Thesis Title

Improving trace Hardy inequalities and Hardy inequalities for fractional Laplacians on bounded domains

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Βελτίωση Hardy ανισοτήτων ίχνους και ανισοτήτων Hardy για κλασματικές Λαπλασιανές σε φραγμένα χωρία

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Dedicated to my family

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Abstract

This thesis is devoted to inequalities which interpolate weighted Hardy and trace Hardy inequalities.We first derive a sharp interpolation between the two Hardy inequalities. Then we proceed to improving these inequalities by adding in the least hand side correction terms, that amount to positive integrals of the functions under consideration. In particular, we concern with integrals, over the half space or its boundary, which involve either the critical Hardy potential or the critical Sobolev exponent. In all cases, it turns out that correction terms of such type can be added at the expense of a logarithmic corrective weight, which is optimal in the sense that the inequality fails for smaller powers of this weight. Furthermore, we show that the aforementioned inequalities can be repeatedly improved, obtaining an infinite correction series.

The results in the two borderline cases of these interpolation inequalities yield refinements of the weighted Hardy and the trace weighted Hardy inequality respectively, thus unify and extend some earlier results. In particular, it follows that the trace Hardy and the Hardy weighted inequalities admit the same infinite improvement.

Moreover, we apply the resulted improvements of the trace Hardy inequality with trace remainder terms, to derive refinements of Hardy inequalities associated with two different fractional Laplacians defined on bounded domains.

Περίληψη (abstract in Greek)

Η παρούσα διατριβή είναι αφιερωμένη σε ανισότητες παρεμβολής μεταξύ ανισοτήτων Hardy με βάρος και Hardy ανισοτήτων ίχνους με βάρος. Αρχικά παράγουμε μια βέλτιστη ανισότητα παρεμβολής μεταξύ των δύο ανισοτήτων Hardy. Έπειτα βελτιώνουμε αυτές τις ανισότητες, προσθέτοντας στο μικρότερο μέλος διορθωτικούς όρους, οι οποίοι είναι θετικά ολοκληρώματα των υπο θεώρηση συναρτήσεων. Συγκεκριμένα, ασχολούμαστε με ολοκληρώματα στον ημιχώρο ή στο σύνορο του, τα οποία εμπλέκουν είτε το κρίσιμο δυναμικό Hardy ή τον κρίσιμο εκθέτη Sobolev. Σε όλες τις περιπτώσεις, αποδεικνύεται ότι μπορούν να προστεθούν διορθωτικοί όροι αυτού του τύπου, αλλά με ένα λογαριθμικό διορθωτικό βάρος, το οποίο είναι βέλτιστο υπό την έννοια ότι η ανισότητα αποτυγχάνει για μικρότερες δυνάμεις αυτού του βάρους. Επιπλέον, δείχνουμε ότι οι προαναφερθείσες ανισότητες μπορούν να βελτιωθούν κατ΄ επανάληψη, παίρνοντας μια άπειρη διορθωτική σειρά.

Τα αποτελέσματα στις δύο οριακές περιπτώσεις αυτών των ανισοτήτων παρεμβολής, συνιστούν βελτιώσεις της ανισότητας Hardy με βάρος και της Hardy ανισότητας ίχνους με βάρος αντίστοιχα, ενοποιώντας και επεκτείνωντας έτσι κάποια γνωστά αποτελέσματα. Ειδικότερα, προκύπτει ότι οι ανισότητες Hardy και οι Hardy ανισοτήτες ίχνους με βάρος, επιδέχονται την ίδια βελτίωση με άπειρους όρους.

Επιπλέον, εφαρμόζουμε τις προκύπτουσες βελτιώσεις της Hardy ανισότητας ίχνους στις οποίες οι διορθωτικοί όροι είναι ολοκληρώματα του ίχνους, ώστε να πάρουμε βελτιώσεις των Hardy ανισοτήτων που σχετίζονται με δύο διαφορετικές κλασματικές Λαπλασιανές οι οποίες ορίζονται σε φραγμένα χωρία.

Notation

Throughout the present work we adapt the following notation.

- $\mathbb{R}^n = \{x = (x_1, x_2, \dots, x_n),\ \ x_1, \dots, x_n \in \mathbb{R}\}$ = the n dimensional Euclidean space, $n \geq 2$.
- $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.
- A point $x \in \mathbb{R}^n$ is written $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.
- $\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\, x_n > 0\} =$ open upper half-space.
- $\partial\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\, x_n = 0\}$.
- $\overline{\mathbb{R}^n_+} = \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\, x_n \geq 0\} =$ closed upper half-space.
- $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ = the unit sphere in $\mathbb{R}^n$.
- $\mathbb{S}^{n-1}_+ = \{x = (x', x_n) \in \mathbb{R}^n : |x| = 1,\ x_n > 0\}$ = the upper half sphere in $\mathbb{R}^n$.
- $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ = the open ball in $\mathbb{R}^n$ with center at the origin and radius $r > 0$.
- $B'_r = \{x' \in \mathbb{R}^{n-1} : |x'| < r\}$ = the open ball in $\mathbb{R}^{n-1}$ with center at the origin and radius $r > 0$.
- $\partial B'_r = \{x' \in \mathbb{R}^{n-1} : |x'| = r\}$ = the sphere in $\mathbb{R}^{n-1}$, of radius $r > 0$ and center at the origin.
- $\mathbb{S}^{n-2} = \partial B'_1 = \{x' \in \mathbb{R}^{n-1} : |x'| = 1\}$ = the unit sphere in $\mathbb{R}^{n-1}$.
- $\Gamma(a) = \int_0^\infty t^{a-1}\, e^{-t}\, \mathrm{d}t$ = the Gamma function, for any real number a except the non-positive integers.
- $\gamma_n \ = \ \pi^{\frac{n}{2}} \,/\, \Gamma(\frac{n}{2}) \ = \$ the $(n-1)-$ dimensional volume of $\mathbb{S}^{n-1}_+$.
- $\omega_n \ = \ 2\,\pi^{\frac{n-1}{2}} \,/\, \Gamma(\frac{n-1}{2}) \ = \$ the $(n-2)-$ dimensional volume of the unit sphere $\mathbb{S}^{n-2}$.
- For any $U \subset \mathbb{R}^n$, we write $U^+ = \mathbb{R}^n_+ \cap U$.
- For any point $x \in \mathbb{S}^{n-1}_+$ we define $\varphi = \varphi(x) = \arccos x_n$, $\varphi \in [0, \pi/2]$. Notice that $\cos\varphi$ = distance of x to $\partial\mathbb{R}^n_+$.
- $C^\infty(\mathcal{V}) = \{u : \mathcal{V} \to \mathbb{R}\ \mid u \text{ is infinitely differentiable}\}$, where $\mathcal{V} \subseteq \mathbb{R}^n$.
- $C^\infty_0(\mathcal{V})$ = the functions in $C^\infty(\mathcal{V})$ with compact support. In particular, if U is an open set containing the origin, then a function in $C^\infty_0(\overline{\mathbb{R}^n_+} \cap U)$ does not necessary vanish on $\partial\mathbb{R}^n_+ \cap U$.
- $D^{1,2}(U)$ = the completion of $C^\infty_0(\overline{\mathbb{R}^n_+} \cap U)$ with respect to the norm $\|u\|_{D^{1,2}(U)} = (\int_{U^+} |\nabla u|^2\, \mathrm{d}x)^{1/2}$.
- $\int_{\mathbb{S}^{n-1}_+} f(x)\ \mathrm{d}\sigma(x)$ = integral of $f : U \subset \mathbb{R}^n \to \mathbb{R}$, over $\mathbb{S}^{n-1}_+$, with respect to the $(n-1)-$dimensional Lebesgue measure.

- $\int_{\mathbb{S}^{n-2}} f(x',0)\, \mathrm{d}\sigma(x')$ = integral of $f : U \subset \mathbb{R}^n \to \mathbb{R}$, over $\mathbb{S}^{n-2}$, with respect to the $(n-2)-$dimensional Lebesgue measure.

- We employ the letters c, C to denote a positive constant, independent of any function u and may change in each occurrence. Whenever it is necessary, we point out the dependence on the parameters involved with subscripts or parentheses.

- A point in $\mathbb{R}^{n+1}$ is denoted as (x, y), where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y \in \mathbb{R}$.

- $\mathbb{R}^{n+1}_+ = \{(x,y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n,\, y > 0\}$ and $\partial\mathbb{R}^{n+1}_+ = \{(x,y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n,\, y = 0\}$. For any $U \subset \mathbb{R}^{n+1}$, we write $U^+ = \mathbb{R}^{n+1}_+ \cap U$.

- For functions $f : \mathcal{D} \to \mathbb{R}$, $g : \mathcal{D} \to \mathbb{R}$, we write $f \sim g$ in $\mathcal{D}$, when there exist constants $c_1 > 0$, $c_2 > 0$, independent of f and g, such that $c_1\, f(z) \leq g(z) \leq c_2\, f(z)$, $\forall z \in \mathcal{D}$.

- We define $X(\vartheta) = X_1(\vartheta) = \frac{1}{1-\ln\vartheta}$, for $\vartheta \in (0,1]$ and $X_k(\vartheta) = X_1(X_{k-1}(\vartheta))$, for $k = 2, 3, \ldots$.

- For any bounded domain $U \subset \mathbb{R}^n$ we abbreviate $d = \sup_{x \in \mathbb{R}^n_+ \cap U} |x|$.

- In Chapter 6, Ω denotes a bounded domain in $\mathbb{R}^n$ and $D = \sup_{x \in \Omega} |x|$.

Contents

Chapter 1

Introduction

1.1 Classical results

Inequalities involving integrals of functions and the modulus of their gradient in various powers, play a prominent role in the continuous development of the theory, methods and applications of partial differential equations and calculus of variations. In particular, such inequalities have become a standard tool in the study of existence, uniqueness, boundedness, stability, asymptotic behaviour and other qualitative properties of solutions of partial differential equations and optimization problems. In addition, they find various applications in geometric measure theory and many other branches of analysis, geometry and physics. Some of the bibliography on this kind of inequalities is [2], [10], [34], [39], [41] and [42].

In the past few decades, there has been a continuous interest by many scholars, in improvements of such inequalities, when possible. Typically, improvements of such inequalities amount to extra terms on the least hand side that involve integrals of powers either of the function or of its gradient. Such improvements are motivated by certain applications, such as in the study of existence and asymptotic behaviour of solutions of parabolic equations with singular potentials (see for instance [7], [12], [32], [52]), in the study stability of solutions of elliptic (e.g [27], [22]) and parabolic equations (e.g. [11], [33], [43]), as well as in the study of the stability of eigenvalues in elliptic problems (e.g. [20]).

In the following two paragraphs we recall certain scaling invariant inequalities , which are well known and they are directly related with the context of the present thesis.

1.1.1 Hardy inequalities

The classical Hardy inequality asserts that for $n \geq 3$, there holds (cf. [34])

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \, \mathrm{d}x \leq \int_{\mathbb{R}^n} |\nabla u|^2 \, \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n).$$

It is well known that the constant $(n-2)^2/4$ is the best possible.

Passing from the whole space $\mathbb{R}^n$ to an open subset U of $\mathbb{R}^n$ with $n \geq 3$, Hardy inequality asserts that

$$\frac{(n-2)^2}{4} \int_{U} \frac{u^2}{|x|^2} \, \mathrm{d}x \leq \int_{U} |\nabla u|^2 \, \mathrm{d}x, \quad \forall u \in C_0^\infty(U). \tag{1.1}$$

Moreover, if U contains the origin, the constant $(n-2)^2/4$ is the best possible.

Hardy inequality represents a classical subject in which there has been intensive research in the last decades, mainly motivated by its application to pde's and more precisely with the positivity of the Schrödinger operator

$$L_\lambda = -\Delta - \frac{\lambda}{|x|^2}, \ \lambda \in \mathbb{R}.$$

The operator L_λ appears in the linearization of the critical nonlinear pde's, playing a crucial role in the asymptotic behaviour of branches of solutions in bifurcation problems (see e.g. [11], [50], [52]). The operator L_λ also arises in physics and in particular in the relativity theory and quantum mechanics (see [18]). We also refer to other interesting applications in molecular physics [40], quantum cosmology [9], combustion models [32], brownian motion [38].

Hardy inequalities on subsets $U \subset \mathbb{R}^n$ such that the origin, where the potential is singular, lies on the boundary ∂U are also valid but with different in general optimal constant (see e.g. [17], [28]).

On the other hand, we can deduce by standard reflection arguments that inequality (1.1) still holds with the same optimal constant on the upper half space

$$\mathbb{R}^n_+ = \{x = (x',\, x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\, x_n > 0\},$$

without the restriction $u = 0$ on the boundary $\partial\mathbb{R}^n_+$, that is

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{|x|^2}\, \mathrm{d}x \leq \int_{\mathbb{R}^n_+} |\nabla u|^2\, \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n). \tag{1.2}$$

In that case inequality (1.2) does not give any information about the summability properties of the trace of the functions u which do not vanish on the boundary $\partial\mathbb{R}^n_+$. Such summability properties can be deduced from the following trace Hardy inequality, also known in the literature as Kato inequality (cf. [35])

$$H_n \int_{\partial\mathbb{R}^n_+} \frac{u^2(x',0)}{|x'|}\, \mathrm{d}x' \leq \int_{\mathbb{R}^n_+} |\nabla u|^2\, \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n),\;\; n \geq 3, \tag{1.3}$$

where the constant

$$H_n = 2\frac{\Gamma^2(\frac{n}{4})}{\Gamma^2(\frac{n-2}{4})}$$

is the best possible.

Let now U be a generic bounded domain and the origin be an interior point of U. Passing from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+ \cap U$, the trace Hardy inequality (1.3) reads

$$H_n \int_{\partial\mathbb{R}^n_+ \cap U} \frac{u^2(x',0)}{|x'|}\, \mathrm{d}x' \leq \int_{\mathbb{R}^n_+ \cap U} |\nabla u|^2\, \mathrm{d}x, \quad \forall u \in C_0^\infty(U), \tag{1.4}$$

with the same optimal constant H_n, as in (1.3). This is a direct consequence of the invariance under scaling, of inequality (1.3).

Inequalities (1.3), (1.4) have furnished a handy tool to investigate qualitative properties of solutions of parabolic and the associated elliptic differential equations with linear boundary conditions with critical potentials or certain non linear boundary conditions (see e.g. [21], [36]).

In [4], it has been established an interpolation inequality between (1.2) and (1.3). More precisely, it has been proven that for all $u \in C_0^\infty(\mathbb{R}^n)$, $2 \leq b < n$ the following inequality holds

$$C(n,b) \int_{\partial\mathbb{R}^n_+} \frac{u^2}{|x'|}\, \mathrm{d}x' + \frac{(b-2)^2}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{|x|^2}\, \mathrm{d}x \leq \int_{\mathbb{R}^n_+} |\nabla u|^2\, \mathrm{d}x. \tag{1.5}$$

Here the constant

$$C(n,b) = 2\frac{\Gamma(\frac{n+b}{4} - \frac{1}{2})\Gamma(\frac{n-b}{4} + \frac{1}{2})}{\Gamma(\frac{n+b}{4} - 1)\Gamma(\frac{n-b}{4})} \tag{1.6}$$

is optimal. Notice that $C(n,\,2) = H_n$ and $C(n,b) \to 0,$ as $b \to n$. This means that when $b = 2$, then inequality (1.5) reduces to (1.3), while as $b \to n,$ then inequality (1.5) reduces to (1.2).

1.1.2 Sobolev inequalities

Sobolev inequalities are among the most useful functional inequalities in analysis. The fundamental role that Sobolev inequalities have played in the study of partial differential equations is well known. They have been studied by many authors and it is by now a classical subject.

The foremost example is the following Sobolev inequality valid for $n \geq 3$, which states the existence of a universal constant $S_n > 0$ such that (see [6], [48])

$$S_n \left(\int_{\mathbb{R}^n} |u|^{2^*} \, \mathrm{d}x \right)^{2/2^*} \leq \int_{\mathbb{R}^n} |\nabla u|^2 \, \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n),$$

where we use the conventional notation 2^* standing for the so called critical Sobolev exponent $2^* = 2n/(n-2)$.

On the other hand, the trace Sobolev inequalities, express a strong integrability property for the trace of a function in terms of some integrability property for its derivatives. Such inequalities are relevant for the study of boundary value problems for differential operators and they have been intensively studied in many contexts, in the last two decades. Here we shall mention only the one, which is useful in study of second order differential quasi-linear equations. It states the existence of a universal constant $\mathrm{s}_n > 0$ such that (cf. [8], [24])

$$\mathrm{s}_n \left(\int_{\partial\mathbb{R}^n_+} |u|^{\frac{2(n-1)}{n-2}} \, \mathrm{d}x' \right)^{\frac{n-2}{n-1}} \leq \int_{\mathbb{R}^n_+} |\nabla u|^2 \, \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n). \tag{1.7}$$

1.2 Main results

Improvements of Hardy inequality (1.1) are useful in the study of existence, uniqueness, boundedness, stability, and other qualitative properties of solutions of parabolic or elliptic semi-linear partial differential equations. Similarly, the trace Hardy inequality (1.4) and its various improvements can be used in the study of certain qualitative properties of solutions of boundary value problems of parabolic or elliptic type. Such type of problems have been considered e.g. in [5], [19], [31], [37], [45], [46].

The aim of this work is to establish certain improvements of these inequalities and at the same time unifying and extending some earlier works. In particular, passing from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+ \cap U$, we refine (1.5) by adding correction terms of several types in the left hand side. The resulted inequalities are useful in the study of diffusion problems with reaction terms, involving critical potentials or certain nonlinearities, both in the equation and in the boundary conditions.

Moreover, we will address this issue in a more general setting, considering weighted integrals, involving a power of the distance to the boundary $\partial\mathbb{R}^n_+$. Such types of weighted integral estimates, are useful in the study of singular/degenerate semi-linear elliptic and parabolic boundary values problems. The derived inequalities can be also combined with a recently developed technique used in order to deal with fractional powers of the Laplacian (see e.g. [25], [26]).

Firstly, we present an extension of (1.3) to more general inequalities involving the distance from the boundary $\partial\mathbb{R}^n_+ \cap U$.

Proposition I (Weighted Kato type inequality). *Let $\alpha \in (-1, 1)$ and $n + \alpha - 2 > 0$. Then for all $u \in C_0^\infty(\mathbb{R}^n)$ there holds*

$$H(n, \alpha) \int_{\partial\mathbb{R}^n_+} \frac{u^2(x', 0)}{|x'|^{1-\alpha}} \, \mathrm{d}x' \leq \int_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x, \tag{1.8}$$

where

$$H(n,\,\alpha) = (1-\alpha)\frac{\Gamma^2(\frac{n-\alpha}{4})\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{3-\alpha}{2})\Gamma^2(\frac{n+\alpha-2}{4})}.$$

The constant $H(n,\alpha)$ is sharp.

When $\alpha = 0$ inequality (1.8) reduces to (1.3) and the inequality fails if $|\alpha| \geq 1$. Note that this inequality can be obtained combining the results in [35] concerning the fractional Laplacian and the relation connecting the energy of the fractional Laplacian and the energy of the related extension problem (see [15], [25]). It can be also shown that (1.8) cannot be improved in the usual sense (see Section 3.3). In Section 3.1 we will give a different proof of this inequality which yields a sharper version obtaining remainder terms, when $\mathbb{R}^n_+$ is replaced by U.

The same situation holds for the weighted Hardy inequality

$$\frac{(\alpha+n-2)^2}{4}\int_{\mathbb{R}^n_+}\frac{x_n^\alpha\, u^2}{|x|^2}\,\mathrm{d}x \leq \int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x, \quad \forall u\in C_0^\infty(\mathbb{R}^n). \tag{1.9}$$

Next we obtain a class of inequalities which interpolate the weighted trace Hardy inequality (1.8) and the weighted Hardy inequality (1.9).

Theorem I (Sharp interpolation of weighted Kato - Hardy inequalities)**.** *Let $\alpha \in (-1,1)$, $2-\alpha \leq b < n$. Then for all $u \in C_0^\infty(\mathbb{R}^n)$ the following inequality holds*

$$K(n,\alpha,b)\int_{\partial\mathbb{R}^n_+}\frac{u^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x' + \frac{(\alpha+b-2)^2}{4}\int_{\mathbb{R}^n_+}\frac{x_n^\alpha\,u^2}{|x|^2}\,\mathrm{d}x \leq \int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x, \tag{1.10}$$

where

$$K(n,\alpha,\,b) = (1-\alpha)\frac{\Gamma(\frac{n-2\alpha-b+2}{4})\Gamma(\frac{n+b-2}{4})\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{3-\alpha}{2})\Gamma(\frac{n-b}{4})\Gamma(\frac{n+2\alpha+b-4}{4})}. \tag{1.11}$$

The constant $K(n,\alpha,\,b)$ is optimal.

Let us point out explicitly that $K(n,\alpha,b) = H(n,\alpha)$, when $b = 2-\alpha$ and $K(n,\alpha,b)\to 0,$ as $b \to n$. This means that when $b = 2-\alpha$, then inequality (1.10) reduces to (1.8), while as $b\to n$ then inequality (1.10) reduces to (1.9). Note also that one can deduce (1.10), simply considering a convex combination of (1.8) and (1.9), however the constants obtained by this argument are not in general sharp.

As in the borderline cases of inequality (1.10), namely (1.8) and (1.9), it can be shown that no L^p norm of u can be added in the left hand side. We refer to Section 3.3 for a precise statement and proof of this claim.

Let now U a generic bounded domain and the origin be an interior point of U. Passing from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+\cap U$, the trace Hardy inequality (1.8) and the weighted Hardy inequality (1.9) state respectively

$$H(n,\,\alpha)\int_{\partial\mathbb{R}^n_+\cap U}\frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' \leq \int_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x, \quad \forall u\in C_0^\infty(U) \tag{1.12}$$

and

$$\frac{(\alpha+n-2)^2}{4}\int_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,u^2}{|x|^2}\,\mathrm{d}x \leq \int_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x, \quad \forall u\in C_0^\infty(U). \tag{1.13}$$

The constants still remain the best possible. In view of (1.10) we obtain the following sharp interpolation inequality between (1.12) and (1.13)

$$K(n,\alpha,b)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x'+\frac{(\alpha+b-2)^2}{4}\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,u^2}{|x|^2}\,\mathrm{d}x\le\int\limits_{\mathbb{R}^n_+\cap U}x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x,\quad\forall u\in C_0^\infty(U).\tag{1.14}$$

Contrary to (1.10), several type of correction terms can be added in the left hand side of (1.14). We will present in the next three sections several refinements of (1.14).

1.2.1 Sobolev type remainder terms

The following result states that (1.14) can be improved by adding a Sobolev type correction term involving a singular weight. This weight is optimal in the sense that the improved inequality holds for this weight but fails for any weight more singular than this one.

Theorem II. *Let $\alpha\in(-1,1)$, $2-\alpha\le b<n$ and U be a bounded domain in $\mathbb{R}^n$. Then there exists a constant $c>0$, depending only on n and α, such that for all $u\in C_0^\infty(U)$ there holds*

$$K(n,\alpha,b)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x'\;+\;\frac{(\alpha+b-2)^2}{4}\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,u^2}{|x|^2}\,\mathrm{d}x+c\left(\int\limits_{\mathbb{R}^n_+\cap U}X^{\frac{2n-2+\alpha}{n-2+\alpha}}\,|u|^{\frac{2n}{n-2+\alpha}}\,\mathrm{d}x\right)^{\frac{n-2+\alpha}{n}}$$
$$\le\int\limits_{\mathbb{R}^n_+\cap U}x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x,\tag{1.15}$$

where $X=X(|x|/d)$, $X(\vartheta)=(1-\ln\vartheta)^{-1}$, $0<\vartheta\le 1$, $d=\sup_{x\in\mathbb{R}^n_+\cap U}|x|$. Moreover the logarithmic correction $X^{\frac{2n-2+\alpha}{n-2+\alpha}}$ cannot be replaced by a smaller power of X.

The nonweighted case, where $\alpha=0$, is of special interest. Let us consider in particular the two limiting cases of (1.15), namely the cases $b=2$ and $b\to n$. When b goes to n, inequality (1.15) reduces to an improvement of Hardy inequality obtained, among others, in [27].

On the other hand, when $b=2$ estimate (1.15) reduces to the following estimate

$$H_n\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2}{|x'|}\,\mathrm{d}x'\;+\;c\left(\int\limits_{\mathbb{R}^n_+\cap U}X^{\frac{2n-2}{n-2}}\,|u|^{\frac{2n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}}\le\int\limits_{\mathbb{R}^n_+\cap U}|\nabla u|^2\,\mathrm{d}x,\;\forall u\in C_0^\infty(U).\tag{1.16}$$

It is worth to mention that in [21], it has been proved that there exists a constant $c=c(n,p,U)>0$ such that

$$H_n\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2(x',0)}{|x'|}\,\mathrm{d}x'+c\left(\int\limits_{\mathbb{R}^n_+\cap U}|\nabla u|^p\,\mathrm{d}x\right)^{2/p}\le\int\limits_{\mathbb{R}^n_+\cap U}|\nabla u|^2\,\mathrm{d}x,\tag{1.17}$$

for all $u\in C_0^\infty(U)$ and $1\le p<2$. In view of the Sobolev inequality $\|u\|_{L^{p^*}(U)}\le c_{n,p}\,\|\nabla u\|_{L^p(U)}$, where $p^*=\frac{np}{n-p}$, we conclude that the following inequality also holds

$$H_n\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2(x',0)}{|x'|}\,\mathrm{d}x'+c(n,p)\left(\int\limits_{\mathbb{R}^n_+\cap U}|u|^{p^*}\,\mathrm{d}x\right)^{2/p^*}\le\int\limits_{\mathbb{R}^n_+\cap U}|\nabla u|^2\,\mathrm{d}x,\quad u\in C_0^\infty(U).\tag{1.18}$$

Then the improvement (1.16) , corresponds to the limiting case of (1.18) as $p \to 2$. Note that inequality (1.16) involves the critical exponent but contrary to (1.18) it has a logarithmic weight which cannot be removed.

As a consequence of Theorem II we obtain, by means of Hölder inequality, the following improvement of (1.14).

$$K(n,\alpha,b) \int\limits_{\partial\mathbb{R}^n_+\cap U} \frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' \quad + \quad \frac{(\alpha+b-2)^2}{4} \int\limits_{\mathbb{R}^n_+\cap U} \frac{x_n^\alpha u^2}{|x|^2}\,\mathrm{d}x + C \int\limits_{\mathbb{R}^n_+\cap U} V(x)\,u^2\,\mathrm{d}x$$
$$\leq \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x,$$

for some positive constant $C = C(n,\alpha,U)$, where the potential $V \geq 0$ is such that

$$\int\limits_{\mathbb{R}^n_+\cap U} V^{\frac{n}{2-\alpha}} X^{\frac{2-\alpha-2n}{2-\alpha}}\,\mathrm{d}x < \infty.$$

Actually, the above improvement still holds for even more singular potentials V, as the following result states.

Theorem III. *Let $\alpha \in (-1,1)$, $2-\alpha \leq b < n$ and U be a bounded domain in $\mathbb{R}^n$. Then there exists a constant $C > 0$, depending only on n, α, such that for all $u \in C_0^\infty(U)$ there holds*

$$K(n,\alpha,b) \int\limits_{\partial\mathbb{R}^n_+\cap U} \frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' \quad + \quad \frac{(\alpha+b-2)^2}{4} \int\limits_{\mathbb{R}^n_+\cap U} \frac{x_n^\alpha u^2}{|x|^2}\,\mathrm{d}x + C \int\limits_{\mathbb{R}^n_+\cap U} \frac{X^2}{|x|^{2-\alpha}}\,u^2\,\mathrm{d}x \leq \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x,$$

where $X = X(|x|/d)$, $X(\vartheta) = (1-\ln\vartheta)^{-1}$, $0 < \vartheta \leq 1$, $d = \sup_{x\in\mathbb{R}^n_+\cap U}|x|$. The weight X^2 cannot be replaced by a smaller power of X..

1.2.2 Infinite improvement

Next we present an improvement of (1.14) by adding correction terms of Hardy type with a singular logarithmic weight. This weight is optimal in the sense that the inequality fails for more singular weights. In the two borderline cases of these interpolation inequalities we obtain refinements of the weighted Hardy and the trace weighted Hardy inequality respectively.

Before stating the result we need to introduce some notation. For $\vartheta \in (0,1]$ we define recursively the functions

$$X_1(\vartheta) = \frac{1}{1-\ln\vartheta}, \quad X_k(\vartheta) = X_1(X_{k-1}(\vartheta)),\ k = 2,3,\ldots.$$

Our result is stated as follows:

Theorem IV ([51])**.** *Let $\alpha \in (-1,1)$, $2-\alpha \leq b < n$ and U is a bounded domain in $\mathbb{R}^n$. Then for all $u \in C_0^\infty(U)$ there holds*

$$K(n,\alpha,b) \int\limits_{\partial\mathbb{R}^n_+\cap U} \frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' + \frac{(\alpha+b-2)^2}{4} \int\limits_{\mathbb{R}^n_+\cap U} \frac{x_n^\alpha\,u^2}{|x|^2}\,\mathrm{d}x \quad + \quad \frac{1}{4}\sum_{i=1}^{\infty} \int\limits_{\mathbb{R}^n_+\cap U} \frac{x_n^\alpha\,X_1^2\cdots X_i^2}{|x|^2}\,u^2\,\mathrm{d}x$$
$$\leq \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x. \tag{1.19}$$

Here the constant $K(n,\alpha,b)$ is given in (1.11) and $X_i = X_i(|x|/d)$, with $d = \sup_{x\in\mathbb{R}^n_+\cap U}|x|$. For fixed b, the constants $\frac{1}{4}$ are optimal, that is for $k = 1,2,\dots$ there holds

$$\frac{1}{4} = \inf_{u\in C_0^\infty(U)} \frac{\int\limits_{U^+} x_n^\alpha |\nabla u|^2\,\mathrm{d}x - K(n,\alpha,b)\int\limits_{\partial\mathbb{R}^n_+\cap U} \frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' - \frac{(\alpha+b-2)^2}{4}\int\limits_{U^+}\frac{x_n^\alpha u^2}{|x|^2}\mathrm{d}x - \frac{1}{4}\sum\limits_{i=1}^{k-1}\int\limits_{U^+}\frac{x_n^\alpha X_1^2\cdots X_i^2}{|x|^2}u^2\,\mathrm{d}x}{\int\limits_{U^+}\frac{x_n^\alpha X_1^2X_2^2\cdots X_k^2}{|x|^2}u^2\,\mathrm{d}x}.$$

Moreover, for each $i = 1,2,\dots,$ the logarithmic correction X_i^2 cannot be replaced by a smaller power of X_i.

Let us state explicitly the result of Theorem IV in the special case where $\alpha = 0$. For any $2 \le b < n$ and $u \in C_0^\infty(U)$, there holds

$$C(n,b)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2}{|x'|}\,\mathrm{d}x' + \frac{(b-2)^2}{4}\int\limits_{\mathbb{R}^n_+\cap U}\frac{u^2}{|x|^2}\,\mathrm{d}x + \frac{1}{4}\sum_{i=1}^\infty\int\limits_{\mathbb{R}^n_+\cap U}\frac{X_1^2\cdots X_i^2}{|x|^2}u^2\,\mathrm{d}x \le \int\limits_{\mathbb{R}^n_+\cap U}|\nabla u|^2\,\mathrm{d}x, \tag{1.20}$$

where the constant $C(n,b)$ is given in (1.6). Inequality (1.20) constitutes an improvement of (1.5) in $\mathbb{R}^n_+\cap U$. The two limiting cases of (1.20), namely the cases $b=2$ and $b\to n$, are of special interest. When $b\to n$, then (1.20) reduces to the following improvement of Hardy inequality obtained, among others, in [27]

$$\frac{(n-2)^2}{4}\int\limits_{\mathbb{R}^n_+\cap U}\frac{u^2}{|x|^2}\,\mathrm{d}x + \frac{1}{4}\sum_{i=1}^\infty\int\limits_{\mathbb{R}^n_+\cap U}\frac{X_1^2\cdots X_i^2}{|x|^2}u^2\,\mathrm{d}x \le \int\limits_{\mathbb{R}^n_+\cap U}|\nabla u|^2\,\mathrm{d}x, \quad \forall u\in C_0^\infty(U). \tag{1.21}$$

On the other hand, when $b=2$ estimate (1.20) reduces to the following infinite improvement of (1.4)

$$H_n\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2}{|x'|}\,\mathrm{d}x' + \frac{1}{4}\sum_{i=1}^\infty\int\limits_{\mathbb{R}^n_+\cap U}\frac{X_1^2\cdots X_i^2}{|x|^2}u^2\,\mathrm{d}x \le \int\limits_{\mathbb{R}^n_+\cap U}|\nabla u|^2\,\mathrm{d}x, \quad \forall u\in C_0^\infty(U). \tag{1.22}$$

It is worth pointing out that both the Hardy inequality (1.21) and the trace Hardy inequality (1.22) admit the same sharp infinite improvement.

If we cut the series in (1.19) at the k term, we obtain the following result.

Theorem V. *Let $\alpha\in(-1,1)$, $2-\alpha\le b<n$ and U be a bounded domain in $\mathbb{R}^n$. Then there exists a constant $c>0$, depending only on n and α, such that*

$$\begin{aligned}K(n,\alpha,b)&\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' + \frac{(\alpha+b-2)^2}{4}\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha u^2}{|x|^2}\,\mathrm{d}x + \frac{1}{4}\sum_{i=1}^k\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha X_1^2\cdots X_i^2}{|x|^2}u^2\,\mathrm{d}x\\ &+ c\left(\int\limits_{\mathbb{R}^n_+\cap U}(X_1\cdots X_kX_{k+1})^{\frac{2n-2+\alpha}{n-2+\alpha}}\,|u|^{\frac{2n}{n-2+\alpha}}\,\mathrm{d}x\right)^{\frac{n-2+\alpha}{n}} \le \int\limits_{\mathbb{R}^n_+\cap U}x_n^\alpha|\nabla u|^2\,\mathrm{d}x, \quad \forall u\in C_0^\infty(U).\end{aligned} \tag{1.23}$$

Here the constant $K(n,\alpha,b)$ is given in (1.11) and $X_i = X_i(|x|/d)$, with $d = \sup_{x\in\mathbb{R}^n_+\cap U}|x|$. Moreover, the logarithmic correction $(X_1\cdots X_{k+1})^{(2n-2+\alpha)/(n-2+\alpha)}$ cannot be replaced by smaller powers of $X_1,\cdots,X_{k+1}$.

Note that applying Hölder inequality to the Sobolev term in (1.23), we obtain the following improvement of (1.14)

$$K(n,\alpha,b) \int\limits_{\partial\mathbb{R}^n_+\cap U} \frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' \;+\; \frac{(\alpha+b-2)^2}{4} \int\limits_{\mathbb{R}^n_+\cap U} \frac{x_n^\alpha u^2}{|x|^2}\,\mathrm{d}x + \frac{1}{4}\sum_{i=1}^{k} \int\limits_{\mathbb{R}^n_+\cap U} \frac{x_n^\alpha X_1^2\cdots X_i^2}{|x|^2}\,u^2\,\mathrm{d}x$$
$$+\; C \int\limits_{\mathbb{R}^n_+\cap U} V(x)\,u^2(x)\,\mathrm{d}x \le \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x, \tag{1.24}$$

for some positive constant $C = C(n,\alpha,U)$, where the potential $V \ge 0$ is such that

$$\int\limits_{\mathbb{R}^n_+\cap U} V^{\frac{n}{2-\alpha}}\,(X_1\cdots X_k X_{k+1})^{\frac{2-\alpha-2n}{2-\alpha}}\,\mathrm{d}x < \infty.$$

However, the improvement obtained in Theorem IV is stronger than (1.24), in the sense that the k-th remainder term involves a more singular potential as well as the series in the left hand side does not terminate.

1.2.3 Trace remainder terms

Let us now return to inequality (1.4) and discuss the problem as to whether this inequality can be refined by adding remainder terms that involve trace L^p norms of u.

Note first that we can deduce, by (1.17) and the trace Sobolev inequalities, that for any $1 \le p < 2$ there holds

$$H_n \int\limits_{\partial\mathbb{R}^n_+\cap U} \frac{u^2(x',0)}{|x'|}\,\mathrm{d}x' + c\left(\int\limits_{\partial\mathbb{R}^n_+\cap U} |u|^{p_*}\,\mathrm{d}x'\right)^{2/p_*} \le \int\limits_{\mathbb{R}^n_+\cap U} |\nabla u|^2\,\mathrm{d}x, \quad \forall u \in C_0^\infty(U), \tag{1.25}$$

where $p_* = p(n-1)/(n-p)$.

We point out that in (1.25), it is excluded the critical trace Sobolev exponent $2_* = 2(n-1)/(n-2)$, appearing in (1.7). In view of this, we address the problem whether there is scope for improving inequality (1.4) by adding a trace term involving the critical exponent 2_*. We will then show that such an improvement does hold but contrary to the subcritical case (1.25), the remaining term has a logarithmic correction which cannot be removed. We establish the result in a more general setting.

Theorem VI. *Let $\alpha \in (-1,1)$, $2-\alpha \le b < n$ and Ω be a bounded domain in $\mathbb{R}^{n-1}$. Then there exists a constant $C > 0$, depending only on n,α, such that for all $u \in C_0^\infty(\mathbb{R}^n)$, with $u(x',0) = 0$, for $x' \notin \Omega$, there holds*

$$K(n,\alpha,b)\int\limits_{\Omega} \frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' \;+\; \frac{(\alpha+b-2)^2}{4}\int\limits_{\mathbb{R}^n_+} \frac{x_n^\alpha\,u^2}{|x|^2}\,\mathrm{d}x + C\left(\int\limits_{\Omega} X^{\frac{2n-3+\alpha}{n-2+\alpha}}\,|u|^{\frac{2(n-1)}{n-2+\alpha}}\,\mathrm{d}x'\right)^{\frac{n-2+\alpha}{n-1}}$$
$$\le \int\limits_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x, \tag{1.26}$$

where $X = X(|x'|/D)$, $X(\vartheta) = (1-\ln\vartheta)^{-1}$, $0 < \vartheta \le 1$, $D = \sup_{x'\in\Omega}|x'|$. The weight $X^{\frac{2n-3+\alpha}{n-2+\alpha}}$ cannot be replaced by a smaller power of X.

An application of Hölder inequality to the Sobolev term in (1.26), leads to the following improvement of (1.14)

$$K(n,\alpha,b)\int\limits_{\Omega}\frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x'+\frac{(\alpha+b-2)^2}{4}\int\limits_{\mathbb{R}^n_+}\frac{x_n^\alpha u^2}{|x|^2}\,\mathrm{d}x+C\int\limits_{\Omega}V(x')\,u^2(x',0)\,\mathrm{d}x'\leq\int\limits_{\mathbb{R}^n_+}x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x,$$

for some positive constant $C=C(n,\alpha,\Omega)$, where the potential $V\geq 0$ is such that

$$\int\limits_{\Omega}V^{\frac{n-1}{1-\alpha}}(x')\,X^{\frac{3-\alpha-2n}{1-\alpha}}(|x'|/D)\,\mathrm{d}x'<\infty.$$

Actually, the above improvement still holds for even more singular potentials V, as the following result states.

Theorem VII. *Let $\alpha\in(-1,1)$, $2-\alpha\leq b<n$ and Ω be a bounded domain in $\mathbb{R}^{n-1}$. Then there exists a constant $C>0$, depending only on n,α, such that for all $u\in C_0^\infty(\mathbb{R}^n)$, with $u(x',0)=0$, for $x'\notin\Omega$, there holds*

$$K(n,\alpha,b)\int\limits_{\Omega}\frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x'\quad+\quad\frac{(\alpha+b-2)^2}{4}\int\limits_{\mathbb{R}^n_+}\frac{x_n^\alpha\,u^2}{|x|^2}\,\mathrm{d}x+C\int\limits_{\Omega}\frac{X^2}{|x'|^{1-\alpha}}\,u^2\,\mathrm{d}x'\leq\int\limits_{\mathbb{R}^n_+}x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x,$$

where $X=X(|x'|/D)$, $X(\vartheta)=(1-\ln\vartheta)^{-1}$, $0<X(\vartheta)\leq 1$, $D=\sup_{x'\in\Omega}|x'|$. The weight X^2 cannot be replaced by a smaller power of X.

1.3 Applications to Hardy inequalities for fractional Laplacians on bounded domains

In the last decade there has been a growing interest in the study of the so-called fractional Laplace operators both for the pure mathematical research and in view of concrete real-world applications, such as, among the others, obstacle problems, financial market, phase transitions and anomalous diffusion.

In the literature, there are several fractional Laplacians defined on bounded domains. In this work we deal with the *spectral* one (see e.g. [14], [16], [47] and references therein) and the *Dirichlet* one (or *integral* or *regional*, see e.g. [15], [29], [30] and references therein). In the next two sections we present improvements of Hardy inequalities for these operators.

1.3.1 The spectral fractional Laplacian

We first introduce the *spectral fractional Laplacian* which will be denoted by A_s, $0<s<1$. Let Ω be a bounded domain in $\mathbb{R}^n$, with $n>2s$. The spectral fractional Laplacian A_s is defined through the spectral decomposition using the s^{th} powers of the eigenfunctions of the conventional Laplacian $-\Delta$.

To give a precise definition, we consider an orthonormal basis of $L^2(\Omega)$, denoted by $\{\phi_k\}_{k=0}^\infty$, consisting of eigenfunctions of $-\Delta$ with homogeneous Dirichlet boundary conditions, associated to the eigenvalues $\{\lambda_k\}_{k=0}^\infty$, which repeated according to their finite multiplicity, are written

$$0<\lambda_1<\lambda_2\leq\lambda_3\leq\cdots\quad\text{with}\quad\lambda_k\to\infty.$$

Namely,

$$\begin{cases}-\Delta\phi_k=\lambda_k\phi_k, & \text{in }\Omega,\\ \phi_k=0, & \text{on }\partial\Omega.\end{cases}$$

Then the operator A_s is defined by

$$A_s f = \sum_{k=1}^{\infty} \lambda_k^s \, c_k \, \phi_k, \quad \forall f \in C_0^\infty(\Omega),$$

where

$$f = \sum_{k=1}^{\infty} c_k \phi_k \; \text{ and } \; c_k = \int_\Omega f \phi_k \, \mathrm{d}x.$$

In fact, the operator A_s can be extended by density for f in the Hilbert space

$$H = \left\{ f = \sum_{k=1}^{\infty} f_k \phi_k \in L^2(\Omega) : \|f\|_H = \left(\sum_{k=1}^{\infty} \lambda_k^s f_k^2 \right)^{1/2} < \infty \right\}$$

and

$$A_s f = \sum_{k=1}^{\infty} \lambda_k^s \, c_k \, \phi_k, \quad \forall f \in H.$$

Regarding the corresponding quadratic form for A_s, we have

$$(A_s f, \, f) := \int_\Omega f A_s f \, \mathrm{d}x = \sum_{k=1}^{\infty} \lambda_k^s \, c_k^2.$$

The Hardy inequality, involving the distance to the origin, associated with the spectral Laplacian A_s reads as follows.

Theorem VIII (Hardy inequality for the spectral fractional Laplacian)**.** *Let $s \in (0,1)$ and Ω be a bounded domain in $\mathbb{R}^n$ with $n > 2s$. Then for all $f \in C_0^\infty(\Omega)$ there holds*

$$h_{s,n} \int_\Omega \frac{f^2(x)}{|x|^{2s}} \, \mathrm{d}x \le (A_s f, \, f), \quad \text{where} \quad h_{s,n} = 2^{2s} \frac{\Gamma^2(\frac{n+2s}{4})}{\Gamma^2(\frac{n-2s}{4})}. \tag{1.27}$$

Moreover, the constant $h_{s,n}$ is sharp if $0 \in \Omega$.

Concerning fractional elliptic problems involving power type nonlinearities and singular potentials (e.g. [49]) the following inequality is useful. We show that (1.27) may be improved by adding a critical Sobolev norm with a logarithmic corrective weight, which cannot be removed.

Theorem IX (Hardy-Sobolev inequality for the spectral fractional Laplacian)**.** *Let $s \in (0,1)$ and Ω be a bounded domain in $\mathbb{R}^n$ with $n > 2s$. Then there exists a positive constant $C = C(n,s)$ such that for all $f \in C_0^\infty(\Omega)$ there holds*

$$h_{s,n} \int_\Omega \frac{f^2(x)}{|x|^{2s}} \, \mathrm{d}x + C \left(\int_\Omega X^{\frac{2(n-s)}{n-2s}}(|x|/D) \, |f(x)|^{\frac{2n}{n-2s}} \, \mathrm{d}x \right)^{\frac{n-2s}{n}} \le (A_s f, \, f), \tag{1.28}$$

where $D = \sup_{x \in \Omega} |x|$. The exponent $\frac{2(n-s)}{n-2s}$ of X cannot be replaced by a smaller one.

As a consequence of Theorem IX we obtain, by Hölder inequality, the following improvement of (1.27),

$$h_{s,n} \int_\Omega \frac{f^2(x)}{|x|^{2s}} \, \mathrm{d}x + C \int_\Omega V(x) \, f^2(x) \, \mathrm{d}x \le (A_s f, \, f), \tag{1.29}$$

for some positive constant $C = C(n, s, \Omega)$, where the potential $V \geq 0$ is such that

$$\int_\Omega V^{\frac{n}{2s}} X^{\frac{s-n}{s}} \, \mathrm{d}x < \infty.$$

In fact the improvement (1.29) of Hardy inequality, still holds for even more singular potentials V, as the following Theorem states.

Theorem X (Improved Hardy inequality for the spectral fractional Laplacian)**.** *Let $s \in (0,1)$ and Ω be a bounded domain in $\mathbb{R}^n$ with $n > 2s$. Then there exists a positive constant $C = C(n,s)$ such that the following inequality is valid*

$$h_{s,n} \int_\Omega \frac{f^2(x)}{|x|^{2s}} \, \mathrm{d}x + C \int_\Omega f^2(x) \, \frac{X^2(|x|/D)}{|x|^{2s}} \, \mathrm{d}x \leq (A_s f, \, f), \quad \forall f \in C_0^\infty(\Omega),$$

where $D = \sup_{x \in \Omega} |x|$. The weight X^2 cannot be replaced by a smaller power of X.

1.3.2 The Dirichlet fractional Laplacian

Let $0 < s < 1$. The fractional Laplacian $(-\Delta)^s$ of a function f in the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^n$, is defined via Fourier transform by

$$(-\Delta)^s f = \mathcal{F}^{-1}\left(|\xi|^{2s}(\mathcal{F}f)\right), \quad \forall \xi \in \mathbb{R}^n. \tag{1.30}$$

Here, as usual, $\mathcal{F}f$ denotes the Fourier transform of f

$$\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) \, \mathrm{d}x.$$

Equivalently, the operator $(-\Delta)^s$ can be defined by the following pointwise formula

$$(-\Delta)^s f(x) = c(n,s) \lim_{\epsilon \to 0^+} \int_{\{|x-y|>\epsilon\}} \frac{f(x) - f(y)}{|x-y|^{n+2s}} \, \mathrm{d}y, \quad \forall x \in \mathbb{R}^n, \tag{1.31}$$

where

$$c(n,s) = \frac{s 2^{2s}}{\pi^{n/2}} \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)}. \tag{1.32}$$

Note that in literature, different definitions of the fractional Laplacian consider different normalizing constant $c(n,s)$. Here the constant $c(n,s)$ is chosen so that the above definition is equivalent with the one via the Fourier transform given by (1.30).

