# On The Geometry Of Quasi-Fuchsian Space Of A Hyperbolic Surface

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# Thesis

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# Contents

# Introduction

To a closed Riemann surface $S$ of genus $g > 1$, we associate the following spaces of structures of $S$ :

-Quasifuchsian space of $S$, the space of quasi-Fuchsian structures of $S$ and

-Teichmüller space of $S$, the space of conformal structures on $S$.

These spaces have rich analytic and geometric characteristics and our aim is to study the connection between them.

Quasifuchsian space $QF(S)$, can be naturally imbedded into a complex number space and is a complex manifold [B4].

Teichmüller space can be imbedded into a complex Banach space and in this way becomes a complex manifold [B2]. This manifold is denoted by $Teich(S)$, and it can be viewed as a complex submanifold of $QF(S)$.

On the other hand, Teichmüller space is in a natural way a real analytic manifold [Kr], denoted by $\widetilde{F}(S)$, which is isomorphic to a real analytic submanifold of $QF(S)$.

It is known that $Teich(S)$ is a Kählerian manifold, its Kählerian metric induced by the Weil-Petersson hermitian form and $\widetilde{F}(S)$ is a real symplectic manifold [W1,2].

In this thesis:

a) We prove that $QF(S)$ is a complex symplectic manifold, by constructing a complex symplectic structure which is natural from the point of view of hyperbolic geometry and can be seen as the complexification of the real symplectic structure of Teichmüller space.

b) We define a new complex structure on $QF(S)$, with respect to which, $\widetilde{F}(S)$ is a complex submanifold of $QF(S)$.

c) We define the Weil-Petersson hermitian form on $QF(S)$. We prove that the two complex structures for $QF(S)$, together with the metric induced by the Weil-Petersson form, give rise to a Hyperkählerian structure on $QF(S)$ which unifies all the above structures and provides a new perspective for their study.

### Previous Results

We fix a closed (that is compact and without boundary) Riemann surface $S$ of genus $g > 1$, and identify its fundamental group $\pi_1(S)$ with the Fuchsian group $\Gamma$ acting on the upper half plane $\mathbb{U} = \{z \in \mathbb{C}, Imz > 0\}$ which is such that $S = \mathbb{U}/\Gamma$. We denote by $\overline{S}$ the surface with the opposite orientation.

**Teichmüller space.** Teichmüller space of $S$, is the set of marked Riemann surfaces $[X]$, where $X$ is a Riemann surface homeomorphic to $S$ by a sense preserving homeomorphism, and a marking on $X$ is a choice of isomorphism of the fundamental groups $\Gamma$ of $S$ and $\Gamma_X$ of $X$.

Teichmüller space appears already in an implicit way in works of Klein and Poincaré. Fricke, Fenchel and Nielsen were among the first who studied its real analytic structure. Real analytic coordinates are provided locally by the hyperbolic lengths $l_i$, $i = 1, ..., 6g-6$ of certain simple closed geodesics on $S$. In this way, Teichmüller space becomes a real analytic $6g-6$ manifold denoted by $\widetilde{F}(S)$. [W1]

In the 1950's, L. Ahlfors and L. Bers developed the theory of quasiconformal mappings, which has proven a powerful tool for the study of Teichmüller space.

Complex structure for this space was firstly defined by O. Teichmüller, but this structure was not the "right" one.

Ahlfors defined a complex structure for Teichmüller space and L. Bers showed a little later that this structure is natural: Teichmüller space can be naturally imbedded into an open set of a $3g-3$ complex Banach space, and thus inherit the structure of a $3g-3$ complex Banach manifold [B2]. This manifold is denoted by $Teich(S)$.

The group of biholomorphisms of $Teich(S)$ is discrete [R], and is called the modular group $Mod(S)$. This group identifies two points $[X], [Y]$ of $Teich(S)$ when there is a conformal mapping from $X$ to $Y$.

A natural hermitian form for $Teich(S)$ was originally introduced by Weil and Petersson. This form is obtained by a hermitian product which, for every point $[X]$, $X = \mathbb{U}/\Gamma_X$, is defined in the complex Banach space of quadratic differentials $Q(\Gamma_X, \mathbb{U})$. This is the space of integrable holomorphic functions $\phi$ defined on $\mathbb{U}$ which satisfy the condition $\phi(\gamma(z))(\gamma'(z))^2 = \phi(z)$ for all $\gamma \in \Gamma_X$ and for all $z \in \mathbb{U}$. For any such $\phi, \psi$ the Weil-Petersson (W-P) hermitian product $h_{WP}$ is given by

$$h_{WP}(\phi, \psi) = \int_X \phi\overline{\psi}.$$

$Q(\Gamma_X, \mathbb{U})$ can be identified to the holomorphic cotangent space of $Teich(S)$ at $[X]$. Ahlfors proved that the Riemanian metric $g_{WP}$ which is induced from

$h_{WP}$ is Kählerian. In 1974, H. Royden proved that its group of isometries is $Mod(S)$ [R], and in 1976, Scott Wolpert showed that $g_{WP}$ is incomplete, (see [W2] and the reference there).

The real form $\omega_{WP}$ induced by the Kählerian metric in $Teich(S)$, is a real symplectic form for $\widetilde{F}(S)$, and thus $(\widetilde{F}(S), \omega_{WP})$ is a symplectic manifold.

In the 1980's, Wolpert studied extensively the W-P symplectic geometry of Teichmüller space. Wolpert described the symplectic form $\omega_{WP}$ in terms of real analytic coordinates $l_i$, and Fenchel-Nielsen (F-N) or twist vector fields [W1,4]. A twist vector field $t_\alpha$ associated to a simple closed geodesic $\alpha$ of $S$, is by definition the infinitesimal generator of a 1-parameter group of diffeomorphisms of Teichmüller space, where such a diffeomorphism is obtained from the following deformation of $S$ : Cut $S$ along $\alpha$, rotate the one part of the surface by an anle $\theta$, and then glue the two pieces to obtain a new Riemann surface. This deformation is known as the F-N or twist deformation of $S$. Wolpert's formula for $\omega_{WP}$ is

$$\omega_{WP}(t_\alpha, t_\beta) = \sum_{p\in\alpha\cap\beta} \cos\phi(\alpha,\beta)_p$$

where $t_\alpha, t_\beta$ are twist vectors corresponding to $\alpha, \beta$ and $\phi(\alpha,\beta)$ is the angle of geodesics $\alpha, \beta$.

Teichmüller space $\widetilde{F}(S)$ admits global (F-N) real analytic coordinates $l_i, \tau_i$, $i = 1, ..., 3g-3$, where $l_i$ are hyperbolic length functions corresponding to $3g-3$ simple closed geodesics $\gamma_i$ which form a maximal partition of $S$, and $\tau_i$ the associated twist functions. According to [W2], these coordinates are canonical for the symplectic manifold $(\widetilde{F}(S), \omega_{WP})$ :

$$\omega_{WP} = \sum_{i=1}^{3g-3} dl_i \wedge d\tau_i.$$

**Quasifuchsian space.** A quasi-Fuchsian deformation of $\Gamma = \pi_1(S)$ is a homomorphism $\rho$ of $\Gamma$ into $\mathbb{PSL}(2,\mathbb{C})$ which is obtained by conjugation by a quasiconformal mapping of the complex plane $\mathbb{C}$. Images $\rho(\Gamma)$ of quasi-Fuchsian deformations are quasi-Fuchsian groups. If $G$ is such a group, then its limit set is a Jordan curve and it acts properly discontinuously on the complement $D_G = D_1 \cup D_2$ of this curve in $\mathbb{C}$. The action of $G$ on $D_1, D_2$ induces two Riemann surfaces $X = D_1/G$ and $Y = D_2/G$, which are homeomorphic by an orientation reversing homeomorphism. $G$ also acts on the upper half space $\mathbb{U}^3$, and the quotient $(\mathbb{U}^3 \cup D_G)/G$ is a 3-manifold (a quasi-Fuchsian manifold), diffeomorphic to $S \times [0,1]$.

Quasifuchsian space $QF(S)$ of $S$, is the quotient of the set of quasi-Fuchsian deformations with the action of $\mathbb{PSL}(2,\mathbb{C})$ by inner automorphisms. It can be also seen as the set of marked quasi-Fuchsian manifolds $[M]$, where a marking on $M$ is a choice of isomorphism between $\Gamma$ and $\pi_1(M)$.

$QF(S)$ has a natural complex structure as a space of representations into a complex Lie group. Topologically it is a ball of dimension $12g-12$, and L. Bers proved that it is a $6g-6$ complex manifold [B4].

Let $[\rho]$ be a point of $\in QF(S)$, $\rho(\Gamma)=G$. Recall that two Riemann surfaces $X, Y$ are associated to $G$. To the point $[\rho]$ we correspond the pair of marked Riemann surfaces $([X],[Y]) \in Teich(S) \times Teich(\overline{S})$.

This correspondence is bijective and compatible with complex structures: The mapping

$$\Psi : QF(S) \to Teich(S) \times Teich(\overline{S})$$

sending $[\rho]$ to $([X],[Y])$ is biholomorphic.

Complex submanifolds of the form $B_Y(S) = \Psi^{-1}(Teich(S) \times \{[Y]\})$ and $B_X(\overline{S}) = \Psi^{-1}(\{[X]\} \times Teich(\overline{S}))$, are biholomorphic to $Teich(S)$ and $Teich(\overline{S})$ respectively. These submanifolds are called the Bers' slices of $QF(S)$.

Another subset of Quasifuchsian space of particular interest, is the Fuchsian space $F(S)$ of $S$. $F(S)$ contains points $[\rho]$, with $\rho(\Gamma)$ Fuchsian, and thus the corresponding Riemann surfaces $X, Y$ satisfy the condition: $Y=\overline{X}$. It is a real analytic $6g-6$ submanifold of $QF(S)$, and can be identified real analytically with the Teichmüller space $\widetilde{F}(S)$.(See [Kr]). Denote by $I_T$ the almost complex operator acting on the tangent space of $Teich(S)$. It also acts on the tangent space of $\widetilde{F}(S)$, and in this manner $\widetilde{F}(S)$ can be considered as an *almost* complex manifold. From the identification of $F(S)$ with $\widetilde{F}(S)$, we may give $F(S)$ an almost complex manifold structure, but its natural imbedding into $QF(S)$ is not holomorphic.

C. Kourouniotis defined holomorphic coordinates for $QF(S)$ [K2]. These coordinates are given locally at each point by the complex lengths $\lambda_i$, $i = 1, ..., 6g-6$ of certain geodesics of the corresponding quasi-Fuchsian manifold. If the point is Fuchsian, then complex coordinates are reduced to real analutic coordinates of Teichmüller space $\widetilde{F}(S)$.

Kourouniotis defined the bending vector fields on Quasifuchsian space. To any simple closed curve $\alpha$ of $S$, there is associated a holomorphic vector field $T_\alpha$, which by definition is the infinitesimal generator of a 1-complex parameter group of biholomorphisms of $QF(S)$, where such a biholomorphism is obtained at each point $[\rho]$, by bending the corresponding 3-manifold along the geodesic $\rho(\alpha)$. On the tangent subbundle of Fuchsian space, the real part of $T_\alpha$ is just the twist field $t_\alpha$.

**A heuristic picture.** Bers' mapping $\Psi$, gives us the following heuristic picture for $QF(S)$ and its submanifolds:

One may think of Quasifuchsian space as an open connected subset $A$ of the complex space $\mathbb{C}^{6g-6}$. Then she may draw Bers' slices $B_S(\overline{S})$ and $B_{\overline{S}}(S)$ as two $3g-3$ complex axes inside $A$ which meet at the point "zero", which is $\Psi^{-1}([S]\times[\overline{S}])$. For convenience she may assume that these axes are perpendicular at 0, but since there is no metric defined, the word "perpendicular" is for the moment meaningless. In this figure, Fuchsian space $F(S)$ is the segment of the $6g-6$ real diagonal of $\mathbb{C}^{6g-6}$ contained in $A$. She can also think of $A$ as a subset of $\mathbb{C}^{6g-6}$, but the latter endowed with the three quatenionic complex structures $I, J, K$ where $I^2 = J^2 = K^2 = IJK = -id$. If $I$ corresponds to the natural complex structure, then the diagonal is a real submanifold of $(A, I)$ but a complex submanifold of $(A, J)$.

Then, the following questions may come to her mind:

a) Which must be the appropriate metric that $QF(S)$ must be endowed with, so that Bers' slices be perpendicular at 0?

b) Which must be the appropriate complex structure, analogue to $J$, so that $F(S)$ is a complex submanifold of $QF(S)$ with respect to this structure?

Mainly, our work answers to these questions.

## Exposition Of Results

The first part of our work is devoted to the study of the complex symplectic geometry of $QF(S)$ which arises from the hyperbolic geometry of quasi-Fuchsian manifolds.

**Complex symplectic manifolds.** A $2n-$complex dimensional complex manifold $M$ is called complex symplectic, if there exists a non-degenerate, closed, $(2,0)-$holomorphic form $\Omega$ defined everywhere on $M$.

The first and the second variation of the complex length of a geodesic under bending admit a neat geometrical description [K3]. We use this description to define a holomorphic $(2,0)$ form for $QF(S)$. We prove (Theorem 3.1.3)

Theorem. *There exists a non degenerate closed holomorphic $(2,0)$ form $\Omega$ defined everywhere on $QF(S)$ turning $QF(S)$ into a complex symplectic manifold. The form $\Omega$ is given at each $[\rho] \in QF(S)$ by the formula:*

$$\Omega_{([\rho])}(T_\alpha, T_\beta) = \sum_{p\in\alpha\cap\beta} \cosh\sigma(\rho(\alpha), \rho(\beta))_p$$

*where $T_\alpha, T_\beta$ are bending vectors corresponding to simple closed geodesics $\alpha, \beta$ on $S$, and $\sigma(\rho(\alpha), \rho(\beta))$ is the complex distance of geodesics $\rho(\alpha), \rho(\beta)$.*

The form $\Omega$ can be regarded as the complexification of the Weil-Petersson real symplectic form of $\widetilde{F}(S)$.

Quasifuchsian space $QF(S)$ admits global complex (complex F-N) coordinates $\lambda_i, \beta_i$, $i = 1, ..., 3g-3$, [K2], where $\lambda_i$ are complex length functions corresponding to the geodesics of the partition of $S$ and $\beta_i$ are the associated bending functions. We prove (Theorem 3.1.10)

THEOREM. *Complex Fenchel-Nielsen coordinates are canonical for the complex symplectic manifold* $(QF(S), \Omega)$ :

$$\Omega = \sum_{i=1}^{3g-3} d\lambda_i \wedge d\beta_i .$$

**Hyperkählerian manifolds.** A $4n$-dimensional Riemannian manifold $(M, g)$ is called Hyperkählerian if there exist two complex operators $I, J$ defined on $M$ such that $IJ + JI = 0$ and $g$ is a Kähler metric for both $I$ and $J$. Note that the existence of such $I, J$ implies the existence of a third operator $K$, which is also such that $(M, K, g)$ is a Kählerian manifold.

A Hyperkählerian manifold is automatically a complex symplectic manifold: in the first place it is symplectic with respect to all three symplectic forms $\omega, \omega_1, \omega_2$ induced by the three operators $I, J, K$ repectively. Considering $M$ as an $I$-complex manifold then the $I$ -holomorphic $(2,0)$-form

$$\Omega = \omega_1 + i\omega_2$$

defines a complex symplectic structure for $(M, I)$. The converse is known to be true when $M$ is compact.

Having already defined a complex symplectic structure for $QF(S)$ it is natural to ask whether there exists a Hyperkählerian structure, from which this complex symplectic structure is obtained. Since our form $\Omega$ is actually the complexification of the W-P symplectic structure $\omega_{WP}$, we are led to W-P geometry, which we study in the second part of our work.

Firstly, and in analogy with Teichmüller space, we define the W-P hermitian product at each point $[\rho]$ of $QF(S)$. The holomorphic cotangent space of $QF(S)$ at $[\rho]$ is the complex Banach space of quadratic differentials $Q(G)$, $G = \rho(\Gamma)$. This is the space of integrable holomorphic functions $\phi$ defined on $D_G$, the region of discontinuity of $G$, which satisfy the condition $\phi(g(z))(g'(z))^2 = \phi(z)$ for all $g \in G$ and for all $z \in D_G$. For any such $\phi, \psi$ the W-P hermitian product $h_{WP}$ is given by

$$h^Q(\phi, \psi) = \int_{X \cup Y} \phi\overline{\psi}.$$

From this product, we obtain the W-P Riemannian metric $g^Q$ defined on Quasifuchsian space. Our following result establishes the W-P Kählerian structure for Quasifuchsian space: (Theorem 3.2.2)

THEOREM. *Let $\Psi$ be the biholomorphic Bers' mapping from $QF(S)$ onto $\widehat{T}(S)$. The following relation holds:*

$$g^Q = (\Psi^*) g^{\widehat{T}}$$

*The triple $(QF(S), I_Q, g^Q)$ defines a Kählerian manifold. The metric is incomplete and the full group of biholomorphic isometries is the modular group $Mod_Q(S)$.*

$I_Q$ is the almost complex operator in the tangent space of $QF(S)$, from which its standard complex structure arises. The group $Mod_Q(S)$ is a discrete subgroup of biholomorphisms of $QF(S)$ isomorphic to the cartesian product $Mod(S) \times Mod(\overline{S})$.

We construct a new almost complex operator $J_Q$ in the tangent space of $QF(S)$, with the following properties:

a) $J_Q$ is skew-commuting with $I_Q$ and

b) its restriction to the tangent subbundle of $F(S)$ is just the almost complex operator $I_T$ of Teichmüller space.

We show that the almost complex operators $J_Q, K_Q = I_Q J_Q$ are complex and parallel with respect to $g^Q$. In this manner we obtain our main result: (Theorems 3.2.5 and 3.2.8)

THEOREM. *The space $QF(S)$ with complex structures $I_Q, J_Q$, and the Weil-Petersson riemannian metric $g^Q$ is a Hyperkählerian manifold. Teichmüller space $F(S)$ is a complex submanifold of $(QF(S), J_Q)$.*

Results concerning the symplectic geometry of Quasifuchsian space follow from our main theorem. Denote by $\omega^Q, \omega_1^Q, \omega_2^Q$ the three Kählerian real symplectic forms corresponding to $I_Q, J_Q, K_Q$ respectively. The following describes the symplectic behaviour of Teichmüller space inside $QF(S)$ : (Theorem 3.2.7)

THEOREM. *i) Teichmüller space as the set of Fuchsian deformations $F(S)$ is a Lagrangian submanifold of $QF(S)$ with respect to $\omega^Q$ and $\omega_2^Q$ and a symplectic submanifold of $QF(S)$ with respect to $\omega_1^Q$. ii) Bers' slices are symplectic submanifolds of $QF(S)$ with respect to $\omega^Q$ and Lagrangian with respect to $\omega_1^Q$ and $\omega_2^Q$.*

The complex symplectic form induced from the Hyperkählerian structure of $QF(S)$ is

$$\Omega^Q = \omega_1^Q + i\omega_2^Q$$

It is natural to ask about the relation of $\Omega^Q$ and $\Omega$. We prove: (Theorem 3.3.1)

THEOREM. *The complex symplectic form $\Omega^Q$ induced from $g^Q$ is equal to* $2\Omega$.

REMARK. Quasifuchsian space is an open subset of the space $V(S)$, the quotient of the space of irreducible representations of the fundamental group of a surface $S$ into $\mathbb{PSL}(2,\mathbb{C})$ by the action of $\mathbb{PSL}(2,\mathbb{C})$ by inner automorphisms. Spaces of the form $Hom^{irr}(\pi_1(S) \to G)/\sim$, where $G$ is a semi-simple Lie group with an invariant inner product on its Lie algebra, are known to be symplectic if the Lie group $G$ is real and complex symplectic if $G$ is complex, by a construction of W.Goldman [G1,2]. Using the same methods, N. Hitchin proved that in the case where $G$ is complex, then these spaces admit a Hyperkählerian structure [H]. Our constructions and proofs follow an entirely different route. They are based on the theory of quasi-conformal mappings and hyperbolic geometry, and thus reveal the analytical and geometrical perspective of the subject.

This work is divided in chapters.

Chapter I is preliminary, and is divided in three sections.

In section 1.1 we revise briefly Möbius transformations and Kleinian groups. A short discussion about quasiconformal mappings lies in 1.2.1 and Ahlfors-Bers existence theorem is stated in 1.2.2.

In section 1.2, we are concerned with Riemann surfaces, and their connection with quasiconformal mappings (subsections 1.3.1 and 1.3.2). In 1.3.3 we describe Bers' Simultaneous Uniformisation Theorem.

Chapter II is divided in two sections.

Section 2.1 reviews topics on Teichmüller theory.

In subection 2.1.1 we give the descriptions of Teichmüller space which we use for the rest of our work. A quite detailed discussion about the complex structure and the Kählerian structure induced by the W-P hermitian product is presented here. (Subsections 2.1.2 and 2.1.3). In subsection 2.1.4, we state some of the results concerning the F-N real analytic coordinate description and the symplectic geometry of Teichmüller space.

In section 2.2 we introduce the Quasifuchsian space $QF(S)$. Subsection 2.2.1 is devoted to topics on the theory of general deformation spaces of Kleinian groups which we use in our further discussion. In 2.2.2 we restrict ourselves to the Quasifuchsian space, describing explicitely its holomorphic tangent and cotangent spaces at each point and highlighting the idea of introducing $J_Q$. In subsections 2.2.3-2.2.6 we review in brief the notions of complex length, complex distance and bending in Quasifuchsian space. In 2.2.7 we are concerned with the holomorphic nature of bending vector fields.

Our main work lies on Chapter III, which is divided in three sections.

In section 3.1 we construct the complex symplectic form $\Omega$ in $QF(S)$. Our construction yields a duality formula which describes the Hamiltonian nature of bending vector fields. In 3.1.3, by using an analytic continuation argument we prove that complex F-N coordinates for $(QF(S), \Omega)$ are canonical.

In section 3.2 we are concerned with W-P geometry of $QF(S)$. In subsection 3.2.1 we define explicitely the W-P metric in $QF(S)$ and prove that it is Kählerian. The real symplectic structure obtained from the metric is also described there. Subsection 3.2.2 contains the proof of the main theorem in three steps:

In the first step we define the almost complex operators $J_Q, K_Q$ acting on the tangent space of each point.

In the second step, we prove that these almost complex operators are almost hermitian: The W-P metric remains invariant under their action.

In the third step we prove that $J_Q$ (and therefore $K_Q$ ) are integrable. We deduce that $QF(S)$ is a complex manifold for both $J_Q, K_Q$ and further that these operators are parallel with respect to the metric. Consequences of the main theorem follow.

Finally, in section 3.3 we prove that the complex symplectic structure $\Omega$ is the one obtained by the Hyperkählerian W-P metric.

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This work is dedicated to women in my life.

CHAPTER 1

# Kleinian groups and Riemann surfaces

## 1.1. Kleinian groups

Definitions and results presented in subsections 1.1.1 and 1.1.2 are standard in bibliography. Here we follow [B1] and [B2].

**1.1.1. Möbius transformations.** A *Möbius transformation* $g$ is a conformal automorphism of the Riemannian sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. It can be represented as the complex function

$$g(z) = \frac{az+b}{cz+d},\ ad-bc=1,\ a,b,c,d \in \mathbb{C}$$

Denote by $\mathbb{SL}(2,\mathbb{C})$ the group consisting of $2 \times 2$ complex matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with determinant 1. The set of Möbius transformations is a group isomorphic to $\mathbb{PSL}(2,\mathbb{C}) = \mathbb{SL}(2,\mathbb{C})/\{\mathbb{I}, -\mathbb{I}\}$, where $\mathbb{I}$ is the identity matrix. Its subgroup consisting of transformations $g$ with $a,b,c,d \in \mathbb{R}$ is denoted by $\mathbb{PSL}(2,\mathbb{R})$.

Möbius transformations other than the identity can be classified as follows:

i) *Parabolic*: They are conjugate to $g(z) = z+1$

ii) *Elliptic*: They are conjugate to $g(z) = \lambda z,\ \mid \lambda \mid = 1$ and

iii) *Loxodromic*: They are conjugate to $g(z) = \lambda z,\ \mid \lambda \mid \neq 1$. A loxodromic transformation is called *hyperbolic* if $\lambda > 0$.

