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Hardy And Hardy-Sobolev Inequalities And Their Applications

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to my parents,
Thomas and Vasiliki

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Abstract

This thesis is separated in three main themes:

A) Let $\Omega \subset \boldsymbol{R}^n$ be an open domain that contains the origin. We find conditions on the potential V which ensure the nonexistence of $H^1(\Omega)$ positive solutions for linear elliptic problems with Hardy-type potentials. In particular, we prove the nonexistence of nontrivial solutions in $H^1(\Omega)$ for the equation

$$-\Delta u = \frac{(n-2)^2}{4}\frac{u}{|x|^2} + bVu,$$

where $b > 0$ is the best constant in the inequality

$$\int_\Omega |\nabla \phi|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{\phi^2}{|x|^2} dx + b\int_\Omega V\phi^2 dx, \quad \forall\ \phi \in C_0^\infty(\Omega).$$

The results depend on an integral assumption on the potential V and what is really of interest is that under the same assumption on V, there is no improvement of the inequality. This result goes against the folklore fact that if there is no minimizer for an inequality, then we can improve it. We also give an example establishing that this integral assumption on V is optimal (see Chapter 3).

B) We prove Hardy and Hardy-Sobolev inequalities involving distance to the boundary of domains with infinite inner radius. More precisely we deal with exterior domains, i.e. complements of smooth compact domains not containing the origin. We introduce the following new geometric condition on Ω

$$-\Delta d(x) + (n-1)\frac{\nabla d(x)\cdot x}{|x|^2} \geq 0,$$

where d denotes the distance function to the boundary of Ω. We prove that under this condition the following Hardy-Sobolev inequality for $n \geq 4$ holds:

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx \geq C\Big(\int_\Omega |u|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_0^\infty(\Omega).$$

The case $n = 3$ is different, we need to assume that Ω satisfies the above condition with strict inequality. Then we prove the following Hardy-Sobolev type inequality

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx \geq C\Big(\int_\Omega X^4(\frac{|x|}{D})|u|^6 dx\Big)^{\frac{1}{3}}, \quad \forall\ u \in C_c^\infty(\Omega),$$

where $X(t) = (1 + \ln t)^{-1},\ 0 < D < \inf\{|x| :\ x \in \partial\Omega\}$. Moreover, the power 4 on X can not be replaced by a smaller power.

We also obtain Hardy and Hardy-Sobolev inequalities for domains above the graph of a $C^{1,1}$ function (see Chapter 4).

C) We prove boundary Harnack type inequalities for positive solutions of the problem

$$\begin{aligned} u_t &= \Delta u + \frac{1}{4}\frac{u}{d^2} && \text{in} \quad \Omega \times (0, T] \\ u &= 0 && \text{on} \quad \partial\Omega \\ u(0, x) &= u_0(x) && \text{in} \quad \Omega, \end{aligned}$$

when Ω is an exterior domain without posing any geometric assumption on Ω. Then we prove heat kernel estimates for this problem for small times (see Chapter 5).

Περίληψη

Αυτή η διατριβή χωρίζεται σε τρία βασικά μέρη:

A) Έστω $\Omega \subset \boldsymbol{R}^n$ ένα ανοιχτό σύνολο το οποίο περιέχει το 0. Βρίσκουμε τις βέλτιστες συνθήκες που πρέπει να ικανοποιεί το δυναμικό V, έτσι ώστε να μην έχουμε $H^1(\Omega)$ λύσεις για ένα γραμμικό ελλειπτικό τελεστή, που περιέχει δυναμικά τύπου Hardy. Ειδικότερα, αποδεικνύουμε την μη ύπαρξη $H^1(\Omega)$ θετικών λύσεων για το πρόβλημα

$$-\Delta u = \frac{(n-2)^2}{4}\frac{u}{|x|^2} + bVu,$$

όπου $b > 0$ είναι η βέλτιστη δυνατή σταθερά για την ανισότητα

$$\int_\Omega |\nabla\phi|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{\phi^2}{|x|^2}dx + b\int_\Omega V\phi^2 dx, \quad \forall\, \phi \in C_0^\infty(\Omega).$$

Τα αποτελέσματα εξαρτώνται από ολοκληρωτικές υποθέσεις για το δυναμικό V. Αξίζει να σημειωθεί εδώ ότι κάτω από τις ίδιες υποθέσεις για το δυναμικό V έχουμε ότι η ιδέα που επικρατεί, ‶οτι μια ανισότητα μπορεί να βελτιωθεί εάν ο «ελαχιστοποιητής» δεν είναι συνάρτηση στον ενεργειακό χώρο¨, είναι λανθασμένη. Τέλος δίνουμε ένα αντιπαράδειγμα που δείχνει το βέλτιστο των υποθέσεων μας για το δυναμικό V (δες Κεφ. 3).

B) Αποδεικνύουμε ανισότητες Hardy και Hardy – Sobolev που περιλαμβάνουν την συνάρτηση απόστασης από το σύνορο ενός συνόλου με άπειρη εσωτερική ακτίνα. Ειδικότερα ασχολούμαστε με χωρία που ορίζονται σαν το εξωτερικό ενός συμπαγούς. Εισάγουμε την νέα γεωμετρική συνθήκη για αυτά τα χωρία Ω:

$$-\Delta d(x) + (n-1)\frac{\nabla d(x)\cdot x}{|x|^2} \geq 0,$$

όπου d συμβολίζει την απόσταση από το σύνορο του Ω. Για τα χωρία που ικανοποιούν αυτή την συνθήκη, αποδεικνύουμε την ανισότητα Hardy – Sobolev για $n \geq 4$:

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx \geq C\left(\int_\Omega |u|^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}}, \quad \forall u \in C_0^\infty(\Omega).$$

Η περίπτωση $n = 3$ είναι διαφορετική. Πρέπει να υποθέσουμε επιπλέον, ότι τα χωρία Ω ικανοποιούν την παραπάνω συνθήκη με γνήσια ανισότητα. Τότε αποδεικνύουμε την ανισότητα τύπου Hardy – Sobolev

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx \geq C\left(\int_\Omega X^4\left(\frac{|x|}{D}\right)|u|^6 dx\right)^{\frac{1}{3}}, \quad \forall\, u \in C_c^\infty(\Omega),$$

όπου $X(t) = (1+\ln t)^{-1},\ 0 < D < \inf\{|x| : \ x \in \partial\Omega\}$. Επιπλέον, η δύναμη 4 στο X δεν μπορεί να αντικατασταθεί από μια μικρότερη δύναμη.

Επίσης αποδεικνύουμε ανισότητες Hardy και Hardy – Sobolev για χωρία που ορίζονται πάνω από το γράφημα μιας $C^{1,1}$ συνάρτησης (δες Κεφ. 4).

Γ) Αποδεικνύουμε ανισότητες Harnack μέχρι και το σύνορο, για τις θετικές λύσεις του παραβολικού προβλήματος

$$\begin{aligned} u_t &= \Delta u + \frac{1}{4}\frac{u}{d^2} && \text{in} \quad \Omega \times (0, T] \\ u &= 0 && \text{on} \quad \partial\Omega \\ u(0, x) &= u_0(x) && \text{in} \quad \Omega, \end{aligned}$$

όπου Ω το εξωτερικό ενός συμπαγούς (Δεν υποθέτουμε καμία γεωμετρική συνθήκη για το Ω). Τέλος αποδεικνύουμε εκτιμήσεις, για μικρούς χρόνους, για τον αντίστοιχο πυρήνα θερμότητας του παραπάνω παραβολικού προβλήματος (δες Κεφ. 5).

Contents

Chapter 1

Overview

In this chapter we present our basic results in this thesis. In section 1.1 we present theorems on non-existence of solutions to elliptic problems involving Hardy type potentials with the distance taken from a point. In particular we state our main results in [Gk]. In section 1.2 we will state new Hardy and Hardy-Sobolev inequalities involving distance to the boundary of domains having infinite inner radius. Finally, in section 1.3 we present the parabolic problems involving Hardy type potentials with the distance taken from the boundary of domains having infinite inner radius. We present new sharp two side estimates for the heat kernel of these problems.

1.1 Distance From a Point

In this section we assume that Ω is an open bounded domain in $\mathbb{R}^n$; $n \geq 3$, containing the origin.
The classical Sobolev inequality asserts that

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq S_n \left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad \forall\, u \in C_0^\infty(\mathbb{R}^n),$$

where the constant $S_n = \pi n(n-2) \left(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \right)^{\frac{2}{n}}$ is optimal.
The classical Sobolev inequality (for some constant $c_n < S_n$) can be proved by the usage of the classical Hardy inequality

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \left(\frac{n-2}{2} \right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx, \quad \forall\, u \in C_0^\infty(\mathbb{R}^n),$$

where the constant $(\frac{n-2}{2})^2$ is optimal. The proof of the classical Hardy inequality is very simple:

$$0 \leq \int_{\mathbb{R}^n} \left| \nabla u + \frac{n-2}{2} \frac{x}{|x|^2} \right|^2 = \int_{\mathbb{R}^n} |\nabla u|^2 dx - \left(\frac{n-2}{2} \right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx, \quad \forall\, u \in C_0^\infty(\mathbb{R}^n).$$

We consider now the following minimizing problem

$$\lambda_1 = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{u^2}{|x|^2} dx}.$$

Then it is well known that, $\lambda_1 = \frac{(n-2)^2}{4}$ and this constant is not attained in $H_0^1(\Omega)$ or equivalently the corresponding Euler-Lagrange equation

$$-\Delta u = \frac{(n-2)^2}{4} \frac{u}{|x|^2}, \quad in \ \ \Omega$$
$$u \gneqq 0 \ \ in \ \ \Omega,$$

has no nontrivial $H_0^1(\Omega)$ solutions.
The fact that the best constant is not attained suggests that one might look for an error term in

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx, \quad \forall\, u \in C_0^\infty(\Omega). \tag{1.1.1}$$

Indeed, Brezis and Vázquez [BV] improved inequality (1.1.1) by adding a positive term in the right hand side.

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + C_\Omega \int_\Omega u^2 dx, \quad \forall\, u \in C_0^\infty(\Omega), \tag{1.1.2}$$

and

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + K \left(\int_\Omega |u|^p dx\right)^{\frac{2}{p}}. \tag{1.1.3}$$

In (1.1.3) we assume that $1 < p < \frac{2n}{n-2}$. The constant C_Ω in (1.1.2) is given by

$$C_\Omega = z_0^2 w_n^{\frac{2}{n}} |\Omega|^{-\frac{2}{n}},$$

where w_n and $|\Omega|$ denote the volume of the unit ball and Ω, respectively, and $z_0 = 2.4048...$ denotes the first zero of the Bessel function $J_0(z)$. The constant C_Ω is optimal when Ω is a ball. But again the minimizing problem

$$C_\Omega = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx}{\int_\Omega u^2 dx},$$

has no $H_0^1(\Omega)$ minimizer or equivalently the corresponding Euler-Lagrange equation

$$\begin{aligned} &-\Delta u = \tfrac{(n-2)^2}{4} \tfrac{u}{|x|^2} + C_\Omega u, \quad in \;\; \Omega \\ &u \gneqq 0 \;\; in \;\; \Omega, \end{aligned}$$

has no $H_0^1(\Omega)$ solutions.
Hardy inequalities as well as their improved versions are used in the study of the solutions of semi-linear elliptic equations. More precisely, Brezis and Vazquez [BV] firstly applied these inequalities in Gelfand problem

$$\begin{aligned} &-\Delta u = \lambda e^u, \quad in \;\; \Omega \\ &u \gneqq 0 \;\; in \;\; \Omega, \end{aligned} \tag{1.1.4}$$

where λ is a positive parameter. It is well known that there exist a positive number $\lambda^* > 0$ such that the problem (1.1.4) has $H_0^1(\Omega)$ solution for $0 < \lambda \leq \lambda^*$, while no $H_0^1(\Omega)$ solutions exist for $\lambda > \lambda^*$. In addition, for $\lambda = 2(n-2)$ and $\Omega = B(0,1)$; the unite ball with center at the origin, one has the singular solution

$$u_1(x) = -2 \ln |x| \in H_0^1(B).$$

The "linearization" of problem (1.1.4) leads to the operator involving Hardy-type potential

$$L_{u_1} u = -\Delta u - \frac{2(n-2)}{|x|^2} u.$$

The authors in [BV] observed that $2(n-2) \leq \frac{(n-2)^2}{4}$, for $n \geq 10$ and they used this fact and (1.1.2) to prove that u_1 is an extremal solution i.e. it is a solution of (1.1.4) with $\lambda^* = 2(n-2)$. For the case $n \leq 9$, they showed by (1.1.1) that $2(n-2) < \lambda^*$ i.e. u_1 is not an extremal solution.

Another problem which use Hardy inequalities is the study the solutions of

(1.1.5)
$$\begin{aligned} &-\Delta u = \lambda |u|^p \ \ ; p > \tfrac{n}{n-2}, \ \ in \ \ B(0,1) \\ &u \gneqq 0 \ \ in \ \ \Omega. \end{aligned}$$

The above problem has the singular solution for $\lambda = \frac{2}{p-1}\left(n - \frac{2p}{p-1}\right)$

$$u_2 = |x|^{-\frac{2}{p-1}} - 1.$$

The "linearization" of problem (1.1.5) leads to operator

$$L_{u_2} u = -\Delta u - \frac{2p}{p-1}\left(n - \frac{2p}{p-1}\right)\frac{u}{|x|^2}.$$

The authors in [BV] used again the improved Hardy inequality (1.1.2) to prove whether or not the singular solution u_2 is an extremal solution of (1.1.5). For further applications of (1.1.2) see [VZ] and [DD1]. Also the improved Hardy inequalities (1.1.2) and (1.1.3) have been useful in the existence and asymptotic behavior of the heat equation with singular potentials see [VZ] and [DD1].

Brezis and Vazquez [BV] posed the following questions (cf. Problem 2, Section 8): *In case* Ω *is a ball centered at zero, are the two terms on the right-hand side of (1.1.2) just the first two terms of a series? Is there a further improvement of (1.1.3)?*

The answer was given by Filippas and Tertikas in [FT]. In particular, they proved the following Hardy-Sobolev type inequality with critical exponent

(1.1.6)
$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + C\Big(\int_\Omega X_1^{\frac{2(n-1)}{n-2}}\Big(\frac{|x|}{D}\Big)|u|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}}, \quad \forall\, u \in C_0^\infty(\Omega),$$

where $X_1(t) = (1 - \ln t)^{-1}$ and $D = \sup_{x\in\Omega}|x|$. Also, they showed that the estimate in (1.1.6) is optimal in the sense, that $X_1^{\frac{2(n-1)}{n-2}}$ can not be replaced by a smaller power of X_1. In addition, it has been recently established in [AFT] that the optimal constant C in (1.1.6) is

$$C = (n-2)^{-\frac{2(n-1)}{n}} S_n,$$

where $S_n = \pi n(n-2)\left(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\right)^{\frac{2}{n}}$ is the Sobolev best constant.

By inequality (1.1.6), the authors in [FT] showed that for each non-negative potential V that satisfies

(1.1.7)
$$\int_\Omega |V|^{\frac{n}{2}} X_1^{1-n}\Big(\frac{|x|}{D}\Big) dx < \infty,$$

there exists a positive constant b such that the following inequality is valid

(1.1.8)
$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + b\int_\Omega V u^2 dx, \quad \forall\, u \in C_0^\infty(\Omega).$$

Also in [FT], it has been proved that there is no further improvement of (1.1.8) with a nonnegative potential W that satisfies (1.1.7).

Since we have no improvement of (1.1.8), one would expect that there exists a non-negative potential V that satisfies the condition (1.1.7) and the minimizing problem

$$b = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx}{\int_\Omega V u^2 dx},$$

has $H_0^1(\Omega)$ minimizer. Or equivalently the corresponding Euler-Lagrange equation

$$(1.1.9) \qquad \begin{aligned} &-\Delta u = \frac{(n-2)^2}{4} \frac{u}{|x|^2} + bVu, \quad in \ \ \Omega \setminus \{0\} \\ &u \gneqq 0 \ \ in \ \ \Omega, \end{aligned}$$

has $H_0^1(\Omega)$ solution. This reasoning is wrong. The authors in [FT] proved the following more general result
Let $V \in L^p_{loc}(\Omega)$; $p > \frac{n}{2}$, $V^- = \max(-V, 0) \in L^p(\Omega)$; $p > \frac{n}{2}$, and $V^+ = \max(V, 0) \in L^{\frac{n}{2},\infty}(\Omega)$ where $L^{\frac{n}{2},\infty}(\Omega)$ denotes the Lorentz space with norm

$$\|u\|_{L^{\frac{n}{2},\infty}(\Omega)} = \sup_{s>0}(s|\{x \in \Omega : |u| > s\}|^{\frac{n}{2}}).$$

Then, the problem (1.1.9) has no $H_0^1(\Omega)$ solutions.
By the above result we note here that the existence or not of further correction terms in these inequalities does not follow from the non-achievement of the best constants in $H_0^1(\Omega)$.
In our work [Gk] we managed to relax the condition on V^- and moreover we found the optimal one. Our result reads as follows

Theorem 1.1.1. *Suppose for some $p > \frac{n}{2}$, the potential $V \in L^p_{loc}(\Omega \setminus \{0\})$ is such that (1.1.8) holds. We also assume that $V^+ \in L^{\frac{n}{2},\infty}(\Omega)$ and V^- satisfies the condition*

$$\int_\Omega |V^-|^{\frac{n}{2}} X_1^{1-n} dx < \infty.$$

Then, problem (1.1.9) has no nontrivial $H^1(\Omega)$ solution. Moreover the assumptions on the potential V are optimal.

The optimality is meant by the fact that we provide a potential V which satisfies,

$$(1.1.10) \qquad \int_\Omega |V^-|^{\frac{n}{2}} X_1^a dx < \infty, \quad \forall \ a > 1 - n$$

but

$$\int_\Omega |V^-|^{\frac{n}{2}} X_1^{1-n} dx = \infty,$$

in which case the problem (1.1.9) has a solution $\phi \in H^1(\Omega)$ (see Example 1 in Section 3.1).
Note that, in problem (1.1.9), we have assumed without loss of generality that V has a strong irregularity only at zero, since otherwise we could apply the same analysis in any such point. Also, notice that our assumption (1.1.7) on V^- implies that V^- has milder irregularity than $\frac{1}{|x|^2}$ near the origin.
Next, setting

$$u = |x|^{-\frac{n-2}{2}} v,$$

it is not difficult to check that the inequality (1.1.8) is equivalent to

$$(1.1.11) \qquad \int_\Omega |x|^{2-n} |\nabla v|^2 dx \geq b \int_\Omega |x|^{2-n} V(x) v^2 dx,$$

where the constant $b > 0$ in (1.1.11) continues to be optimal. By Theorem 1.1.1, the best constant in (1.1.8) can not be achieved by some $u \in H^1(\Omega)$. However, Filippas and Tertikas [FT] proved that the best constant $b > 0$ in (1.1.11) is achieved for some function $v \in W_0^{1,2}(\Omega; |x|^{-n+2})$. We denote here by $W_0^{1,2}(\Omega; |x|^{-n+2})$ the completion of $C_0^\infty(\Omega)$ under the norm

$$\Big(\int_\Omega |x|^{-(n-2)} |\nabla w|^2 dx + \int_\Omega |x|^{-(n-2)} |w|^2 dx \Big)^{\frac{1}{2}}.$$

We note here that the space $W_0^{1,2}(\Omega; |x|^{-n+2})$ is slightly larger space than $H_0^1(\Omega)$ (see [FT]).
In [FT], the authors obtained the following result:

Proposition 1.1.2. *Let V satisfy the condition (1.1.7) and let b be the best constant in inequality (1.1.11). Then inequality (1.1.11) becomes equality for some $v \in W_0^{1,2}(\Omega; |x|^{-n+2})$. That is, there exists a nontrivial function $v \in W_0^{1,2}(\Omega; |x|^{-n+2})$ which solves the corresponding Euler-Lagrange*

$$\begin{aligned} &\operatorname{div}(|x|^{-(n-2)}\ \nabla v)\ + |x|^{-(n-2)} b V v = 0 \qquad on\ \Omega \setminus \{0\} \\ &v \gneqq 0 \ \ in\ \ \Omega. \end{aligned} \tag{1.1.12}$$

We note here by above Proposition and Theorem 1.1.1, we have that the minimizer of

$$b = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx}{\int_\Omega V u^2 dx},$$

belongs to $W_0^{1,2}(\Omega; |x|^{-n+2})$ but it does not belong to $H_0^1(\Omega)$.
We next consider the improved Hardy inequalities for $n \geq 3$, which are established in [FT]

$$\int_\Omega |\nabla u|^2 dx \geq \frac{(n-2)^2}{4} \int_\Omega \frac{|u|^2}{|x|^2} dx + \frac{1}{4} \int_\Omega \frac{|u|^2}{|x|^2} X_1^2(\frac{|x|}{D}) dx, \quad \forall u \in\ H_0^1(\Omega),$$

$$\begin{aligned} \int_\Omega |\nabla u|^2 dx \geq \frac{(n-2)^2}{4} \int_\Omega \frac{|u|^2}{|x|^2} dx \quad &+ \quad \frac{1}{4} \int_\Omega \frac{|u|^2}{|x|^2} X_1^2(\frac{|x|}{D}) dx \\ &+ \quad C\Big(\int_\Omega |u|^{\frac{2n}{n-2}} \Big(X_1(\frac{|x|}{D}) X_1 \Big(X_1(\frac{|x|}{D}) \Big) \Big)^{\frac{2(n-1)}{n-2}} dx \Big)^{\frac{n-2}{n}} \quad \forall u \in\ H_0^1(\Omega) \end{aligned} \tag{1.1.13}$$

where the constant $\frac{1}{4}$ is optimal.
In [FT], we have again by (1.1.13) that if the non nonnegative potential V satisfies the following condition

$$\int_\Omega |V|^{\frac{n}{2}} \Big(X_1(\frac{|x|}{D}) X_1 \Big(X_1(\frac{|x|}{D}) \Big) \Big)^{1-n} dx < \infty, \tag{1.1.14}$$

then the following inequality is valid for all $u \in H_0^1(\Omega)$

$$\begin{aligned} \int_\Omega |\nabla u|^2 dx \geq \frac{(n-2)^2}{4} \int_\Omega \frac{|u|^2}{|x|^2} dx \quad &+ \quad \frac{1}{4} \int_\Omega \frac{|u|^2}{|x|^2} X_1^2(\frac{|x|}{D}) dx \\ &+ \quad b \int_\Omega V u^2 dx, \qquad \forall u \in\ H_0^1(\Omega), \end{aligned} \tag{1.1.15}$$

where the constant $b > 0$ is optimal.
Setting

$$u = |x|^{-\frac{n-2}{2}} v,$$

then the inequality (1.1.15) becomes equivalent to

$$\begin{aligned} \int_\Omega |x|^{-(n-2)}|\nabla v|^2 dx \quad &\geq \quad \frac{1}{4}\int_\Omega \frac{|v|^2}{|x|^n} X_1^2(\frac{|x|}{D})dx \\ &+ \quad b\int_\Omega V|x|^{-(n-2)}v^2 dx, \quad \forall v \in W_0^{1,2}(\Omega; |x|^{-(n-2)}), \end{aligned} \tag{1.1.16}$$

for the same optimal constant $b > 0$ as in (1.1.15).
The parallel result to Theorem 1.1.1 is

Theorem 1.1.3. *Suppose for some $p > \frac{n}{2}$, the potential $V \in L^p_{loc}(\Omega \setminus \{0\})$ is such that (1.1.8) holds. We also assume that $V^+ \in L^{\frac{n}{2},\infty}(\Omega)$ and V^- satisfies the condition*

$$\int_\Omega |V^-|^{\frac{n}{2}} \left(X_1(\frac{|x|}{D})X_1\left(X_1(\frac{|x|}{D})\right)\right)^{1-n} dx < \infty$$

then problem

$$\begin{aligned} &\mathrm{div}(|x|^{-(n-2)}\ Dv) \ + \tfrac{1}{4}X_1^2\tfrac{v}{|x|^n} + |x|^{-(n-2)}bVv = 0 \qquad \text{on } \Omega\setminus\{0\} \\ &v \gneqq 0 \ \ \text{in} \ \ \Omega. \end{aligned}$$

has no nontrivial $W^{1,2}(\Omega; |x|^{-n+2})$ solution. Moreover the assumptions on the potential V are optimal.

The optimality is meant by the fact that we provide a potential V which satisfies,

$$\int_\Omega |V|^{\frac{n}{2}} \left(X_1(\frac{|x|}{D})\right)^{1-n} \left(X_1\left(X_1(\frac{|x|}{D})\right)\right)^a dx < \infty, \ \forall a > 1-n$$

but

$$\int_\Omega |V|^{\frac{n}{2}} \left(X_1(\frac{|x|}{D})X_1\left(X_1(\frac{|x|}{D})\right)\right)^{1-n} dx = \infty,$$

in which case the problem (1.1.9) has a solution $\phi \in H^1(\Omega)$ (see Example 1 in Section 3.2).
The above result can be inducted. Set first

$$\phi_k(|x|) = |x|^{-\frac{n-2}{2}} X_1^{-\frac{1}{2}}(\frac{|x|}{D})X_2^{-\frac{1}{2}}(\frac{|x|}{D})\cdots X_k^{-\frac{1}{2}}(\frac{|x|}{D}),$$

and $\phi_0(x) = \frac{1}{|x|^{\frac{n-2}{2}}}$, where $X_k(t) := X_1(X_{k-1}(t))$, for $k \geq 2$. We next introduce a new function space which is the appropriate setting in our analysis. We denote by $W_0^{1,2}(\Omega;\ \phi_{k-1}^2)$ the Hilbert space which is the completion of $C_0^\infty(\Omega)$ under the norm

$$\left(\int_\Omega \phi_{k-1}^2 u^2 dx + \int_\Omega \phi_{k-1}^2 |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

Then we have

Theorem 1.1.4. *Suppose for some $p > \frac{n}{2}$ the potential $V \in L^p_{loc}(\Omega \setminus \{0\})$ is such that (**??**) holds. We also assume that $V^+ \in L^{\frac{n}{2},\infty}(\Omega)$ and V^- satisfies the condition*

$$\int_\Omega |V^-|^{\frac{n}{2}}(\prod_{i=1}^{k+1} X_i)^{1-n} dx < \infty.$$

Then the problem

$$(1.1.17)\qquad \begin{aligned} &-\mathrm{div}(\phi_{k-1}^2\nabla v) = \tfrac{1}{4}X_k^2X_{k-1}\cdots X_1\tfrac{v}{|x|^n} + bV\phi_{k-1}^2 v, \quad in \ \ \Omega\setminus\{0\}\\ &v \gneqq 0 \ \ in \ \ \Omega, \end{aligned}$$

has no nontrivial $W^{1,2}(\Omega;\phi_{k-1}^2)$ solution. Moreover the assumptions on the potential V are optimal.

The optimality is meant by the fact that we provide a potential V which satisfies,

$$\int_{B_1(0)} |V^-|^{\frac{n}{2}}(\prod_{i=1}^{k} X_i)^{1-n}X_{k+1}^a dx < \infty \ \ \forall a > 1-n,$$

but

$$\int_\Omega |V^-|^{\frac{n}{2}}(\prod_{i=1}^{k+1} X_i)^{1-n} dx = \infty,$$

in which case the problem (1.1.17) has a solution $\phi \in W^{1,2}(\Omega;\phi_{k-1}^2)$ (see Example 2 in Section 3.2).
Let us mentioned that the main tool in proving the above Theorem is the following k-improved Hardy-Sobolev inequality obtained in [FT]

$$\begin{aligned} \int_\Omega |\nabla u|^2 dx \ &\geq \ \frac{(n-2)^2}{4}\int_\Omega \frac{|u|^2}{|x|^2}dx + \frac{1}{4}\sum_{i=1}^{k}\int_\Omega \frac{|u|^2}{|x|^2}X_1^2(\frac{|x|}{D})X_2^2(\frac{|x|}{D})\cdots X_i^2(\frac{|x|}{D})dx\\ &+ \ C\Big(\int_\Omega |u|^{\frac{2n}{n-2}}\Big(X_1(\frac{|x|}{D})X_2(\frac{|x|}{D})\cdots X_{k+1}(\frac{|x|}{D})\Big)^{\frac{2(n-1)}{n-2}}dx\Big)^{\frac{n-2}{n}}, \quad \forall u\in \ H_0^1(\Omega), \end{aligned}$$

where the constant $\frac{1}{4}$ is optimal.

