UNIVERSITY OF CRETE

DEPARTMENT OF ECONOMICS

MASTER PROGRAMME IN ECONOMIC THEORY AND POLICY

RETHYMNO

2005-2006

Master thesis

Measuring the Technical Efficiency of von Liebig Model

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Introduction

Among the various specifications of production function models is the von Liebig model, established in the 1840s. According to the law of minimum the production level is a proportional function of the scarcest input applied in the plant. Although this model has received many critics about its appropriateness to determine input-output relationship with real data, several empirical studies have proved its capability of representing crop responses to fertilizers as well as the polynomial specification does.

Despite the fact that the von Liebig model has been present in the literature for many years, it recently begins to be recognized as a successful model for response analysis and that's why it attracts our interest. However, this paper does not challenge to verify the characteristics of the von Liebig model or to proceed to the comparison of the model with other production functions, but our main focus is the measurement of technical efficiency inside the framework of the von Liebig model.

The analysis of the technical efficiency of von Liebig model can be succeeded through the stochastic frontier von Liebig function. The stochastic frontier production function gives the level of production, which is a function of a set of inputs, technical efficiency term and random error. The technical efficiency term indicate if the economic agents are technically efficient.

In the same context lies the work of Holloway and Paris, who estimated the frontier von Liebig function and reached the conclusion that the particular model fits to the data, indicating its appropriateness. In this paper, the estimation of the frontier von Liebig function is based on different methodology and specification compared to the one developed by Holloway and Paris. The difference lies on the fact that efficiency is estimated via an input oriented measure and we put an efficiency term in each response function, in contrast to Holloway and Paris who employed an output oriented measure of efficiency with one inefficiency term outside the min operator. Additionally, instead of employing the Bayesian techniques for the estimation of the frontier production function that Holloway and Paris applied, we use the maximum likelihood estimation techniques extending Maddala's and Nelson's approach.

In our analysis the von Liebig specification has an error term for each regime, in contrast to Holloway and Paris' specification that has one error term outside the min operator. Such an error term represents the producer's weakness to control perfectly the applied inputs, because the levels of inputs vary across the field due to nonuniformity in the existing levels of the nutrients. That's why we assume that each regime is a random variable with different variance. According to Paris and Knapp, this specification of von Liebig model is defined as a special case of disequilibrium or switching regression models.

The structure of the paper proceeds as follows. The first section provides a historical review of the von Liebig model and presents some of the main results on this model. The second section presents the theoretical von Liebig model and states its main characteristics to facilitate our understanding on the model structure. In the third section the method that is employed to estimate the linear and non linear von Liebig models and the main results are presented. In the forth section the frontier von Liebig function is formulated and estimated with the techniques employed in the previous section. The data set used for the estimation of the von Liebig function and the frontier von Liebig function were published by the Heady and Pesek. Finally, the sixth section concludes and provides some suggestions for further research.

Historical review

Traditionally, crop yield response was specified as a polynomial function such as the quadratic or square root forms. These functions exhibit diminishing marginal productivity and input substitution. These properties plus the fact that the functions are linear in parameters and are relatively easy to estimate contributed to their historical popularity. Recently, they have been criticized because they force input substitution, do not allow for plateau growth and often over-estimate the optimal fertilizer quantity.

Agricultural economists thus continue to analyze and estimate alternative crop response models. Much of interest has focused is the "law of the minimum" which was first expressed by Carl Sprengel in 1828. He stated, *"… when a plant needs 12 substance to develop, it will not grow if any one of these is missing, and it will always grow poorly, when one of these is not available in a sufficiently large amount as requires by the nature of the plant"*. Some years later, Justus von Liebig (1803-1873) re-examined Sprengel's ideas and formulate the "law of minimum". In his book "The Natural Laws of Husbandry" (1863) it is written, *"Every field contains a maximum of one or several, and a minimum of one or several different nutrients. The yield of crops stand in relation to this minimum be it lime, potash, nitrogen, phosphoric acid, magnesia, or any other mineral constituent; this minimum governs and control the level and the persistence of yields. Should this minimum for example be lime or magnesia, the yield of grain and straw, of turnips, potatoes or clover will remain the same and be no greater even though the amount of potash, silica, phosphoric acid, etc., already in the soil be increased a hundred times. The crop yields on this field, however, will be increased by a simple fertilization with lime"*.

The law of minimum posits two crucial characteristics, first a yield plateau and second nonsubstitution between nutrients. The nonsubstitution characteristic indicates that the continually increasing of the nonlimiting nutrients quantity does not affect the yield, like the Leontief production function. An application of this property is that the isoquants of the crop production function have vertical and horizontal legs that joint at right angles.

When the law of minimum first appeared the mathematical tools to analyze it were not significantly developed that’s why it did not receive a complete analysis and empirical testing. The first mathematical expression of the von Liebig hypothesis was linearity between the scarcest fertilizers and yields, and after 1951, with the evolution of mathematics, it could also be expressed as a nonlinear response function. Statistical and mathematical instruments for estimating von Liebig model and for comparing the different models among them were not available until 1977.

Previous work

Until today, many studies have been made in order to examine and test empirically the von Liebig hypothesis. Ackello-Ogutu, Paris and William (1985) were the first compared the von Liebig response function against the polynomial functions using a nonnested test. Using data for a dryland corn-soybean-wheat-hay rotation in Indiana, they found that the von Liebig function was preferred to both the square root and quadratic forms and that the square root was slightly better than the quadratic.

Frank, Beattie and Embleton (1990), using nonnested hypothesis, compared the Von Liebig function with the quadratic and Mitscherlich-Baule functional forms. The Mitscherlich-Baule allows plateau growth and input substitution in contrast to the von Liebig and the quadratic does not include either a growth plateau neither input substitution. In this study they used the data from Heady and Pesek for dryland corn production using 114 observations and the results favor the Mitscherlich-Baule model.

Later on, Paris (1992) used the same data set to compare five alternative crop response functions, quadratic, square root, linear von Liebig, Mitscherlich-Baule and nonlinear von Liebig. The results from the nonnested hypothesis tests among the five alternative functions show that the data are represented better with von Liebig with Mitscherlich regimes that allows diminishing marginal productivity.

Recently, Chambers and Lichtenberg (2001) developed dual representation of von Liebig technology and used nonparametric methods to test the two properties of the law of minimum, plateau yield and nonsubstitution of nutrients. The results of the tests were favoring the law of minimum. In the contrary, Bercks, Geoghegan and Stohs that also used nonparametric methods rejected the law of minimum.

Representation of the von Liebig Model

The law of minimum states that crop yields are proportional to the total availability of a limited nutrient up to the point where another nutrient becomes limiting. The most general formulation of the law of minimum is specified as

$$y_i = \min\left\{ f_N\left(N_i\right), f_P\left(P_i\right), ..., f_C\left(C_i\right)\right\} \qquad (1)$$

where y_i is the actual level of crop production, $f_N\left(N_i\right)$, $f_p\left(P_i\right)$ up to $f_C\left(C_i\right)$ are the increasing individual response functions of the various fertilizers, N, P and C are the levels of fertilizer nutrients (i.e. Nitrogen, Phosphorus and Calcium) where $i = 1, 2, ..., N$ observations.

The potential yield function f_N, f_P and f_C can be expressed by a wide variety of functional forms. For many years the von Liebig hypothesis was represented as a linear relation because of the misinterpretation of the world *relation*. Von Liebig used the world *relation* to describe the dependence of crop yield on limiting nutrients but some analyzers confused his words and they gave a stricter meaning of the word, they interpreted it as a *linear relation* of crop yield with the limited nutrients. After some studies it was proved that the potential function could have a non-linear or any other functional form. The linear specification gives rise to the linear-response and plateau model (LRP) and by focusing upon two nutrients, say Phosphorus and Nitrogen, we restated the von Liebig hypothesis as

$$y_i = \min\left\{a_P + \beta_P P_i, a_N + \beta_N N_i, m\right\} \qquad (2)$$

where y_i is corn yield, P_i and N_i are the applied quantities of the corresponding nutrients, phosphorus and nitrogen. The intercept a_P, in the potential yield function for phosphorus in (2), is a proportional function of the nutrient available in the soil and it takes only positive values. The same explanation applies to the other nutrient, the nitrogen. The parameters β_P and β_N show the slope of the corresponding response function of the two fertilizers. Finally, m is the asymptotic plateau or in other words is the maximum corn yield. After some level of application of the two nutrients, say P^* and N^*, the plant will no longer respond to the extra-applied quantity of these nutrients. Paris expresses the plateau as $m = \min\left(f_K(\bar{K}), f_W(\bar{W}), ..., f_L(\bar{L}) \right)$ where $\bar{K}$, $\bar{W}$,…, $\bar{L}$ are the fixed levels of the growth factors that are beyond the scope of the study. So, in the point m the plant reaches the maximum growth by the use of phosphorus and nitrogen and after this point the plant depends on an input that is not included in our study.

