dr^2}{r^2} + \frac{\tilde{\ell}^2}{r^2} \left(\frac{16}{\tilde{\ell}^4 T_{(2)0}^2} (d\psi + \cos\theta d\phi)^2 + \frac{4}{\tilde{\ell}^2 T_{(2)0}} r d\Omega^2 \right) \tag{B.36}$$ RGF35

As we can see, this metric does not correspond to a standard FG boundary. The different parts of the metric scale at different rates and the metric that would be $\hat{g}$ in the symbolism of appendix (A) is degenerate in the limit $r \to 0$.

The B_3 solution

Similarly to the $\nu_\pm$ solutions, by defining the constant $\tilde{\alpha}$ as:

$$\tilde{\alpha} = -\frac{3+\sqrt{9-12\Delta_+\Delta_-}}{\Delta_+\Delta_-}\ ,\ \Delta_+ > 3, \tag{B.37}$$ RGF36

the expansions for the B_3 solution are:

$$S = x\left(\frac{2\sqrt{3}}{\ell\tilde{\alpha}} + \frac{\sqrt{3}\ell\tilde{\alpha}V_3}{\tilde{\alpha}+6}x + \mathcal{O}(x^2)\right) + \frac{\ell}{8\sqrt{3}(4+3\tilde{\alpha})}\tilde{T}_{(1)0}x^{1+\tilde{\alpha}}(1+\mathcal{O}(x)) + \tag{B.38}$$ EGF37

$$+\frac{\mathcal{C}}{3(\tilde{a}+2)}x^{1+\frac{\tilde{\alpha}}{2}}(1+\mathcal{O}(x)),$$

$$W_1 = -\frac{4\sqrt{3}}{\ell} + \frac{(2-\tilde{\alpha})(4+\tilde{\alpha})(12\tilde{\alpha}^{-2}+\Delta_+\Delta_-)}{2\sqrt{3}(\tilde{\alpha}-4)(\tilde{\alpha}+2)\ell}x^2 + \mathcal{O}(x^3) - \tag{B.39}$$ RGF38

$$-\frac{\ell\tilde{T}_{(1)0}}{4\sqrt{3}}x^{\tilde{\alpha}}(1+\mathcal{O}(x)) - \frac{2}{3}\mathcal{C}x^{\tilde{\alpha}/2}(1+\mathcal{O}(x)),$$

$$W_2 = \frac{2}{\sqrt{3}\ell}\frac{12\tilde{\alpha}^{-2}+\Delta_+\Delta_-}{\tilde{\alpha}-4}x^2 + \mathcal{O}(x^3) + \frac{\ell\tilde{T}_{(1)0}}{4\sqrt{3}}x^{\tilde{\alpha}}(1+\mathcal{O}(x)) + \mathcal{C}x^{\tilde{\alpha}/2}(1+\mathcal{O}(x)), \tag{B.40}$$ RGF39

$$T_1 = \tilde{T}_{(1)0}x^{\tilde{\alpha}}(1+\mathcal{O}(x)) + \frac{48-36\tilde{\alpha}+16\tilde{\alpha}^2+11\tilde{\alpha}^3}{3\sqrt{3}(\tilde{\alpha}+2)^2(\tilde{\alpha}-4)\ell\tilde{\alpha}}\mathcal{C}x^{2+\frac{\tilde{\alpha}}{2}}(1+\mathcal{O}(x)), \tag{B.41}$$ RGF40

$$T_2 = -\frac{12}{\ell^2} + \frac{12\tilde{\alpha}^{-2}+\Delta_+\Delta_-}{\ell^2(4\tilde{\alpha}^{-1}-1)}x^2 + \mathcal{O}(x^3) -$$

$$-\frac{\tilde{T}_{(1)0}}{4}x^{\tilde{\alpha}}(1+\mathcal{O}(x)) - \frac{2\sqrt{3}}{\ell}\mathcal{C}x^{\tilde{\alpha}/2}(1+\mathcal{O}(x)). \tag{B.42}$$ RGF41

Again, from (B.11) we find the expansion of the scalar to leading order:

$$\varphi = \varphi_0 + e^{\frac{2\sqrt{3}}{\tilde{\alpha}}\frac{u-u_0}{\ell}}\left(1+\ldots\right), \tag{B.43}$$ RGF42

From (B.9) and (B.10) the exponentials are to leading order:

$$e^{2A_2} = -\frac{\ell^2}{3L^2}\left(1+\mathcal{O}(e^{\sqrt{3}\frac{u-u_0}{\ell}})\right), \tag{B.44}$$ RGF43

$$e^{2A_1} = \frac{\ell^4\tilde{T}_{(1)0}}{36L^2}e^{2\sqrt{3}\frac{u-u_0}{\ell}}\left(1+\mathcal{O}(e^{\sqrt{3}\frac{u-u_0}{L}})\right). \tag{B.45}$$ RGF44

Finally, the metric near φ_0 is (we define $\tilde{\ell} = \frac{\ell}{\sqrt{3}}$):

$$ds^2 \approx du^2 + \frac{\ell^4\tilde{T}_{(1)0}}{36}e^{2\sqrt{3}\frac{u-u_0}{\ell}}\left(d\psi+\cos\theta d\phi\right)^2 - \frac{\ell^2}{3}d\Omega^2 \quad \Rightarrow \tag{B.46}$$ RGF45

$$ds^2 \approx \frac{\ell^2}{3r^2}dr^2 + \frac{\ell^4\tilde{T}_{(1)0}}{36r^2}\left(d\psi + \cos\theta d\phi\right)^2 - \frac{\ell^2}{3}d\Omega^2 \quad \Rightarrow \tag{B.47}$$ RGF46

$$ds^2 \approx \frac{\tilde{\ell}^2}{r^2}dr^2 + \frac{\tilde{\ell}^2}{r^2}\left(\frac{\tilde{\ell}^2\tilde{T}_{(1)0}}{4}\left(d\psi + \cos\theta d\phi\right)^2 - r^2 d\Omega^2\right) \tag{B.48}$$ RGF47

This metric has the same problems as the one in (B.36).

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