1.2 Distance From The Boundary

The Hardy inequality in half space $\mathbb{R}_+^n = \{(x', x_n) : \ x_n > 0\}$; $n \geq 2$ asserts that

$$\int_{\mathbb{R}_+^n} |\nabla u|^2 dx \geq \frac{1}{4}\int_{\mathbb{R}_+^n} \frac{u^2}{x_n^2}dx, \quad \forall u \in C_0^\infty(\mathbb{R}_+^n),$$

where the constant $\frac{1}{4}$ is optimal. Note here that $x_n = d(x)$ is the distance function in $\mathbb{R}_+^n$. If we now restrict in an open set Ω with Lipschitz boundary the Hardy inequality reads as

$$\int_\Omega |\nabla u|^2 dx \geq \mu_\Omega \int_\Omega \frac{u^2}{d^2}dx, \quad \forall u \in C_0^\infty(\Omega),$$

where the constant $\mu_\Omega \in (0, \frac{1}{4}]$ (see [MS] and [MMP]). We note here that there exist domains Ω such that $\mu_\Omega < \frac{1}{4}$ (see [MMP]). However if the domain Ω is convex then the constant $\mu_\Omega = \frac{1}{4}$ (see [MS] and [MMP]). Also Davies in [D2] introduced the weight function

$$(1.2.18)\qquad D_\Omega(x) = \left(n|\mathbb{S}^{n-1}|^{-1}\int_{\mathbb{S}^{n-1}} \frac{1}{d_e(x)^2}de\right)^{-\frac{1}{2}},$$

where $d_e(x) := \inf\{|t| : x + te \in \Omega^c\}$ for $e \in \mathbb{S}^{n-1}$ and he proved for any domain $\Omega \subsetneq \mathbb{R}^n$

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{D_\Omega^2}dx \quad \forall u \in C_0^\infty(\Omega).$$

The relation between D_Ω and the distance function d is

$$d(x) \le D_\Omega(x) \qquad \text{if } \Omega \text{ is convex.}$$

This follows by some elementary geometric considerations.
It is clear that Hardy inequality holds in an open domain Ω with the best constant $\frac{1}{4}$, if and only if we make some geometric assumption on Ω. But it is not clear if Hardy inequality with best constant $\frac{1}{4}$ is valid only for convex domain. Indeed Barbatis, Filippas and Tertikas [BFT2] relaxed the assumption of convexity for the domain Ω and they introduced a global geometric condition on Ω

$$-\Delta d \ge 0.$$

And they showed that if Ω satisfies the above condition then Hardy inequality is valid for $\mu_\Omega = \frac{1}{4}$. We note here that the above condition is equivalent to the convexity of the domain Ω for $n = 2$, but it is a much weaker condition than convexity of Ω for $n \ge 3$. Also it has been recently proved that the condition $-\Delta d \ge 0$ is equivalently with the fact that the mean curvature of the boundary of Ω is non-negative see ([P] and [LLL]).
Brezis and Marcus [BM] have established an improved version of Hardy inequality. They showed that for an open, convex and bounded domain Ω with smooth boundary, there exists a constant

$$\lambda(\Omega) \ge \frac{1}{4\text{diam}^2(\Omega)} \tag{1.2.19}$$

such that

$$\int_\Omega |\nabla u|^2 dx \ge \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx + \lambda(\Omega)\int_\Omega u^2 dx, \quad \forall u \in C_0^\infty(\Omega). \tag{1.2.20}$$

In this paper Brezis and Marcus asked whether the diameter of Ω in (1.2.19) can be replaced by an expression depending on $|\Omega| := vol\Omega$, namely, whether $\lambda(\Omega) \ge c|\Omega|^{-\frac{2}{n}}$ with some $c = c(n) > 0$. The answer was given by M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and Laptev [HHL]. They showed that

$$\lambda(\Omega) \ge \frac{c(n)}{|\Omega|^{\frac{2}{n}}}; \qquad c(n) = \frac{n^{\frac{n-2}{n}}|\mathbb{S}^{n-1}|^{\frac{2}{n}}}{4}.$$

In particular if $n = 2$ then $c(2) = \frac{\pi}{2}$.
Filippas, Maz'ya and Tertikas [FMaT2] proved inequality (1.2.20) for an open convex domain with smooth boundary which has finite inner radius. In particular, they showed that (1.2.20) is valid for some constant $\lambda(\Omega)$. Also they showed that there exist $c_1(n) > 0$ and $c_2(n) > 0$ such that

$$c_1(n)\frac{1}{\sup_{x\in\Omega} d^2(x)} \le \lambda(\Omega) \le c_2(n)\frac{1}{\sup_{x\in\Omega} d^2(x)}.$$

The question here is, if there exist domains Ω which satisfies a geometric assumption (e.g. convexity) such that we can add a Sobolev term with critical exponent in the right hand side of the Hardy inequality. For instance in half space $\mathbb{R}^n_+$; $n \ge 3$, Maz'ya [Ma] proved the Hardy-Sobolev inequality

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx \ge \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2}dx + C_n\Big(\int_{\mathbb{R}^n_+} |u|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_0^\infty(\mathbb{R}^n_+),$$

for some constant $C_n > 0$ which depends only on n.
Filippas, Maz'ya and Tertikas [FMaT1] managed to prove this amazing result for a family of open sets. More precisely,

they showed that if Ω is an open domain with finite inner radius and it satisfies $-\Delta d \geq 0$. Then there exist a constant C_Ω such that

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx + C_\Omega\Big(\int_\Omega |u|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_0^\infty(\Omega), \tag{1.2.21}$$

(see Theorem 3.4 of [FMaT1]). Recently in [FL] the authors used the weight function $D_\Omega(x)$ (1.2.18) to prove that the constant C_Ω is independent on Ω, if Ω is convex. In particular, they proved that there exist a constant $K_n > 0$ such that

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{D_\Omega^2}dx + K_n\Big(\int_\Omega |u|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_0^\infty(\Omega).$$

We recall here again that $d(x) \leq D_\Omega(x)$ if Ω is convex.
It is clear here that the Hardy and Hardy-Sobolev inequalities are valid for convex domains or for domains satisfying $-\Delta d \geq 0$ and having finite inner radius. But it is not clear that these geometric conditions on domains are necessary. For instance there is no answer if the Hardy and Hardy-Sobolev inequalities hold in $\Omega = B_1^c(0)$; $B_1(0)$ is the unit ball with center at the origin.
In this thesis we prove the analogue inequalities for domains having different geometric conditions from them which we have presented above. In particular we deal with two different types of such domains.
Firstly, we deal with exterior domains, i.e. complements of smooth compact domains. For our purposes here, smooth means C^2 and we consider exterior domains not containing the origin, for instance $B_1^c(0)$. We note here that an exterior domain Ω can not satisfy the condition $-\Delta d \geq 0$. Thus we need a new condition on Ω. For this we introduce the following

$$-\Delta d(x) + (n-1)\frac{\nabla d(x)\cdot x}{|x|^2} \geq 0 \tag{1.2.22}$$

Note that this condition is satisfied in case $\Omega = B_1^c(0)$.

First we state the Hardy inequality under condition (1.2.22)

Theorem 1.2.1. *Let Ω be an exterior domain in $\mathbb{R}^n$ ($n \geq 3$) not containing the origin and satisfying condition (1.2.22). Then*

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx, \qquad \forall\ u \in C_0^\infty(\Omega).$$

The constant $\frac{1}{4}$ is sharp.

We note here that the above inequality for $n = 2$ does not hold, not even with some positive constant in front of the integral term of the right hand side (see example 2 in section 4.1.2). Intuitively, this happens because for large values of $|x|$ the distance function to the boundary behaves like the distance to the origin, and thus by (1.1.1) it fails.
Let us now state the Hardy-Sobolev inequalities which we will prove in this thesis.

Theorem 1.2.2. *Let $n \geq 4$ and Ω be an exterior domain not containing the origin and satisfying condition (1.2.22). Then the following inequality is valid.*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx \geq C\Big(\int_\Omega |u|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_0^\infty(\Omega), \tag{1.2.23}$$

where the constant $C > 0$ depends only on Ω and the dimension n.

We stress again that the domains referred in the above theorem are of infinite inner radius.

The case $n = 3$ is different, as we can see from the following Theorem.

Theorem 1.2.3. *Let $n = 3$ and Ω be an exterior domain not containing the origin and satisfying condition (1.2.22) with strict inequality i.e.*

$$-\Delta d(x) + 2\frac{\nabla d(x)\cdot x}{|x|^2} \gneqq 0. \tag{1.2.24}$$

Then the following inequality is valid.

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx \geq C\Big(\int_\Omega X^4(\frac{|x|}{D})|u|^6 dx\Big)^{\frac{1}{3}}, \quad \forall\, u \in C_c^\infty(\Omega), \tag{1.2.25}$$

where $X(t) = (1+\ln t)^{-1}$, $0 < D < \inf\{|x| : x \in \partial\Omega\}$ and the constant $C > 0$ depends only on Ω. Moreover, the power 4 on X can not be replaced by a smaller power.

We note here that the condition (1.2.24) is necessary, since in the case where $\Omega = B_1^c(0)$, the inequality (1.2.25) does not hold (see Example 3 in Section 4.1.2).

Let us now assume that Ω is an open bounded domain with smooth boundary. Filippas, Maz'ya and Tertikas [FMaT1] showed the following inequality

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx + M\int_\Omega u^2 dx \geq C\Big(\int_\Omega |u|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_c^\infty(\Omega), \tag{1.2.26}$$

where the constant C depends only on n while M depends on n and Ω.
In this thesis we prove an analogue inequality for the exterior domains. Again the inequalities are different in cases $n \geq 4$ and $n = 3$ as we can see in the following Theorems.

Theorem 1.2.4. *Let $n \geq 4$, $\sigma > 0$ and Ω be an exterior domain not containing the origin. Then there exist constants $C(\Omega, n)$ and $C'(\Omega, n, \sigma)$ such that the following inequality is valid,*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx + C'\int_\Omega \frac{u^2}{1+d^{2+\sigma}}dx \geq C\Big(\int_\Omega |u|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_c^\infty(\Omega) \tag{1.2.27}$$

where $\sigma > 0$.

Theorem 1.2.5. *Let $n = 3$, $\sigma > 0$ and Ω be an exterior domain not containing the origin. Then there exist constants $C(\Omega)$ and $C'(\Omega, \sigma)$ such that*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx + C'\int_\Omega \frac{u^2}{1+d^{2+\sigma}}dx \geq C\Big(\int_\Omega X^4(\frac{|x|}{\rho})u^6 dx\Big)^{\frac{1}{3}}, \quad \forall u \in C_c^\infty(\Omega), \tag{1.2.28}$$

where $X(t) = (1+\ln t)^{-1}$, $\rho = \inf\{|x| : x \in \partial\Omega\}$. Moreover, the power 4 on X can not be replaced by a smaller power.

Note again that the domains considered in the above Theorems are of infinite inner radius.

Next, we deal with **domains above the graph of a $C^{1,1}$ function**. More precisely, let $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying the conditions $|\nabla\Gamma| < \lambda$ and $\Gamma \in C^{1,1}(\mathbb{R}^{n-1})$. We then call the set

$$\Omega = \{(x', x_n) \in \mathbb{R}^n : x_n > \Gamma(x')\},$$

a domain above the graph of a $C^{1,1}$ function. Note that again such domains have infinite inner radius. An example of such domain is the half space $\mathbb{R}^n_+$ for $\Gamma(x') = 0$. As we noted above, the Hardy-Sobolev inequality is valid in the half space for $n = 3$. This fact leads us to consider domains above the graph of a $C^{1,1}$ function as a separate case. Another reason is that the distance function satisfies

$$\frac{1}{1+\lambda}(x_n - \Gamma(x')) \leq d(x) \leq (x_n - \Gamma(x')),$$

that is, the distance function does not behave as distance to a point as x_n goes to infinity.
Thus, we have

Theorem 1.2.6. *Let $n \geq 3$ and Ω be a domain above the graph of $C^{1,1}$ function which satisfies $-\Delta d \geq 0$ in the sense of distributions. Then the following inequality is valid*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx \geq C(n,\lambda)\Big(\int_\Omega |u|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}}, \qquad \forall u \in C_c^\infty(\Omega).$$

Observe that the constant in front of the critical Sobolev term depends only on the dimension n and λ.

1.3 Harnack Inequalities and Heat Kernels Estimates

Harnack inequalities have been extremely useful in the study of solutions of elliptic and parabolic equations. They are used to prove Hölder continuity of solutions, strong maximum principles, Liouville properties, as well as sharp two-sided heat kernel estimates. In particular in parabolic problems, Harnack inequalities are equivalent to sharp two• sided heat kernel estimates. See for instance the books [Gr1], [Z1] and [SC2].
Consider the following parabolic problem

$$u_t = \Delta u \quad \text{in} \quad \Omega \times (0,T] \tag{1.3.29}$$

Then, we have the following interior parabolic Harnack inequality,

Proposition 1.3.1 ([Mo]). *Let $u \geq 0$ be a solution of (1.3.29) and Ω' be a convex subdomain of Ω, such that $d = dist(\Omega', \partial\Omega) > 0$. Then, there exists a positive constant C, depending only on n such that*

$$u(y,s) \leq u(x,t)\exp\Big[C\Big(\frac{|x-y|^2}{t-s} + \frac{t-s}{k} + 1\Big)\Big],$$

for all $x,\ y \in \Omega'$ and all $s,\ t$ satisfying $0 < s < t \leq T$, where $k = \min(1,\ s,\ d^2)$.

As we can note in the following proposition, the boundary Harnack inequality is different than interior Harnack inequality

Proposition 1.3.2 ([S]). *Let $Q = \Omega \times (0,T]$ be a Lipschitz cylinder and Γ a compact subset of $\partial_p Q = (\partial\Omega \times [0,T]) \cup (\Omega \times \{0\})$. Suppose $Q' = \Omega' \times (s,t]$, $0 < s < t < T$, is a subcylinder of Q such that $\partial_p Q \cap \partial_p Q'$ is compactly contained in Γ and (X_0, T_0) is a fixed point in Q, with $T_0 > t$.*
Then, for every nonnegative weak solution of $u_t = \Delta u$ in Q vanishing on Γ, we have

$$u(x,t) \leq Cu(x_0, T_0), \quad \textit{for all } (x,t) \in Q',$$

where C is a constant depending only on $T_0 - t$ and $\partial\Omega$. Moreover s can be chosen equal to zero if $\Gamma \cap (\Omega \times \{0\}) \neq \emptyset$.

A reason that we have not a closed formula for these two cases, is that in the proofs of the above propositions, the authors have not used the properties of the minimizer of

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}. \tag{1.3.30}$$

It is well know that the minimizer $\phi \in H_0^1(\Omega)$ behaves as the distance function $d(x)$ near the boundary of Ω. This fact was used by [Z2] to prove the sharp two side heat kernel estimate for this problem, that is:
Let u be a solution of

$$\begin{aligned} u_t &= \Delta u \quad \text{in} \quad \Omega \times (0, T] \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega \times (0, T] \\ u(x,0) &= u_0 \quad \text{in} \quad \Omega, \end{aligned} \tag{1.3.31}$$

Then there exists a heat kernel $h(t, x, y)$ such that (see [D1])

$$u(t, x) = \int_\Omega h(t, x, y) u_0(y) dy,$$

and $h(t, x, y)$ satisfies

$$\begin{aligned} h_t = \Delta_x h &= \Delta_y h \quad \text{in} \quad \Omega \times (0, \infty) \\ h(t, x, y) &= 0 \quad \text{if} \ \ (x, t) \in \partial\Omega \times (0, \infty) \text{ or } (y, t) \in \partial\Omega \times (0, \infty) \\ h(0, x, y) &= \delta_{x,y} \quad \text{in} \quad \Omega. \end{aligned} \tag{1.3.32}$$

We then have the two side heat kernels estimates

Proposition 1.3.3 ([Z2])**.** *Let Ω be an open set with smooth boundary and $h(t, x, y)$ be the respective heat kernel of the problem (1.3.32). Then there exist positive constants C_1 and C_2 such that*

$$\frac{1}{C_1}\Big(\frac{d(x)}{\sqrt{t} \wedge 1} \wedge 1\Big)\Big(\frac{d(y)}{\sqrt{t} \wedge 1} \wedge 1\Big)\frac{1}{t^{\frac{n}{2}}} \exp(-\frac{|x-y|^2}{C_2 t}) e^{-\lambda_1 t}$$

$$\le h(t, x, y) \le C_1\Big(\frac{d(x)}{\sqrt{t} \wedge 1} \wedge 1\Big)\Big(\frac{d(y)}{\sqrt{t} \wedge 1} \wedge 1\Big)\frac{1}{t^{\frac{n}{2}}} \exp(-\frac{C_2|x-y|^2}{t}) e^{-\lambda_1 t}, \qquad \forall\ x,\ y \in \Omega. \tag{1.3.33}$$

We note here again that the usage of the eigenfunction ϕ is crucial. Also we note that the asymptotic behavior of the heat kernel is different for small time than it is for large time.
Using of a minimizer problem like (1.3.30), we can prove boundary Harnack type inequality and then two side heat kernel estimates for parabolic problems with singular potential.
More precisely, let $n \ge 3$ and $0 \in \Omega$ be an open set with smooth boundary. We consider the following parabolic problem

$$\begin{aligned} u_t = \Delta u + \frac{(n-2)^2}{4}\frac{u}{|x|^2} \quad &\text{in} \quad \Omega \times (0, T] \\ u = 0 \quad &\text{on} \quad \partial\Omega \\ u(0, x) = u_0(x) \quad &\text{in} \quad \Omega. \end{aligned} \tag{1.3.34}$$

We note here that in order to investigate the properties of the solutions of the above problem, we need to investigate the following minimizing problem

$$\lambda_1 = \inf_{u \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx - \frac{(n-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2}}{\int_\Omega u^2 dx}. \tag{1.3.35}$$

where $\lambda_1 > 0$ (see [BV]).
It is well known that (see e.g [DD]) there exists a ground state function $\phi \in H^1_{loc}(\Omega \setminus \{0\})$ which solves the corresponding Euler-Lagrange of (1.3.35)

$$-\Delta\phi - \frac{(n-2)^2}{4} \frac{\phi}{|x|^2} = \lambda_1 \phi, \qquad \text{in } \Omega, \quad \phi(x) = 0, \quad \text{on } \partial\Omega.$$

in the weak sense. Also, due to the results in Lemma 7 in [DD] and using Theorem 7.1 in [DS] on one hand, and elliptic regularity on the other, there exist two positive constants C_1, C_2 such that

$$C_1 d(x)|x|^{-\frac{n-2}{2}} \leq \phi(x) \leq C_2 d(x)|x|^{-\frac{n-2}{2}}.$$

The authors in [FMoT3] used this fact and they proved the boundary Harnack type inequality

Proposition 1.3.4. *Let u be a non-negative solution of (1.3.34). Then there exist a positive constant A such that the following estimate is valid for all $x,\ y \in \Omega$ and all $0 < s < t < T$.*

$$\frac{u(s,y)}{\phi(y)} \leq \frac{u(t,x)}{\phi(x)} \exp\left(A\left(1 + \frac{t-s}{R^2} + \frac{t-s}{s} + \frac{|x-y|^2}{t-s}\right)\right),$$

where the constant $R > 0$ is small enough and depends only on $\partial\Omega$.

(In particular this result is a corollary of Theorem 2.11 in [FMoT3]).
By the boundary Harnack type inequality we have the following two side heat kernel estimates for small times

Proposition 1.3.5 ([FMoT3])**.** *Let $n \geq 3$ and $0 \in \Omega$ be an open bounded domain with smooth boundary. Let $h(t,x,y)$ be the respective heat kernel of the problem (1.3.34). Then there exist positive constants $C_1,\ C_2,\ A_1, A_2$ and $T > 0$ depending on Ω such that*

$$C_1 \min\left((|x| + \sqrt{t})^{\frac{n-2}{2}}(|y| + \sqrt{t})^{\frac{n-2}{2}}, \frac{d(x)d(y)}{t}\right)(|x||y|)^{\frac{2-n}{2}} t^{-\frac{n}{2}} \exp(-A_1 \frac{|x-y|^2}{t}) \leq$$

$$\leq h(t,x,y) \leq$$

$$\leq C_2 \min\left((|x| + \sqrt{t})^{\frac{n-2}{2}}(|y| + \sqrt{t})^{\frac{n-2}{2}}, \frac{d(x)d(y)}{t}\right)(|x||y|)^{\frac{2-n}{2}} t^{-\frac{n}{2}} \exp(-A_2 \frac{|x-y|^2}{t}),$$

for all $x,\ y \in \Omega$ and $t \leq T$.

Concerning the large time asymptotic we have:

Proposition 1.3.6 ([FMoT3])**.** *Let $n \geq 3$ and $0 \in \Omega$ be an open bounded domain with smooth boundary. Let $h(t,x,y)$ be the respective heat kernel of the problem (1.3.34). Then there exist positive constants $C_1,\ C_2$ and $t_0 > 0$ depending on Ω such that*

$$C_1 \phi(x)\phi(y) e^{-\lambda_1 t} \leq h(t,x,y) \leq C_2 \phi(x)\phi(y) e^{-\lambda_1 t},$$

for all $x,\ y \in \Omega$ and $t \geq t_0$.

Another parabolic problem which is widely investigated, is

$$
\begin{aligned}
u_t &= \Delta u + \frac{1}{4}\frac{u}{d^2} && \text{in} \quad \Omega \times (0,T] \\
u &= 0 && \text{on} \quad \partial\Omega \\
u(0,x) &= u_0(x) && \text{in} \quad \Omega.
\end{aligned}
\tag{1.3.36}
$$

As we refer above, we need to consider the following minimizing problem

$$
\lambda_1 = \inf_{u \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}}{\int_\Omega u^2 dx}, \tag{1.3.37}
$$

where $\lambda_1 \in \mathbb{R}$ (see [FMaT1]). It is well known that there exists a ground state $\phi \in H^1_{loc}(\Omega)$ which solves the corresponding Euler-Lagrange of (1.3.37)

$$
-\Delta\phi - \frac{1}{4}\frac{\phi}{d^2} = \lambda_1 \phi, \qquad \text{in } \Omega, \quad \phi(x) = 0, \quad \text{on } \partial\Omega,
$$

in the weak sense. Also there exist positive constants C_1 and C_2 such that

$$
C_1 d^{\frac{1}{2}}(x) \leq \phi(x) \leq C_2 d^{\frac{1}{2}}(x),
$$

near to the boundary (see [DD]). Filippas Moschini and Tertikas [FMoT3] used this fact to prove the following boundary Harnack type inequality

Proposition 1.3.7. *Let u be a non-negative solution of (1.3.34). Then there exist a constant A such that the following estimate is valid for all $x,\ y \in \Omega$ and all $0 < s < t < T$.*

$$
\frac{u(s,y)}{\phi(y)} \leq \frac{u(t,x)}{\phi(x)} \exp\left(A\left(1 + \frac{t-s}{R^2} + \frac{t-s}{s} + \frac{|x-y|^2}{t-s}\right)\right),
$$

where the constant $R > 0$ is small enough and depends only on $\partial\Omega$.

(In particular this result is a corollary of Theorem 2.11 in [FMoT3])
Also they proved the following sharp estimates for the heat kernel of problem (1.3.36)

Proposition 1.3.8. *Let Ω be an open bounded set with smooth boundary and $h(t,x,y)$ be the respective heat kernel of the problem (1.3.36). Then there exist positive constant $C_1, C_2,\ A_1,\ A_2$, and t_0 depend on Ω such that*

$$
C_1\Big[\min\Big(\frac{d(x)}{\sqrt{t}},1\Big)\min\Big(\frac{d(y)}{\sqrt{t}},1\Big)\Big]^{\frac{1}{2}} t^{-\frac{n}{2}} \exp\Big(-A_1\frac{|x-y|^2}{t}\Big)
$$

$$
\leq h(t,x,y) \leq C_2\Big[\min\Big(\frac{d(x)}{\sqrt{t}},1\Big)\min\Big(\frac{d(y)}{\sqrt{t}},1\Big)\Big]^{\frac{1}{2}} t^{-\frac{n}{2}} \exp\Big(-A_2\frac{|x-y|^2}{t}\Big),
$$

for any $x,\ y \in \Omega$ and $t \leq t_o$

Also the authors in [FMoT3] proved for convex domains and large enough t the following Proposition

Proposition 1.3.9. *Let Ω be an open bounded set with smooth boundary which satisfies $-\Delta d \geq 0$ and $h(t,x,y)$ be the respective heat kernel of the problem (1.3.36). Then there exist constant $C_1 > 0$ such that*

$$
e^{-\lambda_1 t}\frac{1}{C_1} d^{\frac{1}{2}}(x) d^{\frac{1}{2}}(y) \leq h(t,x,y) \leq e^{-\lambda_1 t} C_1 d^{\frac{1}{2}}(x) d^{\frac{1}{2}}(y).
$$

for any $x,\ y \in \Omega$ and $t \leq t_0$.

We note here that the eigenvalue λ_1 of (1.3.37) is positive since Ω is convex (see [BM]).

In this thesis we prove boundary Harnack type inequalities for the solutions of problem (1.3.36) where Ω is an exterior domain not containing the origin. We also prove two side estimates for the heat kernel for small time. We remind here that an exterior domain is the complement of a smooth compact domain. For our purposes here, smooth means C^2 and we consider exterior domains not containing the origin, for instance $B_1^c(0)$.
We note here that the problem (1.3.36) in exterior domains is a combination of (1.3.36) and (1.3.34) in bounded domains. The reason is that for large values of $|x|$ the distance function to the boundary behaves like the distance to the origin.
Also since the exterior domain is unbounded we need to investigate the following minimizing problem

$$\lambda_1 = \inf_{u \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}}{\int_\Omega \frac{u^2}{1+d^{2+\sigma}}}, \tag{1.3.38}$$

where $\sigma > 0$.
We prove in this thesis the following theorem

Theorem 1.3.10. *Let $n \geq 3$ and Ω be an exterior open set with smooth boundary not containing the origin. Then the constant λ_1 of 1.3.38 is finite. Also there exist a ground state $\phi \in H^1_{loc}(\Omega)$ of corresponding Euler-Lagrange of (1.3.38) i.e. it is a weak solution of*

$$-\Delta\phi - \frac{1}{4}\frac{\phi}{d^2} = \lambda_1 \frac{\phi}{1 + d^{2+\sigma}} \qquad \text{in } \Omega, \qquad \phi = 0 \qquad \text{on } \partial\Omega.$$

Finally there exist positive constants C_1, C_2 and $a_n = \frac{(n-1)}{2} + \sqrt{\frac{(n-2)^2}{4} - \frac{1}{4}}$ such that

$$C_1 \frac{d^{\frac{1}{2}}(x)}{|x|^{a_n}} \leq \phi(x) \leq C_2 \frac{d^{\frac{1}{2}}(x)}{|x|^{a_n}},$$

for any $x \in \Omega$.

Clearly this problem is a combination of the problems (1.3.36) and (1.3.34) in bounded domains.
Using the above theorem and the program initiated by A. Grigor'yan and L. Saloff-Coste (see [Gr4], [GSC], [Gr2] and [Gr3]) in non-compact Riemannian manifolds (see also [SC1] for a nice survey), we prove the following boundary Harnack type inequality

Theorem 1.3.11. *Let u be a non-negative solution of (1.3.36). Then there exist constant A such that the following estimate is valid for all $x,\ y \in \Omega$ and all $0 < s < t < T$.*

$$\frac{u(s,y)}{\phi(y)} \leq \frac{u(t,x)}{\phi(x)} \exp\left(A\left(1 + \frac{t-s}{R^2} + \frac{t-s}{s} + \frac{|x-y|^2}{t-s}\right)\right),$$

where the constant $R > 0$ is small enough and depends only on $\partial\Omega$.

With this theorem at hand, we are able to obtain two side estimates for the heat kernel $h(t,x,y)$ of the problem (1.3.36) in an exterior domain.

Theorem 1.3.12. *Let Ω be an exterior open set with smooth boundary not containing the origin and let $h(t,x,y)$ be the respective heat kernel of the problem (1.3.36). Then there exist positive constant $C_1, C_2,\ A_1,\ A_2$, and t_0 depend on Ω*

such that

$$C_1\Big[\min\Big(\frac{d(x)}{\sqrt{t}},1\Big)\min\Big(\frac{d(y)}{\sqrt{t}},1\Big)\Big]^{\frac{1}{2}} t^{-\frac{n}{2}}\exp\Big(-A_1\frac{|x-y|^2}{t}\Big)$$

$$\le h(t,x,y)\le C_2\Big[\min\Big(\frac{d(x)}{\sqrt{t}},1\Big)\min\Big(\frac{d(y)}{\sqrt{t}},1\Big)\Big]^{\frac{1}{2}} t^{-\frac{n}{2}}\exp\Big(-A_2\frac{|x-y|^2}{t}\Big),$$

for any $x,\ y\in\Omega$ *and* $t\le t_0$.

Chapter 2

Some Basic Methods

In this chapter we present some known results and we give some proofs of them for convenience to the reader of this thesis. Especially, in section 2.1 we prove the inequality 1.1.6 which proof is in [FT]. Finally, in section 2.2 we present a simple elliptic problem and we explain the Moser's iteration of this problem.

2.1 Hardy and Hardy-Sobolev Inequalities in Bounded Domain

In chapter 4 we prove Hardy and Hardy-Sobolev type inequalities in unbounded domain. Thus in this section we would like to present some proofs in bounded domains which will help familiarize the reader with the proofs of these type inequalities.

First we prove the following Proposition,

Proposition 2.1.1 ([FT]). *We assume that, $n \geq 3$ and Ω is an open bounded domain which contains the origin. Then the following inequality is valid*

$$\int_\Omega |\nabla u|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{u^2}{|x|^2} dx + C\Big(\int_\Omega X^{\frac{2(n-1)}{n-2}}\big(\frac{|x|}{D}\big)|u|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}}, \quad \forall\, u \in C_0^\infty(\Omega), \tag{2.1.1}$$

where $X(t) = (1 - \ln t)^{-1}$ and $D = \sup_{x\in\Omega} |x|$.

To prove Theorem 2.1.1 we need the next Lemma, the proof of which can be found in [Ma].