The nonlinear specification of response functions f_P and f_N can be represented by a wide spectrum of functional forms, including the Mitscherlich specification according to Paris. Using two inputs, Phosphorus (P) and Nitrogen (N), the model may be written as

$$y = \min\left\{ m\left[1 - k_P \exp\left(-\beta_P P_i\right)\right], m\left[1 - k_N \exp\left(-\beta_N N_i\right)\right]\right\} \qquad (3)$$

where m is the asymptotic plateau common to each potential yield function. The parameters k_P, k_N, β_P and β_N of the response function are expected to be positive. As in the linear specification, this model imposes a yield plateau and non-substitution among inputs, but they differ in the fact that the nonlinear specification allows decreasing marginal productivity and decreasing returns to scale of inputs.

A single-variable-nutrient illustration of the linear response (LRP) is presented in figure 1. In this figure we include a positive Y -axis and a P -axis. The logic of this figure is that the plants obtain the nutrient from two sources, the soil P represented by the negative segment of the P -axis and the fertilizer P represented on the positive segment of the P -axis. When the unobserved soil P and fertilizer P are zero, the yield will be zero as well. When the fertilizer P is zero and the soil P is different zero, the yield would be positive and is represent by the dotted line. In the figure we can observe the point $Y^{\max}$ which is the maximum yield that can be produced, plateau, and if we increase further the amount of the fertilizer P the yield will not affected.

In many cases, the inputs are not controllable with certainty because of the nonuniformity in the distribution of the inputs. The producer, when he applies a fertilizer in its field, he drops more quantity of fertilizer in some plots and in some others less. Therefore, the above production functions do not describe correctly the plant response. For example, assume that a given homogeneous field is depending on the growth factor phosphorus and therefore its yield response would be $y_P = a_P + \beta_P P$. In the case that the plots across the field differ in the quantity of phosphorus, the k^{th} plot receives $P_k + e_{Pk}$ of the fertilizer phosphorus, where e_P is a mean zero random variable. The term P_k is always known because it is the total quantity of phosphorus that is applied in the field divided by the acreage of the field. In contrast, the term e_{Pk} is unobserved. Consequently, the production response function for the k^{th} plot turns to be $y_{Pk} = a_P + \beta_P P_k + \beta_P e_{Pk}$, given that the phosphorus binds. For convenience we set $\varepsilon_{Pk} = \beta_P e_{Pk}$ and so the production response function is

$y_{Pk} = a_P + \beta_P P_k + \varepsilon_{Pk}$. We can make the same assumption for the second nutrient, nitrogen, and the plateau. If the nitrogen is limited, the response function is $y_{Nk} = a_N + \beta_N N_k + \varepsilon_{Nk}$ where $\varepsilon_{Nk} = \beta_N e_{Nk}$ and if the plateau binds the response function is $y_{mk} = m + \varepsilon_{mk}$ (Peter Berck and Gloria Helfand). In contrast, Paris states that the presence of the error term is inspired by the agronomic variability usually observed in vegetative regimes governed by limited nutrients because each nutrient has different implications for the vegetative as well as the maturity and productive stage of a crop. Therefore, the von Liebig hypothesis, respectively, with linear and nonlinear potential crop functions can be expressed as

$$y_i = \min\left\{a_P + \beta_P P_i + \varepsilon_{Pi}, a_N + \beta_N N_i + \varepsilon_{Ni}, m + \varepsilon_{mi}\right\} \tag{4}$$

$$y_i = \min\left\{m\left[1 - k_P \exp\left(-\beta_P P_i\right)\right] + \varepsilon_{Pi}, m\left[1 - k_N \exp\left(-\beta_N N_i\right)\right] + \varepsilon_{Ni}\right\}. \tag{5}$$

For an alternative and more concise specification of the random error assumes that the error associated with the dependent variable is unique and therefore it is not subject to the minimum operator. In this case, the von Liebig models take the following form

$$y_i = \min\left\{a_P + \beta_P P_i, a_N + \beta_N N_i, m\right\} + \varepsilon_i \tag{6}$$

$$y_i = \min\left\{m\left[1 - k_P \exp\left(-\beta_P P_i\right)\right], m\left[1 - k_N \exp\left(-\beta_N N_i\right)\right]\right\} + \varepsilon_i \tag{7}$$

The estimation of the von Liebig specifications (4) and (5) is complex and it requires quite unusual computations because of the switching nature of the production function. The crop yield depends on the limited nutrient, so in each observation one of the several nutrients could be limited. Also, the estimation faces the problem of sample classification as it is not known a priori which one is the limiting nutrient. So, for the estimation of models (4) and (5) we execute the maximum likelihood that is used for the estimation of disequilibrium models as it was first proposed by Maddala and Nelson.

Estimation of the von Liebig model

The von Liebig specification as we said is a special case of disequilibrium or switching regression model in which each regime has its own stochastic disturbance. For estimating the von Liebig model we use the maximum likelihood method of Maddala and Nelson with some adjustments based in the characteristics of the von Liebig model. In the estimation we consider three regimes, phosphorus, nitrogen and plateau, and therefore the model consists of the following equations

$$y_{Pi} = f_P\left(P_i, \beta_P\right) + \varepsilon_{Pi}\,, \qquad (8)$$

$$y_{Ni} = f_N\left(N_i, \beta_N\right) + \varepsilon_{Ni}\,, \qquad (9)$$

$$y_{mi} = m + \varepsilon_{mi} \qquad (10)$$

where y_{Pi} denotes the crop yield of the regime phosphorus, y_{Ni} the crop yield of nitrogen and y_{mi} of the plateau. The terms P_i and N_i are the quantities of each fertilizer respectively, and β_P, β_N are the vectors of parameters. We define y_i the actual crop yield for the observation i

$$y_i = \min\{y_{Pi}, y_{Ni}, y_{mi}\}. \qquad (11)$$

For simplicity, we assume that ε_{Pi}, ε_{Ni} and ε_{mi} are independently and normally distributed with variances σ_P^2, σ_N^2 and σ_m^2 respectively. Also, we assume that they are pairwise independent, $\operatorname{cov}\left(\varepsilon_i^k, \varepsilon_i^{k'}\right) = 0$ where $k \neq k'$ and $k, k' = P, N, m$.

According to the law of minimum, in the case of three regimes, the actual crop yields depend on the scarcest nutrient and therefore there are three possibilities for the observed yield, y_i:

i. The possibility the nutrient phosphorus is the scarcest input and so the crop yields is equal the response function of phosphorus

$$y_i = f_P\left(P_i, \beta_P\right) + \varepsilon_{Pi} \quad \text{If} \quad f_P\left(P_i, \beta_P\right) + \varepsilon_{Pi} < f_N\left(N_i, \beta_N\right) + \varepsilon_{Ni} \quad \text{and}$$

$$f_P\left(P_i, \beta_P\right) + \varepsilon_{Pi} < m + \varepsilon_{mi}$$

ii. The possibility the nutrient nitrogen is the scarcest input and so the crop yields is equal the response function of nitrogen

$$y_i = f_N\left(N_i, \beta_N\right) + \varepsilon_{Ni} \quad \text{If} \quad f_N\left(N_i, \beta_N\right) + \varepsilon_{Ni} < f_P\left(P_i, \beta_P\right) + \varepsilon_{Pi} \quad \text{and}$$

$$f_N\left(N_i, \beta_N\right) + \varepsilon_{Ni} < m + \varepsilon_{mi}$$

iii. The possibility the the crop yields is equal the plateau

$$y_i = m + \varepsilon_{mi} \qquad \text{If} \qquad m + \varepsilon_{mi} < f_P\left(P_i, \beta_P\right) + \varepsilon_{Pi} \qquad \text{and}$$