Notice also that using a standard change of value we obtain the following equivalent pointwise definition of fractional Laplacian,

$$(-\Delta)^s f(x) \;=\; -\frac{1}{2} \, c(n,s) \int_{\mathbb{R}^n} \frac{f(x+y) - 2f(x) + f(x-y)}{|y|^{n+2s}} \, \mathrm{d}y, \quad \forall x \in \mathbb{R}^n,$$

with $c(n,s)$ as in (1.32).

Passing from $\mathbb{R}^n$ to a bounded domain Ω, we introduce the *Dirichlet fractional Laplacian* which will be still denoted by $(-\Delta)^s$. Extending any function $f \in C_0^\infty(\Omega)$ in the whole space $\mathbb{R}^n$ by setting $f(x) = 0$, $x \notin \Omega$, we define $(-\Delta)^s f$ as the conventional fractional Laplacian $(-\Delta)^s$ on the extended function. In particular,

$$(-\Delta)^s f = \mathcal{F}^{-1}\left(|\xi|^{2s}(\mathcal{F}f)\right), \quad \forall \xi \in \mathbb{R}^n.$$

Equivalently, the operator $(-\Delta)^s$ can be defined by the following pointwise formula (cf. (1.31))

$$(-\Delta)^s f(x) \,=\, c(n,s) \lim_{\epsilon\to 0^+} \int_{\{|x-y|>\epsilon\}} \frac{f(x)-f(y)}{|x-y|^{n+2s}}\,\mathrm{d}y, \quad \forall x\in\mathbb{R}^n,$$

where the constant $c(n,s)$ is given in (1.32).

Regarding the corresponding quadratic form for $(-\Delta)^s$, we have (see for example [29, Lemma 3.1])

$$((-\Delta)^s f,\, f) := \int_\Omega f\,(-\Delta)^s f\,\mathrm{d}x \,=\, \int_{\mathbb{R}^n} |\xi|^{2s}\,(\mathcal{F}f)^2(\xi)\,\mathrm{d}\xi \,=\, \frac{c(n,s)}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

We point out that the fractional operators A_s, $(-\Delta)^s$, are different. Indeed, the spectral one depends on the domain Ω considered, since its eigenvalue and eigenfunctions depend on Ω, while the Dirichlet one evaluated at some point is independent on the domain.

The Hardy inequality for the Dirichlet fractional Laplacian $(-\Delta)^s$, involving the distance to the origin, reads as follows.

Theorem XI (Hardy inequality for the Dirichlet fractional Laplacian)**.** *Let $s\in(0,1)$ and Ω be a bounded domain in $\mathbb{R}^n$ with $n>2s$. For all $f\in C_0^\infty(\Omega)$ there holds*

$$h_{s,n}\int_\Omega \frac{f^2(x)}{|x|^{2s}}\,\mathrm{d}x \le ((-\Delta)^s f,\, f). \tag{1.33}$$

The constant $h_{s,n}$ is given by (1.27) and it is sharp if $0\in\Omega$.

Equivalently, for all $f\in C_0^\infty(\Omega)$ there holds

$$k_{n,s}\int_\Omega \frac{f^2(x)}{|x|^{2s}}\,\mathrm{d}x \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y, \quad \textit{where} \quad k_{n,s} = \frac{2\,\pi^{n/2}\,\Gamma(1-s)\,\Gamma^2(\frac{n+2s}{4})}{s\,\Gamma^2(\frac{n-2s}{4})\,\Gamma(\frac{n+2s}{2})}. \tag{1.34}$$

The constant $k_{n,s}$ is sharp if $0\in\Omega$.

It is worth to note that the Hardy inequalities (1.27), (1.33) associated with two different operators share the same best constant. Notice however that when the distance is taken from the boundary, the optimal constants for the corresponding Hardy inequalities are different, as it was shown among others in [25] (cf. (1.40), (1.41) below).

A proof of inequality (1.33) it was given in [35] (see also [53]). Here we give a different proof of this result following an approach in the spirit of [15], which offers a refined version, involving remainder terms.

In the sequel we will present improvements of (1.33) by adding Sobolev and Hardy type correction terms. Before state our first result in this direction, let us notice that Frank, Lieb and Seiringer [29] have shown, among others, that for any $1\le q<2n/(n-2s)$ and any bounded domain $\Omega\subset\mathbb{R}^n$ there exists a positive constant $c=c(n,s,q,|\Omega|)$ such that for all $f\in C_0^\infty(\Omega)$ there holds

$$k_{s,n}\int_\Omega \frac{f^2(x)}{|x|^{2s}}\,\mathrm{d}x + c\left(\int_\Omega |f(x)|^q\,\mathrm{d}x\right)^{2/q} \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y. \tag{1.35}$$

We point out that the exponent q is strictly smaller than the critical fractional Sobolev exponent $q^* := 2n/(n-2s)$ and the inequality fails for $q=q^*$. Next we present an improvement of (1.34), involving the critical exponent q^* missed by logarithmic correction. Our result is stated in the following Theorem.

Theorem XII (Hardy-Sobolev inequality for the Dirichlet fractional Laplacian)**.** *Let $s \in (0,1)$, Ω be a bounded domain in $\mathbb{R}^n$ with $n > 2s$, and $D = \sup_{x\in\Omega}|x|$. Then there exists a positive constant $C = C(n,s)$ such that for all $f \in C_0^\infty(\Omega)$ there holds*

$$h_{s,n}\int_\Omega \frac{f^2(x)}{|x|^{2s}}\,\mathrm{d}x + C\left(\int_\Omega X^{\frac{2(n-s)}{n-2s}}\left(\tfrac{|x|}{D}\right)|f(x)|^{\frac{2n}{n-2s}}\,\mathrm{d}x\right)^{\frac{n-2s}{n}} \le ((-\Delta)^s f,\, f), \tag{1.36}$$

or, equivalently,

$$k_{s,n}\int_\Omega \frac{f^2(x)}{|x|^{2s}}\,\mathrm{d}x + C\left(\int_\Omega X^{\frac{2(n-s)}{n-2s}}\left(\tfrac{|x|}{D}\right)|f(x)|^{\frac{2n}{n-2s}}\,\mathrm{d}x\right)^{\frac{n-2s}{n}} \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(x)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y. \tag{1.37}$$

Moreover, the weight $X^{\frac{2(n-s)}{n-2s}}$ cannot be replaced by a smaller power of X.

Notice that inequality (1.37) involves the critical exponent but contrary to the subcritical case, that is (1.35), it has a logarithmic correction. However inequality (1.37) is sharp in the sense that inequality fails for smaller powers of the logarithmic correction X.

An application of Hölder inequality to the Sobolev term in (1.37) leads to the following improvement of (1.33)

$$h_{s,n}\int_\Omega \frac{f^2(x)}{|x|^{2s}}\,\mathrm{d}x + C\int_\Omega V(x)\,f^2(x)\,\mathrm{d}x \le ((-\Delta)^s f,\, f), \tag{1.38}$$

for some positive constant $C = C(n,s,\Omega)$, where the potential $V \ge 0$ is such that

$$\int_\Omega V^{\frac{n}{2s}} X^{\frac{s-n}{s}}\,\mathrm{d}x < \infty.$$

In the following Theorem we state that the improvement (1.38) of Hardy inequality, still holds for even more singular potentials V.

Theorem XIII. *Let $s \in (0,1)$, Ω be a bounded domain in $\mathbb{R}^n$ with $n > 2s$, and $D = \sup_{x\in\Omega}|x|$. Then there exists a positive constant $C = C(n,s)$ such that for all $f \in C_0^\infty(\Omega)$ there holds*

$$h_{s,n}\int_\Omega \frac{f^2(x)}{|x|^{2s}}\,\mathrm{d}x + C\int_\Omega f^2(x)\,\frac{X^2(\frac{|x|}{D})}{|x|^{2s}}\,\mathrm{d}x \le ((-\Delta)^s f,\, f),$$

or, equivalently,

$$k_{n,s}\int_\Omega \frac{f^2(x)}{|x|^{2s}}\,\mathrm{d}x + C\int_\Omega f^2(x)\,\frac{X^2(\frac{|x|}{D})}{|x|^{2s}}\,\mathrm{d}x \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

Moreover, the weight X^2 cannot be replaced by a smaller power of X.

Let us finally refer the main idea beyond the proof of the aforementioned fractional Hardy inequalities. The fractional Laplacians are non local operators and this raises several technical difficulties. However, both fractional Laplacians $A_s, (-\Delta)^s$, can be determined as operators that map a Dirichlet boundary condition to a Neumann type condition via an appropriate extension problem (see (2.14), (2.16) below). The key point now is the equivalence of our original nonlocal problem with the extended local problem, where local variational techniques can be applied. This argumentation has been already applied recently by Filippas, Moschini and Tertikas [25],[26] to obtain fractional Hardy and Hardy-Sobolev inequalities involving the distance to the boundary.

In particular, regarding the spectral fractional Laplacian A_s, the following Hardy and Hardy-Sobolev inequalities have been established.

Theorem (Hardy-Sobolev-Maz'ya inequality for the spectral fractional Laplacian, [25], [26])**.** *Let $\frac{1}{2} \leq s < 1$, $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded domain and $d(x) = dist(x, \partial\Omega)$.*

(i) If Ω is such that

$$-\Delta d(x) \geq 0,\ x \in \Omega, \tag{1.39}$$

then for all $f \in C_0^\infty(\Omega)$ there holds

$$d_s \int_\Omega \frac{f^2(x)}{d^{2s}(x)}\,\mathrm{d}x \leq (A_s f,\, f), \quad \text{where} \quad d_s = 2^{2s}\frac{\Gamma^2(\frac{3+2s}{4})}{\Gamma^2(\frac{3-2s}{4})}. \tag{1.40}$$

(ii) If there exists a point $x_0 \in \partial\Omega$ and $r > 0$ such that the part of the boundary $\partial\Omega \cap B_r(x_0)$ is C^1 regular, then the constant d_s is optimal.

(iii) If Ω is a Lipschitz domain satisfying (1.39) , then there exists a constant $C > 0$ such that for all $f \in C_0^\infty(\Omega)$ there holds

$$d_s \int_\Omega \frac{f^2(x)}{d^{2s}(x)}\,\mathrm{d}x + C\left(\int_\Omega |f(x)|^{\frac{2n}{n-2s}}\,\mathrm{d}x\right)^{\frac{n-2s}{n}} \leq (A_s f,\, f).$$

On the other hand, concerning the operator $(-\Delta)^s$, there have been established the following Hardy and Hardy-Sobolev inequalities.

Theorem (Hardy-Sobolev-Maz'ya inequality for the fractional Laplacian $(-\Delta)^s$, [25])**.** *Let $\frac{1}{2} \leq s < 1$, $n \geq 2$, $\Omega \subsetneq \mathbb{R}^n$ be a domain, $d(x) = dist(x, \partial\Omega)$ and $R_{in} = \sup_{x\in\Omega} d(x)$, the inner radius of Ω.*

(i) If Ω is convex, then for all $f \in C_0^\infty(\Omega)$ there holds

$$k_s \int_\Omega \frac{f^2(x)}{d^{2s}(x)}\,\mathrm{d}x \leq ((-\Delta)^s f,\, f), \quad \text{where} \quad k_s = \frac{\Gamma^2(\frac{1+2s}{2})}{\pi}. \tag{1.41}$$

(ii) If there exists a point $x_0 \in \partial\Omega$ and $r > 0$ such that the part of the boundary $\partial\Omega \cap B_r(x_0)$ is C^1 regular, then the constant k_s is optimal.

(iii) If Ω is a uniformly Lipschitz and convex domain with $R_{in} < \infty$ and $s \in (\frac{1}{2}, 1)$, then there exists a constant $C > 0$ such that for all $f \in C_0^\infty(\Omega)$ there holds

$$k_s \int_\Omega \frac{f^2(x)}{d^{2s}(x)}\,\mathrm{d}x + C\left(\int_\Omega |f(x)|^{\frac{2n}{n-2s}}\,\mathrm{d}x\right)^{\frac{n-2s}{n}} \leq ((-\Delta)^s f,\, f).$$

Let us finally notice, that contrary to the Hardy-Sobolev inequalities obtained in [25], where the distance is taken to the boundary, the Hardy-Sobolev inequalities which are stated in Theorems IX, XII, where the distance is taken to the origin, miss the critical Sobolev exponent by a logarithmic correction which cannot be removed. Moreover, when the distance is taken from the boundary, then the fractional Laplacians $(-\Delta)^s$, A_s do not satisfy Hardy inequality in the case of smooth bounded domains Ω and $0 < s < \frac{1}{2}$.

The rest of this work is organized as follows. In Chapter 2, we briefly outline some well known results which we shall deeply exploit in the next Chapters, to prove our results. The proofs of Proposition I and Theorem I are presented in Chapter 3. In Chapter 4, we give the proofs of Theorems II - V. Theorems VI, VII are proved in Chapter 5. Finally, Chapter 6 concerns the fractional Laplacians, where we prove Theorems VIII - XIII

Chapter 2

Preliminaries

2.1 The hypergeometric equation

In this section, we collect the main properties of the solutions of the hypergeometric equation that are extensively referred throughout the next chapters. There exists an extensive literature containing useful identities and properties of hypergeometric functions and hypergeometric equations. We indicatively refer to [1, Section 15], [23, Chapter II], [44, Sections 2.1.2-5]. Here we will follow the notation and terminology of these references.

For a complex function ω of the complex variable z, let us consider the hypergeometric differential equation

$$z\,(1-z)\,\frac{d^2\omega}{dz^2}\;+\;[\,\mathrm{c}\;-\;(\mathrm{a}\;+\;\mathrm{b}\;+\;1)z\,]\,\frac{d\omega}{dz}\;-\;\mathrm{a}\,\mathrm{b}\,\omega=0. \tag{2.1}$$

For our purposes, we will limit ourselves to certain conditions on the parameters $\mathrm{a},\mathrm{b},\mathrm{c}$. More precisely, from now on, we will always assume that $\mathrm{a},\mathrm{b},\mathrm{c}\in\mathbb{R}$ such that

$$\mathrm{c}-\mathrm{a}-\mathrm{b}\geq 0,\quad \mathrm{b}>0,\quad \mathrm{c}>0. \tag{2.2}$$

We shall exclude any other condition on these parameters from our discussion below, that is rather technical and beyond the scope of the present work. Interested readers are referred to the aforementioned literature.

The general solution of (2.1), defined in the complex domain cut along the interval $[1,\infty)$ of the real axis, is given by (see [1, 15.5.3, 15.5.4])

$$\omega(z)=\mathcal{C}_1\,F(\mathrm{a},\,\mathrm{b};\,\mathrm{c};\,z)+\mathcal{C}_2\,z^{1-\mathrm{c}}\,F(\mathrm{a}-\mathrm{c}+1,\,\mathrm{b}-\mathrm{c}+1;\,2-\mathrm{c};\,z), \tag{2.3}$$

for arbitrary complex constants $\mathcal{C}_1,\mathcal{C}_2$. Here the hypergeometric function $F(\mathrm{a},\,\mathrm{b};\,\mathrm{c};\,z)$ is defined by the Gauss series (see [1, 15.1.1])

$$F(\mathrm{a},\,\mathrm{b};\,\mathrm{c};\,z)=\sum_{k=0}^{\infty}\frac{(\mathrm{a})_k(\mathrm{b})_k}{(\mathrm{c})_k}\frac{z^k}{k!}, \tag{2.4}$$

in the disk $|z|<1$ and by analytic continuation in the whole of complex plain cut along the interval $[1,\infty)$ of the real axis. We also use the notation $(\mathrm{a})_k=\mathrm{a}(\mathrm{a}+1)\cdots(\mathrm{a}+k)$ and $(\mathrm{a})_0=1$. Obviously, there holds

$$F(\mathrm{a},\,\mathrm{b};\,\mathrm{c};\,z)=F(\mathrm{b},\,\mathrm{a};\,\mathrm{c};\,z).$$

The hypergeometric series (2.4) is absolutely convergent if $|z|<1$. The convergence also extends over the circle $|z|=1$, if $\mathrm{c}-\mathrm{a}-\mathrm{b}>0$, while the series converges at all points of the unit circle except the point $z=1$, when $\mathrm{c}-\mathrm{a}-\mathrm{b}=0$.

In the sequel, we will give explicit expressions of $F(\mathrm{a},\, \mathrm{b};\, \mathrm{c};\, z)$, for all the other possible values of z, namely the analytic continuation of the series (2.4) into the domain $\{z \in \mathbb{C} : |z| > 1,\ z \notin (1, \infty)\}$. To this end, we assume $|z| > 1,\ z \notin (1, \infty)$ and distinguish the following four cases.

Case I: If none of the numbers a, c − b, a − b is equal to a nonpositive integer $m = 0, -1, -2, \dots,$ then we have (see [1, 15.3.7])

$$\begin{aligned} F(\mathrm{a},\, \mathrm{b};\, \mathrm{c};\, z) &= \frac{\Gamma(\mathrm{c})\Gamma(\mathrm{b}-\mathrm{a})}{\Gamma(\mathrm{b})\Gamma(\mathrm{c}-\mathrm{a})}(-z)^{-\mathrm{a}} F(\mathrm{a},\, \mathrm{a}-\mathrm{c}+1;\, \mathrm{a}-\mathrm{b}+1;\, \frac{1}{z}) \\ &+ \frac{\Gamma(\mathrm{c})\Gamma(\mathrm{a}-\mathrm{b})}{\Gamma(\mathrm{a})\Gamma(\mathrm{c}-\mathrm{b})}(-z)^{-\mathrm{b}} F(\mathrm{b},\, \mathrm{b}-\mathrm{c}+1;\, \mathrm{b}-\mathrm{a}+1;\, \frac{1}{z}). \end{aligned} \tag{2.5}$$

Case II: If $\mathrm{a} = \mathrm{b} \neq -m$, for each $m = 0, -1, -2, \dots$ and $\mathrm{c} - \mathrm{a} \neq l$ for all $l = 1, 2, \dots$ we have (see [1, 15.3.13])

$$F(\mathrm{a},\, \mathrm{a};\, \mathrm{c};\, z) = \frac{\Gamma(\mathrm{c})(-z)^{-\mathrm{a}}}{\Gamma(\mathrm{a})\Gamma(\mathrm{c}-\mathrm{a})} \sum_{k=0}^{\infty} \frac{(\mathrm{a})_k (1-\mathrm{c}+\mathrm{a})_k}{(k!)^2} z^{-k} \left[\ln(-z) + 2\Psi(k+1) - \Psi(\mathrm{a}+k) - \Psi(\mathrm{c}-\mathrm{a}-k)\right] \tag{2.6}$$

Here Ψ stands for the logarithmic derivative of the *Gamma* function, that is $\Psi(z) = \Gamma'(z)/\Gamma(z)$.

Case III: Let us now consider the case where $\mathrm{b} - \mathrm{a} = m,\ m = 1, 2, \dots,$ and $\mathrm{a} \neq -k$, for $k = 0, 1, 2, \dots$ If $\mathrm{c} - \mathrm{a} \neq l$ for all $l = 1, 2, \dots$ we have (see [1, 15.3.14])

$$\begin{aligned} F(\mathrm{a},\, \mathrm{a}+m;\, \mathrm{c};\, z) &= \frac{\Gamma(\mathrm{c})(-z)^{-\mathrm{a}-m}}{\Gamma(\mathrm{a}+m)\Gamma(\mathrm{c}-\mathrm{a})} \sum_{k=0}^{\infty} \frac{(\mathrm{a})_{k+m}(1-\mathrm{c}+\mathrm{a})_{k+m}}{(k+m)!\, k!} z^{-k} \big[\ln(-z) + \Psi(1+m+k) + \Psi(1+k) \\ & -\Psi(\mathrm{a}+m+k) - \Psi(\mathrm{c}-\mathrm{a}-m-k)\big] + (-z)^{-\mathrm{a}} \frac{\Gamma(\mathrm{c})}{\Gamma(\mathrm{a}+m)} \sum_{k=0}^{m-1} \frac{\Gamma(m-k)(\mathrm{a})_k}{k!\Gamma(\mathrm{c}-\mathrm{a}-k)} z^{-k} \end{aligned} \tag{2.7}$$

On the other hand, if $\mathrm{c} - \mathrm{a} = l$, where $l = 1, 2, \dots$ such that $l > m$, we have (see (19) in [23, Sec. 2.1.4])

$$\begin{aligned} F(\mathrm{a},\, \mathrm{a}+m;\, \mathrm{a}+l;\, z) = \frac{\Gamma(\mathrm{a}+l)}{\Gamma(\mathrm{a}+m)}(-z)^{-\mathrm{a}} \Bigg[& (-1)^l (-z)^{-m} \sum_{k=l-m}^{\infty} \frac{(\mathrm{a})_{k+m}(k+m-l)!}{(k+m)!\, k!} z^{-k} \\ & + \sum_{k=0}^{m-1} \frac{\Gamma(2m-k-l)!(\mathrm{a})_k}{(l-k-1)!\, k!} z^{-k} + \frac{(-z)^{-m}}{(l-1)!} \sum_{k=0}^{l-m-1} \frac{(\mathrm{a})_{k+m}(1-l)_{k+m}}{(k+m)!k!} z^{-k} \big[\ln(-z) \\ & + \Psi(1+m+k) + \Psi(1+k) - \Psi(\mathrm{a}+m+k) - \Psi(l-m-k)\big] \Bigg]. \end{aligned} \tag{2.8}$$

Case IV: If at least one of the numbers a, c−b equals to a nonpositive integer, then $F(\mathrm{a}, \mathrm{b}; \mathrm{c}; z)$ becomes an elementary function of z. More precisely, if $\mathrm{a} = -m$ with $m = 0, 1, 2, \dots$ then the hypergeometric series in (2.4) reduces to the polynomial (see [1, 15.4.1])

$$F(-m,\, \mathrm{b};\, \mathrm{c};\, z) = \sum_{k=0}^{m} \frac{(-m)_k (\mathrm{b})_k}{(\mathrm{c})_k} \frac{z^k}{k!}. \tag{2.9}$$

On the other hand, if $\mathrm{c} - \mathrm{b} = -l$, with $l = 0, 1, 2, \dots,$ then $F(\mathrm{a},\, \mathrm{b};\, \mathrm{c};\, z)$ is written in the form (see [1, 15.3.3])

$$F(\mathrm{a}, \mathrm{b}; \mathrm{c};\, z) = (1-z)^{-\mathrm{a}-l} F(\mathrm{c}-\mathrm{a},\, -l;\, \mathrm{c};\, z), \tag{2.10}$$

where the hypergeometric function in the right hand side is a polynomial of degree l, according to (2.9).

We conclude this section with the following differentiation formula (see [1, 15.2.1])

$$\frac{d}{dz}F(\mathrm{a},\,\mathrm{b};\,\mathrm{c};\,z) = \frac{\mathrm{a}\,\mathrm{b}}{\mathrm{c}}\;F(\mathrm{a}+1,\,\mathrm{b}+1;\,\mathrm{c}+1;\,z), \tag{2.11}$$

which will be also useful for our analysis in the next chapters.

2.2 Extension problems related to the fractional Laplacians

An important feature of the fractional Laplacian $(-\Delta)^s$ in the whole space, is its nonlocal character, which can be realized as the boundary operator of a suitable extension in the half space $\mathbb{R}^n \times (0, +\infty)$. More precisely, Caffarelli and Silvestre [15] considered the following extended problem

$$\begin{cases} div(y^{1-2s}\nabla u(x,y)) = 0, & x \in \mathbb{R}^n,\ y > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

Then it was shown that

$$(-\Delta)^s f(x) \;=\; C_s \lim_{y\to 0^+} y^{1-2s} u_y(x,y),$$

where $C_s > 0$ is a constant depending only on s. The fact that this constant does not depend on the dimension n, is proved in [15, Section 3.2] and its precise value

$$C_s = -\frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)}, \tag{2.12}$$

appears in several references, for instance in [13], [47].

Regarding the operators A_s, $(-\Delta)^s$, which are defined on bounded domains (cf. Section 1.3), several authors, motivated by the work in [15], have considered equivalent definitions by means of an extra auxiliary variable. In the next two paragraphs we will present the associated extension problems for these two operators.

2.2.1 An extension problem associated with the Dirichlet fractional Laplacian

The Dirichlet Laplacian $(-\Delta)^s$ in Ω, as defined in the introduction, is plainly the fractional Laplacian $(-\Delta)^s$ in the whole space of the functions supported in Ω. Then following [15], the fractional Laplacian $(-\Delta)^s$ is connected with the extended problem

$$\begin{cases} div(y^{1-2s}\nabla u) = 0, & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.13}$$

In particular, the extension function u is related to the fractional Laplacian of the original function f through the pointwise formula

$$(-\Delta)^s f(x) = C_s \lim_{y\to 0^+} y^{1-2s} u_y(x,y), \quad \forall x \in \mathbb{R}^n, \tag{2.14}$$

where the constant C_s is given in (2.12).

2.2.2 An extension problem associated with the spectral fractional Laplacian

Associated to the bounded domain Ω, let us consider the cylinder $\mathcal{C}_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}^{n+1}_+$ and denote the lateral boundary of the cylinder by $\partial_L \mathcal{C}_\Omega = \partial\Omega \times [0,\infty)$. Now, for a function $f \in C_0^\infty(\Omega)$, we define the so-called $2s-$harmonic extension u to the cylinder $\mathcal{C}_\Omega$ as the unique solution of the problem

$$\begin{cases} div(y^{1-2s}\nabla u) = 0, & \text{in } \mathcal{C}_\Omega, \\ u = 0, & \text{on } \partial_L \mathcal{C}_\Omega, \\ u(x,0) = f(x), & x \in \Omega, \end{cases} \tag{2.15}$$

with $\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}y < \infty$. Then the extension function u is related to the spectral Laplacian of the original function f through the pointwise formula (see [14], [16], [25], [47])

$$\left(A_s f\right)(x) = C_s \lim_{y\to 0^+} y^{1-2s} u_y(x,y), \quad \forall x \in \Omega, \tag{2.16}$$

where the constant C_s is given by (2.12).

Chapter 3

Sharp interpolation between Hardy and trace Hardy inequalities

3.1 The weighted trace Hardy inequality

We begin by fixing some notation that will be used throughout. Recall that $x = (x', x_n) \in \mathbb{R}^n$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. We denote by B'_r the ball with radius r in $\mathbb{R}^{n-1}$, that is $B'_r = \{x' \in \mathbb{R}^{n-1} : |x'| < r\}$. Moreover $\int_{\partial B'_r} u \, \mathrm{d}\sigma(x')$ stands for the integral of the function u, with respect to the $(n-2)-$ dimensional Lebesgue measure over $\partial B'_r = \{x' \in \mathbb{R}^{n-1} : |x'| = r\}$. We also denote by ω_n the $(n-2)-$ dimensional volume of the unit sphere $\partial B'_1 = \{x' \in \mathbb{R}^{n-1} : |x'| = 1\}$, namely $\omega_n = 2\pi^{(n-1)/2}/\Gamma((n-1)/2)$.

Next we will give the proof of Proposition I. For the reader's convenience we restate it here.

Proposition 1. *Let $\alpha \in (-1, 1)$ and $n + \alpha - 2 > 0$. Then for all $u \in C_0^\infty(\mathbb{R}^n)$ there holds*

$$H(n, \alpha) \int_{\partial \mathbb{R}^n_+} \frac{u^2(x', 0)}{|x'|^{1-\alpha}} \, \mathrm{d}x' \leq \int_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x, \tag{3.1}$$

where

$$H(n, \alpha) = (1 - \alpha) \frac{\Gamma^2(\frac{n-\alpha}{4}) \Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{3-\alpha}{2}) \Gamma^2(\frac{n+\alpha-2}{4})}. \tag{3.2}$$

The constant $H(n, \alpha)$ is the best possible.

The definition of the best constant for inequality (3.1) is understood as follows. For any $u \in C_0^\infty(\mathbb{R}^n)$, $u \not\equiv 0$ we define the quotient

$$I[u] = \frac{\int_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x}{\int_{\partial \mathbb{R}^n_+} \frac{u^2}{|x'|^{1-\alpha}} \, \mathrm{d}x'}.$$

Then the best constant c such that the following inequality holds

$$c \int_{\partial \mathbb{R}^n_+} \frac{u^2(x', 0)}{|x'|^{1-\alpha}} \, \mathrm{d}x' \leq \int_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n)$$

is the value

$$c = \inf_{\substack{u \in C_0^\infty(\mathbb{R}^n) \\ u \not\equiv 0}} I[u]. \tag{3.3}$$

Before proceed, let us explain informally the idea behind the proof of Proposition 1. Assuming that a positive smooth minimizer u for the problem (3.3) does exist, then satisfies the Euler-Lagrange equations

$$\begin{cases} div(x_n^\alpha \nabla u) = 0, \text{ in } \mathbb{R}^n_+, \\ \lim_{x_n \to 0^+} \frac{x_n^\alpha}{u(x', x_n)} \frac{\partial u(x', x_n)}{\partial x_n} = -\frac{c}{|x'|^{1-\alpha}}. \end{cases} \tag{3.4}$$

Looking at the special structure of this problem we deduce the invariance under the transformation

$$u(x',\, x_n) \to \lambda^{-\gamma}\, u(\lambda|x'|,\, \lambda x_n), \quad \gamma,\, \lambda \in \mathbb{R},\ x' \in \mathbb{R}^{n-1},\ x_n \geq 0.$$

Setting $\lambda = |x'|^{-1}$ we are led naturally to search for solutions having the form

$$u(x',\, x_n) \;=\; |x'|^{-\gamma}\, G(\frac{x_n}{|x'|}), \tag{3.5}$$

for some smooth function $G : [0, \infty) \to \mathbb{R}$.

Substituting the functions given in (3.5) to the problem (3.4), we have a dimension reduction of the problem from n to 1 dimension: we have to find the solutions of the following boundary values problem

$$\begin{cases} t(1+t^2)\, G''(t) + [(2\gamma - n + 4)t^2 + \alpha]\, G'(t) + \gamma\,(\gamma - n + 3)\, t\, G(t) = 0,\ t > 0, & \text{(3.6a)} \\ G(0) = 1, & \text{(3.6b)} \\ \lim_{t\to\infty} t^\gamma\, G(t) \in \mathbb{R}. & \text{(3.6c)} \end{cases}$$

Notice that in $\{(x', x_n) : x' = 0,\, x_n > 0\}$, $u(x', x_n)$ is well defined due to the condition (3.6c). Note also that in the specific case $n = 4$, $\alpha = 0$, $\gamma = 1$, the problem (3.6) can be solved explicitly and we have $G(t) = 1 - \frac{2}{\pi}\arctan(t)$.

For the general case, using the change of variables $z = -t^2$ and defining the new unknown so that $\omega(z) = G(t)$, we have

$$\frac{dG}{dt} = -2t\frac{d\omega}{dz},\ \frac{d^2G}{dt^2} = -2\frac{d\omega}{dz} + 4t^2\frac{d^2\omega}{dz^2}.$$

Then equation (3.6a) becomes

$$z\,(1-z)\,\omega'' \;+\; [\frac{\alpha+1}{2} - \frac{2\gamma - n + 5}{2}z]\,\omega' \;-\; \frac{\gamma}{2}\,\frac{\gamma - n + 3}{2}\,\omega \;=\; 0, \quad -\infty < z < 0. \tag{3.7}$$

Equation (3.7) belongs to the class of hypergeometric equations and the general solution can be expressed in terms of hypergeometric functions (see (2.3)). After some calculations we obtain

$$c(n,\, \alpha,\, \gamma) := -\lim_{t\to 0^+} t^\alpha G'(t) = \frac{2\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\gamma}{2} - \frac{\alpha-1}{2})\Gamma(\frac{n-1}{2} - \frac{\gamma}{2})}{\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{\gamma}{2})\Gamma(\frac{n+\alpha-2}{2} - \frac{\gamma}{2})},$$

hence

$$\lim_{x_n\to 0^+} \frac{x_n^\alpha}{u(x',\, x_n)}\,\frac{\partial u(x',\, x_n)}{\partial x_n} = -c(n, \alpha, \gamma)\frac{1}{|x'|^{1-\alpha}}.$$

By a standard analysis we get that the constant $c = c(n, \alpha, \gamma)$ attains its maximum value $c = H(n, \alpha)$, for $\gamma = \frac{\alpha+n-2}{2}$.

However the function $\phi(x) = |x'|^{-\frac{\alpha+n-2}{2}} G(\frac{x_n}{|x'|})$ does not even belong in $D^{1,2}(\mathbb{R}^n_+)$, since it has not the right summability property. Nevertheless, using the transformation

$$u(x) = \upsilon(x)\,\phi(x)$$

we get inequality (3.1) with the proper constant:

$$\int_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x - H(n,\alpha) \int_{\partial \mathbb{R}^n_+} \frac{u^2}{|x'|^{1-\alpha}} \, \mathrm{d}x' = \int_{\mathbb{R}^n_+} x_n^\alpha \, \phi^2(x) \, |\nabla \upsilon(x)|^2 \, \mathrm{d}x \geq 0.$$

Then, the optimality of the constant that appears in (3.1) can be demonstrated by sequences obtained on truncating functions of the type

$$u(x) = |x|^{-\frac{\alpha+n-2}{2}} G\left(\frac{x_n}{|x'|}\right).$$

Let us now proceed with the proof Proposition 1. As already mentioned, the main ingredient in the proof is the consideration of the function

$$\phi(x',\, x_n) \;=\; |x'|^{-\frac{n+\alpha-2}{2}} \, G(\frac{x_n}{|x'|}), \;\; x' \in \mathbb{R}^{n-1}, \; x_n \geq 0, \; (x',\, x_n) \neq (0,0), \tag{3.8}$$

where the function $G : [0,\infty) \to \mathbb{R}$ is the solution of the following boundary values problem

$$\begin{cases} (t+t^3)\, G''(t) + [(\alpha+2)t^2 + \alpha]\, G'(t) + \dfrac{4-n+\alpha}{2}\dfrac{n+\alpha-2}{2}\, t\, G(t) = 0, \; t > 0, & \text{(3.9a)} \\ G(0) = 1 & \text{(3.9b)} \\ \lim\limits_{t\to\infty} t^{\frac{n+\alpha-2}{2}} G(t) \in \mathbb{R}. & \text{(3.9c)} \end{cases}$$

Note that in $\{(x', x_n) : x' = 0,\, x_n > 0\}$, $\phi(x', x_n)$ is well defined due to the condition (3.9c). For later use, notice also that multiplying by $t^{\alpha-1}$ equation (3.9a) can be written in divergence form

$$(t^\alpha (1+t^2)\, G'(t))' \,+\, \frac{4-n+\alpha}{2}\,\frac{n+\alpha-2}{2}\, t^\alpha\, G(t) \;=\; 0. \tag{3.10}$$

In the following Lemma we collect some properties of G that will be used later on. In order to state these properties, let us abbreviate, for any functions $f, g : \mathcal{D} \to \mathbb{R}$,

$$f \sim g \;\text{ in }\; \mathcal{D} \iff c_1\, g(z) \leq f(z) \leq c_2\, g(z),\; \forall z \in \mathcal{D}, \quad \text{for some constants } c_1, c_2 > 0 \;\text{ independent of }\; y.$$

Lemma 1. *Let $\alpha \in (-1,1)$ $n+\alpha-2 > 0$. Then the boundary value problem (3.9) has a positive decreasing solution G with the following properties.*
(i) $\lim\limits_{t\to 0^+} t^\alpha G'(t) = -H(n,\alpha)$, where $H(n,\alpha)$ is given in (3.2).

(ii) For $t > 0$ we have

$$\begin{aligned} G &\;\sim\; (1+t^2)^{-\frac{n-2+\alpha}{4}} \\ G' &\;\sim\; t^{-\alpha}(1+t^2)^{-\frac{n-\alpha}{4}}. \end{aligned}$$

(iii) There holds $tG' + \frac{n+\alpha-2}{2} G = O(t^{-\frac{n+\alpha+2}{2}})$, as $t \to \infty$.

Proof. Notice first that in the specific case $n = 4$, $\alpha = 0$ the problem (3.9) can be solved explicitly and we have $G(t) = 1 - \frac{2}{\pi}\arctan(t)$.

For the general case, using the change of variables $z = -t^2$ and defining the new unknown so that $\omega(z) = G(t)$, we have

$$\frac{dG}{dt} = -2t\frac{d\omega}{dz}, \;\; \frac{d^2G}{dt^2} = -2\frac{d\omega}{dz} + 4t^2\frac{d^2\omega}{dz^2}.$$

Then problem (3.9) becomes

$$\begin{cases} z\,(1-z)\,\omega'' \;+\; \left[\dfrac{\alpha+1}{2} \;-\; \dfrac{\alpha+3}{2}\,z\right]\,\omega' \;-\; \dfrac{4-n+\alpha}{4}\,\dfrac{n+\alpha-2}{4}\,\omega \;=\; 0, \; -\infty<z<0, & \text{(3.11a)} \\ \omega(0)=1, & \text{(3.11b)} \\ \lim\limits_{z\to-\infty}(-z)^{\frac{n+\alpha-2}{4}}\omega(z)\in\mathbb{R}. & \text{(3.11c)} \end{cases}$$

Equation (3.11a) is of the from (2.1) and according to (2.3), the general solution is given by

$$\omega(z)=C_1F(\frac{4-n+\alpha}{4},\,\frac{n+\alpha-2}{4};\,\frac{\alpha+1}{2};\,z)+C_2\,(-z)^{\frac{1-\alpha}{2}}\,F(\frac{6-n-\alpha}{4},\,\frac{n-\alpha}{4};\,\frac{3-\alpha}{2};\,z), \tag{3.12}$$

for any $z\in\mathbb{C}\setminus[1,\infty)$. Here, in order to simplify the subsequent presentation, we have incorporated the complex exponential $e^{i\pi(1-\alpha)/2}$ in the constant $\mathcal{C}_2$, appearing in (2.3).

Next we proceed with the evaluation of the constants C_1, C_2. Condition (3.11b) implies that $C_1=1$. The constant C_2 will be evaluated by the condition at ∞, that is (3.11c). To this aim, we distinguish between the cases $n\neq 3$ and $n=3$.

Case I: Let us consider first the case where $n\neq 3$. Then, substituting the hypergeometric functions appearing in (3.12), by their expression given in (2.5) and next multiplying by $(-z)^{(n+\alpha-2)/4}$, we arrive at

$$\begin{aligned} (-z)^{\frac{n+\alpha-2}{4}}\omega(z) \;&=\; (-z)^{\frac{n+\alpha-2}{4}}F(\frac{4-n+\alpha}{4},\,\frac{n+\alpha-2}{4};\,\frac{\alpha+1}{2};\,z) \\ &\quad +C_2\,(-z)^{\frac{n-\alpha}{4}}\,F(\frac{6-n-\alpha}{4},\,\frac{n-\alpha}{4};\,\frac{3-\alpha}{2};\,z) \\ &=\; (-z)^{\frac{n-3}{2}}\left[\frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{n-3}{2})}{\Gamma^2(\frac{n+\alpha-2}{4})}+C_2\,\frac{\Gamma(\frac{3-\alpha}{2})\Gamma(\frac{n-3}{2})}{\Gamma^2(\frac{n-\alpha}{4})}\right]F(\frac{4-n+\alpha}{4},\,\frac{6-n-\alpha}{4};\,\frac{5-n}{2};\,\frac{1}{z}) \\ &\quad +\left[\frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{3-n}{2})}{\Gamma^2(\frac{4-n+\alpha}{4})}+C_2\,\frac{\Gamma(\frac{3-\alpha}{2})\Gamma(\frac{3-n}{2})}{\Gamma^2(\frac{6-n-\alpha}{4})}\right]F(\frac{n+\alpha-2}{4},\,\frac{n-\alpha}{4};\,\frac{n-1}{2};\,\frac{1}{z}). \end{aligned} \tag{3.13}$$

For $n>3$, condition (3.11c) yields

$$C_2=-\frac{\Gamma^2(\frac{n-\alpha}{4})\,\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{3-\alpha}{2})\,\Gamma^2(\frac{n+\alpha-2}{4})}, \tag{3.14}$$

whence the part (iii) follows. For $n=2$, $\alpha>0$, the value of C_2 given in (3.14), leads again to the asymptotics (iii). With this choice of C_2 we have

$$\omega(z)=O\left((-z)^{\frac{2-n-\alpha}{4}}\right),\ \text{as } z\to-\infty. \tag{3.15}$$

Case II: If $n=3$, then formula (2.6) gives the analytic continuation of the hypergeometric functions $F(\frac{\alpha+1}{4},\,\frac{\alpha+1}{4};\,\frac{\alpha+1}{2};\,z)$, $F(\frac{3-\alpha}{4},\,\frac{3-\alpha}{4};\,\frac{3-\alpha}{2};\,z)$, appearing in (3.12). Let us abbreviate $a_1=(\alpha+1)/4$, $a_2=(3-\alpha)/4$, $c_1=(\alpha+1)/2$, $c_2=(3-\alpha)/2$. Substituting the hypergeometric functions appearing in (3.12), by their expression given in (2.6) and next multiplying by $(-z)^{(1+\alpha)/4}$, we arrive at

$$\begin{aligned} (-z)^{(1+\alpha)/4}\omega(z)=&\left[\frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)}+\frac{C_2\Gamma(c_2)}{\Gamma(a_2)\Gamma(c_2-a_2)}\right]\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k}{(k!)^2}z^{-k}\big[\ln(-z)+2\Psi(k+1)\big] \\ &-\frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)}\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k}{(k!)^2}z^{-k}\big[\Psi(a_1+k)+\Psi(c_1-a_1-k)\big] \\ &-\frac{C_2\Gamma(c_2)}{\Gamma(a_2)\Gamma(c_2-a_2)}\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k}{(k!)^2}z^{-k}\big[\Psi(a_2+k)+\Psi(c_2-a_2-k)\big]. \end{aligned} \tag{3.16}$$

Then (3.16) jointly with (3.11c), yield (3.14), with $n = 3$ there. For this value of C_2, (3.35) implies (3.15).

Having determined completely the function ω, we are in position to compute the limit $H(n,\alpha) := -\lim_{t\to 0^+} t^\alpha G'(t) = \lim_{z\to 0^-} 2(-z)^{\frac{\alpha+1}{2}}\omega'(z)$. Using the differentiation formula (2.11), we obtain

$$
\begin{aligned}
\omega'(z) \;=\;& \frac{(4-n+\alpha)(n+\alpha-2)}{8(\alpha+1)}\, F\Big(\frac{8-n+\alpha}{4}, \frac{n+\alpha+2}{4}; \frac{\alpha+3}{2}; z\Big) \\
& -\frac{C_2(1-\alpha)}{2}\,(-z)^{-\frac{\alpha+1}{2}}\, F\Big(\frac{6-n-\alpha}{4}, \frac{n-\alpha}{4}; \frac{3-\alpha}{2}; z\Big) \\
& +C_2\,(-z)^{\frac{1-\alpha}{2}}\, \frac{(6-n-\alpha)(n-\alpha)}{8(3-\alpha)}\, F\Big(\frac{10-n+\alpha}{4}, \frac{n-\alpha+4}{4}; \frac{5-\alpha}{2}; z\Big), \quad |z|<1.
\end{aligned}
$$

We then have

$$
H(n,\alpha) \;=\; \lim_{z\to 0^-} 2(-z)^{\frac{\alpha+1}{2}}\omega'(z) = (1-\alpha)\,\frac{\Gamma^2(\frac{n-\alpha}{4})\,\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{3-\alpha}{2})\,\Gamma^2(\frac{n+\alpha-2}{4})}.
$$

This completes the proof of part (i) of the Lemma.

Let us show now the positivity and monotonicity of G. We consider first the case where $4-n+\alpha<0$. The positivity of G follows from the fact that if there exist $t_0>0$ such that $G(t_0)=0$, then since $\lim_{t\to\infty} G(t) = 0$, there exists $t_m > t_0$ where G attains local non negative maximum or local nonpositive minimum, which contradicts the ode (3.9a). Therefore G is positive and the same argument shows that G is decreasing.

Let now $4-n+\alpha \geq 0$. The substitution $f(t) = (1+t^2)^{\frac{n-2+\alpha}{4}} G(t)$ transforms problem (3.9) to

$$
\begin{cases}
t(1+t^2)^2 f''(t) + [\alpha+(4-n)t^2](1+t^2)f'(t) - \dfrac{(n+\alpha-2)^2}{4}\, t\, f(t) = 0, \; t>0, & (3.17a)\\
f(0) = 1, & (3.17b)\\
\lim_{t\to\infty} f(t) \in \mathbb{R}_+. & (3.17c)
\end{cases}
$$

Note that the positivity of the above limit at ∞, follows directly from the explicit expression of $G(t) = \omega(z)$ (cf. (3.13), (3.16)). Now we can apply a minimum principle argument to this problem, to get the non negativity of f. Indeed, if there exists $t_0>0$ such that $f(t_0)<0$, then since $f(0)=1$, $\lim_{t\to\infty} f(t) > 0$, there exists $t_m > t_0$ where f attains local negative minimum, which contradicts the ode (3.17a). It follows that f is non negative, hence G is non negative. Then (3.10) together with the negativity of G' in a neighbourhood of the origin (cf. part (i)) yield the monotonicity and positivity of G.

The part (ii) of the lemma follows by the conditions (3.9b) and (3.15) together with the positivity of G. $\square$

By the asymptotics of G we obtain the following uniform asymptotics for ϕ on bounded domains

$$
\phi \sim |x|^{-\frac{n+\alpha-2}{2}} \text{ in } \mathbb{R}^n_+. \tag{3.18}
$$

Utilizing (3.8) we have that $\nabla\phi\cdot x + \frac{n-2+\alpha}{2}\phi = 0$. This immediately yields $|\nabla\phi| \geq c\,|x|^{-\frac{n+\alpha}{2}}$ in $\mathbb{R}^n_+$. Moreover straightforward calculations show that ϕ is a positive solution of the corresponding Euler Lagrange equations

$$
\begin{cases}
div(x_n^\alpha\,\nabla\phi) = 0, \text{ in } \mathbb{R}^n_+,\\
\lim_{x_n\to 0^+} \frac{x_n^\alpha}{\phi(x',x_n)}\,\frac{\partial\phi(x',x_n)}{\partial x_n} \;=\; -H(n,\alpha)\frac{1}{|x'|^{1-\alpha}}.
\end{cases} \tag{3.19}
$$

We are now ready to proceed with the

Proof of Proposition 1. By a standard approximation argument we can suppose $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$.

Following the approach in [25] (see also [26]), we expand the square and integrate by parts, to get

$$\int_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla u - \frac{\nabla \phi}{\phi} \, u|^2 \, \mathrm{d}x = \int_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x + \int_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla \phi|^2 \left(\frac{u}{\phi}\right)^2 \, \mathrm{d}x - \int_{\mathbb{R}^n_+} x_n^\alpha \, \nabla u^2 \cdot \frac{\nabla \phi}{\phi} \, \mathrm{d}x =$$

$$\int_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x + \int_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla \phi|^2 \left(\frac{u}{\phi}\right)^2 \, \mathrm{d}x + \int_{\mathbb{R}^n_+} u^2 \, div\left(x_n^\alpha \, \frac{\nabla \phi}{\phi}\right) \, \mathrm{d}x + \int_{\partial \mathbb{R}^n_+} \lim_{x_n \to 0^+} \frac{x_n^\alpha \, u^2}{\phi} \frac{\partial \phi(x', \, x_n)}{\partial x_n} \, \mathrm{d}x' =$$

$$\int_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x - H(n, \alpha) \int_{\partial \mathbb{R}^n_+} \frac{u^2}{|x'|^{1-\alpha}} \, \mathrm{d}x'.$$

In the last equation we used equations (3.19). Notice that on $supp\, u$, ϕ does not vanish, so the function $\frac{u}{\phi}$ is well defined. Actually $u/\phi \in C_0^\infty(\overline{\mathbb{R}^n_+} \setminus \{0\})$. Then the result follows immediately.