Lemma 2.1.2. *Let $A(r)$, $B(r)$ be nonnegative functions. Such that $1/A(r)$, $B(r)$ are integrable in (r, ∞) and $(0, r)$, respectively, for all positive $r < \infty$. Then, for $q \geq 2$ the following inequality*

$$\Big[\int_0^s B(t)|u(t)|^q dt\Big]^{\frac{1}{q}} \leq C\Big[\int_0^s A(t)|u'(t)|^2 dt\Big]^{\frac{1}{2}} \quad , \tag{2.1.2}$$

is valid for all $u \in C^1[0, s]$ such that $u(s) = 0$ (or vanish near infinity, if $s = \infty$), if and only if

$$K := \sup_{r\in(0,s)} \Big[\int_0^r B(t)dt\Big]^{\frac{1}{q}} \Big[\int_r^s (A(t))^{-1} dt\Big]^{\frac{1}{2}} < \infty. \tag{2.1.3}$$

The best constant in (2.1.3) satisfies the following inequality

$$K \leq C \leq K\Big(\frac{q}{q-1}\Big)^{\frac{1}{2}} q^{\frac{1}{q}}. \tag{2.1.4}$$

proof of Theorem 2.1.1: Suppose first that $\Omega = B_1(0)$. Following [VZ] we decompose w into spherical harmonics (since $u \in C_0^\infty(B_D(0))$) to get

$$u(x) = \sum_{m=0}^{\infty} u_m(r) f_m(\sigma),$$

where f_m are orthogonal in $L^2(S^{n-1})$ normalized by $\frac{1}{nw_n}\int_{S^{n-1}} f_m(\sigma)f_n(\sigma)dS = \delta_{mn}$. In particular $f_0(\sigma) = 1$ and the first term in the above decomposition is given by

$$u_0(r) = \frac{1}{nw_n r^{n-1}} \int_{\partial B_r} u(x)dS_x.$$

The f_m's are eigenfunctions of the Laplace-Beltrami operator (∇_σ) with corresponding eigenvalues $c_m = m(n-2+m)$, $m \geq 0$. An easy calculation shows that,

$$\int_{B_1} |\nabla u|^2 dx = \sum_{m=0}^{\infty} \int_{B_1} |\nabla u_m|^2 dx + \sum_{m=0}^{\infty} c_m \int_{B_1} \frac{u_m^2}{|x|^2} dx. \tag{2.1.5}$$

We next estimate the nonradial part using the inequality

$$\int_{B_1} |\nabla u_m|^2 dx + \left(c_m - \frac{n-2}{4}\right) \int_{B_1} \frac{u_m^2}{|x|^2} dx \geq \frac{c_m}{c_m + \frac{(n-2)^2}{4}} \left(\int_{B_1} |\nabla u_m|^2 dx + c_m \int_{B_1} \frac{u_m^2}{|x|^2} dx \right), \ m \geq 1.$$

Taking into account that $c_m \geq N-1$ for $m \geq 1$

$$\begin{aligned} \sum_{m=1}^{\infty} \int_{B_1} |\nabla u_m|^2 dx + \sum_{m=1}^{\infty} \left(c_m - \frac{n-2}{4}\right) \int_{B_1} \frac{u_m^2}{|x|^2} dx &\geq \frac{4(n-1)}{n^2} \Big(\int_{B_1} |\nabla(u-u_0)|^2 dx + \int_{B_1} \frac{|u-u_0|^2}{|x|^2} dx \Big) \\ &\geq C \Big(\int_{B_1} X^{\frac{2(n-1)}{n-2}}(|x|) |u-u_0|^{\frac{2n}{n-2}} dx \Big)^{\frac{n-2}{n}}, \end{aligned} \tag{2.1.6}$$

where in the last inequality we have used the Sobolev inequality and the fact that $0 \leq X \leq 1$. Now, setting $u_0 = |x|^{-\frac{n-2}{2}} w$ we can easily check that

$$\int_\Omega |\nabla u|^2 dx - \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{u^2}{|x|^2} dx = \int_\Omega \frac{|\nabla w|^2}{|x|^{n-2}} dx + \frac{1}{2} \int_\Omega \nabla(|x|^{-(n-2)}) \nabla w^2 dx. \tag{2.1.7}$$

We next show that the last integral above is equal to zero. Let $B_\varepsilon = \{x : \ |x| < \varepsilon\}$ and $S_\varepsilon = \{x : \ |x| = \varepsilon\}$. We then write

$$\int_\Omega \nabla(|x|^{-(n-2)}) \nabla w^2 dx = \int_{B_\varepsilon} \nabla(|x|^{-(n-2)}) \nabla w^2 dx + \int_{\Omega \setminus B_\varepsilon} \nabla(|x|^{-(n-2)}) \nabla w^2 dx.$$

The integrand in the above integrals is easily checked to be an L^1 function and therefore the first integral on the right-hand side tends to zero as $\varepsilon \to 0$. Concerning the second-integral, integrating by parts and using the fact that $\Delta |x|^{-(n-2)} = 0$ we end up with

$$\begin{aligned} \int_{\Omega \setminus B_\varepsilon} \nabla(|x|^{-(n-2)}) \nabla w^2 dx = (n-2)\varepsilon^{-n+1} \int_{S_\varepsilon} w^2 dS_x \quad &= \quad \frac{n-2}{\varepsilon} \int_{S_\varepsilon} u_0^2 dS_x \\ &\to 0 \quad \text{as } \varepsilon \to 0, \end{aligned} \tag{2.1.8}$$

since $u_0 \in C_0^\infty(\Omega)$.

It then follows that the last term in (2.1.7) is zero, and the following identity holds:

$$\int_\Omega |\nabla u_0|^2 dx - \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{u_0^2}{|x|^2}dx = \int_\Omega \frac{|\nabla w|^2}{|x|^{n-2}}dx. \tag{2.1.9}$$

Using (2.1.9), inequality (2.1.1) becomes equivalent to

$$\int_\Omega \frac{|\nabla w|^2}{|x|^{n-2}}dx \geq C\Big(\int_\Omega \frac{|w|^{\frac{2n}{n-2}}}{|x|^n} X(\frac{|x|}{D})^{\frac{2(n-1)}{n-2}}dx\Big)^{\frac{n-2}{n}}. \tag{2.1.10}$$

Now since w is a radially symmetric function, inequality (2.1.10) is equivalent to

$$\int_0^1 r|w_r|^2 dr \geq C\Big(\int_0^1 \frac{|w|^{\frac{2n}{n-2}}}{r} X(\frac{r}{D})^{\frac{2(n-1)}{n-2}}dr\Big)^{\frac{n-2}{n}},$$

where $D = \sup_{x\in\partial\Omega}|x|$. We note that the last inequality is valid by Lemma 2.1.2 for $A(r) = r$, $B(r) = \frac{X^{\frac{2(n-1)}{n-2}}(\frac{r}{D})}{r}$ and $q = \frac{2n}{n-2}$. Thus by the last inequality and inequality (2.1.6) the result follows in the case where the Ω is the unit ball. Consider now the case where Ω is a bounded domain. Then, for some $R > 0$ we have that $\Omega \subset B_R$. Since (2.1.1) is true for any $u \in C_0^\infty(B_R)$ it is true in particular for every $u \in C_0^\infty(\Omega)$ □

Finally, in the following proposition we would like to show how someone can use a geometric condition for a domain to prove the Hardy inequality.

Proposition 2.1.3 ([BFT1])**.** *Let $n \geq 2$. We assume that Ω is an open domain. We also assume that Ω satisfies $-\Delta d \geq 0$ in the sense of distributions. Then the following Hardy inequality is valid*

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx, \quad \forall u \in C_0^\infty(\Omega), \tag{2.1.11}$$

where the constant $\frac{1}{4}$ is optimal.

proof: We set $u = d^{\frac{1}{2}}v$, then by straightforward calculation we have

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega d|\nabla v|^2 dx + \frac{1}{4}\int_\Omega \frac{v^2|\nabla d|^2}{d}dx + \frac{1}{2}\int_\Omega \nabla d\cdot\nabla v^2 dx.$$

Now since $|\nabla d| = 1$ a.e and $\int_\Omega \nabla d\cdot\nabla v^2 dx = -\int_\Omega \Delta d v^2 dx \geq 0$, we have the desired result. □

2.2 The Moser Iteration in a Warm up Problem

In this thesis, Moser's iteration technique plays a fundamental role. For this reason we present here the main ideas of this technique in a warm up problem that we took from L. Saloff-Coste [SC1]. As we will see in a moment, it relies on a certain Sobolev type inequality.
We consider the eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u \qquad \text{in} \quad \Omega \\ u &= 0 \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.2.12}$$

where Ω is an open bounded set. Consider also a solution $u \in H_0^1(\Omega)$ in the weak sense i.e. u satisfies

$$\int_\Omega \nabla u \nabla v dx = \lambda \int_\Omega uv dx, \qquad \forall v \in H_0^1(\Omega). \tag{2.2.13}$$

We say that λ is an eigenvalue of $-\Delta$ provided, there exists a non-trivial solution $u \in H_0^1(\Omega)$ of (2.2.13).
It is well known that (see [E]) that the set of eigenvalues is countable. Also we have

$$0 < \lambda_1 \leq \lambda_2 \leq ...$$

and $\lim_{k\to\infty} \lambda_k = \infty$. In addition, there exists an orthonormal basis $\{\phi_k\}_{k=1}^\infty$ of $L^2(\Omega)$ where $0 < \phi_k \in H_0^1(\Omega)$ is an eigenfunction corresponding to λ_k :

$$\begin{aligned} -\Delta\phi_k &= \lambda_k\phi_k \qquad \text{in} \quad \Omega \\ \phi_k &= 0 \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.2.14}$$

for $k = 1, 2, ...$ Finally, we have by standard elliptic regularity that, $\phi_k \in C^\infty(\Omega)$ and ϕ_k is bounded for any $k = 1, 2, ...$ (see [E]). The main goal is to prove the following upper bound

Proposition 2.2.1. *Let $n \geq 1$ and let ϕ_k be an eigenfunction corresponding to λ_k of problem 2.2.14, then there exist a positive constant A_n such that*

$$\sup_{x\in\Omega} \phi_k^2(x) \leq A_\nu \lambda_k^{\frac{\nu}{2}} \int_\Omega |\phi_k|^2 dx, \tag{2.2.15}$$

To prove the upper bound (2.2.15), it is suffices to use only the following Sobolev type inequality which we call Moser inequality.

Proposition 2.2.2. ***Moser Inequality.*** *Let $n \geq 1$. Then, there exists a positive constant C_n, depending only on ν such that*

$$\int_{\mathbb{R}^n} |f|^{2(1+\frac{2}{n})} dx \leq C_n \int_{\mathbb{R}^n} |\nabla f|^2 dx \left(\int_{\mathbb{R}^n} |f|^2 dx \right)^{\frac{2}{n}}, \qquad \forall f \in C_0^\infty(\mathbb{R}^n). \tag{2.2.16}$$

proof: For the proof of proposition we need to use three cases.
First case $n = 1$
Since $f \in C_0^\infty(R)$, we have

$$\begin{aligned} f^2(x) &= 2\int_{-\infty}^x f'(y)f(y)dy \leq 2\left(\int_\mathbb{R} |f'(y)|^2 dy\right)^{\frac{1}{2}} \left(\int_\mathbb{R} |f(y)|^2 dy\right)^{\frac{1}{2}}, \quad \forall\, x \in \mathbb{R} \Rightarrow \\ sup_{x\in\mathbb{R}} f^4(x) &\leq 4\left(\int_\mathbb{R} |f'(y)|^2 dy\right)\left(\int_\mathbb{R} |f|^2(y) dy\right). \end{aligned}$$

Thus by above inequality we have

$$\int_\mathbb{R} |f|^{2(1+2)} dx \leq 4\left(\int_\mathbb{R} |f'(y)|^2 dy\right)\left(\int_\mathbb{R} |f(y)|^2 dy\right)^2,$$

which is the desired result for $n = 1$.
Second case $n = 2$

First, we recall the following inequality

$$\int_{\mathbb{R}^n}|f|^{\frac{n}{n-1}}dx\leq S_n\left(\int_{\mathbb{R}^n}|\nabla u|dx\right)^{\frac{n}{n-1}},\quad \forall\, f\in C_0^\infty(\mathbb{R}^n),$$

where $S_n=n\pi^{\frac{1}{2}}\left(\Gamma(1+\frac{n}{2})\right)^{-\frac{1}{n}}$ (see [Ma]). Then by above inequality we have for $n=2$

$$\int_{\mathbb{R}^2}|f|^{2(1+1)}dx\leq 2S_n\left(\int_{\mathbb{R}^2}|\nabla f||f|dx\right)^2\leq 2S_n\int_{\mathbb{R}^2}|\nabla f|^2dx\int_{\mathbb{R}^2}|f|^2dx,$$

which is the desired result for $n=2$.
Third case $n\geq 3$.
We recall the classical Sobolev inequality

$$\left(\int_{\mathbb{R}^n}|f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\leq C_n\int_{\mathbb{R}^n}|\nabla f|^2dx,\qquad \forall f\in H_0^1(\mathbb{R}^n), \tag{2.2.17}$$

By Hölder's inequality we have for any $f\in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n}|f|^{2(1+\frac{2}{n})}dx=\int_{\mathbb{R}^n}|f|^2|f|^{\frac{4}{n}}dx\leq\left(\int_{\mathbb{R}^n}|f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\left(\int_{\mathbb{R}^n}|f|^2dx\right)^{\frac{2}{n}}.$$

Now by the last inequality and classical Sobolev inequality (2.2.17) we have the Nash inequality

$$\int_{\mathbb{R}^n}|f|^{2(1+\frac{2}{n})}dx\leq C_n\int_{\mathbb{R}^n}|\nabla f|^2dx\left(\int_{\mathbb{R}^n}|f|^2dx\right)^{\frac{2}{n}}.$$

□

We are ready now to prove the upper bound (2.2.15).
proof of Proposition 2.2.1: For $1\leq p<\infty$ we take $v=|\phi_k|^{2p-2}\phi_k$ in (2.2.13). Then by straightforward calculations we have

$$\lambda_k\int_\Omega|\phi_k|^{2p}dx=(2p-1)\int_\Omega|\phi_k|^{2p-2}|\nabla\phi_k|^2dx=\frac{2p-1}{p^2}\int_\Omega|\nabla|\phi_k|^p|^2dx, \tag{2.2.18}$$

Setting $f=|\phi_k|^p$ in (2.2.16), together with (2.2.18) we obtain

$$\int_\Omega|\phi_k|^{2p(1+\frac{2}{n})}dx\leq C_np\lambda_k\left(\int_\Omega|\phi_k|^{2p}dx\right)^{1+\frac{2}{n}}.$$

Finally set in the last inequality $p_i=\left(1+\frac{2}{n}\right)^i$ to get

$$\begin{aligned}\int_\Omega|\phi_k|^{2p_{i+1}}dx &\leq \left(1+\frac{2}{n}\right)^{i+(i-1)p_1}(C_n\lambda_k)^{1+p_1}\left(\int_\Omega|\phi_k|^{2p_{i-1}}dx\right)^{p_2}\\ &\leq \cdots\leq(C_n\lambda_k)^{\sum_{j=0}^i p_j}\left(1+\frac{2}{n}\right)^{\sum_{j=0}^i(i-j)p_j}\left(\int_\Omega\phi_k^2\right)^{p_{i+1}}\Rightarrow\\ \left(\int_\Omega|\phi_k|^{2p_{i+1}}dx\right)^{\frac{1}{p_{i+1}}} &\leq (C_n\lambda_k)^{\sum_{j=1}^{i+1}\frac{1}{p_j}}\left(1+\frac{2}{n}\right)^{\sum_{j=1}^{i+1}(j-1)\frac{1}{p_j}}\int_\Omega\phi_k^2dx.\end{aligned}$$

Note that $\sum_{j=1}^\infty(j-1)\frac{1}{p_j}<\infty$, $\sum_{j=1}^\infty\frac{1}{p_j}=\frac{n}{2}$ and $\left(\int_\Omega|\phi_k|^{2p_{i+1}}dx\right)^{\frac{1}{p_{i+1}}}\to\sup_{x\in\Omega}\phi_k^2$, as $i\to\infty$. The desired conclusion

(2.2.15) follows. □

Chapter 3

Existence and Nonexistence of Energy Solutions

Let $\Omega \subset \boldsymbol{R}^n$ be an open bounded domain that contains the origin. In this chapter we find conditions on the potential V which ensure the nonexistence of positive solutions for linear elliptic problems with Hardy-type potentials. Especially, in section 3.1 we prove the nonexistence of nontrivial solutions in $H^1(\Omega)$ for the equation

$$\text{(3.0.1)} \qquad \begin{array}{ll} -\Delta u = \frac{(n-2)^2}{4}\frac{u}{|x|^2} + bVu, & \text{in } \Omega \setminus \{0\} \\ u \gneqq 0 & \text{in } \Omega. \end{array}$$

We denote here by $H^1(\Omega)$ the Sobolev space which consists of all functions $u : \Omega \to \mathbb{R}$ such that, ∇u exists in the weak sense and

$$||u||^2_{H^1(\Omega)} = \int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 dx < \infty.$$

The results depend on an integral assumption on the potential V

$$\int_\Omega |V^-|^{\frac{n}{2}} X_1^{1-n}(\frac{|x|}{D}) dx < \infty,$$

where $X_1(t) = (1 - \ln t)^{-1}$. We also give an example establishing that this integral assumption on V is optimal.
In section 3.2, we prove the nonexistence of nontrivial solutions in $W^{1,2}(\Omega; \phi_{k-1})$ for the equation

$$\text{(3.0.2)} \qquad \begin{array}{l} -\text{div}(\phi^2_{k-1} Dv) = \frac{1}{4} X_k^2 X_{k-1} \cdots X_1 \frac{v}{|x|^n} + V\phi^2_{k-1} v, \quad \textit{in } \Omega \setminus \{0\} \\ v \gneqq 0 \quad \textit{in } \Omega. \end{array}$$

We denote here by $W^1(\Omega; \phi_{k-1})$ the space which consists of all functions $u : \Omega \to \mathbb{R}$ such that, ∇u exists in the weak sense and

$$\int_\Omega \phi^2_{k-1} u^2 dx + \int_\Omega \phi^2_{k-1} |Du|^2 dx < \infty,$$

where

$$\text{(3.0.3)} \qquad \phi_k(|x|) = |x|^{-\frac{n-2}{2}} X_1^{-\frac{1}{2}}(\frac{|x|}{D}) X_2^{-\frac{1}{2}}(\frac{|x|}{D}) \cdots X_k^{-\frac{1}{2}}(\frac{|x|}{D}),$$

$X_1(t) = (1 - \ln t)^{-1}$, $D = \sup_{x \in \Omega} |x|$, and $X_k(t) := X_1(X_{k-1}(t))$, for $k \geq 2$. Also we set $\phi_0 = \frac{1}{|x|^{n-2}}$. The results depend on an integral assumption on the potential V

$$\int_\Omega |V^-|^{\frac{n}{2}} \Big(\prod_{i=1}^{k+1} X_i\Big)^{1-n} dx < \infty. \tag{3.0.4}$$

We also give an example establishing that this integral assumption on V is optimal.

3.1 Nonexistence $H^1(\Omega)$ solutions

In this section we suppose that $n \geq 3$ and Ω is an open bounded domain which contains the origin. Also we recall the following inequality in [FT]

$$\int_\Omega |\nabla u|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{u^2}{|x|^2} dx + C\Big(\int_\Omega X_1^{\frac{2(n-1)}{n-2}}\Big(\frac{|x|}{D}\Big)|u|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}}, \quad \forall\, u \in C_0^\infty(\Omega), \tag{3.1.5}$$

where $X_1(t) = (1 - \ln t)^{-1}$ and $D = \sup_{x\in\Omega} |x|$. Now we consider a potential $V \in L^p_{loc}(\Omega)$ for $p > \frac{n}{2}$ that has the following properties:

i) there exists $b > 0$ such that:

$$\int_\Omega |\nabla u|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{u^2}{|x|^2} dx + b\int_\Omega V u^2 dx, \quad \forall\, u \in C_0^\infty(\Omega). \tag{3.1.6}$$

ii) $V^+ \in L^{\frac{n}{2},\infty}(\Omega)$ where we denote here by $L^{\frac{n}{2},\infty}(\Omega)$ Lorentz space with norm

$$\|u\|_{L^{\frac{n}{2},\infty}(\Omega)} = \sup_{s>0}\big(s|\{x \in \Omega : |u| > s\}|^{\frac{2}{n}}\big),$$

which is equivalent to the semi-norm

$$\|u\|^*_{L^{\frac{n}{2},\infty}(\Omega)} = \sup_{E\subset\Omega} |E|^{-1+\frac{2}{n}} \int_E |u| dx$$

iii) and V^- satisfies the following condition

$$\int_\Omega |V^-|^{\frac{n}{2}} X_1^{1-n}\Big(\frac{|x|}{D}\Big) dx < \infty. \tag{3.1.7}$$

We next suppose that the constant $b > 0$ in (3.1.6) is optimal. Our main question is whether the best constant $b > 0$ in (3.1.6) is achieved for some function $u \in H^1_0(\Omega)$, or equivalently whether the corresponding Euler-Lagrange equation

$$\begin{cases} -\Delta u = \frac{(n-2)^2}{4}\frac{u}{|x|^2} + bVu, & \text{in } \Omega\setminus\{0\} \\ u \gneqq 0 & \text{in } \Omega, \end{cases} \tag{3.1.8}$$

has $H^1(\Omega)$ solutions. The answer is given in the following Theorem

Theorem 3.1.1. *Suppose for some $p > \frac{n}{2}$, the potential $V \in L^p_{loc}(\Omega\setminus\{0\})$ is such that (3.1.6) holds. We also assume that $V^+ \in L^{\frac{n}{2},\infty}(\Omega)$ and V^- satisfies the condition (3.1.7). Then, problem (3.1.8) has no $H^1(\Omega)$ solutions.*

Note that, in problem (3.1.8), the assumptions on V is optimal. Particularly in the next example, we provide a potential V which satisfies,

$$\int_\Omega |V^-|^{\frac{n}{2}} X_1^a dx < \infty, \quad \forall\, a > 1-n \tag{3.1.9}$$

but

$$\int_{\Omega} |V^-|^{\frac{n}{2}} X_1^{1-n} dx = \infty, \tag{3.1.10}$$

and in which case the problem (3.1.8) has a solution $\phi \in H^1(\Omega)$.

Example 1 We consider the radially symmetric function $u(x) = |x|^{-\frac{n-2}{2}} X_1^{\beta}(|x|)$ for $\beta > \frac{1}{2}$ which belong to $H^1(B_1(0))$. By straightforward calculation we have

$$-\Delta u = -r^{-1-\frac{n}{2}}(\beta(\beta+1)X_1^{\beta+2}(r) - \frac{(n-2)^2}{4} X_1^{\beta}(r)) = \frac{(n-2)^2}{4} \frac{u}{|x|^2} + Vu,$$

with $V(x) = -\beta(\beta+1)\frac{X_1^2}{|x|^2}$. Note here that $u \in H^1(B_1(0))$ is a nontrivial solution of problem (3.1.8) and the potential $V(x)$ satisfies the condition (3.1.9) and (3.1.10).

Before we go to prove Theorem 3.1.1, let us prove the Harnack inequality for the positive solutions of problem 3.1.8 which is crucial to our analysis. But first we need the following proposition which Kurata proved in [Ku]. This proposition give to us Harnack inequality for the solutions of linear elliptic equations which include potentials in local Kato class $K_n(\Omega)$. Let us first define the local Kato class $K_n(\Omega)$. We set

$$\overline{\eta}(f; r; \Omega) = \sup_{x \in \mathbf{R}^n} \int_{B_r(x)} \frac{|f(y)|\chi_\Omega}{|x-y|^{n-2}} dy,$$

then the function f belong to $K_n(\Omega)$ if and only if

$$\lim_{r \to 0} \overline{\eta}(f; r; \Omega) = 0.$$

Theorem 3.1.2. *Let u a nonnegative weak solution of*

$$Lu = -\sum_{i,j=1}^{n} (a_{i,j} u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} + V(x)u(x) = 0 \quad in\ \Omega,$$

where Ω is a bounded open subset of $\mathbf{R}^n$ and the $(a_{i,j})_{i,j=1}^n$ satisfy the following conditions:
1. $a_{i,j} = a_{j,i} \ \ \forall\ i, j = 1, .., n$
2. $\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \quad \forall \xi \in \mathbf{R}^n, \ \ for\ some\ \lambda \in (0, 1]$

and V, $(b_i^2)_{i=1}^n$ belong to the local Kato class $K_n(\Omega)$. Consider also the constant $\eta(n, \lambda) > 0$ be small enough such that

$$\overline{\eta}(V; r; \Omega) + \sum_{i=1}^{n} \overline{\eta}(b_i^2; r; \Omega) < \eta, \quad \forall r \leq r_0,$$

for some $r_0(n, \lambda, \eta) > 0$. Then there exists constant $C = C(n, \lambda, \eta)$ such that :

$$\max_{B_r} u \leq C \min_{B_r} u,$$

for $B_{4r} \subset \Omega$ and $\forall\ r < r_0$.