$$m + \varepsilon_{mi} < f_N\left(N_i, \beta_N\right) + \varepsilon_{Ni}$$

The likelihood function is based on the unconditional density function of the dependent variable, y_i, and it is represented by the sum of three densities depending on which one of P, N or m is binding, as show below,

$$h\left(y_i\right) = h_P\left(y_i = y_{Pi}, y_{Ni} > y_i, y_{mi} > y_i\right) + h_N\left(y_{Pi} > y_i, y_i = y_{Ni}, y_{mi} > y_i\right) +$$
$$h_m\left(y_{Pi} > y_i, y_{Ni} > y_i, y_i = y_{mi}\right) \tag{12}$$

where $h_z(.)$ represents the density of y when input z is limiting, where $z = P, N, m$. Given the assumptions about the error terms in equations (8) to (10), the densities $h_z(.)$ are

$$h_P\left(y_i = y_{Pi}, y_{Ni} > y_i, y_{mi} > y\right) = \phi_P\left(y_i\right) P\left(y_{Ni} > y_i\right) P\left(y_{mi} > y_i\right) \tag{13}$$

$$h_N\left(y_{Pi} > y_i, y_i = y_{Ni}, y_{mi} > y_i\right) = \phi_N\left(y_i\right) P\left(y_{Pi} > y_i\right) P\left(y_{mi} > y_i\right) \tag{14}$$

$$h_m\left(y_{Pi} > y_i, y_{Ni} > y_i, y_i = y_{mi}\right) = \phi_m\left(y_i\right) P\left(y_{Pi} > y_i\right) P\left(y_{Ni} > y_i\right) \tag{15}$$

The function $\phi_z(.)$ is density function for crop yield in regime z. Since ε_P, ε_N and ε_m are normally distributed, the function $\phi_z(.)$ has a form such as

$$\phi_z\left(y_i\right) = \frac{1}{\sqrt{2\pi}\sigma_z} e^{-\frac{\left(y_i - f_z\left(P_i, \beta_z\right)\right)^2}{2\sigma_z^2}} \tag{16}$$

Also, the rest of the probabilities have the following forms

$$P\left(y_{zi} > y_i\right) = \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_z} e^{-\frac{(y - f_z(Z_i, \beta_z))^2}{2\sigma_z^2}} dy \qquad (17)$$

Finally, in order to estimate the parameters of the model we have to maximize the likelihood function

$$\max_{\beta} \; L(\beta) = \sum_{i=1}^{n} h\left(y_i, \beta\right) \qquad (18)$$

where the vector β includes all the parameters we want to estimate, including the variances (**see appendix B for details**). To obtain the estimation of the set of parameters β we take the first derivatives of likelihood and to get the variance-covariance matrix of parameters we take the inverse matrix of the second derivatives of the likelihood.

After we get the estimates of the parameters and of the variances we can obtain estimates of the probabilities that each observation depends on the nutrient phosphorus, or nitrogen or the plateau.

Data and results of the estimation

For the estimation we use the experimental data of Heady and Pesek that involve yield response of corn to the application of phosphorus and nitrogen and consist of 114 observations. These experimental data have become very popular and have been previously used by many to investigate the von Liebig technology and to compare it with other productions functions.

In table 1 we present the estimation results of the linear and non-linear von Liebig functions. The signs of the parameters are positive as expected. The estimated coefficients in both von Liebig specifications are statically significant at the 1% level. The estimated yield plateaus defined by the two von Liebig models are not the same. The linear von Liebig model shows a yield plateau, estimated at 1.2681, which is lower than the yield plateau imposed by the nonlinear von Liebig model, estimated at 1.3731.

The stochastic frontier von Liebig model

The analysis of technical efficiency stands at the heart of microeconomic theory and this is why it is important to measure the technical efficiency of the von Liebig model. For the analysis of efficiency we need the frontier production function that represents the maximum output attainable, given a set of inputs. The measurement of technical efficiency is based upon deviations of observed output from the efficient production frontier. If the actual production point lies on the frontier it is perfectly efficient, but if it lies below the frontier then it is technically inefficient, with the ratio of the actual to the potential production defining the level of efficiency.

The first who attempted to estimate the frontier production function was Farrel but his attempt was not completely successful. Following that, Aigner, Lovell and Schmidt, and Meeusen and Van den Broeck, who carried out a large part of the empirical literature on efficiency in production, proposed the stochastic frontier model or composed error. The stochastic frontier model was based on the idea that deviations from the production frontier might not be entirely under the control of the producer.

Technical efficiency is only one component of overall economic efficiency but it is the main condition of the producer in order to be economically efficient. Also, profit maximization requires a firm to produce the maximum output given the levels of inputs employed (be technically efficient), use the right mix of inputs given the relative price of each inputs (be input allocatively efficient) and produce the exact combination of outputs given the set of prices (be output allocatively efficient). Technical efficiency can be measured in two ways. The first one is to achieve a given level of output by the optimal combination of inputs and it is called *an input-orientation*. The second approach is the opposite; the optimal output is being produced given a set of inputs, and it is called *an output-orientation.*

The concept of technical efficiency can be illustrated using an example with two inputs (P, N) and a production process that expresses the von Liebig model. In figure 2 the individual is producing a given level of output (y_2) using an input combination defined by point A. The same level of output can be produced by reducing the use of both inputs until the point B that lies on the isoquant associated with the minimum level of inputs required to produce y_2. The input-oriented level of technical efficiency for the input phosphorus is defined by the ratio $0P_1 / 0P_1^*$ and for the input nitrogen is $0N_1 / 0N_1^*$. Suppose now that we are producing y_2 output using an input combination defined by point D (E). This point is on the isoquant y_2 but it does not lie in the efficient set of inputs, therefore the technical inefficiency is due to the excess use of the input N (P) and if we decrease its use until the point B and leave constant the input P (N) we produce the same output y_2.

Furthermore, in the figure we can illustrate the output-orientation measure of allocative efficiency. Suppose that the individual uses an input combination defined in point B and produces y_1 output. If the applied inputs are used efficiently by the individual, its output produced at point C, can be expanded very simply to point B so that it would be technically efficient. Hence, the output-oriented measure of technical efficiency can be given by 0C/0B.

Previous work on the estimation of frontier von Liebig function has been done by Holloway and Paris. For the estimation of the technical efficiency they employed the output-orientation allocative efficiency. The form of frontier von Liebig that they give is

$$y_i = \min\left\{a_w + \beta_w W_i, a_N + \beta_N N_i, m\right\} + v + \varepsilon_i$$

where v is the inefficiency term and u_t is the random disturbance. For the estimation they used the empirical data by Hexem and Heady that illustrates the effects of the nutrients water and nitrogen in crop yield. The results of their estimation show that the frontier von Liebig model is an appropriate frontier production function.

In this paper, for the calculation of technical efficiency in the von Liebig model we use the input-oriented measure and we suppose that the response functions are linear. In addition, we estimate three von Liebig frontier functions because there are three possible cases as we previously mentioned. Case 1 is when we are in the point B and we produce y_1 instead of y_2 which is the efficient quantity to produce, and therefore the von Liebig frontier function has the following specification

$$y_i = \min\left\{a_P + \beta_P P_i TE_{Pi} + \varepsilon_{Pi}, a_N + \beta_N N_i TE_{Ni} + \varepsilon_{Ni}, m + \varepsilon_{mi}\right\} \qquad (25)$$

The other two cases appear when only one of the two inputs is inefficiently applied. These cases occur when the producer applies the combinations of inputs indicated by points D and E in the figure. Particularly, in point E nitrogen is used efficiently, in contrast to phosphorus that is applied inefficiently. Hence, in case 2, a technically efficient term is introduced in the response function of phosphorus and the frontier von Liebig function is

$$y_i = \min\left\{a_P + \beta_P P_i TE_{Pi} + \varepsilon_{Pi}, a_N + \beta_N N_i + \varepsilon_{Ni}, m + \varepsilon_{mi}\right\}. \qquad (26)$$

The same can be said when we are in point D where we use excess quantity of the nutrient nitrogen in contrast to phosphorus of which we use the exact quantity needed in order to produce efficiently the quantity y_2. As previously, only nitrogen that is inefficient has an efficiency term in its response function and the frontier function in case 3 is

$$y_i = \min\left\{a_P + \beta_P P_i + \varepsilon_{Pi}, a_N + \beta_N N_i TE_{Ni} + \varepsilon_{Ni}, m + \varepsilon_{mi}\right\} \qquad (27)$$

The terms TE_P and TE_N are the technical efficiencies of the nutrients phosphorus and nitrogen respectively. They show the percentage of the relative input that is used efficient. If the technical efficiency equals unity, the input is used efficiently. In contrast, if the technical efficiency is below unity, the use of input is not efficient. For example, suppose in figure 2 that we use the set of inputs that is implied on the point A and we produce y_1. If we reduce phosphorus and nitrogen by the proportion

$1-TE_P$ and $1-TE_N$ respectively, we reach the point B and, thus, achieve technical efficiency.