It remains to verify the optimality of the constant $H(n, \alpha)$. To this end, let us denote by $D^{1,2}(\mathbb{R}^n_+, \, x_n^\alpha \, \mathrm{d}x)$ the completion of $C_0^\infty(\overline{\mathbb{R}^n_+})$ with respect to the norm $\|u\|_{D^{1,2}(\mathbb{R}^n_+, x_n^\alpha \, \mathrm{d}x)} = (\int_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x)^{1/2}$. We then define for a function $u \in D^{1,2}(\mathbb{R}^n_+, \, x_n^\alpha \, \mathrm{d}x)$ the quotient

$$Q[u] := \frac{\int_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x}{\int_{\partial \mathbb{R}^n_+} \frac{u^2}{|x'|^{1-\alpha}} \, \mathrm{d}x'} = \frac{Q_1[u]}{Q_2[u]}.$$

We will show that there exist functions $u_\epsilon \in D^{1,2}(\mathbb{R}^n_+, \, x_n^\alpha \, \mathrm{d}x)$ such that $\lim\limits_{\epsilon \to 0^+} Q[u_\epsilon] = H(n, \alpha)$.

We fix $\delta > 0$ and let us denote by C_δ the cylinder $\mathrm{C}_\delta = \{(x', x_n) \in \mathbb{R}^n : |x'| < \delta, \, |x_n| < \delta\}$. Let also $\eta \in C_0^1(\mathrm{C}_{2\delta})$, such that $\eta \equiv 1$ in C_δ. We then define the function

$$u_\epsilon(x', x_n) = \begin{cases} \eta(x) \, \phi(x', x_n), & x_n \geq \epsilon, \\ \eta(x) \, \phi(x', \epsilon), & 0 \leq x_n \leq \epsilon. \end{cases}$$

We firstly estimate the denominator $Q_2[u_\epsilon]$:

$$\begin{aligned} Q_2[u_\epsilon] &= \int\limits_{B'_\delta} \frac{\eta^2(x', 0) \, \phi^2(x', \epsilon)}{|x'|^{1-\alpha}} \, \mathrm{d}x' + \int\limits_{B'_{2\delta} \setminus B'_\delta} \frac{\eta^2(x', 0) \, \phi^2(x', \epsilon)}{|x'|^{1-\alpha}} \, \mathrm{d}x' = \int\limits_0^\delta \int\limits_{\partial B'_r} \frac{\phi^2(x', \epsilon)}{r^{1-\alpha}} \, \mathrm{d}\sigma(x') \, \mathrm{d}r + O(1) \\ &= \omega_n \int\limits_0^\delta G^2\left(\frac{\epsilon}{r}\right) \frac{1}{r} \, \mathrm{d}r + O(1) = \omega_n \int\limits_{\epsilon/\delta}^\infty G^2(s) \frac{1}{s} \, \mathrm{d}s + O(1), \text{ as } \epsilon \to 0. \end{aligned} \tag{3.20}$$

As regards the numerator $Q_1[u_\epsilon]$, taking into account that $\eta \equiv 1$ in C_δ it follows that

$$\begin{aligned} Q_1[u_\epsilon] &= \int\limits_{\{0 \leq x_n \leq \epsilon\}} x_n^\alpha \, |\nabla(\eta(x) \phi(x', \epsilon))|^2 \, \mathrm{d}x + \int\limits_{\{\epsilon \leq x_n \leq \delta\}} x_n^\alpha \, |\nabla(\eta \phi)|^2 \, \mathrm{d}x \\ &= \int\limits_{\{\epsilon \leq x_n \leq \delta\}} x_n^\alpha \, |\nabla(\eta \phi)|^2 \, \mathrm{d}x + O(1) = \int\limits_{\mathrm{C}_\delta \cap \{\epsilon \leq x_n \leq \delta\}} x_n^\alpha \, |\nabla \phi|^2 \, \mathrm{d}x + O(1), \end{aligned} \tag{3.21}$$

as $\epsilon \to 0$. In view of (3.8), the integral term in the right hand side equals to

$$\int\limits_{C_\delta\cap\{\epsilon\le x_n\le\delta\}} x_n^\alpha\,|\nabla\phi|^2\,\mathrm{d}x = \int\limits_\epsilon^\delta\int\limits_0^\delta\int\limits_{\partial B_r'} x_n^\alpha\,|\nabla\phi|^2\,\mathrm{d}\sigma(x')\,\mathrm{d}r\,\mathrm{d}x_n = \int\limits_\epsilon^\delta\int\limits_0^\delta\int\limits_{\partial B_1'} x_n^\alpha\,r^{n-2}\,|\nabla\phi|^2\,\mathrm{d}\sigma(x')\,\mathrm{d}r\,\mathrm{d}x_n$$

$$\omega_n\int\limits_\epsilon^\delta\int\limits_0^\delta \frac{x_n^\alpha}{r^{\alpha+2}}\left(\frac{(\alpha+n-2)^2}{4}G^2\left(\frac{x_n}{r}\right)+\left(1+\frac{x_n^2}{r^2}\right)\,G'^2\left(\frac{x_n}{r}\right)+(\alpha+n-2)\,\frac{x_n}{r}\,G\left(\frac{x_n}{r}\right)G'\left(\frac{x_n}{r}\right)\right)\mathrm{d}r\,\mathrm{d}x_n$$

$$=\omega_n\int\limits_\epsilon^\delta\int\limits_{x_n/\delta}^\infty \frac{s^\alpha}{x_n}\left(\frac{(\alpha+n-2)^2}{4}G^2(s)+(1+s^2)\,G'^2(s)+(\alpha+n-2)\,s\,G(s)G'(s)\right)\mathrm{d}s\,\mathrm{d}x_n. \tag{3.22}$$

In the last equality we used the change of variable $r = x_n/s$. Making now partial integration we have

$$\int\limits_{x_n/\delta}^\infty s^{\alpha+1}\,G(s)G'(s)\,\mathrm{d}s = \frac{1}{2}\int\limits_{x_n/\delta}^\infty s^{\alpha+1}\,\left(G^2(s)\right)'\,\mathrm{d}s = -\frac{(\alpha+1)}{2}\int\limits_{x_n/\delta}^\infty s^\alpha\,G^2(s)\,\mathrm{d}s+\frac{1}{2}\left[s^{\alpha+1}\,G^2(s)\right]_{s=\frac{x_n}{\delta}}^\infty,$$

which by virtue of (3.9c), yields

$$\int\limits_\epsilon^\delta\int\limits_{x_n/\delta}^\infty s^{\alpha+1}\,G(s)G'(s)\,\mathrm{d}s\,\mathrm{d}x_n = -\frac{(\alpha+1)}{2}\int\limits_\epsilon^\delta\int\limits_{x_n/\delta}^\infty s^\alpha\,G^2(s)\,\mathrm{d}s\,\mathrm{d}x_n+O(1),\ \text{as } \epsilon\to 0.$$

Substitute this estimate to (3.22), hence (3.21) becomes

$$Q_1[u_\epsilon] = \omega_n\int\limits_\epsilon^\delta\frac{1}{x_n}\int\limits_{x_n/\delta}^\infty s^\alpha(1+s^2)\,G'^2(s)-\frac{\alpha+n-2}{2}\frac{\alpha-n+4}{2}\,s^\alpha\,G^2(s)\,\mathrm{d}s\,\mathrm{d}x_n+O(1),\ \text{as } \epsilon\to 0. \tag{3.23}$$

Next we make again integration by parts in the s variable and then we use equation (3.10). Then (3.23) becomes

$$\begin{aligned} Q_1[u_\epsilon] &= -\omega_n\int_\epsilon^\delta\frac{1}{x_n}\int\limits_{x_n/\delta}^\infty\left(s^\alpha(1+s^2)\,G'(s)\right)'G(s)+\frac{\alpha+n-2}{2}\frac{\alpha-n+4}{2}\,s^\alpha\,G^2(s)\,\mathrm{d}s\,\mathrm{d}x_n\\ &\quad -\omega_n\int_\epsilon^\delta\frac{1}{x_n}\left(\frac{x_n}{\delta}\right)^\alpha\left(1+\left(\frac{x_n}{\delta}\right)^2\right)G'\left(\frac{x_n}{\delta}\right)G\left(\frac{x_n}{\delta}\right)\,\mathrm{d}x_n+O(1)\\ &= -\omega_n\int_\epsilon^\delta\frac{1}{x_n}\left(\frac{x_n}{\delta}\right)^\alpha\left(1+\left(\frac{x_n}{\delta}\right)^2\right)G'\left(\frac{x_n}{\delta}\right)G\left(\frac{x_n}{\delta}\right)\,\mathrm{d}x_n+O(1),\quad\text{as }\ \epsilon\to 0.\end{aligned}$$

We make now the change of variable $s = \frac{x_n}{\delta}$, to concude

$$Q_1[u_\epsilon] = -\omega_n\int\limits_{\epsilon/\delta}^1 s^{\alpha-1}\left(1+s^2\right)\,G'(s)\,G(s)\,\mathrm{d}s+O(1),\quad\text{as }\ \epsilon\to 0. \tag{3.24}$$

Finally, gathering estimates (3.20) and (3.24) and taking into account (3.9b) and Lemma 1(i) we obtain

$$\lim_{\epsilon\to 0}Q[u_\epsilon] \;=\; \lim_{\epsilon\to 0}\frac{-\omega_n\int\limits_{\epsilon/\delta}^1 s^{\alpha-1}\left(1+s^2\right)\,G'(s)\,G(s)\,\mathrm{d}s+O(1)}{\omega_n\int\limits_{\epsilon/\delta}^\infty G^2(s)\frac{1}{s}\,\mathrm{d}s+O(1)} = \lim_{t\to 0}\frac{-t^\alpha\left(1+t^2\right)\,G'(t)}{G(t)} = H(n,\alpha).$$

$\square$

3.2 Sharp interpolation of weighted Hardy and trace Hardy inequality

In this Section we will give the proof of Theorem I, which we restate here.

Theorem 1. *Let $\alpha \in (-1,1)$, $2-\alpha \le b < n$. Then for all $u \in C_0^\infty(\mathbb{R}^n)$ the following inequality holds*

$$K(n,\alpha,b)\int_{\partial\mathbb{R}^n_+} \frac{u^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x' + \frac{(\alpha+b-2)^2}{4}\int_{\mathbb{R}^n_+} \frac{x_n^\alpha u^2}{|x|^2}\,\mathrm{d}x \le \int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x, \tag{3.25}$$

where

$$K(n,\alpha,\,b) = (1-\alpha)\frac{\Gamma(\frac{n-2\alpha-b+2}{4})\Gamma(\frac{n+b-2}{4})\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{3-\alpha}{2})\Gamma(\frac{n-b}{4})\Gamma(\frac{n+2\alpha+b-4}{4})}. \tag{3.26}$$

The constant $K(n,\alpha,\,b)$ is optimal.

To this aim we consider the function

$$\psi(x) \quad = \quad |x|^{\frac{2-\alpha-b}{2}}|x'|^{\frac{b-n}{2}}B(\frac{x_n}{|x'|}),\; x' \in \mathbb{R}^{n-1},\; x_n \ge 0,\; (x',x_n) \ne (0,0), \tag{3.27}$$

where $B : [0,\infty) \to \mathbb{R}$ is the solution of the following boundary values problem

$$\begin{cases} (t+t^3)\,B''(t) + \left[(4-b)t^2+\alpha\right]\,B'(t) + \dfrac{6-n-b}{2}\dfrac{n-b}{2}\,t\,B(t) = 0,\;\; t>0, & \text{(3.28a)}\\ B(0) = 1, & \text{(3.28b)}\\ \lim\limits_{t\to\infty} t^{\frac{n-b}{2}}B(t) \in \mathbb{R}. & \text{(3.28c)} \end{cases}$$

Note that the function $\psi(x',x_n)$ is well defined in the set $\{(x',x_n) : x'=0,\, x_n>0\}$, due to the condition (3.28c).

For later use, notice also that multiplying by $t^{\alpha-1}(1+t^2)^{\frac{2-\alpha-b}{2}}$ equation (3.28a) can be written in divergence form

$$(t^\alpha(1+t^2)^{\frac{4-\alpha-b}{2}}B'(t))' + \frac{6-n-b}{2}\frac{n-b}{2}\,t^\alpha\,(1+t^2)^{\frac{2-b-\alpha}{2}}B(t) = 0. \tag{3.29}$$

In the following Lemma we collect the basic properties of B that will be used later on.

Lemma 2. *Let $\alpha \in (-1,1)$, $2-\alpha \le b < n$. Then the boundary values problem (3.28) has a positive decreasing solution B with the following properties.*
(i) $-\lim\limits_{t\to 0^+} t^\alpha B'(t) = K(n,\alpha,b)$, where $K(n,\alpha,b)$ is given in (3.26).

(ii) For all $t>0$,

$$\begin{aligned} B &\sim (1+t^2)^{\frac{b-n}{4}},\\ B' &\sim -t^{-\alpha}(1+t^2)^{\frac{2\alpha+b-n-2}{4}}. \end{aligned}$$

(iii) There holds $tB' + \frac{n-b}{2}B = O(t^{\frac{b-n-4}{2}})$, as $t\to\infty$.

Proof. Throughout the proof, for the sake of convenience, we abbreviate

$$\begin{aligned} a_1 &= \frac{6-n-b}{4}, \quad b_1 = \frac{n-b}{4}, \quad c_1 = \frac{\alpha+1}{2}, \quad a_2 = a_1 - c_1 + 1 = \frac{8-n-2\alpha-b}{4}, \\ b_2 &= b_1 - c_1 + 1 = \frac{2+n-2\alpha-b}{4}, \quad c_2 = 2 - c_1 = \frac{3-\alpha}{2}. \end{aligned}$$

It is straightforward to verify that both of the sets of parameters $\{a_1, b_1, c_1\}$, $\{a_2, b_2, c_2\}$, satisfy the conditions (2.2). Therefore, we can apply the theory, that is presented in Section 2.1, wherever is needed.

Our first concern is to find an explicit expression of B. Notice that in the specific case $n = 4$, $\alpha = 0$, $b = 2$ the problem (3.28) can be solved explicitly and we have $B(t) = 1 - \frac{2}{\pi}\arctan(t)$.

Moreover, the case where $\alpha = 0$ has been studied in Proposition 1.

For the other cases, we study the ode (3.28a) in the context of complex variables, using the transformation $z = -t^2$, which maps the regular singular points $\pm i$, 0, ∞ to 1, 0, ∞, respectively. Setting $\omega(z) = B(t)$, problem (3.28) is transformed to the problem

$$\begin{cases} z\,(1-z)\,\dfrac{d^2\omega}{dz^2} + [c_1 - (a_1 + b_1 + 1)\,z]\,\dfrac{d\omega}{dz} - a_1\,b_1\,\omega(z) = 0, \quad -\infty < z < 0, & (3.30a) \\ \omega(0) = 1, & (3.30b) \\ \lim\limits_{z\to-\infty} (-z)^{b_1}\omega(z) \in \mathbb{R}. & (3.30c) \end{cases}$$

Equation (3.30a) belongs to the class of hypergeometric equations and according to (2.3), the general solution is given by

$$\omega(z) = C_1\,F(a_1, b_1; c_1; z) + C_2\,(-z)^{1-c_1}\,F(a_2, b_2; c_2; z), \quad z \in \mathbb{C}\setminus(1,+\infty), \quad C_1, C_2 \in \mathbb{C}. \tag{3.31}$$

Here, in order to simplify the subsequent presentation, we have incorporated the complex exponential $e^{i\pi(1-c_1)}$ in the constant $\mathcal{C}_2$, appearing in (2.3). Let us also remark, for the sake of completeness, that the point $z = 1$ is excluded from (3.31), if $b = 2-\alpha$, but this does not affect the subsequent analysis.

Next we proceed with the evaluation of the constants C_1, C_2. Applying condition (3.30b) to (3.31), noting that $F(a_1, b_1; c_2; 0) = F(a_2, b_2; c_2; 0) = 1$, we obtain $C_1 = 1$.

The constant C_2 will be evaluated by the condition at ∞, that is (3.30c). We then need an expression for $\omega(z)$, when $|z| > 1$. To this end, we will distinguish four cases for n, α, b, corresponding to the formulas (2.5) - (2.10), that give the explicit expression for the hypergeometric functions in (3.31). In all cases, we will show that

$$C_2 = -\frac{\Gamma(c_1)\Gamma(b_2)\Gamma(c_2-a_2)}{\Gamma(c_2)\Gamma(b_1)\Gamma(c_1-a_1)} = -\frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{n-2\alpha-b+2}{4})\Gamma(\frac{n+b-2}{4})}{\Gamma(\frac{3-\alpha}{2})\Gamma(\frac{n-b}{4})\Gamma(\frac{n+2\alpha+b-4}{4})}, \tag{3.32}$$

as well as the following asymptotics

$$\omega(z) = O\left((-z)^{\frac{b-n}{4}}\right), \quad \text{as } z \to -\infty. \tag{3.33}$$

In order to prove the claims (3.32), (3.33), we assume that $|z| > 1, z \notin (1,\infty)$ and distinguish the following cases.

Case I: Assume that none of the numbers a_1, a_2, $c_1 - b_1$, $c_2 - b_2$, $a_1 - b_1 = a_2 - b_2 = \frac{3-n}{2}$, is equal to a nonpositive integer. Then, substituting the expressions of $F(a_1, b_1; c_1; z)$, $F(a_2, b_2; c_2; z)$, which are given by (2.5), into (3.31) and next multiplying by $(-z)^{b_1}$, we arrive at

$$\begin{aligned} (-z)^{b_1}\omega(z) &= (-z)^{\frac{n-3}{2}}\left[\frac{\Gamma(c_1)\Gamma(b_1-a_1)}{\Gamma(b_1)\Gamma(c_1-a_1)} + C_2\frac{\Gamma(c_2)\Gamma(b_2-a_2)}{\Gamma(b_2)\Gamma(c_2-a_2)}\right] F(a_1, a_2; \frac{5-n}{2}; \frac{1}{z}) \\ &+ \left[\frac{\Gamma(c_1)\Gamma(a_1-b_1)}{\Gamma(a_1)\Gamma(c_1-b_1)} + C_2\,\frac{\Gamma(c_2)\Gamma(a_2-b_2)}{\Gamma(a_2)\Gamma(c_2-b_2)}\right] F(b_1, b_2; \frac{n-1}{2}; \frac{1}{z}). \end{aligned} \tag{3.34}$$

For $n>3$, we combine (3.34) with (3.30c) to deduce (3.32). Then, the part (iii) follows immediately, using the differentiation formula (2.11). Similarly for $n=2,\ \alpha>0$, the value of C_2 in (3.32) leads again to the asymptotics (iii). Thereafter, (3.33) results upon a substitution of this value of C_2 in (3.34).

Case II: Next we proceed with the case where $a_1-b_1=a_2-b_2=0$, that is $n=3$. In this case the numbers $a_1,\ a_2,\ c_1-b_1,\ c_2-b_2$ are positive non integers. Then, the explicit expression for the functions $F(a_1,a_1;c_1;z)$ and $F(a_2,a_2;c_2;z)$ appearing in (3.31), is given by (2.6). Substituting (2.6) into (3.31) and then multiplying by $(-z)^{b_1}$, taking into account the relations $a_2=1-c_1+a_1$ and $a_1=1-c_2+a_2$, we arrive at

$$
\begin{aligned}
(-z)^{b_1}\omega(z) = {} & \left[\frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)}+\frac{C_2\Gamma(c_2)}{\Gamma(a_2)\Gamma(c_2-a_2)}\right]\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k}{(k!)^2}z^{-k}\big[\ln(-z)+2\Psi(k+1)\big] \\
& -\frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)}\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k}{(k!)^2}z^{-k}\big[\Psi(a_1+k)+\Psi(c_1-a_1-k)\big] \\
& -\frac{C_2\Gamma(c_2)}{\Gamma(a_2)\Gamma(c_2-a_2)}\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k}{(k!)^2}z^{-k}\big[\Psi(a_2+k)+\Psi(c_2-a_2-k)\big].
\end{aligned}
\tag{3.35}
$$

Then (3.35) jointly with (3.30c), yield (3.32), with $n=3$ there. For this value of C_2, (3.35) implies (3.33).

Case III: Now consider the case that none of the numbers $a_1,\ a_2,\ c_1-b_1,\ c_2-b_2$ is equal to a nonpositive integer and $b_1-a_1=m$ i.e. $n=2m+3$, where $m=1,2,...$ Since $a_1-b_1=a_2-b_2$, we also have $b_2-a_2=m$.

We first assume that the numbers $c_1-a_1,\ c_2-a_2$ are not equal to a positive integer. Then, the explicit expression for the functions $F(a_1,\,a_1+m;\,c_1;\,z)$ and $F(a_2,\,a_2+m;\,c_2;\,z)$ appearing in (3.31), is given in (2.7). Therefore, we substitute (2.7) into (3.31) and then multiply by $(-z)^{b_1}$, to arrive at

$$
\begin{aligned}
(-z)^{b_1}\omega(z) = {} & \left[\frac{\Gamma(c_1)}{\Gamma(a_1+m)\Gamma(c_1-a_1)}+\frac{C_2\Gamma(c_2)}{\Gamma(a_2+m)\Gamma(c_2-a_2)}\right]\Bigg[\ln(-z)\sum_{k=0}^{\infty}\frac{(a_1)_{k+m}(a_2)_{k+m}}{k!(k+m)!}z^{-k} \\
& +(-z)^m\sum_{k=0}^{m-1}\frac{\Gamma(m-k)(a_1)_k(a_2)_k}{k!}(-z)^{-k}+\sum_{k=0}^{\infty}\frac{(a_1)_{k+m}(a_2)_{k+m}}{k!(k+m)!}\left(\Psi(1+m+k)+\Psi(1+k)\right)z^{-k}\Bigg] \\
& -\frac{\Gamma(c_1)}{\Gamma(a_1+m)\Gamma(c_1-a_1)}\sum_{k=0}^{\infty}\frac{(a_1)_{k+m}(a_2)_{k+m}}{k!(k+m)!}z^{-k}\Big[\Psi(a_1+m+k)+\Psi(c_1-a_1-m-k)\Big] \\
& -\frac{C_2\Gamma(c_2)}{\Gamma(a_2+m)\Gamma(c_2-a_2)}\sum_{k=0}^{\infty}\frac{(a_1)_{k+m}(a_2)_{k+m}}{k!(k+m)!}z^{-k}\Big[\Psi(a_2+m+k)+\Psi(c_2-a_2-m-k)\Big].
\end{aligned}
$$

Due to (3.30c), the coefficient, in the brackets, of the first summand in the right hand side equals to zero. A direct calculation leads to (3.32) again. Then, (3.33) results upon a substitution of this value of C_2, in the above formula.

If at least one of the numbers c_1-a_1, c_2-a_2 is equal to an integer $l=1,2,\ldots$, then we can use the formula (2.8) to get the expression of $F(a_1,\,a_1+m;\,a_1+l;\,z)$ or $F(a_2,\,a_2+m;\,a_2+l;\,z)$, respectively. Arguing as above, we derive again (3.32) and (3.33).

Case IV: We conclude with the case where at least one of the numbers $a_1,\ a_2,\ c_1-b_1,\ c_2-b_2$ is equal to a nonpositive integer.

Note first that if one of the numbers a_1, a_2 is a nonpositive integer $-m$ with $m=0,1,2,\ldots$, then the first or second respectively hypergeometric function in (3.31) reduces to a polynomial of degree m (see

(2.9)). On the other hand, if one of the numbers $c_1 - b_1 = \frac{2-n+2\alpha+b}{4}$, $c_2 - b_2 = \frac{4-n+b}{4}$ is a nonpositive integer $-l$ with $l = 0, 1, 2, \ldots$, then the first or second respectively hypergeometric function in (3.31) is of the form $(1-z)^{\beta} p_l(z)$, where p_l is a polynomial of degree l and $\beta = \frac{\alpha+b-2}{2}$ (see (2.10)). Again these two hypergeometric functions cannot be both of this form.

We will consider only the case where $a_1 = -m$ for some $m = 0, 1, 2, \ldots$, while none of the two numbers $c_1 - b_1$, $c_2 - b_2$ is a nonpositive integer. The argumentation for the other cases is quite similar.

The expressions of the first and second hypergeometric function in (3.31) are given by formulas (2.9), (2.5), respectively . Substituting (2.5), (2.9) into (3.31) and then multiplying by $(-z)^{b_1}$, we arrive at

$$\begin{aligned} (-z)^{b_1}\omega(z) &= (-z)^{\frac{n-3}{2}}\left[\frac{(b_1)_m}{(c_1)_m} + C_2\,\frac{\Gamma(c_2)\Gamma(b_2-a_2)}{\Gamma(b_2)\Gamma(c_2-a_2)}\right] F(a_2,\, -m;\, \frac{5-n}{2};\, \frac{1}{z}) \\ &+ C_2\,\frac{\Gamma(c_2)\Gamma(a_2-b_2)}{\Gamma(a_2)\Gamma(c_2-b_2)}\, F(b_2,\, b_1;\, \frac{n-1}{2};\, \frac{1}{z}). \end{aligned} \tag{3.36}$$

Here we also used (2.9), to express the function $F(a_2, a_2-c_2+1; a_2-b_2+1; \frac{1}{z}) = F(a_2,\, -m;\, a_2-b_2+1;\, \frac{1}{z})$. For $n > 3$, condition (3.30c) yields (3.32), thereafter (3.33) results upon a substitution of this value of C_2, in (3.36). Then, the part (iii) follows immediately, using the differentiation formula (2.11). Similarly for $n = 2$, $\alpha > 0$, the value of C_2 in (3.32) leads again to the asymptotics (iii) and (3.33).

The proof of (3.32), (3.33), is now completed. At this point we have completely determined the solution ω for all possible values of n, α, b, subject of course to the restrictions of the Lemma.

We are now in position to compute the limit $K(n,\alpha,b) := -\lim_{t\to 0^+} t^{\alpha} B'(t) = 2 \lim_{z\to 0^-} (-z)^{\frac{\alpha+1}{2}}\omega'(z)$. To this aim, we differentiate (3.31), using the differentiation formula (2.11), to obtain

$$\begin{aligned} \omega'(z) &= \frac{a_1\, b_1}{c_1}\, F(a_1+1,\, b_1+1;\, c_1+1;\, z) - C_2\frac{1-\alpha}{2}\,(-z)^{-\frac{\alpha+1}{2}}\, F(a_2,\, b_2;\, c_2;\, z) \\ &- C_2\,\frac{a_2\, b_2}{c_2}\,(-z)^{\frac{1-\alpha}{2}}\, F(a_2+1,\, b_2+1;\, c_2+1;\, z). \end{aligned}$$

Taking into account the explicit value of C_2 we obtain

$$K(n,\alpha,b) = 2\lim_{z\to 0^-}(-z)^{\frac{\alpha+1}{2}}\omega'(z) = (1-\alpha)\frac{\Gamma(\frac{n-2\alpha-b+2}{4})\,\Gamma(\frac{n+b-2}{4})\,\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{3-\alpha}{2})\,\Gamma(\frac{n-b}{4})\,\Gamma(\frac{n+2\alpha+b-4}{4})}.$$

This completes the proof of part (i).

Let us show now the positivity and monotonicity of B. We first assume that $6-n-b < 0$. Then the positivity of B follows from the fact that if there exist $t_0 > 0$ such that $B(t_0) = 0$, then since $\lim_{t\to\infty} B(t) = 0$, there exists $t_m > t_0$ where B attains local non negative maximum or local nonpositive minimum, which contradicts the ode (3.28a). Therefore B is positive and the same argument shows that B is decreasing.

If $6-n-b \geq 0$, then we make the substitution $f(t) = (1+t^2)^{\frac{n-b}{4}} B(t)$ which transforms problem (3.28) to

$$\begin{cases} t\,(1+t^2)^2\, f''(t) + [\alpha + (4-n)\,t^2]\,(1+t^2)\, f'(t) + \dfrac{(n-b)(4-n-2\alpha-b)}{4}\, t\, f(t) = 0, \quad t > 0, & \text{(3.37a)} \\ f(0) = 1, & \text{(3.37b)} \\ \lim_{t\to\infty} f(t) \in \mathbb{R}_+. & \text{(3.37c)} \end{cases}$$

Note that the positivity of the above limit at ∞, follows directly from the explicit expression of $B(t) = \omega(z)$ (cf. (3.34), (3.35)). Now we can apply a minimum principle argument to this problem, to get the non negativity of f. Indeed, if there exists $t_0 > 0$ such that $f(t_0) < 0$, then since $f(0) = 1$, $\lim_{t\to\infty} f(t) \geq 0$, there exists $t_m > t_0$ where f attains local negative minimum, which contradicts the ode (3.37a). It follows that f is non negative, hence B is non negative. Then (3.29) together with the negativity of B' in a neighbourhood of the origin (cf. part (i)) yield the monotonicity and positivity of B.

The asymptotics for B follow by conditions (3.28b) and (3.28c) together with the positivity of B. As regards the asymptotics for B', we differentiate the expression (3.31) using the differentiation formula (2.11).

Finally, part (iii) follows immediately using the explicit expression of $B(t) = \omega(-t^2)$, $t > 1$. □

By mean of the asymptotics of B we obtain the following uniform asymptotics for ψ.

Lemma 3. *Let the function ψ defined in (3.27). Then there holds*

$$\psi \sim |x|^{-\frac{n+\alpha-2}{2}}, \;\; in \;\; \mathbb{R}^n_+. \tag{3.38}$$

Moreover, for $\alpha \in (-1, 0]$, there holds

$$|\nabla\psi| \sim |x|^{-\frac{n+\alpha}{2}}, \;\; in \;\; \mathbb{R}^n_+.$$

If $\alpha \in (0, 1)$, then there holds

$$|\nabla\psi| \sim |x|^{-\frac{n-\alpha}{2}} x_n^{-\alpha}, \;\; in \;\; \mathbb{R}^n_+.$$

Proof. The asymptotics for ψ follows immediately by the asymptotics of B.

As regards the asymptotic behaviour of $|\nabla\psi|$, let us first note that utilizing (3.27) we obtain $\nabla\psi \cdot x + \frac{n-2+\alpha}{2}\psi = 0$. This immediately yields $|\nabla\psi| \geq c|x|^{-\frac{n+\alpha}{2}}$ in $\mathbb{R}^n_+$. In particular $|\nabla\psi|$ is strictly positive.

Retaining the abbreviations $\gamma = \frac{2-\alpha-b}{2}$, $k = \frac{n-b}{2}$ we have

$$\begin{aligned}
|\nabla\psi|^2 &= \gamma\,(\gamma - 2k)\,|x|^{2\gamma-2}|x'|^{-2k}B^2(t) + |x|^{2\gamma}|x'|^{-2k-2}B'^2(t) + |x|^{2\gamma}|x'|^{-2k-2}\left[kB(t) + tB'(t)\right]^2 \\
&= T_1 + T_2 + T_3, \;\; \text{where} \;\; t = x_n/|x'|.
\end{aligned}$$

The asymptotic of B yields immediately $T_1 \sim |x|^{-(n+\alpha)}$. Similarly, the asymptotic of B' yields immediately $T_2 \sim |x|^{\alpha-n}x_n^{-2\alpha}$. Regarding the term T_3, instead of using separately the asymptotic of B and B' we have to use the part (iii) of Lemma 2. Then we have $T_3 \sim |x|^{-(n+\alpha)}$. The result follows combining the estimates for the terms T_1, T_2, T_3. □

An immediate calculation shows that ψ satisfies the Euler Lagrange equations

$$\begin{cases}
div(x_n^\alpha \nabla\psi) + \frac{(\alpha+b-2)^2}{4}\frac{x_n^\alpha\,\psi}{|x|^2} = 0, & \text{in } \mathbb{R}^n_+, \\
\lim\limits_{x_n\to 0^+} x_n^\alpha \frac{\partial\psi(x', x_n)}{\partial x_n} = -K(n,\alpha,b)\frac{\psi}{|x'|^{1-\alpha}}, & \text{on } \partial\mathbb{R}^n_+ \setminus \{0\}.
\end{cases} \tag{3.39}$$

We are now ready to proceed with the

Proof of Theorem 1. By approximation it suffices to prove (3.25) for all $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. Then following the approach in [25], we expand the square and integrate by parts, to get

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} x_n^\alpha \left|\nabla u - \frac{\nabla\psi}{\psi}\,u\right|^2 \mathrm{d}x = \int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x + \int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla\psi|^2\left(\frac{u}{\psi}\right)^2 \mathrm{d}x - \int_{\mathbb{R}^n_+} x_n^\alpha\,\nabla u^2 \cdot \frac{\nabla\psi}{\psi}\,\mathrm{d}x = \\
&\int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x + \int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla\psi|^2\left(\frac{u}{\psi}\right)^2 \mathrm{d}x + \int_{\mathbb{R}^n_+} u^2\,div\left(x_n^\alpha\,\frac{\nabla\psi}{\psi}\right)\mathrm{d}x + \int_{\partial\mathbb{R}^n_+} \lim_{x_n\to 0}\frac{x_n^\alpha\,u^2}{\psi}\frac{\partial\psi(x', x_n)}{\partial x_n}\,\mathrm{d}x' \\
&= \int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x - K(n,\alpha,b)\int_{\partial\mathbb{R}^n_+} \frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' - \frac{(\alpha+b-2)^2}{4}\int_{\mathbb{R}^n_+} \frac{x_n^\alpha u^2}{|x|^2}\,\mathrm{d}x.
\end{aligned} \tag{3.40}$$

In the last equation we used equations (3.39). Notice that on $supp\, u$, ψ does not vanish, so the function u/ψ is well defined. Actually $u/\psi \in C_0^\infty(\overline{\mathbb{R}^n_+} \setminus \{0\})$.

It remains to verify the optimality of the constant $K(n,\alpha,b)$. To this end, we define for a function $u \in D^{1,2}(\mathbb{R}^n_+,\, x_n^\alpha\, \mathrm{d}x)$ the quotient

$$Q[u] := \frac{\int_{\mathbb{R}^n_+} x_n^\alpha\, |\nabla u|^2\, \mathrm{d}x - \frac{(\alpha+b-2)^2}{4} \int_{\mathbb{R}^n_+} \frac{x_n^\alpha u^2}{|x|^2}\, \mathrm{d}x}{\int_{\partial\mathbb{R}^n_+} \frac{u^2}{|x'|^{1-\alpha}}\, \mathrm{d}x'} = \frac{Q_1[u]}{Q_2[u]}.$$

We will show that there exist functions $u_\epsilon \in D^{1,2}(\mathbb{R}^n_+,\, x_n^\alpha\, \mathrm{d}x)$ such that $\lim\limits_{\epsilon\to 0^+} Q[u_\epsilon] = K(n,\alpha,b)$.

Let $\delta > 0$ and $\eta \in C_0^1(\mathrm{C}_{2\delta})$, such that $\eta \equiv 1$ in C_δ. We define

$$u_\epsilon(x', x_n) = \begin{cases} \eta(x)\, \psi(x', x_n), & x_n \geq \epsilon, \\ \eta(x)\, \psi(x', \epsilon), & 0 \leq x_n \leq \epsilon. \end{cases}$$

In the sequel we will make some computations to derive an estimate of $Q[u_\epsilon]$. To this aim we abbreviate $k = \frac{n-b}{2}$, $\gamma = \frac{2-\alpha-b}{2}$. We start with the denominator $Q_2[u_\epsilon]$.

$$\begin{aligned} Q_2[u_\epsilon] &= \int\limits_{B'_\delta} \frac{\eta^2(x',0)\, \psi^2(x',\epsilon)}{|x'|^{1-\alpha}}\, \mathrm{d}x' + \int\limits_{B'_{2\delta}\setminus B'_\delta} \frac{\eta^2(x',0)\, \psi^2(x',\epsilon)}{|x'|^{1-\alpha}}\, \mathrm{d}x' = \int\limits_0^\delta \int\limits_{\partial B'_r} \frac{\psi^2(x',\epsilon)}{r^{1-\alpha}}\, \mathrm{d}\sigma(x')\, \mathrm{d}r + O(1) \\ &= \omega_n \int\limits_0^\delta \left(1 + \frac{\epsilon^2}{r^2}\right)^\gamma B^2\left(\frac{\epsilon}{r}\right) \frac{1}{r}\, \mathrm{d}r + O(1) = \omega_n \int\limits_{\epsilon/\delta}^\infty (1+s^2)^\gamma B^2(s) \frac{1}{s}\, \mathrm{d}s + O(1), \end{aligned} \tag{3.41}$$

as $\epsilon \to 0$. As regards the numerator $Q_1[u_\epsilon]$, taking into account the specific structure of u_ϵ we obtain

$$\begin{aligned} Q_1[u_\epsilon] &= \int\limits_{\{0\leq x_n\leq\epsilon,\, |x'|<2\delta\}} x_n^\alpha\, |\nabla u_\epsilon|^2 - \gamma^2 \frac{x_n^\alpha u_\epsilon^2}{|x|^2}\, \mathrm{d}x + \int\limits_{\{\epsilon\leq x_n\leq\delta,\, |x'|<2\delta\}} x_n^\alpha\, |\nabla u_\epsilon|^2 - \gamma^2 \frac{x_n^\alpha u_\epsilon^2}{|x|^2}\, \mathrm{d}x \\ &= \int\limits_{\{\epsilon\leq x_n\leq\delta,\, |x'|<\delta\}} x_n^\alpha\, |\nabla \psi|^2 - \gamma^2 \frac{x_n^\alpha \psi^2}{|x|^2}\, \mathrm{d}x + O(1) \quad \text{as } \epsilon \to 0. \end{aligned} \tag{3.42}$$

Here we used Lemma 3 together with the estimate

$$\begin{aligned} &\int_0^\epsilon \int_{|x'|\leq\delta} \frac{x_n^\alpha}{(|x'|^2+\epsilon^2)^{\frac{n+\alpha}{2}}}\, \mathrm{d}x'\, \mathrm{d}x_n = \left(\int_0^\epsilon x_n^\alpha\, \mathrm{d}x_n\right) \left(\int_0^\delta \int_{\partial B'_r} \frac{1}{(|x'|^2+\epsilon^2)^{\frac{n+\alpha}{2}}}\, \mathrm{d}\sigma(x')\, \mathrm{d}r\right) \leq \\ &\frac{\epsilon^{\alpha+1}}{\alpha+1} 2^{\frac{n+\alpha}{2}} \omega_n \int_0^\delta \frac{1}{(r+\epsilon)^{2+\alpha}}\, \mathrm{d}r = \frac{\epsilon^{\alpha+1}}{(\alpha+1)^2} \left(\frac{1}{\epsilon^{\alpha+1}} - \frac{1}{(\delta+\epsilon)^{\alpha+1}}\right) = O(1), \text{ as } \epsilon \to 0. \end{aligned}$$

In view of (3.27), the integral term in the right hand side of (3.42) equals to

$$\int\limits_{\{\epsilon\le x_n\le\delta,\ |x'|<\delta\}} x_n^\alpha\,|\nabla\psi|^2-\gamma^2\frac{x_n^\alpha\psi^2}{|x|^2}\,\mathrm{d}x=\int\limits_\epsilon^\delta\int\limits_0^\delta\int\limits_{\partial B_r'} x_n^\alpha\left(|\nabla\psi|^2-\gamma^2\frac{\psi^2}{|x|^2}\right)\mathrm{d}\sigma(x')\,\mathrm{d}r\,\mathrm{d}x_n=\omega_n\int\limits_\epsilon^\delta\int\limits_0^\delta\frac{x_n^\alpha}{r^{\alpha+2}}\times$$
$$\times\left(1+\frac{x_n^2}{r^2}\right)^\gamma\left(\left(k-2\gamma(1+\frac{x_n}{r})^{-1}\right)kB^2(\frac{x_n}{r})+\left(1+\frac{x_n^2}{r^2}\right)B'^2\left(\frac{x_n}{r}\right)+2k\,\frac{x_n}{r}\,B\left(\frac{x_n}{r}\right)B'\left(\frac{x_n}{r}\right)\right)\mathrm{d}r\,\mathrm{d}x_n$$
$$=\omega_n\int\limits_\epsilon^\delta\frac{1}{x_n}\int\limits_{x_n/\delta}^\infty s^\alpha(1+s^2)^\gamma\left(k^2B^2(s)-2\gamma k(1+s^2)^{-1}B^2(s)+(1+s^2)\,B'^2(s)+2k\,s\,B(s)B'(s)\right)\mathrm{d}s\,\mathrm{d}x_n. \tag{3.43}$$

In the last equality we used the change of variable $r=x_n/s$. Note that the inner integral is finite since the integrand is of order $O(s^{-n})$, as $s\to\infty$. This can be seen grouping the terms and taking into account part (iii) of Lemma 2.

Let us first consider the case where $n\neq 3$. To compute the inner integral containing the factors BB', we make partial integration and take into account (3.33) to obtain

$$\begin{aligned}\int\limits_{x_n/\delta}^\infty s^{\alpha+1}\left(1+s^2\right)^\gamma B(s)B'(s)\,\mathrm{d}s&=\frac{1}{2}\int\limits_{x_n/\delta}^\infty s^{\alpha+1}\left(1+s^2\right)^\gamma\left(B^2(s)\right)'\,\mathrm{d}s\\ &=-\frac{(\alpha+1)}{2}\int\limits_{x_n/\delta}^\infty s^\alpha\left(1+s^2\right)^\gamma B^2(s)\,\mathrm{d}s-\gamma\int\limits_{x_n/\delta}^\infty s^{\alpha+2}\left(1+s^2\right)^{\gamma-1}B^2(s)\,\mathrm{d}s\\ &-\frac{1}{2}\left[s^{\alpha+1}\left(1+s^2\right)^\gamma B^2(s)\right]_{s=\frac{x_n}{\delta}},\end{aligned} \tag{3.44}$$

As regards the inner integral in (3.43) containing the factor B'^2, an integration by parts together with (3.29), (3.33) yields

$$\begin{aligned}\int\limits_{x_n/\delta}^\infty s^\alpha(1+s^2)^{\gamma+1}B'^2(s)\,\mathrm{d}s&=-\int\limits_{x_n/\delta}^\infty\left(s^\alpha(1+s^2)^{\gamma+1}B'(s)\right)'B(s)\,\mathrm{d}s+\left[s^\alpha\left(1+s^2\right)^{\gamma+1}B'(s)B(s)\right]_{s=\frac{x_n}{\delta}}^\infty\\ &=\frac{6-n-b}{2}\frac{n-b}{2}\int\limits_{x_n/\delta}^\infty s^\alpha\,(1+s^2)^\gamma B^2(s)\,\mathrm{d}s-\left[s^\alpha\left(1+s^2\right)^{\gamma+1}B'(s)B(s)\right]_{s=\frac{x_n}{\delta}}.\end{aligned} \tag{3.45}$$

Substitute now the estimates (3.44), (3.45) to (3.43) to get

$$Q_1[u_\epsilon]=-\omega_n\int\limits_\epsilon^\delta\frac{1}{x_n}[s^\alpha\left(1+s^2\right)^\gamma B(s)\left[B'(s)+s(sB'(s)+kB(s))\right]_{s=\frac{x_n}{\delta}}\,\mathrm{d}x_n+O(1),\ \text{as }\epsilon\to 0$$

and then we make the change of variable $t=\frac{x_n}{\delta}$, to conclude

$$Q_1[u_\epsilon]=-\omega_n\int\limits_{\epsilon/\delta}^1 t^{\alpha-1}\left(1+t^2\right)^\gamma B(t)\left[B'(t)+t(tB'(t)+kB(t))\right]\mathrm{d}t+O(1),\ \text{as }\epsilon\to 0. \tag{3.46}$$

In the case where $n = 3$, the integrands in the left hand side of (3.44), (3.45) are not summable in the neighbourhood of ∞. To overcome this problem we may use the same procedure to derive estimates (3.44) - (3.45) with $\int_{x_n/\delta}^{R} \mathrm{d}s$ instead of $\int_{x_n/\delta}^{\infty} \mathrm{d}s$ and then we let $R \to \infty$, obtaining again (3.46). Notice also that taking the limit $R \to \infty$, we have to use the part (iii) of Lemma 2 to show that $\lim\limits_{R\to\infty} \left(R^{\alpha}\left(1+R^2\right)^{\gamma} B(R)\left[B'(R)+R(RB'(R)+kB(R))\right]\right) = 0$.

Finally, gathering estimates (3.41) and (3.46), then using L'Hôpital's rule and taking into account (3.28b) and Lemma 2(i) we obtain

$$
\begin{aligned}
\lim_{\epsilon\to 0} Q[u_\epsilon] &= \lim_{\epsilon\to 0} \frac{-\omega_n \int\limits_{\epsilon/\delta}^{1} s^{\alpha-1}\left(1+s^2\right)^{\gamma} B(s)\left[B'(s)+s(sB'(s)+kB(s))\right]\mathrm{d}s + O(1)}{\omega_n \int\limits_{\epsilon/\delta}^{\infty} (1+s^2)^{\gamma} B^2(s)\frac{1}{s}\,\mathrm{d}s + O(1)} \\
&= \lim_{t\to 0} \frac{-t^{\alpha}\left(1+t^2\right)B'(t)}{B(t)} - kt^{\alpha+1} = K(n,\alpha,b).
\end{aligned}
$$

$\square$

3.3 On the non improvement in $\mathbb{R}^n_+$

Here we will show that (3.25) cannot be improved in the sense that there are no constant $C > 0$, nontrivial potential $V \geq 0$ and exponent $p > 0$ such that the following inequality holds

$$C\left(\int_{\mathbb{R}^n_+} V(x)\,|u(x)|^p \,\mathrm{d}x\right)^{2/p} \leq \int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x - K(n,\alpha,b)\int_{\partial\mathbb{R}^n_+}\frac{u^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x' - \frac{(\alpha+b-2)^2}{4}\int_{\mathbb{R}^n_+}\frac{x_n^\alpha u^2}{|x|^2}\,\mathrm{d}x,$$

for all $u \in D^{1,2}(\mathbb{R}^n_+, x_n^\alpha\,\mathrm{d}x)$. It is sufficient to show that there exist functions $u_\varepsilon \in D^{1,2}(\mathbb{R}^n_+, x_n^\alpha\,\mathrm{d}x)$, such that

$$\frac{\int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla u_\varepsilon|^2\,\mathrm{d}x - K(n,\alpha,b)\int_{\partial\mathbb{R}^n_+}\frac{u_\varepsilon^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x' - \frac{(\alpha+b-2)^2}{4}\int_{\mathbb{R}^n_+}\frac{x_n^\alpha u_\varepsilon^2}{|x|^2}\,\mathrm{d}x}{\left(\int_{\mathbb{R}^n_+} V(x)|u_\varepsilon|^p\,\mathrm{d}x\right)^{\frac{2}{p}}} \to 0, \quad \text{as } \varepsilon \to 0.$$

To this aim we define for any $\varepsilon > 0$ the function

$$u_\varepsilon(x) = \begin{cases} \psi(x)|x|^{\varepsilon/2}, & |x| \leq 1, \\ \psi(x)|x|^{-\varepsilon/2}, & |x| \geq 1, \end{cases}$$

where ψ is defined in (3.27). Now we make integration by parts in the domains $\mathbb{R}^n_+ \cap B_1$, $\mathbb{R}^n_+ \cap (B_R \setminus B_1)$, where $R > 1$, then send $R \to \infty$ taking into account that $\nabla u_\varepsilon(x)\cdot x = \frac{2-n-\alpha-\varepsilon}{2}\,u_\varepsilon(x)$, the relations (3.38), (3.39) together with the estimate

$$\int_{\mathbb{R}^n_+\cap\partial B_R} x_n^\alpha\,u_\varepsilon\left(\nabla u_\varepsilon(x)\cdot\frac{x}{|x|}\right)\,\mathrm{d}\sigma = \frac{2-n-\alpha-\varepsilon}{2R}\int_{\mathbb{R}^n_+\cap\partial B_R} x_n^\alpha\,u_\varepsilon^2\,\mathrm{d}\sigma(x) \leq c(n,\alpha)\,R^{-\varepsilon} \overset{R\to\infty}{\longrightarrow} 0,$$

to obtain

$$\int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla u_\varepsilon|^2\,\mathrm{d}x = K(n,\alpha,b)\int_{\partial\mathbb{R}^n_+}\frac{u_\varepsilon^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x' + \left[\frac{(\alpha+b-2)^2}{4} - \frac{\varepsilon^2}{2}\right]\int_{\mathbb{R}^n_+}\frac{x_n^\alpha u_\varepsilon^2}{|x|^2}\,\mathrm{d}x + \varepsilon\int_{\partial B_1\cap\mathbb{R}^n_+} x_n^\alpha\,\psi^2\,\mathrm{d}\sigma.$$

Here $\mathrm{d}\sigma$ stands for the $(n-1)-$dimensional Lebesgue measure over the corresponding spheres $\partial B_R = \{x \in \mathbb{R}^n : |x| = R\}$ or $\partial B_1 = \{x \in \mathbb{R}^n : |x| = 1\}$. Then, letting $\varepsilon \to 0$, we get

$$\int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla u_\varepsilon|^2\,\mathrm{d}x - K(n,\alpha,b)\int_{\partial\mathbb{R}^n_+}\frac{u_\varepsilon^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x' - \frac{(\alpha+b-2)^2}{4}\int_{\mathbb{R}^n_+}\frac{x_n^\alpha u_\varepsilon^2}{|x|^2}\,\mathrm{d}x \longrightarrow 0.$$

Chapter 4

Improving interpolated Hardy and trace Hardy inequalities on bounded domains

This chapter is devoted to establish improvements of (1.14), by adding several types of remainder terms. In particular we will give the proofs of Theorems II-V and then we will derive certain extensions of these results.