Lemma 3.1.3. *Let u be an $H^1(\Omega)$ solution of (3.1.8) where the negative part of the potential V satisfies the assumption (3.1.7) and the positive part of the potential V belong to the Lorentz space $L^{\frac{n}{2},\infty}(\Omega)$. Also assume that $B_{\frac{3}{2}} \subset\subset \Omega$ and* *consider for $\lambda \geq 1$ the domain*

$$D_\lambda = \{\frac{1}{2\lambda} < |x| < \frac{2}{3\lambda}\}.$$

Then there exists a positive constant C such that

$$u(x) \leq Cu(y) \quad \forall x, y \in D_\lambda,$$

where the positive constant C depends only on n, $\|V\|_{L^{\frac{n}{2},\infty}(\Omega)}$ and Ω but does not depend on $\lambda \geq 1$.

proof: We assume that $B_R \subset\subset \Omega$. Then we set $r_1 = dist\{\partial B_R, \partial\Omega\}$ and take $x \in B_R \setminus B_\rho$ for $\rho < R$. Also we take r such that $4r \leq \frac{min\{r_1,\rho\}}{3}$. Then we note that $V \in K_n(B_r(x))$ $\forall x \in B_R \setminus B_\rho$, since $V \in L^p(B_r(x))$ for some $p > \frac{n}{2}$ and

$$\int_{B_r(x)} \frac{V}{|x-y|^{n-2}} dx + \int_{B_r(x)} \frac{C(n)}{|y|^2|x-y|^{n-2}} dx \leq$$

$$\|V\|_{L^p(B_R\setminus B_\rho)}\Big(\int_{B_r(x)} \frac{1}{|x-y|^{\frac{p(n-2)}{p-1}}} dx\Big)^{\frac{p-1}{p}} + (\frac{12}{11\rho})^2 \int_{B_r(x)} \frac{1}{|x-y|^{n-2}} dx$$

$$\leq c(n,\Omega) r^{2-\frac{n}{p}} (\|V\|_{L^p(B_R\setminus B_\rho)} + (\frac{12}{11\rho})^2).$$

But, note that

$$\|V\|_{L^p(B_R\setminus B_\rho)} = \Big(\frac{|B_R \setminus B_\rho|^{1-\frac{2}{np}}}{|B_R \setminus B_\rho|^{1-\frac{2}{np}}} \int_{B_R\setminus B_\rho} |V|^p dx\Big)^{\frac{1}{p}}$$

$$\leq C|\Omega|^{\frac{1}{p}-\frac{2}{np^2}} \||V|^p\|^{\frac{1}{p}}_{L^{\frac{np}{2},\infty}(\Omega)} = C(n,p,\Omega)\|V\|_{L^{\frac{n}{2},\infty}(\Omega)}.$$

Let η as in Theorem 3.1.2, we can choose $r_0 > 0$ such that

$$C(n,p,\Omega)\Big(\|V\|_{L^{\frac{n}{2},\infty}(\Omega)} + (\frac{12}{11\rho})^2\Big) r_0^{2-\frac{n}{p}} \leq \eta \Leftrightarrow$$

$$r_0^{2-\frac{n}{p}} \leq \frac{\eta}{C(n,p,\Omega)(\|V\|_{L^{\frac{n}{2},\infty}(\Omega)} + (\frac{12}{11\rho})^2)}. \tag{3.1.11}$$

Then u satisfies the assumptions of Theorem 3.1.2 and we have that

$$u(z) \leq Cu(y) \quad \forall\ z, y \in B_r(x),$$

$B_{4r} \subset B_R \setminus B_\rho$ and the constants $C,\ r_0$ depend on $(n,\ \eta,\ \rho)$.
We next set $\lambda = 1$, then by (3.1.11) and Theorem 3.1.2 there exists η and r_0 such that

$$r_0^{2-\frac{n}{p}} \leq \frac{\eta}{C(n,p,\Omega)(\|V\|_{L^{\frac{n}{2},\infty}(\Omega)} + (\frac{24}{11})^2)}$$

and

$$u(z) \leq Cu(y) \quad \forall\ z, y \in B_r(x),$$

where $B_{4r} \subset D$ and constants $C,\ r_0$ depends on $(n,\ \eta,\ V)$. Then, since D is compact there exists $N(n)$ and $x_i \in D_1$ such that $D \subset \bigcup_{i=1}^N B_{r_0}(x_i)$. And we obtain obviously

$$u(x) \leq C^N u(y) \quad \forall\ x, y \in D.$$

Let take now the set D_λ for $\lambda \geq 1$, then we have

$$\left(\frac{r_0}{\lambda}\right)^{2-\frac{n}{p}} \leq \frac{\eta}{C(n,p,\Omega)(\|V\|_{L^{\frac{n}{2},\infty}(\Omega)} + (\frac{24\lambda}{11})^2)} \leq \frac{\eta}{C(n,p,\Omega)(\|V\|_{L^{\frac{n}{2},\infty}(\Omega)} + (\frac{24}{11})^2)}$$

and since $\frac{r_0}{\lambda} < r_0$, we note that for the same $x_i,\ r_0,\ N$ as above that

$$D_\lambda \subset \bigcup_{i=1}^{N} B_{r_0}(\frac{x_i}{\lambda}),$$

which imply,

$$u(x) \leq C^N u(y) \qquad \forall\ x, y \in\ D_\lambda,$$

and the proof of Lemma is complete. □

proof of Theorem 3.1.1: For the proof of Theorem, we argue by contradiction. That is, we assume that u is a $H^1(\Omega)$ positive solution of (3.1.8) (then by standard elliptic regularity we know that $u \in W^{2,p}_{loc}(\Omega \setminus \{0\}) \cap C_{loc}(\Omega \setminus \{0\})$ for some $p > 1$). Thus we can take the surface average of u,

$$U(r) = \frac{1}{nw_n r^{n-1}} \int_{\partial B_r} u(x) dS_x, \tag{3.1.12}$$

where w_n denotes the volume of the unit ball in $\mathbb{R}^n$ and without loss of generality, we assume that the unit ball is contained in Ω. Standard calculations show that, U satisfies the O.D.E almost everywhere

$$U''(r) + \frac{n-1}{r} U'(r) + \frac{(n-2)^2}{4} \frac{U(r)}{r^2} = f(r) - g(r) \quad a.e, \tag{3.1.13}$$

where

$$f(r) = \frac{1}{nw_n r^{n-1}} \int_{\partial B_r} V^- u(x) dS_x \tag{3.1.14}$$

and

$$g(r) = \frac{1}{nw_n r^{n-1}} \int_{\partial B_r} V^+ u(x) dS_x. \tag{3.1.15}$$

We next change variable by

$$W(r) = r^{\frac{n-2}{2}} U(r), \tag{3.1.16}$$

thus by equation (3.1.13), W satisfies the following O.D.E

$$(rW')' = r^{\frac{n}{2}} (f(r) - g(r)) \quad a.e. \tag{3.1.17}$$

Hence, by Lemma 3.1.4 (see below), we have that there exists a constant $C > 0$ independent on r such that

$$W(r) \leq C X_1^{\frac{n-2}{2n}}(r). \tag{3.1.18}$$

To reach a contradiction we will find a lower bound for W that is incompatible with (3.1.18). Working in this direction, first we set

$$W(r) = X_1^{\beta}(r)Z(r),$$

where

$$-\frac{1}{2} < \beta < 0.$$

Then, by straightforward calculations we obtain that Z satisfies the following O.D.E a.e

$$rX_1^{\beta}(r)Z''(r) + 2\beta X_1^{\beta+1}(r)Z'(r) + X_1^{\beta}(r)Z'(r) + \beta(\beta+1)\frac{X_1^{\beta+2}(r)}{r}Z(r) = r^{\frac{n}{2}}(f(r) - g(r)) \quad a.e.$$

We next multiply the above equation by X_1^{β} and we obtain that

$$(rX_1^{2\beta}(r)Z(r)')' = X_1^{\beta}(r)r^{\frac{n}{2}}(f(r) - g(r)) - \beta(\beta+1)\frac{X_1^{2\beta+2}(r)}{r}Z(r) \quad a.e. \tag{3.1.19}$$

Next, we set

$$Q(r) = rX_1^{2\beta}(r)\frac{Z'(r)}{Z(r)}, \tag{3.1.20}$$

then by equation (3.1.19), we obtain that Q satisfies the following O.D.E

$$rX_1^{2\beta}(r)Q'(r) + Q^2(r) = F(r) - G(r) - \beta(\beta+1)X_1^{4\beta+2}(r) \quad a.e, \tag{3.1.21}$$

where

$$F(r) = \frac{r^{1+\frac{n}{2}}X_1^{3\beta}(r)f(r)}{Z(r)}, \tag{3.1.22}$$

$$G(r) = \frac{r^{1+\frac{n}{2}}X_1^{3\beta}(r)g(r)}{Z(r)}. \tag{3.1.23}$$

Thus, by Lemmas 3.1.5, 3.1.6 (see below) we have that, given $\varepsilon > 0$ there exist $r_0 > 0$ and $-\frac{1}{2} < \beta_0 < 0$ such that

$$Q(r) \leq \varepsilon X_1^{2\beta+1}(r), \quad \forall\ 0 < r \leq r_0,\ \ \beta_0 \leq \beta < 0$$

that is

$$\frac{Z'(r)}{Z(r)} \leq \varepsilon\frac{X_1(r)}{r}.$$

Integrating this from r to r_0, we obtain,

$$Z(r) \geq CX_1^{\varepsilon}(r),$$

where $C > 0$ is independent on r, which contradicts by the fact that

$$CX_1^{\varepsilon}(r) \leq Z(r) = X_1^{-\beta}(r)W(r) \leq CX_1^{\frac{n-2}{2n}-\beta}(r),$$

if we choose ε and $|\beta|$ small enough. The result follows. □

It remains to prove the three lemmas which we used in the proof of Theorem 3.1.1. At first we have:

Lemma 3.1.4. *Let U, W, f, g be as defined in (3.1.12), (3.1.16), (3.1.14), (3.1.15), respectively, with V as in Theorem 3.1.1 and $u \in H^1(\Omega)$. We also assume that $B_1(0) \subset \Omega$ and W satisfies in $(0, 1]$ the equation (3.1.17). Then*

(i) $\lim_{r\downarrow 0} W(r) = 0$.

(ii) For r $\in (0, 1]$*, the following representation formula holds,*

(3.1.24)
$$W(r) = \int_0^r \frac{1}{t}\int_0^t s^{\frac{n}{2}}(f(s) - g(s))dsdt.$$

(iii) In addition, for r sufficiently small, say $r < r_0$, the following estimate holds:

(3.1.25)
$$W(r) \le CX_1^{\frac{(n-2)}{2n}}(r) \;\; \forall\, 0 < r < r_0,$$

for some positive constant C independent on r.

proof:

(i) For the proof of the first statement of Lemma, we argue by contradiction. We assume that there exists positive constants $C_0 > 0$ and $r_0 > 0$ such that $W(r) > C_0$ for $0 < r < r_0$. Now since $u \in H^1(B_{r_0})$, we have also that $u \in L^{\frac{2n}{n-2}}(B_{r_0})$. Then it follows from the definitions of U and W (using Hölder's inequality) that

$$\begin{aligned}\frac{1}{r^{n-1}}\int_{\partial B_r}|u|^{\frac{2n}{n-2}}dS_x &\ge c\Big(\frac{1}{r^{n-1}}\int_{\partial B_r}udS_x\Big)^{\frac{2n}{n-2}}\\ &= CU(r)^{\frac{2n}{n-2}} = C(r^{-\frac{n-2}{2}}W(r))^{\frac{2n}{n-2}} \ge C\frac{1}{r^n}.\end{aligned}$$

Hence, multiply the last inequality by r^{n-1} and then integrate it from 0 to r_0 to obtain that

$$\int_{B_{r_0}}|u|^{\frac{2n}{n-2}}dx \ge C\int_0^{r_0}\frac{1}{r}dr = \infty.$$

This is a clearly contradiction, since $u \in L^{\frac{2n}{n-2}}(B_{r_0})$ and C is independent from r. The first statement of Lemma follows. (ii) To prove the second statement of Lemma, we note that the W-equation can be easily integrated to yield

(3.1.26)
$$W(r) = C_1 + \int_r^1 \frac{1}{t}(C_2 + \int_t^1 s^{\frac{n}{2}}(f(s) - g(s))ds)dt.$$

First, we will show that the following limit exists

$$\lim_{t\to 0}\int_t^1 s^{\frac{n}{2}}(f(s) - g(s))ds = I_2 < \infty$$

At first we note that $I_2 \neq -\infty$, since otherwise (3.1.26) would contradict the positivity of W. Hence, it is enough to show that

$$\begin{aligned}J &:= \int_0^1 s^{\frac{n}{2}}f(s)ds\\ &= C(n)\int_0^1 r^{\frac{-n+2}{2}}\int_{\partial B_r}V^-(x)u(x)dS_x dr < \infty.\end{aligned}$$

Since $u \in H^1(\Omega)$, we have $u \in L^{\frac{2n}{n-2}}(\Omega)$. Thus by applying Hölder's inequality as follows, we have :

$$\begin{aligned}\int_{\partial B_r} V^-(x)u(x)dS_x &\le C(\int_{\partial B_r}|V^-|^{\frac{n}{2}}dS_x)^{\frac{2}{n}}(\int_{\partial B_r}|u|^{\frac{2n}{n-2}}dS_x)^{\frac{n-2}{2n}}(\int_{\partial B_r}1dS_x)^{\frac{n-2}{2n}}\\ &= r^{(n-1)\frac{n-2}{2n}}X_1^{\frac{2(n-1)}{n}}C\Big(\int_{\partial B_r}|V^-|^{\frac{n}{2}}X_1^{1-n}dS_x\Big)^{\frac{2}{n}}\Big(\int_{\partial B_r}|u|^{\frac{2n}{n-2}}dS_x\Big)^{\frac{n-2}{2n}}.\end{aligned}$$

Hence, taking into account the last inequality in J and use Hölder's inequality once more we obtain :

$$\begin{aligned}J &\le C\int_0^1 r^{-\frac{n-2}{2n}}X_1^{\frac{2(n-1)}{n}}\Big(\int_{\partial B_r}|V^-|^{\frac{n}{2}}X_1^{1-n}dS_x\Big)^{\frac{2}{n}}\Big(\int_{\partial B_r}|u|^{\frac{2n}{n-2}}dS_x\Big)^{\frac{n-2}{2n}}dr\\ &\le C(n)\Big(\int_{\Omega}|V^-|^{\frac{n}{2}}X_1^{1-n}dS_x\Big)^{\frac{2}{n}}\|u\|_{L^{\frac{2n}{n-2}}(\Omega)}\Big(\int_0^1\frac{X_1^{\frac{4(n-1)}{n-2}}}{r}dr\Big)^{\frac{n-2}{2n}} < \infty.\end{aligned}$$

The reason which the last integral in the above inequality is finite follows by noting first $\frac{4(n-1)}{2n} > 1$ and

$$\Big(\int_0^r\frac{X_1^{\frac{4(n-1)}{n-2}}}{s}ds\Big)^{\frac{n-2}{2n}} = \frac{X_1^{\frac{3n-2}{2n}}(r)}{(\frac{3n-2}{n})^{\frac{n-2}{2n}}}$$

Also, note that we have the following estimate

$$\int_0^r s^{\frac{n}{2}}f(s)ds \le CX_1^{\frac{(-3n+2)}{2n}}(r). \tag{3.1.27}$$

We are ready now to compute the constants. In view of the statement (i) of Lemma and the fact that I_2 is finite, we assert that $C_2 = -I_2$. Since otherwise, we would have that, there would exist $t_0 > 0$ such that $\forall\ 0 < t < t_0 < 1$

$$C_2 + \int_t^1 s^{\frac{n}{2}}(f(s)-g(s))ds > \frac{C_2+I_2}{2},\ if\ C_2+I_2>0$$

or

$$C_2 + \int_t^1 s^{\frac{n}{2}}(f(s)-g(s))ds < \frac{C_2+I_2}{2},\ if\ C_2+I_2<0$$

hence,

$$\begin{aligned}&\int_r^1\frac{1}{t}(C_2+\int_t^1 s^{\frac{n}{2}}(f(s)-g(s))ds)dt\\ =&\int_{t_0}^1\frac{1}{t}(C_2+\int_t^1 s^{\frac{n}{2}}(f(s)-g(s))ds)dt+\int_r^{t_0}\frac{1}{t}(C_2+\int_t^1 s^{\frac{n}{2}}(f(s)-g(s))ds)dt.\end{aligned}$$

But if $I_2 + C_2 > 0$ then

$$\begin{aligned}\int_r^{t_0}\frac{1}{t}(C_2+\int_t^1 s^{\frac{n}{2}}(f(s)-g(s))ds)dt &\ge \frac{I_2+C_2}{2}\int_r^{t_0}\frac{1}{t}dt\\ &= \frac{I_2+C_2}{2}(ln(t_0)-lnr)\to\infty\end{aligned}$$

or if $I_2 + C_2 < 0$ then

$$\begin{aligned}\int_r^{t_0}\frac{1}{t}(C_2+\int_t^1 s^{\frac{n}{2}}(f(s)-g(s))ds)dt &\le \frac{I_2+C_2}{2}\int_r^{t_0}\frac{1}{t}dt\\ &= \frac{I_2+C_2}{2}(ln(t_0)-ln(r))\to-\infty.\end{aligned}$$

Thus, in the first case we have a contradiction by the statement (i) of Lemma, and in the second case we have a contradiction by positivity of W. Hence equation (3.1.24) can be written as

$$W(r) = C_1 - \int_r^1 \frac{1}{t} \int_0^1 s^{n/2}(f(s) - g(s))ds)dt.$$

To compute C_1, first we observe (using (3.1.27)) that

$$\int_r^1 \frac{1}{t} \int_0^1 s^{\frac{n}{2}} f(s)dsdt \leq C \int_r^1 t^{-1} X_1^{\frac{(3n-2)}{2n}} dt = X_1^{\frac{n-2}{2n}}|_r^1 < C \tag{3.1.28}$$

where C is independent on r. Now, note that

$$\int_r^1 \frac{1}{t} \int_0^1 s^{\frac{n}{2}} g(s)dsdt < \infty, \tag{3.1.29}$$

since otherwise, we would have

$$\begin{aligned} W(r) &= C_1 - \int_r^1 \frac{1}{t} \int_0^1 s^{\frac{n}{2}}(f(s) - g(s))dsdt \\ &= C_1 - \int_r^1 \frac{1}{t} \int_0^1 s^{\frac{n}{2}} f(s)dsdt + \int_r^1 \frac{1}{t} \int_0^1 s^{\frac{n}{2}} g(s)dsdt, \end{aligned}$$

but the first integral in the above equation is finite by (3.1.28), hence $W(r) \to \infty$, as r go to zero which is a contradiction by the statement (i) of Lemma. Thus, by (3.1.28), (3.1.29) and the statement(i) of Lemma, we choose $C_1 = I_1$ (since otherwise W(r) would not go to zero as r approaches zero), thus with this choice of C_1 the representation formula follows.

(iii) Finally, to prove the third statement(iii) of the Lemma, we use the representation formula and (3.1.27),

$$\begin{aligned} W(r) = \int_0^r \frac{1}{t} \int_0^t s^{\frac{n}{2}}(f(s) - g(s))dsdt &\leq \int_0^r \frac{1}{t} \int_0^t s^{\frac{n}{2}} f(s)dsdt \\ &\leq C \int_0^r s^{-1} X_1^{\frac{3n-2}{2n}} ds = C X_1^{\frac{n-2}{2n}}(r), \end{aligned}$$

and the result follows. □

Let us now prove the O.D.E lemma.

Lemma 3.1.5. *Let Q be a solution of*

$$r X_1^{2\beta} Q'(r) + Q^2(r) = F(r) - G(r) - \beta(\beta+1) X_1^{4\beta+2}(r) \quad a.e, \quad in\ 0 < r \leq 1 \tag{3.1.30}$$

where $F,\ G$ are nonnegative functions, $-1/2 < \beta < 0$ and

$$\int_0^1 \frac{X_1^{-2\beta} F(s)}{s} ds < \infty,$$

then

$$\lim_{r \downarrow 0} Q(r) = 0.$$

Moreover if given any $\varepsilon > 0$ there exist $r_0 > 0$ such that $\forall\ r \le r_0$ to have

$$\int_0^r \frac{X_1^{-2\beta}F(s)}{s}ds \le \varepsilon X_1^{2\beta+1}(r),$$

then for sufficiently small $|\beta|$, we have the following estimate

$$Q(r) \le 2\varepsilon X_1^{2\beta+1}.$$

proof: After multiplying equation (3.1.30) by $\frac{X_1^{-2\beta}}{r}$ and integrating it from r *to* r_0 with $r < r_0$, we have

$$\begin{aligned} Q(r) = \int_r^{r_0} \frac{X_1^{-2\beta}(s)Q^2(s)}{s}ds + Q(r_0) + \int_r^{r_0} \frac{X_1^{-2\beta}(s)G(s)}{s}ds \quad - \quad \int_r^{r_0} \frac{X_1^{-2\beta}(s)F(s)}{s}ds \\ + \quad \beta(\beta+1)\int_r^{r_0} \frac{X_1^{2\beta+2}(s)}{s}ds. \end{aligned} \tag{3.1.31}$$

Note that the last integral in (3.1.31) is finite since $\beta > -\frac{1}{2}$. We next claim that

$$\int_0^1 \frac{X_1^{-2\beta}(s)Q^2(s)}{s}ds < \infty,$$

since otherwise $H(r) = \int_r^1 \frac{X_1^{-2\beta}(s)Q^2(s)}{s}ds \to \infty$ as r approaches zero. But, by equation (3.1.31) $\lim_{r\to 0} Q(r) = \infty$ which implies that we can always find $r_0 > 0$ such that

$$Q(r) > \int_0^1 \frac{X_1^{-2\beta}(s)F(s)}{s}ds - \beta(\beta+1)\int_0^1 \frac{X_1^{2\beta+2}(s)}{s}ds, \quad \forall r \le r_0. \tag{3.1.32}$$

We may then rewrite (3.1.31) as

$$\begin{aligned} (-rX_1^{2\beta}(r)H'(r))^{\frac{1}{2}} = H(r) + Q(r_0) + \int_r^{r_0} \frac{X_1^{-2\beta}G(s)}{s}ds \quad - \quad \int_r^{r_0} \frac{X_1^{-2\beta}F(s)}{s}ds \\ + \quad \beta(\beta+1)\int_r^{r_0} \frac{X_1^{2\beta+2}(s)}{s}ds, \end{aligned}$$

by using (3.1.32) and the fact that $G \ge 0$ we have

$$(-rX_1^{2\beta}H'(r))^{\frac{1}{2}} \ge H(r), \quad \forall r \le r_0.$$

Hence for $r \le r_0$ we have that :

$$-rX_1^{2\beta}(r)H'(r) \ge H^2(r) \Leftrightarrow \left(\frac{1}{H(r)} - \frac{X_1^{-2\beta-1}(r)}{-2\beta-1}\right)' \ge 0.$$

Integrating this from r to r_0 we obtain

$$-\frac{1}{H(r)} + \frac{X_1^{-2\beta-1}(r)}{-2\beta-1} \ge C,$$

where C is a real constant. But, we have a contradiction, since $H(r)$ grows to infinity as r tends to zero and $\lim_{r\to 0} \frac{X_1^{-2\beta-1}(r)}{-2\beta-1} = -\infty$. Hence, $\lim_{r\to 0} H(r) < \infty$.

Now, we have three cases :

1. $Q^2(r) \to \infty$ as $r \to 0$
2. $Q^2(r) \to c > 0$ as $r \to 0$
3. $Q^2(r) \to 0$ as $r \to 0$

Let us assume that the first case is true. Then there are r_0, $M > 0$ such that $\forall\ 0 < r < r_0$ to have $Q^2(r) > M$. Hence,

$$\int_0^1 \frac{X_1^{-2\beta}(s)Q^2(s)}{s}ds \geq \int_{r_0}^1 \frac{X_1^{-2\beta}(s)Q^2(s)}{s}ds + \int_0^{r_0} \frac{X_1^{-2\beta}(s)Q^2(s)}{s}ds$$

$$\geq \int_{r_0}^1 \frac{X_1^{-2\beta}(s)Q^2(s)}{s}ds + M\int_0^{r_0} \frac{X_1^{-2\beta}(s)}{s}ds = \int_{r_0}^1 \frac{X_1^{-2\beta}(s)Q^2(s)}{s}ds + M\int_0^{r_0} (\frac{X_1^{-2\beta-1}(s)}{-2\beta-1})'ds = \infty,$$

which contradicts by the fact that $\int_0^1 \frac{X_1^{-2\beta}(s)Q^2(s)}{s}ds < \infty$.

Respectively, if we assume that the second case is true, then we can choose $r_0 > 0$ such that $\forall\ r < r_0$ to have that $Q^2(r) > c - \varepsilon$, where we have chosen $\varepsilon = \frac{c}{2}$, hence, by the same arguments as the first case we reach to contradiction. Thus, $lim_{r\to 0}Q(r) = 0$ and the first statement of Lemma follows. To prove the second statement, we note first that, since $Q(r)$ approaches zero as r go to zero, the following representation formula hold

$$Q(r) = -\int_0^r \frac{X_1^{-2\beta}(s)Q^2(s)}{s}ds - \int_0^r \frac{X_1^{-2\beta}(s)G(s)}{s}ds + \int_0^r \frac{X_1^{-2\beta}(s)F(s)}{s}ds - \frac{\beta(\beta+1)}{2\beta+1}X_1^{2\beta+1}(r). \quad (3.1.33)$$

Next, given $\varepsilon > 0$ we choose $r_0 > 0$ such that

$$\int_0^r \frac{X_1^{-2\beta}(s)F(s)}{s}ds \leq \varepsilon X_1^{2\beta+1}(r), \quad \forall r \leq r_0$$

and $\beta_0 > -1/2$ such that $-\frac{\beta(\beta+1)}{2\beta+1} \leq \varepsilon,\ \forall\ \beta_0 \leq \beta < 0$. Thus, by representation formula (3.1.33) and using the fact that the first and the second integral are non-positive, we have the second statement of Lemma. □

Lemma 3.1.6. *Given $\varepsilon > 0$ there exists $r_0 > 0$ such that for all $r \leq r_0$ to have*

$$\int_0^r \frac{X_1^{-2\beta}(s)F(s)}{s}ds < \varepsilon X_1^{2\beta+1}(r),$$

where,

$$F(r) = \frac{r^2 X_1^{4\beta}\int_{\partial B_r} V^- u dS_x}{\int_{\partial B_r} u dS_x},$$

V satisfies the assumptions of Theorem (3.1.1) and u is an $H^1(\Omega)$ solution of (3.1.8).

proof: We assume that $B_{3/2} \subset\subset \Omega$ and consider the domain $D_\lambda = \{\frac{1}{2\lambda} < |x| < \frac{2}{3\lambda}\}$, for $\lambda \geq 1$. Then, by Lemma 3.1.3 we have that there exists constant $C > 0$ which depends only on $n,\ V$ and Ω such that

$$u(x) \leq Cu(y) \quad \forall x, y \in D_\lambda,$$

thus by the last inequality we have

$$u(x) \leq Cu(y)\ \ \forall\ x,\ y \in\ K_r = \{|x| = r\}.$$

Then we obtain that,

$$\begin{aligned}
\frac{X_1^{-2\beta}(r)F(r)}{r} &= \frac{rX_1^{2\beta}(r)\int_{\partial B_r} V^- u dS_x}{\int_{\partial B_r} u dS_x} \\
\le Cr^{-n+2}X_1^{2\beta}(r)\int_{\partial B_r} V^- dS_x &\le Cr^{-n+2}X_1^{2\beta+\frac{2(n-1)}{n}}(r)\Big(\int_{\partial B_r} |V^-|^{\frac{n}{2}} X_1^{1-n} dS_x\Big)^{\frac{2}{n}}\Big(\int_0^r ds\Big)^{\frac{n-2}{n}} \\
&= Cr^{-\frac{n-2}{n}}X_1^{2\beta+\frac{2(n-1)}{n}}(r)\Big(\int_{\partial B_r} |V^-|^{\frac{n}{2}} X_1^{1-n} dS_x\Big)^{\frac{2}{n}}
\end{aligned}$$

where have also used Hölder's inequality. Applying Hölders inequality once more we obtain,

$$\begin{aligned}
\int_0^r \frac{X_1^{-2\beta}(s)F(s)}{s} ds &\le C\Big(\int_{B_r} |V^-|^{\frac{n}{2}} X_1^{1-n} dx\Big)^{\frac{2}{n}}\Big(\int_0^r s^{-1} X_1^{\frac{2\beta n}{n-2}+\frac{2(n-1)}{n-2}} ds\Big)^{\frac{n-2}{n}} \\
&= C\Big(\int_{B_r} |V^-|^{\frac{n}{2}} X_1^{1-n} dx\Big)^{\frac{2}{n}} \frac{X_1^{2\beta+1}(r)}{(\frac{(2\beta+1)n}{n-2})^{\frac{n-2}{n}}}
\end{aligned}$$

and the result follows. □

3.2 Nonexistence $W^{1,2}(\Omega;\phi_{k-1}^2)$ solutions

In this section we suppose that $n \ge 3$ and Ω is an open bounded domain which contains the origin. We next introduce a new function space which is the appropriate setting in our analysis. We denote by $W_0^{1,2}(\Omega;\ \phi_{k-1}^2)$ the Hilbert space which is the completion of $C_0^\infty(\Omega)$ under the norm

$$\Big(\int_\Omega \phi_{k-1}^2 u^2 dx + \int_\Omega \phi_{k-1}^2 |Du|^2 dx\Big)^{\frac{1}{2}},$$

where

$$\phi_k(|x|) = |x|^{-\frac{n-2}{2}} X_1^{-\frac{1}{2}}(\frac{|x|}{D}) X_2^{-\frac{1}{2}}(\frac{|x|}{D}) \cdots X_k^{-\frac{1}{2}}(\frac{|x|}{D}), \tag{3.2.34}$$

$X_1(t) = (1-\ln t)^{-1}$, $D = \sup_{x\in\Omega}|x|$, and $X_k(t) := X_1(X_{k-1}(t))$, for $k \ge 2$. Also we set $\phi_0 = \frac{1}{|x|^{n-2}}$. We recall the inequality in [FT]

$$\Big(\int_\Omega \phi_{k-1}^2 |Du|^2 dx\Big)^{\frac{1}{2}} \ge C\Big(\int_\Omega \frac{|u|^{\frac{2n}{n-2}}}{|x|^n}(\prod_{i=1}^{k-1} X_i) X_k^{\frac{2(n-1)}{n-2}} dx\Big)^{\frac{n-2}{2n}}, \qquad \forall u \in W_0^{1,2}(\Omega;\ \phi_{k-1}^2) \tag{3.2.35}$$

Now we consider a potential $V \in L^p_{loc}(\Omega)$ for $p > \frac{n}{2}$ that has the following properties:
i) there exists $b > 0$ such that:

$$\begin{aligned}
\int_\Omega \phi_{k-1}^2 |\nabla v|^2 dx &\ge \frac{1}{4}\int_\Omega \frac{|v|^2}{|x|^2} X_k^2(\frac{|x|}{D})\phi_{k-1}^2(\frac{|x|}{D}) dx \\
&+ b\int_\Omega V\phi_{k-1}^2 v^2 dx, \quad \forall v \in W_0^{1,2}(\Omega;\phi_{k-1}^2),
\end{aligned} \tag{3.2.36}$$

ii) $V^+ \in L^{\frac{n}{2},\infty}(\Omega)$

iii) and V^- satisfies the following condition

$$\int_\Omega |V^-|^{\frac{n}{2}}(\prod_{i=1}^{k+1} X_i)^{1-n} dx < \infty, \tag{3.2.37}$$

We next suppose that the constant $b > 0$ in (3.2.36) is optimal. Our main question is whether the best constant $b > 0$ in (3.2.36) is achieved for some function $u \in W_0^{1,2}(\Omega;\phi_{k-1}^2)$, or equivalently whether the corresponding Euler-Lagrange equation

$$\begin{aligned} &-\mathrm{div}(\phi_{k-1}^2 Dv) = \tfrac{1}{4} X_k^2 X_{k-1}\cdots X_1 \tfrac{v}{|x|^n} + V\phi_{k-1}^2 v, \quad in \;\; \Omega\setminus\{0\} \\ &v \gneqq 0 \;\; in \;\; \Omega, \end{aligned} \tag{3.2.38}$$

has $\in W^{1,2}(\Omega,\ \phi_{k-1}^2)$ solutions. The answer is given in the following theorem

Theorem 3.2.1. *Suppose for some $p > \frac{n}{2}$ the potential $V \in L^p_{loc}(\Omega\setminus\{0\})$ is such that (3.2.36) holds. We also assume that $V^+ \in L^{\frac{n}{2},\infty}(\Omega)$ and V^- satisfies condition (3.2.37). Then problem (3.2.38) has no $W^{1,2}(\Omega;\phi_{k-1}^2)$ nontrivial solutions.*

We note here that the assumption on the potential V is optimal. Particularly in the next example, we provide a potential V which satisfies,

$$\int_{B_1(0)} |V^-|^{\frac{n}{2}}(\prod_{i=1}^{k} X_i)^{1-n} X_{k+1}^a dx < \infty \;\; \forall a > 1-n, \tag{3.2.39}$$

but

$$\int_{B_1(0)} |V^-|^{\frac{n}{2}}(\prod_{i=1}^{k} X_i)^{1-n} X_{k+1}^{1-n} dx = \infty, \tag{3.2.40}$$

and in which case the problem (3.2.38) has a solution $\phi \in W^{1,2}(\Omega;\phi_{k-1}^2)$,

Example 2 We consider the function $u(x) = X_{k+1}^\beta X_k^{-\frac{1}{2}}$ for $\beta > \frac{1}{2}$ which belong to $W^{1,2}(\Omega;\phi_{k-1}^2)$. Then, we obtain by straightforward calculation that

$$\begin{aligned} \mathrm{div}(\frac{1}{|x|^{n-2}} X_1^{-1}\cdots X_{k-1}^{-1} Du) &= \Big(\beta(\beta+1) X_{k+1}^{\beta+2} X_k^{\frac{3}{2}} X_{k-1}\cdots X_1 + \frac{\beta}{2} X_{k+1}^{\beta+1} X_k^{\frac{3}{2}} X_{k-1}\cdots X_1 \\ &\quad - \frac{\beta}{2} X_{k+1}^{\beta+1} X_k^{\frac{3}{2}} X_{k-1}\cdots X_1 - \frac{1}{4} X_{k+1}^{\beta} X_k^{\frac{3}{2}} X_{k-1}\cdots X_1\Big)\frac{1}{|x|^n} \\ &= -bVu\phi_{k-1}^2 - \frac{1}{4} X_k^2 X_{k-1}\cdots X_1 \frac{u}{|x|^2}. \end{aligned}$$

That is the function u is a solution of problem (3.2.38) with potential $V = -\frac{1}{4}\frac{X_k^2 X_{k-1}\cdots X_1}{|x|^2}$. We note here that the potential V satisfies the condition (3.2.39) and (3.2.40) condition (3.2.37).