Denote by $\mathbb{U}$ the Poincaré upper half plane:

$$\mathbb{U} = \{z = x+iy \in \mathbb{C},\ y > 0\}.$$

Every element of $\mathbb{PSL}(2,\mathbb{R})$ maps $\mathbb{U}$ onto itself. $\mathbb{U}$ becomes a model for hyperbolic geometry when equiped with the line element

$$ds = \frac{\mid dz \mid}{y},$$

and thus $\mathbb{PSL}(2,\mathbb{R})$ can be viewed as the group of all conformal self mappings of $\mathbb{U}$ as well as the group of all non-Euclidean motions in the plane (the group of isometries of the Riemannian manifold $(\mathbb{U}, ds^2)$).

As for $\mathbb{PSL}(2,\mathbb{C})$ something similar holds. Denote by $\mathbb{U}^3$ the upper half space: $\mathbb{U}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R}_+\}$.When equiped with the line element

$$ds = \frac{\sqrt{|\ dz\ |^2 + dt^2}}{t}$$

it becomes a model for non-Euclidean 3-space. We identify $(z,t) \in \mathbb{U}^3$ with the quaternion $z + tj = x + iy + tj + 0k$. Then an element $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $\mathbb{PSL}(2,\mathbb{C})$ acts on $\mathbb{U}^3$ by the rule

$$g(z+tj) = [a(z+tj)+b][c(z+tj)+d]^{-1}.$$

$\mathbb{PSL}(2,\mathbb{C})$ is the full group of isometries of $(\mathbb{U}^3, ds^2)$.

**1.1.2. Kleinian and Fuchsian groups.** Let $X$ be a topological space and $G$ a group of homeomorphisms acting on $X$. The action of $G$ is called *properly discontinuous* if for every compact subset $K$ of $X$ the relation $g(K) \cap K \neq \emptyset$ holds for only a finite number of $g \in G$.

*A Kleinian group $G$ is a discrete subgroup of $\mathbb{PSL}(2,\mathbb{C})$ acting properly discontinuously on some open subset of $\widehat{\mathbb{C}}$.*

It can be shown that a discrete subgroup $G$ of $\mathbb{PSL}(2,\mathbb{C})$ always acts properly discontinuously on $\mathbb{U}^3$, and the quotient $\mathbb{U}^3/G$ is a 3-manifold. However, the discretness of $G$ is not sufficient for properly discontinuous action on some open subset of $\widehat{\mathbb{C}}$. An equivalent to the previous definition of a Kleinian group is the following:

*A Kleinian group $G$ is a discrete subgroup of $\mathbb{PSL}(2,\mathbb{C})$ whose limit set $\Lambda = \Lambda(G)$ is not the whole of $\widehat{\mathbb{C}}$.*

The *limit set* $\Lambda$is defined to be the set of accumulation points of the orbits. A *fixed point* of a Möbius transformation is a point $z \in \widehat{\mathbb{C}}$ satisfying the relation $g(z) = z$. It can be shown that

i) If $g$ is parabolic then it has one fixed point
ii) If $g$ is elliptic then it it has infinitely many fixed points in $\widehat{\mathbb{R}^3}$
iii) If $g$ is loxodromic then it has two fixed points in $\widehat{\mathbb{R}^3}$ and
iv) If $g$ is hyperbolic then it has two real fixed points.

We may give now an equivalent to the previous definition of the limit set of a Kleinian group $G$ :

*The limit set $\Lambda$ of a Kleinian group $G$ is the closure of the set of fixed points of non-elliptic elements of $G$.*

The *region of discontiniuity* $\Omega = \Omega(G)$ is the complement of $\Lambda$ in $\widehat{\mathbb{C}}$. It is open and dense in $\widehat{\mathbb{C}}$ and the largest subset of $\widehat{\mathbb{C}}$ on which the action of $G$ is properly discontinuous. The connected components of $\Omega$ are called *components* of $\Omega$. A Kleinian group is called *elementary* if its limit set is finite. We shall not be concerned with these groups. The rest of our discussion shall be about non-elementary Kleinian groups.

*A Fuchsian group $\Gamma$ is a Kleinian group where all loxodromic elements are hyperbolic.*

Its action leaves a disc or a plane fixed. By conjugation in $\mathbb{PSL}(2,\mathbb{C})$ we can always assume that a Fuchsian group leaves fixed the upper half plane $\mathbb{U}$. In this way its limit set can be

i) the extended real line $\widehat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and then $G$ is called of the *first kind,* or
ii) a nowhere dense subset of $\widehat{\mathbb{R}}$ and then $G$ is called of the *second kind.*

Fuchsian groups are the simplest kind of Kleinian groups. The next simple case of Kleinian groups is that of *quasi-Fuchsian* groups, which we shall define in 1.2.3.

## 1.2. Quasiconformal mappings

Intuitively, a quasiconformal mapping of the complex plane is a mapping which in the tangent plane, takes infinitesimal geometric circles into infinitesimal ellipses, with a global bound on the eccentricity. For the exact definition and its variations given in 1.2.1 we follow [E1].

**1.2.1. Quasiconformal mappings of the plane.** Let $D$ and $D'$ be domains in the complex plane $\mathbb{C}$ and $w : D \to D'$ a sense preserving homeomorphism. For each $z \in D$ we consider the numbers

$$L(z,r) = \max\{\mid w(\zeta) - w(z) \mid , \mid \zeta - z \mid = r\}$$

$$l(z,r) = \min\{\mid w(\zeta) - w(z) \mid , \mid \zeta - z \mid = r\}$$

and

$$H(z) = \limsup_{r \to 0} \frac{L(z,r)}{l(z,r)}$$

We say that $w$ is *quasiconformal* (qc from now on) if $H$ is a bounded function in $D$. The qc mapping $w$ is $K-$ *quasiconformal* ($K-$ qc): there exists a finite number $K$ such that $H(z) \le K$ for almost all $z$ in $D$.

Suppose now that $w : D \to D'$ is a sense preserving $C^1$ diffeomorphism. The complex derivatives $w_z = \frac{\partial w}{\partial z}$ and $w_{\overline{z}} = \frac{\partial w}{\partial \overline{z}}$ are given by

$$w_z = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right)$$

$$w_{\overline{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right)$$

Since $w$ is sense preserving, the Jacobian $J_w = | w_z |^2 - | w_{\overline{z}} |^2$ is strictly positive and we can check that

$$L(z,r) = r(| w_z(z) | + | w_{\overline{z}}(z) |) + o(r),$$

$$l(z,r) = r(| w_z(z) | - | w_{\overline{z}}(z) |) + o(r),$$

$$H(z) = \frac{| w_z(z) | + | w_{\overline{z}}(z) |}{| w_z(z) | - | w_{\overline{z}}(z) |}.$$

We obtain then that the function $w$ is $K-$ qc if and only if for all $z$ in $D$ the following relation holds:

$$| w_{\overline{z}}(z) | \leq \frac{K-1}{K+1} | w_z(z) | . \quad (*)$$

The above is a criterion for quasiconformality and it can be extended in the general case.

A function $w$ has distributional derivatives $w_z, w_{\overline{z}}$ in $D$ if the functions $w_z, w_{\overline{z}}$ are in $L^2_{loc}(D)$ and satisfy

$$\int\int(\phi w_z + w\phi_z)dxdy = \int\int(\phi w_{\overline{z}} + w\phi_{\overline{z}})dxdy = 0$$

for every smooth function $\phi$ with compact support in $D$. The definition for qc mappings can be proven equivalent to the following:

*A sense preserving homeomorphism $w : D \to D'$ is $K-$ qc if $w$ has distributional derivatives in $D$ satisfying $(*)$ a.e.*

We list some properties of qc maps. If $w : D \to D'$ is $K-$ qc in $D$ then:

i) $w$ is differentiable a.e.
ii)$| w_z(z) | > 0$ a.e.
iii) $m(w(E)) = \int\int_E J_w dxdy$ for all (Lebesgue) measurable sets $E \subset D$.
iv) If $K = 1$ then $w$ is conformal.
v) $w^{-1} : D' \to D$ is $K-$ qc.
vi) If $u : D' \to D''$ is $K'-$ qc, then $u \circ w$ is $KK'-$ qc.

**1.2.2. Beltrami equations and Ahlfors-Bers theorem.** Let $w : D \to D'$ be a qc mapping. Then it solves in $D$ the Beltrami equation

$$w_{\overline{z}} = \mu w_z$$

where $\mu = \mu(z)$ is a measurable function on $D$ ( the *complex dilatation* of $w$) belonging to the open unit ball of $L^\infty(D)$. This set is called the set of *Beltrami differentials* of $D$, and is denoted by $Belt(D)$. If a Beltrami differential $\mu$ is given, then there exists a qc mapping in $D$ solving the Beltrami equation. In fact the following theorem due to L. Ahlfors and L. Bers [AB] holds:

THEOREM. *Let $\mu \in Belt(\mathbb{C})$ a Beltrami differential. There exists a unique solution $w^\mu : \mathbb{C} \to \mathbb{C}$ to the Beltrami equation $w_{\overline{z}} = \mu w_z$ which is qc in $\mathbb{C}$ and leaves $0, 1, \infty$ fixed. Every other qc solution is of the form $A \circ w^\mu$ where $A$ is a Möbius transformation.*

*If $\mu$ depends holomorphically as an element of the complex Banach space $L^\infty(\mathbb{C})$ on complex parameters, so does $w^\mu$ : For sufficiently small $\varepsilon$ and every $z$ we have*

$$w^{\varepsilon\mu}(z) = z + \varepsilon w'[\mu](z) + o(\varepsilon),\ \varepsilon \to 0$$

*$o(\varepsilon)$ uniform on compact subsets of $\mathbb{C}$ and*

$$w'[\mu](z) = \frac{\partial w^{\varepsilon\mu}(z)}{\partial \varepsilon}\,|_{\varepsilon=0} = -\frac{1}{\pi}\int\int_{\mathbb{C}} \mu(\zeta) R(z,\zeta)\, dxdy$$

*where*

$$R(z,\zeta) = \frac{z(z-1)}{\zeta(\zeta-1)(\zeta-z)}.$$

If now $\mu \in Belt(D)$ where $D$ is any subdomain of $\mathbb{C}$, then we consider $Belt(D)$ as the set of elements of $Belt(\mathbb{C})$ which vanish everywhere outside $D$. Then $w^\mu$ is a qc self mapping of $\mathbb{C}$ which is conformal outside $D$.

Let $D = \mathbb{U}$ the upper half plane, $\mu \in Belt(\mathbb{U})$. Associated to $\mu$ there exists unique qc self-mapping of $\mathbb{U}$ fixing $0, 1, \infty$, which is obtained as follows: Let $\widetilde{\mu} \in Belt(\mathbb{C})$ defined by

$$\widetilde{\mu}(z) = \begin{cases} \mu(z) & z \in \mathbb{U} \\ \overline{\mu(\overline{z})} & z \in \mathbb{L} \end{cases}$$

Then the mapping in question is $f^\mu = w^{\widetilde{\mu}}$ when the latter is restricted in $\mathbb{U}$.

The Beltrami differentials which satisfy the condition

$$\mu(z) = \overline{\mu(\overline{z})}$$

for every $z \in \mathbb{C}$ are called *symmetric* and their space is denoted by $Belt_S(\mathbb{C})$.

**1.2.3. Beltrami differentials for Kleinian groups.** For the following we refer to [E1].

Let $G$ be a Kleinian group with region of discontinuity $\Omega$ and limit set $\Lambda$. Let $w^\mu$ be any qc map of the extended plane. The group

$$G^\mu = w^\mu \circ G \circ (w^\mu)^{-1} = \{w^\mu \circ g \circ (w^\mu)^{-1};\, g \in G\}$$

is again a group of homeomorphisms acting properly discontinuously on $w^\mu(\Omega)$. $G^\mu$ is Kleinian if it is a group of conformal maps. That is, if $w^\mu \circ g \circ (w^\mu)^{-1}$ is conformal for all $g \in G$. A simple computation shows that this is equivalent to the relation

$$\mu(g(z))\overline{g'(z)}/g'(z) = \mu(z)$$

for all $g \in G$.

*The space of $G-$ invariant complex differentials $L^\infty(G)$ is the complex Banach space consisting of measured complex essentially bounded functions $\mu(z)$ defined on $\mathbb{C}$ with support in $\Omega$, satisfying the transformation law*

$$\mu(g(z))\overline{g'(z)}/g'(z) = \mu(z)$$

*for all $g \in G$ and for all $z \in \mathbb{C}$ and $\mu_{|_\Lambda} = 0$. The open unit ball*

$$\{\mu \in L^\infty(G) : \|\mu\|_\infty < 1\}$$

*endowed with the supremum norm is the space of Beltrami differentials for $G$ and shall be denoted by $Belt(G)$.*

For each Beltrami differential $\mu \in Belt(G)$ the group $G^\mu$ is again Kleinian and according to Ahlfors-Bers theorem its elements depend holomorphically on the parameter $\mu$.

*The groups $G^\mu$ are called quasiconformal deformations of $G$.*

Suppose now that $\Gamma$ is a Fuchsian group and denote by $Belt(\Gamma, \mathbb{U})$(resp. $Belt(\Gamma)$) its set of Beltrami differentials when $\Gamma$ is considered acting on $\mathbb{U}$ (resp. on $\mathbb{C} - \mathbb{R}$). Denote also by $Belt_S(\Gamma)$ the subset of $Belt(\Gamma)$ consisting of symmetric Beltrami differentials. If $\mu \in Belt(\Gamma, \mathbb{U})$, then consider $f^\mu$ as in 1.2.2. The group $\Gamma^\mu = f^\mu \circ \Gamma \circ (f^\mu)^{-1}$ is a Fuchsian group acting on $\mathbb{U}$. Note that this group is the same with $\Gamma^{\widetilde{\mu}} = w^{\widetilde{\mu}} \circ \Gamma \circ (w^{\widetilde{\mu}})^{-1}$, $\Gamma^{\widetilde{\mu}}$ acts on the complex plane.

Fuchsian groups obtained in this manner are called *Fuhsian or real deformations* of $\Gamma$. Note that since $\widetilde{\mu}$ depends real analytically on $\mu$, so do the elements of $\Gamma^{\widetilde{\mu}}$.

**Quasi-Fuchsian groups.** Starting from a Fuchsian group $\Gamma$, and for arbitrary $\mu \in Belt(\Gamma)$, the Kleinian group $\Gamma^\mu$ is called *quasi-Fuchsian*. It is Fuchsian if and only if $\mu \in Belt_S(\Gamma)$. Constructing a quasi-Fuchsian group in this way, we lead ourselves easily to the proof of Bers' simultaneous uniformisation theorem (see 1.3.3 below).

## 1.3. Riemann surfaces

In this section we review topics on Riemann surfaces which are useful for our further discussion. The reference is mainly to [E2] and to the book of O. Lehto [L], chpts 4 and 5.

**1.3.1. Riemann surfaces and differentials.** Let $S$ be an oriented smooth surface in $\mathbb{R}^3$. For any such surface the line element is of the form

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2$$

where $E, G, F$ are the classical Gaussian quantities expressed in terms of the coefficient functions of the inverse of a local parameter $f = (f_1, f_2, f_3)$ and $EG - F^2 > 0$. Using the notation $dz = dx + idy$, $d\overline{z} = dx - idy$ we can always write the expression for the line element in the complex form

$$ds = \lambda(z) \mid dz + \mu(z) d\overline{z} \mid . \quad (**)$$

Here $\lambda$ is a positive function and $\mu$ is a complex function with $|\mu(z)| < 1$.

Two such metrics $ds_1, ds_2$ of $S$ are called *conformally equivalent* if they are proportional at every point of $S$, that is, the identity map of $S$ is conformal with respect to these metrics. A *conformal structure* on $S$ is a conformal equivalece class of metrics.

*A Riemann surface is an oriented smooth surface $S$ with a given conformal equivalence class of metrics.*

This definition is equivalent with the following:

*A Riemann surface $S$ is an 1-dimensional connected complex analytic manifold.*

Indeed, if $S$ is oriented and has a given conformal structure, then the sense preserving conformal mappings from open sets of $S$ into the complex Euclidean plane form a complex analytic atlas for $S$. Conversely, every connected one dimensional complex manifold has a natural orientation, and has a Riemannian metric such that the complex coordinate functions on $S$ are conformal mappings into $\mathbb{C}$.

Let $S$ be a Riemann surface with a complex analytic atlas ( i.e conformal structure) $\{U_i, z_i\}$. Suppose that $f : S \to \mathbb{C}$ is a holomorphic function and the local parameters $z_i, z_j$ have overlapping domains. Writing $f_i = f \circ z_i^{-1}, f_j = f \circ z_j^{-1}$ and regarding $z_i, z_j$ as complex variables, then by differentiating the relation $f_i(z_i) = f_j(z_j)$ we have

$$f_i' dz_i = f_j' dz_j$$

This relation defines locally the (invariant) differential of $f$. In general

*A $(m,n)$ differential on a Riemann surface $S$ is a collection $\varphi$ of functions $\varphi_i : U_i \to \mathbb{C}$ satisfying*

$$\varphi_i (\frac{dz_i}{dz_j})^m (\frac{d\overline{z_i}}{d\overline{z_j}})^n = \varphi_j$$

*in $U_i \cap U_j$.*

It is clear that $(m,n)$ differentials are complex tensors for $S$. The most important kinds of differentials are:

*The set of Beltrami differentials $Belt(S)$ is the set consisting of Lebesgue measurable $(-1,1)$ differentials $\mu$ on $S$ satisfying $\| \mu \|_\infty < 1$.*

*The set of quadratic differentials $Q(S)$ consists of the holomorphic $(2,0)$ differentials on $S$.*

We note that Beltrami differentials can be integrated with respect to the Lebesgue measure, the absolute value of a quadratic differential is a $(1,1)$ differential and the tensorial product of a Beltrami differential and a quadratic differential is a $(1,1)$ differential that is an area element for $S$.

**1.3.2. Riemann surfaces and qc mappings.** We give below the definition of a quasiconformal mapping between Riemann surfaces.[L] p.176.

*A homeomorphism $f : S_1 \to S_2$ between two Riemann surfaces is called $K-$ qc if for any local parameters $z_i$ of an atlas on $S_i$, $i = 1,2$, the mapping $z_2 \circ f \circ z_1^{-1}$ is $K-$ qc in the set where it is defined. The mapping $f$ is qc if it is $K-$ qc for some finite $K \geq 1$.*

We call two Riemann surfaces $S_1, S_2$ *quasiconformally equivalent* if there exixts a qc mapping $f : S_1 \to S_2$. Such a mapping defines an element $\mu \in Belt(S_1)$. If $\mu = 0$ then $f$ is conformal (i.e holomorphic) and $S_1, S_2$ are called *conformally equivalent.*

**Riemann surfaces and Fuchsian groups-Uniformisation.** The following theorem due to Ahlfors classifies Riemann surfaces. It is also known as Riemann's mapping theorem for Riemann surfaces. We refer for a slightly modified version to [L] pp.143-145.

THEOREM. *(Ahlfors) Let $S$ be a Riemann surface. Then it is conformally equivalent to one of the following i) $\mathbb{C}$ ii) $\mathbb{C} - \{0\}$ iii) $\widehat{\mathbb{C}}$ iv) $\mathbb{C}/L$ where $L$ is a lattice and v) $\mathbb{U}/\Gamma$ where $\Gamma$ is a Fuchsian group.*

The above theorem answers to the problem of *uniformising* Riemann surfaces i.e to parametrise them by single-valued holomorphic or meromorphic functions.

We list below some results which are standard in the theory. Their proofs can be found in [L] pp.177-181.

**Lifting of qc mappings.** Qc mappings and Beltrami differentials of Riemann surfaces can be lifted to qc mappings of the corresponding covering surfaces and covering groups respectively.

*Let* $S_1, S_2$ *Riemann surfaces and* $f : S_1 \to S_2$ *a qc mapping. Suppose that* $(\widehat{S}, \pi_i)$, *is the universal covering surface of* $S_i$, $i = 1, 2$, *and* $\Gamma_i$ *the covering group of* $\widehat{S}$ *over* $S_i$.*(The covering surface has to be the same for both* $S_i$ *since they are quasiconformally equivalent). Every lift* $\widehat{f} : \widehat{S} \to \widehat{S}$ *of* $f$ *is qc.*

*If* $S_i = \mathbb{U}/\Gamma_i$, $i = 1, 2$, *and* $f : S_1 \to S_2$ *is qc, then the corresponding* $\mu \in Belt(S_1)$ *can be lifted to an element of* $Belt(\Gamma_1, \mathbb{U})$, *which we denote again by* $\mu$.

**Existence.** *Let* $S = \mathbb{U}/\Gamma$ *a Riemann surface and* $\mu \in Belt(S)$. *There exists a qc mapping* $f$ *of* $S$ *onto another Riemann surface with complex dilatation* $\mu$, *determined uniquely up to a conformal mapping.*

For the sake of our following discussion, we give a brief sketch of the proof: Given $\mu \in Belt(S)$ we may consider it as an element $\mu \in Belt(\Gamma, \mathbb{U})$. As in 1.2.3 we obtain a qc mapping $f^\mu : \mathbb{U} \to \mathbb{U}$ which is unique up to a Möbius transformation. The group $\Gamma^\mu = f^\mu \circ \Gamma \circ (f^\mu)^{-1}$ is Fuchsian and $S^\mu = \mathbb{U}/\Gamma^\mu$ is a Riemann surface. ($S^\mu$ is the surface $S$ with conformal structure the one obtained by $\mu$). If $pr : \mathbb{U} \to S$, $pr' : U \to S^\mu$ are the canonical projections, then $f$ is defined by the relation

$$f \circ pr = pr' \circ f^\mu.$$

Uniqueness of $f$ is deduced by uniqueness of $f^\mu$.

**Conformal structures and Beltrami differentials.** An important consequence of the existence theorem is that $Belt(S)$ is in one-to-one correspondence with the set of conformal structures on the Riemann surface $S$. Indeed, an element of $Belt(S)$ is defined from the form of the metric (**). Conversely, given an element $\mu$ of $Belt(S)$, then by the existence theorem, we obtain a conformal structure for $S$.

**Group isomorphisms induced by qc mappings-Homotopy.** The following results play an important role in Teichmüller theory:

*Let* $S = \mathbb{U}/\Gamma$ *and* $S' = \mathbb{U}/\Gamma'$ *two Riemann surfaces and* $f_i : S \to S'$, $i = 1, 2$, *two qc mappings. Let* $\widehat{f_1}$ *a lift of* $f_1$. *Then* $f_1$ *and* $f_2$ *induce the same group isomorphism between* $\Gamma$ *and* $\Gamma'$ *if there is a lift* $\widehat{f_2}$ *of* $f_2$ *which agrees with* $\widehat{f_1}$ *on the limit set of* $\Gamma$.

*Let* $S = \mathbb{U}/\Gamma$ *a Riemann surface and* $f : S \to S$ *a conformal mapping homotopic to the identity. Then* $f$ *is the identity mapping.*

Up to now, all results are valid for general Riemann surfaces. In the sequel, we are going to be concerned with *closed* Riemann surfaces, i.e compact Riemann surfaces without boundary. A closed Riemann surface of genus $g > 1$, is a quotient $\mathbb{U}/\Gamma$ where $\Gamma$ is a Fuchsian group of the first kind consisting only of hyperbolic elements.

*Any two closed Riemann surfaces of the same genus are quasiconformally equivalent.*

*Let* $S = \mathbb{U}/\Gamma$ *and* $S' = \mathbb{U}/\Gamma'$ *two closed Riemann surfaces of genus* $g > 1$ *and* $f_i : S \to S'$, $i = 1, 2$, *two qc mappings. Then* $f_1$ *is homotopic to* $f_2$ *if there exist qc lifts* $\widehat{f_1}$ *and* $\widehat{f_2}$ *which agree on* $\partial U$.