Retaining the notation introduced in the previous chapter, let us fix some extra notation, that will be used in the proofs. We denote by B_r the ball with radius r in $\mathbb{R}^n$, that is $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ and we abbreviate $B_r^+ = B_r \cap \mathbb{R}^n_+$. Let us also define the spherical surfaces ∂B_r, $\mathbb{S}^{n-1}_+$, by

$$\begin{aligned} \partial B_r &= \{x \in \mathbb{R}^n : |x| = r\} \\ \mathbb{S}^{n-1}_+ &= \{\theta = (\theta', \theta_n),\ \theta' \in \mathbb{R}^{n-1},\ \theta_n > 0 \ :\ |\theta| = 1\}. \end{aligned}$$

Moreover, $\int_{\partial B_r \cap \mathbb{R}^n_+} u \, \mathrm{d}\sigma(x)$, $\int_{\mathbb{S}^{n-1}_+} u \, \mathrm{d}\sigma$ denotes the $(n-1)-$dimensional Lebesgue integration of the function u over $\partial B_r \cap \mathbb{R}^n_+$ and $\mathbb{S}^{n-1}_+$, respectively.

Moreover, for any point $\theta \in \mathbb{S}^{n-1}_+$ we set $\varphi = \varphi(\theta) = \arccos \theta_n$, $\varphi \in [0, \pi/2]$. Notice that $\cos\varphi$ equals to the distance from the point θ to $\partial\mathbb{R}^n_+$.

4.1 Sobolev remainder term

We start with Theorem II :

Theorem 2. *Let $\alpha \in (-1,1)$, $2 - \alpha \leq b < n$ and U be a bounded domain in $\mathbb{R}^n$. Then there exists a constant $c > 0$, depending only on n and α, such that*

$$K(n,\alpha,b) \int\limits_{\partial\mathbb{R}^n_+ \cap U} \frac{u^2}{|x'|^{1-\alpha}} \, \mathrm{d}x' \ + \ \frac{(\alpha+b-2)^2}{4} \int\limits_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha u^2}{|x|^2} \, \mathrm{d}x + c \left(\int\limits_{\mathbb{R}^n_+ \cap U} X^{\frac{2n-2+\alpha}{n-2+\alpha}} \, |u|^{\frac{2n}{n-2+\alpha}} \, \mathrm{d}x \right)^{\frac{n-2+\alpha}{n}}$$

$$\leq \int\limits_{\mathbb{R}^n_+ \cap U} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x, \quad \forall u \in C_0^\infty(U), \tag{4.1}$$

where $X = X(|x|/d)$, $X(\vartheta) = (1 - \ln\vartheta)^{-1}$, $0 < \vartheta \leq 1$, $d = \sup_{x \in \mathbb{R}^n_+ \cap U} |x|$. Moreover the exponent $\frac{2n-2+\alpha}{n-2+\alpha}$ of the weight function cannot be improved.

proof of Theorem 2. By standard approximation, it suffices to prove the result for $u \in C_0^\infty(U \setminus \{0\})$. Indeed, let $\epsilon > 0$, and consider the functions $u_\epsilon = u\eta_\epsilon$, where $u \in C_0^\infty(U)$, $\eta_\epsilon \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, $\eta_\epsilon(x) = 1$, for $|x| \geq 1$ and $|\nabla \eta_\epsilon| \leq c/\epsilon$. Then, by the Lebesgue dominated theorem, we have

$$\int_{\partial\mathbb{R}^n_+ \cap U} \frac{u_\epsilon^2}{|x'|^{1-\alpha}}\, \mathrm{d}x' \to \int_{\partial\mathbb{R}^n_+ \cap U} \frac{u^2}{|x'|^{1-\alpha}}\, \mathrm{d}x' \quad \text{and} \quad \int_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha\, u_\epsilon^2}{|x|^2}\, \mathrm{d}x \to \int_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha\, u^2\, \mathrm{d}x}{|x|^2}, \quad \text{as } \epsilon \to 0 \tag{4.2}$$

and

$$\int_{\mathbb{R}^n_+ \cap U} X^{\frac{2n-2+\alpha}{n-2+\alpha}}\, |u_\epsilon|^{\frac{2n}{n-2+\alpha}}\, \mathrm{d}x \to \int_{\mathbb{R}^n_+ \cap U} X^{\frac{2n-2+\alpha}{n-2+\alpha}}\, |u|^{\frac{2n}{n-2+\alpha}}\, \mathrm{d}x, \quad \text{as } \epsilon \to 0. \tag{4.3}$$

Similarly we have

$$\int_{\mathbb{R}^n_+ \cap U} x_n^\alpha\, \eta_\epsilon^2\, |\nabla u|^2\, \mathrm{d}x \to \int_{\mathbb{R}^n_+ \cap U} x_n^\alpha\, |\nabla u|^2\, \mathrm{d}x, \quad \text{as } \epsilon \to 0.$$

Moreover, taking into account $|\nabla \eta_\epsilon| \leq c/\epsilon$, we get

$$\int_{\mathbb{R}^n_+ \cap U} x_n^\alpha\, |\nabla \eta_\epsilon|^2 u^2\, \mathrm{d}x \leq c\epsilon^{n-2+\alpha} \to 0, \quad \text{as } \epsilon \to 0,$$

hence

$$\int_{\mathbb{R}^n_+ \cap U} x_n^\alpha\, |\nabla u_\epsilon|^2\, \mathrm{d}x \to \int_{\mathbb{R}^n_+ \cap U} x_n^\alpha\, |\nabla u|^2\, \mathrm{d}x, \quad \text{as } \epsilon \to 0. \tag{4.4}$$

Gathering (4.2), (4.3), (4.4) we conclude that it suffices to prove the result for $u \in C_0^\infty(U \setminus \{0\})$.

As in the proof of Theorem 1, we expand the square and integrate by parts to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n_+ \cap U} x_n^\alpha\, |\nabla u - \frac{\nabla \psi}{\psi}\, u|^2\, \mathrm{d}x = \int_{\mathbb{R}^n_+ \cap U} x_n^\alpha\, |\nabla u|^2\, \mathrm{d}x + x_n^\alpha\, |\nabla \psi|^2 \left(\frac{u}{\psi}\right)^2 \mathrm{d}x - x_n^\alpha\, \nabla u^2 \cdot \frac{\nabla \psi}{\psi}\, \mathrm{d}x = \\
&\int_{\mathbb{R}^n_+ \cap U} x_n^\alpha\, |\nabla u|^2\, \mathrm{d}x + \int_{\mathbb{R}^n_+ \cap U} x_n^\alpha\, |\nabla \psi|^2 \left(\frac{u}{\psi}\right)^2 \mathrm{d}x + \int_{\mathbb{R}^n_+ \cap U} u^2\, div \left(x_n^\alpha\, \frac{\nabla \psi}{\psi}\right) \mathrm{d}x \\
&+ \int_{\partial\mathbb{R}^n_+ \cap U} \lim_{x_n \to 0^+} x_n^\alpha\, \frac{\partial \psi(x', x_n)}{\partial x_n}\, \frac{u^2}{\psi}\, \mathrm{d}x' = \int_{\mathbb{R}^n_+ \cap U} x_n^\alpha\, |\nabla u|^2\, \mathrm{d}x - K(n, \alpha, b) \int_{\partial\mathbb{R}^n_+ \cap U} \frac{u^2}{|x'|^{1-\alpha}}\, \mathrm{d}x' \\
&- \frac{(\alpha + b - 2)^2}{4} \int_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha\, u^2}{|x|^2}\, \mathrm{d}x.
\end{aligned}$$

In the last equation we used equations (3.39). Notice also that on $supp\, u$, the function ψ is uniformly bounded by some positive constant hence the function u/ψ is well defined. Actually $u/\psi \in C_0^\infty(\overline{\mathbb{R}^n_+} \cap U \setminus \{0\})$.

We have arrived at the following equality

$$K(n, \alpha, b) \int_{\partial\mathbb{R}^n_+ \cap U} \frac{u^2}{|x'|^{1-\alpha}}\, \mathrm{d}x' + \frac{(\alpha + b - 2)^2}{4} \int_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha u^2}{|x|^2}\, \mathrm{d}x + \int_{\mathbb{R}^n_+ \cap U} x_n^\alpha\, |\nabla \frac{u}{\psi}|^2\, \psi^2\, \mathrm{d}x = \int_{\mathbb{R}^n_+ \cap U} x_n^\alpha\, |\nabla u|^2\, \mathrm{d}x. \tag{4.5}$$

The third term in the left hand side yields the correction term in (4.1). More precisely, we have to show that there exists a positive constant $C = C(\alpha, n)$ such that for all $u \in C_0^\infty(U \setminus \{0\})$ there holds

$$C \left(\int\limits_{\mathbb{R}^n_+ \cap U} X^{\frac{2n-2+\alpha}{n-2+\alpha}} |u|^{\frac{2n}{n-2+\alpha}} \, \mathrm{d}x \right)^{\frac{n-2+\alpha}{n}} \leq \int\limits_{\mathbb{R}^n_+ \cap U} x_n^\alpha \, |\nabla \frac{u}{\psi}|^2 \, \psi^2 \, \mathrm{d}x. \tag{4.6}$$

Note that $U \subseteq B$, where we denote by B the ball $B = \{x \in \mathbb{R}^n : |x| \leq d\}$. Moreover, taking into account that $\psi \sim |x|^{-\frac{n-2+\alpha}{2}}$ in $\mathbb{R}^n_+$ (cf. (3.38)) and making the substitution $u = \upsilon \, \psi$, we conclude that (4.6) will follow after establishing the following inequality

$$c \left(\int\limits_{\mathbb{R}^n_+ \cap B} \frac{X^{\frac{2n-2+\alpha}{n-2+\alpha}} |\upsilon|^{\frac{2n}{n-2+\alpha}}}{|x|^n} \, \mathrm{d}x \right)^{\frac{n-2+\alpha}{n}} \leq \int\limits_{\mathbb{R}^n_+ \cap B} \frac{x_n^\alpha \, |\nabla \upsilon|^2}{|x|^{n-2+\alpha}} \, \mathrm{d}x, \quad \forall \upsilon \in C_0^\infty(B), \tag{4.7}$$

for some constant $c = c(n, \alpha) > 0$. Thus it is sufficient to prove inequality (4.7). To this aim we consider the minimization problem

$$c_{n,\alpha} = \inf_{\substack{\upsilon \in C_0^\infty(B) \\ \upsilon \not\equiv 0}} I[\upsilon], \quad \text{where} \quad I[\upsilon] = \frac{\int\limits_{\mathbb{R}^n_+ \cap B} \frac{x_n^\alpha}{|x|^{n-2+\alpha}} \, |\nabla \upsilon|^2 \, \mathrm{d}x}{\left(\int\limits_{\mathbb{R}^n_+ \cap B} \frac{X^{\frac{2n-2+\alpha}{n-2+\alpha}} \left(\frac{|x|}{d} \right)}{|x|^n} \, |\upsilon|^{\frac{2n}{n-2+\alpha}} \, \mathrm{d}x \right)^{\frac{n-2+\alpha}{n}}} = \frac{I_1[\upsilon]}{I_2[\upsilon]}.$$

Inspired by an idea in [3], we will relate the constant $c_{n,\alpha}$ with the weighted Sobolev constant $S_{n,\alpha}$, depending only on n and α, which is defined by

$$S_{n,\alpha} = \inf_{\substack{\upsilon \in C_0^\infty(B_1) \\ \upsilon \not\equiv 0}} Q[\upsilon], \quad \text{where} \quad Q[\upsilon] = \frac{\int\limits_{\mathbb{R}^n_+ \cap B_1} x_n^\alpha \, |\nabla \upsilon|^2 \, \mathrm{d}x}{\left(\int\limits_{\mathbb{R}^n_+ \cap B_1} |\upsilon|^{\frac{2n}{n-2+\alpha}} \, \mathrm{d}x \right)^{\frac{n-2+\alpha}{n}}} = \frac{Q_1[\upsilon]}{Q_2[\upsilon]}. \tag{4.8}$$

We express the numerator of the quotient $Q[\upsilon]$ in terms of polar coordinates, writing $\upsilon(x) = \upsilon(r, \theta)$, where

$$r = |x|, \;\; \theta \, = \, \frac{x}{|x|} \, \in \mathbb{S}^{n-1}_+.$$

Then we make the change of $r-$variable, setting

$$t = r^{2-n-\alpha} \quad \text{and} \quad \upsilon(r, \theta) = h(t, \theta),$$

thus

$$\mathrm{d}r \, = \, -\frac{r^{n-1+\alpha}}{(n-2+\alpha)} \, \mathrm{d}t, \quad \upsilon_r \, = \, -\frac{(n-2+\alpha)}{r^{n-1+\alpha}} \, h_t, \quad r \, = \, t^{-\frac{1}{(n-2+\alpha)}}.$$

Therefore we have

$$Q_1[v] = \int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\,|\nabla v(x)|^2\,\mathrm{d}x = \int\limits_0^1 \int\limits_{\partial B_r\cap\mathbb{R}^n_+} x_n^\alpha\,|\nabla v(x)|^2\,\mathrm{d}\sigma(x)\,\mathrm{d}r = \int\limits_0^1 \int\limits_{\mathbb{S}^{n-1}_+} r^{n-1+\alpha}\cos^\alpha\varphi\,|\nabla v|^2\,\mathrm{d}\sigma\,\mathrm{d}r$$
$$= \int\limits_0^1 \int\limits_{\mathbb{S}^{n-1}_+} r^{n-1+\alpha}\cos^\alpha\varphi\,\left(v_r^2 + \frac{1}{r^2}|\nabla_\theta v|^2\right)\mathrm{d}\sigma\,\mathrm{d}r$$
$$= \int\limits_0^1 \int\limits_{\mathbb{S}^{n-1}_+} r^{n-1+\alpha}\cos^\alpha\varphi\,\left(\frac{(n-2+\alpha)^2}{r^{2(n-1+\alpha)}}\,h_t^2 + \frac{1}{r^2}|\nabla_\theta h|^2\right)\mathrm{d}\sigma\,\mathrm{d}r$$
$$= \frac{1}{(n-2+\alpha)}\int\limits_1^\infty \int\limits_{\mathbb{S}^{n-1}_+} r^{2(n-1+\alpha)}\cos^\alpha\varphi\,\left(\frac{(n-2+\alpha)^2}{r^{2(n-1+\alpha)}}h_t^2 + \frac{1}{r^2}|\nabla_\theta h|^2\right)\mathrm{d}\sigma\,\mathrm{d}t$$
$$= (n-2+\alpha)\int\limits_1^\infty \int\limits_{\mathbb{S}^{n-1}_+} \cos^\alpha\varphi\,\left(h_t^2 + (n-2+\alpha)^{-2}\,r^{2(n-2+\alpha)}\,|\nabla_\theta h|^2\right)\mathrm{d}\sigma\,\mathrm{d}t$$
$$= (n-2+\alpha)\int\limits_1^\infty \int\limits_{\mathbb{S}^{n-1}_+} \cos^\alpha\varphi\,\left(h_t^2 + (n-2+\alpha)^{-2}\,t^{-2}\,|\nabla_\theta h|^2\right)\mathrm{d}\sigma\,\mathrm{d}t. \tag{4.9}$$

Similarly, transforming the denominator $Q_2[v]$, we have

$$\begin{aligned} Q_2[v] &= \left(\int\limits_{\mathbb{R}^n_+\cap B_1} |v(x)|^{\frac{2n}{n-2+\alpha}}\,\mathrm{d}x\right)^{\frac{n-2+\alpha}{n}} = \left(\int\limits_0^1 \int\limits_{\partial B_r\cap\mathbb{R}^n_+} |v(x)|^{\frac{2n}{n-2+\alpha}}\,\mathrm{d}\sigma(x)\,\mathrm{d}r\right)^{\frac{n-2+\alpha}{n}} \\ &= \left(\int\limits_0^1 \int\limits_{\mathbb{S}^{n-1}_+} r^{n-1}|v|^{\frac{2n}{n-2+\alpha}}\,\mathrm{d}\sigma\,\mathrm{d}r\right)^{\frac{n-2+\alpha}{n}} \\ &= \frac{1}{(n-2+\alpha)^{\frac{n-2+\alpha}{n}}}\left(\int\limits_1^\infty \int\limits_{\mathbb{S}^{n-1}_+} t^{-\frac{2n-2+\alpha}{n-2+\alpha}}\,|h|^{\frac{2n}{n-2+\alpha}}\,\mathrm{d}\sigma\,\mathrm{d}t\right)^{\frac{n-2+\alpha}{n}}. \end{aligned}$$

Thereforewe have

$$(n-2+\alpha)^{\frac{2-2n-\alpha}{n}}S_{n,\alpha} = \inf_{\substack{h\in C^\infty([1,\infty)\times\mathbb{S}^{n-1}_+)\\ h(1,\theta)=0}} \frac{\int\limits_1^\infty\int\limits_{\mathbb{S}^{n-1}_+}\cos^\alpha\varphi\,\left(h_t^2 + ((n-2+\alpha)t)^{-2}\,|\nabla_\theta h|^2\right)\mathrm{d}\sigma\,\mathrm{d}t}{\left(\int\limits_1^\infty\int\limits_{\mathbb{S}^{n-1}_+} t^{-\frac{2n-2+\alpha}{n-2+\alpha}}\,|h|^{\frac{2n}{n-2+\alpha}}\,\mathrm{d}\sigma\,\mathrm{d}t\right)^{\frac{n-2+\alpha}{n}}}. \tag{4.10}$$

Similarly, we transform the quotient $I[v]$ in terms of polar coordinates and then we make the change of

r-variable, setting

$$t = \frac{1}{X(r/d)} = 1 - \ln\left(\frac{r}{d}\right), \quad \upsilon(r,\theta) = w(t,\theta), \quad \text{thus} \quad \mathrm{d}r \;=\; -r\,\mathrm{d}t \quad \text{and} \quad \upsilon_r \;=\; -\frac{1}{r}\,w_t.$$

Then for any $\upsilon \in C_0^\infty(B_d)$, we have

$$\begin{aligned} I_1[\upsilon] &= \int\limits_{\mathbb{R}^n_+\cap B} \frac{x_n^\alpha\,|\nabla\upsilon|^2}{|x|^{n-2+\alpha}}\,\mathrm{d}x = \int\limits_0^d \int\limits_{\partial B_r\cap\mathbb{R}^n_+} \frac{x_n^\alpha\,|\nabla\upsilon(x)|^2}{r^{n-2+\alpha}}\,\mathrm{d}\sigma(x)\,\mathrm{d}r = \int\limits_0^d \int\limits_{\mathbb{S}^{n-1}_+} r\,\cos^\alpha\varphi\,|\nabla\upsilon|^2\,\mathrm{d}\sigma\,\mathrm{d}r \\ &= \int\limits_0^d \int\limits_{\mathbb{S}^{n-1}_+} r\,\cos^\alpha\varphi\,\left(\upsilon_r^2 \;+\; \frac{1}{r^2}\,|\nabla_\theta\upsilon|^2\right)\,\mathrm{d}\sigma\,\mathrm{d}r \\ &= \int\limits_0^d \int\limits_{\mathbb{S}^{n-1}_+} r\,\cos^\alpha\varphi\,\left(\frac{1}{r^2}w_t^2 \;+\; \frac{1}{r^2}\,|\nabla_\theta w|^2\right)\,\mathrm{d}\sigma\,\mathrm{d}r \\ &= \int\limits_1^\infty \int\limits_{\mathbb{S}^{n-1}_+} \cos^\alpha\varphi\,\left(w_t^2 \;+\; |\nabla_\theta w|^2\right)\,\mathrm{d}\sigma\,\mathrm{d}t. \end{aligned}$$

Similarly for the denominator we have

$$\begin{aligned} I_2[\upsilon] &= \left(\int\limits_{\mathbb{R}^n_+\cap B} \frac{X^{\frac{2n-2+\alpha}{n-2+\alpha}}\,|\upsilon(x)|^{\frac{2n}{n-2+\alpha}}}{|x|^n}\,\mathrm{d}x\right)^{\frac{n-2+\alpha}{n}} = \left(\int\limits_0^d \int\limits_{\partial B_r\cap\mathbb{R}^n_+} \frac{X^{\frac{2n-2+\alpha}{n-2+\alpha}}\,|\upsilon(x)|^{\frac{2n}{n-2+\alpha}}}{r^n}\,\mathrm{d}\sigma(x)\,\mathrm{d}r\right)^{\frac{n-2+\alpha}{n}} \\ &= \left(\int\limits_0^d \int\limits_{\mathbb{S}^{n-1}_+} \frac{X^{\frac{2n-2+\alpha}{n-2+\alpha}}(r)\,|\upsilon|^{\frac{2n}{n-2+\alpha}}}{r}\,\mathrm{d}\sigma\,\mathrm{d}r\right)^{\frac{n-2+\alpha}{n}} \\ &= \left(\int\limits_1^\infty \int\limits_{\mathbb{S}^{n-1}_+} t^{-\frac{2n-2+\alpha}{n-2+\alpha}}\,|w|^{\frac{2n}{n-2+\alpha}}\,\mathrm{d}\sigma\,\mathrm{d}t\right)^{\frac{n-2+\alpha}{n}}. \end{aligned}$$

Therefore we have

$$c_{n,\alpha} \;=\; \inf_{\substack{w\in C^\infty([1,\infty)\times\mathbb{S}^{n-1}_+)\\ w(1,\theta)=0}} \frac{\int\limits_1^\infty \int\limits_{\mathbb{S}^{n-1}_+} \cos^\alpha\varphi\,\left(w_t^2 \;+\; |\nabla_\theta w|^2\right)\,\mathrm{d}\sigma\,\mathrm{d}t}{\left(\int\limits_1^\infty \int\limits_{\mathbb{S}^{n-1}_+} t^{-\frac{2n-2+\alpha}{n-2+\alpha}}\,|w|^{\frac{2n}{n-2+\alpha}}\,\mathrm{d}\sigma\,\mathrm{d}t\right)^{\frac{n-2+\alpha}{n}}}.$$

Then an immediate comparison yields

$$c_{n,\alpha} \geq\; \tau_{n,\alpha}\,S_{n,\alpha} > 0, \quad \textit{where} \quad \tau_{n,\alpha} = \begin{cases} (n-2+\alpha)^{\frac{2-2n-\alpha}{n}}, & n+\alpha\geq 3 \\ (n-2+\alpha)^{\frac{2-\alpha}{n}}, & 2<n+\alpha<3, \end{cases}$$

and the proof of the desired inequality is complete.

Next, to complete the proof of the Theorem, we will verify that the weight function $X^{\frac{2n-2+\alpha}{n-2+\alpha}}$ cannot be replaced by a smaller power of X. More precisely, abbreviating $q := \frac{2n}{n-2+\alpha},\ p := \frac{2n-2+\alpha}{n-2+\alpha}$, we will show that there are no constants $0<\epsilon<p,\ c>0,$ such that the following inequality is valid

$$K(n,\alpha,b)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' \;+\; \frac{(\alpha+b-2)^2}{4}\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,u^2}{|x|^2}\,\mathrm{d}x + c\left(\int\limits_{\mathbb{R}^n_+\cap U}X^{p-\epsilon}\,|u|^q\,\mathrm{d}x\right)^{2/q}$$
$$\le \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x,\quad \forall u\in C_0^\infty(U).$$

Note also that it suffices to prove the claim, only for the case $0<\epsilon<p-1$, since $X^{p-\epsilon_0}(\vartheta)>X^{p-\epsilon}(\vartheta)$, $\forall\epsilon_0>\epsilon,\ \vartheta\in(0,1]$.

The result will follow after showing that there exists a sequence $\{u_l\}_{l=0}^\infty\subset C_0^\infty(U)$, such that

$$\frac{\int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u_l|^2\,\mathrm{d}x - K(n,\alpha,b)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u_l^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' - \frac{(\alpha+b-2)^2}{4}\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,u_l^2}{|x|^2}\,\mathrm{d}x}{\left(\int\limits_{\mathbb{R}^n_+\cap U}X^{p-\epsilon}\,|u_l|^q\,\mathrm{d}x\right)^{2/q}} \xrightarrow{m\to\infty} 0.$$

Notice that U contains a ball B_r centered at the origin, and without loss of generality we can assume that $r=1$. Furthermore, noting that $\psi\sim|x|^{-\frac{n+\alpha-2}{2}}$, in $\mathbb{R}^n_+$ (see (3.38)) and making the change of variable $u_l=\upsilon_l\,\psi$, it is sufficient to show that there exists a sequence $\{\upsilon_l\}\subset C_0^\infty(U\setminus\{0\})$ such that (cf. (4.5), (4.6))

$$J[\upsilon_l] \;:=\; \frac{N[\upsilon_l]}{D[\upsilon_l]} \;:=\; \frac{\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^\alpha\,|\nabla\upsilon_l|^2}{|x|^{n-2+\alpha}}\,\mathrm{d}x}{\left(\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{X^{p-\epsilon}\,|\upsilon_l|^q}{|x|^n}\,\mathrm{d}x\right)^{2/q}} \longrightarrow 0,\quad \text{as}\ \ l\to\infty. \tag{4.11}$$

Let us recall the notation $B_r^+=B_r\cap\mathbb{R}^n_+$, abbreviate $V(x)=\frac{X^{p-\epsilon}}{|x|^n}$, $w(x)=\frac{x_n^\alpha}{|x|^{n+\alpha-2}}$ and define the space $D_0^{1,2}(B_1,w(x)\mathrm{d}x)$ as the completion of $C_0^\infty(\overline{\mathbb{R}^n_+}\cap B_1^+)$ with respect to the norm $\|\upsilon\|=(\int_{B_1^+}|\nabla\upsilon|^2\,w(x)\,\mathrm{d}x)^{1/2}$. Then, by a standard approximation, it suffices to fix a sequence $\{\upsilon_l\}\subset D_0^{1,2}(B_1,w(x)\mathrm{d}x)$ with $\int_{B_1^+}V(x)\,|\upsilon_l|^q\,\mathrm{d}x<\infty$, such that $J[\upsilon_l]\to 0$, as $l\to\infty$.

To this end, we choose δ such that $0<\epsilon<\delta<p-1$, which eventually will be sent to ϵ, we set $R_m=e^{1-m}$ so that

$$\frac{1}{m}\le X(|x|)\le 1 \Leftrightarrow R_m\le|x|\le 1,\qquad m=1,2,3,\ldots$$

and define the functions f_m as follows

$$f_m(x)=\begin{cases} X^{\frac{\delta}{q}-\frac12}(|x|), & R_m\le|x|\le 1,\\ m^{\frac32-\frac{\delta}{q}}X(|x|), & |x|\le R_m,\end{cases}\quad \text{hence}\ \ \nabla f_m(x)=\begin{cases}\left(\frac{\delta}{q}-\frac12\right)X^{\frac{\delta}{q}+\frac12}(|x|)\,\frac{x}{|x|^2}, & R_m<|x|\le 1,\\ m^{\frac32-\frac{\delta}{q}}\,X^2(|x|)\,\frac{x}{|x|^2}, & 0<|x|<R_m.\end{cases} \tag{4.12}$$

Then, we have

$$D^{q/2}[f_m] = \int\limits_{B_1^+ \setminus B_{R_m}^+} \frac{X^{\delta-\epsilon+1}}{|x|^n}\, \mathrm{d}x \,+\, m^{3q/2-\delta} \int\limits_{B_{R_m}^+} \frac{X^{p-\epsilon+q}}{|x|^n}\, \mathrm{d}x \,=:\, D_1(m) + D_2(m)$$

and

$$N[f_m] = \left(\frac{\delta}{q} - \frac{1}{2}\right)^2 \int\limits_{B_1^+ \setminus B_{R_m}^+} \frac{x_n^\alpha X^{\frac{2\delta}{q}+1}(|x|)}{|x|^{n+\alpha}}\, \mathrm{d}x \,+\, m^{3-\frac{2\delta}{q}} \int\limits_{B_{R_m}^+} \frac{x_n^\alpha X^4(|x|)}{|x|^{n+\alpha}}\, \mathrm{d}x \,=:\, N_1(m) + N_2(m).$$

We will next estimate the terms D_1, D_2, N_1, N_2, using polar coordinates . More precisely, making the change of variable

$$t \,=\, X(r), \quad \text{thus} \quad \mathrm{d}t \,=\, \frac{X^2(r)}{r}\, \mathrm{d}r$$

and setting $\mathcal{C}_{n,\alpha} = \int_{\mathbb{S}_+^{n-1}} x_n^\alpha \,\mathrm{d}\sigma(x)$, $\gamma_n = \int_{\mathbb{S}_+^{n-1}} 1 \,\mathrm{d}\sigma(x)$, we have

$$\begin{aligned}
D_1(m) &= \int\limits_{\mathbb{S}_+^{n-1}} 1\, \mathrm{d}\sigma \int\limits_{R_m}^{1} \frac{X^{\delta-\epsilon+1}(r)}{r}\, \mathrm{d}r = \gamma_n \int\limits_{1/m}^{1} t^{\delta-\epsilon-1}\, \mathrm{d}t = \frac{\gamma_n\,(1 - m^{\epsilon-\delta})}{\delta - \epsilon}, \\
D_2(m) &= m^{3q/2-\delta} \int\limits_{\mathbb{S}_+^{n-1}} 1\, \mathrm{d}\sigma \int\limits_{0}^{R_m} \frac{X^{p-\epsilon+q}(r)}{r}\, \mathrm{d}r = \gamma_n\, m^{3q/2-\delta} \int\limits_{0}^{1/m} t^{p-\epsilon+q-2}\, \mathrm{d}t = \frac{\gamma_n\, m^{\epsilon-\delta}}{3(p-1)-\epsilon}, \\
N_1(m) &= \left(\frac{\delta}{q} - \frac{1}{2}\right)^2 \int\limits_{\mathbb{S}_+^{n-1}} x_n^\alpha\, \mathrm{d}\sigma(x) \int\limits_{R_m}^{1} \frac{X^{\frac{2\delta}{q}+1}(r)}{r}\, \mathrm{d}r = \mathcal{C}_{n,\alpha} \left(\frac{\delta}{q} - \frac{1}{2}\right)^2 \int\limits_{1/m}^{1} t^{\frac{2\delta}{q}-1}\, \mathrm{d}t \\
&= \mathcal{C}_{n,\alpha} \left(\frac{\delta}{q} - \frac{1}{2}\right)^2 \frac{q(1 - m^{-2\delta/q})}{2\delta} = \mathcal{C}_{n,\alpha}\, (\delta + 1 - p)^2\, \frac{1 - m^{-2\delta/q}}{2q\delta}, \\
N_2(m) &= m^{3-\frac{2\delta}{q}} \int\limits_{\mathbb{S}_+^{n-1}} x_n^\alpha\, \mathrm{d}\sigma(x) \int\limits_{0}^{R_m} \frac{X^4(r)}{r}\, \mathrm{d}r = \mathcal{C}_{n,\alpha}\, m^{3-\frac{2\delta}{q}} \int\limits_{0}^{1/m} t^2\, \mathrm{d}t = \frac{\mathcal{C}_{n,\alpha}\, m^{-2\delta/q}}{3}.
\end{aligned}$$

We conclude that

$$J[f_m] = \frac{\mathcal{C}_{n,\alpha}}{\gamma_n^{2/q}}\, \frac{(\delta + 1 - p)^2\, \frac{1-m^{-2\delta/q}}{2q\delta} + \frac{m^{-2\delta/q}}{3}}{\left(\frac{1-m^{\epsilon-\delta}}{\delta-\epsilon} + \frac{m^{\epsilon-\delta}}{3(p-1)-\epsilon}\right)^{2/q}}.$$

We then take a sequence $\delta_l \searrow \epsilon$ and choose m_l sufficiently large so that $m_l^{\epsilon-\delta_l} < 1/2$. It follows that $J[f_{m_l}] \to 0$, as $l \to \infty$.

Given now a function $\eta \in C_0^\infty(U)$, which is constant, not zero, in a neighbourhood of the origin, it is straightforward to verify that the sequence $\upsilon_l = f_{m_l}\,\eta$, satisfies $J[\upsilon_l] \to 0$, as $l \to \infty$, that is the condition (4.11). $\square$

4.2 Infinite improvement

We proceed now with the proof of Theorem IV.

Theorem 3. *Let $\alpha \in (-1,1)$, and $2-\alpha \le b < n$. Then for all $u \in C_0^\infty(U)$ there holds*

$$K(n,\alpha,b) \int\limits_{\partial\mathbb{R}^n_+ \cap U} \frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' + \frac{(\alpha+b-2)^2}{4} \int\limits_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha\, u^2}{|x|^2}\,\mathrm{d}x \;+\; \frac{1}{4}\sum_{i=1}^{\infty} \int\limits_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha X_1^2 \cdots X_i^2}{|x|^2}\, u^2\,\mathrm{d}x$$
$$\le \int\limits_{\mathbb{R}^n_+ \cap U} x_n^\alpha\, |\nabla u|^2\,\mathrm{d}x. \tag{4.13}$$

Here the constant $K(n,\alpha,b)$ is given in (1.11) and $X_i = X_i(|x|/d)$, with $d = \sup_{x \in \mathbb{R}^n_+ \cap U} |x|$. For fixed b, the constants $\frac{1}{4}$ are optimal, that is for $k = 1,2,\dots$ there holds

$$\frac{1}{4} = \inf_{u \in C_0^\infty(U)} \frac{\int\limits_{U^+} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x - K(n,\alpha,b)\int\limits_{\partial\mathbb{R}^n_+ \cap U} \frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' - \frac{(\alpha+b-2)^2}{4}\int\limits_{U^+} \frac{x_n^\alpha\, u^2}{|x|^2}\,\mathrm{d}x - \frac{1}{4}\sum_{i=1}^{k-1}\int\limits_{U^+} \frac{x_n^\alpha X_1^2 \cdots X_i^2}{|x|^2}\, u^2\,\mathrm{d}x}{\int\limits_{U^+} \frac{x_n^\alpha\, X_1^2 X_2^2 \cdots X_k^2}{|x|^2}\, u^2\,\mathrm{d}x}.$$

Moreover, for each $i = 1,2,\dots,$ the logarithmic correction X_i^2 cannot be replaced by a smaller power of X_i.

Proof of Theorem 3. For the reader's convenience, we divide the proof in three parts. Firstly we will establish inequality (4.13), next we will verify the optimality of the coefficients $1/4$ of the remainder terms and finally we will prove the optimality of the powers of the logarithmic weights.

An essential role, in all three parts of the proof, will play the function ψ_k, defined by

$$\psi_k(x) \;=\; \psi(x)\, X_1^{-1/2}(r)\, X_2^{-1/2}(r) \cdots X_k^{-1/2}(r) \;=\; \psi(x)\, P(r), \quad r = |x|, \tag{4.14}$$

where the function ψ is defined in (3.27). It can be shown that ψ_k satisfies the problem

$$\begin{cases} div(x_n^\alpha \nabla\psi_k) + \dfrac{(\alpha+b-2)^2}{4}\dfrac{x_n^\alpha \psi_k}{|x|^2} + \dfrac{x_n^\alpha\,\psi_k}{4|x|^2}\displaystyle\sum_{i=1}^{k} X_1^2 \cdots X_i^2 = 0, \quad \text{in } \mathbb{R}^n_+ \cap U, & (4.15a) \\[2ex] -\displaystyle\lim_{x_n \to 0^+} \dfrac{x_n^\alpha \partial\psi_k(x',x_n)}{\partial x_n} = K(n,\alpha,b)\dfrac{\psi_k(x',0)}{|x'|^{1-\alpha}}, \quad \text{on } \partial\mathbb{R}^n_+ \cap U \setminus \{0\}. & (4.15b) \end{cases}$$

Indeed, the boundary condition (4.15b) is directly checked, whereas to prove (4.15a), we will use the relation

$$X_i'(r) = \frac{1}{r} X_1(r) \cdots X_{i-1}(r) X_i^2(r),\; i = 1,\dots,k, \tag{4.16}$$

hence

$$\nabla P(r) = \frac{P'(r)}{r} x = -\frac{1}{2r^2} P(r) \left(\sum_{j=1}^{k} X_1 X_2 \cdots X_j \right) x.$$

Then we have $\nabla\psi_k = P(r)\nabla\psi - \frac{1}{2r^2}\psi P(r)\left(\sum_{j=1}^k X_1X_2\cdots X_j\right)x$ hence

$$\begin{aligned}
&\Delta\psi_k = P(r)\Delta\psi - \frac{1}{r^2}P(r)\left(\sum_{j=1}^k X_1X_2\cdots X_j\right)\nabla\psi\cdot x - \frac{1}{2r^2}\psi P(r)\left(\sum_{j=1}^k\left(\sum_{l=1}^j X_1^2X_2^2\cdots X_l^2X_{l+1}\cdots X_j\right)\right)\\
&+\frac{1}{4r^2}\psi P(r)\left(\sum_{j=1}^k X_1X_2\cdots X_j\right)^2 + \frac{1}{r^2}\psi P(r)\left(\sum_{j=1}^k X_1X_2\cdots X_j\right) - \frac{n}{2r^2}\psi P(r)\left(\sum_{j=1}^k X_1X_2\cdots X_j\right)\\
= \;& P(r)\Delta\psi - \frac{P(r)}{r^2}\left[\left(\sum_{j=1}^k X_1X_2\cdots X_j\right)(\nabla\psi\cdot\mathbf{x} + \frac{n-2}{2}\psi) + \frac{1}{4}\psi\sum_{j=1}^k X_1^2X_2^2\cdots X_j^2\right]\\
= \;& P(r)\Delta\psi - \frac{\alpha}{r}\psi P'(r) - \frac{1}{4r^2}\psi_k\sum_{j=1}^k X_1^2X_2^2\cdots X_j^2.
\end{aligned}$$

In the last equation we used that $\nabla\psi\cdot x + \frac{n+\alpha-2}{2}\psi = 0$. Then it follows that

$$\begin{aligned}
&div(x_n^\alpha\nabla\psi_k) + \frac{(\alpha+b-2)^2}{4}\frac{x_n^\alpha\psi_k}{|x|^2} + \frac{x_n^\alpha\,\psi_k}{4|x|^2}\sum_{i=1}^k X_1^2\cdots X_i^2\\
&= x_n^\alpha P(r)\Delta\psi - \frac{\alpha}{r}x_n^\alpha\psi P'(r) + \alpha x_n^{\alpha-1}\psi\frac{\partial P}{\partial x_n} + \alpha x_n^{\alpha-1}P(r)\frac{\partial\psi}{\partial x_n} + \frac{(\alpha+b-2)^2}{4}\frac{x_n^\alpha\psi}{|x|^2}P(r)\\
&= P(r)\left(x_n^\alpha\Delta\psi + \alpha x_n^{\alpha-1}\frac{\partial\psi}{\partial x_n} + \frac{(\alpha+b-2)^2}{4}\frac{x_n^\alpha\psi}{|x|^2}\right) - \frac{\alpha}{r}x_n^\alpha\psi P'(r) + \frac{\alpha}{r}x_n^\alpha\psi P'(r) = 0.
\end{aligned}$$

Part I: Derivation of the estimate (4.13). By standard approximation, it suffices to prove the result for $u\in C_0^\infty(U\setminus\{0\})$. Indeed, let $\epsilon>0$, $\mathrm{C}_\epsilon = \{x\in\mathbb{R}^n : \epsilon<|x|<1-\epsilon\}$ and consider the functions $u_\epsilon = u\eta_\epsilon$, where $u\in C_0^\infty(U)$, $\eta_\epsilon\in C_0^\infty(U\setminus\{0\})$, $\eta_\epsilon\equiv 1$ in C_ϵ and $|\nabla\eta_\epsilon|\le c/\epsilon$. Then, by the Lebesgue dominated theorem, we have

$$\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u_\epsilon^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' \to \int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' \quad\text{and}\quad \int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,u_\epsilon^2}{|x|^2}\,\mathrm{d}x \to \int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,u^2\,\mathrm{d}x}{|x|^2},\quad\text{as}\quad\epsilon\to 0 \tag{4.17}$$

and for $i=1,2,\ldots$

$$\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,X_1^2\cdots X_i^2}{|x|^2}\,u_\epsilon^2\,\mathrm{d}x \to \int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,X_1^2\cdots X_i^2}{|x|^2}\,u^2\,\mathrm{d}x,\quad\text{as}\quad\epsilon\to 0. \tag{4.18}$$

Similarly we have

$$\int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,\eta_\epsilon^2\,|\nabla u|^2\,\mathrm{d}x \to \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x,\quad\text{as}\quad\epsilon\to 0.$$

Moreover, taking into account $|\nabla\eta_\epsilon|\le c/\epsilon$, we get

$$\int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla\eta_\epsilon|^2u^2\,\mathrm{d}x \le c\epsilon^{n-2+\alpha}\to 0,\quad\text{as}\quad\epsilon\to 0,$$

hence

$$\int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u_\epsilon|^2\,\mathrm{d}x \to \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x, \quad \text{as } \epsilon\to 0. \tag{4.19}$$

Gathering (4.17), (4.18), (4.19) we conclude that it suffices to prove the result for $u \in C_0^\infty(U\setminus\{0\})$.

Thus, in the sequel we suppose that $u \in C_0^\infty(U\setminus\{0\})$. As in the proof of Theorem 1, we expand the square and integrate by parts to obtain

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u - \frac{\nabla\psi_k}{\psi_k}\,u|^2\,\mathrm{d}x = \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x + x_n^\alpha\,|\nabla\psi_k|^2\left(\frac{u}{\psi_k}\right)^2\mathrm{d}x - x_n^\alpha\,\nabla u^2\cdot\frac{\nabla\psi_k}{\psi_k}\,\mathrm{d}x = \\
&\int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x + \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla\psi_k|^2\left(\frac{u}{\psi_k}\right)^2\mathrm{d}x + \int\limits_{\mathbb{R}^n_+\cap U} u^2\,div\left(x_n^\alpha\,\frac{\nabla\psi_k}{\psi_k}\right)\mathrm{d}x \\
&+\int\limits_{\partial\mathbb{R}^n_+\cap U} \lim_{x_n\to 0^+} x_n^\alpha\,\frac{\partial\psi_k(x',x_n)}{\partial x_n}\,\frac{u^2}{\psi_k}\,\mathrm{d}x' = \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x - K(n,\alpha,b)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' \\
&-\frac{(\alpha+b-2)^2}{4}\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,u^2}{|x|^2}\,\mathrm{d}x - \frac{1}{4}\sum_{i=1}^k\int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,\frac{X_1^2\cdots X_i^2\,u^2}{|x|^2}\,\mathrm{d}x.
\end{aligned} \tag{4.20}$$

In the last equation we used equations (4.15). Notice also that on $supp\,u$, the function u/ψ_k is well defined. Actually $u/\psi_k \in C_0^\infty(\overline{\mathbb{R}^n_+}\cap U\setminus\{0\})$. We then conclude that

$$K(n,\alpha,b)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' + \frac{(\alpha+b-2)^2}{4}\int\limits_{U^+}\frac{x_n^\alpha u^2}{|x|^2}\,\mathrm{d}x + \frac{1}{4}\sum_{i=1}^k\int\limits_{U^+}\frac{x_n^\alpha X_1^2\cdots X_i^2\,u^2}{|x|^2}\,\mathrm{d}x \le \int\limits_{U^+} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x,$$

for all $u \in C_0^\infty(U\setminus\{0\})$, therefore by approximation, for all $u\in C_0^\infty(U)$. Then taking the limit $k\to\infty$ we obtain inequality (4.13).

Part II: Optimality of the constants $\frac{1}{4}$. Next we will verify the optimality of the constants $\frac14$, appearing in (4.13). To this aim, we fix $k\in\mathbb{N}$ and set $\varepsilon = (\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_k)$, where $\varepsilon_0>0$, $\varepsilon_1>0,\ldots,\varepsilon_k>0$. Then it is sufficient to show that there exist functions $u_\varepsilon\in C_0^\infty(U\setminus\{0\})$ such that

$$\frac{\int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u_\varepsilon|^2\,\mathrm{d}x - K(n,\alpha,b)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u_\varepsilon^2\,\mathrm{d}x'}{|x'|^{1-\alpha}} - \frac{(\alpha+b-2)^2}{4}\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{x_n^\alpha u_\varepsilon^2}{|x|^2}\,\mathrm{d}x - \frac14\sum_{i=1}^{k-1}\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,X_1^2\cdots X_i^2u_\varepsilon^2}{|x|^2}\,\mathrm{d}x}{\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,X_1^2\cdots X_k^2u_\varepsilon^2}{|x|^2}\,\mathrm{d}x} \xrightarrow{\varepsilon\to 0} \frac14.$$

Setting $u_\varepsilon = \psi_k\,v_\varepsilon$ and utilizing equation (4.20) we have

$$\begin{aligned}
&\frac{\int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u_\varepsilon|^2\,\mathrm{d}x - K(n,\alpha,b)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u_\varepsilon^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' - \frac{(\alpha+b-2)^2}{4}\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{x_n^\alpha u_\varepsilon^2}{|x|^2}\,\mathrm{d}x - \frac14\sum_{i=1}^{k-1}\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,X_1^2\cdots X_i^2u_\varepsilon^2}{|x|^2}\,\mathrm{d}x}{\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,X_1^2\cdots X_k^2u_\varepsilon^2}{|x|^2}\,\mathrm{d}x} \\
&= \frac{\int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,\psi_k^2|\nabla v_\varepsilon|^2\,\mathrm{d}x}{\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,X_1^2\cdots X_k^2\psi_k^2\,v_\varepsilon^2}{|x|^2}\,\mathrm{d}x} + \frac14.
\end{aligned} \tag{4.21}$$

Hence it is sufficient to show that there exist functions $\upsilon_\varepsilon \in C_0^\infty(U \setminus \{0\})$ such that

$$\frac{\int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,\psi_k^2|\nabla\upsilon_\varepsilon|^2\,\mathrm{d}x}{\int\limits_{\mathbb{R}^n_+\cap U} \frac{x_n^\alpha\,X_1^2\cdots X_k^2\psi_k^2\,\upsilon_\varepsilon^2}{|x|^2}\,\mathrm{d}x} \xrightarrow{\varepsilon\to 0} 0. \tag{4.22}$$

Note that there exists a ball $B_r \subset U$ for some $r>0$ and without loss of generality we may assume that $r=1$. We then fix some $0<\delta<1/2$ and we consider the functions $\upsilon_\varepsilon(x) = r^{\varepsilon_0}X_1^{\varepsilon_1}\cdots X_k^{\varepsilon_k}\eta(x)$, $r=|x|$, where $\eta \in C_0^\infty(B_{2\delta})$ with $\eta\equiv 1$ in B_δ. In the sequel we will show that υ_ε satisfy condition (4.22).