We assume that the technical efficiency can be also written as $TE=\exp(-u)$ where u derives from a half normal distribution, $u \sim N^{+}\left(0,\sigma_u^2\right)$. The term u takes only positive values, $u \geq 0$, and therefore, technical efficiency satisfies the condition of technical efficiency $0 < TE \leq 1$ (when $u=0 \Rightarrow \exp(-0)=1 \Rightarrow$ there is not technical inefficiency and when $u>0 \Rightarrow 0<\exp(-u)<1 \Rightarrow$ there is technical inefficiency).

Estimation of frontier von Liebig function

We have three von Liebig frontier functions that we must estimate, each one illustrating a different case. For the estimation of these stochastic frontier von Liebig models we use the same method as in the previous estimation of von Liebig model, the maximum likelihood method. The difference between these estimations is that we have two independent random variables in the response functions of phosphorus and nitrogen with different distribution: the error components ε_P and ε_N and the terms u_P and u_N. Hence, the probability function arises from the sum of the two different distributions and is expressed as

$$\phi(y)=\frac{2}{2\pi\sigma_{\varepsilon z}\sigma_{uz}}\int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{\left(y_i-a_z-\beta_z Z_i TE_z\right)^2}{\sigma_{\varepsilon z}^2}+\frac{\left(-\ln TE_z\right)^2}{\sigma_{uz}^2}\right]\right\}\frac{1}{TE_z}dTE_z \qquad (29)$$

where $\sigma_{\varepsilon z}$ and σ_{uz} are the standard deviations of the error component and of the u_z respectively, where $z=P,N$. We note that the probability function (29) represents only the response functions that contain a technically efficient term. If the response function does not contain a technical efficient term, the distribution function would like the (9**)(see Appendix C for details)**.

Results

For the estimation of the von Liebig frontier functions we use the same experimental data Heady and Pesek that we used in the previous estimation of the von Liebig function.

Table 2 contains the estimations of the three different von Liebig frontier functions. The signs of the parameters for the three models are positive as we expected to be. All the estimated coefficients and intercepts are statistically significant at the 0.05% level. Also, we notify that the estimations of the three models are similar with the estimations of the linear von Liebig model in which there are no technically efficient terms.

In the first model, where both response functions of the inputs, P and N, have a technically efficient term, the standard deviation of the technical efficient term for phosphorus is statistically significant at the 0.05% level. In contrast, the standard deviation of the technically efficient term for nitrogen is not significantly different from zero. That means that only the input phosphorus is technical inefficiency.

The estimation of the second model is the same with the estimation of the first model with the difference that we do not estimate the standard deviation of the technically efficient term for nitrogen. On the contrary, the estimation of the third model is not exactly the same; only the estimated value of the plateau is the same along the three models. In addition, the estimated standard deviation of technical efficiency of nitrogen in the third model is not significantly different from zero. This means that the input nitrogen is used efficiently.

Measurement of the technical efficiency

The second step in the analysis of the stochastic frontier model is to calculate the technical efficiency for each observation of the experimental data. For the calculation we use the estimated parameter values from the von Liebig frontier function.

In order to measure the technical efficiency we take the conditional expectation of technical efficiency given the value of crop yield, which is

$$E\left(TE_{zi} \left| y_i \right.\right) = \frac{\int_0^1 TE_z f_z\left(TE_z, y_i\right) dTE_z}{h\left(y\right)}$$

where $h\left(y\right)$ is the value of likelihood function got evaluated at estimates and $f\left(TE, y\right)$ is the joint density of the technical efficiency and the crop yield and is express as

$$f\left(TE_{zi}, y_i\right) = \varphi\left(TE_{zi}, y_{zi} = y, y_{z'i} > y_i, y_{mi} > y_i\right) + \varphi\left(TE_{zi}, y_{zi} > y_i, y_{z'i} = y_i, y_{mi} > y_i\right) +$$

$$\varphi\left(TE_{zi}, y_{zi} > y_i, y_{z'i} > y_i, y_{mi} = y_i\right)$$

where $\phi(.)$ is the joint probability function of TE_{zi} with the y_i which is bound to the limited input. Because we assume that the error terms are independently distributed the joint density can be also written as

$$f\left(TE_{zi}, y_i\right) = \varphi\left(TE_{zi}, y_{zi} = y_i\right) P\left(y_{z'i} > y_i\right) P\left(y_{mi} > y_i\right) + \tag{32}$$

$$P\left(TE_{zi}, y_{zi} > y_i\right) \varphi\left(y_{z'i} = y_i\right) P\left(y_{mi} > y_i\right) + P\left(TE_{zi}, y_{zi} > y_i\right) P\left(y_{z'i} > y_i\right) \phi\left(y_{mi} > y_i\right)$$

The forms of the joint density is

$$\varphi\left(TE_{zi}, y_{zi} = y_i\right) = \frac{2}{2\pi\sigma_{\varepsilon z}\sigma_{uz}} \int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{\left(y_i - a_z - \beta_z Z_i TE_z\right)^2}{\sigma_{\varepsilon z}^2} + \frac{\left(-\ln TE_z\right)^2}{\sigma_{uz}^2}\right]\right\} dTE_z \tag{33}$$

where $z, z' = P, N$ and $z \neq z'$. We use this procedure for measuring the technical efficiency for the three cases **(see appendix D for details)**.

The tables (3) and (4) contain the results of the term technical efficiency for the three case we estimated. We note that the results of case 1 are similar with the case 2 and 3. The nutrient phosphorus is technical efficient only for one of the 114 observations of the experimental data and the most observations use only the 80% of the nutrient phosphorus efficient. In contrast, the nutrient nitrogen is used technical efficiency from all the observations of the data.

Conclusion

In the last year the von Liebig model attracts the interest of the agricultural economist and it was proved that it could perform as well as the polynomial functions. In this paper we analyze the von Liebig model and we estimate the von Liebig production function and the stochastic frontier von Liebig functions in order to measure the technical efficiency. For the estimation we use an extension of maximum likelihood of Maddala and Nelson that it was constructed for the estimation of the disequilibrium models.

We estimate three stochastic frontier von Liebig functions, each one describe different case, and for the measurement of technical efficiency we use the input orientation. In the first case we assume that both inputs are technical inefficient, in the second and in the third we assume that one of the inputs are technical inefficient. The results show that the nutrient phosphorus is technical inefficient, in contrast to nitrogen that is technical efficient. Also, the estimations of the case 3 in which we assume that the phosphorus is technical efficient and the nitrogen is technical inefficient, are exactly the same with the estimations of the von Liebig model without technical efficient term because the results show that the nitrogen is technical efficient.

Further research in this area should include the measure of technical efficiency using different distribution of half normal. Also, the measure of allocative efficiency of inputs is very interest. If the prices of inputs are available, we can find input combinations minimizing the cost of production.