Before we prove the Theorem 3.2.1, let us give a Harnack type inequality for positive solutions of problem (3.2.38).

Lemma 3.2.2. *Let u be a $W^{1,2}(\Omega;\phi_{k-1}^2)$ solution of (3.2.38) where the potential V satisfies the assumptions of Theorem*

3.2.1 and assume that $B_{\frac{3}{2}} \subset\subset \Omega$. Then there exists a positive constant C such that

$$u(x) \le Cu(y) \qquad \forall x, y \in K_r = \{z \in \Omega : |z| = r\} \text{ and } r \le \frac{3}{2},$$

where constant C depends only on n, $\|V\|_{L^{\frac{n}{2},\infty}(\Omega)}$ and Ω.

proof: We set $u = \phi_{k-1}^{-1}\upsilon$. Then

$$-\Delta\upsilon = \overline{V}\upsilon, \qquad in \ \ \Omega \setminus \{0\}$$

where

$$\overline{V} = V + \frac{(n-2)^2/4 + \sum_{i=1}^{n} X_1^2 \cdots X_i^2}{|x|^2}.$$

Thus as Lemma 3.1.3 there exist a constant $C > 0$ independent on r such that $\forall\ x,\ y \in \{z : |z| = r\}$ to have

$$\upsilon(x) \le C\upsilon(y) \Rightarrow \phi_{k-1}u(x) \le C\phi_{k-1}u(y).$$

□

proof of Theorem 3.2.1: As in Theorem 3.1.1, we may assume that u is a $W^{1,2}(\Omega;\ \phi_{k-1}^2)$ nontrivial positive solution of (3.2.38) (then by standard elliptic regularity we know that $u \in W_{loc}^{2,p}(\Omega \setminus \{0\}) \cap C_{loc}(\Omega \setminus \{0\})$ for some $p > 1$). We next take the surface average of u :

$$U(r) = \frac{1}{nw_n r^{n-1}} \int_{\partial B_r} u(x)dS_x, \tag{3.2.41}$$

where w_n denotes the volume of the unit ball in R^n. Without loss of generality, we may assume that the unit ball is contained in Ω. As in Theorem 3.1.1, we show by straightforward calculation that U satisfies the following O.D.E,

$$rz_{k-1}^{-1}U'' + U'(z_{k-1}^{-1} - rz_{k-1}^{-2}z_{k-1}') + \frac{1}{4}\frac{X_k^2 z_{k-1}}{r} + rz_{k-1}^{-1}(g(r) - f(r)) = 0 \ \ a.e, \tag{3.2.42}$$

where $z_k = X_1 \cdots X_k$ and f, g defined as (3.1.14) and (3.1.15) respectively. Next, we set

$$U = X_k^{-\frac{1}{2}}W, \tag{3.2.43}$$

thus the equation (3.2.42) becomes

$$-X_k^{\frac{1}{2}}W' + r(z_{k-1}^{-1})'X_k^{-\frac{1}{2}}W' + z_{k-1}^{-1}X_k^{-\frac{1}{2}}W' + rz_{k-1}^{-1}X_k^{-\frac{1}{2}}W'' = r\varphi_{k-1}^2(f-g) \ \ a.e.$$

Finally, if we multiply the last equation by $X_k^{-\frac{1}{2}}$, we can easily obtain that:

$$(rz_k^{-1}W')' = rz_{k-1}^{-1}X_k^{-\frac{1}{2}}(f(r) - g(r)) \quad a.e. \tag{3.2.44}$$

Hence, by Lemma 3.2.3 (see below), we have that there exists a constant $C > 0$ independent on r such that

$$W(r) \le CX_{k+1}^{\frac{n-2}{2n}}(r). \tag{3.2.45}$$

To reach a contradiction we will find a lower bound for W that is incompatible with (3.2.45). Working in this direction, we set $W(r) = X_{k+1}^{\beta} Z(r)$ for $-\frac{1}{2} < \beta < 0$. We can easily check that Z satisfies the following O.D.E

$$(rX_{k+1}^{2\beta} z_k^{-1} Z')' = rX_{k+1}^{\beta} z_{k-1}^{-1} X_k^{-\frac{1}{2}} (f(r) - g(r)) - \beta(\beta+1) \frac{X_{k+1}^{2\beta+2} z_k}{r} \quad a.e. \tag{3.2.46}$$

Next we set $Q = \frac{rX_{k+1}^{2\beta} z_k^{-1} Z'}{Z}$ and by simple calculations, we note that Q satisfies the following O.D.E

$$\begin{aligned} rX_{k+1}^{2\beta} z_k^{-1} Q'(r) + Q^2(r) &= \frac{r^2 X_{k+1}^{3\beta} X_k^{-\frac{3}{2}} z_{k-1}^{-2} f(r)}{Z} - \frac{r^2 X_k^{-\frac{3}{2}} z_{k-1}^{-2} g(r)}{Z} - \beta(\beta+1) X_{k+1}^{4\beta+2} \\ &:= F(r) - G(r) - \beta(\beta+1) X_{k+1}^{4\beta+2} \quad a.e. \end{aligned} \tag{3.2.47}$$

Thus, by Lemmas 3.2.4, 3.2.5 (see below) we obtain that, given $\varepsilon > 0$ there exist $r_0 > 0$ and $-1/2 < \beta_0 < 0$ such that

$$Q(r) \le \varepsilon X_{k+1}^{2\beta+1}(r), \quad \forall 0 < r \le r_0, \ \ \beta_0 \le \beta < 0$$

that is

$$\frac{Z'(r)}{Z(r)} \le \varepsilon \frac{X_{k+1}(r) z_k(r)}{r}.$$

Integrating this from r to r_0, we obtain,

$$Z(r) \ge C X_{k+1}^{\varepsilon}(r),$$

where $C > 0$ is independent on r, which contradicts by the fact that

$$C X_{k+1}^{\varepsilon}(r) \le Z(r) = X_{k+1}^{-\beta}(r) W(r) \le C X_{k+1}^{\frac{n-2}{2n} - \beta}(r),$$

if we choose ε and $|\beta|$ small enough. The result follows. □

Lemma 3.2.3. *Let U, W, f, g be as defined in (3.2.41), (3.2.43), (3.1.14), (3.1.15) respectively. We also assume that* $B_1(0) \subset \Omega$, *V is as in Theorem 3.2.1,* $u \in W^{1,2}(\Omega; \phi_{k-1}^2)$ *and W satisfies in* $(0,1]$ *the equation (3.2.46) almost everywhere. Then*

(i) $\lim_{r \downarrow 0} W(r) = 0$.

(ii) For all r $\in (0,1]$, *the following representation formula holds,*

$$W(r) = \int_0^r \frac{z_k}{t} \int_0^t s X_k^{-1/2} z_{k-1}^{-1} (f(s) - g(s)) ds dt. \tag{3.2.48}$$

(iii) In addition, for r sufficiently small, say $r < r_0$, there exists a positive constant independent on r such that,

$$W(r) \le C X_{k+1}^{\frac{n-2}{2n}}(r) \quad 0 < r < r_0. \tag{3.2.49}$$

proof: For the proof of statement (i) of Lemma, we argue by contradiction. We assume that $W(t) > C_0 > 0\ \forall t \in (0, r_0)$. Then it follows from the definitions of U and W (using Hölder's inequality) that

$$\begin{aligned} \frac{1}{r^{n-1}} \int_{\partial B_r} \frac{|u|^{\frac{2n}{n-2}} z_{k-1} X_k^{\frac{2(n-1)}{n-2}}}{|r|^n} dS_x &\ge c \Big(\frac{1}{r^{n-1}} \int_{\partial B_r} \frac{|u| z_{k-1}^{\frac{n-2}{2n}} X_k^{\frac{n-1}{n}}}{|r|^{\frac{n-2}{2}}} dS_x \Big)^{\frac{2n}{n-2}} \\ &= C \frac{X_k^{\frac{2(n-1)}{n-2}}(r) z_{k-1}(r)}{r^n} \Big(W(r) X_k^{-\frac{1}{2}}(r) \Big)^{\frac{2n}{n-2}} \ge C \frac{z_k(r)}{r^n}, \end{aligned}$$

for some positive constant C independent on r. Hence, multiply the last inequality by r^{n-1} and integrating it from 0 to r_0, we obtain that

$$\int_{B_{r_0}} \frac{|u|^{\frac{2n}{n-2}} z_{k-1} X_k^{\frac{2(n-1)}{n-2}}}{|x|^n} dx \geq C \int_0^{r_0} (\log X_k)' dr = \infty.$$

This is clearly a contradiction, since in view of (3.2.35) we have that $\int_{B_{r_0}} \frac{|u|^{\frac{2n}{n-2}} z_{k-1} X_k^{\frac{2(n-1)}{n-2}}}{|x|^n} dx < \infty$. The first statement of lemma follows.

(ii)

To prove the second statement we note that the W-equation can be easily integrated to yield

$$W(r) = C_1 + \int_r^1 \frac{z_k(t)}{t}(C_2 + \int_t^1 sX_k^{-\frac{1}{2}} z_{k-1}^{-1}(f(s) - g(s))ds)dt. \tag{3.2.50}$$

First, we will show that the following limit exists

$$\lim_{t \to 0} \int_t^1 sX_k^{-\frac{1}{2}} z_{k-1}^{-1}(f(s) - g(s))ds = I_2 < \infty$$

At first we note that $I_2 \neq -\infty$, since otherwise (3.2.50) would contradict the positivity of W. Hence, it is enough to show that

$$\begin{aligned} J &:= \int_0^1 sX_k^{-\frac{1}{2}} z_{k-1}^{-1} f(s)ds \\ &= C(n) \int_0^1 r^{-n+2} X_k^{-\frac{1}{2}} z_{k-1}^{-1} \int_{\partial B_r} V^-(x)u(x)dS_x dr < \infty. \end{aligned}$$

By applying Hölder's inequality as follows, we obtain :

$$\begin{aligned} \int_{\partial B_r} V^-(x)u(x)dS_x &\leq C\Big(\int_{\partial B_r} |V^-|^{\frac{n}{2}} dS_x\Big)^{\frac{2}{n}} \Big(\int_{\partial B_r} |u|^{\frac{2n}{n-2}} dS_x\Big)^{\frac{n-2}{2n}} \Big(\int_{\partial B_r} 1 dS_x\Big)^{\frac{n-2}{2n}} \\ &= r^{\frac{(n-2)(2n-1)}{2n}} X_{k+1}^{\frac{2(n-1)}{n}} X_k^{\frac{n-1}{n}} z_{k-1}^{\frac{3n-2}{2n}} \times \\ &\times C\Big(\int_{\partial B_r} |V^-|^{\frac{n}{2}} z_{k+1}^{1-n} dS_x\Big)^{\frac{2}{n}} \Big(\int_{\partial B_r} \frac{|u|^{\frac{2n}{n-2}} z_{k-1} X_k^{\frac{2(n-1)}{n-2}}}{r^n} dS_x\Big)^{\frac{n-2}{2n}}. \end{aligned}$$

Hence, taking into account the last inequality in J and use Hölder's inequality once more we obtain :

$$J \leq C(n)\Big(\int_\Omega |V^-|^{\frac{n}{2}} z_{k+1}^{1-n} dx\Big)^{\frac{2}{n}} \Bigg(\int_{L^1(B_{r_0})} |x|^{-n} |u|^{\frac{2n}{n-2}} z_{k-1} X_k^{\frac{2(n-1)}{n-2}}\Bigg)^{\frac{n-2}{2n}} \Bigg(\int_0^1 r^{-1} X_{k+1}^{\frac{4(n-1)}{n-2}} z_k dr\Bigg)^{\frac{n-2}{2n}} < \infty.$$

Since $u \in W^{1,2}(\Omega; \phi_{k-1}^2)$ we have in view of (3.2.35)

$$\Bigg(\int_{L^1(B_{r_0})} |x|^{-n} |u|^{\frac{2n}{n-2}} z_{k-1} X_k^{\frac{2(n-1)}{n-2}}\Bigg)^{\frac{n-2}{2n}} < \infty.$$

Also, the last integral above is finite since,

$$\Bigg(\int_0^r s^{-1} X_{k+1}^{\frac{4(n-1)}{n-2}} z_k ds\Bigg)^{\frac{n-2}{2n}} = \frac{X_{k+1}^{\frac{3n-2}{2n}}(r)}{(\frac{3n-2}{n})^{\frac{n-2}{2n}}} \tag{3.2.51}$$

Thus $I_2 < \infty$. The representation formula (3.2.48) follows by the same arguments as in Lemma 3.1.4.
(iii) To prove now the third statement of Lemma, we use the representation formula (3.2.48), the fact that $g \geq 0$ and the estimate (3.2.51)

$$W(r) = \int_0^r \frac{z_k}{t}\int_0^t sX_k^{-\frac{1}{2}} z_{k-1}^{-1}(f(s)-g(s))dsdt \leq \int_0^r \frac{z_k}{t}\int_0^t sX_k^{-\frac{1}{2}} z_{k-1}^{-1} f(s)dsdt$$
$$\leq C\int_0^r \frac{z_k X_{k+1}^{\frac{3n-2}{2n}}}{s}ds = CX_{k+1}^{\frac{n-2}{2n}}(r)$$

□

Let us now give the analogue O.D.E lemma as lemma (3.1.5).

Lemma 3.2.4. *Let Q be a solution of*

(3.2.52) $$rX_{k+1}^{2\beta} z_k^{-1} Q'(r) + Q^2(r) = F(r) - G(r) - \beta(\beta+1)X_{k+1}^{4\beta+2} \quad a.e, \quad in\ 0 < r \leq 1$$

where $F,\ G$ are nonnegative functions, $-1/2 < \beta < 0$ and

$$\int_0^1 \frac{X_{k+1}^{-2\beta} z_k F(s)}{s}ds < \infty.$$

Then

$$\lim_{r\downarrow 0} Q(r) = 0.$$

Moreover if given any $\varepsilon > 0$ there exist $r_0 > 0$ such that $\forall\ r \leq r_0$ to have

$$\int_0^r \frac{X_{k+1}^{-2\beta} z_k F(s)}{s}ds \leq \varepsilon X_{k+1}^{2\beta+1}(r),$$

then for sufficiently small $|\beta|$, we have the following estimate

$$Q(r) \leq 2\varepsilon X_{k+1}^{2\beta+1}(r).$$

proof: After multiplying equation (3.2.52) by $\frac{z_k X_{k+1}^{-2\beta}}{r}$ and integrating it from r *to* r_0 with $r < r_0$, we have

$$Q(r) = \int_r^{r_0} \frac{z_k X_{k+1}^{-2\beta} Q^2(s)}{s}ds + Q(r_0) + \int_r^{r_0} \frac{X_{k+1}^{-2\beta} z_k G(s)}{s}ds - \int_r^{r_0} \frac{X_{k+1}^{-2\beta} z_k F(s)}{s}ds$$
(3.2.53) $$+ \beta(\beta+1)\int_r^{r_0} \frac{X_{k+1}^{2\beta+2} z_k}{s}ds.$$

Note that the last integral in (3.2.53) is finite since $\beta > -\frac{1}{2}$. We next claim that

$$\int_0^1 \frac{X_{k+1}^{-2\beta} z_k Q^2(s)}{s}ds < \infty,$$

since otherwise $H(r) = \int_r^1 \frac{X_{k+1}^{-2\beta} z_k Q^2(s)}{s}ds \to \infty$ as r approaches zero. But, by equation (3.2.53) $\lim_{r\to 0} Q(r) = \infty$ which implies that we can always find $r_0 > 0$ such that

(3.2.54) $$Q(r) > \int_0^1 \frac{X_{k+1}^{-2\beta} z_k F(s)}{s}ds - \beta(\beta+1)\int_0^1 \frac{X_{k+1}^{2\beta+2} z_k}{s}ds, \quad \forall r \leq r_0.$$

We may then rewrite (3.2.53) as

$$(-rX_{k+1}^{2\beta}(r)z_k^{-1}(r)H'(r))^{\frac{1}{2}} = H(r) + Q(r_0) + \int_r^{r_0} \frac{X_{k+1}^{-2\beta} z_k G(s)}{s} ds \quad - \quad \int_r^{r_0} \frac{X_{k+1}^{-2\beta} z_k F(s)}{s} ds$$
$$+ \quad \beta(\beta+1) \int_r^{r_0} \frac{X_{k+1}^{2\beta+2} z_k}{s} ds,$$

by using (3.2.54) and the fact that $G \geq 0$ we have

$$(-rX_{k+1}^{2\beta}(r)z_k^{-1}(r)H'(r))^{\frac{1}{2}} \geq H(r), \quad \forall r \leq r_0.$$

Hence for $r \leq r_0$ we have that :

$$-rX_{k+1}^{2\beta}(r)z_k^{-1}(r)H'(r) \geq H^2(r) \Leftrightarrow \Big(\frac{1}{H(r)} - \frac{X_{k+1}^{-2\beta-1}(r)}{-2\beta-1}\Big)' \geq 0.$$

Integrating this from r to r_0 we obtain

$$-\frac{1}{H(r)} + \frac{X_{k+1}^{-2\beta-1}(r)}{-2\beta-1} \geq C,$$

where C is a real constant. But, we have a contradiction, since $H(r)$ grows to infinity as r tends to zero and $\lim_{r\to 0} \frac{X_{k+1}^{-2\beta-1}(r)}{-2\beta-1} = -\infty$. Hence, $\lim_{r\to 0} H(r) < \infty$.

Now, we have three cases:

1. $Q^2(r) \to \infty \quad as \quad r \to 0$
2. $Q^2(r) \to c > 0 \quad as \quad r \to 0$
3. $Q^2(r) \to 0 \quad as \quad r \to 0$

Let us assume that the first case is true. Then there are r_0 and $M > 0$ such that $\forall\ 0 < r < r_0$ to have $Q^2(r) > M$. Hence,

$$\int_0^1 \frac{X_{k+1}^{-2\beta} z_k Q^2(s)}{s} ds \geq \int_{r_0}^1 \frac{X_{k+1}^{-2\beta} z_k Q^2(s)}{s} ds + \int_0^{r_0} \frac{X_{k+1}^{-2\beta} z_k Q^2(s)}{s} ds$$
$$\geq \int_{r_0}^1 \frac{X_{k+1}^{-2\beta} z_k Q^2(s)}{s} ds + M \int_0^{r_0} \frac{X_{k+1}^{-2\beta} z_k}{s} ds = \int_{r_0}^1 \frac{X_{k+1}^{-2\beta} z_k Q^2(s)}{s} ds + M \int_0^{r_0} (\frac{X_{k+1}^{-2\beta-1}}{-2\beta-1})' ds = \infty,$$

which contradicts by the fact that $\int_0^1 \frac{X_{k+1}^{-2\beta} z_k Q^2(s)}{s} ds < \infty$.

Respectively, if we assume that the second case is true, then we can choose $r_0 > 0$ such that $\forall\ r < r_0$ to have that $Q^2(r) > c - \varepsilon$, where we have chosen $\varepsilon = \frac{c}{2}$, hence, by the same arguments as the first case we reach to contradiction. Thus, $lim_{r\to 0} Q(r) = 0$ and the first statement of Lemma follows. To prove the second statement, we note first that, since $Q(r)$ approaches zero as r go to zero, the following representation formula hold

$$Q(r) = -\int_0^r \frac{X_{k+1}^{-2\beta} z_k Q^2(s)}{s} ds - \int_0^r \frac{X_{k+1}^{-2\beta} z_k G(s)}{s} ds + \int_0^r \frac{X_{k+1}^{-2\beta} z_k F(s)}{s} ds - \frac{\beta(\beta+1)}{2\beta+1} X_{k+1}^{2\beta+1}(r). \tag{3.2.55}$$

Next, given $\varepsilon > 0$ we choose $r_0 > 0$ such that

$$\int_0^r \frac{X_{k+1}^{-2\beta} z_k F(s)}{s} ds \leq \varepsilon X_{k+1}^{2\beta+1}(r), \quad \forall r \leq r_0$$

and $\beta_0 > -1/2$ such that $-\frac{\beta(\beta+1)}{2\beta+1} \leq \varepsilon,\ \forall\ \beta_0 \leq \beta < 0$. Thus, by representation formula (3.2.55) and using the fact that the first and the second integral are non-positive, we have the second statement of Lemma. □

Finally we have

Lemma 3.2.5. *Given $\varepsilon>0$ there exist r_0 such that*

$$\int_0^r \frac{X_{k+1}^{-2\beta} z_k F(s)}{s} ds < \varepsilon X_{k+1}^{2\beta+1}(r), \quad \forall 0<r\le r_0,$$

where

$$F(r)=\frac{r^2 X_{k+1}^{4\beta} z_k^{-2}\int_{\partial B_r} V^- u dS_x}{\int_{\partial B_r} u dS_x},$$

V satisfies the assumptions of Theorem 3.2.1 and u is a $W^{1,2}(\Omega)$ solution of (3.2.38).

proof: By Lemma 3.2.2, we have

$$u(x)\le Cu(y)\ \forall\ x,\ y\in\{x:|x|=r\},$$

where the positive constant C depends only on n, V and Ω. Then we obtain that,

$$\begin{aligned}
\frac{X_{k+1}^{-2\beta} z_k F(r)}{r} &= \frac{r X_{k+1}^{2\beta} z_k^{-1}\int_{\partial B_r} V^- u dS_x}{\int_{\partial B_r} u dS_x} \le C r^{-n+2} r X_{k+1}^{2\beta} z_k^{-1}\int_{\partial B_r} V^- dS_x \\
&\le C r^{-n+2} X_{k+1}^{2\beta+\frac{2(n-1)}{n}}(r) z_k^{\frac{n-2}{n}}(r)\Big(\int_{\partial B_r} |V^-|^{\frac{n}{2}} z_{k+1}^{1-n} dS_x\Big)^{\frac{2}{n}}\Big(\int_0^r ds\Big)^{\frac{n-2}{n}} \\
&= C r^{-\frac{n-2}{n}} X_{k+1}^{2\beta+\frac{2(n-1)}{n}}(r) z_k^{\frac{n-2}{n}}(r)\Big(\int_{\partial B_r} |V^-|^{\frac{n}{2}} z_{k+1}^{1-n} dS_x\Big)^{\frac{2}{n}}
\end{aligned}$$

where have also used Hölder's inequality. Applying Hölders inequality once more we obtain,

$$\begin{aligned}
\int_0^r \frac{X_{k+1}^{-2\beta} z_k F(s)}{s} ds &\le C\Big(\int_{B_r} |V^-|^{\frac{n}{2}} z_{k+1}^{1-n} dx\Big)^{\frac{2}{n}}\Big(\int_0^r s^{-1} X_{k+1}^{\frac{2\beta n}{n-2}+\frac{2(n-1)}{n-2}} z_k ds\Big)^{\frac{n-2}{n}} \\
&= C\Big(\int_{B_r} |V^-|^{\frac{n}{2}} z_{k+1}^{1-n} dx\Big)^{\frac{2}{n}} \frac{X_{k+1}^{2\beta+1}(r)}{(\frac{(2\beta+1)n}{n-2})^{\frac{n-2}{n}}}
\end{aligned}$$

and the result follows. □

Chapter 4

Hardy and Hardy-Sobolev Inequalities in Unbounded Domains

In this chapter, we will prove Hardy and Hardy-Sobolev inequalities in domains with infinite inner radius.
In particular, in subsection 4.1.1 we deal with exterior domains, i.e. complements of smooth compact domains. For our purposes here, smooth means C^2 and we consider exterior domains not containing the origin, for instance $\mathbb{R}^n \setminus B_1(0)$.
Also, we suppose that Ω satisfies the following geometric condition

$$-\Delta d(x) + (n-1)\frac{\nabla d(x) \cdot x}{|x|^2} \geq 0, \tag{4.0.1}$$

in the sense of distributions. Here we denote by $d(x) = \inf_{y \in \partial\Omega} |x - y|$.
Note that this condition is satisfied in case $\Omega = \mathbb{R}^n \setminus B_1(0)$.
First we state the Hardy-Sobolev inequality inequality under condition (4.0.1).

Theorem 4.0.6. *Let $n \geq 4$ and Ω be an exterior domain not containing the origin and satisfying condition (4.0.1). Then the following inequality is valid.*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx \geq C\Big(\int_\Omega |u|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_0^\infty(\Omega),$$

where the constant $C > 0$ depends only on Ω and the dimension n.

We stress again that the domains referred the above theorem are of infinite inner radius.
The case $n = 3$ is different, as we can see from the following Theorem.

Theorem 4.0.7. *Let $n = 3$ and Ω be an exterior domain not containing the origin and satisfying condition (4.0.1) with strict inequality i.e.*

$$-\Delta d(x) + 2\frac{\nabla d(x) \cdot x}{|x|^2} \gneqq 0.$$

Then the following inequality is valid.

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx \geq C\Big(\int_\Omega X^4\big(\frac{|x|}{D}\big)|u|^6 dx\Big)^{\frac{1}{3}}, \quad \forall\, u \in C_c^\infty(\Omega),$$

where $X(t) = (1 + \ln t)^{-1},\ 0 < D < \inf\{|x| :\ x \in \partial\Omega\}$ and the constant $C > 0$ depends only on Ω. Moreover, the power 4 on X can not be replaced by a smaller power.

In subsection 4.1.2, we give three examples where the Hardy or Hardy-Sobolev inequality does not hold. Especially, we give two examples in dimension $n = 2$ (for the sets $\Omega = B_1^c(0)$ and $\Omega = \mathbb{R}^2 \setminus \{-1 \le x \le 1\}$) for which the Hardy inequality

$$\int_\Omega |\nabla u|^2 dx - c\int_\Omega \frac{u^2}{d^2}dx \ge 0, \quad \forall\ u \in C_0^\infty(\Omega),$$

is not valid even for some constant $c < \frac{1}{4}$.
Finally, we give an example for $\Omega = \mathbb{R}^3 \setminus B_1(0)$ for which the Hardy-Sobolev inequality does not hold.

$$\int_{B_1^c} |\nabla u|^2 dx - \frac{1}{4}\int_{B_1^c} \frac{u^2}{d^2}dx \ge c\Big(\int_{B_1^c} u^6 X^a(|x|)dx\Big)^{\frac{n-2}{n}},$$

where $d = |x| - 1$, $X(t) = (1 + ln(t))^{-1}$ and $a > 1$.
In subsection 4.1.3, we still assume that Ω is an exterior domain and we prove the following theorems without assuming a geometric condition on Ω.

Theorem 4.0.8. *Let $n \ge 4$, $\sigma > 0$ and Ω be an exterior domain not containing the origin. Then there exist positive constants $C(\Omega, n)$ and $C'(\Omega, n, \sigma)$ such that the following inequality is valid,*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx + C'\int_\Omega \frac{u^2}{1 + d^{2+\sigma}}dx \ge C\Big(\int_\Omega |u|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_c^\infty(\Omega).$$

Theorem 4.0.9. *Let $n = 3$, $\sigma > 0$ and Ω be an exterior domain not containing the origin. Then there exist positive constants $C(\Omega, n)$ and $C'(\Omega, n, \sigma)$ such that*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx + C'\int_\Omega \frac{u^2}{1 + d^{2+\sigma}}dx \ge C\Big(\int_\Omega X^4(\frac{|x|}{\rho})u^6 dx\Big)^{\frac{1}{3}}, \quad \forall u \in C_c^\infty(\Omega),$$

where $X(t) = (1 + \ln t)^{-1}$, $\rho = \inf\{|x| :\ x \in \partial\Omega\}$. Moreover, the power 4 on X can not be replaced by a smaller power.