First, we estimate the denominator in (4.22). Since there exist constants $c_1>0$, $c_2>0$ (see (3.38)) such that

$$c_1|x|^{-\frac{n-2+\alpha}{2}} \le \psi(x) \le c_2|x|^{-\frac{n-2+\alpha}{2}}, \text{ in } \mathbb{R}^n_+\cap U, \tag{4.23}$$

we calculate

$$\begin{aligned}&\int\limits_{\mathbb{R}^n_+\cap U} \frac{x_n^\alpha\,X_1^2\cdots X_k^2\psi_k^2\,\upsilon_\varepsilon^2}{|x|^2}\,\mathrm{d}x = \int\limits_{\mathbb{R}^n_+\cap B_{2\delta}} \frac{x_n^\alpha\,X_1^{1+2\varepsilon_1}\cdots X_k^{1+2\varepsilon_k}\,\psi^2\,\eta^2}{|x|^{2-2\varepsilon_0}}\,\mathrm{d}x \ge c_1^2 \int\limits_{\mathbb{R}^n_+\cap B_\delta} \frac{x_n^\alpha\,X_1^{1+2\varepsilon_1}\cdots X_k^{1+2\varepsilon_k}}{|x|^{n+\alpha-2\varepsilon_0}}\,\mathrm{d}x\\ &+ \int\limits_{\mathbb{R}^n_+\cap\{B_{2\delta}\setminus B_\delta\}} \frac{x_n^\alpha\,X_1^{1+2\varepsilon_1}\cdots X_k^{1+2\varepsilon_k}\,\psi^2\,\eta^2}{|x|^{2-2\varepsilon_0}}\,\mathrm{d}x = C(n,\alpha)\int_0^\delta \frac{X_1^{1+2\varepsilon_1}(r)\cdots X_k^{1+2\varepsilon_k}(r)}{r^{1-2\varepsilon_0}}\,\mathrm{d}r + O_\varepsilon(1).\end{aligned}$$

Next we take successively the limits $\varepsilon_0,\ldots,\varepsilon_k\to 0$, and then utilize the relation (4.16) to obtain

$$\begin{aligned}\lim_{\varepsilon_k\to 0}\cdots\lim_{\varepsilon_1\to 0}\lim_{\varepsilon_0\to 0}\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,X_1^2\cdots X_k^2\psi_k^2\,\upsilon_\varepsilon^2}{|x|^2}\,\mathrm{d}x &\ge C(n,\alpha)\lim_{\varepsilon_k\to 0}\int_0^\delta \frac{X_1(r)\cdots X_{k-1}(r)X_k^{1+2\varepsilon_k}(r)}{r}\,\mathrm{d}r + O(1)\\ &= C(n,\alpha)\lim_{\varepsilon_k\to 0}\frac{1}{2\varepsilon_k}X_k^{2\varepsilon_k}(\delta) + O(1) = \infty. \end{aligned}\tag{4.24}$$

Next we will estimate the numerator in (4.22). We use (4.23) and then polar coordinates to get

$$\begin{aligned}&\int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,\psi_k^2\,|\nabla\upsilon_\varepsilon|^2\,\mathrm{d}x = \int\limits_{\mathbb{R}^n_+\cap B_\delta} x_n^\alpha\,\psi_k^2\,|\nabla\upsilon_\varepsilon|^2\,\mathrm{d}x + \int\limits_{\mathbb{R}^n_+\cap\{B_\delta\setminus B_{2\delta}\}} x_n^\alpha\,\psi_k^2\,|\nabla\upsilon_\varepsilon|^2\,\mathrm{d}x\\ &= \int\limits_{\mathbb{R}^n_+\cap B_\delta} x_n^\alpha\,\psi_k^2\,|\nabla(|x|^{\varepsilon_0}X_1^{\varepsilon_1}\cdots X_k^{\varepsilon_k})|^2\,\mathrm{d}x + O(1) \le c_2^2 \int\limits_{\mathbb{R}^n_+\cap B_\delta} \frac{x_n^\alpha\,X_1^{-1}\cdots X_k^{-1}}{|x|^{n+\alpha-2}}\left|\nabla(|x|^{\varepsilon_0}X_1^{\varepsilon_1}\cdots X_k^{\varepsilon_k})\right|^2\mathrm{d}x\\ &+O(1) = c(n,\alpha)\int\limits_0^\delta rX_1^{-1}\cdots X_k^{-1}\left[\frac{d}{dr}(r^{\varepsilon_0}X_1^{\varepsilon_1}(r)\cdots X_k^{\varepsilon_k}(r))\right]^2\mathrm{d}r + O(1)\\ &= c(n,\alpha)\int\limits_0^\delta r^{2\varepsilon_0-1}X_1^{2\varepsilon_1-1}\cdots X_k^{2\varepsilon_k-1}\left(\varepsilon_0+\sum_{i=1}^k \varepsilon_i X_1(r)\cdots X_i(r)\right)^2\mathrm{d}r + O(1)\end{aligned}$$

$$
\begin{aligned}
&= c(n,\alpha)\varepsilon_0^2 \int\limits_0^\delta r^{2\varepsilon_0-1} X_1^{2\varepsilon_1-1}\cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r \\
&+ c(n,\alpha)\sum_{i=1}^k \varepsilon_i^2 \int\limits_0^\delta r^{2\varepsilon_0-1} X_1^{2\varepsilon_1+1}\cdots X_i^{2\varepsilon_i+1} X_{i+1}^{2\varepsilon_{i+1}-1}\cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r \\
&+ 2c(n,\alpha)\varepsilon_0 \sum_{i=1}^k \varepsilon_i \int\limits_0^\delta r^{2\varepsilon_0-1} X_1^{2\varepsilon_1}\cdots X_i^{2\varepsilon_i} X_{i+1}^{2\varepsilon_{i+1}-1}\cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r \\
&+ 2c(n,\alpha)\sum_{i=1}^{k-1}\sum_{j=i+1}^k \varepsilon_i\varepsilon_j \int\limits_0^\delta r^{2\varepsilon_0-1} X_1^{2\varepsilon_1+1}\cdots X_i^{2\varepsilon_i+1} X_{i+1}^{2\varepsilon_{i+1}}\cdots X_j^{2\varepsilon_j} X_{j+1}^{2\varepsilon_{j+1}-1}\cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r + O(1) \\
&= c(n,\alpha)(I_1+I_2+I_3+I_4)+O(1). \qquad (4.25)
\end{aligned}
$$

Next, we estimate the limit of I_1, I_3, as $\varepsilon_0 \to 0$. To this end, we integrate by parts to get

$$
\begin{aligned}
I_1 &= \varepsilon_0^2 \int\limits_0^\delta r^{2\varepsilon_0-1} X_1^{2\varepsilon_1-1}\cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r \frac{\varepsilon_0}{2}\int_0^\delta (r^{2\varepsilon_0})' X_1^{2\varepsilon_1-1}\cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r = \\
&= \frac{\varepsilon_0}{2}\sum_{i=1}^k (1-2\varepsilon_i)\int_0^\delta r^{2\varepsilon_0-1} X_1^{2\varepsilon_1}\cdots X_i^{2\varepsilon_i} X_{i+1}^{2\varepsilon_{i+1}-1}\cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r + O(1).
\end{aligned}
$$

To estimate the first term in the right hand side we integrate by parts once again, to get for $i = 1, \ldots, k-1$

$$
\begin{aligned}
&\varepsilon_0 \int_0^\delta r^{2\varepsilon_0-1} X_1^{2\varepsilon_1}\cdots X_i^{2\varepsilon_i} X_{i+1}^{2\varepsilon_{i+1}-1}\cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r = \frac{1}{2}\int_0^\delta (r^{2\varepsilon_0})' X_1^{2\varepsilon_1}\cdots X_i^{2\varepsilon_i} X_{i+1}^{2\varepsilon_{i+1}-1}\cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r = \\
&-\sum_{j=1}^i \varepsilon_j \int_0^\delta r^{2\varepsilon_0-1} X_1^{2\varepsilon_1+1}\cdots X_j^{2\varepsilon_j+1} X_{j+1}^{2\varepsilon_{j+1}}\cdots X_i^{2\varepsilon_i} X_{i+1}^{2\varepsilon_{i+1}-1}\cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r \\
&+\sum_{j=i+1}^k \left(\frac{1}{2}-\varepsilon_j\right)\int_0^\delta r^{2\varepsilon_0-1} X_1^{2\varepsilon_1+1}\cdots X_i^{2\varepsilon_i+1} X_{i+1}^{2\varepsilon_{i+1}}\cdots X_j^{2\varepsilon_j} X_{j+1}^{2\varepsilon_{j+1}-1}\cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r + O(1). \qquad (4.26)
\end{aligned}
$$

Thus

$$
\begin{aligned}
I_1 &= \sum_{i=1}^k\sum_{j=1}^i \varepsilon_j(\varepsilon_i-\frac{1}{2})B_{ji} + \sum_{i=1}^{k-1}\sum_{j=i+1}^k (\frac{1}{2}-\varepsilon_i)(\frac{1}{2}-\varepsilon_j)B_{ij} + O(1) \\
&= 2\sum_{i=1}^{k-1}\sum_{j=i+1}^k \varepsilon_i\varepsilon_j B_{ij} + \sum_{i=1}^k \varepsilon_i^2 B_{ii} - \frac{1}{2}\sum_{i=1}^k \varepsilon_i B_{ii} - \sum_{i=1}^{k-1}\sum_{j=i+1}^k \varepsilon_i B_{ij} + \frac{1}{2}\sum_{i=1}^{k-1}\sum_{j=i+1}^k (\frac{1}{2}-\varepsilon_j)B_{ij} + O(1),
\end{aligned}
\qquad (4.27)
$$

where we abbreviate for $j \le i$

$$
B_{ji} = \int_0^\delta r^{2\varepsilon_0-1} X_1^{2\varepsilon_1+1}\cdots X_j^{2\varepsilon_j+1} X_{j+1}^{2\varepsilon_{j+1}}\cdots X_i^{2\varepsilon_i} X_{i+1}^{2\varepsilon_{i+1}-1}\cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r.
$$

In particular, we set $B_{ii} = \int_0^\delta r^{2\varepsilon_0 - 1} X_1^{2\varepsilon_1+1} \cdots X_i^{2\varepsilon_i+1} X_{i+1}^{2\varepsilon_{i+1}-1} \cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r$. Note that $B_{ji} = B_{ji}(\varepsilon_0, \ldots, \varepsilon_k)$. Moreover by (4.26) we have

$$\begin{aligned} I_3 &= 2\varepsilon_0 \sum_{i=1}^{k} \varepsilon_i \int_0^\delta r^{2\varepsilon_0-1} X_1^{2\varepsilon_1} \cdots X_i^{2\varepsilon_i} X_{i+1}^{2\varepsilon_{i+1}-1} \cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r \\ &= -2\sum_{i=1}^{k}\sum_{j=1}^{i} \varepsilon_i\varepsilon_j B_{ji} + 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \varepsilon_i(\frac{1}{2} - \varepsilon_j) B_{ij} + O(1) \\ &= -4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \varepsilon_i\varepsilon_j B_{ij} - 2\sum_{i=1}^{k} \varepsilon_i^2 B_{ii} + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \varepsilon_i B_{ij} + O(1). \end{aligned} \tag{4.28}$$

Using the same notation we write

$$I_2 = \sum_{i=1}^{k} \varepsilon_i^2 \int_0^\delta r^{2\varepsilon_0-1} X_1^{2\varepsilon_1+1} \cdots X_i^{2\varepsilon_i+1} X_{i+1}^{2\varepsilon_{i+1}-1} \cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r = \sum_{i=1}^{k} \varepsilon_i^2 B_{ii} \tag{4.29}$$

and

$$I_4 = 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \varepsilon_i\varepsilon_j \int_0^\delta r^{2\varepsilon_0-1} X_1^{2\varepsilon_1+1} \cdots X_i^{2\varepsilon_i+1} X_{i+1}^{2\varepsilon_{i+1}} \cdots X_j^{2\varepsilon_j} X_{j+1}^{2\varepsilon_{j+1}-1} \cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r = 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \varepsilon_i\varepsilon_j B_{ij}. \tag{4.30}$$

Then we combine estimates (4.25), (4.27), (4.28), (4.29), (4.30) to estimate, after a simplification, the numerator in (4.22) :

$$\int_{\mathbb{R}^n_+ \cap U} x_n^\alpha\, \psi_k^2\, |\nabla v_\varepsilon|^2\,\mathrm{d}x \le \frac{c(n,\alpha)}{2} \left[\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \left(\frac{1}{2} - \varepsilon_j\right) B_{ij} - \sum_{i=1}^{k} \varepsilon_i B_{ii} \right] + O(1). \tag{4.31}$$

It is clear now that the first term in the right hand side has a limit as $\varepsilon_0 \to 0$, which is computed by setting $\varepsilon_0 = 0$ in the terms B_{ij}.

Next, with $\varepsilon_0 = 0$, we will take the limit $\varepsilon_{11} \to 0$. We then have to estimate the limit of the terms $B_{1j}, j = 2, \ldots, k$ as well as $\varepsilon_1 B_1$, when $\varepsilon_1 \to 0$, that cannot be computed immediately by setting $\varepsilon_1 = 0$. To this aim we integrate by parts to get

$$\begin{aligned} \varepsilon_1 B_{11} &= \varepsilon_1 \int_0^\delta r^{-1} X_1^{2\varepsilon_1+1} X_2^{2\varepsilon_2-1} \cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r = \frac{1}{2}\int_0^\delta \left(X_1^{2\varepsilon_1}\right)' X_2^{2\varepsilon_2-1} \cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r \\ &= \sum_{i=2}^{k} \left(\frac{1}{2} - \varepsilon_i\right) \int_0^\delta r^{-1} X_1^{2\varepsilon_1+1} X_2^{2\varepsilon_2} \cdots X_i^{2\varepsilon_i} X_{i+1}^{2\varepsilon_{i+1}-1} \cdots X_k^{2\varepsilon_k-1}\,\mathrm{d}r + O(1) \\ &= \sum_{i=2}^{k} \left(\frac{1}{2} - \varepsilon_i\right) B_{1i} + O(1). \end{aligned}$$

We then have

$$\begin{aligned}\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\left(\frac{1}{2}-\varepsilon_j\right)B_{ij}-\sum_{i=1}^{k}\varepsilon_i B_{ii} &= \sum_{i=2}^{k-1}\sum_{j=i+1}^{k}\left(\frac{1}{2}-\varepsilon_j\right)B_{ij}-\sum_{i=2}^{k}\varepsilon_i B_{ii}+\sum_{j=2}^{k}\left(\frac{1}{2}-\varepsilon_j\right)B_{1j}-\varepsilon_1 B_{11}\\ &= \sum_{i=2}^{k-1}\sum_{j=i+1}^{k}\left(\frac{1}{2}-\varepsilon_j\right)B_{ij}-\sum_{i=2}^{k}\varepsilon_i B_{ii}+O(1).\end{aligned}$$

Now we can take the limit as $\varepsilon_1 \to 0$ by setting $\varepsilon_1 = 0$. We repeat the same argument, taking successively the limits $\varepsilon_2 \to 0, \dots, \varepsilon_k \to 0$ to conclude that

$$\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\left(\frac{1}{2}-\varepsilon_j\right)B_{ij}-\sum_{i=1}^{k}\varepsilon_i B_{ii}=O(1). \tag{4.32}$$

Then (4.22) follows combining (4.24), (4.31), (4.32).

Part III: Optimality of the exponent of the weight function. To complete the proof of the Theorem, it remains to verify that the weight functions in the correction terms cannot be replaced by smaller powers of X_i. More precisely, for each $k = 1, 2, \dots,$ we will show that there are no constants $0 < \epsilon < 2,\ c > 0$, such that the following inequality is valid

$$\begin{aligned}K(n,\alpha,b)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' &+ \frac{(\alpha+b-2)^2}{4}\int\limits_{U^+}\frac{x_n^\alpha u^2}{|x|^2}\,\mathrm{d}x+\frac{1}{4}\sum_{i=1}^{k-1}\int\limits_{U^+}\frac{x_n^\alpha X_1^2\cdots X_i^2}{|x|^2}u^2\,\mathrm{d}x\\ &+ c\int\limits_{U^+}\frac{x_n^\alpha X_1^2\cdots X_{k-1}^2X_k^{2-\epsilon}}{|x|^2}u^2\,\mathrm{d}x\le\int\limits_{U^+}x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x,\quad\forall u\in C_0^\infty(U).\end{aligned}$$

Here, the summation $\sum_{i=1}^{k-1}$ denotes zero if $k = 1$. Note also that it suffices to prove the claim, only for the case $0 < \epsilon < 1$, since $X_k^{2-\epsilon_0} > X_k^{2-\epsilon}$, $\forall\epsilon_0 > \epsilon$.

The result will follow after showing that there exists a sequence $\{u_m\} \subset C_0^\infty(U)$, such that

$$\frac{\int\limits_{U^+}x_n^\alpha\,|\nabla u_m|^2\,\mathrm{d}x-K(n,\alpha,b)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u_m^2}{|x'|^{1-\alpha}}\,\mathrm{d}x'-\frac{(\alpha+b-2)^2}{4}\int\limits_{U^+}\frac{x_n^\alpha u_m^2}{|x|^2}\,\mathrm{d}x-\frac{1}{4}\sum\limits_{i=1}^{k-1}\int\limits_{U^+}\frac{x_n^\alpha X_1^2\cdots X_i^2u_m^2}{|x|^2}\,\mathrm{d}x}{\int\limits_{U^+}\frac{x_n^\alpha X_1^2\cdots X_{k-1}^2X_k^{2-\epsilon}u_m^2}{|x|^2}\,\mathrm{d}x}\xrightarrow{m\to\infty}0.$$

Let now $\upsilon \in C_0^\infty(U \setminus \{0\})$. Making the substitution $u = \upsilon\,\psi_{k-1}$, we infer by (4.20), that the numerator of the above quotient is equal to $\int_{U^+} x_n^\alpha\,|\nabla\upsilon|^2\,\psi_{k-1}^2\,\mathrm{d}x$. Then in view of (3.38), that is $\psi \sim |x|^{\frac{2-n-\alpha}{2}}$ in $\mathbb{R}^n_+$, it suffices to fix a sequence $\{\upsilon_m\} \subset C_0^\infty(U \setminus \{0\})$ such that

$$I[\upsilon_m]=\frac{N[\upsilon_m]}{D[\upsilon_m]}:=\frac{\int_{U^+}\frac{x_n^\alpha X_1^{-1}\cdots X_{k-1}^{-1}|\nabla\upsilon_m|^2}{|x|^{n+\alpha-2}}\,\mathrm{d}x}{\int_{U^+}\frac{x_n^\alpha X_1\cdots X_{k-1}X_k^{2-\epsilon}\upsilon_m^2}{|x|^{n+\alpha}}\,\mathrm{d}x}\longrightarrow 0,\quad\text{as } m\to\infty. \tag{4.33}$$

In order to construct an appropriate sequence satisfying the condition (4.33), we need to introduce some notation. We abbreviate $V(x) = \frac{x_n^\alpha X_1\cdots X_{k-1}X_k^{2-\epsilon}}{|x|^{n+\alpha}}$, $w(x) = \frac{x_n^\alpha X_1^{-1}\cdots X_{k-1}^{-1}}{|x|^{n+\alpha-2}}$ and define the space $D_0^{1,2}(U \setminus \{0\}, w(x)\mathrm{d}x)$ as the completion of $C_0^\infty(U\setminus\{0\})$ with respect to the norm $\|u\| = (\int_{U^+} |\nabla u|^2 w(x)\,\mathrm{d}x)^{1/2}$. By a standard approximation, it suffices to fix a sequence $\{\upsilon_m\} \subset D_0^{1,2}(U\setminus\{0\}, w(x)\mathrm{d}x)$ with $\int_{U^+} V(x)\upsilon_m^2(x)\,\mathrm{d}x < \infty$, such that $I[\upsilon_m] \to 0$, as $m \to \infty$.

To this aim, we choose δ such that $0<\epsilon<\delta<1$, which eventually will be sent to ϵ. Furthermore, we define recursively $R_1(m)=e^{1-m}$, $R_{i+1}(m)=R_i(e^{m-1})$, $i=1,2,\dots,k$, to obtain finally the parameter $R_m:=R_{k+1}(m)$. It can be easily seen that

$$\frac{1}{m}\le X_k(|x|)\le 1 \Leftrightarrow R_m\le |x|\le 1.$$

We then consider the functions f_m, defined by

$$f_m(x)=\begin{cases} X_k^{\frac{\delta-1}{2}}(|x|), & R_m\le |x|\le 1,\\ m^{\frac{3-\delta}{2}}X_k(|x|), & |x|\le R_m,\end{cases}$$

whence

$$\nabla f_m(x)=\begin{cases} \frac{\delta-1}{2}X_1\cdots X_{k-1}X_k^{\frac{\delta+1}{2}}(|x|)\,\frac{x}{|x|^2},\ R_m\le |x|\le 1,\\ m^{\frac{3-\delta}{2}}\,X_1\cdots X_{k-1}\,X_k^2(|x|)\,\frac{x}{|x|^2},\ |x|\le R_m.\end{cases}$$

We then have

$$\begin{aligned} N[f_m] &= \left(\frac{\delta-1}{2}\right)^2\int_{U^+\setminus B_{R_m}}\frac{x_n^\alpha X_1\cdots X_{k-1}X_k^{\delta+1}}{|x|^{n+\alpha}}\,\mathrm{d}x \;+\; m^{3-\delta}\int_{B_{R_m}^+}\frac{x_n^\alpha X_1\cdots X_{k-1}X_k^4}{|x|^{n+\alpha}}\,\mathrm{d}x\\ &= T_1(m)+T_2(m)\end{aligned}$$

and

$$\begin{aligned} D[f_m] &= \int_{U^+\setminus B_{R_m}}\frac{x_n^\alpha X_1\cdots X_{k-1}X_k^{1+\delta-\epsilon}}{|x|^{n+\alpha}}\,\mathrm{d}x \;+\; m^{3-\delta}\int_{B_{R_m}^+}\frac{x_n^\alpha X_1\cdots X_{k-1}X_k^{4-\epsilon}}{|x|^{n+\alpha}}\,\mathrm{d}x\\ &= T_3(m)+T_4(m).\end{aligned}$$

Next we will estimate the terms T_1, T_2, T_3, T_4, using polar coordinates as well as (4.16) when differentiating f_m. More precisely, we have

$$\begin{aligned} T_1(m) &= \left(\frac{\delta-1}{2}\right)^2\int_{\mathbb{S}_+^{n-1}}y_n^\alpha\,\mathrm{d}\sigma(y)\int_{R_m}^1 X_k^{\delta-1}\,\frac{X_1\cdots X_{k-1}X_k^2(r)}{r}\,\mathrm{d}r\\ &= \mathcal{C}_{n,\alpha}\left(\frac{\delta-1}{2}\right)^2\int_{1/m}^1 t^{\delta-1}\,\mathrm{d}t=\mathcal{C}_{n,\alpha}\,\frac{(\delta-1)^2}{4\delta}(1-m^{-\delta})\end{aligned}$$

and

$$T_2(m) = m^{3-\delta}\int_{\mathbb{S}_+^{n-1}}y_n^\alpha\,\mathrm{d}\sigma(y)\int_0^{R_m}X_k^2\,\frac{X_1\cdots X_{k-1}X_k^2(r)}{r}\,\mathrm{d}r=\mathcal{C}_{n,\alpha}\,m^{3-\delta}\int_0^{1/m}t^2\,\mathrm{d}t=\mathcal{C}_{n,\alpha}\,\frac{m^{-\delta}}{3}.$$

Similarly we get

$$T_3(m) = \int_{\mathbb{S}_+^{n-1}}y_n^\alpha\,\mathrm{d}\sigma(y)\int_{R_m}^1 X_k^{\delta-\varepsilon-1}\,\frac{X_1\cdots X_{k-1}X_k^2(r)}{r}\,\mathrm{d}r=\mathcal{C}_{n,\alpha}\int_{1/m}^1 t^{\delta-\varepsilon-1}\,\mathrm{d}t=\mathcal{C}_{n,\alpha}\,\frac{1-m^{\varepsilon-\delta}}{\delta-\varepsilon}$$

and

$$\begin{aligned} T_4(m) &= m^{3-\delta}\int_{\mathbb{S}_+^{n-1}}y_n^\alpha\,\mathrm{d}\sigma(y)\int_0^{R_m}X_k^{2-\varepsilon}\,\frac{X_1\cdots X_{k-1}X_k^2(r)}{r}\,\mathrm{d}r\\ &= \mathcal{C}_{n,\alpha}\,m^{3-\delta}\int_0^{1/m}t^{2-\varepsilon}\,\mathrm{d}t=\mathcal{C}_{n,\alpha}\,\frac{m^{\varepsilon-\delta}}{3-\varepsilon}.\end{aligned}$$

Here $\mathcal{C}_{n,\alpha} = \int_{\mathbb{S}_+^{n-1}} x_n^\alpha \, \mathrm{d}\sigma(x)$, where $\mathrm{d}\sigma(x)$ stands for the $(n-1)-$ dimensional Lebesgue measure on the upper half sphere $\mathbb{S}_+^{n-1} = \{x \in \mathbb{R}_+^n : |x| = 1\}$. Therefore, we obtain

$$N[f_m] = \mathcal{C}_{n,\alpha} \left(\frac{(\delta-1)^2}{4\delta}(1-m^{-\delta}) + \frac{m^{-\delta}}{3} \right) \qquad \text{and} \qquad D[f_m] = \mathcal{C}_{n,\alpha} \left(\frac{1-m^{\epsilon-\delta}}{\delta-\epsilon} + \frac{m^{\epsilon-\delta}}{3-\epsilon} \right).$$

We then take a sequence $\delta_i \searrow \epsilon$ and choose m_i sufficiently large so that $m_i^{\epsilon-\delta_i} < 1/2$. It follows that $I[f_{m_i}] \to 0$, as $i \to \infty$.

Given now a function $\eta \in C_0^\infty(U)$, which is constant, not zero, in a neighbourhood of the origin, it is straightforward to verify that the sequence $\upsilon_i = f_{m_i} \eta$, satisfies $I[\upsilon_i] \to 0$, as $i \to \infty$, that is the condition (4.33). $\square$

We proceed with the proof of Theorem V, which we restate here.

Theorem 4. *Let $\alpha \in (-1,1)$, $2-\alpha \le b < n$ and U be a bounded domain in $\mathbb{R}^n$. Then there exists a constant $c > 0$, depending only on n and α, such that*

$$K(n,\alpha,b) \int_{\partial\mathbb{R}_+^n \cap U} \frac{u^2}{|x'|^{1-\alpha}} \, \mathrm{d}x' + \frac{(\alpha+b-2)^2}{4} \int_{\mathbb{R}_+^n \cap U} \frac{x_n^\alpha \, u^2}{|x|^2} \, \mathrm{d}x + \frac{1}{4} \sum_{i=1}^k \int_{\mathbb{R}_+^n \cap U} \frac{x_n^\alpha X_1^2 \cdots X_i^2}{|x|^2} u^2 \, \mathrm{d}x$$
$$+ c \left(\int_{\mathbb{R}_+^n \cap U} (X_1 \cdots X_k X_{k+1})^{\frac{2n-2+\alpha}{n-2+\alpha}} \, |u|^{\frac{2n}{n-2+\alpha}} \, \mathrm{d}x \right)^{\frac{n-2+\alpha}{n}} \le \int_{\mathbb{R}_+^n \cap U} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x, \quad \forall u \in C_0^\infty(U). \tag{4.34}$$

Here the constant $K(n,\alpha,b)$ is given in (1.11) and $X_i = X_i(|x|/d)$, with $d = \sup_{x \in \mathbb{R}_+^n \cap U} |x|$. Moreover, the logarithmic correction $(X_1 \cdots X_{k+1})^{(2n-2+\alpha)/(n-2+\alpha)}$ cannot be replaced by smaller powers of $X_1, \cdots, X_{k+1}$.

Proof. By equation (4.20) we have

$$\int_{\mathbb{R}_+^n \cap U} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x \; - \; K(n,\alpha,b) \int_{\partial\mathbb{R}_+^n \cap U} \frac{u^2}{|x'|^{1-\alpha}} \, \mathrm{d}x' - \frac{(\alpha+b-2)^2}{4} \int_{\partial\mathbb{R}_+^n \cap U} \frac{x_n^\alpha u^2}{|x|^2} \, \mathrm{d}x$$
$$- \; \frac{1}{4} \sum_{i=1}^k \int_{\mathbb{R}_+^n \cap U} \frac{x_n^\alpha \, X_1^2 \cdots X_i^2 u^2}{|x|^2} \, \mathrm{d}x = \int_{\mathbb{R}_+^n \cap U} x_n^\alpha \, \psi_k^2 \, |\nabla(u/\psi_k)|^2 \, \mathrm{d}x,$$

where ψ_k is defined in (4.14). The term in the right hand side yields the expected Sobolev type remainder term in (4.34). More precisely, we have to show that there exists a constant $C > 0$, depending only on n and α, such that for all $u \in C_0^\infty(U \setminus \{0\})$ there holds

$$C \left(\int_{\mathbb{R}_+^n \cap U} (X_1 \cdots X_k X_{k+1})^{\frac{2n-2+\alpha}{n-2+\alpha}} \, |u|^{\frac{2n}{n-2+\alpha}} \, \mathrm{d}x \right)^{\frac{n-2+\alpha}{n}} \le \int_{\mathbb{R}_+^n \cap U} x_n^\alpha \, \psi_k^2 \, |\nabla(u/\psi_k)|^2 \, \mathrm{d}x. \tag{4.35}$$

Now, taking into account that $\psi \sim |x|^{-\frac{n-2+\alpha}{2}}$ in $\mathbb{R}_+^n$ (cf. (3.38)) and making the substitution $u = \psi_k \, \upsilon$, we conclude that the validity of (4.35) follows by the existence of a positive constant $c = c(n,\alpha)$ such that the following inequality is valid

$$c\left(\int\limits_{\mathbb{R}^n_+\cap U}\frac{X_1\cdots X_k\,X_{k+1}^{\frac{2n-2+\alpha}{n-2+\alpha}}}{|x|^n}\,|v|^{\frac{2n}{n-2+\alpha}}\,\mathrm{d}x\right)^{\frac{n-2+\alpha}{n}}\le\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,X_1^{-1}\cdots X_k^{-1}}{|x|^{n-2+\alpha}}\,|\nabla v|^2\,\mathrm{d}x,\quad\forall v\in C_0^\infty(U).\quad(4.36)$$

Note that $U\subseteq B$, where we denote by B the ball $B=\{x\in\mathbb{R}^n:|x|\le d\}$. Therefore, (4.36) will follow on its turn after showing the existence of a positive constant $C=C(n,\alpha)$, independent of d, such that for all $v\in C_0^\infty(B)$, there holds

$$c\left(\int\limits_{\mathbb{R}^n_+\cap B}\frac{X_1\cdots X_k\,X_{k+1}^{\frac{2n-2+\alpha}{n-2+\alpha}}}{|x|^n}\,|v|^{\frac{2n}{n-2+\alpha}}\,\mathrm{d}x\right)^{\frac{n-2+\alpha}{n}}\le\int\limits_{\mathbb{R}^n_+\cap B}\frac{x_n^\alpha\,X_1^{-1}\cdots X_k^{-1}}{|x|^{n-2+\alpha}}\,|\nabla v|^2\,\mathrm{d}x.$$

To this aim we consider the minimization problem

$$c_{n,\alpha}=\inf_{\substack{v\in C_0^\infty(B)\\ v|\not\equiv 0}}I[v],\quad\text{where}\quad I[v]=\frac{\int\limits_{\mathbb{R}^n_+\cap B}\frac{x_n^\alpha\,X_1^{-1}\left(\frac{|x|}{d}\right)\cdots X_k^{-1}\left(\frac{|x|}{d}\right)}{|x|^{n-2+\alpha}}\,|\nabla v|^2\,\mathrm{d}x}{\left(\int\limits_{\mathbb{R}^n_+\cap B}\frac{X_1\left(\frac{|x|}{d}\right)\cdots X_k\left(\frac{|x|}{d}\right)X_{k+1}^{\frac{2n-2+\alpha}{n-2+\alpha}}\left(\frac{|x|}{d}\right)}{|x|^n}\,|v|^{\frac{2n}{n-2+\alpha}}\,\mathrm{d}x\right)^{\frac{n-2+\alpha}{n}}}=\frac{I_1[v]}{I_2[v]}.$$

We will compare the constant $c_{n,\alpha}$ with the weighted Sobolev constant $S_{n,\alpha}$ defined by

$$S_{n,\alpha}=\inf_{\substack{v\in C_0^\infty(B_1)\\ v|\not\equiv 0}}Q[v],\quad\text{where}\quad Q[v]=\frac{\int\limits_{\mathbb{R}^n_+\cap B_1}x_n^\alpha\,|\nabla v|^2\,\mathrm{d}x}{\left(\int\limits_{\mathbb{R}^n_+\cap B_1}|v|^{\frac{2n}{n-2+\alpha}}\,\mathrm{d}x\right)^{\frac{n-2+\alpha}{n}}}=\frac{Q_1[v]}{Q_2[v]}.$$

We express the numerator of the quotient $Q[v]$ in terms of polar coordinates, writing $v(x)=v(r,\theta)$, where

$$r=|x|,\quad\theta=\frac{x}{|x|}\in\mathbb{S}^{n-1}_+.$$

Then we make the change of $r-$variable, setting

$$t=r^{2-n-\alpha}\quad\text{and}\quad v(r,\theta)=h(t,\theta),$$

to obtain (cf. (4.8),(4.10))

$$(n-2+\alpha)^{\frac{2-2n-\alpha}{n}}\,S_{n,\alpha}=\inf_{\substack{h\in C^\infty([1,\infty)\times\mathbb{S}^{n-1}_+)\\ h(1,\theta)=0}}\frac{\int\limits_1^\infty\int\limits_{\mathbb{S}^{n-1}_+}\cos^\alpha\varphi\,\left(h_t^2+(n-2+\alpha)^{-2}\,t^{-2}\,|\nabla_\theta h|^2\right)\,\mathrm{d}\sigma\,\mathrm{d}t}{\left(\int\limits_1^\infty\int\limits_{\mathbb{S}^{n-1}_+}t^{-\frac{2n-2+\alpha}{n-2+\alpha}}\,|h|^{\frac{2n}{n-2+\alpha}}\,\mathrm{d}\sigma\,\mathrm{d}t\right)^{\frac{n-2+\alpha}{n}}}.$$

Next we express the quotient I in terms of polar coordinates and then we make the change of variable in the r variable setting

$$v(r,\theta)=w(t,\theta),\ t=\frac{1}{X_{k+1}(\frac{r}{d})},\qquad\text{thus}\quad\frac{\mathrm{d}t}{\mathrm{d}r}=\frac{X_1\left(\frac{r}{d}\right)\cdots X_k\left(\frac{r}{d}\right)}{r}\quad\text{(cf. (4.16)).}$$

We also define recursively

$$R_1(t) = e^{1-t},\ R_{i+1}(t) = R_i(e^{t-1}),\ i = 1, 2, \ldots, k-1, \quad \text{so that} \quad X_1(\frac{r}{d}) = R_1(t), \ldots, X_k(\frac{r}{d}) = R_k(t).$$

Therefore, we have

$$\begin{aligned}
I_1[\upsilon] &= \int\limits_{\mathbb{R}^n_+ \cap B} \frac{x_n^\alpha\, X_1^{-1}\left(\frac{|x|}{d}\right) \cdots X_k^{-1}\left(\frac{|x|}{d}\right)}{|x|^{n-2+\alpha}}\, |\nabla \upsilon(x)|^2 \,\mathrm{d}x \\
&= \int\limits_0^d \int\limits_{\partial B_r \cap \mathbb{R}^n_+} \frac{x_n^\alpha\, X_1^{-1}\left(\frac{r}{d}\right) \cdots X_k^{-1}\left(\frac{r}{d}\right)}{r^{n-2+\alpha}}\, |\nabla \upsilon(x)|^2 \,\mathrm{d}\sigma(x)\,\mathrm{d}r \\
&= \int\limits_0^d \int\limits_{\mathbb{S}^{n-1}_+} r\, \cos^\alpha \varphi\, X_1^{-1}\left(\frac{r}{d}\right) \cdots X_k^{-1}\left(\frac{r}{d}\right) \left(\upsilon_r^2 + \frac{1}{r^2} |\nabla_\theta \upsilon|^2\right) \mathrm{d}\sigma\,\mathrm{d}r \\
&= \int\limits_0^d \int\limits_{\mathbb{S}^{n-1}_+} \cos^\alpha \varphi\, \frac{X_1\left(\frac{r}{d}\right) \cdots X_k\left(\frac{r}{d}\right)}{r} \left(w_t^2 + \left(X_1^{-2}\left(\frac{r}{d}\right) \cdots X_k\left(\frac{r}{d}\right)\right)^{-2} |\nabla_\theta w|^2\right) \mathrm{d}\sigma\,\mathrm{d}r \\
&= \int\limits_1^\infty \int\limits_{\mathbb{S}^{n-1}_+} \cos^\alpha \varphi\, \left(w_t^2 + (R_1(t) \cdots R_k(t))^{-2}\, |\nabla_\theta w|^2\right) \mathrm{d}\sigma\,\mathrm{d}t.
\end{aligned}$$

Similarly for the denominator we have

$$\begin{aligned}
I_2[\upsilon] &= \left(\int\limits_{\mathbb{R}^n_+ \cap B} \frac{X_1\left(\frac{|x|}{d}\right) \cdots X_k\left(\frac{|x|}{d}\right)\, X_{k+1}^{\frac{2n-2+\alpha}{n-2+\alpha}}\left(\frac{|x|}{d}\right)}{|x|^n}\, |\upsilon|^{\frac{2n}{n-2+\alpha}} \,\mathrm{d}x \right)^{\frac{n-2+\alpha}{n}} \\
&= \left(\int\limits_0^d \int\limits_{\partial B_r \cap \mathbb{R}^n_+} \frac{X_1\left(\frac{r}{d}\right) \cdots X_k\left(\frac{r}{d}\right)\, X_{k+1}^{\frac{2n-2+\alpha}{n-2+\alpha}}\left(\frac{r}{d}\right)}{r^n}\, |\upsilon(x)|^{\frac{2n}{n-2+\alpha}} \,\mathrm{d}\sigma(x)\,\mathrm{d}r \right)^{\frac{n-2+\alpha}{n}} \\
&= \left(\int\limits_0^d \int\limits_{\mathbb{S}^{n-1}_+} \frac{X_1\left(\frac{r}{d}\right) \cdots X_k\left(\frac{r}{d}\right)\, X_{k+1}^{\frac{2n-2+\alpha}{n-2+\alpha}}\left(\frac{r}{d}\right)}{r}\, |\upsilon|^{\frac{2n}{n-2+\alpha}} \,\mathrm{d}\sigma\,\mathrm{d}r \right)^{\frac{n-2+\alpha}{n}} \\
&= \left(\int\limits_1^\infty \int\limits_{\mathbb{S}^{n-1}_+} t^{-\frac{2n-2+\alpha}{n-2+\alpha}}\, |w|^{\frac{2n}{n-2+\alpha}} \,\mathrm{d}\sigma\,\mathrm{d}t \right)^{\frac{n-2+\alpha}{n}}.
\end{aligned}$$

Therefore we have

$$c_{n,\alpha} = \inf_{\substack{w \in C^\infty([1,\infty) \times \mathbb{S}^{n-1}_+) \\ w(1,\theta)=0}} \frac{\int\limits_1^\infty \int\limits_{\mathbb{S}^{n-1}_+} \cos^\alpha \varphi \left(w_t^2 + (R_1(t) \cdots R_k(t))^{-2}\, |\nabla_\theta w|^2\right) \mathrm{d}\sigma\,\mathrm{d}t}{\left(\int\limits_1^\infty \int\limits_{\mathbb{S}^{n-1}_+} t^{-\frac{2n-2+\alpha}{n-2+\alpha}}\, |w|^{\frac{2n}{n-2+\alpha}} \,\mathrm{d}\sigma\,\mathrm{d}t \right)^{\frac{n-2+\alpha}{n}}}.$$

Then an immediate comparison, noting that $R_1 < 1, \dots, R_k < 1$, yields

$$c_{n,\alpha} \geq \tau_{n,\alpha}\, S_{n,\alpha} > 0, \quad \textit{where} \quad \tau_{n,\alpha} = \begin{cases} (n-2+\alpha)^{\frac{2-2n-\alpha}{n}}, & n+\alpha \geq 3 \\ (n-2+\alpha)^{\frac{2-\alpha}{n}}, & 2 < n+\alpha < 3, \end{cases}$$

and the proof of the desired inequality is complete.

The optimality of the exponent $p = (2n-2+\alpha)/(n-2+\alpha)$ of the weight $(X_1 \cdots X_k X_{k+1})^p$, can be deduced in a way quite similar to that presented in Theorem 2, using the functions (cf. (4.12))

$$f_m(x) = \begin{cases} X_k^{\frac{\delta+1-p}{2(p-1)}}(|x|), & R_m \leq |x| \leq 1, \\ m^{\frac{3p-3-\delta}{2(p-1)}} X_k(|x|), & |x| \leq R_m, \quad m \in \mathbb{N},\ \delta < 1, \end{cases}$$

so we may refrain from giving details here. □

Remark. *The optimality of the exponent* $p := \frac{2n-2+\alpha}{n-2+\alpha}$ *of the logarithmic weights in Theorem 4 can be also deduced by the optimality of the exponent of the weight* $(X_1 \cdots X_{k+1})^2$*, appearing in Theorem 3, jointly with Hölder inequality, as follows.*

Notice first, that it suffices to prove the optimality of the power of X_{k+1}*, that is the term* X_{k+1}^p *cannot be replaced by* $X_{k+1}^{p-\epsilon}$*, for any* $p > \epsilon > 0$*. Indeed, after having shown this, the optimality of the exponent* p *of the terms* $X_i,\ i = 1, \dots, k$*, results upon the estimate* $X_{k+1}^{p-\epsilon} X_i^p \leq X_{k+1}^p X_i^{p-\epsilon}$*, since* $X_{k+1} \geq X_i$*.*

In order to verify the optimality of the power of X_{k+1}*, let us abbreviate*

$$I_k[u] := \int_{\mathbb{R}^n_+ \cap U} x_n^\alpha |\nabla u|^2 \, dx \;-\; K(n,\alpha,b) \int_{\partial \mathbb{R}^n_+ \cap U} \frac{u^2}{|x'|^{1-\alpha}} \, dx' - \frac{(\alpha+b-2)^2}{4} \int_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha u^2}{|x|^2} \, dx$$
$$- \frac{1}{4} \sum_{i=1}^k \int_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha X_1^2 \cdots X_i^2}{|x|^2} u^2 \, dx,$$

and suppose, towards contradiction, that there exists a constant $c > 0$*, such that the following inequality holds*

$$c \left(\int_{\mathbb{R}^n_+ \cap U} (X_1 \cdots X_k)^p \; X_{k+1}^{p-\epsilon} |u|^{\frac{2n}{n-2+\alpha}} \, dx \right)^{\frac{n-2+\alpha}{n}} \leq I_k[u], \quad \forall u \in C_0^\infty(U). \tag{4.37}$$

In the left hand side, we will employ the Hölder's inequality with conjugate exponents $q = \frac{n}{2-\alpha}$*,* $q' = \frac{n}{n-2+\alpha}$*. More precisely, we choose* $0 < \delta < 1$ *and noting that* $p = q'+1$ *we get*

$$\int_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha \;(X_1 \cdots X_k)^2 \; X_{k+1}^{2-\epsilon(1-\delta)/q'}}{|x|^2} u^2 \, dx = \int_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha \;(X_1 \cdots X_k)^{\frac{1}{q}} \; X_{k+1}^{\frac{1}{q}+\frac{\epsilon\delta}{q'}}}{|x|^2} \;(X_1 \cdots X_k)^{1+\frac{1}{q'}} \; X_{k+1}^{1+\frac{1}{q'}-\frac{\epsilon}{q'}} u^2 \, dx$$
$$\leq \left(\int_{\mathbb{R}^n_+ \cap U} \frac{x_n^{\alpha q} \;(X_1 \cdots X_k)\; X_{k+1}^{1+\epsilon\delta q/q'}}{|x|^{2q}} \, dx \right)^{1/q} \left(\int_{\mathbb{R}^n_+ \cap U} (X_1 \cdots X_k)^p \; X_{k+1}^{p-\epsilon} |u|^{\frac{2n}{n-2+\alpha}} \, dx \right)^{1/q'}. \tag{4.38}$$

Combining (4.37), (4.38), we infer the existence of a positive constant C *such that*

$$C \int_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha \;(X_1 \cdots X_k)^2 \; X_{k+1}^{2-\epsilon(1-\delta)/q'}}{|x|^2} u^2 \, dx \;\leq\; I_k[u], \quad \forall u \in C_0^\infty(U).$$

This inequality contradicts to Theorem 3, where it is stated that the weight $(X_1 \cdots X_{k+1})^2$ *in the inequality*

$$\frac{1}{4} \int\limits_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha \, (X_1 \cdots X_{k+1})^2}{|x|^2} \, u^2 \, \mathrm{d}x \;\leq\; I_k[u],$$

cannot be replaced by powers of $X_1, \cdots, X_{k+1}$ *with exponents being smaller than 2.*

4.3 Weighted Hardy inequalities

In this section we establish some weighted Hardy inequalities that will be used later on. In fact, we will derive more general results that are of independent interest.

Lemma 4. *Let* $n \geq 2$, $R > 0$ *and let* A, B, Γ, *be real numbers such that* $A + 1 > 0$, $B + n - 1 > 0$, $A + B + n - \Gamma > 0$. *Then, the following inequality holds*

$$\frac{(B+n-1)(A+B+n-\Gamma^+)}{(A+B+n)} \int\limits_{\mathbb{R}^n_+ \cap B_R} \frac{x_n^A \, |x'|^B \, |\upsilon|}{|x|^\Gamma} \, \mathrm{d}x \leq \int\limits_{\mathbb{R}^n_+ \cap B_R} \frac{x_n^A \, |x'|^{B+1} \, |\nabla \upsilon|}{|x|^\Gamma} \, \mathrm{d}x, \quad \forall \upsilon \in C_0^1(B_R),$$

where $\Gamma^+ = \max\{\Gamma, 0\}$.