Appendix A

Figure 1: LRP representation of crop yield to Phosphorus

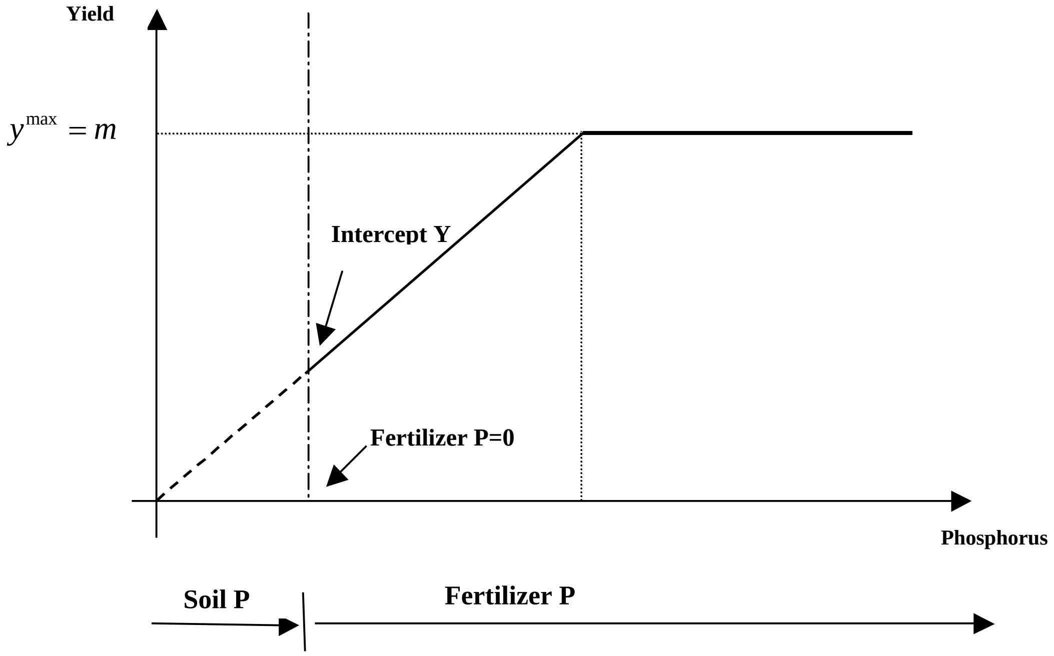

Figure 2: Input and output oriented efficiency measures

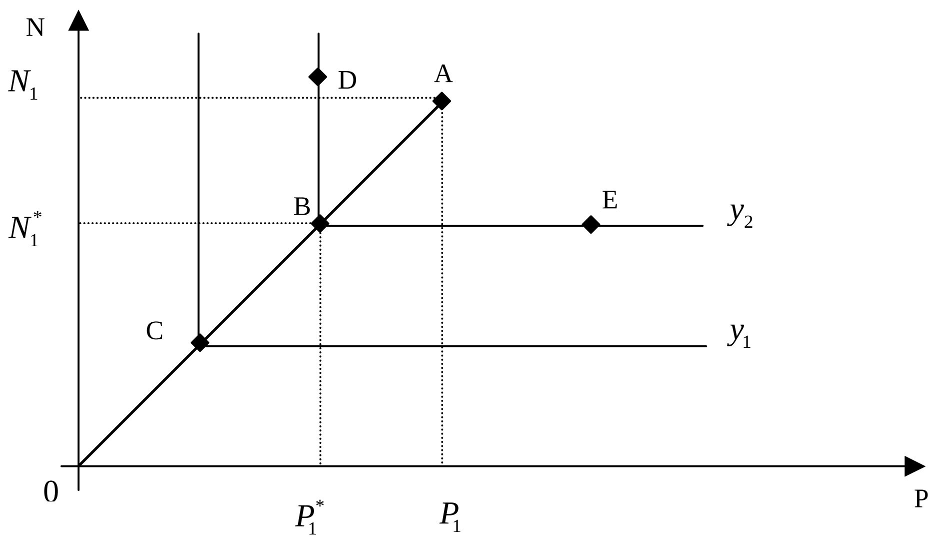

Table 1

Estimated von Liebig function

Linear von Liebig

$$y_i = \min\{0.2641 + 0.9395 * P_i + \varepsilon_{Pi}, 0.2895 + 0.9824 * N_i + \varepsilon_{Ni}, 1.2681 + \varepsilon_{mi}\}$$
$$(0.0534)\ (0.1016) \qquad (0.0307)\ (0.0697) \qquad (0.0144)$$

$$\hat{\sigma}_P = 0.2359, \quad \hat{\sigma}_N = 0.1294, \quad \hat{\sigma}_m = 0.0935, \quad \log L = 79.7382$$
$$(0.0304) \qquad (0.0168) \qquad (0.0104)$$

Von Liebig with Mitscherlich regimes

$$y_i = \min\{1.3731(1 - 0.8729 * \exp(-1.9617 * P_i)) + \varepsilon_{Pi}, 1.3731(1 - 0.802 * \exp(-1.5431 * N_i)) + \varepsilon_{Ni}\}$$
$$(0.0227)\ (0.0248) \qquad (0.2345) \qquad (0.0227)\ (0.0217) \qquad (0.1431)$$

$$\hat{\sigma}_P = 0.1426, \quad \hat{\sigma}_N = 0.1225, \quad \log L = 70.6505$$
$$(0.0151) \qquad (0.0126)$$

Table 2

Estimated frontier von Liebig functions

Parameters	Frontier von Liebig Function (case1)	Frontier von Liebig Function (case2)	Frontier von Liebig Function (case3)
a_P	0.2201 (0.0302)	0.2201 (0.0302)	0.2641 (0.0534)
a_N	0.2909 (0.0306)	0.2910 (0.0307)	0.2894 (0.0307)
β_P	1.5266 (0.0894)	1.5266 (0.0894)	0.9395 (0.1016)
β_N	0.9875 (0.0724)	0.9785 (0.0693)	0.9914 (0.0728)
m	1.2681 (0.0144)	1.2681 (0.0144)	1.2681 (0.0144)
$\sigma_{\varepsilon P}$	0.1333 (0.0248)	0.1333 (0.0248)	0.2359 (0.0304)
$\sigma_{\varepsilon N}$	0.1293 (0.0168)	0.1294 (0.0168)	0.1293 (0.0168)
$\sigma_{\varepsilon m}$	0.0926 (0.0106)	0.0926 (0.0106)	0.0935 (0.0104)
σ_{uP}	0.4406 (0.0880)	0.4406 (0.0880)	-
σ_{uN}	0.0114 (0.0261)	-	0.0114 (0.0264)

Table 3

***Case 1**: Nutrient Phosphorus*

Levels of TE	Number of obs
0.1	0
0.2	0
0.3	0
0.4	1
0.5	2
0.6	2
0.7	1
0.8	97
0.9	10
1	1

Mean	Median
0.72885	0.72673

***Case 1:** Nutrient Nitrogen*

Levels of TE	Number of obs
0.1	0
0.2	0
0.3	0
0.4	0
0.5	0
0.6	0
0.7	0
0.8	0
0.9	0
1	114

Mean	Median
0.99097	0.99097

Table 3

***Case2:** Nutrient Phosphorus*

Levels of TE	Number of obs
0.1	0
0.2	0
0.3	0
0.4	1
0.5	2
0.6	2
0.7	1
0.8	97
0.9	10
1	1

Mean	Median
0.72885	0.72673

***Case3:** Nutrient Nitrogen*

Levels of TE	Number of obs
0.1	0
0.2	0
0.3	0
0.4	0
0.5	0
0.6	0
0.7	0
0.8	0
0.9	0
1	114

Mean	Median
0.99094	0.99097

Appendix B

Estimation of von Liebig function

In the estimation we consider three regimes, phosphorus, nitrogen and plateau, and therefore the model consists of the following equations

$$y_{Pi} = f_P\left(P_i, \beta_P\right) + \varepsilon_{Pi}\,, \qquad (1)$$

$$y_{Ni} = f_N\left(N_i, \beta_N\right) + \varepsilon_{Ni}\,, \qquad (2)$$

$$y_{mi} = m + \varepsilon_{mi} \qquad (3)$$

where
y_{Pi}, y_{Ni}, y_{mi} : crop yields of the regime phosphorus, nitrogen and plateau respectively

P_i, N_i : quantities of fertilizer phosphorus and nitrogen respectively,

β_P, β_N : vectors of parameters
m : plateau

We define y_i the actual crop yield for the observation i

$$y_i = \min\left\{y_{Pi}, y_{Ni}, y_{mi}\right\}. \qquad (4)$$

We assume that ε_{Pi}, ε_{Ni} and ε_{mi} are independently and normally distributed with variances σ_P^2, σ_N^2 and σ_m^2 respectively, and that they are pairwise independent, $\mathrm{cov}\left(\varepsilon_i^k, \varepsilon_i^{k'}\right) = 0$ where $k \neq k'$ and $k, k' = P, N, m$.