Not every sense preserving homeomorphism between two Riemann surfaces is homotopic to a qc mapping. But in the case of closed surfaces we have the following due to Teichmüller:

*Let* $S = \mathbb{U}/\Gamma$ *and* $S' = \mathbb{U}/\Gamma'$ *two closed Riemann surfaces of the same genus* $g > 1$. *Then every homotopy class of sense preserving homeomorphisms of* $S$ *onto* $S'$ *contains a qc mapping.*

**1.3.3. Simultaneous uniformisation.** Let $S_1, S_2$be closed Riemann surfaces of genus $g > 1$. Due to uniformisation we can choose a Fuchsian group $\Gamma_1$ so that $\mathbb{U}/\Gamma_1 = \overline{S_1}$, the mirror image of $S_1$. It is evident that then, $S_1 = \mathbb{L}/\Gamma_1$, where $\mathbb{L} = \{z = x + iy,\ y < 0\}$ is the lower half plane. We choose another Fuchsian group $\Gamma_2$, so that $S_2 = \mathbb{U}/\Gamma_2$. Let $f : \overline{S_1} \to S_2$ a qc mapping. Then $f$ can be lifted to a qc self mapping of $\mathbb{U}$. For simplicity we denote the lift again by $f$. Consider the function $\mu = f_{\overline{z}}/f_z$ in $\mathbb{U}$, and the element $\mu^*$ of $Belt(\Gamma_1, \mathbb{U})$ defined by the relations: $\mu^* = \mu$ in $\mathbb{U}$ and $\mu^* = 0$ elsewhere. Then $G = \Gamma_1^{\mu^*}$ is a quasi-Fuchsian group. It is obvious that

$$f^{\mu^*}(\mathbb{U})/G = S_2 \text{ and } f^{\mu^*}(\mathbb{L})/G = S_1$$

and $S_1, S_2$ are homeomorphic by a sense reversing homeomorphism. We say that the quasi-Fuchsian group $G$ *represents* the Riemann surfaces $S_1, S_2$ or equivalently the Fuchsian groups $\Gamma_1, \Gamma_2$. We refer to Bers [B3], for further details.

CHAPTER 2

# Teichmüller and Quasifuchsian space

## 2.1. Teichmüller space

We fix once and for all a closed Riemann surface $S = \mathbb{U}/\Gamma$ of genus $g > 1$.

### 2.1.1. Definitions of Teichmüller space.

**Teichmüller space of a Riemann surface.** [W1] We consider triples $(S', [f], S)$ where $[f]$ is the homotopy class of a quasiconformal mapping $f$ of the surface $S$ to $S'$. We define an equivalence relation on the space of these triples as follows:

-$(S_1, [f_1], S) \sim (S_2, [f_2], S)$ *if there is a conformal mapping of* $S_1$ *to* $S_2$ *homotopic to* $f_2 \circ f_1^{-1}$.

An equivalence class $[X] = [(X, [f], S)]$ is a *marked Riemann surface*, and the Teichmüller space $Teich(S)$ is the set of marked Riemann surfaces. The equivalence class of $(S, [id], S)$ is called the *origin* of $Teich(S)$ and shall be denoted by $[id]$.

**Teichmüller space of a Fuchsian group.** [W1] Consider triples $(\Gamma', \rho, \Gamma)$ where $\Gamma'$ is a Fuchsian group acting on $\mathbb{U}$ and $\rho : \Gamma \to \Gamma'$ is an orientation and type-preserving isomorphism of Fuchsian groups. We define an equivalence relation on the space of these pairs as follows:

-$(\Gamma_1, \rho_1, \Gamma) \sim (\Gamma_2, \rho_2, \Gamma)$ *if there exists a real Möbius transformation* $A$ *such that* $A\Gamma_1 A^{-1} = \Gamma_2$, $\rho_2 = AdA \circ \rho_1$.

The equivalence class $[\Gamma'] = [(\Gamma', \rho, \Gamma)]$ is a *marked Fuchsian group*, and Teichmüller space $Teich(\Gamma, \mathbb{U})$ is the space of marked Fuchsian groups. The origin $[id]$ here is the equivalence class of the triple $(\Gamma, id, \Gamma)$.

**Isomorphism:** [B2] theorem VII. *If* $S = \mathbb{U}/\Gamma$ *then* $Teich(S)$ *and* $Teich(\Gamma, \mathbb{U})$ *are canonically isomorphic.*

**Teichmüller spaces and Beltrami differentials.** Let $L^\infty(\Gamma, \mathbb{U})$ be the space of $\Gamma$-invariant differentials, and $Belt(\Gamma, \mathbb{U})$ the set of $\Gamma$-invariant Beltrami differentials.

Let $\mu \in Belt(\Gamma, \mathbb{U})$. Consider the Beltrami equations

$$w_{\overline{z}} = \begin{cases} \mu w_z & z \in \mathbb{U} \\ 0 & z \in \mathbb{L} \end{cases} \quad (1)$$

$$f_{\overline{z}} = \begin{cases} \mu f_z & z \in \mathbb{U} \\ \overline{\mu(\overline{z})} f_z & z \in \mathbb{L} \end{cases} \quad (2)$$

Denote by $w^\mu, f^\mu$, the unique normalised quasiconformal solutions to (1) and (2) respectively.

From the above stated definitions we can easily see that:

1. $Teich(\Gamma, \mathbb{U})$ is the space of equivalence classes of quasiconformal mappings $f : \mathbb{U} \to \mathbb{U}$ which are such that $f \circ \Gamma \circ f^{-1} = \Gamma'$ is again a Fuchsian group, where two such qc self mappings $f_1, f_2$ of $\mathbb{U}$ are equivalent if they coincide on the real axis. We have seen in 1.2.3 that such a mapping induces an isomorphism $\rho_f : \Gamma \to \Gamma'$. If $f$ is the restriction in $\mathbb{U}$ of $f^\mu$, a solution of (2), then $\Gamma'$ is always a Fuchsian group. To any equivalent class $[f]$ associate the point $[\Gamma'] = [(\Gamma', \rho_f, \Gamma)] \in Teich(\Gamma, \mathbb{U})$. Any triple $(\Gamma', \rho, \Gamma)$ is equivalent with one of the form $(\Gamma', \rho_f, \Gamma)$.

2. $Teich(\Gamma, \mathbb{U})$ is the space of equivalence classes of Beltrami differentials. Suppose that $f_1, f_2$ are the restrictions in $\mathbb{U}$ of $f^{\mu_1}$ and $f^{\mu_2}$ solutions of (2) respectively. We see that equivalence relation between mappings establishes an equivalence relation between Beltrami differentials: $\mu_1 \sim \mu_2$ if and only if $f^{\mu_1} \sim f^{\mu_2}$.Therefore an equivalence class $[\mu]$ also represents a point of $Teich(\Gamma, \mathbb{U})$.

3. To each $[\mu]$, correspond the point $[S^\mu] = [(S^\mu, [id], S)]$ of $Teich(S)$. Any triple $(X, [f], S)$ is equivalent to one of the form $(S^\mu, [id], S)$. Therefore $Teich(S)$ is the space of equivalence classes $[S^\mu]$.

**2.1.2. Complex structure.** Teichmüller space of $S$ is a complex manifold of complex dimension $3g - 3$. In the following we shall describe its complex structure. We shall work with the space $T(\Gamma, \mathbb{U})$ and refer to [W2] for what follows.

We consider $Q(\Gamma, \mathbb{U})$ the vector space of $\Gamma-$*invariant integrable quadratic differentials.* An element $\phi \in Q(\Gamma, \mathbb{U})$ is a holomorphic function $\phi(z)$ on $\mathbb{U}$ satisfying the transformation law: $\phi(\gamma(z))(\gamma'(z))^2 = \phi(z)$ for all $\gamma \in \Gamma$ and for all $z \in \mathbb{U}$ and also the condition

$$\int_S \mid \phi \mid < \infty$$

Any element of $Q(\Gamma, \mathbb{U})$ defines a $(2, 0)$ differential on $S$.

Given $\mu \in L^\infty(\Gamma, \mathbb{U})$ and $\phi \in Q(\Gamma, \mathbb{U})$ the natural pairing

$$(\mu, \phi) = \int_S \mu\phi$$

is well defined since the product $\mu\phi$ is a $\Gamma-$invariant area form. Let

$$Q(\Gamma, \mathbb{U})\perp = N(\Gamma, \mathbb{U})$$

be the null space of the pairing. The dual vector spaces $R(\Gamma, \mathbb{U}) = L^\infty(\Gamma, \mathbb{U})/N(\Gamma, \mathbb{U})$ and $Q(\Gamma, \mathbb{U})$ are $3g-3$ complex dimensional.

**Complex structure.** We consider the surjective map

$$\widehat{\Phi} : Belt(\Gamma, \mathbb{U}) \to Teich(\Gamma, \mathbb{U})$$

sending each $\mu \in Belt(\Gamma, \mathbb{U})$ to the marked Fuchsian group $[f^\mu\Gamma(f^\mu)^{-1}] \in Teich(\Gamma, \mathbb{U})$, for $f^\mu$ solution of (2). We define a complex structure on $Teich(\Gamma, \mathbb{U})$ by requiring $\widehat{\Phi}$ to be holomorphic.

FACT 2.1.1. *For $\mu \in L^\infty(\Gamma, \mathbb{U})$ the relation $\widehat{\Phi}'_{(0)}(\mu) = 0$ is equivalent to $(\mu, \phi) = 0$ for every $\phi \in Q(\Gamma, \mathbb{U})$. The kernel of the differential map at the origin is the space $N(\Gamma, \mathbb{U})$.*

FACT 2.1.2. *Holomorphic tangent and cotangent spaces at the origin are realised by the dual spaces $R(\Gamma, \mathbb{U})$ and $Q(\Gamma, \mathbb{U})$ respectively.*

The almost complex operator $I_T$ defined on the holomorphic tangent space at the origin is obtained by multiplication by $i$ in $R(\Gamma, \mathbb{U})$. Each coset of $R(\Gamma, \mathbb{U})$ has a unique representative in the subspace $L^\infty_c(\Gamma, \mathbb{U})$ of $L^\infty(\Gamma, \mathbb{U})$ consisting of *canonical* differentials, that is differentials of the form

$$(Imz)^2\overline{\phi}, \phi \in Q(\Gamma, \mathbb{U}).$$

There exists a natural projection operator $P : L^\infty(\Gamma, \mathbb{U}) \to L^\infty_c(\Gamma, \mathbb{U})$ given by

$$P[\mu](z) = \frac{12(Imz)^2}{\pi} \int\int_{\mathbb{U}} \frac{\mu(\zeta)}{(\zeta - \overline{z})^4} dxdy, \quad z \in \mathbb{U}.$$

The projection operator induces the inverse of the isomorphism

$$L^\infty_c(\Gamma, \mathbb{U}) \to R(\Gamma, \mathbb{U}).$$

Consequently $L^\infty_c(\Gamma, \mathbb{U})$ provides another model for the holomorphic tangent space at the origin.

Local complex coordinates can be described in a neighborhood of the origin as follows:[W3] We choose

$$\mu_1, ..., \mu_{3g-3} \in L^\infty(\Gamma, \mathbb{U})$$

whose $N(\Gamma, \mathbb{U})$ cosets form a basis of the holomorphic tangent space at the origin. Given any $\mu \in Belt(\Gamma, \mathbb{U})$ we define the map

$$\widetilde{a^\mu} : Belt(\Gamma, \mathbb{U}) \to Belt(\Gamma^\mu, \mathbb{U})$$

so that it satisfies the relation

$$f^{\widetilde{a^\mu}(\nu)} = f^\nu \circ (f^\mu)^{-1}$$

for every $\nu \in Belt(\Gamma, \mathbb{U})$. Explicitely

$$\widetilde{a^\mu}(\nu) = \left( \frac{\nu - \mu}{1 - \overline{\mu}\nu} \frac{f_z^\mu}{\overline{f_z^\mu}} \right) \circ (f^\mu)^{-1}.$$

This map is a biholomorphic (in the sense of Frechet derivatives) bijection of $Belt(\Gamma, \mathbb{U})$ onto $Belt(\Gamma^\mu, \mathbb{U})$ The induced mapping

$$a^\mu \ : Teich(\Gamma, \mathbb{U}) \to Teich(\Gamma^\mu, \mathbb{U})$$

defined by

$$a^\mu([\nu]) = [\widetilde{a^\mu}(\nu)]$$

maps $[\mu] \in Teich(\Gamma, \mathbb{U})$ to the origin of $Teich(\Gamma^\mu, \mathbb{U})$. We choose $V$, a neighborhood of zero in $\mathbb{C}^{3g-3}$ sufficiently small so that if $t = (t_1, ..., t_{3g-3}) \in V$ then

$$\mu(t) = \sum_{j=1}^{3g-3} t_j \mu_j$$

satisfies $\| \mu(t) \|_\infty < 1$. Consider the complex linear bijection

$$L^\mu : L^\infty(\Gamma, \mathbb{U}) \to L^\infty(\Gamma^\mu, \mathbb{U})$$

given by

$$L^\mu(\nu) = \left( \frac{\nu}{1 - |\mu|^2} \frac{f_z^\mu}{\overline{f_z^\mu}} \right) \circ (f^\mu)^{-1}.$$

(Note that $L^\mu(\nu) = \frac{\partial}{\partial t} \widetilde{a^\mu}(\mu + t\nu) \,|_{t=0}$cf. [A1]). The coordinate mapping

$$\widetilde{\Phi} : V \to Teich(\Gamma, \mathbb{U})$$

is then given by $\widetilde{\Phi}(t) = [\Gamma^{\mu(t)}]$. Holomorphic tangent vectors are described via this map by

$$\frac{\partial}{\partial t_j(\mu)} \,|_t = L^{\mu(t)}(\mu_j) mod N(\Gamma^{\mu(t)}, \mathbb{U}) \in R(\Gamma^{\mu(t)}, \mathbb{U}).$$

We mention here the following, [L] p. 211:

FACT 2.1.3. *Quasiconformally equivalent Riemann surfaces have biholomorphic Teichmüller spaces.*

**Complex structure again.** Although we have required the projection mapping $\widehat{\Phi}$ to be holomorphic, the complex structure arising on the Teichmüller space is natural. This can be seen through *Bers' imbedding*[B2] which we shall describe in the sequel.

We have seen that two Beltrami differentials $\mu, \nu$ are equivalent if and only if $f^\mu = f^\nu$ on the extended real axis $\widehat{\mathbb{R}}$.

The following is due to Bers:

FACT 2.1.4. $f^\mu = f^\nu$ *on* $\widehat{\mathbb{R}}$ *if and only if* $w^\mu = w^\nu$ *on* $\mathbb{L}$ *the latter being solutions of* (1).

Teichmüller space can be canonically identified with a bounded domain in the complex vector space $B(\Gamma, \mathbb{L})$ of $\Gamma-$invariant *bounded quadratic differentials.* An element $\phi \in B(\Gamma, \mathbb{L})$ is a bounded holomorphic function $\phi(z)$ on $\mathbb{L}$, that is

$$\| \phi \|_* = sup\{\frac{\phi(z)}{y^2},\, z \in \mathbb{L}\} < \infty$$

and satisfying the transformation law

$$\phi(\gamma(z))(\gamma'(z))^2 = \phi(z)$$

for all $\gamma \in \Gamma$ and for all $z \in \mathbb{L}$. Let $\phi^\mu$ be the *schwarzian derivative* of the conformal map $w^\mu$ on $\mathbb{L}$ :

$$\phi^\mu = \left(\frac{(w^\mu)''}{(w^\mu)'}\right)' - \frac{1}{2}\left(\frac{(w^\mu)''}{(w^\mu)'}\right)^2$$

THEOREM. *(AHLFORS-WEILL-BERS): a)* $\phi^\mu$ *is an element of* $B(\Gamma, \mathbb{L})$ *and* $\| \phi^\mu \|_* < 6$. *b) The mapping*

$$B : Belt(\Gamma, \mathbb{U}) \to B(\Gamma, \mathbb{L})$$

*defined by* $B(\mu) = \phi^\mu$ *is an open and holomorphic map of Banach spaces and has a holomorphic section* $\sigma_\mu$ *on the ball* $B(0,2)$ *of* $B(\Gamma, \mathbb{L})$ : *If* $\phi \in B(0,2)$ *then* $\sigma_\mu(\phi)$ *defined by* $\sigma_\mu(\phi)(z) = -2y^2\phi(\overline{z})$, $z \in \mathbb{U}$ *is an element of* $Belt(\Gamma, \mathbb{U})$.

For the proof see for instance [L] p. 207. The map $B$ induces a mapping

$$\Phi_B : Teich(\Gamma, \mathbb{U}) \to B(\Gamma, \mathbb{L})$$

where

$$\Phi_B([\mu]) = \varphi^\mu.$$

$\Phi_B$ is injective since by fact 2.1.4, $Teich(\Gamma, \mathbb{U})$ is in natural one-to-one correspondence with the set of conformal maps $w^\mu$. The identification of $Teich(\Gamma, \mathbb{U})$ is with the set

$$\{\phi^\mu, \mu \in Belt(\Gamma, \mathbb{U})\},$$

in $B(\Gamma, \mathbb{L})$.The complex structure obtained in this way, is the same with the one defined in the previous paragraph. We refer to [L], p. 208-211, for further details.

**Modular group.** The *modular group* (or the *mapping class group*) $Mod(S)$ or $Mod(\Gamma, \mathbb{U})$ is the group of all qc mappings $h$ of $\mathbb{U}$ such that $h \circ \gamma \circ h^{-1} \in \Gamma$ for all $\gamma \in \Gamma$ modulo the group of these mappings which satisfy $h \circ \gamma \circ h^{-1} = \gamma$ for all $\gamma \in \Gamma$. There is a homomorphism of this group into the group of biholomorphic self-mappings of Teichmüller space: if $h$ is a representative of a coset, then the coset is mapped into the self-biholomorphism $\gamma_h$, where

$$\gamma_h([f^\mu]) = [f^\mu \circ h^{-1}]$$

for every $[f^\mu]$.The modular group is the full set of such biholomorphisms [R].

**Real tangent space.** In the following we shall give a descrirtion of the real tangent space, and the action of the almost complex operator $I_T$ on it. The description is via the space of symmetric differentials $L^\infty_S(\Gamma)$.

$L^\infty_S(\Gamma)$ is isomorphic to $L^\infty(\Gamma, \mathbb{U})$ when the latter is considered as a real vector space.

The isomorphism $L^\infty(\Gamma, \mathbb{U}) \to L^\infty_S(\Gamma)$ is given by

$$\mu \to \widetilde{\mu} = \begin{cases} \mu(z) & z \in \mathbb{U} \\ \overline{\mu(\overline{z})} & z \in \mathbb{L} \end{cases}$$

Note that the isomorphism maps $Belt(\Gamma, \mathbb{U})$ to $Belt_S(\Gamma)$. We define a complex operator in $L^\infty_S(\Gamma)$ as follows: For $\widetilde{\mu}$ symmetric, the operator $I_S$ is given by

$$(I_S(\widetilde{\mu}))(z) = \begin{cases} i\widetilde{\mu}(z) & z \in \mathbb{U} \\ -i\widetilde{\mu}(z) & z \in \mathbb{L} \end{cases}$$

It is easy to see that if $\mu \in L^\infty(\Gamma, \mathbb{U})$, then

$$\mu = \frac{1}{2}(\widetilde{\mu} - iI_S(\widetilde{\mu})).$$

Now if $\mu \in L^\infty(\Gamma, \mathbb{U})$ is a representative of a holomorphic tangent vector $\frac{\partial}{\partial z(\mu)}$, and $\frac{\partial}{\partial x(\mu)}$ the corresponding real vector then the above can be viewed as the isomorphism of the Lie algebra of infinitesimal automorphisms of the complex structure $I_T$ with the Lie algebra of holomorphic vector fields, that is

$$\frac{\partial}{\partial z(\mu)} = \frac{1}{2}\left(\frac{\partial}{\partial x(\mu)} - iI_T\frac{\partial}{\partial x(\mu)}\right).$$

The complex operator defined in this way differs in sign from the one defined in [G]. There, the complex structure of the Teichmüller space is described via the Hilbert transform.

**2.1.3. Weil-Petersson geometry.** For the following we refer to [W2] and the references given there. Teichmüller space can be given a natural hermitian structure. For $\phi, \psi \in Q(\Gamma, \mathbb{U})$ we define

$$h^*(\phi, \psi) = \int_S \phi\overline{\psi}$$

which is the *Weil-Petersson* hermitian product in the cotangent space at the origin. We obtain the corresponding description in the tangent space using the fact that $KerP = N(\Gamma, \mathbb{U})$ : Indeed

$$\int_S \mu\phi = \int_S P[\mu]\phi$$

for all $\mu \in L^\infty(\Gamma, \mathbb{U})$ and $\phi \in Q(\Gamma, \mathbb{U})$. The Weil-Petersson hermitian product in the holomorphic tangent space $L_c^\infty(\Gamma, \mathbb{U})$ at the origin is then given simply by

$$h(\mu, \nu) = \int_S \mu\overline{\nu}.$$

The hermitian form in the holomorphic tangent space at the origin of $Teich(\Gamma, \mathbb{U})$ is given by

$$h\left(\frac{\partial}{\partial z(\mu)}, \frac{\partial}{\partial z(\nu)}\right) = \int_S P[\mu]\overline{P[\nu]}.$$

and the induced riemannian metric in the real tangent space at the origin is just

$$g\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) = 2Re\{\int_S P[\mu]\overline{P[\nu]}\}.$$

The metric is Káhlerian, of negative holomorphic sectional curvature, incomplete and its group of biholomorphic isometries is $Mod(S)$. The real symplectic W-P form $\omega_{WP}$ induced by the metric is given by

$$\omega_{WP}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) = g\left(I_T\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) = -2Im\{\int_S P[\mu]\overline{P[\nu]}\}.$$

The W-P hermitian form can be everywhere defined in the following manner: If $[\xi]$, $\xi \in Belt(\Gamma, \mathbb{U})$, is any point of $Teich(\Gamma, \mathbb{U})$, then it can be considered as the origin of the Teichmüller space $Teich(\Gamma^\xi, \mathbb{U})$. The mapping

$$a^{\xi} : Teich(\Gamma, \mathbb{U}) \to Teich(\Gamma^{\xi}, \mathbb{U})$$

is biholomorphic and maps $[\xi] \in Teich(\Gamma, \mathbb{U})$ to the origin $[id_{\xi}]$ of $Teich(\Gamma^{\xi}, \mathbb{U})$. Having defined the W-P product $h^{\xi}$ at the origin of $Teich(\Gamma^{\xi}, \mathbb{U})$, we have the product $h$ at $[\xi] \in Teich(\Gamma, \mathbb{U})$ by pulling back $h^{\xi}$ to $[\xi]$. Explicitely, for $\mu, \nu \in L^{\infty}(\Gamma, \mathbb{U})$,

$$h_{([\xi])}\left(\frac{\partial}{\partial z(\mu)}, \frac{\partial}{\partial z(\nu)}\right) = h^{\xi}\left(\frac{\partial}{\partial z(L^{\xi}\mu)}, \frac{\partial}{\partial z(L^{\xi}\nu)}\right) = ((a^{\xi})^{*}h^{\xi})_{([\xi])}\left(\frac{\partial}{\partial z(\mu)}, \frac{\partial}{\partial z(\nu)}\right).$$

**2.1.4. Real analytic structure and symplectic geometry of Teichmüller space.** In this subsection we list some known results about the real analytic and the symplectic structure of Teichmüller space.

Teichmüller space as a real analytic manifold will be denoted again by $Teich(S)$. For the notions of *geodesic length* and *twist* functions as well as for the proper definition of the *Fenchel-Nielsen twist* field we refer to [W1,2]. Theorems 2.1.5 and 2.1.6 provide real analytic coordinates for Teichmüller space. Theorem 2.1.7 describes a local basis for the real tangent space. A complete description of the real symplectic form induced by the W-P metric is given in Theorem 2.1.8.