In subsection 4.1.4, we deal with the minimizing problem

$$\lambda_1 = \inf_{u \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}}{\int_\Omega \frac{u^2}{1+d^{2+\sigma}}},$$

where $\sigma > 0$. First we prove that $\lambda_1 \in \mathbb{R}$ and then we prove the existence of a ground state function $\phi \in H^1_{loc}(\Omega)$ which solves the corresponding Euler-Lagrange problem in the sense of the weak solutions

$$-\Delta\phi - \frac{1}{4}\frac{\phi}{d^2} = \lambda_1 \frac{\phi}{1 + d^{2+\sigma}} \qquad \text{in } \Omega.$$

Finally, we prove the following estimate for the function ϕ

$$C_1 \frac{d^{\frac{1}{2}}(x)}{|x|^{a_n}} \le \phi(x) \le C_2 \frac{d^{\frac{1}{2}}(x)}{|x|^{a_n}}, \quad \text{where} \quad a_n = \frac{n-1}{2} + \sqrt{\frac{(n-2)^2}{4} - \frac{1}{4}}.$$

Finally in section 4.2, we assume that the set Ω is above the graph of a $C^{1,1}$ function i.e.

$$\Omega = \{(x', x_n) \in \mathbb{R}^n : x_n > \Gamma(x')\},$$

where $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfies the conditions $|\nabla\Gamma| < \lambda$ and $\Gamma \in C^{1,1}(\mathbb{R}^{n-1})$ and we prove the following Hardy-Sobolev inequality

Theorem 4.0.10. *Let $n \geq 3$ and Ω be a domain above the graph of $C^{1,1}$ function which satisfies $-\Delta d \geq 0$ in the sense of distributions. Then there exists a positive constant $C(n, \lambda)$ such that the following inequality is valid*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx \geq C(n,\lambda)\Big(\int_\Omega |u|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}}, \qquad \forall u \in C_c^\infty(\Omega).$$

Observe that the constant in front of critical Sobolev term depends only on dimension n and λ.

4.1 Hardy and Hardy-Sobolev Type Inequalities in Exterior Domains

4.1.1 Hardy and Hardy-Sobolev Type Inequalities in Exterior Domains Special Case

In this section we prove Hardy and Hardy Sobolev type inequalities in exterior domains under the condition (4.1.2). We call Ω an exterior domain if it is the complement of smooth compact domain. For our purposes here, smooth means C^2 and we consider exterior domains not containing the origin (i.e. there exists $\rho > 0$ such that $B_\rho(0) \subset\subset \Omega^c$). The main assumption which we use for Ω is in terms of the distance function $d(x) = \inf\{|x-y| : y \in \partial\Omega\}$. More specifically, we assume that

$$-\Delta d(x) + (n-1)\frac{\nabla d(x)\cdot x}{|x|^2} \geq 0 \tag{4.1.2}$$

in the sense of distributions i.e.

$$\int_\Omega \left(-\Delta d(x) + (n-1)\frac{\nabla d(x)\cdot x}{|x|^2}\right) u dx \geq 0, \quad \forall\ 0 \leq u \in C_0^\infty(\Omega).$$

Note that in the case where $\Omega = B_R^c(0)$ then inequality (4.1.2) becomes equality. Also note that in the case where Ω is the exterior of ellipse then assumption (4.1.2) is not satisfied.
First, let us show that the inequality (4.1.2) becomes equality if and only if K is a ball centered at zero.

Lemma 4.1.1. *Assume that $0 \in K$, where K has smooth enough boundary. Assume also that the following equality holds for each $x \in \partial K$*

$$-\Delta d(x) + (n-1)\frac{\nabla d(x)\cdot x}{|x|^2} = 0.$$

Then K is a ball centered at zero.

proof: Let $x \in \partial K$. By a rotation of coordinates, we map x to $\tilde{x}$ such that $\tilde{x} = (0, .., \tilde{x}_n)$ and $|\tilde{x}_n| = |x|$. Then the unit outer normal is $(0, .., 1)$ and $-\Delta d = (n-1)H(x) = (n-1)H(\tilde{x})$, since the mean curvature ($H(x)$) is invariant under the change of coordinate system. Then we have

$$-\Delta d(\tilde{x}) + (n-1)\frac{\nabla d(\tilde{x})\cdot\tilde{x}}{|\tilde{x}|^2} = (n-1)H(\tilde{x}) + (n-1)\frac{1}{|\tilde{x}_n|} \Leftrightarrow$$

$$H(\tilde{x}) = -\frac{1}{|\tilde{x}_n|} = -\frac{1}{|x|} = H(x).$$

Returning now to the initial coordinate system we obtain that

$$-\frac{1}{|x|} + \frac{\nabla d\cdot x}{|x|^2} = 0 \Leftrightarrow \nabla d\cdot x = |x| \Leftrightarrow \nabla d = \frac{x}{|x|}.$$

Since the $x \in \partial K$ is arbitrary the last equality holds for each $x \in \partial K$. Thus K is a ball centered at zero. □

Theorem 4.1.2. *Let $n \geq 4$ and Ω be an exterior domain not containing the origin which satisfies the condition (4.1.2). Then the following inequality is valid.*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx \geq C\Big(\int_\Omega |u|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_0^\infty(\Omega) \tag{4.1.3}$$

where the constant $C > 0$ depends only on Ω and dimension n.

proof: We set

$$u = |x|^{-\frac{n-1}{2}} d^{\frac{1}{2}} v,$$

then by straightforward calculations, we have

$$\begin{aligned}\int_\Omega |\nabla u|^2 dx &= \int_\Omega \frac{d|\nabla v|^2}{|x|^{n-1}}dx + \frac{1}{4}\int_\Omega \frac{|v|^2}{|x|^{n-1}d}dx + \frac{(n-1)^2}{4}\int_\Omega \frac{d|v|^2}{|x|^n}dx \\ &- \frac{n-1}{2}\int_\Omega \frac{\nabla d\cdot x|v|^2}{|x|^{n+1}}dx - \frac{n-1}{2}\int_\Omega \frac{dx\cdot\nabla v^2}{|x|^{n+1}}dx + \frac{1}{2}\int_\Omega \frac{\nabla d\cdot\nabla v^2}{|x|^{n-1}}dx.\end{aligned}$$

Also, we note that

$$\begin{aligned}\int_\Omega \frac{dx\cdot\nabla v^2}{|x|^{n+1}}dx &= -\int_\Omega \frac{\nabla d\cdot x|v|^2}{|x|^{n+1}}dx + \int_\Omega \frac{d|v|^2}{|x|^n}dx, \\ \int_\Omega \frac{\nabla d\cdot\nabla v^2}{|x|^{n-1}}dx &= \int_\Omega\Big(-\Delta d(x) + (n-1)\frac{\nabla d(x)\cdot x}{|x|^2}\Big)\frac{v^2}{|x|^{n-1}}dx \geq 0,\end{aligned}$$

where in the last inequality we have used the condition (4.1.2). Taking into account the last calculations we have

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx \geq \int_\Omega \frac{d|\nabla v|^2}{|x|^{n-1}}dx + \frac{(n-1)(n-3)}{4}\int_\Omega \frac{d|v|^2}{|x|^{n+1}}dx.$$

Thus by the above inequality it is enough to show that the following inequality is valid

$$\int_\Omega \frac{d|\nabla v|^2}{|x|^{n-1}}dx + \frac{(n-1)(n-3)}{4}\int_\Omega \frac{d|v|^2}{|x|^{n+1}}dx \geq \Big(\int_\Omega \frac{d^{\frac{n}{n-2}}|v|^{\frac{2n}{n-2}}}{|x|^{n\frac{n-1}{n-2}}}dx\Big)^{\frac{n-2}{n}}, \quad \forall v \in C_0^\infty(\Omega) \tag{4.1.4}$$

Now let $\Omega_\delta = \{x \in \Omega : d(x) \leq \delta\}$ for some $\delta > 0$ sufficiently small and $\Omega_\delta^c = \mathbb{R}^n \setminus \Omega_\delta$. Then note that

$$\frac{\delta}{\rho} \leq \frac{d}{|x|} \leq 1 \ \forall\ x \in \Omega_\delta^c \quad \text{and} \quad \rho' \leq |x| \leq \rho + \delta \ \forall\ x \in \Omega_\delta \tag{4.1.5}$$

where $\rho = \sup\{|x| : x \in \partial\Omega\}$ and $\rho' = \inf\{|x| : x \in \partial\Omega\}$. To prove inequality (4.1.4), we need to define cutoff functions supported near to the boundary. Let $a(t) \in C^\infty([0,\infty))$ be a nondecreasing function such that $a(t) = 1$ for $t \in [0, \frac{1}{2})$, $a(t) = 0$ for $t \geq 1$ and $a'(t) \leq C_0$. For δ small we define $\phi_\delta(x) := a(\frac{d(x)}{\delta}) \in C^{1,1}(\Omega)$. Note that $\phi_\delta = 1$ on $\Omega_{\frac{\delta}{2}}$, $\phi_\delta = 0$ on Ω_δ^c and $|\nabla\phi_\delta| = |a'(\frac{d(x)}{\delta})|\frac{|\nabla d|}{\delta} \leq \frac{C_0}{\delta}$ with C_0 a universal constant.

By (4.1.5) and then Sobolev inequality we have

$$\begin{aligned}
\int_{\Omega^c_{\frac{\delta}{2}}} \frac{d|\nabla((1-\phi_\delta)v)|^2}{|x|^{n-1}}dx &+ \frac{(n-1)(n-3)}{4}\int_{\Omega^c_{\frac{\delta}{2}}} \frac{d|(1-\phi_d)v|^2}{|x|^{n+1}}dx \\
&\geq c(\delta,\rho)\Big(\int_{\Omega^c_{\frac{\delta}{2}}} \frac{|\nabla((1-\phi_\delta)v)|^2}{|x|^{n-2}}dx + \frac{(n-1)(n-3)}{4}\int_{\Omega^c_{\frac{\delta}{2}}} \frac{|(1-\phi_\delta)v|^2}{|x|^n}dx\Big) \\
&\geq C(\delta,\rho)\Big(\int_{\Omega^c_{\frac{\delta}{2}}} \frac{|(1-\phi_\delta)v|^{\frac{2n}{n-2}}}{|x|^n}dx\Big)^{\frac{n-2}{n}} \\
&\geq C(\delta,\rho)\Big(\int_{\Omega^c_\delta} \frac{d^{\frac{n}{n-2}}|(1-\phi_\delta)v|^{\frac{2n}{n-2}}}{|x|^{n\frac{n-1}{n-2}}}dx\Big)^{\frac{n-2}{n}}, \qquad (4.1.6)
\end{aligned}$$

where in the last inequality we have used again (4.1.5) and the fact that $\Omega^c_\delta \subset \Omega^c_{\frac{\delta}{2}}$.

Now by Theorem 2.4 in [FMaT1] and (4.1.5) for sufficiently small $\delta > 0$ we have

$$\begin{aligned}
\int_{\Omega_\delta} \frac{d|\nabla(\phi_\delta v)|^2}{|x|^{n-1}}dx &+ \frac{(n-1)(n-3)}{4}\int_{\Omega_\delta} \frac{d|\phi_\delta v|^2}{|x|^{n+1}}dx \\
&\geq C(\delta,\rho,\rho')\Big(\int_{\Omega_\delta} \frac{d^{\frac{n}{n-2}}|\phi_\delta v|^{\frac{2n}{n-2}}}{|x|^{\frac{n(n-1)}{n-2}}}dx\Big)^{\frac{n-2}{n}}. \qquad (4.1.7)
\end{aligned}$$

We add (4.1.6) and (4.1.7) to obtain

$$\begin{aligned}
&C(\delta,\rho,\rho')\Big(\int_{\Omega_\delta} \frac{d^{\frac{n}{n-2}}|\phi_\delta v|^{\frac{2n}{n-2}}}{|x|^{\frac{n(n-1)}{n-2}}}dx\Big)^{\frac{n-2}{n}} + C(\delta,\rho)\Big(\int_{\Omega^c_\delta} \frac{d^{\frac{n}{n-2}}|(1-\phi_\delta)v|^{\frac{2n}{n-2}}}{|x|^{\frac{n(n-1)}{n-2}}}dx\Big)^{\frac{n-2}{n}} \\
&\leq \int_{\Omega_\delta} \frac{d|\nabla(\phi_\delta v)|^2}{|x|^{n-1}}dx + \int_{\Omega^c_{\frac{\delta}{2}}} \frac{d|\nabla((1-\phi_\delta)v)|^2}{|x|^{n-1}}dx + 2\frac{(n-1)(n-3)}{4}\int_\Omega \frac{d|v|^2}{|x|^{n+1}}dx \\
&\leq C'\Big(\int_{\Omega_\delta\setminus\Omega_{\frac{\delta}{2}}} \frac{dv^2}{|x|^{n-1}}dx\Big) + C(n)\Big(\int_\Omega \frac{d|\nabla(v)|^2}{|x|^{n-1}}dx + \frac{(n-1)(n-3)}{4}\int_\Omega \frac{d|v|^2}{|x|^{n+1}}dx\Big), \qquad (4.1.8)
\end{aligned}$$

where in the last inequality we have use the fact that $\nabla\phi_\delta \neq 0$ only for any $x \in \Omega_\delta \setminus \Omega_{\frac{\delta}{2}}$.
Thus in view of (4.1.8), to complete the proof of theorem we need the following inequality

$$\int_{\Omega_\delta\setminus\Omega_{\frac{\delta}{2}}} \frac{dv^2}{|x|^{n-1}}dx \leq C\Big(\int_\Omega \frac{d|\nabla v|^2}{|x|^{n-1}}dx + \frac{(n-1)(n-3)}{4}\int_\Omega \frac{d|v|^2}{|x|^{n+1}}dx\Big). \qquad (4.1.9)$$

The last inequality is simple to prove because by (4.1.5) we have

$$\int_{\Omega_\delta\setminus\Omega_{\frac{\delta}{2}}} \frac{dv^2}{|x|^{n-1}}dx \leq (\rho+\delta)^2\Big(\int_{\Omega_\delta\setminus\Omega_{\frac{\delta}{2}}} \frac{dv^2}{|x|^{n+1}}dx\Big).$$

□

The case $n = 3$ is different, as we can see from the following Theorem.

Theorem 4.1.3. *Let $n = 3$ and Ω be an exterior domain not containing the origin and satisfies the condition (4.1.2)* *with strictly inequality i.e.*

$$-\Delta d(x) + 2\frac{\nabla d(x)\cdot x}{|x|^2} \gneqq 0. \tag{4.1.10}$$

Then the following inequality is valid.

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx \ge C\Big(\int_\Omega X^4(\frac{|x|}{D})u^6 dx\Big)^{\frac{1}{3}}, \quad \forall\ u \in C_c^\infty(\Omega) \tag{4.1.11}$$

where $X(t) = (1+\ln t)^{-1},\ 0 < D < \inf\{|x| :\ x \in \partial\Omega\}$ *and the constant* $C > 0$ *depends only on* Ω*. Moreover, the power* 4 *on* X *can not be replaced by a smaller power.*

The condition 4.1.10 is equivalent with the fact that there exists $\varepsilon > 0$ and a ball of radius $\rho > 0$ with center at x_0 and $B_\rho(x_0) \subset \Omega$ such that

$$\int_{B_\rho(x_0)} \left(-\Delta d(x) + 2\frac{\nabla d(x)\cdot x}{|x|^2}dx\right) u dx \ge \varepsilon \int_{B_\rho(x_0)} u dx, \quad \forall\ 0 \le u \in C_0^\infty(B_\rho(x_0)).$$

To prove Theorem 4.1.3 we need the following Lemma.

Lemma 4.1.4. *Let* $n \ge 3$ *and* Ω *be an exterior domain not containing the origin. Then the following inequality is valid*

$$\int_\Omega \frac{|\nabla u|^2}{|x|^{n-2}}dx \ge C\Big(\int_\Omega \frac{u^{\frac{2n}{n-2}}}{|x|^n} X(\frac{|x|}{D})^{\frac{2(n-1)}{n-2}} dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_0^\infty(\Omega) \tag{4.1.12}$$

where $X(t) = \frac{1}{1+\ln(t)}$, $0 < D < \inf_{x\in\partial\Omega}|x|$ *and* $C > 0$ *depends only on* Ω *and* n*. Moreover, the power* $\frac{2(n-1)}{n-2}$ *on* X *can not be replaced by a smaller power.*

To prove Lemma 4.1.4 we need the following lemma which the proof is in [Ma].

Lemma 4.1.5. *Let* $A(r),\ B(r)$ *nonnegative functions. Such that* $1/A(r),\ B(r)$ *are integrable in* $(0,r)$ *and* (r,∞)*, respectively, for all positive* $r < \infty$*. Then, for* $q \ge 2$ *the Sobolev inequality*

$$\Big[\int_0^s B(t)|u(t)-u(0)|^q dt\Big]^{\frac{1}{q}} \le C\Big[\int_0^s A(t)|u'(t)|^2 dt\Big]^{\frac{1}{2}} \quad , \tag{4.1.13}$$

is valid for all $u \in C^1[0,s]$ *such that* $u(s) = 0$ *(or vanish near infinity, if* $s = \infty$*), if and only if*

$$K = \sup_{r\in(0,s)} \Big[\int_r^s B(t)dt\Big]^{\frac{1}{q}} \Big[\int_0^r (A(t))^{-1}dt\Big]^{\frac{1}{2}} \tag{4.1.14}$$

be finite. The best constant in (4.1.13) satisfies the following inequality

$$K \le C \le K\Big(\frac{q}{q-1}\Big)^{\frac{1}{2}} q^{\frac{1}{q}}. \tag{4.1.15}$$

proof of Lemma 4.1.4: First we assume that u is a radially symmetric function. Then inequality (4.1.12) is equivalent to

$$\int_\rho^\infty r|u_r|^2 dr \ge C\Big(\int_\rho^\infty \frac{|u|^{\frac{2n}{n-2}}}{r} X(\frac{r}{D})^{\frac{2(n-1)}{n-2}} dr\Big)^{\frac{n-2}{n}}, \tag{4.1.16}$$

where $\rho = \inf_{x\in\partial\Omega}|x|$. We note that the last inequality is valid by Lemma 4.1.5 for $A(r) = r$, $B(r) = \frac{X^{\frac{2(n-1)}{n-2}}(\frac{r}{D})}{r}$ and $q = \frac{2n}{n-2}$. Suppose first that $\rho < 1$. Following [VZ] we decompose u into spherical harmonics (since $u \in (B^c_\rho(0))$) to get

$$u(x) = \sum_{m=0}^{\infty} u_m(r) f_m(\sigma),$$

where f_m are orthogonal in $L^2(S^{n-1})$ normalized by $\frac{1}{nw_n}\int_{S^{n-1}} f_m(\sigma)f_n(\sigma)dS = \delta_{mn}$. In particular $f_0(\sigma) = 1$ and the first term in the above decomposition is given by

$$u_0(r) = \frac{1}{nw_n r^{n-1}}\int_{\partial B_r} u(x)dS_x.$$

The f_m's are eigenfunctions of the Laplace-Beltrami operator (∇_σ) with corresponding eigenvalues $c_m = m(n-2+m)$, $m \geq 0$. An easy calculation shows that,

$$\int_{B_1^c} |\nabla u|^2 dx = \sum_{m=0}^{\infty}\int_{B_1^c}\frac{|\nabla u_m|^2}{|x|^{n-2}}dx + \sum_{m=0}^{\infty} c_m \int_{B_1^c}\frac{u_m^2}{|x|^n}dx, \tag{4.1.17}$$

Now note that

$$\begin{aligned}\sum_{m=1}^{\infty}\int_{B_1^c}\frac{|\nabla u_m|^2}{|x|^{n-2}}dx + \sum_{m=1}^{\infty} c_m\int_{B_1^c}\frac{u_m^2}{|x|^n}dx &\geq \frac{1}{2}\Big(\int_{B_1^c}\frac{|\nabla(u-u_0)|^2}{|x|^{n-2}}dx + \int_{B_1^c}\frac{|u-u_0|^2}{|x|^n}dx\Big)\\ &\geq C\Big(\int_{B_1^c} X^{\frac{2(n-1)}{n-2}}(|x|)\frac{|u-u_0|^{\frac{2n}{n-2}}}{|x|^n}dx\Big)^{\frac{n-2}{n}},\end{aligned} \tag{4.1.18}$$

Also, since u_0 is radially symmetric we have by (4.1.16), that u_0 satisfies

$$\int_{B_1^c}\frac{|\nabla u_0|^2}{|x|^{n-2}}dx \geq C\Big(\int_{B_1^c}\frac{|u_0|^{\frac{2n}{n-2}}}{|x|^n}X(\frac{|x|}{D})^{\frac{2(n-1)}{n-2}}dx\Big)^{\frac{n-2}{n}}, \tag{4.1.19}$$

thus by (4.1.18) and (4.1.19) the proof is complete. □

proof of Theorem 4.1.3: We set

$$u = |x|^{-1}d^{\frac{1}{2}}v$$

as in Theorem 4.1.2, then the inequality (4.1.11) becomes equivalent to

$$\int_\Omega \frac{d|\nabla v|^2}{|x|^2}dx + \int_\Omega\Big(-\Delta d(x) + 2\frac{\nabla d(x)\cdot x}{|x|^2}\Big)\frac{v^2}{|x|^2}dx \geq C\Big(\int_\Omega\frac{d^3X^4(\frac{|x|}{D})|v|^6}{|x|^6}dx\Big)^{\frac{1}{3}}. \tag{4.1.20}$$

To prove (4.1.20) we need the cutoff (ϕ_δ) which we used in Theorem 4.1.2. Also we recall that $\Omega_\delta = \{x \in \Omega : d(x) \leq \delta\}$ for some $\delta > 0$ sufficiently small and $\Omega^c_\delta = \mathbb{R}^n \setminus \Omega_\delta$. Then note that

$$\frac{\delta}{\rho} \leq \frac{d}{|x|} \leq 1 \;\forall\; x \in \Omega^c_\delta \quad \text{and} \quad \rho' \leq |x| \leq \rho + \delta \;\forall\; x \in \Omega_\delta \tag{4.1.21}$$

where $\rho = \sup\{|x| : x \in \partial\Omega\}$ and $\rho' = \inf\{|x| : x \in \partial\Omega\}$. Then by Theorem 2.4 in [FMaT1], Lemma 4.1.4, (4.1.10) and (4.1.21) the following inequalities are valid

$$\int_{\Omega_\delta}\frac{d|\nabla(\phi_\delta v)|^2}{|x|^2}dx + \int_{\Omega_\delta}\Big(-\Delta d(x) + 2\frac{\nabla d(x)\cdot x}{|x|^2}\Big)\frac{|\phi_\delta v|^2}{|x|^2}dx \geq C(\delta,\rho,\rho')\Big(\int_{\Omega_\delta}\frac{d^3X^4(\frac{|x|}{D})|\phi_\delta v|^6}{|x|^6}dx\Big)^{\frac{1}{3}},$$

$$\int_{\Omega^c_{\frac{\delta}{2}}} \frac{d|\nabla(1-\phi_\delta)v|^2}{|x|^2}dx \quad + \quad \int_{\Omega^c_{\frac{\delta}{2}}} \Big(-\Delta d(x) + 2\frac{\nabla d(x)\cdot x}{|x|^2}\Big)\frac{|(1-\phi_\delta)v|^2}{|x|^2}dx$$

$$\geq \quad C(\delta,\rho)\Big(\int_{\Omega^c_{\frac{\delta}{2}}} \frac{|(1-\phi_\delta)v|^6}{|x|^3}X(\frac{|x|}{D})^4 dx\Big)^{\frac{1}{3}}, \tag{4.1.22}$$

Now by (4.1.10) there exists $\varepsilon > 0$ and a ball radius $\rho > 0$ with center x_0 and $B_\rho(x_0) \subset \Omega$ such that

$$\int_{B_\rho(x_0)} \left(-\Delta d(x) + 2\frac{\nabla d(x)\cdot x}{|x|^2}dx\right) u dx \geq \varepsilon \int_{B_\rho(x_0)} u dx, \quad \forall\ 0 \leq u \in C_0^\infty(B_\rho(x_0)).$$

Consider now $\eta \in C_0^\infty(B_\rho(0))$, $0 \leq \eta \leq 1$ and $\eta(x) = 1$ in $B_{\frac{\rho}{2}}$ Also consider a $B_R(0) \supset\supset \Omega^c$ such that $B_\rho(x_0) \subset B_R(0)$. Then we have

$$\begin{aligned}
\int_{B_R(0)} \left(-\Delta d(x) + 2\frac{\nabla d(x)\cdot x}{|x|^2}dx\right) v^2 dx &\geq \int_{B_R(0)} \left(-\Delta d(x) + 2\frac{\nabla d(x)\cdot x}{|x|^2}dx\right) v^2 dx \\
&\geq \int_{B_\rho(x_0)} \left(-\Delta d(x) + 2\frac{\nabla d(x)\cdot x}{|x|^2}dx\right) \eta v^2 dx \\
&\geq \varepsilon \int_{B_\rho(x_0)} \eta v^2 dx \\
&\geq \varepsilon \int_{B_{\frac{\rho}{2}}(x_0)} v^2 dx
\end{aligned} \tag{4.1.23}$$

Now in view of Theorem 4.1.2 and (4.1.23), we only need to show the following inequality

$$\int_{\Omega_\delta \backslash \Omega_{\frac{\delta}{2}}} \frac{dv^2}{|x|^{n-1}}dx \leq C'\Big(\int_\Omega \frac{d|\nabla v|^2}{|x|^{n-1}}dx + \varepsilon \int_{B_{\frac{\rho}{2}}(x_0)} \frac{v^2}{|x|^{n-1}}dx\Big)dx, \tag{4.1.24}$$

We will prove inequality (4.1.24) by contradiction. Specifically, we will prove that the following inequality is valid

$$\int_{B_r(0)\backslash \Omega_{\frac{\delta}{2}}} v^2 dx \leq C'\Big(\int_{B_r \backslash \Omega_{\frac{\delta}{2}}} |\nabla v|^2 dx + \varepsilon \int_{B_R(0)} v^2 dx\Big), \tag{4.1.25}$$

where B_r is a ball radius r such that $B_R(0) \subset B_r(0)$ and $\Omega_\delta \subset B_r$. Where the stated estimate false, there would exist for each integer $k = 1, ...$ a function $v_k \in H^1$ satisfying

$$\int_{B_r(0)\backslash \Omega_{\frac{\delta}{2}}} v_k^2 dx \geq k \int_{B_r(0)\backslash \Omega_{\frac{\delta}{2}}} |\nabla v_k|^2 dx + \varepsilon \int_{B_{\frac{\rho}{2}}(x_0)} v_k^2 dx.$$

We re-normalize v_k such that

$$\int_{B_r(0)\backslash \Omega_{\frac{\delta}{2}}} v_k^2 dx = 1, \tag{4.1.26}$$

which implies

$$\int_{B_r \backslash \Omega_{\frac{\delta}{2}}} |\nabla v_k|^2 dx + \varepsilon \int_{B_{\frac{\rho}{2}}(x_0)} v_k^2 dx \leq \frac{1}{k}. \tag{4.1.27}$$

In particular the functions $\{u_k\}$ are bounded in H^1. Thus by Rellich-Kondrachov Theorem, there exists a subsequence

$\{v_{k_j}\} \subset \{v_k\}$ and a function $v \in L^2$ such that

$$v_{k_j} \to v, \quad in\ L^2.$$

But then

$$\int_{B_r(0)\setminus\Omega_{\frac{\delta}{2}}} v^2 dx = 1. \tag{4.1.28}$$

On the other hand by (4.1.27) we have that $Dv = 0$ *a.e* and $v = 0$ *a.e* in $B_{\frac{\rho}{2}}(x_0)$ which implies that $v = 0$ *a.e* in $B_r(0) \setminus \Omega_{\frac{\delta}{2}}$. Where we have clearly a contradiction by (4.1.28). □

4.1.2 Examples Where we have not Hardy or Hardy-Sobolev Inequality for $n = 2$ and $n = 3$

In this subsection we will give some examples where Hardy and Hardy-Sobolev inequality is not valid in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

Example 1 *Consider the set* $K_a = \{-a \le x \le a\}$ *for some positive constant* a. *Then, there does not exist constant* $c > 0$, *such that the following inequality to be valid*

$$\int_{\mathbb{R}^2\setminus K_a} |\nabla u|^2 dx - c\int_{\mathbb{R}^2\setminus K_a} \frac{u^2}{d^2} dx \ge 0, \ \forall u \in C_0^\infty(\mathbb{R}^2 \setminus K)$$

where $d(x) = \inf\{|x-y| : \ y \in K_a, \ x \in \mathbb{R}^2\}$.

proof: We will show it by contradiction. We assume that $\mathbb{R}^2 \setminus K_a$ has the Hardy property, that is, there exist a positive constant c such that

$$\int_{\mathbb{R}^2\setminus K_a} |\nabla u|^2 dx \ge c\int_{\mathbb{R}^2\setminus K_a} \frac{u^2}{d^2} dx, \ \forall u \in C_0^\infty(\mathbb{R}^2 \setminus K_a). \tag{4.1.29}$$

Now set $u(x) = v(\tilde{x})$, where $\tilde{x} = \frac{x}{a}$. Then $d(x) = ad(\tilde{x})$ (where $d(\tilde{x}) = \inf\{|x-y| : \ y \in K_1, \ x \in \mathbb{R}^2\}$) and $\nabla_x u = \frac{\nabla_{\tilde{x}} v}{a}$. Then inequality (4.1.29) becomes equivalent to

$$\int_{\mathbb{R}^2\setminus K_1} |\nabla v|^2 d\tilde{x} \ge c\int_{\mathbb{R}^2\setminus K_1} \frac{v^2}{d^2} d\tilde{x}, \ \forall v \in C_0^\infty(\mathbb{R}^2 \setminus K_1). \tag{4.1.30}$$

By (4.1.30) we obtain that the constant c is independent on a. Next in (4.1.29), send a at zero to obtain that

$$\int_{\mathbb{R}^2\setminus\{0\}} |\nabla u|^2 dx \ge c\int_{\mathbb{R}^2\setminus\{0\}} \frac{u^2}{d^2} dx, \ \forall u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}).$$

Which is clearly a contradiction, since the Hardy inequality in $\mathbb{R}^2$ is not valid. □

Example 2 *There does not exist constant* $0 < c \le \frac{1}{4}$ *such that*

$$\int_{B_1^c} |\nabla u|^2 dx - c\int_{B_1^c} \frac{u^2}{d^2} dx \ge 0, \quad \forall\ u \in C_0^\infty(\mathbb{R}^2 \setminus B_1), \tag{4.1.31}$$

where $d = |x| - 1$.

proof: We will show it by contradiction. We assume that there exists a constant $0 < c \le \frac{1}{4}$ such that (4.1.31) is valid.