The likelihood function is

$$h\left(y_i\right) = h_P\left(y_i = y_{Pi}, y_{Ni} > y_i, y_{mi} > y_i\right) + h_N\left(y_{Pi} > y_i, y_i = y_{Ni}, y_{mi} > y_i\right) +$$
$$h_m\left(y_{Pi} > y_i, y_{Ni} > y_i, y_i = y_{mi}\right) \qquad (5)$$

where $h_z(.)$ represents the density of y when input i is limiting, where $z = P, N, \pi$.

Given the assumptions about the error terms the likelihood function can be written as

$$h(y_i) = \phi_P(y_i) P(y_{Ni} > y_i) P(y_{mi} > y_i) + \phi_N(y_i) P(y_{Pi} > y_i) P(y_{mi} > y_i) +$$
$$\phi_m(y_i) P(y_{Pi} > y_i) P(y_{Ni} > y_i) \tag{6}$$

The function $\phi_z(.)$ is density function for crop yield in regime z . Since ε_P , ε_N and ε_m are normally distributed, the function $\phi_z(.)$ has a form such as

$$\phi_P(y_i) = \frac{1}{\sqrt{2\pi}\sigma_P} e^{-\frac{\left(y_i - f_P\left(P_i, \beta_P\right)\right)^2}{2\sigma_P^2}} \tag{7}$$

$$\phi_N(y_i) = \frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{\left(y_i - f_N\left(N_i, \beta_N\right)\right)^2}{2\sigma_N^2}} \tag{8}$$

$$\phi_m(y_i) = \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{(y_i - m)^2}{2\sigma_m^2}} \tag{9}$$

Also, the rest of the probabilities have the following forms

$$P(y_{Pi} > y_i) = \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_P} e^{-\frac{(y - f_P(P_i, \beta_P)_i)^2}{2\sigma_P^2}} dy \tag{10}$$

$$P(y_{Ni} > y_i) = \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{(y - f_N(N_i, \beta_N))^2}{2\sigma_N^2}} dy \tag{11}$$

$$P(y_{mi} > y_i) = \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{(y - m)^2}{2\sigma_m^2}} dy \tag{12}$$

Finally, in order to estimate the parameters of the model we have to maximize the likelihood function

$$\max_{\beta} \quad L(\beta) = \sum_{i=1}^{n} h\left(y_i, \beta\right) \qquad (13)$$

$$\max_{\beta} \quad L\left(\beta\right) = \phi_P\left(y_i\right) = \frac{1}{\sqrt{2\pi}\sigma_P} e^{-\frac{\left(y_i - f_P\left(P_i, \beta_P\right)\right)^2}{2\sigma_P^2}} * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{(y - f_N(N_i, \beta_N))^2}{2\sigma_N^2}} dy * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{(y-m)^2}{2\sigma_m^2}} dy +$$

$$\frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{\left(y_i - f_N\left(N_i, \beta_N\right)\right)^2}{2\sigma_N^2}} * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_P} e^{-\frac{(y - f_P(P_i, \beta_P))^2}{2\sigma_P^2}} dy * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{(y-m)^2}{2\sigma_m^2}} dy +$$

$$\frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{(y_i - m)^2}{2\sigma_m^2}} * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_P} e^{-\frac{(y - f_P(P_i, \beta_P))^2}{2\sigma_P^2}} dy * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{(y - f_N(N_i, \beta_N))^2}{2\sigma_N^2}} dy$$

where the vector β includes all the parameters we want to estimate, including the variances. To obtain the estimation of the set of parameters β we take the first derivatives of likelihood and to get the variance-covariance matrix of parameters we take the inverse matrix of the second derivatives of the likelihood.

Appendix C

Estimation of frontier von Liebig function

Case 1:

The model consists of the following equations

$$y_{Pi} = a_P + \beta_P P_i TE_{Pi} + \varepsilon_{Pi}\,, \tag{14}$$

$$y_{Ni} = a_N + \beta_N N_i TE_{Ni} + \varepsilon_{Ni}\,, \tag{15}$$

$$y_{mi} = m + \varepsilon_{mi} \tag{16}$$

where a_P , a_N : intercepts of response function of phosphorus and nitrogen respectively

β_P , β_N : coefficients of response functions of phosphorus and nitrogen respectively

P_i , N_i : apply quantities of phosphorus and nitrogen respectively

TE_{Pi} , TE_{Ni} : technical efficiency of nutrients phosphorus and nitrogen respectively

m : plateau

We define y_i the actual crop yield for the observation i

$$y_i = \min\{y_{Pi}, y_{Ni}, y_{mi}\} \tag{17}$$

The likelihood function is the same with the likelihood function we use for the estimation of von Liebig function,

$$h(y_i) = \varphi_P(y_i)P(y_{Ni} > y_i)P(y_{mi} > y_i) + \varphi_N(y_i)P(y_{Pi} > y_i)P(y_{mi} > y_i) + \varphi_m(y_i)P(y_{Pi} > y_i)P(y_{Ni} > y_i) \tag{18}$$

Because the response functions of phosphorus and nitrogen contain two random variables, the density of crop yield is the sum of the two distributions of these random variables and is expressed as

$$\varphi_z(y) = \frac{2}{2\pi\sigma_{\varepsilon z}\sigma_{uz}} \int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{(y_i - a_z - \beta_z Z_i TE_z)^2}{\sigma_{\varepsilon z}^2} + \frac{(-\ln TE_z)^2}{\sigma_{uz}^2}\right]\right\} \frac{1}{TE_z} dTE_z \quad (19)$$

$$P_z(y_{zi} > y_i) = \frac{2}{2\pi\sigma_{\varepsilon z}\sigma_{uz}} \int_{y_i}^{\infty} \int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{(y - a_z - \beta_z Z_i TE_z)^2}{\sigma_{\varepsilon z}^2} + \frac{(-\ln TE_z)^2}{\sigma_{uz}^2}\right]\right\} \frac{1}{TE_z} dTE_z dy \quad (20)$$

where $z = P, N$,

For the response function of plateau that contains only error term, the density function is

$$\phi_m(y_i) = \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{(y_i - m)^2}{2\sigma_m^2}} \quad (21)$$

$$P(y_{mi} > y_i) = \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{(y-m)^2}{2\sigma_m^2}} dy \quad (22)$$

Therefore the likelihood function that we maximize is

$$\max_{\beta} \; L(\beta) = \sum_{i=1}^{n} h(y_i, \beta) \quad (23)$$

$$\max_{\beta} \; L(\beta) = \frac{2}{2\pi\sigma_{\varepsilon P}\sigma_{uP}} \int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{(y_i - a_P - \beta_P P_i TE_P)^2}{\sigma_{\varepsilon P}^2} + \frac{(-\ln TE_P)^2}{\sigma_{uP}^2}\right]\right\} \frac{1}{TE_P} dTE_P \; *$$

$$\frac{2}{2\pi\sigma_{\varepsilon N}\sigma_{uN}} \int_{y_i}^{\infty} \int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{(y - a_N - \beta_N N_i TE_N)^2}{\sigma_{\varepsilon N}^2} + \frac{(-\ln TE_N)^2}{\sigma_{uN}^2}\right]\right\} \frac{1}{TE_N} dTE_N dy * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{(y-m)^2}{2\sigma_m^2}} dy +$$

$$\frac{2}{2\pi\sigma_{\varepsilon N}\sigma_{uN}} \int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{(y_i - a_N - \beta_N N_i TE_N)^2}{\sigma_{\varepsilon N}^2} + \frac{(-\ln TE_N)^2}{\sigma_{uN}^2}\right]\right\} \frac{1}{TE_N} dTE_N \; *$$

$$\frac{2}{2\pi\sigma_{\varepsilon P}\sigma_{uP}}\int_{y_i}^{\infty}\int_{0}^{1}\exp\left\{-\frac{1}{2}\left[\frac{\left(y-a_P-\beta_P P_i TE_P\right)^2}{\sigma_{\varepsilon P}^2}+\frac{\left(-\ln TE_P\right)^2}{\sigma_{uP}^2}\right]\right\}\frac{1}{TE_P}dTE_P dy*\int_{y_i}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_m}e^{-\frac{(y-m)^2}{2\sigma_m^2}}dy+$$

$$\frac{1}{\sqrt{2\pi}\sigma_m}e^{-\frac{(y_i-m)^2}{2\sigma_m^2}}*\frac{2}{2\pi\sigma_{\varepsilon P}\sigma_{uP}}\int_{y_i}^{\infty}\int_{0}^{1}\exp\left\{-\frac{1}{2}\left[\frac{\left(y-a_P-\beta_P P_i TE_P\right)^2}{\sigma_{\varepsilon P}^2}+\frac{\left(-\ln TE_P\right)^2}{\sigma_{uP}^2}\right]\right\}\frac{1}{TE_P}dTE_P dy*$$

$$\frac{2}{2\pi\sigma_{\varepsilon N}\sigma_{uN}}\int_{y_i}^{\infty}\int_{0}^{1}\exp\left\{-\frac{1}{2}\left[\frac{\left(y-a_N-\beta_N N_i TE_N\right)^2}{\sigma_{\varepsilon N}^2}+\frac{\left(-\ln TE_N\right)^2}{\sigma_{uN}^2}\right]\right\}\frac{1}{TE_N}dTE_N dy$$