THEOREM 2.1.5. *[W1]: Given a partition of the surface $S$ by $3g-3$ simple closed geodesics $\gamma_i$, there exist geodesic length functions $l_{\gamma_i} : Teich(S) \to \mathbb{R}$ and twist functions $\tau_i : Teich(S) \to \mathbb{R}$ $i = 1, ..., 3g-3$ which form a system of global real analytic coordinates for $Teich(S)$. (Fenchel-Nielsen coordinates for Teichmüller space).*

THEOREM 2.1.6. *[W1]: Given a partition of the surface $S$ by $3g-3$ simple closed geodesics $\gamma_i$, then for every $[\rho] \in Teich(S)$ there exist a neighborhood $V([\rho])$ and $3g-3$ simple closed curves $\alpha_i$ with $\gamma_i \cap \alpha_j = \emptyset$ if $i \neq j$ such that the geodesic length functions $l_{\gamma_i}, l_{\alpha_i}$ $i = 1, ..., 3g-3$ form a system of local real analytic coordinates for $Teich(S)$.*

THEOREM 2.1.7. *[W1]: Let $\gamma_i, \alpha_i$ as in Theorem 2.1.2. The twist vector fields $t_{\gamma_i}, t_{\alpha_i}$ form a local basis of the real tangent space of $Teich(S)$.*

THEOREM 2.1.8. *[W2,4]: The real form $\omega_{WP}$ induced by the W-P metric turns $Teich(S)$ into a symplectic manifold. The form $\omega_{WP}$ satisfies the following:*

i) $(\omega_{WP})_{([\rho])}(t_\alpha, t_\beta) = \sum_{p\in\alpha\cap\beta} \cos\theta(\rho(\alpha), \rho(\beta))_p$

*where* $t_\alpha, t_\beta$ *are twist vectors corresponding to geodesics* $\alpha, \beta$ *on* $S$, *and* $\theta(\rho(\alpha), \rho(\beta))_p$ *denotes the angle between geodesics in* $\mathbb{U}$.

ii) *The twist field* $t_\alpha$ *is hamiltonian for the geodesic length function* $l_\alpha$*,that is*

$$\omega_{WP}(t_\alpha, \cdot) = -dl_\alpha$$

iii) *The expression of* $\omega_{WP}$ *in global Fenchel-Nielsen coordinates is*

$$\omega_{WP} = \sum_{i=1}^{3g-3} dl_{\gamma_i} \wedge d\tau_i$$

## 2.2. Quasifuchsian space

**2.2.1. Deformation spaces of Kleinian groups and the theorem of Bers.** Let $G$ be a finitely generated non elementary Kleinian group with region of discontinuity $\Omega$ and limit set $\Lambda$. The quotient $\Omega/G$ is a complex manifold, a finite disjoint union of Riemann surfaces [A1].

Consider the spaces of $G$-invariant differentials $L^\infty(G)$ and its unit ball $Belt(G)$. Let $\mu \in Belt(G)$. By $w^\mu$ we denote the unique normalised solution to the Beltrami equation

$$w_{\overline{z}} = \mu w_z \quad (3)$$

As we have seen in 1.2.3. that the function $w^\mu$ induces a quasiconformal isomorfism of $G$ onto another Kleinian group, that is $\rho_w : G \to w^\mu G (w^\mu)^{-1}$.We call two such isomorphisms $\rho_{w_1}, \rho_{w_2}$ equivalent if there exists an element $A$ of $\mathbb{PSL}(2,\mathbb{C})$ such that $\rho_{w_1} = AdA \circ \rho_{w_2}$. The equivalence of isomorphisms is the same as the equivalence of Beltrami differentials, where two such differentials $\mu, \nu$ are equivalent if $w^\mu = w^\nu$ on $\Lambda$.

Definition. The set of equivalence classes

$$Def(G) = Hom_{qc}(G \to \mathbb{PSL}(2,\mathbb{C}))/\sim$$

is the *deformation space* of $G$ [Kr].

Deformation space $Def(G)$ admits a natural complex structure. Consider the following surjective mapping

$$\Phi : Belt(G) \to Def(G)$$

sending $\mu \in Belt(G)$ to $[\mu] = [w^\mu] \in Def(G)$. The mapping $\Phi$ is holomorphic since by Ahlfors-Bers theorem if $\mu \in Belt(G)$ depends holomorphically on complex parameters, the same holds for $w^\mu$.

**Bers' theorem.** The following theorem modified here for our purposes is due to L. Bers [B4], [Kr]:

THEOREM 2.2.1. *Let $G$ be a finitely generated Kleinian group with region of discontinuity $\Omega$ consisting of simply connected components. Then $Def(G)$ is a connected complex manifold, biholomorphic to a cartesian product of Teichmüller spaces.*

Denote by $Q(G)$ the vector space of $G-$*invariant quadratic differentials.* An element $\phi \in Q(G)$ is a holomorphic function $\phi(z)$ on $\Omega$ satisfying the condition: $\phi(g(z)(g'(z))^2 = \phi(z)$ for all $g \in G$ and $z \in \Omega$ and is such that

$$\int_{\Omega/G} \mid \phi \mid < \infty$$

A natural pairing can be defined as follows: Given $\mu \in L^\infty(G)$ and $\phi \in Q(G)$ then

$$(\mu, \phi) = \int_{\Omega/G} \mu\phi$$

Denote by $N(G) \subset L^\infty(G)$ the null space of the pairing. The spaces $R(G) = L^\infty(G)/N(G)$ and $Q(G)$ are finite dimensional and dual with respect to the pairing[A1] . Further we have [B5]:

FACT 2.2.2. *Let $\Phi : Belt(G) \to Def(G)$ be the canonical surjection $\xi \to [\xi]$ and $\mu \in L^\infty(G)$. Then $\Phi'_{(0)}(\mu) = 0$ if and only if $(\mu, \phi) = 0$ for every $\phi \in Q(G)$. Accordingly, the holomorphic tangent and cotangent spaces at the origin of $Def(G)$ can be identified with the finite dimensional vector spaces $R(G)$ and $Q(G)$ respectively.*

We may now carry out the same procedure as in the case of Teichmüller space to describe local complex coordinates in a neighborhood of the origin: Let $d = dim_C(R(G))$. Again we choose $\mu_1, ..., \mu_d \in L^\infty(G)$ whose $N(G)$ cosets form a basis of the holomorphic tangent space at the origin. Given any $\mu \in Belt(G)$ we define the map

$$\widetilde{a^\mu} : Belt(G) \to Belt(G^\mu)$$

by the relation

$$w^{\widetilde{a^\mu}(\nu)} = w^\nu \circ (w^\mu)^{-1}$$

where $G^\mu = w^\mu \circ G \circ (w^\mu)^{-1}$, $w^\mu$ is the normalised solution of (3), $\nu \in Belt(G)$. $\widetilde{a^\mu}$ is given by

$$\widetilde{a^\mu}(\nu) = \left(\frac{\nu - \mu}{1 - \overline{\mu}\nu}\frac{w^\mu_z}{\overline{w^\mu_z}}\right) \circ (w^\mu)^{-1}.$$

This map is a holomorphic bijection of $Belt(G)$ onto $Belt(G^\mu)$. The induced mapping

$$a^\mu : Def(G) \to Def(G^\mu)$$

defined by

$$a^\mu([\nu]) = [\widetilde{a^\mu}(\nu)]$$

maps $[\mu] \in Def(G)$ to the origin of $Def(G^\mu)$.

We choose $V$ a neighborhood of 0 in $C^d$ sufficiently small so that if $t = (t_1, ..., t_d) \in V$ then

$$\mu(t) = \sum_{j=1}^{d} t_j \mu_j$$

satisfies $\|\mu(t)\|_\infty < 1$. Consider the complex linear bijection

$$L^\mu : L^\infty(G) \to L^\infty(G^\mu)$$

given by

$$L^\mu(\nu) = \left(\frac{\nu}{1-|\mu|^2}\frac{w^\mu_z}{\overline{w^\mu_z}}\right) \circ (w^\mu)^{-1}.$$

The coordinate mapping $\widetilde{\Phi} : V \to Def(G)$ is then given by $\widetilde{\Phi}(t) = [\mu(t)]$. The coordinate holomorphic tangent vectors are

$$\frac{\partial}{\partial t_j(\mu)}\Big|_t = L^{\mu(t)}(\mu_j) mod N(G^{\mu(t)}) \in R(G^{\mu(t)}).$$

FACT 2.2.3. *[B4] Quasiconformally equivalent Kleinian groups have biholomorphic deformation spaces.*

### 2.2.2. Quasifuchsian space.

**2.2.2. Quasifuchsian space.** Let $\Gamma$ be a Fuchsian group. We maintain our assumptions for $\Gamma$ i.e that is finitely generated, identified with the fundamental group of a closed Riemann surface $S$ of genus $g > 1$, and we consider it acting on $\mathbb{U} \cup \mathbb{L}$.

DEFINITION. The *space $QF(S)$ of quasi-Fuchsian structures of $S$* (the *Quasifuchsian space* from now on) is simply $Def(\Gamma)$.

Consider $\mu \in Belt(\Gamma)$ and $w^\mu$ the unique normalised solution of (3). Let

$$\rho_w : \Gamma \to \Gamma^\mu$$

where $\Gamma^\mu = w^\mu \Gamma (w^\mu)^{-1}$, be the induced group homomorphism. Recall that $\Gamma^\mu$ is quasi-Fuchsian, (when $\mu$ is symmetric $\Gamma^\mu$ is Fuchsian) and also that if $\Omega_{\mathbb{U}} = w^\mu(\mathbb{U})$, $\Omega_{\mathbb{L}} = w^\mu(\mathbb{L})$ are the disgoint invariant components of its region of discontinuouity, then $\Omega_{\mathbb{U}}/\Gamma^\mu, \Omega_{\mathbb{L}}/\Gamma^\mu$ are Riemann surfaces which are homeomorphic by an orientation reversing homeomorphism.

Bers' theorem 2.2.1 can be rephrased in the case of the Quasifuchsian space as the following:

THEOREM 2.2.4. *$QF(S)$ is a connected complex manifold of complex dimension $6g-6$. There exists a biholomorphism*

$$\Psi : QF(S) \to Teich(S) \times Teich(\overline{S}).$$

We may describe $\Psi$ as follows (we follow [Kr]). We consider

$$\eta_\Gamma : L^\infty(\Gamma) \to L^\infty(\Gamma, \mathbb{U}) \times L^\infty(\Gamma, \mathbb{L})$$

mapping each $\mu \in L^\infty(\Gamma)$ to $(\mu_{\mathbb{U}} = \mu_{|\mathbb{U}}, \mu_{\mathbb{L}} = \mu_{|\mathbb{L}}) \in L^\infty(\Gamma, \mathbb{U}) \times L^\infty(\Gamma, \mathbb{L})$.

The mapping $\Psi$ is induced by $\eta_\Gamma$ when the latter is restricted to the open unit ball and maps $[\mu]$ to $([\mu_{\mathbb{U}}], [\mu_{\mathbb{L}}]) \in Teich(S) \times Teich(\overline{S})$.

NOTE 2.2.5. We can regard the Quasifuchsian space in the following manner [M]: Let $\mathbb{U}^3$ be the hyperbolic upper half-space. If $\Omega$ is the region of discontinuity of $G$ then the hyperbolic 3-manifold $M = (\mathbb{U}^3 \cup \Omega)/G$ is diffeomorphic to $S \times [0,1]$. We call $M$ a *quasi-Fuchsian manifold.* $M$ carries a hyperbolic structure on its interior and a projective structure on its boundary. Therefore a point of $QF(S)$ determines a pair of projective surfaces whose union is the boundary of $M$. The pair of underlying conformal structures on these surfaces denoted by $([\partial_c M], [\overline{\partial_c} M])$ is an element of the product of Teichmüller spaces $Teich(S) \times Teich(\overline{S})$. The space $QF(S)$ is the space of *marked quasi-Fuchsian manifolds*: A marking $[M]$ of $M$ is a choice of isomorphism between its fundamental group $\pi_1(M)$ and $\Gamma \equiv \pi_1(S)$. Speaking in these terms, Bers' mapping $\Psi$ maps each $[M]$ to $([\partial_c M], [\overline{\partial_c} M])$.

REMARK 2.2.6. *(Bers slices)* Teichmüller space $Teich(S)$ (resp. $Teich(\overline{S})$) is complex isomorphic to the complex submanifold of $QF(S)$ consisting of marked manifolds $[M]$ where $\overline{\partial_c} M = \overline{S}$ (resp.the complex submanifold of $QF(S)$ consisting of marked manifolds $[M]$ where $\partial_c M = S$.)

Teichmüller space as the space of Fuchsian deformations is a real analytic submanifold of $QF(S)$ : Let $\iota$ be the involution

$$\iota(z) = \overline{z},\ z \in \mathbb{C}.$$

It induces an involution $\beta$ of $Belt(\Gamma)$, $\mu \to \iota \circ \mu \circ \iota$ and accordingly an involution $\widetilde{\beta}$ of $QF(S)$

$$\widetilde{\beta}([w^{\mu}]) = [\beta(\mu)].$$

Teichmüller space can be identified real analytically with the fixed point set of this involution [K-M]. This set is the subset of Fuchsian deformations $F(S)$ of $QF(S)$ :

$$F(S) = Hom_{qc}(\Gamma \to \mathbb{PSL}(2,\mathbb{R}))/\sim$$

or in terms of the above note

$$F(S) = \{[M] \in QF(S) :\ \overline{\partial_c}M = \overline{\partial_c M}\}.$$

**Modular group.** The *modular group* $Mod_Q(S)$ or $Mod_Q(\Gamma)$ is the group of all quasiconformal homeomorphisms $h$ of the complex plane such that $h \circ \gamma \circ h^{-1} \in \Gamma$ for all $\gamma \in \Gamma$ modulo the group of those homeomorphisms which satisfy $h \circ \gamma \circ h^{-1} = \gamma$ for all $\gamma \in \Gamma$. As in the case of Teichmüller space, there is a homomorphism of this group into the group of biholomorphic self-mappings of $QF(S)$ as follows: if $h$ is a representative of a coset, then the coset is mapped into the self-biholomorphism $\gamma_h$, where

$$\gamma_h([w^{\mu}]) = [w^{\mu} \circ h^{-1}]$$

for every $[w^{\mu}] \in QF(S)$.

**Holomorphic tangent and cotangent spaces at the origin.** The holomorphic tangent and cotangent spaces at the origin of $QF(S)$ are identified according to our discussion in 2.2.1 with $R(\Gamma) = L^{\infty}(\Gamma)/N(\Gamma)$ and $Q(\Gamma)$ repectively. The alternative description for the holomorphic tangent space is that of the space $L^{\infty}_c(\Gamma)$ of canonical differentials. An element $\mu \in L^{\infty}(\Gamma)$ is canonical if $\mu = (Imz)^2\overline{\phi},\ \ \phi \in Q(\Gamma)$. We can easily verify the next:

PROPOSITION 2.2.7. *$\mu \in L^{\infty}_c(\Gamma)$ if and only if $\mu_{\mathbb{U}} \in L^{\infty}_c(\Gamma,\mathbb{U}), \mu_{\mathbb{L}} \in L^{\infty}_c(\Gamma,\mathbb{L})$.*

The projection operator $P_{\mathbb{C}} : L^{\infty}(\Gamma) \to L^{\infty}_c(\Gamma)$ is given in the obvious manner :

$$P_{\mathbb{C}}[\mu](z) = \begin{cases} P_{\mathbb{U}}[\mu_{\mathbb{U}}](z) & z \in \mathbb{U} \\ P_{\mathbb{L}}[\mu_{\mathbb{L}}](z) & z \in \mathbb{L} \end{cases}$$

where $\mu_{\mathbb{U}} = \mu_{|\mathbb{U}}$, $\mu_{\mathbb{L}} = \mu_{|\mathbb{L}}$ and $P_{\mathbb{U}}$, $P_{\mathbb{L}}$ are the projection operators $L^{\infty}(\Gamma,\mathbb{U}) \to L^{\infty}_c(\Gamma,\mathbb{U})$, $L^{\infty}(\Gamma,\mathbb{L}) \to L^{\infty}_c(\Gamma,\mathbb{L})$ respectively.

**Real tangent space at the origin.** We need the following preparatory lemma:

LEMMA 2.2.8. *There exists an isomorphism:*

$$L^{\infty}(\Gamma) \simeq L^{\infty}_{S}(\Gamma) \oplus_R iL^{\infty}_{S}(\Gamma) = L^{\infty}_{S}(\Gamma) \otimes \mathbb{C}$$

PROOF. Firstly observe that the direct sum is meaningful since $L^{\infty}_{S}(\Gamma)$ is a real vector space. We define a symmetrisation operator $S : L^{\infty}(\Gamma) \to L^{\infty}_{S}(\Gamma)$ sending each $\mu \in L^{\infty}(\Gamma)$ to $S(\mu) \in L^{\infty}_{S}(\Gamma)$ where

$$S(\mu)(z) = \frac{\mu(z) + \overline{\mu(\overline{z})}}{2}$$

for every $z \in \mathbb{C}$. One may check trivially that $S\Big|_{L^{\infty}_{S}(\Gamma)} = id$, and $S^2 = S$, the latter denoting that $S$ is a projection operator. Moreover $S$ is bijective when restricted to the sets $L^{\infty}(\Gamma, \mathbb{U})$ and $L^{\infty}(\Gamma, \mathbb{L})$ respectively. Now if $\mu \in L^{\infty}(\Gamma)$ we can write

$$\mu = S(\mu) + iS(-i\mu)$$

and the isomorphism in question is defined by the above relations. □

Let $\mu$ be a representative of a holomorphic tangent vector $\frac{\partial}{\partial z(\mu)}$and denote by $I$ the multiplication by $i$ in $L^{\infty}_{c}(\Gamma)$ defining the complex structure operator $I_Q$ in the holomorphic tangent space at the origin of $QF(S)$. The relation $\mu = S(\mu) + iS(-i\mu)$ implies that for each tangent direction $\frac{\partial}{\partial x(\mu)}$corresponding to $\frac{\partial}{\partial z(\mu)}$ in the underlying real tangent space, associated there is a pair

$$\left(\frac{\partial}{\partial x(S(\mu))}, \frac{\partial}{\partial x(S(-i\mu))}\right)$$

of "Fuchsian" tangent directions at the origin. We may write

$$\frac{\partial}{\partial x(\mu)} = \frac{\partial}{\partial x(S(\mu))} + I_Q \frac{\partial}{\partial x(S(-i\mu))}.$$

In view of the mapping $\Psi$, the relation $\eta_{\Gamma}(\mu) = (\mu_{\mathbb{U}}, \mu_{\mathbb{L}})$ implies

$$\Psi_* \left(\frac{\partial}{\partial z(\mu)}\right) = \left(\frac{\partial}{\partial z(\mu_{\mathbb{U}})}, \frac{\partial}{\partial z(\mu_{\mathbb{L}})}\right)$$

where

$$\frac{\partial}{\partial z(\mu_{\mathbb{U}})}, \frac{\partial}{\partial z(\mu_{\mathbb{L}})}$$

are holomorphic tangent vectors at the origin of $Teich(S)$, $Teich(\overline{S})$ respectively. Let

$$\frac{\partial}{\partial x(\mu_{\mathbb{U}})}, \frac{\partial}{\partial x(\mu_{\mathbb{L}})}$$

be the corresponding real tangent vectors. Since $\Psi$ is holomorphic, we have

$$\Psi_*\left(\frac{\partial}{\partial x(\mu)}\right) = \Psi_*\left(\frac{\partial}{\partial x(S(\mu))}, \frac{\partial}{\partial x(S(-i\mu))}\right) = \left(\frac{\partial}{\partial x(\mu_{\mathbb{U}})}, \frac{\partial}{\partial x(\mu_{\mathbb{L}})}\right).$$

In section 2.1.1 we saw that the complex operator $I_T$ of the Teichmüller space is arising from the complex operator $I_S$ of $L^\infty_S(\Gamma)$. However, $I_S$ is *not* $I$ in the complexification: if $\mu = S(\mu) + iS(-i\mu) \in L^\infty(\Gamma)$ then

$$I_S(\mu)(z) = I_S(S(\mu))(z) + iI_S(S(-i\mu))(z) = \begin{cases} i\mu(z) & z \in \mathbb{U} \\ -i\mu(z) & z \in \mathbb{L} \end{cases}$$

whereas $I(\mu) = i\mu$ everywhere on $\mathbb{C}$. This reflects the fact that Teichmüller space, when considered as the space of Fuchsian deformations, is a real submanifold of $QF(S)$ (cf. Remark 2.2.4).

On the other hand, it is easy to check that $I_S$ is equal to the complex structure of $L^\infty(\Gamma, \mathbb{U})$ when we are restricted to $L^\infty(\Gamma, \mathbb{U})$, and is equal to the conjugate complex operator of $L^\infty(\Gamma, \mathbb{L})$ when we are restricted to $L^\infty(\Gamma, \mathbb{L})$. We may trivially verify that $I_S$ commutes with $I$ in $L^\infty(\Gamma)$.

We consider the operator $\overline{I_S}$ defined for each $\mu \in L^\infty(\Gamma)$ by

$$\overline{I_S}(\mu)(z) = I_S(S(\mu))(z) - iI_S(S(-i\mu))(z)$$

which we shall denote from here on by $J$. It is easy to verify that $J$ *skew-commutes* with $I$ :

$$IJ + JI = 0.$$

The explicit form for $J$ is given by

$$J(\mu)(z) = \begin{cases} i\overline{\mu(\overline{z})} & z \in \mathbb{U} \\ -i\overline{\mu(\overline{z})} & z \in \mathbb{L} \end{cases}$$

Our goal is to show that $J$ gives rise to a complex operator $J_Q$ on $QF(S)$, turning $QF(S)$ into a $J_Q-$ complex manifold. We shall deal with this in Chapter III. For the moment we turn our discussion todeformation spaces of quasi-Fuchsian groups.

**Deformation space of a quasi-Fuchsian group.** Suppose that $G$ is a finitely generated quasi-Fuchsian group and $\Omega = \Omega_{\mathbb{U}} \cup \Omega_{\mathbb{L}}$ is its region of discontinuity. There exist conformal Riemannian mappings

$$\varphi : U \to \Omega_{\mathbb{U}},\ \chi : L \to \Omega_{\mathbb{L}}$$

such that the groups

$$\Gamma_{\mathbb{U}} = (\varphi)^{-1} \circ G \circ \varphi, \ \ \Gamma_{\mathbb{L}} = (\chi)^{-1} \circ G \circ \chi$$

are Fuchsian groups acting on $\mathbb{U}$ and $\mathbb{L}$ respectively. We state Bers' theorem for the case of a deformation space of a quasi-Fuchsian group:

THEOREM 2.2.9. *Let $G$ be a finitely generated quasi-Fuchsian group. $Def(G)$ is a connected complex manifold. There exists a biholomorphism*

$$\Psi_G : Def(G) \to Teich(\Gamma_U, \mathbb{U}) \times Teich(\Gamma_L, \mathbb{L})$$

Consider $\eta_G : L^\infty(G) \to L^\infty(\Gamma_U, \mathbb{U}) \times L^\infty(\Gamma_L, \mathbb{L})$ which sends each $\mu \in L^\infty(G)$ to $(\mu_{\mathbb{U}}, \mu_{\mathbb{L}}) \in L^\infty(\Gamma_{\mathbb{U}}, \mathbb{U}) \times L^\infty(\Gamma_{\mathbb{L}}, \mathbb{L})$ where

$$\mu_{\mathbb{U}}(z) = (\mu \circ \varphi)(z) \frac{\overline{\varphi'(z)}}{\varphi'(z)}, \ \ z \in \mathbb{U},$$

$$\mu_{\mathbb{L}}(z) = (\mu \circ \chi)(z) \frac{\overline{\chi'(z)}}{\chi'(z)}, \ \ z \in \mathbb{L}.$$

$\eta_G$ maps $Belt(G)$ bijectively onto $Belt(\Gamma_{\mathbb{U}}, \mathbb{U}) \times Belt(\Gamma_{\mathbb{L}}, \mathbb{L})$. $\Psi_G$ is defined so that it maps each $[\mu]$ to $([\mu_{\mathbb{U}}], [\mu_{\mathbb{L}}])$.