Proof. It suffices to prove the inequality only for $R = 1$, as it is scaling invariant. Integration by parts in x' variables yields

$$\begin{aligned}
(B+1) \int\limits_{\mathbb{R}^n_+ \cap B_1} \frac{x_n^A \, |x'|^B}{|x|^\Gamma} |\upsilon| \, \mathrm{d}x &= \int\limits_{\mathbb{R}^n_+ \cap B_1} \frac{x_n^A \, |\upsilon|}{|x|^\Gamma} \, (\nabla |x'| \cdot \nabla |x'|^{B+1}) \, \mathrm{d}x \\
= (2-n) \int\limits_{\mathbb{R}^n_+ \cap B_1} \frac{x_n^A \, |x'|^B}{|x|^\Gamma} \, |\upsilon| \, \mathrm{d}x &+ \Gamma \int\limits_{\mathbb{R}^n_+ \cap B_1} \frac{x_n^A \, |x'|^{B+2}}{|x|^{\Gamma+2}} \, |\upsilon| \, \mathrm{d}x - \int\limits_{\mathbb{R}^n_+ \cap B_1} \frac{x_n^A \, |x'|^{B+1}}{|x|^\Gamma} \, (\nabla |x'| \cdot \nabla_{x'} |\upsilon|) \, \mathrm{d}x.
\end{aligned}$$

There isn't boundary term on $\{x_n = 0\}$ due to the presence of the integrand $x' \cdot \mathbf{0} = 0$. We then have

$$(B+n-1) \int\limits_{\mathbb{R}^n_+ \cap B_1} \frac{x_n^A \, |x'|^B}{|x|^\Gamma} |\upsilon| \, \mathrm{d}x \leq \Gamma \int\limits_{\mathbb{R}^n_+ \cap B_1} \frac{x_n^A \, |x'|^{B+2} \, |\upsilon|}{|x|^{\Gamma+2}} \, \mathrm{d}x + \int\limits_{\mathbb{R}^n_+ \cap B_1} \frac{x_n^A \, |x'|^{B+1} \, |\nabla \upsilon|}{|x|^\Gamma} \, \mathrm{d}x. \tag{4.39}$$

If $\Gamma \leq 0$, then the result follows immediately. Consider now the case $\Gamma > 0$. We will estimate the first term of the right hand side. Define the vector field

$$\mathbf{F} = \frac{x_n^A \, |x'|^{B+2}}{|x|^{\Gamma+2}} \, (x', \, x_n).$$

Then we have

$$|\mathbf{F}| = \frac{x_n^A \, |x'|^{B+2}}{|x|^{\Gamma+1}} \leq \frac{x_n^A \, |x'|^{B+1}}{|x|^\Gamma}$$

and

$$\begin{aligned}
div \mathbf{F} \;&=\; \frac{x_n^A \, |x'|^{B+2}}{|x|^{\Gamma+2}} \, div(x', \, x_n) + \nabla \left(\frac{x_n^A \, |x'|^{B+2}}{|x|^{\Gamma+2}} \right) \cdot (x', \, x_n) = n \, \frac{x_n^A \, |x'|^{B+2}}{|x|^{\Gamma+2}} \\
&\quad + \Big(\frac{x_n^A |x'|^B}{|x|^{\Gamma+2}} [(B+2) - (\Gamma+2)|x'|^2 |x|^{-2}] \, x', \, \frac{x_n^{A-1} \, |x'|^{B+2}}{|x|^{\Gamma+2}} \left(A - (\Gamma+2) x_n^2 |x|^{-2} \right) \Big) \cdot (x', \, x_n) \\
&=\; (A+B+n+2) \frac{x_n^A \, |x'|^{B+2}}{|x|^{\Gamma+2}} - (\Gamma+2) \frac{x_n^A \, |x'|^{B+2}}{|x|^{\Gamma+2}} = (A+B+n-\Gamma) \frac{x_n^A \, |x'|^{B+2}}{|x|^{\Gamma+2}}.
\end{aligned}$$

Next we use the inequality

$$\int_{\mathbb{R}^n_+\cap B_1} div\mathbf{F}\,|v|\,\mathrm{d}x = -\int_{\mathbb{R}^n_+\cap B_1} \mathbf{F}\cdot\nabla|v|\,\mathrm{d}x \le \int_{\mathbb{R}^n_+\cap B_1} |\mathbf{F}|\,|\nabla v|\,\mathrm{d}x.$$

There isn't boundary term on $\{t=0\}$ due to the presence of the integrand $(x',\,x_n)\cdot(\mathbf{0},\,-1)=0$ and the assumptions on A, B, Γ. We then get

$$(A+B+n-\Gamma)\int_{\mathbb{R}^n_+\cap B_1} \frac{x_n^A\,|x'|^{B+2}}{|x|^{\Gamma+2}}\,|v|\,\mathrm{d}x \le \int_{\mathbb{R}^n_+\cap B_1} \frac{x_n^A\,|x'|^{B+1}}{|x|^{\Gamma}}\,|\nabla v|\,\mathrm{d}x.$$

The result follows combining this estimate with inequality (4.39). □

The following Lemma, is a non-trivial substitute of Lemma 4, in the case where $A+B+n=\Gamma$.

Lemma 5. *Let $n\ge 2$, $R>0$ and A, B be real numbers such that $A+1>0$, $B+n-1>0$. Then there holds the following inequality*

$$\frac{B+n-1}{A+B+n+1}\int_{\mathbb{R}^n_+\cap B_R} \frac{x_n^A\,|x'|^B\,X^2}{|x|^{A+B+n}}\,|v|\,\mathrm{d}x + \frac{2}{A+B+n+1}\int_{\mathbb{R}^n_+\cap B_R} \frac{x_n^A\,|x'|^{B+2}\,X^3}{|x|^{A+B+n+2}}\,|v|\,\mathrm{d}x$$
$$\le \int_{\mathbb{R}^n_+\cap B_R} \frac{x_n^A\,|x'|^{B+1}\,X}{|x|^{A+B+n}}\,|\nabla v|\,\mathrm{d}x, \quad \forall v\in C^1_0(B_R),$$

where $X=X(|x|/R)$, $X(\vartheta)=(1-\ln\vartheta)^{-1}$, $0<\vartheta\le 1$.

Proof. We can assume that $R=1$, since the inequality is scaling invariant. Moreover, to simplify the notation we abbreviate $\Gamma=A+B+n$. Integration by parts in x' variables yields

$$(B+1)\int_{\mathbb{R}^n_+\cap B_1} \frac{x_n^A\,|x'|^B\,X^2}{|x|^{\Gamma}}\,|v|\,\mathrm{d}x + 2\int_{\mathbb{R}^n_+\cap B_1} \frac{x_n^A\,|x'|^{B+2}\,X^3}{|x|^{\Gamma+2}}\,|v|\,\mathrm{d}x = \int_{\mathbb{R}^n_+\cap B_1} \frac{x_n^A}{|x|^{\Gamma}}\,|v|\,(\nabla|x'|\cdot\nabla_{x'}(X^2\,|x'|^{B+1}))\,\mathrm{d}x$$
$$= (2-n)\int_{\mathbb{R}^n_+\cap B_1} \frac{x_n^A\,|x'|^B\,X^2}{|x|^{\Gamma}}\,|v|\,\mathrm{d}x + \Gamma\int_{\mathbb{R}^n_+\cap B_1} \frac{x_n^A\,|x'|^{B+2}\,X^2}{|x|^{\Gamma+2}}\,|v|\,\mathrm{d}x - \int_{\mathbb{R}^n_+\cap B_1} \frac{x_n^A\,|x'|^{B+1}\,X^2}{|x|^{\Gamma}}\,(\nabla|x'|\cdot\nabla_{x'}|v|)\,\mathrm{d}x,$$

hence

$$(B+n-1)\int_{\mathbb{R}^n_+\cap B_1} \frac{x_n^A\,|x'|^B\,X^2}{|x|^{\Gamma}}\,|v|\,\mathrm{d}x + 2\int_{\mathbb{R}^n_+\cap B_1} \frac{x_n^A\,|x'|^{B+2}\,X^3}{|x|^{\Gamma+2}}\,|v|\,\mathrm{d}x$$
$$\le \Gamma\int_{\mathbb{R}^n_+\cap B_1} \frac{x_n^A\,|x'|^{B+2}\,X^2}{|x|^{\Gamma+2}}\,|v|\,\mathrm{d}x + \int_{\mathbb{R}^n_+\cap B_1} \frac{x_n^A\,|x'|^{B+1}\,X}{|x|^{\Gamma}}\,|\nabla v|\,\mathrm{d}x. \tag{4.40}$$

We will estimate the first term of the right hand side. Define the vector field

$$\mathbf{F} = \frac{x_n^A\,|x'|^{B+2}\,X}{|x|^{\Gamma+2}}\,(x',\,x_n).$$

Then we have

$$|\mathbf{F}| = \frac{x_n^A\,|x'|^{B+2}\,X}{|x|^{\Gamma+1}} \le \frac{x_n^A\,|x'|^{B+1}\,X}{|x|^{\Gamma}}$$

and

$$div\mathbf{F} = \frac{x_n^A\,|x'|^{B+2}\,X}{|x|^{\Gamma+2}}\,div(x',\,x_n) + \nabla\left(\frac{x_n^A\,|x'|^{B+2}\,X}{|x|^{\Gamma+2}}\right)\cdot(x',\,x_n) = n\,\frac{x_n^A\,|x'|^{B+2}\,X}{|x|^{\Gamma+2}}$$
$$+\left(\frac{x_n^A\,|x'|^{B}\,X}{|x|^{\Gamma+2}}\left[(B+2)+\frac{|x'|^2\,X}{|x|^2}-(\Gamma+2)\frac{|x'|^2}{|x|^2}\right]x',\,\frac{x_n^{A-1}\,|x'|^{B+2}\,X}{|x|^{\Gamma+2}}\left[A+\frac{x_n^2\,X}{|x|^2}-(\Gamma+2)\frac{x_n^2}{|x|^2}\right]\right)\cdot(x',\,x_n)$$
$$=(A+B+n+2)\frac{x_n^A\,|x'|^{B+2}\,X}{|x|^{\Gamma+2}}-(\Gamma+2)\frac{x_n^A\,|x'|^{B+2}\,X}{|x|^{\Gamma+2}}+\frac{x_n^A\,|x'|^{B+2}\,X^2}{|x|^{\Gamma+2}}$$
$$=(A+B+n-\Gamma)\frac{x_n^A\,|x'|^{B+2}\,X}{|x|^{\Gamma+2}}+\frac{x_n^A\,|x'|^{B+2}\,X^2}{|x|^{\Gamma+2}}=\frac{x_n^A\,|x'|^{B+2}\,X^2}{|x|^{\Gamma+2}}.$$

Next we apply the inequality

$$\int\limits_{\mathbb{R}^n_+\cap B_1} div\mathbf{F}\,|v|\,\mathrm{d}x = -\int\limits_{\mathbb{R}^n_+\cap B_1}\mathbf{F}\cdot\nabla|v|\,\mathrm{d}x \le \int\limits_{\mathbb{R}^n_+\cap B_1}|\mathbf{F}|\,|\nabla v|\,\mathrm{d}x,$$

to get

$$\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^A\,|x'|^{B+2}\,X^2}{|x|^{\Gamma+2}}\,|v|\,\mathrm{d}x \le \int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^A\,|x'|^{B+1}\,X}{|x|^{\Gamma}}\,|\nabla v|\,\mathrm{d}x.$$

The result follows combining this estimate with inequality (4.40). □

Actually, a similar argumentation to the proof of Lemma 5, leads to the following generalization.

Lemma 6. *Let $n \ge 2$, $R > 0$ and A, B, γ be real numbers such that $A+1>0$, $B+n-1>0$, $\gamma>1$. Then there holds the following inequality*

$$\frac{(B+n-1)(\gamma-1)}{A+B+n+\gamma-1}\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^A\,|x'|^B\,X^\gamma}{|x|^{A+B+n}}\,|v|\,\mathrm{d}x+\frac{\gamma(\gamma-1)}{A+B+n+\gamma-1}\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^A\,|x'|^{B+2}\,X^{\gamma+1}}{|x|^{A+B+n+2}}\,|v|\,\mathrm{d}x$$
$$\le\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^A\,|x'|^{B+1}\,X^{\gamma-1}}{|x|^{A+B+n}}\,|\nabla v|\,\mathrm{d}x,\quad \forall v\in C_0^1(B_R),$$

where $X = X(|x|/R)$, $X(\vartheta) = (1-\ln\vartheta)^{-1}$, $0<\vartheta\le 1$.

We pass now to the corresponding L^2 estimates.

Lemma 7. *Let $n \ge 2$, $R>0$ and A, B be real numbers such that $A+1>0$, $B+n-1>0$. Then there holds the following inequality*

$$\frac{(B+n-1)^2}{4(A+B+n+1)^2}\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^A\,|x'|^B\,X^2}{|x|^{A+B+n}}\,u^2\,\mathrm{d}x+\frac{B+n-1}{(A+B+n+1)^2}\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^A\,|x'|^{B+2}\,X^3}{|x|^{A+B+n+2}}\,u^2\,\mathrm{d}x$$
$$\le\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^A\,|x'|^{B+2}}{|x|^{A+B+n}}\,|\nabla u|^2\,\mathrm{d}x,\quad \forall u\in C_0^1(B_R),$$

where $X = X(|x|/R)$, $X(\vartheta) = (1-\ln\vartheta)^{-1}$, $0<\vartheta\le 1$.

Proof. We apply Lemma 5 to $\upsilon = u^2$, to get

$$\frac{B+n-1}{A+B+n+1}\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^A\,|x'|^B\,X^2}{|x|^{A+B+n}}\,u^2\,\mathrm{d}x+\frac{2}{A+B+n+1}\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^A\,|x'|^{B+2}\,X^3}{|x|^{A+B+n+2}}\,u^2\,\mathrm{d}x$$
$$\le\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{2\,x_n^A\,|x'|^{B+1}\,X}{|x|^{A+B+n}}\,|u|\,|\nabla u|\,\mathrm{d}x$$
$$\le\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{\epsilon\,x_n^A\,|x'|^B\,X^2}{|x|^{A+B+n}}\,u^2\,\mathrm{d}x+\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^A\,|x'|^{B+2}}{\epsilon|x|^{A+B+n}}\,|\nabla u|^2\,\mathrm{d}x.$$

In the last inequality we employed the Cauchy's inequality

$$ab\le\epsilon a^2+\frac{b^2}{4\epsilon},\qquad a>0,\ b>0,\ \epsilon>0. \tag{4.41}$$

The result follows setting $\epsilon=\frac{B+n-1}{2(A+B+n+1)}$. □

Lemma 8. *Let $n\ge 2$, $R>0$ and A, B be real numbers such that $A+1>0$, $B+n-1>0$. Then there holds the following inequality*

$$\frac{A+1}{A+B+n+1}\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^A\,|x'|^B\,X^2}{|x|^{A+B+n}}\,|\upsilon|\,\mathrm{d}x+\frac{2}{A+B+n+1}\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^{A+2}\,|x'|^B\,X^3}{|x|^{A+B+n+2}}\,|\upsilon|\,\mathrm{d}x$$
$$\le\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^{A+1}\,|x'|^B\,X}{|x|^{A+B+n}}\,|\nabla\upsilon|\,\mathrm{d}x,\quad\forall\upsilon\in C_0^1(B_R),$$

where $X=X(|x|/R)$, $X(\vartheta)=(1-\ln\vartheta)^{-1}$, $0<\vartheta\le 1$.

Proof. We can assume that $R=1$, since the inequality is invariant under scaling. For convenience in calculations we set $\varGamma=A+B+n$. Integration by parts in the x_n variable yields

$$(A+1)\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^A\,|x'|^B\,X^2}{|x|^{\varGamma}}\,|\upsilon|\,\mathrm{d}x=-2\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^{A+2}\,|x'|^B\,X^3}{|x|^{\varGamma+2}}\,|\upsilon|\,\mathrm{d}x$$
$$+(A+B+n)\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^{A+2}\,|x'|^B\,X^2}{|x|^{\varGamma+2}}\,|\upsilon|\,\mathrm{d}x-\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^{A+1}\,|x'|^B\,X^2}{|x|^{\varGamma}}\,|\upsilon|_{x_n}\,\mathrm{d}x,$$

hence

$$(A+1)\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^A\,|x'|^B\,X^2}{|x|^{\varGamma}}\,|\upsilon|\,\mathrm{d}x+2\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^{A+2}\,|x'|^B\,X^3}{|x|^{\varGamma+2}}\,|\upsilon|\,\mathrm{d}x$$
$$\le(A+B+n)\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^{A+2}\,|x'|^B\,X^2}{|x|^{\varGamma+2}}\,|\upsilon|\,\mathrm{d}x+\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^{A+1}\,|x'|^B\,X^2}{|x|^{\varGamma}}\,|\nabla\upsilon|\,\mathrm{d}x. \tag{4.42}$$

We will estimate the first term of the right hand side. Define the vector field

$$\mathbf{F} = \frac{x_n^{A+2}\,|x'|^B\,X}{|x|^{\Gamma+2}}\,(x',\,x_n).$$

Then we have

$$|\mathbf{F}| = \frac{x_n^{A+2}\,|x'|^B\,X}{|x|^{\Gamma+1}} \leq \frac{x_n^{A+1}\,|x'|^B\,X}{|x|^{\Gamma}}$$

and

$$\begin{aligned}
div\mathbf{F} &= \frac{x_n^{A+2}\,|x'|^B\,X}{|x|^{\Gamma+2}}\,div(x',\,x_n) + \nabla\left(\frac{x_n^{A+2}\,|x'|^B\,X}{|x|^{\Gamma+2}}\right)\cdot(x',\,x_n) = \frac{n\,x_n^{A+2}\,|x'|^B\,X}{|x|^{\Gamma+2}} \\
&+ \left(\frac{x_n^{A+2}\,|x'|^{B-2}\,X}{|x|^{\Gamma+2}}\left[B + \frac{|x'|^2\,X}{|x|^2} - (\Gamma+2)\frac{|x'|^2}{|x|^2}\right]x',\, \frac{x_n^{A+1}\,|x'|^B\,X}{|x|^{\Gamma+2}}\left[A+2+\frac{x_n^2\,X}{|x|^2} - (\Gamma+2)\frac{x_n^2}{|x|^2}\right]\right)\cdot(x',\,x_n) \\
&= (A+B+n+2)\frac{x_n^{A+2}\,|x'|^B\,X}{|x|^{\Gamma+2}} - (\Gamma+2)\frac{x_n^{A+2}\,|x'|^B\,X}{|x|^{\Gamma+2}} + \frac{x_n^{A+2}\,|x'|^B\,X^2}{|x|^{\Gamma+2}} \\
&= (A+B+n-\Gamma)\frac{x_n^{A+2}\,|x'|^B\,X}{|x|^{\Gamma+2}} + \frac{x_n^{A}\,|x'|^{B+2}\,X^2}{|x|^{\Gamma+2}} = \frac{x_n^{A+2}\,|x'|^B\,X^2}{|x|^{\Gamma+2}}.
\end{aligned}$$

Next we apply the inequality

$$\int\limits_{\mathbb{R}^n_+\cap B_1} div\mathbf{F}\,|v|\,\mathrm{d}x = -\int\limits_{\mathbb{R}^n_+\cap B_1}\mathbf{F}\cdot\nabla|v|\,\mathrm{d}x \leq \int\limits_{\mathbb{R}^n_+\cap B_1}|\mathbf{F}|\,|\nabla v|\,\mathrm{d}x,$$

to get

$$\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^{A+2}\,|x'|^B\,X^2}{|x|^{\Gamma+2}}\,|v|\,\mathrm{d}x \leq \int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^{A+1}\,|x'|^B\,X}{|x|^{\Gamma}}\,|\nabla v|\,\mathrm{d}x.$$

The result follows combining this estimate with inequality (4.42). □

Lemma 9. *Let $n \geq 2$, $R > 0$ and A, B be real numbers such that $A+1>0$, $B+n-1>0$. Then there holds the following inequality*

$$\frac{1}{4}\left(\frac{A+1}{A+B+n+1}\right)^2\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^{A}\,|x'|^B\,X^2}{|x|^{A+B+n}}\,u^2\,\mathrm{d}x \leq \int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^{A+2}\,|x'|^B}{|x|^{A+B+n}}\,|\nabla u|^2\,\mathrm{d}x,\quad \forall u\in C_0^1(B_R),$$

where $X = X(|x|/R)$, $X(\vartheta) = (1-\ln\vartheta)^{-1}$, $0<\vartheta\leq 1$.

Proof. We apply Lemma 8 to $v = u^2$ and then we employ Cauchy's inequality (4.41), to get

$$\begin{aligned}
\frac{A+1}{A+B+n+1}\int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^{A}\,|x'|^B\,X^2}{|x|^{A+B+n}}\,u^2\,\mathrm{d}x &\leq \int\limits_{\mathbb{R}^n_+\cap B_R}\frac{2x_n^{A+1}\,|x'|^B\,X}{|x|^{A+B+n}}\,|\nabla u||u|\,\mathrm{d}x, \\
\leq \int\limits_{\mathbb{R}^n_+\cap B_R}\frac{x_n^{A}\,|x'|^B\,X^2}{\epsilon|x|^{A+B+n}}\,u^2\,\mathrm{d}x &+ \int\limits_{\mathbb{R}^n_+\cap B_R}\frac{\epsilon x_n^{A+2}\,|x'|^B}{|x|^{A+B+n}}\,|\nabla u|^2\,\mathrm{d}x.
\end{aligned}$$

The result follows setting $\epsilon = 2\,\frac{A+B+n+1}{A+1}$. □

Lemma 10. *Let $n \geq 2$, $R > 0$ and A be a real number such that $A+1>0$. Then, there holds*

$$\frac{1}{8}\left(\frac{A+1}{2A+n+3}\right)^2 \int\limits_{\mathbb{R}^n_+\cap B_R} \frac{x_n^A\,|x'|^A\,X^2}{|x|^{2A+n}}\,u^2\,\mathrm{d}x \leq \int\limits_{\mathbb{R}^n_+\cap B_R} \frac{x_n^{A+2}\,|x'|^{A+2}}{|x|^{2A+n+2}}\,|\nabla u|^2\,\mathrm{d}x, \quad \forall u \in C_0^1(B_R),$$

where $X = X(|x|/R)$, $X(\vartheta) = (1-\ln\vartheta)^{-1}$, $0<\vartheta\leq 1$.

Proof. We apply Lemma 9 with $B = A+2$ there, to get

$$\frac{1}{4}\left(\frac{A+1}{2A+n+3}\right)^2 \int\limits_{\mathbb{R}^n_+\cap B_R} \frac{x_n^A\,|x'|^{A+2}\,X^2}{|x|^{2A+n+2}}\,u^2\,\mathrm{d}x \leq \int\limits_{\mathbb{R}^n_+\cap B_R} \frac{x_n^{A+2}\,|x'|^{A+2}}{|x|^{2A+n+2}}\,|\nabla u|^2\,\mathrm{d}x, \tag{4.43}$$

and similarly, we apply Lemma 7 with $A = B+2$ there, to obtain

$$\frac{(B+n-1)^2}{4(2B+n+3)^2} \int\limits_{\mathbb{R}^n_+\cap B_R} \frac{x_n^{B+2}\,|x'|^{B}\,X^2}{|x|^{2B+n+2}}\,u^2\,\mathrm{d}x \leq \int\limits_{\mathbb{R}^n_+\cap B_R} \frac{x_n^{B+2}\,|x'|^{B+2}}{|x|^{2B+n+2}}\,|\nabla u|^2\,\mathrm{d}x. \tag{4.44}$$

The result follows, setting $B = A$ in (4.44), and then adding (4.44) to (4.43). □

4.4 Hardy type remainder terms

Next we proceed with the proof of Theorem III, which we restate here:

Theorem 5. *Let $\alpha \in (-1,1)$, $2-\alpha \leq b < n$ and U be a bounded domain in $\mathbb{R}^n$. Then there exists a constant $C>0$, depending only on n, α, such that for all $u \in C_0^\infty(U)$ there holds*

$$K(n,\alpha,b) \int\limits_{\partial\mathbb{R}^n_+\cap U} \frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' + \frac{(\alpha+b-2)^2}{4} \int\limits_{\mathbb{R}^n_+\cap U} \frac{x_n^\alpha u^2}{|x|^2}\,\mathrm{d}x + C \int\limits_{\mathbb{R}^n_+\cap U} \frac{X^2}{|x|^{2-\alpha}}\,u^2\,\mathrm{d}x \leq \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x, \tag{4.45}$$

where $X = X(|x|/d)$, $X(\vartheta) = (1-\ln\vartheta)^{-1}$, $0<\vartheta\leq 1$, $d = \sup_{x\in\mathbb{R}^n_+\cap U}|x|$. The exponent 2 of the weight function cannot be improved.

Proof of Theorem 5. As in the proof of Theorem 2, we arrive at the following estimate

$$K(n,\alpha,b) \int\limits_{\partial\mathbb{R}^n_+\cap U} \frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' + \frac{(\alpha+b-2)^2}{4} \int\limits_{\mathbb{R}^n_+\cap U} \frac{x_n^\alpha\,u^2}{|x|^2}\,\mathrm{d}x + \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla \frac{u}{\psi}|^2\,\psi^2\,\mathrm{d}x = \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x, \tag{4.46}$$

valid for all $u \in C_0^\infty(U\setminus\{0\})$. The third term in the left hand side, yields the correction term in (4.45). Therefore, it remains to show that there exists a positive constant $C = C(n,\alpha)$ such that for all $u \in C_0^\infty(U\setminus\{0\})$ there holds

$$C \int\limits_{\mathbb{R}^n_+\cap U} \frac{X^2\,u^2}{|x|^{2-\alpha}}\,\mathrm{d}x \leq \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\,|\nabla \frac{u}{\psi}|^2\,\psi^2\,\mathrm{d}x. \tag{4.47}$$

Note that $U \subseteq B_R$, with $R = d$. Moreover, taking into account (3.38) and setting $u = \upsilon\,\psi$, we conclude that (4.47) will follow on its turn, after establishing the following inequality

$$C \int\limits_{\mathbb{R}^n_+ \cap B_R} \frac{X^2\,\upsilon^2}{|x|^n}\,\mathrm{d}x \le \int\limits_{\mathbb{R}^n_+ \cap B_R} \frac{x_n^\alpha\,|\nabla \upsilon|^2}{|x|^{\alpha+n-2}}\,\mathrm{d}x, \quad \upsilon \in C_0^\infty(U \setminus \{0\}), \tag{4.48}$$

for some positive constant $C = C(n,\alpha)$, independent of R. Now (4.48) follows by Lemma 10 with $A = 0$ there. For $-1 < \alpha < 0$, inequality (4.48) also follows by Lemma 7 with $A = B = 0$.

Next, to complete the proof of the Theorem, we will verify that the weight function X^2 cannot be replaced by a smaller power of X. More precisely, we will show that there are no constants $0 < \epsilon < 2$, $c > 0$, such that the following inequality is valid

$$\begin{aligned} K(n,\alpha,b) \int\limits_{\partial\mathbb{R}^n_+ \cap U} \frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' \;+\; \frac{(\alpha+b-2)^2}{4} \int\limits_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha\,u^2}{|x|^2}\,\mathrm{d}x + c \int\limits_{\mathbb{R}^n_+ \cap U} \frac{X^{2-\epsilon}}{|x|^{2-\alpha}}\,u^2\,\mathrm{d}x \\ \le \int\limits_{\mathbb{R}^n_+ \cap U} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x, \quad \forall u \in C_0^\infty(U). \end{aligned}$$

Note also that it suffices to prove the claim, only for the case $0 < \epsilon < 1$, since $X^{2-\epsilon_0}(\vartheta) > X^{2-\epsilon}(\vartheta)$, $\forall \epsilon_0 > \epsilon$, $\vartheta \in (0,1]$.

The result will follow after showing that there exists a sequence $\{u_l\}_{l=0}^\infty \subset C_0^\infty(U)$, such that

$$\frac{\int\limits_{\mathbb{R}^n_+ \cap U} x_n^\alpha\,|\nabla u_l|^2\,\mathrm{d}x - K(n,\alpha,b) \int\limits_{\partial\mathbb{R}^n_+ \cap U} \frac{u_l^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' - \frac{(\alpha+b-2)^2}{4} \int\limits_{\mathbb{R}^n_+ \cap U} \frac{x_n^\alpha\,u_l^2}{|x|^2}\,\mathrm{d}x}{\int\limits_{\mathbb{R}^n_+ \cap U} \frac{X^{2-\epsilon}}{|x|^{2-\alpha}}\,u_l^2\,\mathrm{d}x} \xrightarrow{m\to\infty} 0.$$

Notice that U contains a ball B_r centered at the origin, and without loss of generality we can assume that $r = 1$. Furthermore, noting that $\psi \sim |x|^{-\frac{n+\alpha-2}{2}}$, in $\mathbb{R}^n_+$ (see (3.38)) and making the change of variable $u_l = \upsilon_l\,\psi$, it is sufficient to show that there exists a sequence $\{\upsilon_l\} \subset C_0^\infty(U \setminus \{0\})$ such that (cf. (4.46), (4.48))

$$J[\upsilon_l] \;:=\; \frac{N[\upsilon_l]}{D[\upsilon_l]} \;:=\; \frac{\int\limits_{\mathbb{R}^n_+ \cap B_1} \frac{x_n^\alpha\,|\nabla \upsilon_l|^2}{|x|^{n-2+\alpha}}\,\mathrm{d}x}{\int\limits_{\mathbb{R}^n_+ \cap B_1} \frac{X^{2-\epsilon}\,\upsilon_l^2}{|x|^n}\,\mathrm{d}x} \longrightarrow 0, \ \text{ as } \ l \to \infty. \tag{4.49}$$

Let us recall the notation $B_r^+ = B_r \cap \mathbb{R}^n_+$, abbreviate $V(x) = \frac{X^{2-\epsilon}}{|x|^n}$, $w(x) = \frac{x_n^\alpha}{|x|^{n+\alpha-2}}$ and define the space $D_0^{1,2}(B_1, w(x)\mathrm{d}x)$ as the completion of $C_0^\infty(\overline{\mathbb{R}^n_+} \cap B_1)$ with respect to the norm $\|\upsilon\| = (\int_{B_1^+} |\nabla \upsilon|^2\,w(x)\,\mathrm{d}x)^{1/2}$. Then, by a standard approximation, it suffices to fix a sequence $\{\upsilon_l\} \subset D_0^{1,2}(B_1, w(x)\mathrm{d}x)$ with $\int_{B_1^+} V(x)\,\upsilon_l^2\,\mathrm{d}x < \infty$, such that $J[\upsilon_l] \to 0$, as $l \to \infty$.

To this end, we choose δ such that $0 < \epsilon < \delta < 1$, which eventually will be sent to ϵ, we set $R_m = e^{1-m}$ so that

$$\frac{1}{m} \le X(|x|) \le 1 \Leftrightarrow R_m \le |x| \le 1, \quad m = 1,2,3,\ldots$$

and define the functions f_m as follows

$$f_m(x) = \begin{cases} X^{\frac{\delta-1}{2}}(|x|), & R_m \le |x| \le 1, \\ m^{\frac{3-\delta}{2}}\,X(|x|), & |x| \le R_m, \end{cases} \quad \text{hence} \ \ \nabla f_m(x) = \begin{cases} \left(\frac{\delta-1}{2}\right) X^{\frac{\delta+1}{2}}(|x|)\,\frac{x}{|x|^2}, & R_m < |x| \le 1, \\ m^{\frac{3-\delta}{2}}\,X^2(|x|)\,\frac{x}{|x|^2}, & 0 < |x| < R_m. \end{cases}$$

Then, we have

$$D[f_m] = \int\limits_{B_1^+ \setminus B_{R_m}^+} \frac{X^{\delta-\epsilon+1}}{|x|^n}\,\mathrm{d}x \,+\, m^{3-\delta} \int\limits_{B_{R_m}^+} \frac{X^{4-\epsilon}}{|x|^n}\,\mathrm{d}x \;=:\; D_1(m) + D_2(m)$$

and

$$N[f_m] = \left(\frac{\delta-1}{2}\right)^2 \int\limits_{B_1^+ \setminus B_{R_m}^+} \frac{x_n^\alpha X^{\delta+1}(|x|)}{|x|^{n+\alpha}}\,\mathrm{d}x \,+\, m^{3-\delta} \int\limits_{B_{R_m}^+} \frac{x_n^\alpha X^4(|x|)}{|x|^{n+\alpha}}\,\mathrm{d}x \;=:\; N_1(m) + N_2(m).$$

We will next estimate the terms D_1, D_2, N_1, N_2, using polar coordinates . More precisely, making the change of variable

$$t \;=\; X(r), \quad \text{thus} \quad \mathrm{d}t \;=\; \frac{X^2(r)}{r}\,\mathrm{d}r$$

and setting $\mathcal{C}_{n,\alpha} = \int_{\mathbb{S}_+^{n-1}} x_n^\alpha \,\mathrm{d}\sigma(x)$, $\gamma_n = \int_{\mathbb{S}_+^{n-1}} 1 \,\mathrm{d}\sigma(x)$, we have

$$\begin{aligned}
D_1(m) &= \int\limits_{\mathbb{S}_+^{n-1}} 1\,\mathrm{d}\sigma \int\limits_{R_m}^{1} \frac{X^{\delta-\epsilon+1}(r)}{r}\,\mathrm{d}r = \gamma_n \int\limits_{1/m}^{1} t^{\delta-\epsilon-1}\,\mathrm{d}t = \frac{\gamma_n\,(1-m^{\epsilon-\delta})}{\delta-\epsilon},\\
D_2(m) &= m^{3-\delta} \int\limits_{\mathbb{S}_+^{n-1}} 1\,\mathrm{d}\sigma \int\limits_{0}^{R_m} \frac{X^{4-\epsilon}(r)}{r}\,\mathrm{d}r = \gamma_n\, m^{3-\delta} \int\limits_{0}^{1/m} t^{2-\epsilon}\,\mathrm{d}t = \frac{\gamma_n\, m^{\epsilon-\delta}}{3-\epsilon},\\
N_1(m) &= \left(\frac{\delta-1}{2}\right)^2 \int\limits_{\mathbb{S}_+^{n-1}} x_n^\alpha\,\mathrm{d}\sigma(x) \int\limits_{R_m}^{1} \frac{X^{\delta+1}(r)}{r}\,\mathrm{d}r = \mathcal{C}_{n,\alpha} \left(\frac{\delta-1}{2}\right)^2 \int\limits_{1/m}^{1} t^{2\delta-1}\,\mathrm{d}t\\
&= \mathcal{C}_{n,\alpha} \left(\frac{\delta-1}{2}\right)^2 \frac{(1-m^{-\delta})}{\delta} = \mathcal{C}_{n,\alpha}\,(\delta-1)^2\,\frac{1-m^{-\delta}}{4\delta},\\
N_2(m) &= m^{3-\delta} \int\limits_{\mathbb{S}_+^{n-1}} x_n^\alpha\,\mathrm{d}\sigma(x) \int\limits_{0}^{R_m} \frac{X^4(r)}{r}\,\mathrm{d}r = \mathcal{C}_{n,\alpha}\, m^{3-\delta} \int\limits_{0}^{1/m} t^2\,\mathrm{d}t = \frac{\mathcal{C}_{n,\alpha}\, m^{-\delta}}{3}.
\end{aligned}$$

We conclude that

$$J[f_m] = \frac{\mathcal{C}_{n,\alpha}}{\gamma_n}\;\frac{(\delta-1)^2\,\frac{1-m^{-\delta}}{4\delta} + \frac{m^{-\delta}}{3}}{\frac{1-m^{\epsilon-\delta}}{\delta-\epsilon} + \frac{m^{\epsilon-\delta}}{3-\epsilon}}.$$

We then take a sequence $\delta_l \searrow \epsilon$ and choose m_l sufficiently large so that $m_l^{\epsilon-\delta_l} < 1/2$. It follows that $J[f_{m_l}] \to 0$, as $l \to \infty$.

Given now a function $\eta \in C_0^\infty(U)$, which is constant, not zero, in a neighbourhood of the origin, it is straightforward to verify that the sequence $\upsilon_l = f_{m_l}\,\eta$, satisfies $J[\upsilon_l] \to 0$, as $l \to \infty$, that is the condition (4.49). $\square$

4.5 Inequalities with more general weights

In this section we generalize the results obtained in the previous sections. We start with the following extension of Proposition I.

Theorem 6. *Let $n \geq 3$ and α, β real numbers such that $\alpha \in (-1,1)$ and $\alpha+\beta+n-2>0$. Then for all $u \in W^{1,2}(\mathbb{R}^n_+,\, x_n^\alpha\,|x'|^\beta\,\mathrm{d}x)$ there holds*

$$H(n,\,\alpha,\,\beta)\int\limits_{\partial\mathbb{R}^n_+}\frac{u^2(x',0)}{|x'|^{1-\alpha-\beta}}\,\mathrm{d}x' \leq \int\limits_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,|\nabla u|^2\,\mathrm{d}x, \tag{4.50}$$

where

$$H(n,\,\alpha,\,\beta) = (1-\alpha)\frac{\Gamma^2(\frac{n-\alpha+\beta}{4})\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{3-\alpha}{2})\Gamma^2(\frac{\alpha+\beta+n-2}{4})}.$$

The constant $H(n,\alpha,\,\beta)$ is sharp.

The main ingredient in the proof of these Theorems is the consideration of the solution ϕ for the corresponding Euler Lagrange equations

$$\begin{cases} div(x_n^\alpha\,|x'|^\beta\nabla\phi)=0, \text{ in } \mathbb{R}^n_+, \\ \lim\limits_{x_n\to 0}\frac{x_n^\alpha\,|x'|^\beta}{\phi(x',x_n)}\frac{\partial\phi(x',x_n)}{\partial x_n} = -H(n,\alpha,\beta)\frac{1}{|x'|^{1-\alpha-\beta}}. \end{cases} \tag{4.51}$$

To this aim we study the following boundary value problem

$$(s+s^3)\,g''(s)+[(\alpha+2)s^2+\alpha]\,g'(s)+\frac{\alpha-\beta+4-n}{2}\frac{\alpha+\beta+n-2}{2}\,s\,g(s)=0, \tag{4.52}$$

with the boundary conditions

$$g(0)=1 \tag{4.53}$$

and

$$\lim_{s\to\infty} s^{\frac{n-2+\alpha+\beta}{2}}g(s) \text{ exists.} \tag{4.54}$$

For later use, notice that multiplying by $s^{\alpha-1}$ equation (4.52) can be written in divergence form

$$(s^\alpha(1+s^2)g'(s))'+\frac{\alpha-\beta+4-n}{2}\frac{\alpha+\beta+n-2}{2}\,s^\alpha\,g(s)=0. \tag{4.55}$$

Using the change of variables $z=-s^2$ and defining the new unknown so that $\omega(z)=g(s)$, we have

$$\frac{\mathrm{d}g}{\mathrm{d}s}-2s\omega',\ \frac{d^2g}{ds^2}=-2\frac{d\omega}{dz}+4s^2\frac{d^2\omega}{dz^2}.$$

Then equation (4.52) becomes

$$z(1-z)\omega''+[\frac{\alpha+1}{2}-\frac{\alpha+3}{2}z]\omega'-\frac{\alpha-\beta+4-n}{4}\frac{\alpha+\beta+n-2}{4}\omega=0. \tag{4.56}$$

Equation (4.56) belongs to the class of hypergeometric equations and the general solution can be expressed in terms of hypergeometric functions (see [1, section 15]) in the neighborhood of the singular point z=0 :

$$\begin{aligned}\omega(z) &= c_1\, F(\frac{\alpha-\beta+4-n}{4}, \frac{\alpha+\beta+n-2}{4}, \frac{\alpha+1}{2}; z) \\ &\quad + c_2\, z^{\frac{1-\alpha}{2}}\, F(\frac{6-n-\alpha}{4}, \frac{n-\alpha}{4}, \frac{3-\alpha}{2}; z),\end{aligned}$$

hence

$$\begin{aligned}g(s) &= c_1\, F(\frac{\alpha-\beta+4-n}{4}, \frac{\alpha+\beta+n-2}{4}, \frac{\alpha+1}{2}; -s^2) \\ &\quad + c_2\, s^{1-\alpha}\, e^{\frac{i\pi(1-\alpha)}{2}}\, F(\frac{6-n-\alpha}{4}, \frac{n-\alpha}{4}, \frac{3-\alpha}{2}; -s^2).\end{aligned}$$

Condition (4.53) implies that $c_1 = 1$. The constant c_2 will be evaluated by the condition (4.54). To this aim we will use formula 15.3.7 in [1] to arrive at

$$\begin{aligned}g(s) &= \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{n-3+\beta}{2})}{\Gamma^2(\frac{n-2+\alpha+\beta}{4})} s^{\frac{n-4-\alpha+\beta}{2}}\, F(\frac{\alpha+4-n-\beta}{4}, \frac{6-n-\alpha-\beta}{4}, \frac{5-n-\beta}{2}; -\frac{1}{s^2}) \\ &+ \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{3-n-\beta}{2})}{\Gamma^2(\frac{\alpha+4-n-\beta}{4})} s^{\frac{2-n-\alpha-\beta}{2}}\, F(\frac{\alpha+n-2+\beta}{4}, \frac{n-\alpha+\beta}{4}, \frac{n+\beta-1}{2}; -\frac{1}{s^2}) \\ &+ c_2\, \frac{\Gamma(\frac{3-\alpha}{2})\Gamma(\frac{n-3+\beta}{2})}{\Gamma^2(\frac{n-\alpha+\beta}{4})} e^{\frac{i\pi(1-\alpha-\beta)}{2}}\, s^{\frac{\beta-\alpha+n-4}{2}}\, F(\frac{6-n-\alpha-\beta}{4}, \frac{\alpha-\beta-n+4}{4}, \frac{5-n-\beta}{2}; -\frac{1}{s^2}) \\ &+ c_2\, \frac{\Gamma(\frac{3-\alpha}{2})\Gamma(\frac{3-n-\beta}{2})}{\Gamma(\frac{6-n-\alpha}{4})\Gamma(\frac{6-n-\alpha-\beta}{4})} e^{\frac{i\pi(1-\alpha-\beta)}{2}}\, s^{\frac{2-\alpha-\beta-n}{2}}\, F(\frac{n-\alpha+\beta}{4}, \frac{\alpha+\beta+n-2}{4}, \frac{n+\beta-1}{2}; -\frac{1}{s^2}).\end{aligned}$$

Then condition (4.54) yields

$$c_2 = -e^{-\frac{i\pi(1-\alpha-\beta)}{2}} \frac{\Gamma^2(\frac{n-\alpha+\beta}{4})\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{3-\alpha}{2})\Gamma^2(\frac{\alpha+\beta+n-2}{4})}.$$

With this choice of c_2 we have

$$g(s) = O(s^{\frac{2-n-\alpha-\beta}{2}}), \text{ as } s \to \infty, \tag{4.57}$$

and in particular

$$\lim_{s\to\infty} s^{\frac{\alpha+\beta}{2}} g(s) = 0. \tag{4.58}$$

Next we compute the limit

$$H(n, \alpha, \beta) := \lim_{s\to 0^+} -s^{\alpha} g'(s).$$

Using the differentiation formula 15.2.1 in [1] we obtain

$$\begin{aligned}g'(s) &= -2s\, \frac{(\alpha-\beta+4-n)(\alpha+\beta+n-2)}{8(\alpha+1)}\, F(\frac{\alpha-\beta+8-n}{4}, \frac{\alpha+\beta+n+2}{4}, \frac{\alpha+3}{2}; -s^2) + \\ &\quad -(1-\alpha)c_2\, s^{-\alpha}\, e^{\frac{i\pi(1-\alpha)}{2}}\, F(\frac{6-n-\alpha}{4}, \frac{n-\alpha}{4}, \frac{3-\alpha}{2}; -s^2) + \\ &\quad -2c_2\, s^{2-\alpha} \frac{(6-n-\alpha)(n-\alpha)}{8(3-\alpha)}\, e^{\frac{i\pi(1-\alpha)}{2}}\, F(\frac{10-n+\alpha}{4}, \frac{n-\alpha}{4}, \frac{3-\alpha}{2}; -s^2).\end{aligned}$$

We then have

$$H(n,\,\alpha,\,\beta)=\lim_{s\to 0^+}-s^\alpha g'(s)=(1-\alpha)\frac{\Gamma^2(\frac{n-\alpha+\beta}{4})\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{3-\alpha}{2})\Gamma^2(\frac{\alpha+\beta+n-2}{4})}. \tag{4.59}$$

Set now

$$\phi(x',x_n)=|x'|^{-\frac{n-2+\alpha+\beta}{2}}g(\frac{x_n}{|x'|}),\ x'\in\mathbb{R}^{n-1},\ x_n\ge 0,\ (x',x_n)\neq(0,0). \tag{4.60}$$

Note that in $\{(x',x_n)\,:\,x'=0,\,x_n>0\}$, $\phi(x',x_n)$ is well defined due to the condition (4.54). Then ϕ satisfies equations (4.51). Indeed the boundary condition in (4.51) is a consequence of (4.53) and (4.59), while $div(x_n^\alpha\,|x'|^\beta\nabla\phi)=0$ is equivalent to (4.52), with $s=\frac{x_n}{|x'|}$. Moreover, gathering conditions (4.53) and (4.57) we obtain

$$g(s)\sim(1+s^2)^{-\frac{n-2+\alpha+\beta}{4}},\ s>0$$

thus

$$\phi\sim|x|^{-\frac{n-2+\alpha+\beta}{2}}\ \text{in}\ \mathbb{R}^n_+. \tag{4.61}$$

Proof of Theorem 6. By a standard density argument it suffices to prove the result for $u\in C_0^\infty(\mathbb{R}^n)$. Moreover, by approximation we can suppose $u\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$. Indeed, let $\epsilon>0$, $\mathrm{C}_\epsilon=\{(x',x_n)\in\mathbb{R}^n:|x'|<\epsilon,\,|x_n|<\epsilon\}$ and consider the functions $u_\epsilon=u\eta_\epsilon$, where $u\in C_0^\infty(\mathbb{R}^n)$, $\eta_\epsilon\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$, $\eta_\epsilon\equiv 1$ in $supp\,u\setminus\mathrm{C}_\epsilon$ and $|\nabla\eta_\epsilon|\le c/\epsilon$. Then we have

$$\int_{\partial\mathbb{R}^n_+}\frac{u_\epsilon^2}{|x'|^{1-\alpha-\beta}}\,\mathrm{d}x'\to\int_{\partial\mathbb{R}^n_+}\frac{u^2}{|x'|^{1-\alpha-\beta}}\,\mathrm{d}x',\ \text{ as }\ \epsilon\to 0,$$

by the Lebesgue dominated theorem and similarly

$$\int_{\mathbb{R}^n_+}x_n^\alpha\,|x'|^\beta\,\eta_\epsilon^2|\nabla u|^2\,\mathrm{d}x\to\int_{\mathbb{R}^n_+}x_n^\alpha\,|x'|^\beta\,|\nabla u|^2\,\mathrm{d}x,\ \text{ as }\ \epsilon\to 0.$$

Moreover we have

$$\int_{\mathbb{R}^n_+}x_n^\alpha\,|x'|^\beta\,|\nabla\eta_\epsilon|^2|u|^2\,\mathrm{d}x\le c\epsilon^{\alpha+\beta+n-2}\to 0,\ \text{ as }\ \epsilon\to 0,$$

hence

$$\int_{\mathbb{R}^n_+}x_n^\alpha\,|x'|^\beta\,|\nabla u_\epsilon|^2\,\mathrm{d}x\to\int_{\mathbb{R}^n_+}x_n^\alpha\,|x'|^\beta\,|\nabla u|^2\,\mathrm{d}x,\ \text{ as }\ \epsilon\to 0.$$

Following the approach in [25] (see also [26]), we expand the square and integrate by parts, to get

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,|\nabla u - \frac{\nabla\phi}{\phi}\,u|^2\,\mathrm{d}x = \\
&\int_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,|\nabla u|^2\,\mathrm{d}x + \int_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,|\nabla\phi|^2\left(\frac{u}{\phi}\right)^2\mathrm{d}x - \int_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,\nabla u^2\cdot\frac{\nabla\phi}{\phi}\,\mathrm{d}x = \\
&\int_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta|\nabla u|^2\,\mathrm{d}x + \int_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,|\nabla\phi|^2\left(\frac{u}{\phi}\right)^2\mathrm{d}x + \int_{\mathbb{R}^n_+} u^2\,div\left(x_n^\alpha\,|x'|^\beta\,\frac{\nabla\phi}{\phi}\right)\mathrm{d}x \\
&+\int_{\partial\mathbb{R}^n_+}\lim_{x_n\to 0}\frac{x_n^\alpha\,|x'|^\beta\,u^2}{\phi}\frac{\partial\phi(x',\,x_n)}{\partial x_n}\,\mathrm{d}x' = \\
&\int_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,|\nabla u|^2\,\mathrm{d}x - H(n,\alpha,\,\beta)\int_{\partial\mathbb{R}^n_+}\frac{u^2}{|x'|^{1-\alpha-\beta}}\,\mathrm{d}x'.
\end{aligned}
\tag{4.62}
$$

In the last equation we used equations (4.51). Notice that on $supp\,u$, ϕ does not vanish, so the function $\upsilon = \frac{u}{\phi}$ is well defined. Actually $\upsilon \in C_0^\infty(\mathbb{R}^n\setminus\{0\})$. Then equation (4.62) is equivalent to

$$
H(n,\alpha,\,\beta)\int_{\partial\mathbb{R}^n_+}\frac{u^2}{|x'|^{1-\alpha-\beta}}\,\mathrm{d}x' + \int_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,|\nabla\frac{u}{\phi}|^2\,\phi^2\,\mathrm{d}x = \int_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,|\nabla u|^2\,\mathrm{d}x \tag{4.63}
$$

and the result follows immediately.