Case 2:

The model consists of the following equations

$$y_{Pi}=a_P+\beta_P P_i TE_{Pi}+\varepsilon_{Pi}\,, \qquad (24)$$

$$y_{Ni}=a_N+\beta_N N_i+\varepsilon_{Ni}\,, \qquad (25)$$

$$y_{mi}=m+\varepsilon_{mi} \qquad (26)$$

We define y_i the actual crop yield for the observation i

$$y_i=\min\left\{y_{Pi},y_{Ni},y_{mi}\right\} \qquad (27)$$

The difference with the case 1 is that there is a technical efficient term only in response function of phosphorus because we assume that the use of the other nutrient (nitrogen) is technical efficiency.

The likelihood function is

$$h(y_i)=\varphi_P\left(y_i\right)P\left(y_{Ni}>y_i\right)P\left(y_{mi}>y_i\right)+\varphi_N\left(y_i\right)P\left(y_{Pi}>y_i\right)P\left(y_{mi}>y_i\right)+$$
$$\varphi_m\left(y_i\right)P\left(y_{Pi}>y_i\right)P\left(y_{Ni}>y_i\right) \qquad (26)$$

The form of $\varphi_P(y_{Pi} = y_i)$, $P\left(y_{Pi} > y_i\right)$ are the same as in the case 1 because the response function of phosphorus continues having two random variables, the technical efficient term and the disturbance term. At the contrary, the $\phi_N\left(y_{Ni} = y_i\right)$ and $P\left(y_{Ni} = y_i\right)$ have the form of normal distribution function, like the (8) and (11). Therefore the maximum likelihood is

$$\max_{\beta} \; L(\beta) = \sum_{i=1}^{n} h\left(y_i, \beta\right) \qquad (27)$$

$$L\left(\beta\right) = \frac{2}{2\pi\sigma_{\varepsilon P}\sigma_{uP}} \int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{\left(y_i - a_P - \beta_P P_i TE_P\right)^2}{\sigma_{\varepsilon P}^2} + \frac{\left(-\ln TE_P\right)^2}{\sigma_{uP}^2}\right]\right\} \frac{1}{TE_P} dTE_P * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{(y - a_N - \beta_N N_i)^2}{2\sigma_N^2}} dy * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{(y-m)^2}{2\sigma_m^2}} dy +$$

$$\frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{\left(y_i - a_N - \beta_N N_i\right)^2}{2\sigma_N^2}} * \frac{2}{2\pi\sigma_{\varepsilon P}\sigma_{uP}} \int_{y_i}^{\infty} \int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{\left(y - a_P - \beta_P P_i TE_P\right)^2}{\sigma_{\varepsilon P}^2} + \frac{\left(-\ln TE_P\right)^2}{\sigma_{uP}^2}\right]\right\} \frac{1}{TE_P} dTE_P dy * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{(y-m)^2}{2\sigma_m^2}} dy +$$

$$\frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{\left(y_i - m\right)^2}{2\sigma_m^2}} * \frac{2}{2\pi\sigma_{\varepsilon P}\sigma_{uP}} \int_{y_i}^{\infty} \int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{\left(y - a_P - \beta_P P_i TE_P\right)^2}{\sigma_{\varepsilon P}^2} + \frac{\left(-\ln TE_P\right)^2}{\sigma_{uP}^2}\right]\right\} \frac{1}{TE_P} dTE_P dy * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{(y - a_N - \beta_N N_i)^2}{2\sigma_N^2}} dy$$

Case 3:

The model consists of the following equations

$$y_{Pi} = a_P + \beta_P P_i + \varepsilon_{Pi}\,, \qquad (28)$$

$$y_{Ni} = a_N + \beta_N N_i TE_{Ni} + \varepsilon_{Ni}\,, \qquad (29)$$

$$y_{mi} = m + \varepsilon_{mi} \qquad (30)$$

We define y_i the actual crop yield for the observation i

$$y_i = \min\left\{y_{Pi}, y_{Ni}, y_{mi}\right\} \qquad (31)$$

We put a technical efficient term only in response function of nitrogen because we assume that the use of the other nutrient (phosphorus) is technical efficiency.

The likelihood function is

$$h(y_i) = \varphi_P\left(y_i\right)P\left(y_{Ni} > y_i\right)P\left(y_{mi} > y_i\right) + \varphi_N\left(y_i\right)P\left(y_{Pi} > y_i\right)P\left(y_{mi} > y_i\right) +$$

$$\varphi_m\left(y_i\right)P\left(y_{Pi} > y_i\right)P\left(y_{Ni} > y_i\right) \qquad (32)$$

The form of $\varphi_N(y_{Ni} = y_i)$, $P\left(y_{Ni} > y_i\right)$ is like the (19) and (20) respectively, because the response function of nitrogen has two random variables, the technical efficient term and the disturbance term. At the contrary, the $\phi_P\left(y_{Pi} = y_i\right)$ and $P\left(y_{Pi} = y_i\right)$ have the form of normal distribution function, like the (7) and (10) respectively. Therefore the maximum likelihood is

$$\max_{\beta} \; L(\beta) = \sum_{i=1}^{n} h\left(y_i, \beta\right) \qquad (33)$$

$$L(\beta) = \frac{1}{\sqrt{2\pi}\sigma_P} e^{-\frac{\left(y_i - a_P - \beta_P P_i\right)^2}{2\sigma_P^2}} * \frac{2}{2\pi\sigma_{\varepsilon N}\sigma_{uN}} \int_{y_i}^{\infty} \int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{\left(y - a_N - \beta_N N_i TE_N\right)^2}{\sigma_{\varepsilon N}^2} + \frac{\left(-\ln TE_N\right)^2}{\sigma_{uN}^2}\right]\right\} \frac{1}{TE_N} dTE_N dy * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{(y-m)^2}{2\sigma_m^2}} dy +$$

$$\frac{2}{2\pi\sigma_{\varepsilon N}\sigma_{uN}} \int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{\left(y_i - a_N - \beta_N N_i TE_N\right)^2}{\sigma_{\varepsilon N}^2} + \frac{\left(-\ln TE_N\right)^2}{\sigma_{uN}^2}\right]\right\} \frac{1}{TE_N} dTE_N * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_P} e^{-\frac{\left(y - a_P - \beta_P P_i\right)^2}{2\sigma_P^2}} dy * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{(y-m)^2}{2\sigma_m^2}} dy +$$

$$\frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{\left(y_i - m\right)^2}{2\sigma_m^2}} * \frac{2}{2\pi\sigma_{\varepsilon N}\sigma_{uN}} \int_{y_i}^{\infty} \int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{\left(y - a_N - \beta_N N_i TE_N\right)^2}{\sigma_{\varepsilon N}^2} + \frac{\left(-\ln TE_N\right)^2}{\sigma_{uN}^2}\right]\right\} \frac{1}{TE_N} dTE_N dy * \int_{y_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_P} e^{-\frac{\left(y - a_P - \beta_P P_i\right)^2}{2\sigma_P^2}} dy$$