Holomorphic tangent and cotangent spaces to $Def(G)$ at the origin are the spaces $R(G)$ and $Q(G)$ respectively. An alternative description for the holomorphic tangent space is that of the space $L_c^\infty(G)$ of canonical Beltrami differentials on $\Omega$. An element $\mu \in L^\infty(G)$ is called canonical if $\mu = (\lambda_\Omega)^{-2}\overline{\phi}, \quad \phi \in Q(G)$, and $\lambda_\Omega$ is the hyperbolic metric in $\Omega$. The following is easy to verify:

PROPOSITION 2.2.10. *$\mu \in L_c^\infty(G)$ if and only if $\mu_{\mathbb{U}}, \mu_{\mathbb{L}}$ are elements of $L_c^\infty(\Gamma_{\mathbb{U}}, \mathbb{U}), L_c^\infty(\Gamma_{\mathbb{L}}, \mathbb{L})$ respectively.*

*The projection opertator $P_\Omega : L^\infty(G) \to L_c^\infty(G)$ is*

$$P_\Omega = (\eta_G)^{-1} \circ (P_{\mathbb{U}} \times P_{\mathbb{L}}) \circ \eta_G$$

*where $P_{\mathbb{U}}, P_{\mathbb{L}}$ are the projection operators of $L_c^\infty(\Gamma_{\mathbb{U}}, \mathbb{U}), L_c^\infty(\Gamma_{\mathbb{L}}, \mathbb{L})$ respectively.*

PROOF. Since $\eta_G$ maps bijectively $L_c^\infty(G)$ onto $L_c^\infty(\Gamma_{\mathbb{U}}, \mathbb{U}) \times L_c^\infty(\Gamma_{\mathbb{L}}, \mathbb{L})$, $P_\Omega$ is well defined. Also, since $P_{\mathbb{U}}^2 = P_{\mathbb{U}}$ and $P_{\mathbb{L}}^2 = P_{\mathbb{L}}$, it follows that $P_\Omega^2 = P_\Omega$. ∎

**Tangent spaces of $QF(S)$ at any point.** Let $\xi \in Belt(\Gamma)$ and $[\xi] \in QF(S)$. As usual we denote by $w^\xi$ the unique normalised solution of (3). The group $\Gamma^\xi = w^\xi \circ \Gamma \circ (w^\xi)^{-1}$ is quasi-Fuchsian.

The mapping

$$a^\xi : QF(S) \to Def(\Gamma^\xi)$$

(see 2.2.1) is biholomorphic.We have that

i) the holomorphic tangent space of $Def(\Gamma^\xi)$ at the origin is

$$L_c^\infty(\Gamma^\xi) = (\widetilde{a^\xi})_*(L_c^\infty(\Gamma))$$

and

ii) the holomorphic cotangent space at the origin is

$$Q(\Gamma^\xi) = ((a^\xi)^{-1})^*(Q(\Gamma)).$$

Therefore if $\mu \in L^\infty(\Gamma)$ then

$$\left(\frac{\partial}{\partial z(\mu)}\right)_{([\xi])} = P_{\Omega^\xi}[L^\xi(\mu)] = (a_*^\xi)^{-1}_{([id_\xi])}\left(\frac{\partial}{\partial z(L^\xi(\mu))}\right)_{([id_\xi])}$$

where $[id_\xi]$ is the origin of $Def(\Gamma^\xi)$.

Let $\mu = S(\mu) + iS(-i\mu) \in L^\infty(\Gamma)$. Then

$$\left(\frac{\partial}{\partial x(\mu)}\right)_{([\xi])} = (a_*^\xi)^{-1}_{([id_\xi])}\left(\frac{\partial}{\partial x(L^\xi(S(\mu)))}, \frac{\partial}{\partial x(L^\xi(S(-i\mu)))}\right)_{([id_\xi])}$$

**2.2.3. Complex distance and length.** To obtain geometric parameters for $QF(S)$, we use the notion of complex distance (see [K3]): if $\alpha, \beta$ are two geodesics in the upper half space $\mathbb{U}^3$, and $\gamma$ is their common perpendicular, then the *complex distance* between $\alpha$ and $\beta$ is $\sigma = d + i\phi$, where $d$ is the hyperbolic distance between $\alpha$ and $\beta$, and $\phi$is the dihedral angle between the plane containing $\alpha$ and $\gamma$ and the plane containing $\beta$ and $\gamma$. If $h$ is a non parabolic isometry of $\mathbb{U}^3$, and $\alpha$ is a geodesic perpendicular to the axis of $h$, then the *complex displacement* of $h$ is the complex distance between $\alpha$ and $h(\alpha)$. If $\alpha$ is a simple closed geodesic on $S$, and $h$ is an element of $\pi_1(S) = \Gamma$ corresponding to $\alpha$, then the *complex length* of $\alpha$ at a point $[\rho] \in QF(S)$, denoted by $\lambda_\alpha([\rho])$, is just the complex displacement of $\rho(h)$. For each such $\alpha$, the *complex length function*

$$\lambda_\alpha : QF(S) \to \mathbb{C}$$

defined in this way is a holomorphic function of $QF(S)$. We mention that

$$\lambda_\alpha = l_\alpha + i\vartheta_\alpha$$

where $l_\alpha$ is the geodesic length function and $\vartheta_\alpha$ is an angle function, obtained by the imaginary part of complex displacement.

**2.2.4. Bending in the Quasifuchsian space.** In [K1,3] C. Kourouniotis defined a holomorphic transformation of $QF(S)$ called the bending deformation, which is a generalisation on $QF(S)$ of the Fenchel-Nielsen as well as of the quakebending deformation[E-M] defined for the Teichmüller space. Kourouniotis' construction is described in detail in [K1]. We review this construction in brief. Suppose that $\alpha$ is a simple closed curve on $S$. For any $[\rho] \in QF(S)$ and $t$ in a neighborhood of 0 in $\mathbb{C}$, we obtain a new quasi-Fuchsian structure $B_\alpha(t,\rho)$ as follows: assume for simplicity that $[\rho] \in F(S)$, that is $\rho(\Gamma) = \Gamma'$ Fuchsian, and $\widetilde{\alpha}$ is the geodesic of the upper half plane $\mathbb{U}$ corresponding to $\alpha$. All the translates $\Gamma'(\widetilde{\alpha})$ lie on $\mathbb{U} \subset \mathbb{U}^3$. We want to map each component of $\mathbb{U} - \Gamma'(\widetilde{\alpha})$ isometrically to a flat 2-dimensional piece in $\mathbb{U}^3$, in a way such that each component is moved with respect to its neighbours by a complex distance $t$. If $t$ is sufficiently small, then we can construct a quasiconformal homeomorphism $w : \mathbb{U}^3 \to \mathbb{U}^3$, which does exactly that, and which defines a quasi-Fuchsian structure $B_\alpha(t,\rho) : \Gamma \to \mathbb{PSL}(2,\mathbb{C})$ (bending $\rho$ along $\alpha$) mapping each $\gamma$ to $w \circ \rho(\gamma) \circ w^{-1}$. For the non-Fuchsian case, we only note that all the necessary information is encoded in the order of the endpoints of geodesics around the limit set of the corresponding quasi-Fuchsian group. Furthermore, and under assumptions of discreteness, $B_\alpha(t,\rho)$ defines a local holomorphic flow of $QF(S)$.

**2.2.5. Variations of complex length functions of geodesics.** Associated to bending along $\alpha$ there is a holomorphic vector field (cf. prop. 2.2.12 below)

$$T_\alpha([\rho]) = \frac{d}{dt}(0)(B_\alpha(t,\rho))$$

called bending vector field (associated to $\alpha$). Bending vector fields are related to complex length functions by identities which are generalisations to $QF(S)$ of analogue identities which were firstly given by S. Wolpert.

Let $\alpha$ be a simple closed geodesic on $S$ and $\lambda_\alpha$ its complex length function. Suppose that $\beta$ is another simple closed geodesic of $S$. The first variation $T_\beta\lambda_\alpha$ of $\lambda_\alpha$ under bending along $\beta$ is given by

$$T_\beta\lambda_\alpha = \frac{d}{dt}(0)(\lambda_\alpha(B_\beta(t,\rho)))$$

If $\gamma$ is a simple closed geodesic on $S$, then the second variation $T_\gamma T_\beta\lambda_\alpha$ of $\lambda_\alpha$ under bending along $\beta$ and $\gamma$ is given by

$$T_\gamma T_\beta\lambda_\alpha = \frac{\partial^2}{\partial s \partial t}(0,0)(\lambda_\alpha(B_\beta(t, B_\gamma(s,\rho))))$$

Variations of complex length have the following geometric interpretation [K3]:

$$T_\beta \lambda_\alpha = \sum_{p \in \alpha \cap \beta} \cosh \sigma(\rho(\beta), \rho(\alpha))_p$$

$$T_\gamma T_\beta \lambda_\alpha =$$

$$\frac{\sum_{p\in\alpha\cap\beta}\sum_{q\in\alpha\cap\gamma} \sinh\sigma(\rho(\alpha),\rho(\beta))_p \sinh\sigma(\rho(\alpha),\rho(\gamma))_q \cosh(\frac{1}{2}\lambda_\alpha - \sigma(p,q)}{2\sinh\frac{1}{2}\lambda_\alpha} +$$

$$+ \frac{\sum_{p\in\alpha\cap\beta}\sum_{r\in\beta\cap\gamma} \sinh\sigma(\rho(\alpha),\rho(\beta))_p \sinh\sigma(\rho(\beta),\rho(\gamma))_r \cosh(\frac{1}{2}\lambda_\alpha - \sigma(p,r)}{2\sinh\frac{1}{2}\lambda_\beta}.$$

The following identities due to C. Kourouniotis [K3], are crucial for our construction of the complex symplectic structure of $QF(S)$ given in the next chapter:

1. $T_\alpha \lambda_\beta + T_\beta \lambda_\alpha = 0$

2. $T_\alpha T_\beta \lambda_\gamma + T_\beta T_\alpha \lambda_\gamma + T_\gamma T_\beta \lambda_\alpha = 0$

**2.2.6. Complex coordinates.** The following theorems provide complex coordinates for $QF(S)$ [K2]:

THEOREM 2.2.11. *Given a partition of the surface $S$ by $3g-3$ simple closed geodesics $\gamma_i$, there exist complex length functions $\lambda_{\gamma_i} : QF(S) \to \mathbb{C}$ and bending functions $\beta_i : QF(S) \to \mathbb{C}$ $i = 1, ..., 3g-3$ which form a system of global holomorphic coordinates for $QF(S)$.( complex F-N coordinates).*

THEOREM 2.2.12. *Given a partition of the surface $S$ by $3g-3$ simple closed geodesics $\gamma_i$, then for every $[\rho] \in QF(S)$ there exist a neighborhood $V([\rho])$ and $3g-3$ simple closed curves $\alpha_i$ with $\gamma_i \cap \alpha_j = \emptyset$ if $i \neq j$ such that the complex length functions $\lambda_{\gamma_i}, \lambda_{\alpha_i}$ $i = 1, ..., 3g-3$ form a system of local holomorphic coordinates for $QF(S)$.*

**2.2.7. Holomorphic nature of bending vector fields.** The following is in [K2] (Prop. 3.10):

PROPOSITION 2.2.13. *Let $\alpha$ be a simple closed geodesic on $S$. $T_\alpha$ is a holomorphic vector field of $QF(S)$.*

Since $T_\alpha$ is holomorphic, it can be written as

$$T_\alpha = \frac{1}{2}(F_\alpha - i I_Q F_\alpha)$$

where $F_\alpha$ is a real vector field and $I_Q$ is the complex operator defined on the tangent space of of $QF(S)$.

PROPOSITION 2.2.14. *Let $\alpha, \beta$ be simple closed geodesics on $S$ and $T_\alpha$ the corresponding holomorphic bending vector field on $QF(S)$. Then in the tangent space of $F(S)$ we have*

$$F_\alpha l_\beta = t_\alpha l_\beta, \quad (I_Q F_\alpha) l_\beta = 0$$

*where $t_\alpha$ is a twist vector field corresponding to $\alpha$ and $l_\beta$ is the geodesic length function corresponding to $\beta$.*

PROOF. Making elementary calculations we have

$$T_\alpha \lambda_\beta = \frac{1}{2}(F_\alpha - iI_Q F_\alpha)(l_\beta + i\vartheta_\beta) = \frac{1}{2}(F_\alpha l_\beta + (I_Q F_\alpha)\vartheta_\beta) + \frac{i}{2}(F_\alpha \vartheta_\beta - (I_Q F_\alpha) l_\beta)$$

On the other hand

$$T_\alpha \lambda_\beta = \sum_{p\in\alpha\cap\beta} \cosh \sigma(\rho(\alpha), \rho(\beta))_p =$$

$$\sum_{p\in\alpha\cap\beta} \cosh d(\rho(\alpha), \rho(\beta))_p \cos \phi(\rho(\alpha), \rho(\beta))_p +$$

$$i \sum_{p\in\alpha\cap\beta} \sinh d(\rho(\alpha), \rho(\beta))_p \sin \phi(\rho(\alpha), \rho(\beta))_p$$

Equating real and imaginary parts and applying Cauchy-Riemann equations we obtain:

$$F_\alpha l_\beta = (I_Q F_\alpha)\vartheta_\beta = \sum_{p\in\alpha\cap\beta} \cosh d(\rho(\alpha), \rho(\beta))_p \cos \phi(\rho(\alpha), \rho(\beta))_p$$

$$F_\alpha \vartheta_\beta = -(I_Q F_\alpha) l_\beta = \sum_{p\in\alpha\cap\beta} \sinh d(\rho(\alpha), \rho(\beta))_p \sin \phi(\rho(\alpha), \rho(\beta))_p$$

At Fuchsian points we have that $d(\rho(\alpha), \rho(\beta)) = 0$, since $\rho(\alpha), \rho(\beta)$ are intersecting geodesics in $\mathbb{U}^3$ and therefore

$$F_\alpha l_\beta = t_\alpha l_\beta = \sum_{p\in\alpha\cap\beta} \cos \phi(\rho(\alpha), \rho(\beta))_p$$

and

$$(I_Q F_\alpha) l_\beta = 0.$$

□

COROLLARY 2.2.15. *When restricted to the tangent subbundle of $F(S)$, $F_\alpha$ is the twist vector field $t_\alpha$.*

CHAPTER 3

# Differential Geometries of Quasifuchsian space

## 3.1. Complex symplectic geometry

In this section we describe in detail the construction of a complex symplectic form for $QF(S)$. We firstly discuss some general aspects on complex symplectic manifolds.

### 3.1.1. Complex symplectic manifolds.

DEFINITION 3.1.1. Let $M$ be a $2n$ complex manifold. We say that $M$ is a *complex symplectic* manifold if it carries a complex symplectic structure i.e there is a non degenerate, closed $(2,0)$- form $\Omega$ defined everywhere on $M$.

It is simple to check that a complex symplectic manifold is also a real symplectic manifold: If $\Omega$ is the complex symplectic form on $M$, then $\Omega = \omega + i\varphi$ and $\omega, \varphi$ are non-degenerate real closed 2-forms defining symplectic structures on $M$.

We state some definitions and properties holding on complex symplectic manifolds. Their proof is analogue to that in the real case and can be found for instance in [L-M].

1. The holomorphic form $\Omega$ defines an isomorphism between holomorphic tangent and cotangent bundles of $M$. For every $p \in M$ and every holomorphic vector $Z \in T^{(1,0)}_{(p)}(M)$, the isomorphism is described by the holomorphic 1-form $\Omega(Z)$ where

$$\Omega(Z)_{(p)} = \Omega_{(p)}(Z, \cdot)$$

2. Let $f$ be a holomorphic function on $M$. A holomorphic vector field $H^{\mathbb{C}}_f$which satisfies

$$\Omega(H^{\mathbb{C}}_f)(\Xi) = \Omega(H^{\mathbb{C}}_f, \Xi) = -d'f(\Xi) = -\Xi f.$$

for every holomorphic vector field $\Xi$ defined on $M$ shall be called the *complex Hamiltonian* of $f$. (Here by $d'$ we denote the holomorphic differential operator on $M$. If $d$ is the usual differential then $d = d' + d''$ where $d''$ is the antiholomorphic differential on $M$ ).

3. If $\phi_t$ is the local holomorphic flow of $H_f^{\mathbb{C}}$, then for sufficiently small $t$ we have

$$\phi_t^* \Omega = \Omega.$$

4. If $L_Z$ denotes the Lie derivative with respect to a holomorphic vector field $Z$, then

$$L_{H_f^{\mathbb{C}}}(\Omega) = 0.$$

5. Suppose that in a neighborhood of a point $p$ of a complex symplectic manifold $(M, \Omega)$ there exist coordinates $(z_1, ..., z_{2n})$ such that $\Omega$ can be expressed as

$$\Omega = \sum_{i=1}^{n} d' z_i \wedge d' z_{i+n}.$$

Then $(z_1, ..., z_{2n})$ are called canonical coordinates for $M$ in the neighborhood of $p$.

**3.1.2. Construction of the complex symplectic structure.** We start with the following:

PROPOSITION 3.1.2. *Let $\gamma_i, \alpha_i$ be as in Theorem 2.2.11. The bending vector fields $T_{\gamma_i}, T_{\alpha_i}$ form a local basis of the holomorphic tangent space of $QF(S)$.*

PROOF. We consider an open neighborhood $V([\rho_0])$ of $[\rho_0] \in QF(S)$ and local coordinates

$$(\lambda_{\gamma_1}, ..., \lambda_{\gamma_{3g-3}}, \lambda_{\alpha_1}, ..., \lambda_{\alpha_{3g-3}})$$

The bending vectors $T_{\gamma_i}, T_{\alpha_i}$ are linear combinations of the vectors $\frac{\partial}{\partial \lambda_{\gamma_i}}, \frac{\partial}{\partial \lambda_{\alpha_i}}$ : At each $[\rho] \in V([\rho_0])$ we have

$$[T] = \begin{bmatrix} 0 & A \\ -A & B \end{bmatrix} \left[ \ \frac{\partial}{\partial \lambda} \ \right]$$

where:

$$[T] = [T_{\gamma_1}, ..., T_{\gamma_{3g-3}}, T_{\alpha_1}, ..., T_{\alpha_{3g-3}}]^T,$$

$$\left[ \ \frac{\partial}{\partial \lambda} \ \right] = \left[ \frac{\partial}{\partial \lambda_{\gamma_1}}, ..., \frac{\partial}{\partial \lambda_{\gamma_{3g-3}}}, \frac{\partial}{\partial \lambda_{\alpha_1}}, ..., \frac{\partial}{\partial \lambda_{\alpha_{3g-3}}} \right]^T,$$

the matrix $B$ is a non zero $(3g-3) \times (3g-3)$ complex matrix with entries $T_{\alpha_i} \lambda_{\alpha_j}$ or 0, and the matrix $A$ is a diagonal $(3g-3) \times (3g-3)$ matrix with entries

$$T_{\gamma_i}\lambda_{\alpha_i} = \sum_{p\in\gamma_i\cap\alpha_i} \cosh\sigma(\rho(\gamma_i),\rho(\alpha_i))_p$$

Thus the determinant of the transformation is equal to $(-1)^{3g-3}(\det A)^2$ which is different from zero, since at each $[\rho] \in V([\rho_0])$ we have $T_{\gamma_i}\lambda_{\alpha_i} \neq 0$ for every $i = 1, ..., 3g-3$. □

THEOREM 3.1.3. *There exists a non degenerate closed holomorphic* $(2,0)$ *form* $\Omega$ *defined everywhere on* $QF(S)$ *turning* $QF(S)$ *into a complex symplectic manifold. The form* $\Omega$ *is given at each* $[\rho] \in QF(S)$ *by the formula:*

$$\Omega_{([\rho])}(T_\alpha, T_\beta) = \sum_{p\in\alpha\cap\beta} \cosh\sigma(\rho(\alpha),\rho(\beta))_p$$

*where* $T_\alpha, T_\beta$ *are bending vectors corresponding to simple closed geodesics* $\alpha, \beta$ *on* $S$*, and* $\sigma(\rho(\alpha),\rho(\beta))$ *is the complex distance of geodesics* $\rho(\alpha), \rho(\beta)$*.*

The proof will be given in steps:

STEP 1. *Local definition of the form.*

We firstly define locally a closed holomorphic 2-form in the holomorphic tangent space of $QF(S)$. Let $V([\rho_0])$ be an open neighborhood of $[\rho_0] \in QF(S)$, $(\lambda_{\gamma_1}, ..., \lambda_{\gamma_{3g-3}}, \lambda_{\alpha_1}, ..., \lambda_{\alpha_{3g-3}})$ local coordinates for $V([\rho_0])$ and $T_{\gamma_i}, T_{\alpha_i}$ $i = 1, ..., 3g-3$ the basis of the holomorphic tangent space consisting of bending vectors. For indices $\alpha, \beta$ running through indices $\gamma_i, \alpha_i$ we set for each $[\rho] \in V([\rho_0])$

$$\Omega_{([\rho])}(T_\alpha, T_\beta) = T_{\alpha_{([\rho])}}\lambda_\beta = \sum_{p\in\alpha\cap\beta} \cosh\sigma(\rho(\alpha),\rho(\beta))_p$$

By Kourouniotis' first identity, the skew symmetry of $\Omega$ is instantly obtained. On the other hand $\Omega$ is a holomorphic form, since the quantities $T_\alpha\lambda_\beta$ are all holomorphic functions in $V([\rho_0])$.

STEP 2. *Independence of the choice of coordinates.*

The form $\Omega$ does not depend on the choice of local coordinates: To see this, we consider $\alpha, \beta$ simple closed geodesics on $S$ and we establish that the above formula is also valid in this case. Indeed by Prop.3.1.2 there exist holomorphic functions $f_\alpha^i, g_\alpha^i$ and $f_\beta^i, g_\beta^i$ $i = 1, ..., 3g-3$ defined on $V([\rho_0])$ such that

$$T_\alpha = f_\alpha^i T_{\gamma_i} + g_\alpha^i T_{\alpha_i}$$

$$T_\beta = f^i_\beta T_{\gamma_i} + g^i_\beta T_{\alpha_i}$$

where upper and lower indices denote summation. We now calculate straight forward:

$$\Omega(T_\alpha, T_\beta) = \Omega(f^i_\alpha T_{\gamma_i} + g^i_\alpha T_{\alpha_i}, f^i_\beta T_{\gamma_i} + g^i_\beta T_{\alpha_i}) =$$

$$f^i_\alpha f^j_\beta \Omega(T_{\gamma_i}, T_{\gamma_j}) + f^i_\alpha g^j_\beta \Omega(T_{\gamma_i}, T_{\alpha_j}) + g^i_\alpha f^j_\beta \Omega(T_{\alpha_i}, T_{\gamma_j}) + g^i_\alpha g^j_\beta \Omega(T_{\alpha_i}, T_{\alpha_j}) =$$

$$f^i_\alpha f^j_\beta T_{\gamma_i} \lambda_{\gamma_j} + f^i_\alpha g^j_\beta T_{\gamma_i} \lambda_{\alpha_j} + g^i_\alpha f^j_\beta T_{\alpha_i} \lambda_{\gamma_j} + g^i_\alpha g^j_\beta T_{\alpha_i} \lambda_{\alpha_j} =$$

$$f^j_\beta (f^i_\alpha T_{\gamma_i} + g^i_\alpha T_{\alpha_i}) \lambda_{\gamma_j} + g^j_\beta (f^i_\alpha T_{\gamma_i} + g^i_\alpha T_{\alpha_i}) \lambda_{\alpha_j} = f^j_\beta T_\alpha \lambda_{\gamma_j} + g^j_\beta T_\alpha \lambda_{\alpha_j} =$$

$$= -f^j_\beta T_{\gamma_j} \lambda_\alpha - g^j_\beta T_{\alpha_j} \lambda_\alpha = -T_\beta \lambda_\alpha = T_\alpha \lambda_\beta .$$

STEP 3. *$\Omega$ is closed.*

Consider bending vector fields $T_\alpha, T_\beta, T_\gamma$ on simple closed curves $\alpha, \beta, \gamma$ of $S$ respectively. Then

$$d'\Omega(T_\alpha, T_\beta, T_\gamma) = T_\alpha \Omega(T_\beta, T_\gamma) - T_\beta \Omega(T_\alpha, T_\gamma) + T_\gamma \Omega(T_\alpha, T_\beta) -$$

$$-\Omega([T_\alpha, T_\beta], T_\gamma) + \Omega([T_\alpha, T_\gamma], T_\beta) - \Omega([T_\beta, T_\gamma], T_\alpha) =$$

$$T_\alpha T_\beta \lambda_\gamma - T_\beta T_\alpha \lambda_\gamma + T_\gamma T_\alpha \lambda_\beta - [T_\alpha, T_\beta] \lambda_\gamma + [T_\alpha, T_\gamma] \lambda_\beta - [T_\beta, T_\gamma] \lambda_\alpha =$$

$$T_\alpha T_\beta \lambda_\gamma - T_\beta T_\alpha \lambda_\gamma - T_\gamma T_\beta \lambda_\alpha - T_\alpha T_\beta \lambda_\gamma + + T_\beta T_\alpha \lambda_\gamma + T_\alpha T_\gamma \lambda_\beta - T_\gamma T_\alpha \lambda_\beta -$$

$$-T_\beta T_\gamma \lambda_\alpha + T_\gamma T_\beta \lambda_\alpha = 0.$$

STEP 4. *$\Omega$ is non degenerate.*

Suppose that there exists a holomorphic vector field $Z$ such that $\Omega(Z, W) = 0$ for all holomorphic vector fields $W$. This equation is equivalent locally to the $6g - 6$ equations

$$\Omega(Z, T_\alpha) = 0$$

where $\alpha$ is an index running among indices $\gamma_i, \alpha_i$ $i = 1, ..., 3g-3$. Since an expression for $Z$ in local coordinates is

$$Z = f^i T_{\gamma_i} + g^i T_{\alpha_i}$$

for some holomorphic functions $f^i, g^i$, we obtain from the equations

$$\Omega(Z, T_{\gamma_i}) = 0$$

that $g^i = 0$, since

$$\Omega(T_{\gamma_i}, T_{\gamma_j}) = 0$$

and

$$\Omega(T_{\alpha_i}, T_{\gamma_j}) = \begin{cases} \sum_{p \in \alpha_i \cap \gamma_j} \cosh \sigma(\rho(\alpha_i), \rho(\gamma_j))_p & i = j \\ 0 & i \neq j \end{cases}$$

Therefore $Z = f^i T_{\gamma_i}$, and by the same reasoning we get from the equations $\Omega(Z, T_{\alpha_i}) = 0$ that $f^i = 0$. Thus $Z = 0$ and $\Omega$ is non-degenerate.
The proof is therefore completed.