We set $u = \frac{(r-1)^{\frac{1}{2}}}{r^{\frac{1}{2}}} v$, where

$$v(r) = \begin{cases} r-1 & , \ 1 < r \leq 2 \\ 2^{\varepsilon} r^{-\varepsilon} & , \ 2 < r. \end{cases}$$

Observe that $u \in \mathfrak{D}_0^{1,2}(B_1^c)$. Then by straightforward calculations in 4.1.31, we have

$$\int_{B_1^c} \frac{d|\nabla v|^2}{|x|} dx - \frac{1}{4} \int_{B_1^c} \frac{d|v|^2}{|x|^3} dx + (\frac{1}{4} - c) \int_{B_1^c} \frac{|v|^2}{d|x|} dx \geq 0. \tag{4.1.32}$$

Now note that

$$\int_{B_2^c} \frac{d|\nabla v|^2}{|x|} dx = 2\pi\varepsilon^2 (\frac{2^{-2\varepsilon}}{2\varepsilon} - \frac{2^{-2\varepsilon-1}}{2\varepsilon+1}), \tag{4.1.33}$$

$$\int_{B_2^c} \frac{d|v|^2}{|x|^3} dx = 2\pi (\frac{2^{-2\varepsilon}}{2\varepsilon} - \frac{-2\varepsilon-1}{2\varepsilon+1}) \tag{4.1.34}$$

and

$$\int_{B_2^c} \frac{|v|^2}{d|x|} dx \leq \frac{\pi}{\varepsilon}. \tag{4.1.35}$$

But,

$$\lim_{\varepsilon \to 0} (\frac{1}{4} - c) \frac{1}{2\varepsilon} - \frac{1}{4} \frac{2^{-2\varepsilon}}{2\varepsilon} = -\infty,$$

which is a contradiction by (4.1.32), (4.1.33), (4.1.34), (4.1.35). □

Finally, we will show that on the exterior of a unit ball in $\mathbb{R}^3$, Hardy-Sobolev inequality does not hold.

Example 3 *There is not any constant $c > 0$ such that the following inequality to be valid,*

$$\int_{B_1^c} |\nabla u|^2 dx - \frac{1}{4} \int_{B_1^c} \frac{u^2}{d^2} dx \geq c \Big(\int_{B_1^c} u^6 X^a(|x|) dx \Big)^{\frac{n-2}{n}}, \tag{4.1.36}$$

where $d = |x| - 1$, $X(t) = (1 + ln(t))^{-1}$ and $a > 1$.

proof: We will show it by contradiction. We assume that there exists $c > 0$ such that (4.1.36) is valid. We set $u = \frac{(r-1)^{\frac{1}{2}}}{r} v$, where

$$v(r) = \begin{cases} (r-1)^{\varepsilon} & , \ 1 < r \leq 2 \\ 2^{\varepsilon} r^{-\varepsilon} & , \ 2 < r. \end{cases}$$

Observe that $u \in \mathfrak{D}_0^{1,2}(B_1^c)$. Then by straightforward calculations in 4.1.36, we have

$$\int_{B_1^c} \frac{d|\nabla v|^2}{|x|^2} dx \geq c \Big(\int_{B_1^c} \frac{d^3 v^6 X^a(|x|)}{|x|^6} dx \Big)^{\frac{1}{3}}. \tag{4.1.37}$$

Now note that,

$$\int_{B_2} \frac{d|\nabla v|^2}{|x|^2} dx = c_1 \varepsilon, \tag{4.1.38}$$

$$\int_{B_2^c} \frac{d|\nabla v|^2}{|x|^2} dx = c_1 2^{-2\varepsilon}\varepsilon, \tag{4.1.39}$$

$$\int_{B_2} \frac{d^3 v^6 X^a(|x|)}{|x|^6} dx \geq \frac{c_2}{4+6\varepsilon}, \tag{4.1.40}$$

and

$$\int_{B_2^c} \frac{d^3 v^6 X^a(|x|)}{|x|^6} dx = \int_2^\infty \frac{(r-1)^3 r^{-6\varepsilon} X^a(r)}{r^4} dr. \tag{4.1.41}$$

Letting $\varepsilon \to 0$, by (4.1.37), (4.1.38), (4.1.39), (4.1.40) and (4.1.41) we have clearly a contradiction. □

4.1.3 Hardy Inequalities in Exterior Domain General Case

In this subsection we will prove Hardy-Sobolev type inequalities without the assumption of (4.1.2). Before we prove the Hardy-Soblolev inequality, we need a theorem for the following space:

Definition 4.1.6. *Let $n \geq 3$ and Ω be an exterior domain not containing the origin. We denote by $\mathfrak{D}_0^{1,2}(\Omega;|x|)$ the completion of $C_0^\infty(\Omega)$ function under the norm*

$$||u||^2_{\mathfrak{D}_0^{1,2}(\Omega;|x|)} = \int_\Omega \frac{|\nabla u|^2}{|x|^{n-2}} dx + \int_\Omega \frac{u^2}{(1+|x|^{2+\sigma})|x|^{n-2}} dx, \tag{4.1.42}$$

where σ is a non-negative constant.
Also, we denote by $\mathfrak{D}^{1,2}(\Omega;|x|)$ the completion of $C^\infty(\overline{\Omega})$ with compact support at infinity under the norm (4.1.42).

Theorem 4.1.7. *Let Ω be a exterior domain not containing the origin. Select an exterior domain V such that $\overline{\Omega} \subset V$. Then for each $u \in \mathfrak{D}^{1,2}(\Omega;|x|)$ there exists a function $\widetilde{u} \in \mathfrak{D}_0^{1,2}(V;|x|)$ such that*

(i) $|u| \leq |\widetilde{u}| \leq (N+1)u$ a.e in Ω

(ii) $\widetilde{u}$ has support in V
and
(iii) $||\widetilde{u}||^2_{\mathfrak{D}_0^{1,2}(V;|x|)} \leq C||u||^2_{\mathfrak{D}_0^{1,2}(\Omega;|x|)}$

where the constant N depends on $\partial\Omega$ and constant C depends only on n, Ω, N, σ and V.

proof: Let $r_0 = \inf_{x\in\partial\Omega}|x|$ and $R_0 = \sup_{x\in\partial\Omega}|x|$. Fix $x \in \partial\Omega$, then there exists a $r < \frac{r_0}{4}$ and a C^1 function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that (upon relabeling and reorienting the coordinates axes if necessary) we have

$$\Omega \cap B(x,r) = \{x \in B(x,r) : x_n > \gamma(x')\}.$$

Then we define $y_i = x_i =: \Phi^i(x)$ for $i = 1, ..., n-1$ and $y_n = x_n - \gamma(x') = \Phi^n(x)$. Similarly we set $x_i = y_i =: Y^i(y)$ for $i = 1, ..., n-1$ and $x_n = y_n + \gamma(y') =: Y^n(y)$.
Then $\Phi = Y^{-1}$ and the mapping $x \to \Phi(x) = y$ "straightens out $\partial\Omega$" near to x. Observe also $det\Phi = detY = 1$. Now let $x^i \in \partial\Omega$ and fix r_i small enough such that for the ball $B(\Phi(x^i), r) = B(y^i, r)$, we have that if $x \in W^i = Y(B(y^i, r))$ then

$|x| > \frac{3r_0}{4}$. Now, for $u \in C^\infty(\overline{\Omega})$ with compact support at infinity, we have

$$\int_{W^i} \frac{|\nabla u|^2}{|x|^{n-2}} dx + \int_{W^i} \frac{u^2}{(1+|x|^{2+\sigma})|x|^{n-2}} dx \geq \frac{1}{(R_0)^{n-2}(1+R^{2+\sigma})}\Big(\int_{W^i} |\nabla u|^2 dx + \int_{W^i} u^2 dx\Big). \tag{4.1.43}$$

Now set $u'(y) = u(Y(y))$, $B^+ = B(y^i) \cap \{y_n \geq 0\}$ and $B^- = B(y^i) \cap \{y_n \leq 0\}$. We define

$$\widetilde{u}(y) = \begin{cases} u'(y) & \text{for } y \in B^+ \\ -3u'(y', -y_n) + 4u'(y', \frac{-y_n}{2}) & \text{for } y \in B^-. \end{cases}$$

Then $\widetilde{u} \in C^1(B)$. To check this let us write $u^+ = \widetilde{u}|_{B^+}$ and $u^- = \widetilde{u}|_{B^-}$. We demonstrate first

$$u^-_{y_n} = u^+_{y_n} \qquad \text{on} \qquad \{y_n = 0\}.$$

Now since $u^-(y', 0) = u^+(y', 0)$ we have

$$u^-_{y_i} = u^+_{y_i} \qquad \text{on} \qquad \{y_n = 0\} \quad \text{for,} \quad i = 1, ..., n-1.$$

Thus we have

$$\int_B |\nabla \widetilde{u}|^2 dy + \int_B |\widetilde{u}|^2 dy \leq C\Big(\int_{B^+} |\nabla u'|^2 dy + \int_{B^+} |u'|^2 dy\Big).$$

Now since $c_1|\nabla_x u| \leq |\nabla_y u| \leq c_2|\nabla_x u|$ for some constants c_1, c_2 which depend on γ, we have

$$\int_{W^i} |\nabla \widetilde{u}|^2 dx + \int_{W^i} |\widetilde{u}|^2 dx \leq C\Big(\int_{Y(B^+)} |\nabla u|^2 dx + \int_{Y(B^+)} |u|^2 dx\Big) \Rightarrow$$

$$\Big(\min\{\frac{3r_0}{4}, 1\}\Big)^{n-2}\Big(1 + (\frac{r_0}{4})^{2+\sigma}\Big)\Big(\int_{W^i} \frac{|\nabla \widetilde{u}|^2}{|x|^{n-2}} dx + \int_{W^i} \frac{|\widetilde{u}|^2}{|x|^{n-2}(1+|x|^{2+\sigma})} dx\Big)$$

$$\leq R_0^{n-2}\Big(1 + R_0^{2+\sigma}\Big) C\Big(\int_{Y(B^+)} \frac{|\nabla u|^2}{|x|^{n-2}} dx + \int_{Y(B^+)} \frac{|u|^2}{|x|^{n-2}(1+|x|^{2+\sigma})} dx\Big), \tag{4.1.44}$$

where we have use the fact that $|x| > \frac{3r_0}{4}$ and (4.1.43).
Since $\partial\Omega$ is compact, there exist finitely many points $x^i \in \partial\Omega$, open sets W^i and extensions $\widetilde{u}_i$ of u to W^i ($i = 1, ..., N$), as above, such that $\partial\Omega \subset \bigcup_{i=1}^N W^i$. Take $\overline{W}^0 = B_{4R_0} \setminus \Omega^c$ and let $\{\zeta_i\}_{i=0}^N$ be an associates partition of unity of $(B_{2R_0} \setminus \Omega^c) \cup \bigcup_{i=1}^N W^i \subset \bigcup_{i=0}^N W^i$. Consider now the C^1 function $a(t) = 1$ if $t \leq 1$ and $a(t) = 0$ if $t \geq 2$ and set $\zeta_{N+1} = 1 - a(\frac{t}{R})$.
Write $\widetilde{u} := \sum_{i=0}^{N+1} \zeta_i \widetilde{u}_i$, where $\widetilde{u}_0 = u$ and $\widetilde{u}_{N+1} = u$. Then utilizing estimate (4.1.44) (with u_i in place u, $\widetilde{u}_i$ in place $\widetilde{u}$) we obtain the bound

$$\|\widetilde{u}\|^2_{\mathfrak{D}_0^{1,2}(U;|x|)} \leq C\|u\|^2_{\mathfrak{D}^{1,2}(\Omega;|x|)}, \tag{4.1.45}$$

where $U = \Omega \cup \bigcup_{i=1}^N W^i$ and C positive constant which depends only on Ω, n, N, σ but not on u. Furthermore we can arrange for the support of $\widetilde{u}$ to lie within $V \supset \overline{U}$. □

Remark: In view of the above Theorem we can prove by similar way that

$$\int_\Omega \frac{|\nabla \tilde{u}|^2}{|x|^{n-2}} dx + \int_\Omega \frac{\tilde{u}^2}{(1+|x|^{2+\sigma})|x|^{n-2}} dx \leq C(n, \Omega) \int_\Omega \frac{|\nabla u|^2}{|x|^{n-2}} dx + C'(n, \Omega, \sigma) \int_\Omega \frac{u^2}{(1+|x|^{2+\sigma})|x|^{n-2}} dx.$$

Theorem 4.1.8. *Let $n = 3$, $\sigma > 0$ and Ω be an exterior set not containing the origin. Then there exist constants $C(\Omega)$ and $C'(\Omega, \sigma)$ such that the following inequality is valid,*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx + C' \int_\Omega \frac{u^2}{1+d^{2+\sigma}} dx \geq C\Big(\int_\Omega X^4(\frac{|x|}{\rho}) u^6 dx\Big)^{\frac{1}{3}}, \quad \forall u \in C_c^\infty(\Omega) \tag{4.1.46}$$

where $X(t) = (1+\ln t)^{-1}$, $\rho = \inf\{|x| : \ x \in \partial\Omega\}$. Moreover, the power 4 on X can not be replaced by a smaller power.

proof: Let $R_0 = \sup_{x\in\partial\Omega}|x|$ and $\eta \in C^2(\Omega)$ such that $\eta(x) = d^{\frac{1}{2}}(x)$ $\forall x \in \Omega_{\varepsilon_0}$ where ε_0 is small enough, $\eta(x) = \frac{(|x|-2R_0)^{\frac{1}{2}}}{|x|}$ $\forall x \in B^c(0,4R_0)$ and $c_1 \leq \eta \leq c_2$ otherwise, where c_1 and c_2 are positive constants. Set $u = \eta v$ then we have

$$\begin{aligned}\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx \quad &+ \quad C'\int_\Omega \frac{u^2}{1+d^{2+\sigma}}dx \\ &= \int_\Omega \eta^2|\nabla v|^2 dx - \int_\Omega (\eta\Delta\eta + \frac{\eta^2}{4d^2})v^2 dx + C'\int_\Omega \frac{\eta^2 v^2}{1+d^{+2+\sigma}} =\end{aligned}$$

$$= \int_{\Omega_{\varepsilon_0}} \eta^2|\nabla v|^2 dx - \int_{\Omega_{\varepsilon_0}} (\eta\Delta\eta + \frac{\eta^2}{4d^2})v^2 dx + C'\int_{\Omega_{\varepsilon_0}} \frac{\eta^2 v^2}{1+d^{2+\sigma}}dx \tag{4.1.47}$$

$$+ \int_{B(0,4R_0)\setminus\Omega^c} \eta^2|\nabla v|^2 dx - \int_{B(0,4R_0)\setminus\Omega^c} (\eta\Delta\eta + \frac{\eta^2}{4d^2})v^2 dx + C'\int_{B(0,4R_0)\setminus\Omega^c} \frac{\eta^2 v^2}{1+d^{2+\sigma}}dx \tag{4.1.48}$$

$$+ \int_{B^c(0,4R_0)} \eta^2|\nabla v|^2 dx - \int_{B^c(0,4R_0)} (\eta\Delta\eta + \frac{\eta^2}{4d^2})v^2 dx + C'\int_{B^c(0,4R_0)} \frac{\eta^2 v^2}{1+d^{2+\sigma}}dx \tag{4.1.49}$$

$$= I_1 + I_2 + I_3, \tag{4.1.50}$$

where I_1, I_2 and I_3 are the terms in (4.1.47), (4.1.48) and (4.1.49) respectively.
Now for I_1 we have by Theorem 2.4 in [FMaT1]

$$\begin{aligned} I_1 = \int_{\Omega_{\varepsilon_0}} d|\nabla v|^2 dx \quad &- \quad \frac{1}{2}\int_{\Omega_{\varepsilon_0}} \Delta d v^2 dx + C'\int_{\Omega_{\varepsilon_0}} \frac{dv^2}{1+d^{2+\sigma}}dx \\ &\geq \quad C\Big(\int_{\Omega_{\varepsilon_0}} d^3 v^6 dx\Big)^{\frac{1}{3}} \geq C\Big(\int_{\Omega_{\varepsilon_0}} X^4(\frac{|x|}{\rho})\eta^3 v^6 dx\Big)^{\frac{1}{3}}, \end{aligned} \tag{4.1.51}$$

where in the last inequality we have used the fact that $0 \leq X(t) \leq 1$.
For I_2 we first note

$$\int_{B(0,4R_0)\setminus\Omega^c} \eta^2|\nabla v|^2 dx \geq c_1^2 \int_{B(0,4R_0)\setminus\Omega^c} |\nabla v|^2 dx, \tag{4.1.52}$$

$$|\int_{B(0,4R_0)\setminus\Omega^c} (\eta\Delta\eta + \frac{\eta^2}{4d^2})v^2 dx| \leq C_0 \int_{B(0,4R_0)\setminus\Omega^c} v^2 dx, \tag{4.1.53}$$

$$C'\int_{B(0,4R_0)\setminus\Omega^c} \frac{\eta^2 v^2}{1+d^{2+\sigma}}dx \geq \frac{C'c_1^2}{1+(4R)^{2+\sigma}} \int_{B(0,4R_0)\setminus\Omega^c} v^2 dx. \tag{4.1.54}$$

Thus if we choose $C' \geq 2\frac{C_0(1+(4R)^{2+\sigma})}{c_1^2}$ we have by (4.1.52), (4.1.53), (4.1.54) and the Sobolev inequality

$$\begin{aligned} I_2 \geq C\left(\int_{B(0,4R_0)\setminus\Omega^c} |\nabla v|^2 dx + \int_{B(0,4R_0)\setminus\Omega^c} v^2 dx\right) &\geq C\Big(\int_{B(0,4R_0)\setminus\Omega^c} v^6 dx\Big)^{\frac{1}{3}} \\ &\geq C\Big(\int_{B(0,4R_0)\setminus\Omega^c} X^4(\frac{|x|}{\rho})\eta^6 v^6 dx\Big)^{\frac{1}{3}}. \end{aligned} \tag{4.1.55}$$

For I_3 first we note that

$$-\int_{B^c(0,4R_0)} (\eta\Delta\eta + \frac{\eta^2}{4d^2})v^2 dx \geq 0, \tag{4.1.56}$$

since $d(x) \geq |x| - 2R_0$ in $B^c(0,4R_0)$. Also we note that $\frac{1}{2} \leq \frac{d}{|x|} \leq 1$ for each $x \in B^c(0,4R_0)$.Thus we have by (4.1.56)

$$\begin{aligned} I_3 &\geq C\Big(\int_{B^c(0,4R_0)} \frac{|\nabla v|^2}{|x|} dx + \int_{B^c(0,4R_0)} \frac{|v|^2}{|x|(1+|x|^{2+\sigma})} dx\Big) \\ &\geq C(\Omega)\int_{B^c(0,2R_0)} \frac{|\nabla \widetilde{v}|^2}{|x|} dx + C'(\Omega,\sigma)\int_{B^c(0,4R_0)} \frac{|\widetilde{v}|^2}{|x|(1+|x|^{2+\sigma})} dx, \end{aligned} \tag{4.1.57}$$

where $\widetilde{v}$ is the function as in Theorem 4.1.7 (see remark bellow). Thus since $\widetilde{v} \in C_c^1(B^c(0,2R_0))$ we have by Lemma 4.1.4

$$\begin{aligned} \int_{B^c(0,2R_0)} \frac{|\nabla \widetilde{v}|^2}{|x|} dx &\geq C\Big(\int_{B^c(0,2R_0)} X^4(\frac{|x|}{\rho})\frac{\widetilde{v}^6}{|x|^3} dx\Big)^{\frac{1}{3}} \\ &\geq C\Big(\int_{B^c(0,4R_0)} X^4(\frac{|x|}{\rho})\frac{\widetilde{v}^6}{|x|^3} dx\Big)^{\frac{1}{3}} \\ &= C\Big(\int_{B^c(0,4R_0)} X^4(\frac{|x|}{\rho})\frac{v^6}{|x|^3} dx\Big)^{\frac{1}{3}}, \end{aligned} \tag{4.1.58}$$

where in the last inequality we have used the fact that $\widetilde{v} = v \quad \forall x \in B^c(0,4R_0)$. Thus by (4.1.56), (4.1.57) and (4.1.58) we have

$$I_3 \geq C\Big(\int_{B^c(0,4R_0)} X^4(\frac{|x|}{\rho})\eta^6 v^6 dx\Big)^{\frac{1}{3}}. \tag{4.1.59}$$

And the proof follows by (4.1.51), (4.1.55), (4.1.59) and (4.1.50) □

Theorem 4.1.9. *Let $n \geq 4$, $\sigma > 0$ and Ω be an exterior set not containing the origin. Then there exist constants $C(\Omega,n)$ and $C'(\Omega,n,\sigma)$ such that the following inequality is valid,*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx + C'\int_\Omega \frac{u^2}{1+d^{2+\sigma}} dx \geq C\Big(\int_\Omega |u|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_c^\infty(\Omega). \tag{4.1.60}$$

proof: Let $R_0 = \sup_{x\in\partial\Omega} |x|$ and $\eta \in C^2(\Omega)$ such that $\eta(x) = d^{\frac{1}{2}}(x)\ \forall x \in \Omega_{\varepsilon_0}$ where ε_0 is small enough, $\eta(x) = \frac{(|x|-2R_0)^{\frac{1}{2}}}{|x|^{\frac{n-1}{2}}}$

$\forall x \in B^c(0,4R_0)$ and $c_1 \leq \eta \leq c_2$ otherwise, where c_1 and c_2 are positive constants. Set $u = \eta v$ then we have

$$\begin{aligned}
\int_\Omega |\nabla u|^2 dx &- \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx + C'\int_\Omega \frac{u^2}{1+d^{2+\sigma}}dx \\
&= \int_\Omega \eta^2|\nabla v|^2 dx - \int_\Omega (\eta\Delta\eta + \frac{\eta^2}{4d^2})v^2 dx + C'\int_\Omega \frac{\eta^2 v^2}{1+d^{+2+\sigma}}dx \\
&= \int_{\Omega_{\varepsilon_0}} \eta^2|\nabla v|^2 dx - \int_{\Omega_{\varepsilon_0}} (\eta\Delta\eta + \frac{\eta^2}{4d^2})v^2 dx + C'\int_{\Omega_{\varepsilon_0}} \frac{\eta^2 v^2}{1+d^{2+\sigma}}dx \qquad (4.1.61)\\
&+ \int_{B(0,4R_0)\setminus\Omega^c} \eta^2|\nabla v|^2 dx - \int_{B(0,4R_0)\setminus\Omega^c} (\eta\Delta\eta + \frac{\eta^2}{4d^2})v^2 dx + C'\int_{B(0,4R_0)\setminus\Omega^c} \frac{\eta^2 v^2}{1+d^{2+\sigma}}dx \qquad (4.1.62)\\
&+ \int_{B^c(0,4R_0)} \eta^2|\nabla v|^2 dx - \int_{B^c(0,4R_0)} (\eta\Delta\eta + \frac{\eta^2}{4d^2})v^2 dx + C'\int_{B^c(0,4R_0)} \frac{\eta^2 v^2}{1+d^{2+\sigma}}dx \qquad (4.1.63)\\
&= I_1 + I_2 + I_3, \qquad (4.1.64)
\end{aligned}$$

where $I_1,\ I_2$ and I_3 are the terms in (4.1.61), (4.1.62) and (4.1.63) respectively.
Now for I_1 we have by Theorem 2.4 in [FMaT1]

$$\begin{aligned}
I_1 = \int_{\Omega_{\varepsilon_0}} d|\nabla v|^2 dx &- \frac{1}{2}\int_{\Omega_{\varepsilon_0}} \Delta d v^2 dx + C'\int_{\Omega_{\varepsilon_0}} \frac{dv^2}{1+d^{2+\sigma}}dx \\
&\geq C\Big(\int_{\Omega_{\varepsilon_0}} d^{\frac{n}{n-2}}|v|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}} \geq C\Big(\int_{\Omega_{\varepsilon_0}} \eta^{\frac{2n}{n-2}}|v|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}. \qquad (4.1.65)
\end{aligned}$$

For I_2 we first note

$$\int_{B(0,4R_0)\setminus\Omega^c} \eta^2|\nabla v|^2 dx \geq c_1^2 \int_{B(0,4R_0)\setminus\Omega^c} |\nabla v|^2 dx, \qquad (4.1.66)$$

$$|\int_{B(0,4R_0)\setminus\Omega^c} (\eta\Delta\eta + \frac{\eta^2}{4d^2})v^2 dx| \leq C_0 \int_{B(0,4R_0)\setminus\Omega^c} v^2 dx, \qquad (4.1.67)$$

$$C'\int_{B(0,4R_0)\setminus\Omega^c} \frac{\eta^2 v^2}{1+d^{2+\sigma}}dx \geq \frac{C' c_1^2}{1+(4R)^{2+\sigma}}\int_{B(0,4R_0)\setminus\Omega^c} v^2 dx. \qquad (4.1.68)$$

Thus if we choose $C' \geq 2\frac{C_0(1+(4R)^{2+\sigma})}{c_1^2}$ we have by (4.1.66), (4.1.67), (4.1.68) and the Sobolev inequality

$$\begin{aligned}
I_2 \geq C(\int_{B(0,4R_0)\setminus\Omega^c} |\nabla v|^2 dx + \int_{B(0,4R_0)\setminus\Omega^c} v^2 dx) &\geq C\Big(\int_{B(0,4R_0)\setminus\Omega^c} |v|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{2n}} \\
&\geq C\Big(\int_{B(0,4R_0)\setminus\Omega^c} \eta^{\frac{2n}{n-2}}|v|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{2n}}. \qquad (4.1.69)
\end{aligned}$$

For I_3 first we note that

$$-\int_{B^c(0,4R_0)} (\eta\Delta\eta + \frac{\eta^2}{4d^2})v^2 dx \geq \frac{(n-1)(n-3)}{4}\int_{B^c(0,4R_0)} \frac{(|x|-2R_0)v^2}{|x|^{n+1}}dx, \qquad (4.1.70)$$

since $d(x) \geq |x| - 2R_0$ in $B^c(0, 4R_0)$. Also we note that $\frac{1}{2} \leq \frac{d}{|x|} \leq 1$ for each $x \in B^c(0, 4R_0)$.Thus we have by (4.1.70)

$$
\begin{aligned}
I_3 &\geq C\Big(\int_{B^c(0,4R_0)} \frac{|\nabla v|^2}{|x|^{n-2}}dx + \frac{(n-1)(n-3)}{4}\int_{B^c(0,4R_0)} \frac{v^2}{|x|^n}dx\Big) \\
&\geq C'\Big(\int_{B^c(0,2R_0)} \frac{|\nabla \widetilde{v}|^2}{|x|^{n-2}}dx + \frac{(n-1)(n-3)}{4}\int_{B^c(0,4R_0)} \frac{\widetilde{v}^2}{|x|^n}dx\Big), \qquad (4.1.71)
\end{aligned}
$$

where $\widetilde{v}$ is the function as in Theorem 4.1.7. Thus since $\widetilde{v} \in C_c^1(B^c(0, 2R_0))$ we have by Sobolev inequality

$$
\begin{aligned}
\int_{B^c(0,2R_0)} \frac{|\nabla \widetilde{v}|^2}{|x|^{n-2}}dx + \frac{(n-1)(n-3)}{4}\int_{B^c(0,4R_0)} \frac{\tilde{v}^2}{|x|^{n+1}}dx &\geq C\Big(\int_{B^c(0,2R_0)} \frac{|\widetilde{v}|^{\frac{2n}{n-2}}}{|x|^n}dx\Big)^{\frac{n-2}{n}} \\
&\geq C\Big(\int_{B^c(0,4R_0)} \frac{|\widetilde{v}|^{\frac{2n}{n-2}}}{|x|^n}dx\Big)^{\frac{n-2}{n}} \\
&= C\Big(\int_{B^c(0,4R_0)} \frac{|v|^{\frac{2n}{n-2}}}{|x|^n}dx\Big)^{\frac{n-2}{n}}, \qquad (4.1.72)
\end{aligned}
$$

where in the last equality we have used the fact that $\widetilde{v} = v \;\; \forall x \in B^c(0, 4R_0)$. Thus by (4.1.70), (4.1.71) and (4.1.72) we have

$$
I_3 \geq C\Big(\int_{B^c(0,4R_0)} |\eta|^{\frac{2n}{n-2}} |v|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{2n}}. \qquad (4.1.73)
$$

And the proof follows by (4.1.65), (4.1.69), (4.1.73) and (4.1.64) □

Finally, we will prove two theorems which are useful for the next subsection.