In order to verify the optimality of the constant $H(n,\alpha,\,\beta)$ we define for a function $u \in W^{1,2}(\mathbb{R}^n_+)$ the quotient

$$
Q[u] := \frac{\int_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,|\nabla u|^2\,\mathrm{d}x}{\int_{\partial\mathbb{R}^n_+}\frac{u^2}{|x'|^{1-\alpha-\beta}}\,\mathrm{d}x'} = \frac{Q_1[u]}{Q_2[u]}. \tag{4.64}
$$

We will show that there exist functions $u_\epsilon \in W^{1,2}(\mathbb{R}^n_+)$ such that

$$
\lim_{\epsilon\to 0^+} Q[u_\epsilon] = H(n,\alpha,\,\beta).
$$

Let $\delta > 0$, and $\eta \in C_0^1(\mathrm{C}_{2\delta})$, such that $\eta \equiv 1$ in C_δ. We define

$$
u_\epsilon(x', x_n) = \begin{cases} \eta(x)\,\phi(x', x_n), & x_n \ge \epsilon,\\ \eta(x)\,\phi(x',\epsilon), & 0 \le x_n \le \epsilon. \end{cases}
$$

We firstly estimate the denominator $Q_2[u_\epsilon]$. To this aim we set $B'_r = \{x' \in \mathbb{R}^{n-1} : |x'| < 1\}$. Then we have

$$
\begin{aligned}
Q_2[u_\epsilon] &= \int_{B'_\delta}\frac{\eta^2(x',0)\,\phi^2(x',\epsilon)}{|x'|^{1-\alpha-\beta}}\,\mathrm{d}x' + \int_{B'_{2\delta}\setminus B'_\delta}\frac{\eta^2(x',0)\,\phi^2(x',\epsilon)}{|x'|^{1-\alpha-\beta}}\,\mathrm{d}x' = \int_0^\delta\int_{B'_r}\frac{\phi^2(x',\epsilon)}{r^{1-\alpha-\beta}}\,\mathrm{d}\sigma(x')\,\mathrm{d}r + O(1)\\
&= \int_0^\delta g^2\left(\frac{\epsilon}{r}\right)\frac{1}{r}\,\mathrm{d}r + O(1) = \int_{\epsilon/\delta}^\infty g^2(s)\frac{1}{s}\,\mathrm{d}s + O(1),
\end{aligned}
\tag{4.65}
$$

as $\epsilon \to 0$. As regards the numerator $Q_1[u_\epsilon]$, taking into account that $\eta \equiv 1$ in C_δ it follows that

$$
\begin{aligned}
Q_1[u_\epsilon] &= \int\limits_{\{0\le x_n\le\epsilon\}} x_n^\alpha\,|x'|^\beta\,|\nabla(\eta(x)\phi(x',\epsilon))|^2\,\mathrm{d}x + \int\limits_{\{\epsilon\le x_n\le\delta\}} x_n^\alpha\,|x'|^\beta\,|\nabla(\eta\phi)|^2\,\mathrm{d}x \\
&= \int\limits_{\{\epsilon\le x_n\le\delta\}} x_n^\alpha\,|x'|^\beta\,|\nabla(\eta\phi)|^2\,\mathrm{d}x + O(1) \\
&= \int\limits_{\mathrm{C}_\delta\cap\{\epsilon\le x_n\le\delta\}} x_n^\alpha\,|x'|^\beta\,|\nabla\phi|^2\,\mathrm{d}x + O(1),
\end{aligned}
\tag{4.66}
$$

as $\epsilon \to 0$. In view of (4.60), we have

$$
\int\limits_{\mathrm{C}_\delta\cap\{\epsilon\le x_n\le\delta\}} x_n^\alpha\,|x'|^\beta\,|\nabla\phi|^2\,\mathrm{d}x = \int\limits_\epsilon^\delta\int\limits_0^\delta\int\limits_{\partial B_r'} \eta^2(x')\frac{x_n^\alpha}{r^{\alpha+n}}\left(\frac{(\alpha+n-2)^2}{4}g^2\left(\frac{x_n}{r}\right)\right.
$$

$$
\left. + \frac{x_n^2}{r^2}\,g'^2\left(\frac{x_n}{r}\right) + \frac{\alpha+n-2}{r}\,x_n\,g\left(\frac{x_n}{r}\right)g'\left(\frac{x_n}{r}\right) + g'^2\left(\frac{x_n}{r}\right)\right)\mathrm{d}\sigma(x')\,\mathrm{d}r\,\mathrm{d}x_n =
$$

$$
\int\limits_\epsilon^\delta\int\limits_0^\delta \frac{x_n^\alpha}{r^{\alpha+2}}\left(\frac{(\alpha+n-2)^2}{4}g^2\left(\frac{x_n}{r}\right) + \frac{x_n^2}{r^2}\,g'^2\left(\frac{x_n}{r}\right) + (\alpha+n-2)\,\frac{x_n}{r}\,g\left(\frac{x_n}{r}\right)g'\left(\frac{x_n}{r}\right) + g'^2\left(\frac{x_n}{r}\right)\right)\mathrm{d}r\,\mathrm{d}x_n =
$$

$$
\int\limits_\epsilon^\delta\int\limits_{x_n/\delta}^\infty \frac{s^\alpha}{x_n}\left(\frac{(\alpha+n-2)^2}{4}g^2(s) + (1+s^2)\,g'^2(s) + (\alpha+n-2)\,s\,g(s)g'(s)\right)\mathrm{d}s\,\mathrm{d}x_n. \tag{4.67}
$$

In the last equality we used the change of variable $r = x_n/s$. Making now partial integration we have

$$
(\alpha+n-2)\int\limits_{x_n/\delta}^\infty s^{\alpha+1}\,g(s)g'(s)\,\mathrm{d}s = \frac{\alpha+n-2}{2}\int\limits_{x_n/\delta}^\infty s^{\alpha+1}\,\left(g^2(s)\right)'\,\mathrm{d}s =
$$

$$
-\frac{(\alpha+1)(\alpha+n-2)}{2}\int\limits_{x_n/\delta}^\infty s^\alpha\,g^2(s)\,\mathrm{d}s + \frac{a+n-2}{2}\left[s^{\alpha+1}\,g^2(s)\right]_{s=\frac{x_n}{\delta}}^\infty,
$$

which by virtue of (4.57), yields

$$
\begin{aligned}
(\alpha+n-2)\int\limits_\epsilon^\delta\int\limits_{x_n/\delta}^\infty s^{\alpha+1}\,g(s)g'(s)\,\mathrm{d}s\,\mathrm{d}x_n \quad &= \quad -\frac{(\alpha+1)(\alpha+n-2)}{2}\int\limits_\epsilon^\delta\int\limits_{x_n/\delta}^\infty s^\alpha\,g^2(s)\,\mathrm{d}s \\
& mathrmdx_n + O(1),
\end{aligned}
$$

as $\epsilon \to 0$. Substitute this estimate to (4.67), hence (4.66) becomes

$$
Q_1[u_\epsilon] = \int_\epsilon^\delta \frac{1}{x_n}\int\limits_{x_n/\delta}^\infty s^\alpha(1+s^2)\,g'^2(s) - \frac{\alpha+\beta+n-2}{2}\,\frac{\alpha-\beta-n+4}{2}\,s^\alpha\,g^2(s)\,\mathrm{d}s\,\mathrm{d}x_n + O(1), \tag{4.68}
$$

as $\epsilon \to 0$. Next we make again integration by parts in the s variable and then we use equation (4.55). Then (4.68) yields

$$\begin{aligned} Q_1[u_\epsilon] &= -\int_\epsilon^\delta \frac{1}{x_n} \int\limits_{x_n/\delta}^{\infty} \left(s^\alpha (1+s^2)\, g'(s) \right)' g(s) + \frac{\alpha+\beta+n-2}{2} \frac{\alpha-\beta-n+4}{2} s^\alpha\, g^2(s)\, \mathrm{d}s\, \mathrm{d}x_n \\ &\quad - \int_\epsilon^\delta \frac{1}{x_n} \left(\frac{x_n}{\delta}\right)^\alpha \left(1 + \left(\frac{x_n}{\delta}\right)^2\right) g'\left(\frac{x_n}{\delta}\right) g\left(\frac{x_n}{\delta}\right) \mathrm{d}x_n + O(1) \\ &= -\int_\epsilon^\delta \frac{1}{x_n} \left(\frac{x_n}{\delta}\right)^\alpha \left(1 + \left(\frac{x_n}{\delta}\right)^2\right) g'\left(\frac{x_n}{\delta}\right) g\left(\frac{x_n}{\delta}\right) \mathrm{d}x_n + O(1), \end{aligned}$$

as $\epsilon \to 0$. We make now the change of variable $s = \frac{x_n}{\delta}$, to concude

$$Q_1[u_\epsilon] = -\int\limits_{\epsilon/\delta}^{1} s^{\alpha-1} \left(1+s^2\right) g'(s)\, g(s)\, \mathrm{d}s + O(1), \tag{4.69}$$

as $\epsilon \to 0$. Finally, gathering estimates (4.65) and (4.69) and taking into account (4.59) and (4.53) we obtain

$$\lim_{\epsilon\to 0} Q[u_\epsilon] = \lim_{\epsilon \to 0} \frac{-\int\limits_{\epsilon/\delta}^{1} s^{\alpha-1}\left(1+s^2\right) g'(s)\, g(s)\, \mathrm{d}s + O(1)}{\int\limits_{\epsilon/\delta}^{\infty} g^2(s) \frac{1}{s}\, \mathrm{d}s + O(1)} = \lim_{t\to 0} \frac{-t^\alpha \left(1+t^2\right) g'(t)}{g(t)} = H(n,\alpha,\beta).$$

□

Theorem 7. *Let α, β real numbers such that $\alpha \in (-1,1)$ and $\alpha+\beta+n-2>0$. Then for all $u \in C_0^\infty(U)$ there holds*

$$H(n,\alpha,\beta) \int\limits_{\partial\mathbb{R}^n_+ \cap U} \frac{u^2\, \mathrm{d}x'}{|x'|^{1-\alpha-\beta}} + \frac{1}{4} \int\limits_{\mathbb{R}^n_+\cap U} \frac{x_n^\alpha\, |x'|^\beta X^2\, u^2}{|x|^2}\, \mathrm{d}x \le \int\limits_{\mathbb{R}^n_+\cap U} x_n^\alpha\, |x'|^\beta\, |\nabla u|^2\, \mathrm{d}x, \tag{4.70}$$

where $X = X(|x|/d)$. The constant $\frac{1}{4}$ is optimal.

Proof of Theorem 7. By a standard density argument it suffices to prove the result for $u \in C_0^\infty(B_1)$. Moreover, by approximation we can suppose $u \in C_0^\infty(B_1 \setminus \{0\})$. Indeed, let $\epsilon > 0$, $\mathrm{C}_\epsilon = \{x \in \mathbb{R}^n : \epsilon < |x| < 1-\epsilon\}$ and consider the functions $u_\epsilon = u\eta_\epsilon$, where $u \in C_0^\infty(B_1)$, $\eta_\epsilon \in C_0^\infty(B_1\setminus\{0\})$, $\eta_\epsilon \equiv 1$ in C_ϵ and $|\nabla \eta_\epsilon| \le c/\epsilon$. Then, by the Lebesgue dominated theorem, we have

$$\int\limits_{\partial\mathbb{R}^n_+\cap B_1} \frac{u_\epsilon^2}{|x'|^{1-\alpha-\beta}}\, \mathrm{d}x' \to \int\limits_{\partial\mathbb{R}^n_+\cap B_1} \frac{u^2}{|x'|^{1-\alpha-\beta}}\, \mathrm{d}x', \ \text{ as } \ \epsilon \to 0 \tag{4.71}$$

and

$$\int\limits_{\mathbb{R}^n_+\cap B_1} \frac{x_n^\alpha\, |x'|^\beta\, X^2}{|x|^2}\, u_\epsilon^2\, \mathrm{d}x \to \int\limits_{\mathbb{R}^n_+\cap B_1} \frac{x_n^\alpha\, |x'|^\beta X^2}{|x|^2}\, u^2\, \mathrm{d}x, \ \text{ as } \ \epsilon \to 0. \tag{4.72}$$

Similarly we have

$$\int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\, |x'|^\beta\, \eta_\epsilon^2\, |\nabla u|^2\, \mathrm{d}x \to \int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\, |x'|^\beta\, |\nabla u|^2\, \mathrm{d}x, \ \text{ as } \ \epsilon \to 0.$$

Moreover, taking into account $|\nabla\eta_\epsilon| \le c/\epsilon$, we get

$$\int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\,|x'|^\beta|\nabla\eta_\epsilon|^2|u|^2\,\mathrm{d}x \le c(n,\alpha,\beta)\epsilon^{n-2+\alpha+\beta} \to 0, \text{ as } \epsilon\to 0,$$

hence

$$\int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\,|x'|^\beta\,|\nabla u_\epsilon|^2\,\mathrm{d}x \to \int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\,|x'|^\beta\,|\nabla u|^2\,\mathrm{d}x, \text{ as } \epsilon\to 0. \tag{4.73}$$

Gathering (4.71), (4.72), (4.73) we conclude that it suffices to prove the result for $u \in C_0^\infty(B_1\setminus\{0\})$.

Thus, in the sequel we suppose that $u\in C_0^\infty(B_1\setminus\{0\})$. We define the function

$$\psi(x) \;=\; \phi(x)X^{-1/2}(r),\, r=|x|.$$

We will show that ψ satisfies the equations

$$\begin{cases} div(x_n^\alpha\,|x'|^\beta\nabla\psi) + \frac{x_n^\alpha\,|x'|^\beta X^2\,\psi}{4|x|^2} = 0, & \text{in } \mathbb{R}^n_+\cap B_1,\\ \lim\limits_{x_n\to 0} \frac{x_n^\alpha\,|x'|^\beta}{\psi(x',x_n)}\,\frac{\partial\psi(x',x_n)}{\partial x_n} = -H(n,\alpha,\beta)\frac{1}{|x'|^{1-\alpha-\beta}}. \end{cases} \tag{4.74}$$

Indeed, we have

$$\nabla\psi = X^{-1/2}(r)\nabla\phi - \frac{1}{2r^2}\phi X^{1/2}(r)x,$$

hence

$$\begin{aligned}\Delta\psi &= X^{-1/2}\Delta\phi - \frac{X^{1/2}}{r^2}\nabla\phi\cdot x - \left[\frac{1}{4r^2}X^{3/2} + \frac{n-2}{2r^2}X^{1/2}\right]\phi\\ &= X^{-1/2}\Delta\phi - \frac{X^{1/2}}{r^2}(\nabla\phi\cdot x + \frac{n-2}{2}\phi) - \frac{1}{4r^2}X^{3/2}\phi.\end{aligned}$$

Then it follows that

$$\begin{aligned} &div(x_n^\alpha\,|x'|^\beta\,\nabla\psi) + \frac{x_n^\alpha\,|x'|^\beta\,X^2\,\psi}{4r^2} = x_n^\alpha\,|x'|^\beta\,\Delta\psi + \alpha x_n^{\alpha-1}\,|x'|^\beta\,\psi_{x_n} + \beta x_n^\alpha\,|x'|^{\beta-2}\,\nabla_{\mathbf{x}'}\psi\cdot\mathbf{x}' + \frac{x_n^\alpha\,|x'|^\beta X^2\,\psi}{4r^2}\\ &= x_n^\alpha|x'|^\beta X^{-1/2}\Delta\phi - x_n^\alpha|x'|^\beta\frac{X^{1/2}}{r^2}(\nabla\phi\cdot x + \frac{n-2}{2}\phi) - \frac{x_n^\alpha|x'|^\beta}{4r^2}X^{3/2}\phi + \alpha x_n^{\alpha-1}|x'|^\beta\phi_{x_n}X^{-1/2}\phi X^{1/2}\\ &- \frac{\alpha x_n^\alpha|x'|^\beta}{2r^2} + \beta x_n^\alpha\,|x'|^{\beta-1}\,X^{-1/2}\nabla_{\mathbf{x}'}\phi\cdot\mathbf{x}' - \frac{\beta\, x_n^\alpha\,|x'|^\beta\,X^{1/2}}{2r^2}\phi + \frac{x_n^\alpha|x'|^\beta X^{3/2}\,\phi}{4r^2}\\ &= X^{-\frac{1}{2}}(x_n^\alpha|x'|^\beta\Delta\phi + \alpha x_n^{\alpha-1}|x'|^\beta\phi_{x_n} - \beta\,x_n^\alpha\,|x'|^{\beta-2}\nabla_{\mathbf{x}'}\phi\cdot\mathbf{x}') - \frac{x_n^\alpha\,|x'|^\beta\,X^{1/2}}{r^2}(\nabla\phi\cdot x + \frac{n-2+\alpha+\beta}{2}\phi) = 0.\end{aligned}$$

In the last equation we used that $div(x_n^\alpha\,|x'|^\beta\,\nabla\phi) = 0$ as well as $\nabla\phi\cdot x + \frac{n-2+\alpha+\beta}{2}\phi = 0$. The latter can be directly shown utilizing (4.60).

Following the approach in [27], we expand the square and integrate by parts to obtain

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,|\nabla u - \frac{\nabla\phi}{\phi}\,u|^2\,\mathrm{d}x = \\
&\int\limits_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,|\nabla u|^2\,\mathrm{d}x + \int\limits_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,|\nabla\phi|^2\left(\frac{u}{\phi}\right)^2\mathrm{d}x - \int\limits_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,\nabla u^2\cdot\frac{\nabla\phi}{\phi}\,\mathrm{d}x = \\
&\int\limits_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta|\nabla u|^2\,\mathrm{d}x + \int\limits_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,|\nabla\phi|^2\left(\frac{u}{\phi}\right)^2\mathrm{d}x + \int\limits_{\mathbb{R}^n_+} u^2\,div\left(x_n^\alpha\,|x'|^\beta\,\frac{\nabla\phi}{\phi}\right)\mathrm{d}x \\
&+\int\limits_{\partial\mathbb{R}^n_+}\lim_{x_n\to 0}\frac{x_n^\alpha\,|x'|^\beta\,u^2}{\phi}\frac{\partial\phi(x',\,x_n)}{\partial x_n}\,\mathrm{d}x' = \\
&\int\limits_{\mathbb{R}^n_+} x_n^\alpha\,|x'|^\beta\,|\nabla u|^2\,\mathrm{d}x - H(n,\alpha,\,\beta)\int\limits_{\partial\mathbb{R}^n_+}\frac{u^2}{|x'|^{1-\alpha-\beta}}\,\mathrm{d}x' - \int\limits_{\mathbb{R}^n_+}\frac{x_n^\alpha\,|x'|^\beta X^2(|x|)\,u^2}{|x|^2}\,\mathrm{d}x.
\end{aligned}
\tag{4.75}
$$

In the last equation we used equations (4.74). Then inequality (4.70) follows directly.

Next we will verify the optimality of the constant $\frac{1}{4}$. Let $\varepsilon > 0$, $\varepsilon_1 > 0$. Then it is sufficient to show that there exist functions $u_{\varepsilon,\varepsilon_1} \in W_0^{1,\,2}(B_1,\,x_n^\alpha|x'|^\beta\,\mathrm{d}x)$ such that

$$
\frac{\int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\,|x'|^\beta|\nabla u_{\varepsilon,\varepsilon_1}|^2\,\mathrm{d}x - H(n,\alpha,\beta)\int\limits_{\partial\mathbb{R}^n_+\cap B_1}\frac{u^2_{\varepsilon,\varepsilon_1}}{|x'|^{1-\alpha-\beta}}\,\mathrm{d}x'}{\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^\alpha\,|x'|^\beta X^2(|x|)\,u^2_{\varepsilon,\varepsilon_1}}{|x|^2}\,\mathrm{d}x} \overset{\varepsilon,\varepsilon_1\to 0}{\longrightarrow} \frac{1}{4}.
$$

Setting $u_{\varepsilon,\varepsilon_1} = \psi\,v_{\varepsilon,\varepsilon_1}$ and utilizing equation (4.75) we have

$$
\frac{\int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\,|x'|^\beta|\nabla u_{\varepsilon,\varepsilon_1}|^2\,\mathrm{d}x - H(n,\alpha,\beta)\int\limits_{\partial\mathbb{R}^n_+\cap B_1}\frac{u^2_{\varepsilon,\varepsilon_1}}{|x'|^{1-\alpha-\beta}}\,\mathrm{d}x'}{\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^\alpha\,|x'|^\beta X^2(|x|)\,u^2_{\varepsilon,\varepsilon_1}}{|x|^2}\,\mathrm{d}x} = \frac{\int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\,|x'|^\beta\psi^2|\nabla v_{\varepsilon,\varepsilon_1}|^2\,\mathrm{d}x}{\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^\alpha\,|x'|^\beta X^2(|x|)\,\psi^2\,v^2_{\varepsilon,\varepsilon_1}}{|x|^2}\,\mathrm{d}x} + \frac{1}{4}.
$$

Hence it is sufficient to show that there exist functions $v_{\varepsilon,\varepsilon_1}$ such that $\psi\,v_{\varepsilon,\varepsilon_1} \in W_0^{1,2}(B_1,\,x_n^\alpha|x'|^\beta\,\mathrm{d}x)$ and

$$
\frac{\int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\,|x'|^\beta\psi^2|\nabla v_{\varepsilon,\varepsilon_1}|^2\,\mathrm{d}x}{\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^\alpha\,|x'|^\beta\,X^2(|x|)\,\psi^2\,v^2_{\varepsilon,\varepsilon_1}}{|x|^2}\,\mathrm{d}x} \overset{\varepsilon,\varepsilon_1\to 0}{\longrightarrow} 0. \tag{4.76}
$$

Let $0 < \delta < 1$. We set $v_{\varepsilon,\varepsilon_1}(x) = r^\varepsilon X^{\varepsilon_1}\eta(x)$, $r = |x|$, where $\eta \in C_0^\infty(B_{2\delta})$ with $\eta \equiv 1$ in B_δ. In the sequel we will show that $v_{\varepsilon,\varepsilon_1}$ satisfy condition (4.76).

First, we estimate the denominator in (4.76). Since there exist constants c_1, c_2 such that

$$
c_1|x|^{-\frac{n-2+\alpha+\beta}{2}} \le \phi(x) \le c_2|x|^{-\frac{n-2+\alpha+\beta}{2}}, \text{ in } \mathbb{R}^n_+\cap B_1,
$$

we calculate

$$\begin{aligned}\int\limits_{\mathbb{R}^n_+\cap B_1} \frac{x_n^\alpha\,|x'|^\beta\,X^2\,\psi^2\,\upsilon^2_{\varepsilon,\varepsilon_1}}{|x|^2}\,\mathrm{d}x &= \int\limits_{\mathbb{R}^n_+\cap B_{2\delta}} \frac{x_n^\alpha\,|x'|^\beta\,X^{1+2\varepsilon_1}\,\psi^2\eta^2}{|x|^{2-2\varepsilon}}\,\mathrm{d}x = c\int\limits_{\mathbb{R}^n_+\cap B_\delta} \frac{x_n^\alpha\,|x'|^\beta X^{1+2\varepsilon_1}}{|x|^{n-2\varepsilon+\alpha+\beta}}\,\mathrm{d}x + O_{\varepsilon,\varepsilon_1}(1)\\ &= C(n,\alpha,\beta)\int_0^\delta \frac{X^{1+2\varepsilon_1}(r)}{r^{1-2\varepsilon}}\,\mathrm{d}r + O_{\varepsilon,\varepsilon_1}(1).\end{aligned}$$

Therefore

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^n_+\cap B_1} \frac{x_n^\alpha\,|x'|^\beta\,X^2(|x|)\,\psi^2\,\upsilon^2_{\varepsilon,\varepsilon_1}}{|x|^2}\,\mathrm{d}x = C\int_0^\delta \frac{X^{1+2\varepsilon_1}(r)}{r}\,\mathrm{d}r + O_{\varepsilon_1}(1) = \frac{C}{2\varepsilon_1}X^{2\varepsilon_1}(\delta) + O_{\varepsilon_1}(1). \tag{4.77}$$

Concerning the numerator in (4.76), we have

$$\begin{aligned}&\int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\,|x'|^\beta\,\psi^2\,|\nabla\upsilon_{\varepsilon,\varepsilon_1}|^2\,\mathrm{d}x = \int\limits_{\mathbb{R}^n_+\cap B_\delta} x_n^\alpha\,|x'|^\beta\,\psi^2\,|\nabla(|x|^\varepsilon X^{\varepsilon_1})|^2\,\mathrm{d}x + O(1)\\ &\le c_2\int\limits_{\mathbb{R}^n_+\cap B_\delta} x_n^\alpha\,|x'|^\beta\,\frac{X^{-1}}{|x|^{n-2+\alpha+\beta}}\,\left|\nabla(|x|^\varepsilon X^{\varepsilon_1})\right|^2\,\mathrm{d}x + O_{\varepsilon,\varepsilon_1}(1)\\ &= c(n,\alpha,\beta)\int_0^\delta \frac{X^{2\varepsilon_1-1}}{r^{1-2\varepsilon}}\,\left(\varepsilon^2+\varepsilon_1^2X^2+2\varepsilon\varepsilon_1X\right)\,\mathrm{d}r + O_{\varepsilon,\varepsilon_1}(1) = c(n,\alpha,\beta)(I_1+I_2+I_3) + O_{\varepsilon,\varepsilon_1}(1).\end{aligned} \tag{4.78}$$

To estimate the first term I_1, we integrate by parts to get

$$\begin{aligned}I_1 &= \varepsilon^2\int_0^\delta \frac{X^{2\varepsilon_1-1}}{r^{2\varepsilon-1}}\,\mathrm{d}r = \frac{\varepsilon}{2}\int_0^\delta X^{2\varepsilon_1-1}(r^{2\varepsilon})'\,\mathrm{d}r = -\varepsilon\left(-\frac{1}{2}+\varepsilon_1\right)\int_0^\delta X^{2\varepsilon_1}r^{2\varepsilon-1}\,\mathrm{d}r + O_{\varepsilon,\varepsilon_1}(1)\\ &= -\frac{1}{2}\left(-\frac{1}{2}+\varepsilon_1\right)\int_0^\delta X^{2\varepsilon_1}(r^{2\varepsilon})'\,\mathrm{d}r + O_{\varepsilon,\varepsilon_1}(1) = \varepsilon_1\left(-\frac{1}{2}+\varepsilon_1\right)\int_0^\delta X^{2\varepsilon_1+1}r^{2\varepsilon-1}\,\mathrm{d}r + O_{\varepsilon,\varepsilon_1}(1).\end{aligned} \tag{4.79}$$

Similarly we have

$$I_3 = 2\varepsilon\varepsilon_1\int_0^\delta X^{2\varepsilon_1}r^{2\varepsilon-1}\,\mathrm{d}r = \varepsilon_1\int_0^\delta X^{2\varepsilon_1}(r^{2\varepsilon})'\,\mathrm{d}r = -2\varepsilon_1^2\int_0^\delta X^{2\varepsilon_1+1}r^{2\varepsilon-1}\,\mathrm{d}r + O_{\varepsilon,\varepsilon_1}(1). \tag{4.80}$$

Combining (4.78), (4.79), (4.80) we obtain

$$\int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\,|x'|^\beta\,\psi^2\,|\nabla\upsilon_{\varepsilon,\varepsilon_1}|^2\,\mathrm{d}x = -\frac{c\varepsilon_1}{2}\int_0^\delta X^{2\varepsilon_1+1}r^{2\varepsilon-1}\,\mathrm{d}r + O_{\varepsilon,\varepsilon_1}(1),$$

thus

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\,|x'|^\beta\,\psi^2\,|\nabla\upsilon_{\varepsilon,\varepsilon_1}|^2\,\mathrm{d}x = -\frac{c}{4}X^{2\varepsilon_1}(\delta) = O_{\varepsilon_1}(1). \tag{4.81}$$

Finally combining (4.77) with (4.81) we conclude that

$$\lim_{\varepsilon_1\to 0}\lim_{\varepsilon\to 0}\frac{\int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\,|x'|^\beta\,\psi^2|\nabla\upsilon_{\varepsilon,\varepsilon_1}|^2\,\mathrm{d}x}{\int\limits_{\mathbb{R}^n_+\cap B_1}\frac{x_n^\alpha\,|x'|^\beta\,X^2(|x|)\,\psi^2\,\upsilon^2_{\varepsilon,\varepsilon_1}}{|x|^2}\,\mathrm{d}x} = 0.$$

□

Chapter 5

Trace remainder terms

In this chapter, we will give the proof of Theorems VI, VII. Before proceed to the proofs, let us fix some notation which will be used. We recall that $B'_r = \{x' \in \mathbb{R}^{n-1} : |x'| < r\}$ and we define the sphere $\mathbb{S}^{n-2} = \partial B'_1 = \{\theta' \in \mathbb{R}^{n-1} : \ |\theta'| = 1\}$. Moreover, $\int_{\partial B'_r} u \, \mathrm{d}\sigma(x')$, and $\int_{\mathbb{S}^{n-2}} u \, \mathrm{d}\sigma$ denote the $(n-2)-$dimensional Lebesgue integral of the function u, over $\partial B'_r$ and $\mathbb{S}^{n-2}$ respectively. We also retain the notation that has been introduced in the previous chapters.

We are now ready to start with the proof of Theorem VI which we restate here.

Theorem 8. *Let $\alpha \in (-1,1)$, $2-\alpha \le b < n$ and Ω be a bounded domain in $\mathbb{R}^{n-1}$. Then there exists a constant $C > 0$, depending only on n, α, such that for all $u \in C_0^\infty(\mathbb{R}^n)$, with $u(x',0) = 0$, for $x' \notin \Omega$, there holds*

$$K(n,\alpha,b) \int\limits_{\Omega} \frac{u^2}{|x'|^{1-\alpha}} \, \mathrm{d}x' \quad + \quad \frac{(\alpha+b-2)^2}{4} \int\limits_{\mathbb{R}^n_+} \frac{x_n^\alpha \, u^2}{|x|^2} \, \mathrm{d}x + C \left(\int\limits_{\Omega} X^{\frac{2n-3+\alpha}{n-2+\alpha}} \, |u|^{\frac{2(n-1)}{n-2+\alpha}} \, \mathrm{d}x' \right)^{\frac{n-2+\alpha}{n-1}}$$
$$\le \int\limits_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x, \tag{5.1}$$

where $X = X(|x'|/D)$, $X(\vartheta) = (1-\ln\vartheta)^{-1}$, $0 < \vartheta \le 1$, $D = \sup_{x' \in \Omega} |x'|$. The weight $X^{\frac{2n-3+\alpha}{n-2+\alpha}}$ cannot be replaced by a smaller power of X.

proof of Theorem 8. For the reader's convenience, we divide the proof in two parts. Firstly we will establish inequality (5.1) and next we will prove the optimality of the power of the logarithmic weight X. A key role, in both parts of the proof, will play the function ψ, defined by (3.27).

Part I: Derivation of the estimate (5.1). By standard approximation, it suffices to prove the result for functions u with the further restriction $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ (cf. (4.2), (4.3), (4.4)).

Working as in Theorem 1, we arrive at the following equality

$$K(n,\alpha,b) \int\limits_{\Omega} \frac{u^2}{|x'|^{1-\alpha}} \, \mathrm{d}x' + \frac{(\alpha+b-2)^2}{4} \int\limits_{\mathbb{R}^n_+} \frac{x_n^\alpha \, u^2}{|x|^2} \, \mathrm{d}x + \int\limits_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla \frac{u}{\psi}|^2 \, \psi^2 \, \mathrm{d}x = \int\limits_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x,$$

valid for any $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, with $u(x',0) = 0$, for $x' \notin \Omega$. Therefore it remains to show that for such functions u, there exists a positive constant $C = C(n,\alpha)$ such that the following inequality is valid

$$C \left(\int\limits_{\Omega} X^{\frac{2n-3+\alpha}{n-2+\alpha}} \, |u|^{\frac{2(n-1)}{n-2+\alpha}} \, \mathrm{d}x' \right)^{\frac{n-2+\alpha}{n-1}} \le \int\limits_{\mathbb{R}^n_+} x_n^\alpha \, |\nabla \frac{u}{\psi}|^2 \, \psi^2 \, \mathrm{d}x. \tag{5.2}$$

Taking into account that $\psi \sim |x|^{-\frac{n-2+\alpha}{2}}$ in $\mathbb{R}^n_+$ (cf. (3.38)) and making the change of variable $u = \upsilon\,\psi$, we conclude that (5.2) will follow after establishing the following inequality

$$c\left(\int\limits_{\Omega} \frac{X^{\frac{2n-3+\alpha}{n-2+\alpha}}(|x'|/D)}{|x'|^{n-1}}\ |\upsilon|^{\frac{2(n-1)}{n-2+\alpha}}\,\mathrm{d}x'\right)^{\frac{n-2+\alpha}{n-1}} \le \int\limits_{\mathbb{R}^n_+} \frac{x_n^{\alpha}}{|x|^{n-2+\alpha}}\ |\nabla\upsilon|^2\,\mathrm{d}x,\quad \forall \upsilon\in C_0^{\infty}(\mathbb{R}^n), \tag{5.3}$$

with $\upsilon(x',0)=0$, for $x'\notin\Omega$ and $c = c(n,\alpha) > 0$.

Note that for such υ there exists $R \ge D$, depending on υ, such that $\upsilon\in C_0^{\infty}(B_R)$, where we denote by B_R the ball $B_R = \{x\in\mathbb{R}^n : |x|\le R\}$. Therefore, (5.3) will follow on its turn after showing the existence of a positive constant C depending only on n and α, such that for all $\upsilon\in C_0^{\infty}(B_R)$, with $\upsilon(x',0)=0,\ |x'|\ge D,$ where $0<D\le R$, there holds

$$C\left(\int\limits_{\partial\mathbb{R}^n_+\cap B_R} \frac{X^{\frac{2n-3+\alpha}{n-2+\alpha}}(|x'|/D)}{|x'|^{n-1}}\ |\upsilon|^{\frac{2(n-1)}{n-2+\alpha}}\,\mathrm{d}x'\right)^{\frac{n-2+\alpha}{n-1}} \le \int\limits_{\mathbb{R}^n_+\cap B_R} \frac{x_n^{\alpha}}{|x|^{n-2+\alpha}}\ |\nabla\upsilon|^2\,\mathrm{d}x. \tag{5.4}$$

To this aim, we study the minimization problem

$$C_{n,\alpha} = \inf_{\substack{\upsilon\in C_0^{\infty}(B_R)\\ \upsilon|_{B'_R}\not\equiv 0,\ \upsilon(x',0)=0,\ |x'|\ge D}} I[\upsilon],\quad \text{where}\quad I[\upsilon] = \frac{\int\limits_{\mathbb{R}^n_+\cap B_R} \frac{x_n^{\alpha}}{|x|^{n-2+\alpha}}\ |\nabla\upsilon|^2\,\mathrm{d}x}{\left(\int\limits_{\{|x'|\le D\}} \frac{X^{\frac{2n-3+\alpha}{n-2+\alpha}}(|x'|/D)}{|x'|^{n-1}}\ |\upsilon|^{\frac{2(n-1)}{n-2+\alpha}}\,\mathrm{d}x'\right)^{\frac{n-2+\alpha}{n-1}}} = \frac{I_1[\upsilon]}{I_2[\upsilon]}.$$

We will compare the constant $C_{n,\alpha}$ with the weighted trace Sobolev constant $s_{n,\alpha}$, which depends only on n and α and it is defined by

$$s_{n,\alpha} = \inf_{\substack{\upsilon\in C_0^{\infty}(B_1)\\ \upsilon|_{B'_1}\not\equiv 0}} Q[\upsilon],\quad \text{where}\quad Q[\upsilon] = \frac{\int\limits_{\mathbb{R}^n_+\cap B_1} x_n^{\alpha}\ |\nabla\upsilon|^2\,\mathrm{d}x}{\left(\int\limits_{\partial\mathbb{R}^n_+\cap B_1} |\upsilon|^{\frac{2(n-1)}{n-2+\alpha}}\,\mathrm{d}x'\right)^{\frac{n-2+\alpha}{n-1}}} = \frac{Q_1[\upsilon]}{Q_2[\upsilon]}. \tag{5.5}$$

We express the numerator of the quotient $Q[\upsilon]$ in terms of polar coordinates, writing $\upsilon(x) = \upsilon(r,\theta)$, where

$$r = |x|,\quad \theta = \frac{x}{|x|}\in\mathbb{S}^{n-1}_+.$$

Then we make the change of $r-$variable, setting

$$t = r^{2-n-\alpha}\quad \text{and}\quad \upsilon(r,\theta) = h(t,\theta),$$

thus

$$\mathrm{d}r = -\frac{r^{n-1+\alpha}}{n-2+\alpha}\,\mathrm{d}t,\quad \upsilon_r = -\frac{n-2+\alpha}{r^{n-1+\alpha}}\,h_t,\quad r = t^{-\frac{1}{n-2+\alpha}}.$$

After straightforward manipulations we obtain (cf. (4.9))

$$Q_1[\upsilon] = (n-2+\alpha)\int\limits_1^{\infty}\int\limits_{\mathbb{S}^{n-1}_+} \cos^{\alpha}\varphi\ \left(h_t^2 + (n-2+\alpha)^{-2}\,t^{-2}\,|\nabla_{\theta}h|^2\right)\,\mathrm{d}\sigma\,\mathrm{d}t. \tag{5.6}$$

Next we will express the denominator $Q_2[v]$, in terms of polar coordinates in $\mathbb{R}^{n-1}$. We have

$$\begin{aligned}
Q_2[v] &= \left(\int\limits_{B_1'} |v(x',0)|^{\frac{2(n-1)}{n-2+\alpha}}\,\mathrm{d}x'\right)^{\frac{n-2+\alpha}{n-1}} = \left(\int\limits_0^1\int\limits_{\partial B_r'} |v(x',0)|^{\frac{2(n-1)}{n-2+\alpha}}\,\mathrm{d}\sigma(x')\,\mathrm{d}r\right)^{\frac{n-2+\alpha}{n-1}}\\
&= \left(\int\limits_0^1\int\limits_{\mathbb{S}^{n-2}} r^{n-2}\,|v|^{\frac{2(n-1)}{n-2+\alpha}}\,\mathrm{d}\sigma\,\mathrm{d}r\right)^{\frac{n-2+\alpha}{n-1}}\\
&= \frac{1}{(n-2+\alpha)^{\frac{n-2+\alpha}{n-1}}}\left(\int\limits_1^\infty\int\limits_{\mathbb{S}^{n-2}} t^{-\frac{2n-3+\alpha}{n-2+\alpha}}\,|h|^{\frac{2(n-1)}{n-2+\alpha}}\,\mathrm{d}\sigma\,\mathrm{d}t\right)^{\frac{n-2+\alpha}{n-1}}.
\end{aligned}$$

Therefore we have

$$(n-2+\alpha)^{\frac{3-2n-\alpha}{n-1}}s_{n,\alpha} = \inf_{\substack{h\in C^\infty([1,\infty)\times\mathbb{S}_+^{n-1})\\ h(1,\theta)=0}} \frac{\int\limits_1^\infty\int\limits_{\mathbb{S}_+^{n-1}}\cos^\alpha\varphi\,\left(h_t^2+((n-2+\alpha)\,t)^{-2}\,|\nabla_\theta h|^2\right)\,\mathrm{d}\sigma\mathrm{d}t}{\left(\int\limits_1^\infty\int\limits_{\mathbb{S}^{n-2}} t^{-\frac{2n-3+\alpha}{n-2+\alpha}}\,|h|^{\frac{2(n-1)}{n-2+\alpha}}\,\mathrm{d}\sigma\,\mathrm{d}t\right)^{\frac{n-2+\alpha}{n-1}}}. \tag{5.7}$$

Similarly, we transform the terms of the quotient $I[v]$ by means of polar coordinates and then we make the change of r-variable, setting

$$t=\frac{1}{X(r/D)} = 1-\ln\left(\frac{r}{D}\right),\quad v(r,\theta)=w(t,\theta),\quad \text{thus}\quad \mathrm{d}r\;=\;-r\,\mathrm{d}t\quad\text{and}\quad v_r\;=\;-\frac{1}{r}\,w_t.$$

Then for any $v\in C_0^\infty(B_R)$, we have

$$\begin{aligned}
I_1[v] &= \int\limits_{\mathbb{R}_+^n\cap B_R}\frac{x_n^\alpha\,|\nabla v|^2}{|x|^{n-2+\alpha}}\,\mathrm{d}x = \int\limits_0^R\int\limits_{\partial B_r\cap\mathbb{R}_+^n}\frac{x_n^\alpha\,|\nabla v(x)|^2}{r^{n-2+\alpha}}\,\mathrm{d}\sigma(x)\,\mathrm{d}r = \int\limits_0^R\int\limits_{\mathbb{S}_+^{n-1}} r\,\cos^\alpha\varphi\,|\nabla v|^2\,\mathrm{d}\sigma\,\mathrm{d}r\\
&= \int\limits_0^R\int\limits_{\mathbb{S}_+^{n-1}} r\,\cos^\alpha\varphi\,\left(v_r^2+\frac{1}{r^2}\,|\nabla_\theta v|^2\right)\,\mathrm{d}\sigma\,\mathrm{d}r\\
&= \int\limits_0^R\int\limits_{\mathbb{S}_+^{n-1}} r\,\cos^\alpha\varphi\,\left(\frac{1}{r^2}w_t^2+\frac{1}{r^2}\,|\nabla_\theta w|^2\right)\,\mathrm{d}\sigma\,\mathrm{d}r\\
&= \int\limits_{1-\ln(\frac{R}{D})}^\infty\int\limits_{\mathbb{S}_+^{n-1}}\cos^\alpha\varphi\,\left(w_t^2+|\nabla_\theta w|^2\right)\,\mathrm{d}\sigma\,\mathrm{d}t.
\end{aligned}\tag{5.8}$$

Similarly for the denominator I_2, for any $v \in C_0^\infty(B_R)$, with $v(x',0)=0$, if $|x'| \geq D$, we have

$$
\begin{aligned}
I_2[v] &= \left(\int\limits_{\{|x'|\leq D\}} \frac{X^{\frac{2n-3+\alpha}{n-2+\alpha}}(|x'|/D)}{|x'|^{n-1}} \, |v(x',0)|^{\frac{2(n-1)}{n-2+\alpha}} \, \mathrm{d}x' \right)^{\frac{n-2+\alpha}{n-1}} \\
&= \left(\int\limits_0^D \int\limits_{\partial B'_r} \frac{X^{\frac{2n-3+\alpha}{n-2+\alpha}}(r/D)}{r^{n-1}} \, |v(x',0)|^{\frac{2(n-1)}{n-2+\alpha}} \, \mathrm{d}\sigma(x') \, \mathrm{d}r \right)^{\frac{n-2+\alpha}{n-1}} \\
&= \left(\int\limits_0^D \int\limits_{\mathbb{S}^{n-2}} \frac{X^{\frac{2n-3+\alpha}{n-2+\alpha}}(r/D)}{r} \, |v|^{\frac{2(n-1)}{n-2+\alpha}} \, \mathrm{d}\sigma \, \mathrm{d}r \right)^{\frac{n-2+\alpha}{n-1}} \\
&= \left(\int\limits_1^\infty \int\limits_{\mathbb{S}^{n-2}} t^{-\frac{2n-3+\alpha}{n-2+\alpha}} \, |w|^{\frac{2(n-1)}{n-2+\alpha}} \, \mathrm{d}\sigma \, \mathrm{d}t \right)^{\frac{n-2+\alpha}{n-1}} .
\end{aligned}
$$

Therefore, abbreviating $\varrho = 1 - \ln(\frac{R}{D})$, we obtain

$$
C_{n,\alpha} = \inf_{\substack{w \in C^\infty([\varrho,\infty)\times\mathbb{S}_+^{n-1}) \\ w(1,\theta)=0,\, w(t,\theta',0)=0,\, t\leq 1}} \frac{\int\limits_\varrho^\infty \int\limits_{\mathbb{S}_+^{n-1}} \cos^\alpha\varphi \, \left(w_t^2 + |\nabla_\theta w|^2\right) \, \mathrm{d}\sigma \, \mathrm{d}t}{\left(\int\limits_1^\infty \int\limits_{\mathbb{S}^{n-2}} t^{-\frac{2n-3+\alpha}{n-2+\alpha}} \, |w|^{\frac{2(n-1)}{n-2+\alpha}} \, \mathrm{d}\sigma \, \mathrm{d}t \right)^{\frac{n-2+\alpha}{n-1}}} . \tag{5.9}
$$

Comparing the quotients in (5.7), (5.9) and noting that $\varrho < 1$, we conclude that

$$
C_{n,\alpha} \geq \tau_{n,\alpha} \, s_{n,\alpha} > 0, \quad \textit{where} \quad \tau_{n,\alpha} = \begin{cases} (n-2+\alpha)^{\frac{3-2n-\alpha}{n-1}}, & n+\alpha \geq 3 \\ (n-2+\alpha)^{\frac{1-\alpha}{n-1}}, & 2 < n+\alpha < 3. \end{cases}
$$

This proves (5.4), whereafter (5.1) follows.