Two immediate consequences of Theorem 3.1.3 follow:

COROLLARY 3.1.4. *Let $B_t$ be the local holomorphic flow induced by bending. Then*

$$B_t^* \Omega = \Omega.$$

COROLLARY 3.1.5. *For each bending vector field $T_\alpha$, the following holds:*

$$L_{T_\alpha}(\Omega) = 0.$$

The next theorem is also obvious and generalises Wolpert's Duality formula (Theorem 2.1.4. ii)):

THEOREM 3.1.6. ***(Duality formula)****: Let $\alpha$ be a simple closed geodesic on the surface $S$. The bending vector field $T_\alpha$ is the complex Hamiltonian of the complex length function $\lambda_\alpha$ :*

$$H^{\mathbb{C}}_{\lambda_\alpha} = T_\alpha$$

*that is*

$$\Omega(T_\alpha, \cdot) = -d' \lambda_\alpha$$

**3.1.3. Expression of $\Omega$ in global complex F-N coordinates.** In this section we prove an analogue of S. Wolpert's Theorem 2.1.4. iii). We need an analytic continuation argument first.

LEMMA 3.1.7. *Let $D$ be an open connected subset of $\mathbb{C}^n$, and $F : D \to \mathbb{C}$ a holomorphic function,*

$$F = F(z_1, ...z_n),\ z_i = x_i + iy_i\ i = 1, ..., n.$$

*Suppose that $D$ contains segments of*

$$\mathbb{R}^i = \{\overline{x_i} \in \mathbb{C}^n, \overline{x_i} = (0, ..., x_i, ..., 0, 0, ..., 0)\}$$

*for every $i = 1, ..., n$, and $F(x_1, ...x_n, 0, ..., 0) = 0$ for every $(x_1, ...x_n, 0, ..., 0) \in D$. Then $F \equiv 0$ in $D$.*

PROOF. We fix a polydisk $\Delta = \Delta(\overline{x_1^0}, r_1) \times ... \times \Delta(\overline{x_n^0}, r_n)$ where

$$\overline{x_i^0} = (0, ..., x_i^0, ..., 0, 0, ..., 0) \in D$$

for every $i = 1, ..., n$ and $r_i > 0$ are small enough so that $\Delta \subset D$. Now $F_{|\Delta}(x_1, ...x_n, 0, ..., 0) = 0$ and therefore $F_{|\Delta} = 0$. Since $F$ vanishes in an open subset of $D$, it has to vanish everywhere in $D$. □

Let now $\Omega = \omega_1 + i\omega_2$. Consider the real symplectic form $\omega_{WP}$ of the real submanifold $F(S) \cong Teich(S)$ of $QF(S)$.

THEOREM 3.1.8. *Let $\omega_{WP}$ be the symplectic form of $Teich(S)$ and $\omega_1 = Re\Omega, \omega_2 = Im\Omega$ restricted to the tangent subbundle of Fuchsian space. Then $\omega_1 = \omega_{WP}$ and $\omega_2 = 0$.*

PROOF. Let $\alpha, \beta$ be simple closed curves on $S$ and $t_\alpha, t_\beta$ the corresponding twist vector fields. At a point $[\rho]$ of $F(S)$ we have

$$(\omega_{WP})_{([\rho])}(t_\alpha, t_\beta) = \sum_{p \in \alpha \cap \beta} \cos\theta(\rho(\alpha), \rho(\beta))_p = \frac{d}{dx}\Big|_{x=0} (l_\beta(B_\alpha(x, \rho)) =$$

by the holomorphic character of bending

$$= Re\left[\frac{d}{dt}\Big|_{t=x+iy=0} (\lambda_\beta(B_\alpha(t, \rho))\right] = Re[T_{\alpha_{([\rho])}} \lambda_\beta] =$$

by definition of $\Omega$

$$= Re[\Omega_{([\rho])}(T_\alpha, T_\beta)] = \omega_{1_{([\rho])}}(F_\alpha, F_\beta) =$$

by corollary 2.2.14

$$= \omega_{1_{([\rho])}}(t_\alpha, t_\beta).$$

In an analogous manner we prove the second relation. At Fuchsian points we have:

$$\omega_{1_{([\rho])}}(t_\alpha, t_\beta) = \omega_{2_{([\rho])}}(F_\alpha, F_\beta) = Im[T_{\alpha_{([\rho])}} \lambda_\beta] = 0.$$

□

The previous theorem provides informations concerning the symplectic behaviour of $Teich(S)$ inside $QF(S)$. Let $(M, \omega)$ a symplectic manifold of dimension $2n$. Recall that a submanifold $N$ of dimension $n$ is called: a) symplectic if $\omega_N$, the restriction of $\omega$ to the tangent subbundle of $N$ is a symplectic form for $N$, and b) Lagrangian if $\omega_N = 0$.

Therefore, from Theorem 3.1.8 we have the following

COROLLARY 3.1.9. *Teichmüller space is: a) an $\omega_1$−symplectic submanifold of $QF(S)$ and b) an $\omega_2$−Lagrangian submanifold of $QF(S)$.*

We are ready now to prove the main theorem of this section:

THEOREM 3.1.10. *The expression of $\Omega$ in global complex F-N coordinates is*

$$\Omega = \sum_{i=1}^{3g-3} d'\lambda_{\gamma_i} \wedge d'\beta_i$$

*Also*

$$\Omega\left(\frac{\partial}{\partial\beta_i}, \cdot\right) = -d'\lambda_{\gamma_i} \quad \Omega\left(\frac{\partial}{\partial\lambda_{\gamma_i}}, \cdot\right) = d'\beta_i$$

*and therefore the holomorphic vector fields $\frac{\partial}{\partial\beta_i}, \frac{\partial}{\partial\lambda_{\gamma_i}}$ are complex Hamiltonian for the form $\Omega$.*

NOTE 3.1.11. The coefficients of $\Omega$ in the complex F-N coordinates do not depend on the bending parameters. Recall that if $\Omega$ is a (2,0) form and $Z, Z_1, Z_2$ are holomorphic vector fields on a complex manifold $M$, then for the Lie derivative $L_Z$ of $\Omega$ we have:

$$L_Z\Omega(Z_1, Z_2) = Z\Omega(Z_1, Z_2) - \Omega([Z, Z_1], Z_2) - \Omega(Z_1, [Z, Z_2]).$$

Set $Z = \frac{\partial}{\partial\beta_j}$ for some $j$ and $Z_1, Z_2 \in \{\frac{\partial}{\partial\beta_i}, \frac{\partial}{\partial\lambda_{\gamma_i}}\}$. Since $\frac{\partial}{\partial\beta_j} = H^{\mathbb{C}}_{\lambda_{\gamma_j}}$, we obtain that $L_{\frac{\partial}{\partial\beta_j}}\Omega = 0$. Since $Z, Z_1, Z_2$ are coordinate vector fields, they commute and therefore we have:

$$\frac{\partial}{\partial\beta_j}\Omega(\frac{\partial}{\partial\beta_i}, \frac{\partial}{\partial\lambda_{\gamma_i}}) = 0$$

a relation which proves that the coeefficients of $\Omega$ are independent of the bending parameters.

PROOF. By duality formula of Theorem 3.1.6, we easily see that the following hold:

$$\Omega\left(\frac{\partial}{\partial\beta_i},\frac{\partial}{\partial\lambda_{\gamma_j}}\right)=-\delta_{ij},$$

and

$$\Omega\left(\frac{\partial}{\partial\beta_i},\frac{\partial}{\partial\beta_j}\right)=0,$$

for each $i,j=1,...,3g-3$. Therefore,

$$\Omega=\sum_{i=1}^{3g-3}d'\lambda_{\gamma_i}\wedge d'\beta_i+\sum_{1\le i<j\le 3g-3}\Omega\left(\frac{\partial}{\partial\lambda_{\gamma_i}},\frac{\partial}{\partial\lambda_{\gamma_j}}\right)d'\lambda_{\gamma_i}\wedge d'\lambda_{\gamma_j}$$

Straightforward calculations lead us to the following expressions for the real and imaginary part of $\Omega$ :

$$\omega_1=Re\Omega=\sum_{i=1}^{3g-3}(dl_{\gamma_i}\wedge d\tau_i-d\vartheta_i\wedge d\psi_i)+$$

$$+\sum_{1\le i<j\le 3g-3}Re\left[\Omega\left(\frac{\partial}{\partial\lambda_{\gamma_i}},\frac{\partial}{\partial\lambda_{\gamma_j}}\right)\right](dl_{\gamma_i}\wedge dl_{\gamma_j}-d\vartheta_i\wedge d\vartheta_j)-$$

$$-\sum_{1\le i<j\le 3g-3}Im\left[\Omega\left(\frac{\partial}{\partial\lambda_{\gamma_i}},\frac{\partial}{\partial\lambda_{\gamma_j}}\right)\right](d\vartheta_i\wedge dl_{\gamma_j}-dl_{\gamma_i}\wedge d\vartheta_j)$$

and

$$\omega_2=Im\Omega=\sum_{i=1}^{3g-3}(d\vartheta_i\wedge d\tau_i-dl_{\gamma_i}\wedge d\psi_i)+$$

$$+\sum_{1\le i<j\le 3g-3}Re\left[\Omega\left(\frac{\partial}{\partial\lambda_{\gamma_i}},\frac{\partial}{\partial\lambda_{\gamma_j}}\right)\right](d\vartheta_i\wedge dl_{\gamma_j}-dl_{\gamma_i}\wedge d\vartheta_j)+$$

$$+\sum_{1\le i<j\le 3g-3}Im\left[\Omega\left(\frac{\partial}{\partial\lambda_{\gamma_i}},\frac{\partial}{\partial\lambda_{\gamma_j}}\right)\right](dl_{\gamma_i}\wedge dl_{\gamma_j}-d\vartheta_i\wedge d\vartheta_j)$$

where

$$Re\left[\Omega\left(\frac{\partial}{\partial\lambda_{\gamma_i}},\frac{\partial}{\partial\lambda_{\gamma_j}}\right)\right]=\omega_1\left(\frac{\partial}{\partial l_{\gamma_i}},\frac{\partial}{\partial l_{\gamma_j}}\right)$$

and

$$Im\left[\Omega\left(\frac{\partial}{\partial\lambda_{\gamma_i}},\frac{\partial}{\partial\lambda_{\gamma_j}}\right)\right]=\omega_2\left(\frac{\partial}{\partial l_{\gamma_i}},\frac{\partial}{\partial l_{\gamma_j}}\right).$$

When restricting ourselves on the tangent bundle of $F(S)$ the functions $\vartheta_i,\psi_i$ are all zero, therefore in this case

$$\Omega=\sum_{i=1}^{3g-3}dl_{\gamma_i}\wedge d\tau_i+\sum_{1\le i<j\le 3g-3}\Omega\left(\frac{\partial}{\partial\lambda_{\gamma_i}},\frac{\partial}{\partial\lambda_{\gamma_j}}\right)dl_{\gamma_i}\wedge dl_{\gamma_j}.$$

We shall show that the holomorphic functions

$$\Omega_{ij}([\rho])=\Omega_{([\rho])}\left(\frac{\partial}{\partial\lambda_{\gamma_i}},\frac{\partial}{\partial\lambda_{\gamma_j}}\right)$$

are real valued when restricted to $F(S)$. Let $[\rho]$ be a Fuchsian point. Then by Prop. 3.1.2, we have that in a neighbourhood of $[\rho]$

$$\frac{\partial}{\partial\lambda_{\gamma_i}}=f^j_{\gamma_i}T_{\gamma_j}+g^j_{\gamma_j}T_{\alpha_j}$$

where $f^j_{\gamma_i},g^j_{\gamma_j}$ are holomorphic functions defined on this neighborhood. According to the matrix equation in Prop. 3.1.2, we obtain by multiplying with the inverse matrix that

$$\left[\frac{\partial}{\partial\lambda}\right]=\begin{bmatrix}A^{-1}BA^{-1} & -A^{-1}\\ A^{-1} & 0\end{bmatrix}[T]$$

Therefore

$$\left[\frac{\partial}{\partial\lambda_\gamma}\right]=\begin{bmatrix}A^{-1}BA^{-1} & -A^{-1}\end{bmatrix}[T]$$

where

$$\left[\frac{\partial}{\partial\lambda_\gamma}\right]=\left[\frac{\partial}{\partial\lambda_{\gamma_1}}\cdots\frac{\partial}{\partial\lambda_{\gamma_{3g-3}}}\right]^T$$

Recall that the entries of the diagonal matrix $A$ are of the form $T_{\gamma_i}\lambda_{\alpha_i}$ or 0 wheras the entries of the matrix $B$ are of the form $T_{\alpha_i}\lambda_{\alpha_j}$ or 0. All these quantities (and eventually the quantities $f^j_{\gamma_i},g^j_{\gamma_j}$ which are the entries of the matrices $A^{-1}BA^{-1}$ and $-A^{-1}$ respectively) evaluated on a Fuchsian point are real, for if $[\rho]$ is such a point then we have

$$T_\alpha\lambda_\beta=\sum_{p\in\alpha\cap\beta}\cos\phi(\rho(\alpha),\rho(\beta))_p$$

where $\alpha,\beta$ are indices running over indices $\gamma_i,\alpha_i$, $i=1,...,3g-3$. After this, it is easy to see that at the point $[\rho]$ we have:

$$\Omega_{ij} = \Omega\left(\frac{\partial}{\partial\lambda_{\gamma_i}}, \frac{\partial}{\partial\lambda_{\gamma_j}}\right) =$$

$$\Omega(f^l_{\gamma_i} T_{\gamma_l} + g^l_{\gamma_i} T_{\alpha_l},\ f^k_{\gamma_j} T_{\gamma_k} + g^k_{\gamma_j} T_{\alpha_k}) =$$

$$f^l_{\gamma_i} f^k_{\gamma_j} T_{\gamma_l} \lambda_{\gamma_k} + f^l_{\gamma_i} g^k_{\gamma_j} T_{\gamma_l} \lambda_{\alpha_k} +$$

$$+ g^l_{\gamma_i} f^k_{\gamma_j} T_{\alpha_l} \lambda_{\gamma_k} + g^l_{\gamma_i} g^k_{\gamma_j} T_{\alpha_l} \lambda_{\alpha_k}$$

which is clearly real when evaluated at $[\rho]$. From Theorem 3.1.8 and Wolpert's duality formula for $\omega_{WP}$ we have:

$$Re\left[\Omega\left(\frac{\partial}{\partial\lambda_{\gamma_i}}, \frac{\partial}{\partial\lambda_{\gamma_j}}\right)\right] = \omega_1\left(\frac{\partial}{\partial l_{\gamma_i}}, \frac{\partial}{\partial l_{\gamma_j}}\right) = \omega_{WP}\left(\frac{\partial}{\partial l_{\gamma_i}}, \frac{\partial}{\partial l_{\gamma_j}}\right) = 0$$

and

$$Im\left[\Omega\left(\frac{\partial}{\partial\lambda_{\gamma_i}}, \frac{\partial}{\partial\lambda_{\gamma_j}}\right)\right] = \omega_2\left(\frac{\partial}{\partial l_{\gamma_i}}, \frac{\partial}{\partial l_{\gamma_j}}\right) = 0$$

at points of $F(S)$. The holomorphic functions

$$\Omega_{ij} = \Omega_{ij}(\lambda_{\gamma_1}, ..., \lambda_{\gamma_{3g-3}}, \beta_1, ..., \beta_{3g-3})$$

satisfy the assumptions of Lemma 3.1.7. Therefore they are identically zero in $QF(S)$ and the theorem is proved. □

NOTE 3.1.12. The complex length functions $\lambda_{\gamma_i}$ are completely determined by the geodesics $\gamma_i$ of the partition of the surface $S$. But to define the bending functions $\beta_i$, we need a choice of partition of $S$. We deduce that holomorphic differentials $d'\beta_i$ and complex Hamiltonian vector fields $\frac{\partial}{\partial\lambda_{\gamma_i}}$ depend on the choice of the partition.

## 3.2. Weil-Petersson geometry

**3.2.1. Kählerian geometry.** Let $G$ be a finitely generated quasi-Fuchsian group. Denote as usual by $\Omega = \Omega_{\mathbb{U}} \cup \Omega_{\mathbb{L}}$ its region of discontinuity, and $\Gamma_{\mathbb{U}}, \Gamma_{\mathbb{L}}$ the associated Fuchsian groups induced by the conformal Riemann mappings

$$\varphi : \mathbb{U} \to \Omega_{\mathbb{U}},\ \chi : \mathbb{L} \to \Omega_{\mathbb{L}}.$$

The hyperbolic metric $\lambda_\Omega$ in $\Omega$ is defined so that

$$\lambda_\Omega \mid_{\Omega_{\mathbb{U}}} = \lambda_{\Omega_{\mathbb{U}}} \ \ \lambda_\Omega \mid_{\Omega_{\mathbb{L}}} = \lambda_{\Omega_{\mathbb{L}}}$$

where the hyperbolic metrics in $\Omega_{\mathbb{U}}, \Omega_{\mathbb{L}}$ are defined by

$$\lambda_{\Omega_{\mathbb{U}}} = (\varphi^{-1})^* \lambda_{\mathbb{U}} \quad \lambda_{\Omega_{\mathbb{L}}} = (\chi^{-1})^* \lambda_{\mathbb{L}}$$

and $\lambda_{\mathbb{U}}, \lambda_{\mathbb{L}}$ are the usual hyperbolic metrics in the upper and lower half planes respectively.

Given $\mu \in L^\infty(G)$ and $\phi \in Q(G)$ we consider the pairing

$$(\mu, \phi)_{Def(G)} = \int_{\Omega/G} \mu\phi = \int_{\Omega_{\mathbb{U}}/G} (\mu\,|_{\Omega_{\mathbb{U}}})(\phi\,|_{\Omega_{\mathbb{U}}}) + \int_{\Omega_{\mathbb{L}}/G} (\mu\,|_{\Omega_{\mathbb{L}}})(\phi\,|_{\Omega_{\mathbb{L}}}).$$

From the right hand side of the equation we obtain that the pairing is well defined. The finite dimensional spaces $R(G) \simeq L^\infty_c(G)$ and $Q(G)$ are dual with respect to the pairing.

Let $\mu \in L^\infty(G)$ and $\phi \in Q(G)$. Consider $(\mu_{\mathbb{U}}, \mu_{\mathbb{L}}) = \eta_G(\mu)$ and also $\phi_{\mathbb{U}}(z) = (\phi \circ \varphi)(z)(\varphi'(z))^2$ and $\phi_{\mathbb{L}}(z) = (\phi \circ \chi)(z)(\chi'(z))^2$. Observe then that by changing the variable we obtain

$$(\mu, \phi)_{Def(G)} = \int_{\mathbb{U}/\Gamma_{\mathbb{U}}} \mu_{\mathbb{U}}\phi_{\mathbb{U}} + \int_{L/\Gamma_{\mathbb{L}}} \mu_{\mathbb{L}}\phi_{\mathbb{L}} = (\mu_{\mathbb{U}}, \phi_{\mathbb{U}})_{Teich(\Gamma_{\mathbb{U}})} + (\mu_{\mathbb{L}}, \phi_{\mathbb{L}})_{Teich(\Gamma_{\mathbb{L}})}.$$

**Weil-Petersson hermitian product and metric for $QF(S)$.**

We start by defining the Weil-Petersson hermitian product in the holomorphic tangent and cotangent spaces at the origin of $Def(G)$.

Let $\mu, \nu \in L^\infty_c(G)$ and $\phi, \psi \in Q(G)$. The *Weil-Petersson hermitian product* is defined by

$$h^{Def(G)}(\mu, \nu) = \int_{\Omega/G} \mu\overline{\nu}$$

and the corresponding product on the cotangent space by

$$h^{*Def(G)}(\phi, \psi) = \int_{\Omega/G} \phi\overline{\psi}$$

Denote by $\widehat{T}(G)$ the cartesian product of Teichmüller spaces $Teich(\Gamma_{\mathbb{U}}, \mathbb{U}) \times Teich(\Gamma_{\mathbb{L}}, \mathbb{L})$ and by

$$h^{\widehat{T}(G)} = h^{T(\Gamma_{\mathbb{U}})} \times h^{T(\Gamma_{\mathbb{L}})}$$

the hermitian product, where $h^{T(\Gamma_{\mathbb{U}})}, h^{T(\Gamma_{\mathbb{L}})}$ are the Weil-Petersson hermitian product on $L^\infty_c(\Gamma_{\mathbb{U}}, \mathbb{U}), L^\infty_c(\Gamma_{\mathbb{L}}, \mathbb{L})$ respectively. Let

$$\eta_G : L^\infty_c(G) \to L^\infty_c(\Gamma_{\mathbb{U}}, \mathbb{U}) \times L^\infty_c(\Gamma_{\mathbb{L}}, \mathbb{L})$$

restricted to canonical differentials, and denote by $\eta^*_G$ its dual mapping from $Q(\Gamma_{\mathbb{U}}, \mathbb{U}) \times Q(\Gamma_{\mathbb{L}}, \mathbb{L})$ onto $Q(G)$.

PROPOSITION 3.2.1. $h^{Def(G)} = \eta_G^*(h^{\widehat{T}(G)})$

PROOF. For $\mu, \nu \in L_c^\infty(G)$ we have

$$h^{Def(G)}(\mu, \nu) = \int_{\Omega/G} \mu\overline{\nu} =$$

$$= \int_{(\Omega_{\mathbb{U}})/G} (\mu\mid_{\Omega_{\mathbb{U}}})(\overline{\nu\mid_{\Omega_{\mathbb{U}}}}) + \int_{(\Omega_{\mathbb{L}})/G} (\mu\mid_{\Omega_{\mathbb{L}}})(\overline{\nu\mid_{\Omega_{\mathbb{L}}}}).$$

Changing the variables we deduce that this is simply

$$\int_{\mathbb{U}/\Gamma_{\mathbb{U}}} \mu_{\mathbb{U}}\overline{\nu}_{\mathbb{U}} + \int_{\mathbb{L}/\Gamma_{\mathbb{L}}} \mu_{\mathbb{L}}\overline{\nu}_{\mathbb{L}} = h^{\widehat{T}(G)}(\eta_G(\mu), \eta_G(\nu)) = \eta_G^*(h^{\widehat{T}(G)})(\mu, \nu).$$

□

After this preparatory discussion we are ready to define the Weil-Petersson hermitian form on $QF(S)$. Let $\mu, \nu \in L^\infty(\Gamma)$, $[\xi]$ any point of $QF(S)$, and $(\frac{\partial}{\partial z(\mu)})_{([\xi])}$, $(\frac{\partial}{\partial z(\nu)})_{([\xi])}$the corresponding holomorphic vectors at $[\xi]$.