Appendix D

Measurement of Technical efficiency

Case1

For the measure of technical efficiency of phosphorus in case 1 we estimate the conditional expectation of the technical efficiency for the phosphorus,

$$E\left(TE_{Pi}\left|y_i\right.\right)=\frac{\int_0^1 TE_P\, f\left(TE_P,y_i\right)dTE_p}{h\left(y_i\right)} \tag{34}$$

where $h\left(y\right)$ is the value of likelihood function got evaluated at estimates and $f\left(TE_P,y_i\right)$ is the joint density of the technical efficiency and the crop yield and has the following expression

$$f\left(TE_{Pi},y_i\right)=\varphi\left(TE_{Pi},y_{Pi}=y,y_{Ni}>y_i,y_{mi}>y_i\right)+\varphi\left(TE_{Pi},y_{Pi}>y_i,y_{Ni}=y_i,y_{mi}>y_i\right)+$$
$$\varphi\left(TE_{Pi},y_{Pi}>y_i,y_{Ni}>y_i,y_{mi}=y_i\right) \tag{35}$$

where $\phi\left(.\right)$ is the joint probability function of TE_{Pi} with the y_i which is bound to the limited input. Because we assume that the error terms are independently distributed the joint density can be also written as

$$f\left(TE_{Pi},y_i\right)=\varphi\left(TE_{Pi},y_{Pi}=y_i\right)P\left(y_{Ni}>y_i\right)P\left(y_{mi}>y_i\right)+ \tag{36}$$
$$P\left(TE_{Pi},y_{Pi}>y_i\right)\phi(y_{Ni}=y_i)P\left(y_{mi}>y_i\right)+P\left(TE_{Pi},y_{Pi}>y_i\right)P\left(y_{Ni}>y_i\right)\phi\left(y_{mi}=y_i\right)$$

The forms of the joint density is

$$\varphi\left(TE_{Pi}, y_{Pi}=y_i\right)=\frac{2}{2\pi\sigma_{\varepsilon P}\sigma_{uP}}\int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{\left(y_i-a_P-\beta_P P_i TE_P\right)^2}{\sigma_{\varepsilon P}^2}+\frac{\left(-\ln TE_P\right)^2}{\sigma_{uP}^2}\right]\right\} dTE_P \qquad (37)$$

$$P\left(TE_{Pi}, y_{Pi}=y_i\right)=\frac{2}{2\pi\sigma_{\varepsilon P}\sigma_{uP}}\int_{y_i}^{\infty}\int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{\left(y-a_P-\beta_P P_i TE_P\right)^2}{\sigma_{\varepsilon P}^2}+\frac{\left(-\ln TE_P\right)^2}{\sigma_{uP}^2}\right]\right\} dTE_P dy \qquad (38)$$

The forms of $\phi(y_{Ni}=y_i)$ and $P\left(y_{Ni}>y_i\right)$ are like (19) and (20) respectively, and the forms of $\phi\left(y_{mi}=y_i\right)$ and $P\left(y_{mi}>y_i\right)$ are like (21) and (20) respectively.

The same procedure we use for the measurement of technical efficiency of nitrogen. We estimate the conditional expectation of the technical efficiency for the nitrogen,

$$E\left(TE_{Ni}\left|y_i\right.\right)=\frac{\int_0^1 TE_N f\left(TE_N, y_i\right) dTE_N}{h\left(y_i\right)} \qquad (39)$$

where $h\left(y\right)$ is the value of likelihood function got evaluated at estimates and $f\left(TE_N, y_i\right)$ is the joint density of the technical efficiency and the crop yield and has the following expression

$$f\left(TE_{Ni}, y_i\right)=P\left(y_{Pi}=y_i\right)\varphi\left(TE_{Ni}, y_{Ni}=y_i\right)P\left(y_{mi}>y_i\right)+ \qquad (40)$$
$$\phi\left(TE_{Ni}, y_{Ni}>y_i\right)P(y_{Pi}=y_i)P\left(y_{mi}>y_i\right)+P\left(y_{Pi}>y_i\right)P\left(TE_{Ni}, y_{Ni}>y_i\right)\phi\left(y_{mi}=y_i\right)$$

The forms of the joint density is

$$\varphi\left(TE_{Ni}, y_{Ni} = y_i\right) = \frac{2}{2\pi\sigma_{\varepsilon N}\sigma_{uN}} \int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{\left(y_i - a_N - \beta_N N_i TE_N\right)^2}{\sigma_{\varepsilon N}^2} + \frac{\left(-\ln TE_N\right)^2}{\sigma_{uN}^2}\right]\right\} dTE_N \qquad (41)$$

$$P\left(TE_{Ni}, y_{Ni} = y_i\right) = \frac{2}{2\pi\sigma_{\varepsilon N}\sigma_{uN}} \int_{y_i}^{\infty} \int_0^1 \exp\left\{-\frac{1}{2}\left[\frac{\left(y - a_N - \beta_N N_i TE_N\right)^2}{\sigma_{\varepsilon N}^2} + \frac{\left(-\ln TE_N\right)^2}{\sigma_{uN}^2}\right]\right\} dTE_N dy \qquad (42)$$

The forms of $\phi(y_{Pi} = y_i)$ and $P\left(y_{Pi} > y_i\right)$ are like (19) and (20) respectively, and the forms of $\phi\left(y_{mi} = y_i\right)$ and $P\left(y_{mi} > y_i\right)$ are like (21) and (20) respectively.

Case 2

For the measure of technical efficiency of phosphorus in case 2 we estimate the conditional expectation of the technical efficiency for the phosphorus,

$$E\left(TE_{Pi} \middle| y_i\right) = \frac{\int_0^1 TE_P f\left(TE_P, y_i\right) dTE_p}{h\left(y_i\right)} \qquad (43)$$

where $h\left(y\right)$ is the value of likelihood function got evaluated at estimates and $f\left(TE_P, y_i\right)$ is the joint density of the technical efficiency and the crop yield and has the following expression

$$f\left(TE_{Pi}, y_i\right) = \varphi\left(TE_{Pi}, y_{Pi} = y_i\right) P\left(y_{Ni} > y_i\right) P\left(y_{mi} > y_i\right) + \qquad (44)$$

$$P\left(TE_{Pi}, y_{Pi} > y_i\right)\phi(y_{Ni} = y_i)P\left(y_{mi} > y_i\right) + P\left(TE_{Pi}, y_{Pi} > y_i\right)P\left(y_{Ni} > y_i\right)\phi\left(y_{mi} = y_i\right)$$

The forms of the joint density $\varphi\left(TE_{Pi}, y_{Pi} = y_i\right)$ and $P\left(TE_{Pi}, y_{Pi} > y_i\right)$ are like (37) and (38) and the forms of $\phi(y_{Ni} = y_i)$, $P\left(y_{Ni} > y_i\right)$, $\phi\left(y_{mi} = y_i\right)$ and $P\left(y_{mi} > y_i\right)$ are like (8), (11), (21) and (20) respectively.

<u>*Case 3*</u>

For the measurement of technical efficiency of nitrogen we estimate the conditional expectation of the technical efficiency for the nitrogen,

$$E\left(TE_{Ni} \left| y_i\right.\right) = \frac{\int_0^1 TE_N \, f\left(TE_N, y_i\right) dTE_N}{h\left(y_i\right)} \tag{45}$$

where $h\left(y\right)$ is the value of likelihood function got evaluated at estimates and $f\left(TE_N, y_i\right)$ is the joint density of the technical efficiency and the crop yield and has the following expression

$$f\left(TE_{Ni}, y_i\right) = \varphi\left(y_{Pi} = y_i\right)P\left(TE_{Ni}, y_{Ni} > y_i\right)P\left(y_{mi} > y_i\right) + \tag{46}$$
$$\phi\left(TE_{Ni}, y_{Ni} = y_i\right)P(y_{Pi} > y_i)P\left(y_{mi} > y_i\right) + P\left(y_{Pi} > y_i\right)P\left(TE_{Ni}, y_{Ni} > y_i\right)\phi\left(y_{mi} = y_i\right)$$

The forms of the joint density $\phi\left(TE_{Ni}, y_{Ni} = y_i\right)$ and $P\left(TE_{Ni}, y_{Ni} > y_i\right)$ are like (41) and (42) respectively, and the forms of the $\varphi\left(y_{Pi} = y_i\right)$, $P\left(y_{Pi} > y_i\right)$, $\phi\left(y_{mi} = y_i\right)$ and $P\left(y_{mi} > y_i\right)$ are like (7), (10), (21) and (20) respectively.

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