Part II: Optimality of the exponent of the weight function. To complete the proof of the theorem, it remains to verify that the weight function $X^{\frac{2n-3+\alpha}{n-2+\alpha}}$ cannot be replaced by a smaller power of X. To simplify the calculations let us abbreviate $q := \frac{2(n-1)}{n-2+\alpha}$, $p := \frac{2n-3+\alpha}{n-2+\alpha}$. Moreover, we can assume that $\Omega = B_1$.

In view of (3.38) and making the change of variable $u_m = v_m \, \psi$, it is sufficient to show that for each $0 < \epsilon \leq p$ there exists sequence $\{v_m\} \subset C_0^\infty(B_1 \setminus \{0\})$ such that (cf. (5.2), (5.3), (5.4))

$$
J[v_m] \; := \; \frac{N[v_m]}{D[v_m]} \; := \; \frac{\int\limits_{\mathbb{R}_+^n \cap B_1} \frac{x_n^\alpha \, |\nabla v_m|^2}{|x|^{n-2+\alpha}} \, \mathrm{d}x}{\left(\int\limits_{\partial\mathbb{R}_+^n \cap B_1} \frac{X^{p-\epsilon} \, |v_m|^q}{|x'|^{n-1}} \, \mathrm{d}x' \right)^{2/q}} \longrightarrow 0, \text{ as } m \to \infty. \tag{5.10}
$$

Notice that it suffices to prove the claim, only for the case $0 < \epsilon < p-1$, since $X^{p-\epsilon_0} \geq X^{p-\epsilon}$, $\forall \epsilon_0 \geq \epsilon$.

In order to construct an appropriate sequence satisfying the condition (5.10), we will use a density argument. More precisely, abbreviating $V(x') = \frac{X^{p-\epsilon}(|x'|)}{|x'|^{n-1}}$, $w(x) = \frac{x_n^\alpha}{|x|^{n+\alpha-2}}$ we define the space $D^{1,2}(B_1, w(x)\mathrm{d}x)$ as the completion of $C_0^\infty(\overline{\mathbb{R}_+^n} \cap B_1)$ with respect to the norm $\|v\| = (\int_{B_1^+} |\nabla v|^2 \, w(x) \mathrm{d}x)^{1/2}$.

Then, by a standard approximation, it suffices to fix a sequence $\{\upsilon_m\} \subset D^{1,2}(B_1, w(x)\mathrm{d}x)$ with $\int_{\partial\mathbb{R}^n_+\cap B_1} V(x')\ |\upsilon_m|^q\ \mathrm{d}x' < \infty$, such that $J[\upsilon_m] \to 0$, as $m \to \infty$.

To this end, we choose δ such that $0 < \epsilon < \delta < p-1$, which eventually will be sent to ϵ, we set $R_m = e^{1-m}$ so that

$$\frac{1}{m} \le X(|x|) \le 1 \Leftrightarrow R_m \le |x| \le 1,$$

and define the functions f_m as follows

$$f_m(x) = \begin{cases} X^{\frac{\delta}{q}-\frac{1}{2}}(|x|), & R_m \le |x| \le 1, \\ m^{\frac{3}{2}-\frac{\delta}{q}} X(|x|), & |x| \le R_m, \end{cases} \quad \text{hence} \quad \nabla f_m(x) = \begin{cases} \left(\frac{\delta}{q}-\frac{1}{2}\right) X^{\frac{\delta}{q}+\frac{1}{2}} \frac{x}{|x|^2}, & R_m \le |x| \le 1, \\ m^{\frac{3}{2}-\frac{\delta}{q}} X^2 \frac{x}{|x|^2}, & |x| \le R_m. \end{cases}$$

We then have

$$D^{q/2}[f_m] = \int\limits_{B_1'\setminus B_{R_m}'} \frac{X^{1-\epsilon+\delta}}{|x'|^{n-1}}\,\mathrm{d}x' \ + \ m^{3q/2-\delta} \int\limits_{B_{R_m}'} \frac{X^{p-\epsilon+q}}{|x'|^{n-1}}\,\mathrm{d}x' \ =: D_1 + D_2$$

and

$$N[f_m] = \left(\frac{\delta}{q}-\frac{1}{2}\right)^2 \int\limits_{B_1^+\setminus B_{R_m}^+} \frac{x_n^\alpha\, X^{2\delta/q+1}}{|x|^{n+\alpha}}\,\mathrm{d}x + m^{3-2\delta/q} \int\limits_{B_{R_m}^+} \frac{x_n^\alpha\, X^4}{|x|^{n+\alpha}}\,\mathrm{d}x \ =: N_1 + N_2.$$

We will next estimate the terms D_1, D_2, N_1, N_2, using polar coordinates and taking into account that $X'(r) = X^2(r)/r$. More precisely, setting $\mathcal{C}_{n,\alpha} = \int_{\mathbb{S}_+^{n-1}} x_n^\alpha\,\mathrm{d}\sigma(x)$ and $\omega_n = \int_{\mathbb{S}^{n-2}} 1\,\mathrm{d}\sigma$, we have

$$\begin{aligned}
D_1 &= \int_{\mathbb{S}^{n-2}} 1\,\mathrm{d}\sigma \int_{R_m}^1 \frac{X^{1-\epsilon+\delta}(r)}{r}\,\mathrm{d}r = \omega_n \int_{1/m}^1 t^{\delta-\epsilon-1}\,\mathrm{d}t = \frac{\omega_n\,(1-m^{\epsilon-\delta})}{\delta-\epsilon}, \\
D_2 &= m^{3q/2-\delta} \int_{\mathbb{S}^{n-2}} 1\,\mathrm{d}\sigma \int_0^{R_m} \frac{X^{p-\epsilon+q}(r)}{r}\,\mathrm{d}r = \omega_n\, m^{3q/2-\delta} \int_0^{1/m} t^{p-\epsilon+q-2}\,\mathrm{d}t \\
&= \frac{\omega_n\, m^{\epsilon-\delta}}{p+q-\epsilon-1}, \\
N_1 &= \left(\frac{\delta}{q}-\frac{1}{2}\right)^2 \int_{\mathbb{S}_+^{n-1}} x_n^\alpha\,\mathrm{d}\sigma(x) \int_{R_m}^1 \frac{X^{2\delta/q+1}(r)}{r}\,\mathrm{d}r = \mathcal{C}_{n,\alpha} \left(\frac{\delta}{q}-\frac{1}{2}\right)^2 \int_{1/m}^1 t^{2\delta/q-1}\,\mathrm{d}t \\
&= \mathcal{C}_{n,\alpha} \left(\frac{\delta}{q}-\frac{1}{2}\right)^2 \frac{q(1-m^{-2\delta/q})}{2\delta} \ = \ \mathcal{C}_{n,\alpha}\ (\delta+1-p)^2\, \frac{1-m^{-2\delta/q}}{2q\delta}, \\
N_2 &= m^{3-2\delta/q} \int_{\mathbb{S}_+^{n-1}} x_n^\alpha\,\mathrm{d}\sigma(x) \int_0^{R_m} \frac{X^4(r)}{r}\,\mathrm{d}r = \mathcal{C}_{n,\alpha}\, m^{3-2\delta/q} \int_0^{1/m} t^2\,\mathrm{d}t \ = \ \frac{\mathcal{C}_{n,\alpha}\, m^{-2\delta/q}}{3}.
\end{aligned}$$

We conclude that

$$J[f_m] = \frac{\mathcal{C}_{n,\alpha}}{\omega_n^{2/q}}\, \frac{(\delta+1-p)^2\, \frac{1-m^{-2\delta/q}}{2q\delta} + \frac{m^{-2\delta/q}}{3}}{\left(\frac{1-m^{\epsilon-\delta}}{\delta-\epsilon} + \frac{m^{\epsilon-\delta}}{p+q-\epsilon-1}\right)^{2/q}}.$$

We then take a sequence $\delta_i \searrow \epsilon$ and choose m_i sufficiently large so that $m_i^{\epsilon-\delta_i} < 1/2$. It follows that $J[f_{m_i}] \to 0$, as $i \to \infty$. Given now a function $\eta \in C_0^\infty(B_1)$, which is constant, not zero, in a neighbourhood of the origin, it is straightforward to verify that the sequence $\upsilon_i = f_{m_i}\,\eta$, satisfies $J[\upsilon_i] \to 0$, as $i \to \infty$, that is the condition (5.10). □

Next we proceed with the proof of Theorem VII, which we restate here:

Theorem 9. *Let $\alpha \in (-1,1)$, $2-\alpha \le b < n$ and Ω be a bounded domain in $\mathbb{R}^{n-1}$. Then there exists a constant $C>0$, depending only on n,α, such that for all $u \in C_0^\infty(\mathbb{R}^n)$, with $u(x',0)=0$, for $x' \notin \Omega$, there holds*

$$K(n,\alpha,b)\int_\Omega \frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' + \frac{(\alpha+b-2)^2}{4}\int_{\mathbb{R}^n_+} \frac{x_n^\alpha u^2}{|x|^2}\,\mathrm{d}x + C\int_\Omega \frac{X^2}{|x'|^{1-\alpha}}\,u^2\,\mathrm{d}x' \le \int_{\mathbb{R}^n_+} x_n^\alpha |\nabla u|^2\,\mathrm{d}x, \tag{5.11}$$

where $X = X(|x'|/D)$, $X(\vartheta) = (1-\ln\vartheta)^{-1}$, $0 < X(\vartheta) \le 1$, $D = \sup_{x'\in\Omega}|x'|$. The weight X^2 cannot be replaced by a smaller power of X.

Proof of Theorem 9. We divide the proof in two parts. Firstly we will establish inequality (5.11) and next we will prove the optimality of the exponent 2 of the logarithmic weight X. A key role, in both parts of the proof, will play the function ψ, defined by (3.27).

Part I: Derivation of the estimate (5.11). As in the proof of Theorem 1, we arrive at the following inequality (see (3.40))

$$K(n,\alpha,b)\int_\Omega \frac{u^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x' + \frac{(b-2+\alpha)^2}{4}\int_{\mathbb{R}^n_+} \frac{x_n^\alpha\, u^2}{|x|^2}\,\mathrm{d}x + \int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla \frac{u}{\psi}|^2\,\psi^2\,\mathrm{d}x = \int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x, \tag{5.12}$$

valid for all $u \in C_0^\infty(\mathbb{R}^n\setminus\{0\})$, with $u(x',0)=0$, for $x'\notin\Omega$. The third term in the left hand side yields the correction term in the original inequality. Therefore the result will follow after establishing the following inequality

$$C\int_\Omega \frac{X^2\,u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' \le \int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla\frac{u}{\psi}|^2\,\psi^2\,\mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n\setminus\{0\}),\ \text{with}\ \ u(x',0)=0,\ \text{for}\ x'\notin\Omega. \tag{5.13}$$

for some constant $C>0$, depending only on n and α. Then taking into account (3.38), that is $\psi \sim |x|^{-\frac{n-2+\alpha}{2}}$ in $\mathbb{R}^n_+$, and making the substitution $u = \upsilon\,\psi$, we conclude that (5.13) will follow after showing that there exists a constant $c = c(n,\alpha) > 0$, such that the following inequality is valid

$$c\int_\Omega \frac{X^2\,\upsilon^2}{|x'|^{n-1}}\,\mathrm{d}x' \le \int_{\mathbb{R}^n_+} \frac{x_n^\alpha\,|\nabla\upsilon|^2}{|x|^{\alpha+n-2}}\,\mathrm{d}x, \quad \forall \upsilon \in C_0^\infty(\mathbb{R}^n),\ \text{with}\ \ \upsilon(x',0)=0,\ \text{for}\ x'\notin\Omega. \tag{5.14}$$

Note that for such υ there exists $R \ge D$, depending on υ, such that $\upsilon \in C_0^\infty(B_R)$, where we notate the ball $B_R = \{x \in \partial\mathbb{R}^n : |x| \le 1\}$. Therefore, (5.14) will follow on its turn after showing the existence of a positive constant $C = C(n,\alpha)$, independent of R and D, such that for all $\upsilon \in C_0^\infty(B_R)$, with $\upsilon(x',0)=0$, $|x'| \ge D$, where $0 < D \le R$, there holds

$$c\int_{\partial\mathbb{R}^n_+\cap B_R} \frac{X^2\,\upsilon^2}{|x'|^{n-1}}\,\mathrm{d}x' \le \int_{\mathbb{R}^n_+\cap B_R} \frac{x_n^\alpha\,|\nabla\upsilon|^2}{|x|^{\alpha+n-2}}\,\mathrm{d}x, \quad \upsilon \in C_0^\infty(B_R). \tag{5.15}$$

To this aim we study the minimization problem

$$T_{n,\alpha} = \inf_{\substack{\upsilon\in C_0^\infty(B_R) \\ \upsilon|_{B_R'}\not\equiv 0,\ \upsilon(x',0)=0,\ |x'|\ge D}} I[\upsilon], \quad \text{where} \quad I[\upsilon] := \frac{\displaystyle\int_{\mathbb{R}^n_+\cap B_R} \frac{x_n^\alpha\,|\nabla\upsilon|^2}{|x|^{n-2+\alpha}}\,\mathrm{d}x}{\displaystyle\int_{\{|x'|\le D\}} \frac{X^2\,\upsilon^2}{|x'|^{n-1}}\,\mathrm{d}x'} = \frac{I_1[\upsilon]}{I_2[\upsilon]}.$$

We will compare $T_{n,\alpha}$ with the trace Hardy constant $H(n,\alpha)$ (see Proposition 1) defined by

$$H(n,\alpha)=\inf_{\substack{v\in C_0^\infty(B_1)\\ v\not\equiv 0}} Q[v], \quad \text{where} \quad Q[v]:=\frac{\int\limits_{\mathbb{R}^n_+\cap B_1} x_n^\alpha\,|\nabla v|^2\,\mathrm{d}x}{\int\limits_{\partial\mathbb{R}^n_+\cap B_1}\frac{v^2}{|x'|^{1-\alpha}}\,\mathrm{d}x'}=\frac{Q_1[v]}{Q_2[v]}.$$

We express the numerator of the quotient $Q[v]$ in terms of polar coordinates, writing $v(x)=v(r,\theta)$, where

$$r=|x|,\;\;\theta\;=\;\frac{x}{|x|}\;\in\mathbb{S}^{n-1}_+.$$

Then we make the change of $r-$variable, setting

$$t\;=\;r^{2-n-\alpha}\quad\text{ and }\quad v(r,\theta)=h(t,\theta),$$

thus

$$\mathrm{d}r\;=\;-\frac{r^{n-1+\alpha}}{n-2+\alpha}\,\mathrm{d}t,\quad v_r\;=\;-\frac{n-2+\alpha}{r^{n-1+\alpha}}\,h_t,\quad r\;=\;t^{-\frac{1}{n-2+\alpha}}.$$

Straightforward manipulations yield (see (4.9))

$$Q_1[v]=(n-2+\alpha)\int\limits_1^\infty\int\limits_{\mathbb{S}^{n-1}_+}\cos^\alpha\varphi\,\left(h_t^2\,+\,(n-2+\alpha)^{-2}\,t^{-2}\,|\nabla_\theta h|^2\right)\,\mathrm{d}\sigma\,\mathrm{d}t.$$

Similarly, we will transform the denominator $Q_2[v]$ in terms of polar coordinates in $\mathbb{R}^{n-1}$. We have

$$\begin{aligned}Q_2[v]\;&=\;\int\limits_{B_1'}\frac{v^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x'=\int\limits_0^1\int\limits_{\partial B_r'}\frac{v^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}\sigma(x')\,\mathrm{d}r=\int\limits_0^1\int\limits_{\mathbb{S}^{n-2}} r^{n-3+\alpha}v^2\,\mathrm{d}\sigma\,\mathrm{d}r\\&=\;\frac{1}{n-2+\alpha}\int\limits_1^\infty\int\limits_{\mathbb{S}^{n-2}} t^{-2}\,h^2\,\mathrm{d}\sigma\,\mathrm{d}t.\end{aligned}$$

We then have

$$(n+\alpha-2)^{-2}H(n,\alpha)=\inf_{\substack{h\in C^\infty([1,\infty)\times\mathbb{S}^{n-1}_+)\\ h(1,\theta)=0}}\frac{\int\limits_1^\infty\int\limits_{\mathbb{S}^{n-1}_+}\cos^\alpha\varphi\left(h_t^2\,+\,((n+\alpha-2)t)^{-2}\,|\nabla_\theta h|^2\right)\,\mathrm{d}\sigma\,\mathrm{d}t}{\int\limits_1^\infty\int\limits_{\mathbb{S}^{n-2}} t^{-2}\,h^2\,\mathrm{d}\sigma\,\mathrm{d}t}. \tag{5.16}$$

Next we transform the quotient $I[v]$ in terms of polar coordinates and then we make the change of r-variable, setting

$$t=\frac{1}{X(r/D)}=1-\ln\left(\frac{r}{D}\right),\;\;v(r,\theta)=w(t,\theta),\quad\text{thus}\quad \mathrm{d}r\;=\;-r\,\mathrm{d}t\quad\text{and}\quad v_r\;=\;-\frac{1}{r}\,w_t.$$

Then for any $v\in C_0^\infty(B_R)$, with $v(x',0)=0$, if $|x'|\geq D$, we have (see (5.8))

$$I_1[v]=\int\limits_\varrho^\infty\int\limits_{\mathbb{S}^{n-1}_+}\cos^\alpha\varphi\,\left(w_t^2\,+\,|\nabla_\theta w|^2\right)\,\mathrm{d}\sigma\,\mathrm{d}t,$$

where we abbreviate $\varrho = 1 - \ln(R/D)$. Similarly for the denominator we have

$$\begin{aligned}
I_2[\upsilon] &= \int\limits_{\{|x'|\le D\}} \frac{X^2\,\upsilon^2(x',0)}{|x'|^{n-1}}\,\mathrm{d}x' = \int\limits_0^D \int\limits_{\partial B'_r} \frac{X^2\,\upsilon^2(x',0)}{r^{n-1}}\,\mathrm{d}\sigma(x')\,\mathrm{d}r = \int\limits_0^D \int\limits_{\mathbb{S}^{n-2}} \frac{X^2(r/D)\,\upsilon^2}{r}\,\mathrm{d}\sigma\,\mathrm{d}r \\
&= \int\limits_1^\infty \int\limits_{\mathbb{S}^{n-2}} t^{-2}\,w^2\,\mathrm{d}\sigma\,\mathrm{d}t.
\end{aligned}$$

Therefore we have

$$T_{n,\alpha} = \inf_{\substack{w\in C^\infty([\varrho,\infty)\times\mathbb{S}_+^{n-1}) \\ w(1,\theta)=0,\, w(t,\theta',0)=0,\, t\le 1}} = \frac{\int\limits_\varrho^\infty \int\limits_{\mathbb{S}_+^{n-1}} \cos^\alpha\varphi\,\left(w_t^2 + |\nabla_\theta w|^2\right)\,\mathrm{d}\sigma\,\mathrm{d}t}{\int\limits_1^\infty \int\limits_{\mathbb{S}^{n-2}} t^{-2}\,w^2\,\mathrm{d}\sigma\,\mathrm{d}t}. \tag{5.17}$$

Comparing the quotients in (5.16), (5.17), noting that $\ln(R/D) > 0$, we conclude that

$$T_{n,\alpha} \ge \kappa_{n,\alpha}\,(n-2+\alpha)^{-2}H(n,\alpha) > 0, \quad \textit{where} \quad \kappa_{n,\alpha} = \begin{cases} (n-2+\alpha)^{-2}, & n+\alpha\ge 3 \\ 1, & 2<n+\alpha<3. \end{cases}$$

This proves (5.15), whereafter (5.11) follows.

Part II: Optimality of the weight function X^2. To complete the proof of the theorem, we have to verify that the weight function X^2 cannot be replaced by a smaller power of X. We can assume that $\Omega = B_1$. Let $0 < \varepsilon < 2$. It is sufficient to show that there exists sequence $\{\upsilon_m\} \subset C_0^\infty(B_1\setminus\{0\})$ such that (cf. (5.12), (5.13), (5.14))

$$I[\upsilon_m] := \frac{N[\upsilon_m]}{D[\upsilon_m]} := \frac{\int\limits_{\mathbb{R}^n_+\cap B_1} \frac{x_n^\alpha\,|\nabla\upsilon_m|^2}{|x|^{n-2+\alpha}}\,\mathrm{d}x}{\int\limits_{\partial\mathbb{R}^n_+\cap B_1} \frac{X^{2-\varepsilon}\,\upsilon_m^2}{|x'|^{n-1}}\,\mathrm{d}x'} \longrightarrow 0, \text{ as } m\to\infty.$$

Notice also that it suffices to prove the claim, only for the case $0<\varepsilon<1$, since $X^{2-\varepsilon_0} \ge X^{2-\varepsilon}$, $\forall\varepsilon_0\ge\varepsilon$. To this aim we choose δ such that $\varepsilon<\delta<1$, which will eventually be sent to ε, we set $R_m = e^{1-m}$ so that $1/m = X(|x|) \Leftrightarrow R_m = |x|$, and define the functions f_m as follows

$$f_m(x) = \begin{cases} X^{\frac{\delta-1}{2}}(|x|), & R_m\le|x|\le 1, \\ m^{\frac{3-\delta}{2}}X(|x|), & |x|\le R_m, \end{cases}$$

hence

$$\nabla f_m(x) = \begin{cases} \frac{\delta-1}{2}X^{\frac{\delta+1}{2}}(|x|)\,\frac{x}{|x|^2}, & R_m\le|x|\le 1, \\ m^{\frac{3-\delta}{2}}\,X^2(|x|)\,\frac{x}{|x|^2}, & |x|\le R_m. \end{cases}$$

We then have

$$D[f_m] = \int\limits_{B'_1\setminus B'_{R_m}} \frac{X^{2-\varepsilon}(|x'|)f_m^2}{|x'|^{n-1}}\,\mathrm{d}x' + \int\limits_{B'_{R_m}} \frac{X^{2-\varepsilon}(|x'|)f_m^2}{|x'|^{n-1}}\,\mathrm{d}x' = D_1 + D_2$$

and

$$N[f_m] = \int\limits_{B_1^+ \setminus B_{R_m}^+} \frac{x_n^\alpha \, |\nabla f_m|^2}{|x|^{n-2+\alpha}} \, \mathrm{d}x + \int\limits_{B_{R_m}^+} \frac{x_n^\alpha \, |\nabla f_m|^2}{|x|^{n-2+\alpha}} \, \mathrm{d}x = N_1 + N_2.$$

Next we will estimate the terms D_1, D_2, N_1, N_2, using polar coordinates and taking into account that $X'(r) = X^2(r)/r$. More precisely, setting $\omega_n = \int_{\mathbb{S}^{n-2}} 1 \, \mathrm{d}\sigma$, we have

$$D_1 = \int\limits_{B_1' \setminus B_{R_m}'} \frac{X^{\delta-\varepsilon+1}}{|x'|^{n-1}} \, \mathrm{d}x' = \omega_n \int\limits_{1/m}^{1} t^{\delta-\varepsilon-1} \, \mathrm{d}t = \omega_n \, \frac{1 - m^{\varepsilon-\delta}}{\delta - \varepsilon}$$

and

$$D_2 = m^{3-\delta} \int\limits_{B_{R_m}'} \frac{X^{4-\epsilon}}{|x'|^{n-1}} \, \mathrm{d}x' = \omega_n \, m^{3-\delta} \int\limits_{0}^{1/m} t^{2-\varepsilon} \, \mathrm{d}t = \omega_n \, \frac{m^{\varepsilon-\delta}}{3 - \varepsilon}.$$

Similarly, setting $\mathcal{C}_{n,\alpha} = \int_{\mathbb{S}_+^{n-1}} x_n^\alpha \, \mathrm{d}\sigma(x)$, we have

$$N_1 = \mathcal{C}_{n,\alpha} \left(\frac{\delta - 1}{2} \right)^2 \int\limits_{R_m}^{1} \frac{X^{\delta+1}(r)}{r} \, \mathrm{d}r = \mathcal{C}_{n,\alpha} \left(\frac{\delta - 1}{2} \right)^2 \int\limits_{1/m}^{1} t^{\delta-1} \, \mathrm{d}t = \mathcal{C}_{n,\alpha} \frac{(\delta - 1)^2}{4\delta} (1 - m^{-\delta})$$

and

$$N_2 = \mathcal{C}_{n,\alpha} \, m^{3-\delta} \int\limits_{0}^{R_m} \frac{X^4(r)}{r} \, \mathrm{d}r = \mathcal{C}_{n,\alpha} m^{3-\delta} \int\limits_{0}^{1/m} t^2 \, \mathrm{d}t = \mathcal{C}_{n,\alpha} \frac{m^{-\delta}}{3}.$$

We then take a sequence $\delta_i \searrow \varepsilon$ and choose m_i sufficiently large so that $m_i^{\varepsilon-\delta_i} < 1/2$. It follows that $I[f_{m_i}] \to 0$, as $i \to \infty$. Given now a function $\eta \in C_0^\infty(B_1)$, which is constant, not zero, in a neighbourhood of the origin, it is straightforward to verify that the sequence $\upsilon_i = f_{m_i} \, \eta$, satisfies $I[\upsilon_i] \to 0$, as $i \to \infty$. $\square$

Remark. *We point out that Theorems VI , VII are valid for functions supported in $\mathbb{R}_+^n$. However, if we restrict the attention to functions supported on a bounded domain U, then a stronger result holds. In particular, if we cut the series in (1.19) at the k term, then the resulted inequality can be improved by adding trace remainder terms:*

Let $\alpha \in (-1, 1)$, $2 - \alpha \leq b < n$ and U be a bounded domain in $\mathbb{R}^n$. Then there exists a constant $c > 0$, such that for all $u \in C_0^\infty(U)$, there holds

$$K(n, \alpha, b) \int\limits_{\partial \mathbb{R}_+^n \cap U} \frac{u^2}{|x'|^{1-\alpha}} \, \mathrm{d}x' + \frac{(\alpha + b - 2)^2}{4} \int\limits_{\mathbb{R}_+^n \cap U} \frac{x_n^\alpha \, u^2}{|x|^2} \, \mathrm{d}x + \frac{1}{4} \sum_{i=1}^{k} \int\limits_{\mathbb{R}_+^n \cap U} \frac{x_n^\alpha \, X_1^2(\frac{|x|}{d}) \cdots X_i^2(\frac{|x|}{d})}{|x|^2} \, u^2 \, \mathrm{d}x$$

$$+ \, c \left(\int\limits_{\partial \mathbb{R}_+^n \cap U} \left(X_1 \left(\frac{|x'|}{D} \right) \cdots X_k \left(\frac{|x'|}{D} \right) X_{k+1} \left(\frac{|x'|}{D} \right) \right)^{\frac{2n-3+\alpha}{n-2+\alpha}} |u|^{\frac{2(n-1)}{n-2+\alpha}} \, \mathrm{d}x' \right)^{\frac{n-2+\alpha}{n-1}} \leq \int\limits_{\mathbb{R}_+^n \cap U} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x,$$

where the constant $K(n,\alpha,b)$ is given in (1.11) , $D=\sup_{x\in\partial\mathbb{R}^n_+\cap U}|x|$ *and* $d=\sup_{x\in\mathbb{R}^n_+\cap U}|x|$*. Moreover, the weight* $(X_1\cdots X_{k+1})^{(2n-3+\alpha)/(n-2+\alpha)}$ *cannot be replaced by smaller powers of* $X_1,\cdots,X_{k+1}$*.*

Under the same assumptions, there exists a constant $C>0$*, such that for all* $u\in C_0^\infty(U)$*, there holds*

$$K(n,\alpha,b)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x'+\frac{(\alpha+b-2)^2}{4}\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,u^2}{|x|^2}\,\mathrm{d}x+\frac{1}{4}\sum_{i=1}^{k}\int\limits_{\mathbb{R}^n_+\cap U}\frac{x_n^\alpha\,X_1^2(\frac{|x|}{d})\cdots X_i^2(\frac{|x|}{d})}{|x|^2}\,u^2\,\mathrm{d}x$$

$$C\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{\left(X_1\left(\frac{|x'|}{D}\right)\cdots X_k\left(\frac{|x'|}{D}\right)\,X_{k+1}\left(\frac{|x'|}{D}\right)\right)^2}{|x'|^{1-\alpha}}\,u^2\,\mathrm{d}x'\leq\int\limits_{\mathbb{R}^n_+\cap U}x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x.$$

Moreover, the logarithmic correction $(X_1\cdots X_{k+1})^2$ *cannot be replaced by powers of* $X_1,\cdots,X_{k+1}$*, with exponents smaller than* 2*.*

We can derive these improvements, following a similar argumentation with the one of the proofs of Theorems VI, VII, but working with the function ψ_k *(see (4.14)) instead of* ψ*.*

Chapter 6

Improving Hardy inequalities for fractional Laplacians on bounded domains

Throughout this Chapter we assume that $s \in (0,1)$ and Ω is a bounded domain in $\mathbb{R}^n$ with $n > 2s$. Recall also the fractional Laplacians $A_s,\ (-\Delta)^s$ which are defined in Section 1.3. In the sequel we will apply the improved trace Hardy inequalities, obtained among others in Chapter 5, to the extended problems (see Section 2.2) associated with these operators, to derive relative refined fractional Hardy inequalities.

6.1 Hardy inequalities for the spectral fractional Laplacian

In this section we establish sharp fractional Hardy inequalities with remainder terms, associated with the spectral Laplacian A_s. More precisely, we will give the proof of Theorems VIII, IX, X .

6.1.1 The Hardy inequality for the spectral fractional Laplacian

We start with the proof of Theorem VIII which we restate here, for the reader's convenience.

Theorem 10 (Hardy inequality for the spectral fractional Laplacian A_s)**.** *Let $s \in (0,1)$ and Ω be a bounded domain in $\mathbb{R}^n$ with $n > 2s$. Then for all $f \in C_0^\infty(\Omega)$ there holds*

$$h_{s,n} \int_\Omega \frac{f^2(x)}{|x|^{2s}}\, \mathrm{d}x \leq (A_s f,\, f), \quad \textit{where} \quad h_{s,n} = 2^{2s} \frac{\Gamma^2(\frac{n+2s}{4})}{\Gamma^2(\frac{n-2s}{4})}. \tag{6.1}$$

The constant $h_{s,n}$ is sharp if $0 \in \Omega$.

The basic estimate for the proof of Theorem 10 is the weighted trace Hardy inequality (3.1), which we restate here, in the settings of the present chapter: For any $s \in (0,1)$ with $n > 2s$, there holds

$$H_{n,s} \int_{\mathbb{R}^n} \frac{u^2(x,0)}{|x|^{2s}}\, \mathrm{d}x \leq \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}\, |\nabla u|^2\, \mathrm{d}x\, \mathrm{d}y, \quad \forall u \in C_0^\infty(\mathbb{R}^{n+1}), \tag{6.2}$$

where the constant

$$H_{n,s} = \frac{2s\Gamma^2(\frac{n+2s}{4})\Gamma(1-s)}{\Gamma(1+s)\Gamma^2(\frac{n-2s}{4})} \tag{6.3}$$

is the best possible. As a direct consequence of the scaling invariance of (6.2), the following inequality is also valid

$$H_{n,s} \int_\Omega \frac{u^2(x,0)}{|x|^{2s}}\, \mathrm{d}x \leq \int_0^\infty \int_\Omega y^{1-2s}\, |\nabla u|^2 \,\mathrm{d}x\, \mathrm{d}y, \quad \forall u \in C_0^\infty(\Omega \times \mathbb{R}), \tag{6.4}$$

where the constant $H_{n,s}$ is given in (6.3) and it is optimal if $0 \in \Omega$. With this estimate at hand, we can proceed with the

Proof of Theorem 10 . For any $f \in C_0^\infty(\Omega)$ we consider the function u which is the unique extension of f in $\Omega \times (0,\infty)$, satisfying the problem (2.15), with $\int_{\mathcal{C}_\Omega} y^{1-2s}\, |\nabla u|^2 \,\mathrm{d}x\, \mathrm{d}y < \infty$. We then have (see [25], [47])

$$(A_s f,\, f) = \frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \int_0^\infty \int_\Omega y^{1-2s}\, |\nabla u|^2 \,\mathrm{d}x\, \mathrm{d}y. \tag{6.5}$$

Then substituting this estimate in (6.4), we obtain the desired inequality (6.1).

It remains to prove the optimality of $h_{s,n}$. Let $\{u_k\}_{k=1}^\infty$ be a minimizing sequence for the problem

$$H_{n,s} = \inf_{\substack{u \in C_0^\infty(\Omega\times\mathbb{R}) \\ u|_\Omega \not\equiv 0}} \frac{\int_0^\infty \int_\Omega y^{1-2s} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y}{\int_\Omega \frac{u^2(x,0)}{|x|^{2s}}\,\mathrm{d}x},$$

that is

$$\frac{\int_0^\infty \int_\Omega y^{1-2s} |\nabla u_k|^2 \,\mathrm{d}x\,\mathrm{d}y}{\int_\Omega \frac{u_k^2(x,0)}{|x|^{2s}}\,\mathrm{d}x} \overset{k\to\infty}{\longrightarrow} H_{n,s}.$$

We set $f_k(x) = u_k(x,0)$ and let $\bar{u}_k : \mathbb{R}^{n+1} \to \mathbb{R}$ be the extension of f_k satisfying problem (2.15) for $f = f_k$, $u = u_k$ there. Then we have

$$\int_0^\infty \int_\Omega y^{1-2s}\, |\nabla \bar{u}_k|^2 \,\mathrm{d}x\,\mathrm{d}y \leq \int_0^\infty \int_\Omega y^{1-2s}\, |\nabla u_k|^2 \,\mathrm{d}x\,\mathrm{d}y$$

and since $u_k = \bar{u}_k$ in Ω, we have

$$\frac{\int_0^\infty \int_\Omega y^{1-2s} |\nabla \bar{u}_k|^2 \,\mathrm{d}x\,\mathrm{d}y}{\int_\Omega \frac{\bar{u}_k^2(x,0)}{|x|^{2s}}\,\mathrm{d}x} \overset{k\to\infty}{\longrightarrow} H_{n,s}.$$

Finally, in view of (6.5) we get

$$\frac{(A_s f_k,\, f_k)}{\int_\Omega \frac{f_k^2}{|x|^{2s}}\,\mathrm{d}x} \overset{k\to\infty}{\longrightarrow} h_{s,n}.$$

□

6.1.2 Hardy-Sobolev inequality for the spectral fractional Laplacian

Let us now proceed to derive the Hardy-Sobolev inequality for the regional Laplacian A_s. To this aim, we will need the following improvement of (6.4), concerning the extended problem, which is rather of independent interest.

Theorem 11. *Let $s \in (0,1)$ and $n > 2s$. Then there exists a positive constant $C = C(n,s)$ such that for all $u \in C_0^\infty(\Omega \times \mathbb{R})$ there holds*

$$H_{n,s} \int_\Omega \frac{u^2}{|x|^{2s}}\,\mathrm{d}x + C \left(\int_\Omega X^{\frac{2(n-s)}{n-2s}}\, |u|^{\frac{2n}{n-2s}} \,\mathrm{d}x \right)^{\frac{n-2s}{n}} \leq \int_0^\infty \int_\Omega y^{1-2s} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y, \tag{6.6}$$

where $X = X(|x|/D)$, $X(\vartheta) = (1 - \ln \vartheta)^{-1}$, $0 < \vartheta \leq 1$, $D = \sup_{x\in\Omega} |x|$. *The weight* $X^{\frac{2(n-s)}{n-2s}}$ *cannot be replaced by a smaller power of* X.

Theorem 11 is an immediate consequence of Theorem 8 with $b = 2 - \alpha$ there. We are now ready to prove Theorem IX:

Theorem 12 (Hardy-Sobolev inequality for the spectral fractional Laplacian)**.** *Let* $s \in (0, 1)$ *and* Ω *be a bounded domain in* $\mathbb{R}^n$ *with* $n > 2s$. *Then there exists a positive constant* $C = C(n, s)$ *such that for all* $f \in C_0^\infty(\Omega)$ *there holds*

$$h_{s,n} \int_\Omega \frac{f^2(x)}{|x|^{2s}}\,\mathrm{d}x + C \left(\int_\Omega X^{\frac{2(n-s)}{n-2s}} |f(x)|^{\frac{2n}{n-2s}}\,\mathrm{d}x \right)^{\frac{n-2s}{n}} \leq (A_s f,\, f),$$

where $X = X(|x|/D)$, $X(\vartheta) = (1 - \ln \vartheta)^{-1}$, $0 < \vartheta \leq 1$, $D = \sup_{x\in\Omega} |x|$. *The exponent* $\frac{2(n-s)}{n-2s}$ *of* X *cannot be replaced by a smaller one.*

Proof of Theorem 12 . The result follows immediately, applying Theorem 11 to the extension u of f (cf. (2.13)) and substituting the energy of u through the relation (6.5). □

6.1.3 Improved Hardy inequality for the spectral fractional Laplacian

To establish Theorem X, we will need the following improvement of (6.4), which is an immediate consequence of Theorem 9, with $b = 2 - \alpha$ there.

Theorem 13. *Let* $s \in (0, 1)$ *and* Ω *be a bounded domain in* $\mathbb{R}^n$ *with* $n > 2s$. *Then there exists a positive constant* $C = C(n, s)$ *such that for all* $u \in C_0^\infty(\Omega \times \mathbb{R})$ *there holds*

$$H_{n,s} \int_\Omega \frac{u^2}{|x|^{2s}}\,\mathrm{d}x + C \int_\Omega \frac{X^2}{|x|^{2s}} u^2\,\mathrm{d}x \leq \int_0^\infty \int_\Omega y^{1-2s} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}y,$$

where $X = X(|x|/D)$, $X(\vartheta) = (1 - \ln \vartheta)^{-1}$, $0 < \vartheta \leq 1$, $D = \sup_{x\in\Omega} |x|$. *The exponent* 2 *of* X *cannot be improved.*

Let us now proceed with the proof of Theorem X, which we restate here, for convenience of the reader.

Theorem 14 (Improved Hardy inequality for the spectral fractional Laplacian)**.** *Let* $s \in (0, 1)$ *and* Ω *be a bounded domain in* $\mathbb{R}^n$ *with* $n > 2s$. *Then there exists a positive constant* $C = C(n, s)$ *such that*

$$h_{s,n} \int_\Omega \frac{f^2(x)}{|x|^{2s}}\,\mathrm{d}x + C \int_\Omega f^2(x) \frac{X^2}{|x|^{2s}}\,\mathrm{d}x \leq (A_s f,\, f), \quad \forall f \in C_0^\infty(\Omega),$$

where $X = X(|x|/D)$, $X(\vartheta) = (1 - \ln \vartheta)^{-1}$, $0 < \vartheta \leq 1$, $D = \sup_{x\in\Omega} |x|$. *The weight* X^2 *cannot be replaced by a smaller power of* X.

Proof of Theorem 14 . We apply Theorem 13 to the extension u of f (cf. (2.13)) whence Theorem 14 results upon a substitution of the energy of u through the relation (6.5). □

6.2 Hardy inequalities for the Dirichlet fractional Laplacian

This section is devoted to establish improvements of the fractional Hardy inequality associated with the Dirichlet Laplacian $(-\Delta)^s$, by adding Sobolev and Hardy type correction terms. More precisely, we will give the proof of Theorems XI , XII, XIII.

6.2.1 The Hardy inequality for the Dirichlet fractional Laplacian

We first establish the sharp Hardy inequality which is stated in Theorem XI:

Theorem 15 (Hardy inequality for the fractional Laplacian $(-\Delta)^s$)**.** *Let $s \in (0,1)$ and Ω be a bounded domain in $\mathbb{R}^n$ with $n > 2s$. Then there holds*

$$h_{s,n} \int_\Omega \frac{f^2(x)}{|x|^{2s}}\,\mathrm{d}x \le ((-\Delta)^s f,\, f), \quad \forall f \in C_0^\infty(\Omega). \tag{6.7}$$

The constant $h_{s,n}$ is given by (6.1) and if $0 \in \Omega$, then it is sharp.

Proof of Theorem 15 . Applying standard scaling arguments in (6.2), we see that the same inequality holds when the integral in the right hand side is computed over $\mathbb{R}^n \times (0,\infty)$ and the test functions $u \in C_0^\infty(\mathbb{R}^{n+1})$ with $u(x,0) = 0, x \notin \Omega$, that is

$$H_{n,s} \int_\Omega \frac{u^2(x,0)}{|x|^{2s}}\,\mathrm{d}x \le \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}\,|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}y, \quad \forall u \in C_0^\infty(\mathbb{R}^{n+1}) \;\text{ with }\; u(x,0) = 0, x \notin \Omega. \tag{6.8}$$

We point out that (6.2), (6.8) share the same optimal constant.

We now consider the solution $u = u(x,y)$ of the extended problem (2.13). In this case we have (see [15], [25])

$$((-\Delta)^s f,\, f) = \frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}y. \tag{6.9}$$

Hence, substituting this estimate in (6.8), we obtain (6.7).

It remains to prove the optimality of $h_{s,n}$. Let $\{u_k\}_{k=1}^\infty$ be a minimizing sequence for the problem (cf. (6.8))

$$H_{n,s} = \inf_{\substack{u \in C_0^\infty(\mathbb{R}^{n+1}),\, u(\cdot,0) \in C_0(\Omega) \\ u|_\Omega \not\equiv 0}} \frac{\int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}\,|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}y}{\int_\Omega \frac{u^2(x,0)}{|x|^{2s}}\,\mathrm{d}x},$$

that is

$$\frac{\int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}\,|\nabla u_k|^2\,\mathrm{d}x\,\mathrm{d}y}{\int_\Omega \frac{u_k^2(x,0)}{|x|^{2s}}\,\mathrm{d}x} \overset{k\to\infty}{\longrightarrow} H_{n,s}.$$

We set $f_k(x) = u_k(x,0)$ and let $\bar{u}_k : \mathbb{R}^{n+1} \to \mathbb{R}$ be the extension of f_k satisfying problem (2.13) for $f = f_k$, $u = u_k$. Then we have

$$\int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}\,|\nabla \bar{u}_k|^2\,\mathrm{d}x\,\mathrm{d}y \le \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}\,|\nabla u_k|^2\,\mathrm{d}x\,\mathrm{d}y$$

and since $u_k(x,0) = \bar{u}_k(x,0)$, we have

$$\frac{\int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}|\nabla \bar{u}_k|^2\,\mathrm{d}x\,\mathrm{d}y}{\int_\Omega \frac{\bar{u}_k^2(x,0)}{|x|^{2s}}\,\mathrm{d}x} \overset{k\to\infty}{\longrightarrow} H_{n,s}.$$

Finally, in view of (6.9) we get

$$\frac{((-\Delta)^s f_k,\, f_k)}{\int_\Omega \frac{f_k^2}{|x|^{2s}}\,\mathrm{d}x} \overset{k\to\infty}{\longrightarrow} h_{s,n}.$$

$\square$

6.2.2 Hardy-Sobolev inequality for the Dirichlet fractional Laplacian

The Hardy-Sobolev inequality for the regional Laplacian $(-\Delta)^s$, is strongly connected with the following improvement of (6.8), concerning the extended problem, which is rather of independent interest.

Theorem 16. *Let $s \in (0,1)$ and Ω be a bounded domain in $\mathbb{R}^n$ with $n > 2s$. Then there exists a positive constant $C = C(n,s)$ such that for all $u \in C_0^\infty(\mathbb{R}^{n+1})$, with $u(x,0) = 0$, $x \notin \Omega$, there holds*

$$H_{n,s} \int_\Omega \frac{u^2}{|x|^{2s}} \,\mathrm{d}x + C \left(\int_\Omega X^{\frac{2n-2s}{n-2s}} |u|^{\frac{2n}{n-2s}} \,\mathrm{d}x \right)^{\frac{n-2s}{n}} \leq \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s} |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}y,$$

where $X = X(|x|/D)$, $X(\vartheta) = (1 - \ln \vartheta)^{-1}$, $0 < \vartheta \leq 1$, $D = \sup_{x \in \Omega} |x|$. The exponent $\frac{2n-2s}{n-2s}$ of the weight function cannot be improved.

Theorem 16 is an immediate consequence of Theorem 8, with $b = 2 - \alpha$ there. We are now ready to derive the Hardy - Sobolev inequality for the regional Laplacian $(-\Delta)^s$, stated in Theorem XII:

Theorem 17. *Let $s \in (0,1)$ and Ω be a bounded domain in $\mathbb{R}^n$ with $n > 2s$. Then there exists a positive constant $C = C(n,s)$ such that for all $f \in C_0^\infty(\Omega)$ there holds*

$$h_{s,n} \int_\Omega \frac{f^2(x)}{|x|^{2s}} \,\mathrm{d}x + C \left(\int_\Omega X^{\frac{2(n-s)}{n-2s}} |f(x)|^{\frac{2n}{n-2s}} \,\mathrm{d}x \right)^{\frac{n-2s}{n}} \leq ((-\Delta)^s f,\, f), \tag{6.10}$$

where $X = X(|x|/D)$, $X(\vartheta) = (1 - \ln \vartheta)^{-1}$, $0 < \vartheta \leq 1$, $D = \sup_{x \in \Omega} |x|$. Moreover $X^{\frac{2(n-s)}{n-2s}}$ cannot be replaced by a smaller power of X.

Proof of Theorem 17 . The Theorem results upon an application of Theorem 16 to the extension u of f (cf. (2.15)) and using the relation (6.9). □

6.2.3 Improved Hardy inequality for the Dirichlet fractional Laplacian

For the proof of Theorem XIII we will need an improvement of (6.8), which is associated with the extended problem. More precisely, in view of Theorem 9, with $b = 2 - \alpha$ there, we have the following result.

Theorem 18. *Let $s \in (0,1)$ and Ω be a bounded domain in $\mathbb{R}^n$ with $n > 2s$. Then there exists a positive constant $C = C(n,s)$ such that for all $u \in C_0^\infty(\mathbb{R}^{n+1})$, with $u(x,0) = 0$, $x \notin \Omega$, there holds*

$$H_{n,s} \int_\Omega \frac{u^2}{|x|^{2s}} \,\mathrm{d}x + C \int_\Omega \frac{X^2}{|x|^{2s}} u^2 \,\mathrm{d}x \leq \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s} |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}y,$$

where $X = X(|x|/D)$, $X(\vartheta) = (1 - \ln \vartheta)^{-1}$, $0 < \vartheta \leq 1$, $D = \sup_{x \in \Omega} |x|$.

We conclude with the proof of Theorem XIII:

Theorem 19. *Let $s \in (0,1)$ and Ω be a bounded domain in $\mathbb{R}^n$ with $n > 2s$. Then there exists a positive constant $C = C(n,s)$ such that*

$$h_{s,n} \int_\Omega \frac{f^2(x)}{|x|^{2s}} \,\mathrm{d}x + C \int_\Omega f^2(x) \frac{X^2}{|x|^{2s}} \,\mathrm{d}x \leq ((-\Delta)^s f,\, f), \quad \forall f \in C_0^\infty(\Omega),$$

where $X = X(|x|/D)$, $X(\vartheta) = (1 - \ln \vartheta)^{-1}$, $0 < \vartheta \leq 1$, $D = \sup_{x \in \Omega} |x|$. Moreover X^2 cannot be replaced by a smaller power of X.

Proof of Theorem 19 . We apply Theorem 18 to the extension u of f (cf. (2.15)) whence Theorem 19 results upon a substitution of the energy of u through the relation (6.9). □

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