The *Weil-Petersson hermitian form of* $QF(S)$ is defined by

$$H^Q_{([\xi])}\left(\frac{\partial}{\partial z(\mu)}, \frac{\partial}{\partial z(\nu)}\right) = h^{Def(\Gamma^\xi)}(P_{\Omega^\xi}[L^\xi\mu], P_{\Omega^\xi}[L^\xi\nu])$$

$$= \int_{\Omega^\xi/\Gamma^\xi} P_{\Omega^\xi}[L^\xi\mu]\overline{P_{\Omega^\xi}[L^\xi\nu]}.$$

The *Weil-Petersson Riemannian metric* corresponding to $H^{Q(S)}$ is then given by

$g^Q_{([\xi])}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) =$

$$= 2Re\{H^Q_{([\xi])}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right)\} = 2Re\{\int_{\Omega^\xi/\Gamma^\xi} P_{\Omega^\xi}[L^\xi\mu]\overline{P_{\Omega^\xi}[L^\xi\nu]}\}.$$

The following relates the Weil-Petersson geometries of $QF(S)$ and $\widehat{T}(S) = Teich(S) \times Teich(\overline{S})$, the latter endowed with the product Riemannian metric $g^{\widehat{T}} = g^{T(S)} + g^{T(\overline{S})}$.

THEOREM 3.2.2. *Let $\Psi$ be the biholomorphic Bers' mapping from $QF(S)$ onto $\widehat{T}(S)$. The following relation holds:*

$$g^Q = (\Psi^*)g^{\widehat{T}}$$

*The pair $(QF(S), g^Q)$ realises a Kählerian manifold. The metric is incomplete and the full group of biholomorphic isometries is the modular group $Mod_Q(S)$.*

We need a preparatory lemma first.

LEMMA 3.2.3. *Let $[\xi] \in QF(S)$, $\eta(\xi) = (\xi_{\mathbb{U}}, \xi_{\mathbb{L}})$.*

*The following relation holds:*

$$(L^{\xi_{\mathbb{U}}} \times L^{\xi_{\mathbb{L}}}) \circ \eta_{\Gamma} = \eta_{\Gamma^{\xi}} \circ L^{\xi}.$$

PROOF. Let $\mu \in L^{\infty}(\Gamma)$ and $\varphi^{\xi} : \mathbb{U} \to \Omega^{\xi}_{\mathbb{U}}$, $\chi^{\xi} : \mathbb{L} \to \Omega^{\xi}_{\mathbb{L}}$ be the normalised conformal mappings.Then

$$(\eta_{\Gamma^{\xi}} \circ L^{\xi})(\mu) = \left( (L^{\xi}\mu \circ \varphi^{\xi})(z) \frac{\overline{(\varphi^{\xi})'(z)}}{(\varphi^{\xi})'(z)}, (L^{\xi}\mu \circ \chi^{\xi})(z) \frac{\overline{(\chi^{\xi})'(z)}}{(\chi^{\xi})'(z)} \right).$$

Now,

$$(L^{\xi}\mu \circ \varphi^{\xi})(z) \frac{\overline{(\varphi^{\xi})'(z)}}{(\varphi^{\xi})'(z)} = \frac{\mu((w^{\xi})^{-1} \circ \varphi^{\xi})(z))}{1- \mid \xi((w^{\xi})^{-1} \circ \varphi^{\xi})(z)) \mid^2} \frac{\overline{w^{\xi}_z}}{w^{\xi}} (w^{\xi})^{-1} (\varphi^{\xi}(z)) \frac{\overline{(\varphi^{\xi})'(z)}}{(\varphi^{\xi})'(z)},$$

where $w^{\xi}$ is the unique normalised solution to (3) corresponding to $\xi$. Let $f^{\xi_{\mathbb{U}}}$ be the unique normalised solution to (2) corresponding to $\xi_{\mathbb{U}}$. Then $f^{\xi_{\mathbb{U}}}$ and $(\varphi^{\xi})^{-1} \circ w^{\xi}$ are equal on $\mathbb{U}$ : Observe that the normalised functions $f^{\xi_{\mathbb{U}}}$ and $w^*$ defined so that $w^* = (\varphi^{\xi})^{-1} \circ w^{\xi}$ in $\mathbb{U}$ and $w^* = f^{\xi_{\mathbb{U}}}$ in $\mathbb{L}$ are equal, since they solve the same Beltrami equation in the complex plane. Therefore onat $\mathbb{U}$ we have

$$(w^{\xi})^{-1} \circ \varphi^{\xi} = (f^{\xi_{\mathbb{U}}})^{-1}$$

Consequently,

$$(L^{\xi}\mu \circ \varphi^{\xi})(z) \frac{\overline{(\varphi^{\xi})'(z)}}{(\varphi^{\xi})'(z)} =$$

$$\frac{\mu((f^{\xi_{\mathbb{U}}})^{-1}(z))}{1- \mid \xi_{\mathbb{U}}((f^{\xi_{\mathbb{U}}})^{-1}(z)) \mid^2} \frac{\overline{f^{\xi_{\mathbb{U}}}_z}}{f^{\xi_{\mathbb{U}}}_{\overline{z}}} ((f^{\xi_{\mathbb{U}}})^{-1}(z)) = (L^{\xi_{\mathbb{U}}}\mu_U)(z),$$

and in the same manner one obtains

$$(L^{\xi}\mu \circ \chi^{\xi})(z) \frac{\overline{(\chi^{\xi})'(z)}}{(\chi^{\xi})'(z)} =$$

$$\frac{\mu((f^{\xi_{\mathbb{L}}})^{-1}(z))}{1- \mid \xi((f^{\xi_{\mathbb{L}}})^{-1}(z)) \mid^2} \frac{\overline{f^{\xi_{\mathbb{L}}}_z}}{f^{\xi_{\mathbb{L}}}_{\overline{z}}} ((f^{\xi_{\mathbb{L}}})^{-1}(z)) = (L^{\xi_L}\mu_{\mathbb{L}})(z).$$

$\square$

PROOF OF THEOREM 3.2.2: The induced product metric on $\widehat{T}(S)$ is Kählerian and since $\Psi$ is biholomorphic, so is $(\Psi^*)g^{\widehat{T}}$. We only need to establish the equality.

Let $\xi \in Belt(\Gamma)$, $[\xi] \in QF(S)$, $\eta_\Gamma(\xi) = (\xi_{\mathbb{U}}, \xi_{\mathbb{L}})$ and $\mu \in L^\infty(\Gamma)$, $\eta_\Gamma(\mu) = (\mu_{\mathbb{U}}, \mu_{\mathbb{L}})$. We have that

$$\Psi_*\left(\frac{\partial}{\partial z(\mu)}\right)_{([\xi])} = \left(\frac{\partial}{\partial z(\mu_{\mathbb{U}})}, \frac{\partial}{\partial z(\mu_{\mathbb{L}})}\right)_{([\xi_{\mathbb{U}}],[\xi_{\mathbb{L}}])}$$

For $\mu, \nu \in L^\infty(\Gamma)$

$$g^Q_{([\xi])}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) = 2Re\{H^Q_{([\xi])}\left(\frac{\partial}{\partial z(\mu)}, \frac{\partial}{\partial z(\nu)}\right)\} =$$

$$2Re\{h^{Def(\Gamma^\xi)}(P_{\Omega^\xi}[L^\xi\mu], P_{\Omega^\xi}[L^\xi\nu])\}.$$

By proposition 3.2.1 the latter is equal to

$$2Re\{\eta^*_{\Gamma^\xi}(h^{Def(\Gamma^\xi)})(P_{\Omega^\xi}[L^\xi\mu], P_{\Omega^\xi}[L^\xi\nu])\} =$$

$$2Re\{h^{\widehat{T}(\Gamma^\xi)}(\eta_{\Gamma^\xi}(P_{\Omega^\xi}[L^\xi\mu]), \eta_{\Gamma^\xi}(P_{\Omega^\xi}[L^\xi\nu]))\}.$$

By proposition 2.2.9 this becomes

$$2Re\{h^{\widehat{T}(\Gamma^\xi)}((P_{\mathbb{U}} \times P_{\mathbb{L}})[\eta_{\Gamma^\xi}(L^\xi\mu)], (P_{\mathbb{U}} \times P_{\mathbb{L}})[\eta_{\Gamma^\xi}(L^\xi\nu)])\}$$

which by lemma 3.2.3 is equal to

$$2Re\{h^{\widehat{T}(\Gamma^\xi)}((P_{\mathbb{U}} \times P_{\mathbb{L}})[(L^{\xi_{\mathbb{U}}} \times L^{\xi_{\mathbb{L}}})(\mu_{\mathbb{U}}, \mu_{\mathbb{L}})], (P_{\mathbb{U}} \times P_{\mathbb{L}})[(L^{\xi_{\mathbb{U}}} \times L^{\xi_{\mathbb{L}}})(\mu_{\mathbb{U}}, \mu_{\mathbb{L}})])\} =$$

$$2Re\left\{h^{\widehat{T}}_{(\Psi([\xi]))}\left(\left(\frac{\partial}{\partial z(\mu_{\mathbb{U}})}, \frac{\partial}{\partial z(\mu_{\mathbb{L}})}\right), \left(\frac{\partial}{\partial z(\nu_{\mathbb{U}})}, \frac{\partial}{\partial z(\nu_{\mathbb{L}})}\right)\right)\right\} =$$

$$g^{\widehat{T}}_{(\Psi([\xi]))}\left(\Psi_*\frac{\partial}{\partial x(\mu)}, \Psi_*\frac{\partial}{\partial x(\nu)}\right) = (\Psi^* g^{\widehat{T}})_{([\xi])}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right).$$

The metric $g^Q$ is incomplete due to incompleteness of $g^{\widehat{T}}$. Since the full group of Weil-Petersson isometries in $Teich(S)$ (resp. $Teich(\overline{S})$) is $Mod(S)$ (resp. $Mod(\overline{S})$) and $g^{\widehat{T}}$ is a Riemannian product metric, it follows that the group of biholomorphic isometries of $\widehat{T}(S)$ is just $Mod(S) \times Mod(\overline{S})$. We only have to establish the isomorphism of this group with $Mod_Q(S)$.

Let $h$ be a quasiconformal automorphism of the complex plane and $\gamma_h \in Mod_Q(S)$ acting on $QF(S)$ as follows: For each $[w^\mu] \in QF(S)$

$$\gamma_h([w^\mu]) = ([w^\mu \circ h^{-1}])$$

We define a biholomorphic self mapping $\widetilde{\gamma_h}$ of $Teich(S) \times Teich(\overline{S})$ by the relation

$$\widetilde{\gamma_h} = \Psi \circ \gamma_h \circ \Psi^{-1}$$

The isomorphism in question is $R : Mod_Q(S) \to Mod(S) \times Mod(\overline{S})$ where

$$R(\gamma_h) = \widetilde{\gamma_h}$$

Clearly, for each $\gamma_h$, $\widetilde{\gamma_h} \in Mod(S) \times Mod(\overline{S})$ and $R$ is a group homomorphism. Also $R$ is injective:

$$\ker R = \{\gamma_h : \widetilde{\gamma_h} = \Psi \circ \gamma_h \circ \Psi^{-1} = id\} = \{id\}$$

Finally $R$ is surjective: If $\widetilde{\gamma} \in Mod(S) \times Mod(\overline{S})$ then $\gamma = \Psi^{-1} \circ \widetilde{\gamma} \circ \Psi$ is an element of $Mod_Q(S)$ and $R(\gamma) = \widetilde{\gamma}$. Therefore, the group of biholomorphic isometries of $Q(S)$ is $Mod_Q(S)$.

**Weil-Petersson symplectic geometry**. As for the real symplectic form $\omega^Q$ induced by $g^Q$ we have

$$\omega^Q_{([\xi])}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) = g^Q_{([\xi])}\left(I_Q \frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) =$$

$$-2Im\left\{\int_{\Omega^\xi/\Gamma^\xi} P_{\Omega^\xi}[L^\xi \mu]\overline{P_{\Omega^\xi}[L^\xi \nu]}\right\},$$

which by Theorem 3.2.2 is equal to

$$(\Psi^* \omega^{\widehat{T}})_{([\xi])}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right)$$

where

$$\omega^{\widehat{T}} = \omega^{T(S)}_{WP} + \omega^{T(\overline{S})}_{WP} \equiv Pr_1^* \omega^{T(S)}_{WP} + Pr_2^* \omega^{T(\overline{S})}_{WP}.$$

The mappings

$$Pr_1 : \widehat{T}(S) \to Teich(S) \quad Pr_2 : \widehat{T}(S) \to Teich(\overline{S})$$

are the natural holomorphic projections and $\omega_{WP}^{T(S)}$, $\omega_{WP}^{T(\overline{S})}$ are the real symplectic forms induced by the Weil-Petersson product on $Teich(S), Teich(\overline{S})$ respectively.

For later use we finally prove the following

LEMMA 3.2.4. *Let $[\xi] \in QF(S)$, $\mu, \nu \in L^\infty(\Gamma)$ and $S : L^\infty(\Gamma) \to L^\infty_S(\Gamma)$ the symmetrisation projection operator. Then*

$$g^Q_{([\xi])}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) =$$

$$g^Q_{([\xi])}\left(\frac{\partial}{\partial x(S(\mu))}, \frac{\partial}{\partial x(S(\nu))}\right) + g^Q_{([\xi])}\left(\frac{\partial}{\partial x(S(-i\mu))}, \frac{\partial}{\partial x(S(-i\nu))}\right) +$$

$$+\omega^Q_{([\xi])}\left(\frac{\partial}{\partial x(S(-i\mu))}, \frac{\partial}{\partial x(S(\nu))}\right) - \omega^Q_{([\xi])}\left(\frac{\partial}{\partial x(S(\mu))}, \frac{\partial}{\partial x(S(-i\nu))}\right).$$

PROOF. Recall that every $\mu \in L^\infty(\Gamma)$ can be written as $\mu = S(\mu) + iS(-i\mu)$. Therefore

$$g^Q_{([\xi])}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) = 2Re\left\{\int_{\Omega^\xi/\Gamma^\xi} P_{\Omega^\xi}[L^\xi\mu]\overline{P_{\Omega^\xi}[L^\xi\nu]}\right\} =$$

$$2Re\left\{\int_{\Omega^\xi/\Gamma^\xi} P_{\Omega^\xi}[L^\xi(S(\mu) + iS(-i\mu))]\overline{P_{\Omega^\xi}[L^\xi(S(\nu) + iS(-i\nu))]}\right\} =$$

since $L^\xi$ is complex linear

$$2Re\left\{\int_{\Omega^\xi/\Gamma^\xi} P_{\Omega^\xi}[L^\xi(S(\mu)) + iL^\xi(S(-i\mu))]\overline{P_{\Omega^\xi}[L^\xi(S(\nu)) + iL^\xi(S(-i\nu))]}\right\} =$$

since $P_{\Omega^\xi}$ is complex linear

$$2Re\left\{\int_{\Omega^\xi/\Gamma^\xi} (P_{\Omega^\xi}[L^\xi(S(\mu))] + iP_{\Omega^\xi}[L^\xi(S(-i\mu))])\times\right.$$

$$\left.\times(\overline{P_{\Omega^\xi}[L^\xi(S(\nu))]} - i\overline{P_{\Omega^\xi}[L^\xi(S(-i\nu))]})\right\} =$$

$$2Re\left\{\int_{\Omega^\xi/\Gamma^\xi} P_{\Omega^\xi}[L^\xi(S(\mu))](\overline{P_{\Omega^\xi}[L^\xi(S(\nu))]}\right\} +$$

$$2Re\left\{\int_{\Omega^\xi/\Gamma^\xi} P_{\Omega^\xi}[L^\xi(S(-i\mu))]\overline{P_{\Omega^\xi}[L^\xi(S(-i\nu))]}\right\} -$$

$$2Im\left\{\int_{\Omega^\xi/\Gamma^\xi} P_{\Omega^\xi}[L^\xi(S(-i\mu))]\overline{P_{\Omega^\xi}[L^\xi(S(\nu))]}\right\} +$$

$$2Im\left\{\int_{\Omega^\xi/\Gamma^\xi} P_{\Omega^\xi}[L^\xi(S(\mu))\overline{P_{\Omega^\xi}[L^\xi(S(-i\nu))]}\right\}$$

and the result is obtained. □

**3.2.2. Hyperkählerian geometry.** We refer to [Be], chpt.14, for the definition of Hyperkählerian manifolds and the general results stated below.

A $4n$-dimensional Riemannian manifold $(M, g)$ is called *Hyperkählerian* if and only if there exist two complex structures $I$ and $J$ defined on $M$ such that:
$a) IJ + JI = 0$
$b) g$ is a Kähler metric for each $I$ and $J$.

It is immediate that $K = IJ$ is a complex structure and $g$ is also Kähler for $K$. In general, given $(x, y, z) \in R^3$ satisfying $x^2 + y^2 + z^2 = 1$ then $xI + yJ + zK$ is a complex structure on $M$ parallel with respect to $g$. We choose one of all those complex structures (say $I$) and we consider $M$ as a complex manifold for $I$. An important fact about Hyperkählerian manifolds is that their Ricci curvature tensor is zero.

We also always have that

*A Hyperkählerian manifold is a complex symplectic manifold.*

Indeed the complex 2-form $\Omega$ defined by

$$\Omega(X, Y) = g(JX, Y) + ig(KX, Y)$$

is non-degenerate, parallel with respect to $g$ and $I$-holomorphic.

The converse is true in the case where $M$ is a compact manifold but not known to be always true otherwise. Examples in the non-compact case were given among others by Calabi [C], Hitchin [H].

In this subsection we shall prove

THEOREM 3.2.5. *The space $QF(S)$ with complex structures $I_Q, J_Q$ and the Weil-Petersson Riemannian metric $g^Q$ is a Hyperkählerian manifold.*

PROOF. The proof will be given in steps. Since we already have that $(QF(S), g^Q, I_Q)$ is Kählerian it remains to show the following: □

-There exist complex operators $J_Q, K_Q$ defined on $QF(S)$, skew commuting with $I_Q$.
-These operators are parallel with respect to $g^Q$.

STEP 1. *Almost complex operators.*

We had seen in 2.2.2 that there exists a naturally defined complex operator $J$ on $L^\infty(\Gamma)$ skew commuting with the standard complex operator $I$. Recall that

$$J(\mu)(z) = \begin{cases} i\overline{\mu(\overline{z})} & z \in \mathbb{U} \\ -i\overline{\mu(\overline{z})} & z \in \mathbb{L} \end{cases}$$

for $\mu \in L^\infty(\Gamma)$. We shall define almost complex structures $J_Q, K_Q$ everywhere on $QF(S)$ so that the Riemannian metric $g^Q$ induced by the Weil-Petersson product remains invariant by their action. We focus our attention on $J_Q$. Let $\mu \in L^\infty(\Gamma)$ and $[\xi]$ any point of $QF(S)$. Let

$$\left(\frac{\partial}{\partial x(\mu)}\right)_{([\xi])}, \left(\frac{\partial}{\partial z(\mu)}\right)_{[\xi]}, \left(\frac{\partial}{\partial \overline{z}(\mu)}\right)_{([\xi])}$$

be (see 2.2.2) the associated tangent, holomorphic and antiholomorphic tangent vectors at $[\xi]$ respectively. Speaking in complex terms, we may define $J_Q$ in terms of two operators

$$(J'_Q)_{([\xi])} : T^{(1,0)}_{([\xi])}(QF(S)) \to T^{(0,1)}_{([\xi])}(QF(S))$$

$$(J''_{Q(S)})_{([\xi])} : T^{(0,1)}_{([\xi])}(QF(S)) \to T^{(1,0)}_{([\xi])}(QF(S))$$

satisfying the conditions

$$J'_Q \circ J''_Q = -id_{|_{T^{(0,1)}}}$$

$$J''_Q \circ J'_Q = -id_{|_{T^{(1,0)}}}$$

at each point $[\xi]$. Set

$$(J'_Q)_{([\xi])}\left(\frac{\partial}{\partial z(\mu)}\right)_{([\xi])} = \overline{P_{\Omega^\xi}[L^\xi(J(\mu))]} = \left(\frac{\partial}{\partial \overline{z}(J(\mu))}\right)_{([\xi])}$$

$$(J''_Q)_{([\xi])}\left(\frac{\partial}{\partial \overline{z}(\mu)}\right)_{([\xi])} = P_{\Omega^\xi}[L^\xi(J(\mu))] = \left(\frac{\partial}{\partial z(J(\mu))}\right)_{([\xi])}$$

Speaking now in real terms we set

$$(J_Q)_{([\xi])}\left(\frac{\partial}{\partial x(\mu)}\right)_{([\xi])} = 2Re\{P_{\Omega^\xi}[L^\xi(J(\mu))]\} = \left(\frac{\partial}{\partial x(J_M(\mu))}\right)_{([\xi])}$$

In an analogous manner an almost complex structure $K_Q$ is defined on $QF(S)$ induced by the complex operator $K$ of $L^\infty(\Gamma)$ given by

$$K(\mu)(z) = \begin{cases} -\overline{\mu(\overline{z})} & z \in \mathbb{U} \\ \overline{\mu(\overline{z})} & z \in \mathbb{L} \end{cases}$$

The above lead us to the following:

-$(QF(S), J_Q)$ *and* $(QF(S), K_Q)$ *are almost complex manifolds.*

STEP 2. *Almost hermitian structure.*

An almost complex Riemannian manifold $(M, g, I)$ is called *almost hermitian* if for every two vector fields $X, Y$ of $M$ the following holds:

$$g(IX, IY) = g(X, Y)$$

We shall prove:

-$(QF(S), g^Q, J_Q)$ *and* $(QF(S), g^Q, K_Q)$ *are almost hermitian manifolds.*

PROOF. We only prove our first assertion, the proof of the second one is then immediate. Let $[\xi] \in QF(S)$ and $\mu, \nu \in L^\infty(\Gamma)$. It suffices to show that

$$g^Q_{([\xi])}\left(J_Q\frac{\partial}{\partial x(\mu)}, J_Q\frac{\partial}{\partial x(\nu)}\right) = g^Q_{([\xi])}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right)$$

It is convenient to prove the above relation firstly in the case where $\mu, \nu \in L^\infty_S(\Gamma)$. It is obvious that $J$ is then just $I_S$. Therefore

$$g^Q_{([\xi])}\left(J_Q\frac{\partial}{\partial x(\mu)}, J_Q\frac{\partial}{\partial x(\nu)}\right) = g^Q_{([\xi])}\left(\frac{\partial}{\partial x(I_S(\mu))}, \frac{\partial}{\partial x(I_S(\nu))}\right) =$$

$$(\Psi^* g^{\widehat{T}})_{([\xi])}\left(\frac{\partial}{\partial x(I_S(\mu))}, \frac{\partial}{\partial x(I_S(\nu))}\right) =$$

$$g^{\widehat{T}}_{(\Psi([\xi]))}\left((\frac{\partial}{\partial x(i\mu_{\mathbb{U}})}, \frac{\partial}{\partial x(-i\mu_{\mathbb{L}})}), (\frac{\partial}{\partial x(i\nu_{\mathbb{U}})}, \frac{\partial}{\partial x(-i\nu_{\mathbb{L}})})\right) =$$

$$g^{T(S)}_{([\xi_{\mathbb{U}}])}\left(\frac{\partial}{\partial x(i\mu_U)}, \frac{\partial}{\partial x(i\nu_U)}\right) + g^{T(\overline{S})}_{([\xi_{\mathbb{L}}])}\left(\frac{\partial}{\partial x(-i\mu_{\mathbb{L}})}, \frac{\partial}{\partial x(-i\nu_{\mathbb{L}})}\right) =$$

$$g^{T(S)}_{([\xi_{\mathbb{U}}])}\left(\frac{\partial}{\partial x(\mu_{\mathbb{U}})}, \frac{\partial}{\partial x(\nu_{\mathbb{U}})}\right) + g^{T(\overline{S})}_{([\xi_{\mathbb{L}}])}\left(\frac{\partial}{\partial x(\mu_{\mathbb{L}})}, \frac{\partial}{\partial x(\nu_{\mathbb{L}})}\right) =$$

$$g^{\widehat{T}}_{(\Psi([\xi]))}\left((\frac{\partial}{\partial x(\mu_{\mathbb{U}})}, \frac{\partial}{\partial x(\mu_{\mathbb{L}})}), (\frac{\partial}{\partial x(\nu_{\mathbb{U}})}, \frac{\partial}{\partial x(\nu_{\mathbb{L}})})\right) =$$

$$(\Psi^* g^{\widehat{T}})_{([\xi])}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) = g^{Q}_{([\xi])}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right).$$

Additionally, by the skew-commutativity of $J_Q$ and $I_Q$ and the above relation, we obtain that for $\mu, \nu \in L^{\infty}_S(\Gamma)$ the following holds:

$$\omega^{Q}_{([\xi])}\left(J_Q\frac{\partial}{\partial x(\mu)}, J_Q\frac{\partial}{\partial x(\nu)}\right) = -\omega^{Q}_{([\xi])}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right).$$

We apply Lemma 2.1.4 to get

$$g^{Q}_{([\xi])}\left(J_Q\frac{\partial}{\partial x(\mu)}, J_Q\frac{\partial}{\partial x(\nu)}\right) = g^{Q}_{([\xi])}\left(\frac{\partial}{\partial x(J(\mu))}, \frac{\partial}{\partial x(J(\nu))}\right) =$$

$$= g^{Q}_{([\xi])}\left(\frac{\partial}{\partial x(S(J(\mu)))}, \frac{\partial}{\partial x(S(J(\nu)))}\right) + g^{Q}_{([\xi])}\left(\frac{\partial}{\partial x(S(-iJ(\mu)))}, \frac{\partial}{\partial x(S(-iJ(\nu)))}\right) +$$

$$+\omega^{Q}_{([\xi])}\left(\frac{\partial}{\partial x(S(-iJ(\mu)))}, \frac{\partial}{\partial x(S(J(\nu)))}\right) - \omega^{Q}_{([\xi])}\left(\frac{\partial}{\partial x(S(J(\mu)))}, \frac{\partial}{\partial x(S(-iJ(\nu)))})\right).$$

Since $J$ skew commutes with $I$ this is

$$g^{Q}_{([\xi])}\left(\frac{\partial}{\partial x(S(J(\mu)))}, \frac{\partial}{\partial x(S(J(\nu)))}\right) + g^{Q}_{([\xi])}\left(\frac{\partial}{\partial x(S(J(i\mu)))}, \frac{\partial}{\partial x(S(J(i\nu)))}\right) +$$

$$+\omega^{Q}_{([\xi])}\left(\frac{\partial}{\partial x(S(J(i\mu)))}, \frac{\partial}{\partial x(S(J(\nu)))}\right) - \omega^{Q}_{([\xi])}\left(\frac{\partial}{\partial x(S(J(\mu)))}, \frac{\partial}{\partial x(S(J(i\nu)))}\right).$$

On the other hand $J$ and $S$ are commuting operators: $SJ = JS$. Thus, the previous expression becomes

$$g^Q_{([\xi])}\left(\frac{\partial}{\partial x(J(S(\mu)))},\frac{\partial}{\partial x(J(S(\nu)))}\right)+g^Q_{([\xi])}\left(\frac{\partial}{\partial x(J(S(i\mu)))},\frac{\partial}{\partial x(J(S(i\nu)))}\right)+$$

$$+\omega^Q_{([\xi])}\left(\frac{\partial}{\partial x(J(S(i\mu)))},\frac{\partial}{\partial x(J(S(\nu)))}\right)-\omega^Q_{([\xi])}\left(\frac{\partial}{\partial x(J(S(\mu)))},\frac{\partial}{\partial x(J(S(i\nu)))}\right).$$

The product is invariant and the symplectic form is skew-invariant in the case of symmetric differentials, therefore we conclude that the previous is equal to

$$g^Q_{([\xi])}\left(\frac{\partial}{\partial x(S(\mu))},\frac{\partial}{\partial x(S(\nu))}\right)+g^Q_{([\xi])}\left(\frac{\partial}{\partial x(S(i\mu))},\frac{\partial}{\partial x(S(i\nu))}\right)+$$

$$-\omega^Q_{([\xi])}\left(\frac{\partial}{\partial x(S(i\mu))},\frac{\partial}{\partial x(S(\nu))}\right)+\omega^Q_{([\xi])}\left(\frac{\partial}{\partial x(S(\mu))},\frac{\partial}{\partial x(S(i\nu))}\right).$$

But this is

$$g^Q_{([\xi])}\left(\frac{\partial}{\partial x(S(\mu))},\frac{\partial}{\partial x(S(\nu))}\right)+g^Q_{([\xi])}\left(\frac{\partial}{\partial x(S(-i\mu))},\frac{\partial}{\partial x(S(-i\nu))}\right)+$$

$$+\omega^Q_{([\xi])}\left(\frac{\partial}{\partial x(S(-i\mu))},\frac{\partial}{\partial x(S(\nu))}\right)-\omega^Q_{([\xi])}\left(\frac{\partial}{\partial x(S(\mu))},\frac{\partial}{\partial x(S(-i\nu))}\right)$$

which by Lemma 3.2.4 is just

$$g^Q_{([\xi])}\left(\frac{\partial}{\partial x(\mu)},\frac{\partial}{\partial x(\nu)}\right).$$

The proof of the $J_Q$-invariance of the metric is thus completed. □

STEP 3. *Integrability and Kählerian structure.*

By Newlander-Nirenberg theorem an almost complex structure $J$ on a manifold $M$ is complex and therefore $M$ is a $J$-complex manifold iff $J$ is integrable, that is also equivalent to say that $J$ has no torsion. If the manifold is almost hermitian let $\nabla$ be the Riemannian connection. The connection is said to be almost complex if $\nabla_X J = 0$ for all vector fields $X$ of $M$. If the latter happens, then $M$ is a $J$-complex Kählerian manifold.

Let $\nabla$ be the Riemannian connection corresponding to $g^Q$. We shall prove

-$\nabla$ *is almost complex with respect to* $J_Q$, *that is*

$$\nabla_{\frac{\partial}{\partial x(\mu)}} J_Q = 0$$

*for every* $\mu \in L^\infty(\Gamma)$.

PROOF. It is sufficient to prove the relation

$$\nabla_{\frac{\partial}{\partial x(\mu)}}\left(J_Q\frac{\partial}{\partial x(\nu)}\right)=J_Q\left(\nabla_{\frac{\partial}{\partial x(\mu)}}\frac{\partial}{\partial x(\nu)}\right)$$

for all $\mu,\nu\in L^\infty(\Gamma)$. Start from $\widetilde{\nabla}$, the Riemannian connection of $\widehat{T}(S)$. This is $I_{T(S)}\times I_{T(\overline{S})}$ almost complex. We assert that it is also $I'_{\widehat{T}}=I_{T(S)}\times(-I_{T(\overline{S})})$ almost complex. Indeed, one verifies at once that $I'_{\widehat{T}}$ is a complex operator for $\widehat{T}(S)$ and so, its torsion $N$ is zero. In addition, $g^{\widehat{T}}$ remains invariant under the action of $I'_{\widehat{T}(S)}$ and one verifies that the following holds for the corresponding fundamental form $\Omega'$ :

$$\Omega'=\omega_{WP}^{T(S)}-\omega_{WP}^{T(\overline{S})}$$

and therefore $\Omega'$ is closed. Now if $\widehat{\mu},\widehat{\nu},\widehat{\xi}\in L^\infty(\Gamma,U)\times L^\infty(\Gamma,L)$ then the following condition is satisfied ([K-N], Prop. 4.2, Chpt IX, p.148):

$$4g^{\widehat{T}}\left((\widetilde{\nabla}_{\frac{\partial}{\partial x(\widehat{\mu})}}I'_{\widehat{T}})\frac{\partial}{\partial x(\widehat{\nu})},\frac{\partial}{\partial x(\widehat{\xi})}\right)=6d\Omega'\left(\frac{\partial}{\partial x(\widehat{\mu})},I'_{\widehat{T}}\frac{\partial}{\partial x(\widehat{\nu})},I'_{\widehat{T}}\frac{\partial}{\partial x(\widehat{\xi})}\right)-$$

$$-6d\Omega'\left(\frac{\partial}{\partial x(\widehat{\mu})},\frac{\partial}{\partial x(\widehat{\nu})},\frac{\partial}{\partial x(\widehat{\xi})}\right)+g^{\widehat{T}}\left(N\left(\frac{\partial}{\partial x(\widehat{\nu})},\frac{\partial}{\partial x(\widehat{\xi})}\right),I'_{\widehat{T}(S)}\frac{\partial}{\partial x(\widehat{\mu})}\right).$$

Since $d\Omega'=0$ and $N=0$ we conclude that

$$\widetilde{\nabla}_{\frac{\partial}{\partial x(\widehat{\mu})}}I'_{\widehat{T}}=0.$$

We now distinguish two cases: Suppose first that $\nu$ is a symmetric differential. Then we can then verify straightforward that at every point $[\xi]$ and for every $\mu\in L^\infty(\Gamma)$, the following relation holds:

$$(J_Q)_{([\xi])}\left(\frac{\partial}{\partial x(\nu)}\right)_{([\xi])}=((\Psi_*)^{-1}\circ I'_{\widehat{T}}\circ\Psi_*)_{([\xi])}\left(\frac{\partial}{\partial x(\nu)}\right)_{([\xi])}$$

Since $I'_{\widehat{T}}$ is an integrable operator and $\Psi$ is a biholomorphism, we have

$$\nabla_{\frac{\partial}{\partial x(\mu)}}\left(J_Q\frac{\partial}{\partial x(\nu)}\right)=J_Q\left(\nabla_{\frac{\partial}{\partial x(\mu)}}\frac{\partial}{\partial x(\nu)}\right)$$

for every $\mu\in L^\infty(\Gamma)$, and $\nu\in L^\infty_{sym}(\Gamma)$. Let now $\nu$ be an arbitrary differential. Since

$$\left(\frac{\partial}{\partial x(\nu)}\right)_{([\xi])} = \left(\frac{\partial}{\partial x(S(\nu))}\right)_{([\xi])} + \left(I_Q \frac{\partial}{\partial x(S(-i\nu))}\right)_{([\xi])}$$

we have:

$$\nabla_{\frac{\partial}{\partial x(\mu)}} J_Q \left(\frac{\partial}{\partial x(\nu)}\right) = \nabla_{\frac{\partial}{\partial x(\mu)}} J_Q \left(\frac{\partial}{\partial x(S(\nu))}\right) + \nabla_{\frac{\partial}{\partial x(\mu)}} J_Q \left(I_Q \frac{\partial}{\partial x(S(-i\nu))}\right) =$$

since $I_Q$ and $J_Q$ skew-commute

$$\nabla_{\frac{\partial}{\partial x(\mu)}} J_Q \left(\frac{\partial}{\partial x(S(\nu))}\right) - \nabla_{\frac{\partial}{\partial x(\mu)}} I_Q J_Q \left(\frac{\partial}{\partial x(S(-i\nu))}\right) =$$

since $I_Q$ is integrable

$$\nabla_{\frac{\partial}{\partial x(\mu)}} J_Q \left(\frac{\partial}{\partial x(S(\nu))}\right) - I_Q \left(\nabla_{\frac{\partial}{\partial x(\mu)}} J_Q \left(\frac{\partial}{\partial x(S(-i\nu))}\right)\right) =$$

by the first case

$$J_Q \left(\nabla_{\frac{\partial}{\partial x(\mu)}} \left(\frac{\partial}{\partial x(S(\nu))}\right)\right) - I_Q J_Q \left(\nabla_{\frac{\partial}{\partial x(\mu)}} \left(\frac{\partial}{\partial x(S(-i\nu))}\right)\right) =$$

again by skew-commutativity

$$J_Q \left(\nabla_{\frac{\partial}{\partial x(\mu)}} \left(\frac{\partial}{\partial x(S(\nu))}\right)\right) + J_Q I_Q \left(\nabla_{\frac{\partial}{\partial x(\mu)}} \left(\frac{\partial}{\partial x(S(-i\nu))}\right)\right) =$$

by integrability of $I_Q$

$$= J_Q \left(\nabla_{\frac{\partial}{\partial x(\mu)}} \left(\frac{\partial}{\partial x(S(\nu))}\right)\right) + J_Q \left(\nabla_{\frac{\partial}{\partial x(\mu)}} I_Q \left(\frac{\partial}{\partial x(S(-i\nu))}\right)\right) =$$

$$= J_Q \left(\nabla_{\frac{\partial}{\partial x(\mu)}} \left(\frac{\partial}{\partial x(S(\nu))}\right) + \nabla_{\frac{\partial}{\partial x(\mu)}} I_Q \left(\frac{\partial}{\partial x(S(-i\nu))}\right)\right) =$$

$$= J_Q \left(\nabla_{\frac{\partial}{\partial x(\mu)}} \left(\frac{\partial}{\partial x(S(\nu))} + I_Q \frac{\partial}{\partial x(S(-i\nu)}\right)\right) =$$

$$= J_Q(\nabla_{\frac{\partial}{\partial x(\mu)}} \frac{\partial}{\partial x(\nu)}).$$

It is now evident that the connection is also almost complex with respect to $K_Q$. □

The proof of theorem 3.2.5 is here completed.

A series of consequenses of Theorem 3.2.5 is following: The first one is obvious:

THEOREM 3.2.6. *The fundamental forms corresponding to the Kaehlerian manifolds* $(QF(S), g^Q, J_Q)$, $(QF(S), g^Q, K_Q)$ *respectively given by*

$$\omega^Q_{1_{([\xi])}}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) = g^Q_{([\xi])}\left(J_Q\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right)$$

$$\omega^Q_{2_{([\xi])}}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) = g^Q_{([\xi])}\left(K_Q\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right)$$

*are closed.*

THEOREM 3.2.7. *i) Teichmüller space as the set of Fuchsian deformations* $F(S)$ *is: A Lagrangian submanifold of* $QF(S)$ *with respect to* $\omega^Q$ *and* $\omega^Q_2$ *and a symplectic submanifold of* $QF(S)$ *with respect to* $\omega^Q_1$.

*ii) Bers' slices are symplectic submanifolds of* $QF(S)$ *with respect to* $\omega^Q$ *and Lagrangian submanifolds of* $QF(S)$ *with respect to* $\omega^Q_1$ *and* $\omega^Q_2$.

PROOF. i) Let $\mu, \nu \in L^\infty_S(\Gamma)$ and $[\xi] \in F(S)$ (i.e $\xi$ is a symmetric Beltrami differential). We shall prove that the restriction of $\omega^Q$ to the tangent subbundle of $F(S)$, is equal to 0. We calculate

$$\omega^Q_{([\xi])}(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}) =$$

$$-2Im\left\{\int_{\mathbb{U}/\Gamma^\xi} P_{\mathbb{U}}[L^\xi\mu]\overline{P_{\mathbb{U}}[L^\xi\nu]} + \int_{L/\Gamma^\xi} P_{\mathbb{L}}[L^\xi(\overline{\mu(\overline{z})})]\overline{P_{\mathbb{L}}[L^\xi(\overline{\nu(\overline{z})})]}\right\}.$$

Note that for $\xi \in Belt_s(\Gamma)$, $L^\xi$ maps $L^\infty_S(\Gamma)$ onto $L^\infty_S(\Gamma^\xi)$. Therefore, by changing the variable in the right integral and taking out conjugates, we obtain that the above expression is equal to

$$-2Im\left\{\int_{\mathbb{U}/\Gamma^\xi} P_{\mathbb{U}}[L^\xi\mu]\overline{P_{\mathbb{U}}[L^\xi\nu]} + \overline{\int_{\mathbb{U}/\Gamma^\xi} P_{\mathbb{U}}[L^\xi\mu]\overline{P_{\mathbb{U}}[L^\xi\nu]}}\right\} =$$

$$-2Im\left\{2Re\left\{\int_{\mathbb{U}/\Gamma^\xi} P_{\mathbb{U}}[L^\xi\mu]\overline{P_{\mathbb{U}}[L^\xi\nu]}\right\}\right\} = 0.$$

We prove now that the restriction of $\omega_1^Q$ to the tangent subbundle of $F(S)$, is equal to $2\omega_{WP}$. Recall that if $\mu \in L_S^\infty(\Gamma)$ then $J = I_S$. We have then

$$\omega^Q_{1_{([\xi])}}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) =$$

$$2Re\left\{\int_{\mathbb{U}/\Gamma^\xi} P_{\mathbb{U}}[L^\xi(i\mu)]\overline{P_{\mathbb{U}}[L^\xi\nu]} - \int_{\mathbb{L}/\Gamma^\xi} P_{\mathbb{L}}[L^\xi(i\mu)]\overline{P_{\mathbb{L}}[L^\xi\nu]}\right\} =$$

$$-2Im\left\{\int_{\mathbb{U}/\Gamma^\xi} P_{\mathbb{U}}[L^\xi\mu]\overline{P_{\mathbb{U}}[L^\xi\nu]} - \int_{\mathbb{L}/\Gamma^\xi} P_{\mathbb{L}}[L^\xi\mu]\overline{P_{\mathbb{L}}[L^\xi\nu]}\right\} =$$

$$-4Im\left\{\int_{\mathbb{U}/\Gamma^\xi} P_{\mathbb{U}}[L^\xi\mu]\overline{P_{\mathbb{U}}[L^\xi\nu]}\right\} = 2\omega_{WP_{([\xi])}}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right).$$

Finally, we prove that the restriction of $\omega_2^Q$ to the tangent subbundle of $F(S)$, is equal to 0. Indeed, if $\mu \in L_S^\infty(\Gamma)$ then

$$K(\mu)(z) = \begin{cases} -\mu(z) & z \in \mathbb{U} \\ \mu(z) & z \in \mathbb{L} \end{cases}$$

which is not symmetric and therefore $K_Q = 0$ when restricted to the tangent subbundle of $F(S)$. Thus

$$\omega^Q_{2_{([\xi])}}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) = 0.$$

ii) With no loss of generality we deal only with the Bers' slice

$$B(S) = \Psi^{-1}(Teich(S) \times \{[\overline{S}]\}).$$

Let $\mu, \nu \in L^\infty(\Gamma, \mathbb{U})$ and $[\xi] \in B(S)$ (i.e $\xi = 0$ in $\mathbb{L}$.)

$$\omega^Q_{([\xi])}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) =$$

$$-2Im\left\{\int_{\Omega^\xi_{\mathbb{U}}/\Gamma^\xi} P_{\Omega^\xi_{\mathbb{U}}}[L^\xi\mu]\overline{P_{\Omega^\xi_{\mathbb{U}}}[L^\xi\nu]}\right\} = \omega_{WP_{([\xi])}}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right).$$

In this manner the restriction of $\omega^Q$ to the tangent subbundle of $B(S)$ defines a symplectic structure for $B(S)$. It can be easily checked that

$$\Psi^*(\omega_{WP}^{T(S)} \times \{0\}) = (\omega^Q \mid_{B(S)})$$

where $\Psi$ is here restricted to $B(S)$. Now, if $\mu \in L^\infty(\Gamma, \mathbb{U})$ then

$$J(\mu)(z) = \begin{cases} 0 & z \in \mathbb{U} \\ -i\overline{\mu(\overline{z})} & z \in \mathbb{L} \end{cases}$$

and

$$K(\mu)(z) = \begin{cases} 0 & z \in \mathbb{U} \\ \overline{\mu(\overline{z})} & z \in \mathbb{L} \end{cases}$$

Therefore, $J_Q$ and $K_Q$ are zero when restricted to the tangent subbundle of $B(S)$. Thus

$$\omega^Q_{1_{([\xi])}}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) = \omega^Q_{2_{([\xi])}}\left(\frac{\partial}{\partial x(\mu)}, \frac{\partial}{\partial x(\nu)}\right) = 0.$$

□

THEOREM 3.2.8. *Teichmüller space $F(S)$ is a complex submanifold of $(QF(S), J_Q)$.*

PROOF. Consider the immersion

$$\iota_S : (L^\infty_S(\Gamma), I_S) \to (L^\infty(\Gamma), J)$$

given by

$$\iota_S(\mu) = \mu$$

It is clear that $\iota_S$ is a holomorphic mapping. From $\iota_S$ we obtain the immersion

$$\iota_F : (F(S), I_T) \to (QF(S), J_Q)$$

sending each $[\xi]$ to itself for $\xi \in Belt_S(\Gamma)$. We only have to show that $\iota_F$ is almost complex, i.e satisfies

$$J_Q \circ (\iota_F)_* = (\iota_F)_* \circ I_T$$

Indeed, consider any vector field $\frac{\partial}{\partial x(\mu)}$, $\mu \in L^\infty_S(\Gamma)$. Then

$$(J_Q \circ (\iota_F)_*)\left(\frac{\partial}{\partial x(\mu)}\right) = J_Q\left(\frac{\partial}{\partial x(\iota_S(\mu))}\right) = \frac{\partial}{\partial x(J(\iota_S(\mu)))} =$$

$$= \frac{\partial}{\partial x(\iota_S(I_S(\mu)))} = (\iota_F)_*\left(\frac{\partial}{\partial x(I_S(\mu))}\right) = (\iota_F)_* \circ I_T)\left(\frac{\partial}{\partial x(\mu)}\right).$$

Therefore, (cf. [K-N], Lemma of section 8, Chpt. IX) $\iota_F$ is holomorphic and $(F(S), I_T)$ is a complex submanifold of $(QF(S), J_Q)$. □

## 3.3. Complex symplectic geometry again

In this section we prove that the complex symplectic structure of $QF(S)$ constructed in 3.1 is that obtained from the W-P Hyperkählerian metric.

We denote by $\Omega^Q$ the closed holomorphic $(2,0)$ form induced from the Hyperkählerian metric $g^Q$ by the relation

$$\Omega^Q_{([\xi])}\left(\frac{\partial}{\partial x(\mu)},\frac{\partial}{\partial x(\nu)}\right)=\omega^Q_{1([\xi])}\left(\frac{\partial}{\partial x(\mu)},\frac{\partial}{\partial x(\nu)}\right)+i\omega^Q_{2([\xi])}\left(\frac{\partial}{\partial x(\mu)},\frac{\partial}{\partial x(\nu)}\right)$$

This form, as we have mentioned in the beginning of section 3.2, defines a complex symplectic structure on $QF(S)$. We shall prove here the following

THEOREM 3.3.1. $\Omega^Q = 2\Omega$.

PROOF. By Theorem 3.2.7 and Theorem 3.1.8 we obtain that the holomorphic forms $\Omega^Q$, $2\Omega$ are identical in the tangent subbundle of $F(S)$. In other words if we express the holomorphic form $\Omega^Q - 2\Omega$ in global complex F-N coordinates $\lambda_{\gamma_i} = l_{\gamma_i} + \vartheta_i$, $\beta_i = \tau_1 + \psi_i$, $i = 1, ..., 3g-3$ and let

$$F_{\alpha\beta} = F_{\alpha\beta}(\lambda_{\gamma_1}, ..., \lambda_{\gamma_{3g-3}}, \beta_1, ..., \beta_{3g-3}),$$

be the coordinate holomorphic functions of $\Omega^Q - 2\Omega$, then

$$F_{\alpha\beta}(l_{\gamma_1}, ..., l_{\gamma_{3g-3}}, \tau_1, ..., \tau_{3g-3}, 0, ..., 0, 0, ..., 0) = 0$$

Since $(l_{\gamma_i}, \tau_i) \in (0, +\infty) \times \mathbb{R}$ for each $i = 1, ..., n$,[W4] we apply Lemma 3.1.7 to get $F_{\alpha\beta} \equiv 0$. Thus $\Omega^Q$ and $2\Omega$ coincide everywhere in $QF(S)$. □

DEFINITION. Let $(M, \Omega)$ be a complex symplectic manifold and $N$ a complex submanifold of $M$. We call $N$ *complex symplectic* if the restriction of $\Omega$ to the holomorphic tangent subbundle of $N$ provides a complex symplectic structure for $N$. We call $N$ *complex Lagrangian* if $\Omega$ vanishes identically in the holomorphic tangent subbundle of $N$.

THEOREM 3.3.2. *Bers' slices are complex Lagrangian submanifolds of* $(QF(S), \Omega^Q)$.

PROOF. By Theorem 3.2.7 ii) Bers' slices are Lagrangian submanifolds of $QF(S)$ with respect to both $\omega^Q_1, \omega^Q_2$ and the result follows. □

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