Lemma 4.1.10. *Let $n \geq 4$ and Ω be an exterior domain. Then the following inequality is valid*

$$
\int_\Omega \frac{|\nabla u|^2}{|x|^{2a_n}}dx \geq C\Big(\int_\Omega \frac{|u|^{\frac{2n}{n-2}}}{|x|^{\frac{2a_n n}{n-2}}}dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_0^\infty(\Omega) \qquad (4.1.74)
$$

where $a_n = \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - \frac{1}{4}}$, $C > 0$ depends only on Ω and n.

proof: As in Lemma 4.1.4, we only need to show the inequality for radially symmetric functions. Thus, let u be a symmetric function then inequality becomes equivalent to

$$
\int_\rho^\infty \frac{u_r^2}{r^{2\beta_n - 1}}dr \geq C\Big(\int_\rho^\infty \frac{|u|^{\frac{2n}{n-2}}}{r^{1+\frac{2n\beta_n}{n-2}}}dr\Big)^{\frac{n-2}{n}},
$$

where $\rho = \inf_{x\in\partial\Omega} |x|$ and $\beta_n = \sqrt{\frac{(n-2)^2}{4} - \frac{1}{4}}$. And the lemma follows by Lemma 4.1.5 with $A(r) = \frac{1}{r^{2\beta_n - 1}}$ and $B(r) = \frac{1}{r^{1+\frac{2n\beta_n}{n-2}}}$. □

Theorem 4.1.11. *Let $n \geq 4$ and Ω be an exterior domain not containing the origin. Then the following inequality is valid*

$$
\int_\Omega \frac{d}{|x|^{2a_n+1}}(|\nabla u|^2 + \frac{u^2}{1 + d^{2+\sigma}})dx \geq C\Big(\int_{\Omega^c} \frac{d^{\frac{n}{n-2}} u^{\frac{2n}{n-2}}}{|x|^{(2a_n+1)\frac{n}{n-2}}}dx\Big)^{\frac{n-2}{n}} \quad \forall u \in C_0^\infty(\Omega), \qquad (4.1.75)
$$

where $a_n = \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - \frac{1}{4}}$.

proof: Let $\Omega_\delta = \{x \in \Omega : d(x) \le \delta\}$ for some $\delta > 0$ sufficiently small and $\Omega^c_\delta = \mathbb{R}^n \setminus \Omega_\delta$. Then note that

$$\frac{\delta}{\rho} \le \frac{d}{|x|} \le 1 \;\forall\; x \in \Omega^c_\delta \quad \text{and} \quad \rho' \le |x| \le \rho + \delta \;\forall\; x \in \Omega_\delta, \tag{4.1.76}$$

where $\rho = \sup\{|x| : x \in \partial\Omega\}$ and $\rho' = \inf\{|x| : x \in \partial\Omega\}$ To prove inequality (4.1.75), we need to define cutoff functions supported near to the boundary. Let $a(t) \in C^\infty([0,\infty))$ be a nondecreasing function such that $a(t) = 1$ for $t \in [0, \frac{1}{2})$, $a(t) = 0$ for $t \ge 1$ and $a'(t) \le C_0$. For δ small we define $\phi_\delta(x) := a(\frac{d(x)}{\delta}) \in C^{1,1}(\Omega)$. Note that $\phi_\delta = 1$ on $\Omega_{\frac{\delta}{2}}$, $\phi_\delta = 0$ on Ω^c_δ and $|\nabla\phi_\delta| = |a'(\frac{d(x)}{\delta})|\frac{|\nabla d|}{\delta} \le \frac{C_0}{\delta}$ with C_0 a universal constant.
By (4.1.76) we have

$$\begin{aligned}
\int_{\Omega^c_{\frac{\delta}{2}}} \frac{d|\nabla((1-\phi_\delta)v)|^2}{|x|^{1+2a_n}}dx + \int_{\Omega^c_{\frac{\delta}{2}}} \frac{d|(1-\phi_d)v|^2}{|x|^{2a_n+1}(1+d^{2+\sigma})}dx &\ge C(\delta,\rho)\Big(\int_{\Omega^c_{\frac{\delta}{2}}} \frac{|\nabla((1-\phi_\delta)v)|^2}{|x|^{2a_n}}dx\Big) \\
&\ge C(\delta,\rho)\Big(\int_{\Omega^c_{\frac{\delta}{2}}} \frac{|(1-\phi_\delta)v|^{\frac{2n}{n-2}}}{|x|^{\frac{2na_n}{n-2}}}dx\Big)^{\frac{n-2}{n}} \\
&\ge C(\delta,\rho)\Big(\int_{\Omega^c_\delta} \frac{d^{\frac{n}{n-2}}|(1-\phi_\delta)v|^{\frac{2n}{n-2}}}{|x|^{(1+2a_n)\frac{n}{n-2}}}dx\Big)^{\frac{n-2}{n}},
\end{aligned} \tag{4.1.77}$$

where in the last inequality we have used again (4.1.76) and the fact that $\Omega^c_\delta \subset \Omega^c_{\frac{\delta}{2}}$.
Now by Theorem 2.4 in [FMaT1] and (4.1.76) for sufficiently small $\delta > 0$ we have

$$\begin{aligned}
\int_{\Omega^c_{\frac{\delta}{2}}} \frac{d|\nabla((1-\phi_\delta)v)|^2}{|x|^{1+2a_n}}dx \quad &+ \quad \int_{\Omega^c_{\frac{\delta}{2}}} \frac{d|(1-\phi_d)v|^2}{|x|^{1+2a_n}(1+d^{2+\sigma})}dx \\
&\ge \quad C(\delta,\rho,\rho')\Big(\int_{\Omega_\delta} \frac{d^{\frac{n}{n-2}}|\phi_\delta v|^{\frac{2n}{n-2}}}{|x|^{(1+2a_n)\frac{n}{n-2}}}dx\Big)^{\frac{n-2}{n}}.
\end{aligned} \tag{4.1.78}$$

Now we add (4.1.77) and (4.1.78) to obtain

$$\begin{aligned}
&C(\delta,\rho,\rho')\Big(\int_{\Omega_\delta} \frac{d^{\frac{n}{n-2}}|\phi_\delta v|^{\frac{2n}{n-2}}}{|x|^{(1+2a_n)\frac{n}{n-2}}}dx\Big)^{\frac{n-2}{n}} + C(\delta,\rho)\Big(\int_{\Omega^c_\delta} \frac{d^{\frac{n}{n-2}}|(1-\phi_\delta)v|^{\frac{2n}{n-2}}}{|x|^{(1+2a_n)\frac{n}{n-2}}}dx\Big)^{\frac{n-2}{n}} \\
&\le \int_{\Omega_\delta} \frac{d|\nabla(\phi_\delta v)|^2}{|x|^{1+2a_n}}dx + \int_{\Omega^c_{\frac{\delta}{2}}} \frac{d|\nabla((1-\phi_\delta)v)|^2}{|x|^{1+2a_n}}dx + 2\int_\Omega \frac{d|v|^2}{|x|^{1+2a_n}(1+d^{2+\sigma})}dx \\
&\le C'\Big(\int_{\Omega_\delta\setminus\Omega_{\frac{\delta}{2}}} \frac{dv^2}{|x|^{1+2a_n}}dx\Big) + C(n)\Big(\int_\Omega \frac{d|\nabla v|^2}{|x|^{1+2a_n}}dx + \int_\Omega \frac{d|v|^2}{|x|^{1+2a_n}(1+d^{2+\sigma})}dx\Big),
\end{aligned} \tag{4.1.79}$$

where in the last inequality we have used the fact that $\nabla\phi_\delta \ne 0$ only $\forall x \in \Omega_\delta \setminus \Omega_{\frac{\delta}{2}}$.
In view of (4.1.79), it suffices to prove

$$\int_{\Omega_\delta\setminus\Omega_{\frac{\delta}{2}}} \frac{dv^2}{|x|^{1+2a_n}}dx \le C\Big(\int_\Omega \frac{d|\nabla(v)|^2}{|x|^{1+2a_n}}dx + \int_\Omega \frac{d|v|^2}{|x|^{1+2a_n}(1+d^{2+\sigma})}dx\Big). \tag{4.1.80}$$

However this follows because by (4.1.76) we have

$$\int_{\Omega_\delta\setminus\Omega_{\frac{\delta}{2}}} \frac{dv^2}{|x|^{1+2a_n}}dx \le (1+\delta)^{2+\sigma}\Big(\int_{\Omega_\delta\setminus\Omega_{\frac{\delta}{2}}} \frac{dv^2}{|x|^{1+2a_n}(1+d^{2+\sigma})}dx\Big).$$

□

Theorem 4.1.12. *Let n=3 and Ω be an exterior domain not containing the origin. Then the following inequality is valid*

$$\int_\Omega \frac{d}{|x|^2}(|\nabla u|^2 + \frac{u^2}{1+d^{2+\sigma}})dx \geq C\Big(\int_\Omega \frac{d^3u^6X^4(\frac{|x|}{\rho})}{|x|^3}dx\Big)^{\frac{1}{3}}, \quad \forall u \in C_0^\infty(\Omega)$$

where $X(t) = (1+\ln t)^{-1}$. *Moreover, the power* 4 *on X can not be replaced by a smaller power.*

proof: The proof of the theorem is the same as in Theorem 4.1.11. The only difference is that, we use here Lemma 4.1.4 instead of Lemma 4.1.10. □

4.1.4 Existence of Minimizer in Suitable Spaces and Their Behavior

In this subsection, we assume the set Ω to be an exterior domain not containing the origin. By Theorems 4.1.9 (for $n = 3$) and 4.1.8 (for $n \geq 4$) we note that there exists a constant $\lambda \in \mathbb{R}$ such that

$$(4.1.81) \qquad -\infty < \lambda = \inf_{u\in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx}{\int_\Omega \frac{u^2}{1+d^{2+\sigma}}dx},$$

where $\sigma > 0$.

The main goal of this section is to prove the existence of a ground state function $\phi \in H^1_{loc}(\Omega)$ which solves the corresponding Euler-Lagrange of 4.1.81 in the weak sense i.e.

$$(4.1.82) \qquad -\Delta\phi - \frac{\phi}{4d^2} = \lambda\frac{\phi}{d^{2+\sigma}} \quad \text{in} \quad \Omega.$$

Also, we would like to know how this function ϕ behaves. The space which we use to prove the existence of ϕ is $\mathfrak{D}_0^{1,2}(\Omega; |x|, d)$ which is the closure of $C_0^\infty(\Omega)$ functions under the norm

$$\|u\|^2_{\mathfrak{D}_0^{1,2}(\Omega;|x|,d)} = \int_\Omega \frac{d}{|x|^{2a_n+1}}\left(|\nabla u|^2 + \frac{u^2}{1+d^{2+\sigma}}\right)dx,$$

where $a_n = \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - \frac{1}{4}}$ and $\sigma > 0$. By Theorems 4.1.12 and 4.1.11, we have for $n = 3$ and $n \geq 4$ respectively the following inequalities

$$(4.1.83) \qquad \|u\|^2_{\mathfrak{D}_0^{1,2}(\Omega;|x|,d)} \geq C\Big(\int_\Omega \frac{d^3u^6X^4(\frac{|x|}{\rho})}{|x|^3}dx\Big)^{\frac{1}{3}}, \quad \forall u \in C_0^\infty(\Omega),$$

where $\rho = \inf\{|x| : x \in \partial\Omega\}$ and $X(t) = (1+\ln t)^{-1}$.

$$(4.1.84) \qquad \|u\|^2_{\mathfrak{D}_0^{1,2}(\Omega;|x|,d)} \geq C\Big(\int_\Omega \frac{d^{\frac{n}{n-2}}u^{\frac{2n}{n-2}}}{|x|^{(2a_n+1)\frac{n}{n-2}}}dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_0^\infty(\Omega).$$

where $a_n = \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - \frac{1}{4}}$.

Theorem 4.1.13. *Let* $n = 3$ *and let Ω be a exterior domain not containing the origin. Then there exists a ground state function* $\phi \in H^1_{loc}(\Omega)$ *such that* ϕ *solves the problem (4.1.82) in the weak sense.*

proof: Let $\eta \in C^2(\Omega)$ be a function such that $\eta(x) = d^{\frac{1}{2}}(x)$ near the boundary, say, $d(x) \leq \varepsilon_0$, and $\eta(x) = |x|^{-\frac{1}{2}}$ away from the boundary, say $|x| > R > 2R_0 = \sup_{x\in\partial\Omega}$ and $c_1 \leq \eta \leq c_2$ otherwise, where $c_1,\ c_2$ are positive constants. Then

$||v||_{W_0^1(\Omega;|x|,d)}$ is equivalent with the norm

$$||v||^2 = \int_\Omega \eta^2(|\nabla v|^2 + \frac{v^2}{1+d^{2+\sigma}})dx.$$

Also we have the following inequality

$$\int_\Omega \eta^2(|\nabla v|^2 + \frac{v^2}{1+d^{2+\sigma}})dx \geq C\Big(\int_\Omega \eta^6 v^6 X^4(\frac{|x|}{\rho})dx\Big)^{\frac{1}{3}}, \tag{4.1.85}$$

by (4.1.83). Changing the variables by $u = \eta v$ in (4.1.81) we have the equivalent problem

$$-\infty < \lambda = \inf_{v\in C_0^\infty(\Omega)} \frac{\int_\Omega \eta^2|\nabla v|^2dx - \int_\Omega(\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})v^2dx}{\int_\Omega \frac{\eta^2v^2}{1+d^{2+\sigma}}dx}. \tag{4.1.86}$$

For R and δ sufficiently large and small respectively we get

$$\begin{aligned}\Big|\int_\Omega(\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})v^2dx\Big| &= \Big|\int_{B_R^c}(\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})v^2dx + \int_{B_R\setminus\Omega_\delta}(\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})v^2dx + \int_{\Omega_\delta}(\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})v^2dx\Big| \\ &\leq |I_1| + |I_2| + |I_3|\end{aligned}$$

First we note for $|x| > R > \max\{2R_0, 1\}$ that

$$\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2} = \frac{1}{4}\left(\frac{1}{d^2|x|} - \frac{1}{|x|^3}\right) \leq \frac{2R_0}{4|x|^2d^2} = \frac{R_0}{2}\frac{\eta^2}{|x|d^2}.$$

Using the last inequality and the fact that $\frac{1}{2} \leq \frac{d}{|x|} \leq 1$ for $|x| > 2R_0$, we have

$$\begin{aligned}|I_1| \leq C(R_0)\int_{B_R^c}\frac{\eta^2v^2}{|x|d^2}dx &\leq C\Big(\int_{B_R^c}X^4(\frac{|x|}{\rho})\eta^6|u|^6dx\Big)^{\frac{1}{3}}\Big(\int_{B_R^c}\frac{X^{-2}(\frac{|x|}{\rho})}{|x|^{\frac{9}{2}}}dx\Big)^{\frac{2}{3}} \\ &\leq C(R_0,n)\frac{1}{R}\int_\Omega \eta^2(|\nabla v|^2 + \frac{v^2}{1+d^{2+\sigma}})dx,\end{aligned} \tag{4.1.87}$$

where in the last two inequalities we have used Hölder inequality and inequality (4.1.85) respectively. Also since $\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2} \in L^\infty(\Omega)$, we have

$$|I_2| \leq C(\delta)(R(1+R)^{2+\sigma})\int_{B_R\setminus\Omega_\delta}\frac{\eta^2v^2}{(1+d^{2+\sigma})}dx. \tag{4.1.88}$$

Finally,

$$|I_3| \leq C\frac{\delta}{X_1(\delta)}\int_{\Omega_\delta}\frac{X_1(\delta)}{\delta}v^2dx \leq C\int_{\Omega_{\varepsilon_0}}\frac{X_1(d)}{d}|\phi_{\varepsilon_0}(d)v|^2dx,$$

where $X_1(d) = (1-\ln d)^{-1}$ and $\phi_{\varepsilon_0}(d)$ is the function as in Theorem 4.1.11.
But by [FMoT1]-Proposition 5.1 we have

$$\int_{\Omega_{\varepsilon_0}}\frac{X_1(d)}{d}|\phi_{\varepsilon_0}v|^2dx \leq C\int_{\Omega_{\varepsilon_0}} d(|\nabla(\phi_{\varepsilon_0}u)|^2 + |\phi_{\varepsilon_0}u|^2)dx,$$

which implies

$$|I_3| \leq C\frac{\delta}{X_1(\delta)}\Big(\int_\Omega \eta^2(|\nabla v|^2 + \frac{v^2}{1+d^{2+\sigma}})dx\Big). \tag{4.1.89}$$

Finally we combine the estimates (4.1.87), (4.1.88) and (4.1.89) to deduce that for any $\varepsilon > 0$ there exist M_ε such that

$$\Big|\int_\Omega (\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})v^2 dx\Big| \leq \varepsilon\int_\Omega \eta^2|\nabla v|^2 dx + M_\varepsilon\int_\Omega \frac{v^2}{1+d^{2+\sigma}}dx. \tag{4.1.90}$$

In the sequel we will establish the existence of a function $\psi_1 \in W_0^1(\Omega; |x|, d)$ which realizes the infimum in (4.1.86). To this end let w_k be a minimizing sequence normalized by $\int_\Omega \frac{v^2}{1+d^{2+\sigma}}dx = 1$. Then using (4.1.90) we can easily obtain (by (4.1.86)) that the sequence w_k is bounded i.e. $\sup_k \|w_k\| < N$. Therefore there exist a subsequence still denoted by w_k such that it converges to $W_0^1(\Omega; |x|, d)-$weakly to ψ_1. Clearly by embedding theorems for $R > 0$ and $\delta > 0$ large and small enough respectively we have

$$\int_{B_R\setminus\Omega_\delta} (\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})w_k^2 dx \to \int_{B_R\setminus\Omega_\delta} (\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})\psi_1^2 dx. \tag{4.1.91}$$

Also, we have by (4.1.87)

$$\Big|\int_{B_R^c} (\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})w_k^2 dx\Big| \leq CN\frac{1}{R}. \tag{4.1.92}$$

By (4.1.89) we have

$$\Big|\int_{\Omega_\delta} (\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})w_k^2 dx\Big| \leq \int_{\Omega_\delta} \frac{\eta^2 w_k^2}{1+d^{2+\sigma}}dx \leq CN\frac{\delta}{X_1(\delta)}, \tag{4.1.93}$$

where $X_1(\delta) = (1 - \ln\delta)^{-1}$. Using now (4.1.91), (4.1.92) and (4.1.93) we have

$$\int_\Omega (\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})w_k^2 dx \to \int_\Omega (\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})\psi_1^2 dx$$

and the result follows by lower semicontinuity of the gradient term of numerator in (4.1.86). □

Theorem 4.1.14. *Let $n \geq 4$ and et Ω be an exterior domain not containing the origin. Then there exists a function $\phi \in H^1_{loc}(\Omega)$ such that ϕ solves the problem (4.1.82) in the weak sense.*

proof: Let $\eta \in C^2(\Omega)$ be a function such that $\eta(x) = d^{\frac{1}{2}}(x)$ near the boundary, say, $d(x) \leq \varepsilon_0$, and $\eta(x) = |x|^{-a_n}$ away from the boundary where $a_n = \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - \frac{1}{4}}$, say $|x| > R > 2R_0 = \sup_{x\in\partial\Omega}$ and $c_1 \leq \eta \leq c_2$ otherwise, where $c_1,\ c_2$ are positive constants. Then $\|v\|_{W_0^1(W_0^1(\Omega;|x|,d)}$ is equivalent with the norm

$$\|v\|^2 = \int_\Omega \eta^2(|\nabla v|^2 + \frac{v^2}{1+d^{2+\sigma}})dx.$$

Also we have the following inequality

$$\int_\Omega \eta^2(|\nabla v|^2 + \frac{v^2}{1+d^{2+\sigma}})dx \geq C\Big(\int_\Omega \eta^{\frac{2n}{n-2}}|v|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{2n}}, \tag{4.1.94}$$

by (4.1.83). Changing the variables by $u = \eta v$ in (4.1.81) we have the equivalent problem

$$(4.1.95) \qquad -\infty < \lambda = \inf_{v \in C_0^\infty(\Omega)} \frac{\int_\Omega \eta^2 |\nabla v|^2 dx - \int_\Omega (\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})v^2 dx}{\int_\Omega \frac{\eta^2 v^2}{1+d^{2+\sigma}} dx}.$$

For R and δ sufficiently large and small respectively we get

$$\begin{aligned}\Big|\int_\Omega (\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})v^2 dx\Big| &= \Big|\int_{B_R^c} (\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})v^2 dx + \int_{B_R\setminus\Omega_\delta} (\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})v^2 dx + \int_{\Omega_\delta} (\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})v^2 dx\Big| \\ &\le |I_1| + |I_2| + |I_3|\end{aligned}$$

First we note for $|x| > R > \max\{2R_0, 1\}$ that

$$\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2} = \frac{1}{4}\Big(\frac{1}{d^2|x|^{2a_n}} - \frac{1}{|x|^{2+2a_n}}\Big) \le \frac{2R_0}{4|x|^{1+2a_n}d^2} = \frac{R_0}{2}\frac{\eta^2}{|x|d^2}.$$

Using the last inequality and the fact that $\frac{1}{2} \le \frac{d}{|x|} \le 1$ for $|x| > 2R_0$, we have

$$|I_1| \le C(R_0)\int_{B_R^c} \frac{\eta^2 v^2}{|x|d^2} dx \le C\Big(\int_{B_R^c} \eta^{\frac{2n}{n-2}}|u|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{2n}}\Big(\int_{B_R^c} \frac{1}{|x|^{\frac{3n}{2}}} dx\Big)^{\frac{2}{n}}$$

$$(4.1.96) \qquad \le C(R_0, n)\frac{1}{R^{\frac{n}{2}}}\int_\Omega \eta^2\Big(|\nabla v|^2 + \frac{v^2}{1+d^{2+\sigma}}\Big)dx,$$

where in the last two inequalities we have used Hölder inequality and inequality (4.1.85) respectively. Also since $\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2} \in L^\infty(\Omega)$, we have

$$(4.1.97) \qquad |I_2| \le C(\delta)(R(1+R)^{2+\sigma})\int_{B_R\setminus\Omega_\delta} \frac{\eta^2 v^2}{(1+d^{2+\sigma})} dx.$$

Finally,

$$|I_3| \le C\frac{\delta}{X_1(\delta)}\int_{\Omega_\delta} \frac{X_1(\delta)}{\delta} v^2 dx \le C\int_{\Omega_{\varepsilon_0}} \frac{X_1(d)}{d}|\phi_{\varepsilon_0}(d)v|^2 dx,$$

where $X_1(d) = (1 - \ln d)^{-1}$ and $\phi_{\varepsilon_0}(d)$ is the function as in Theorem 4.1.11.
But by [FMoT1]-Proposition 5.1 we have

$$\int_{\Omega_{\varepsilon_0}} \frac{X_1(d)}{d}|\phi_{\varepsilon_0} v|^2 dx \le C\int_{\Omega_{\varepsilon_0}} d(|\nabla(\phi_{\varepsilon_0} u)|^2 + |\phi_{\varepsilon_0} u|^2)dx,$$

which implies

$$(4.1.98) \qquad |I_3| \le C\frac{\delta}{X_1(\delta)}\Big(\int_\Omega \eta^2\Big(|\nabla v|^2 + \frac{v^2}{1+d^{2+\sigma}}\Big)dx\Big).$$

Finally for combine the estimates (4.1.96), (4.1.97) and (4.1.98) to deduce that for any $\varepsilon > 0$ there exist M_ε such that

$$(4.1.99) \qquad \Big|\int_\Omega (\eta\Delta\eta + \frac{1}{4}\frac{\eta^2}{d^2})v^2 dx\Big| \le \varepsilon\int_\Omega \eta^2|\nabla v|^2 dx + M_\varepsilon\int_\Omega \frac{v^2}{1+d^{2+\sigma}} dx.$$

In the sequel we will establish the existence of a function $\psi_1 \in W_0^1(\Omega; |x|, d)$ which realizes the infimum in (4.1.86). To this end let w_k be a minimizing sequence normalized by $\int_\Omega \frac{v^2}{1+d^{2+\sigma}} dx = 1$. Then using (4.1.99) we can easily obtain (by

(4.1.95)) that the sequence w_k is bounded i.e. $\sup_k \|w_k\| < N$. Therefore there exist a subsequence still denoted by w_k such that it converges to $W_0^1(\Omega;|x|,d)-$weakly to ψ_1. Clearly by embedding theorems for $R>0$ and $\delta>0$ large and small enough respectively we have

$$\int_{B_R\setminus\Omega_\delta}(\eta\Delta\eta+\frac{1}{4}\frac{\eta^2}{d^2})w_k^2dx\to\int_{B_R\setminus\Omega_\delta}(\eta\Delta\eta+\frac{1}{4}\frac{\eta^2}{d^2})\psi_1^2dx. \tag{4.1.100}$$

Also, we have by (4.1.87)

$$\Big|\int_{B_R^c}(\eta\Delta\eta+\frac{1}{4}\frac{\eta^2}{d^2})w_k^2dx\Big|\le CN\frac{1}{R^{\frac{n}{2}}}. \tag{4.1.101}$$

By (4.1.89) we have

$$\Big|\int_{\Omega_\delta}(\eta\Delta\eta+\frac{1}{4}\frac{\eta^2}{d^2})w_k^2dx\Big|\le\int_{\Omega_\delta}\frac{\eta^2w_k^2}{(1+d^{2+\sigma})}dx\le CN\frac{\delta}{X_1(\delta)}, \tag{4.1.102}$$

where $X_1(\delta)=(1-\ln\delta)^{-1}$. Using now (4.1.100), (4.1.101) and (4.1.102) we have

$$\int_{\Omega}(\eta\Delta\eta+\frac{1}{4}\frac{\eta^2}{d^2})w_k^2dx\to\int_{\Omega}(\eta\Delta\eta+\frac{1}{4}\frac{\eta^2}{d^2})\psi_1^2dx$$

and the result follows by lower semicontinuity of the gradient term of numerator in (4.1.95). □

Theorem 4.1.15. *The asymptotic behavior of ϕ in Theorems 4.1.13 and 4.1.14 is like $d^{\frac{1}{2}}$ near to the boundary and like $|x|^{-a_n}$ away from the boundary, where $a_n=\frac{n-2}{2}+\sqrt{\frac{(n-2)^2}{4}-\frac{1}{4}}$.*

proof: Assume first $n\ge 4$. It is well known that the eigenfunction $\phi\sim d^{\frac{1}{2}}$ (see [DD] for a lower bound and see in [FMoT1] for a upper bound). Thus we will focus away from the boundary such that ψ_1 is the minimizer of 4.1.95. For $|x|>R$ where R is large enough, ψ_1 solves the problem

$$L\psi_1=-div(\frac{1}{|x|^{2a_n}}\nabla\psi_1)+\frac{1}{4}(\frac{1}{|x|^{2+2a_n}}-\frac{1}{d^2|x|^{2a_n}})\psi_1=\lambda\frac{\psi}{|x|^{2a_n}(1+d^{2+\sigma})}. \tag{4.1.103}$$

First we show the lower bound. Let $M\ge\lambda_1$. Consider the function $1+C_1|x|^{-\sigma}$, then

$$(L+\frac{M}{|x|^{2a_n(1+d^{2+\sigma})}})(1+C_1|x|^{-\sigma})$$

$$\le\sigma(-\sqrt{\frac{(n-2)^2}{4}-\frac{1}{4}}-\sigma)C_1\frac{1}{|x|^{2a_n+\sigma+2}}+\frac{MC_1}{(1+d^{2+\sigma})|x|^{2a_n+\sigma}}+\frac{M}{(1+d^{2+\sigma})|x|^{2a_n}}\le 0$$

for $C_1>0$ and $|x|$ large enough. On the other hand the first eigenfunction ψ_1 of L satisfies

$$\Big(L+\frac{M}{|x|^{2a_n}(1+d^{2+\sigma})}\Big)\psi_1=\Big(\frac{\lambda}{|x|^{2a_n}(1+d^{2+\sigma})}\Big)\psi_1+\Big(\frac{M}{|x|^{2a_n}(1+d^{2+\sigma})}\Big)\psi_1\ge 0.$$

Now since both function ψ_1 and $1+C_1|x|^{-\sigma}$ are smooth away from the boundary, we can select constant ε such that $\varepsilon(1+C_1|x|^{-\sigma})-\psi_1\le 0$ on ∂B_R^c. Let $g(x)=\varepsilon(1+C_1|x|^{-\sigma})-\psi_1\le 0$ and $g^+=\max\{g,0\}$. Thus we can take g^+ as test function to obtain (see remark below)

$$\int_{B_R^c} \frac{1}{|x|^{2a_n}} \nabla g \nabla g^+ dx + \frac{1}{4}(\frac{1}{|x|^{2a_n+2}} - \frac{1}{d^2|x|^{2a_n}}) g g^+ dx + M\frac{gg^+}{|x|^{2a_n}(1+d^{2+\sigma})} \leq 0 \Rightarrow$$

$$\int_{B_R^c} \frac{1}{|x|^{2a_n}} |\nabla g^+|^2 dx + \frac{1}{4}(\frac{1}{|x|^{2a_n+2}} - \frac{1}{d^2|x|^{2a_n}}) |g^+|^2 dx + M\frac{|g^+|^2}{|x|^{2a_n}(1+d^{2+\sigma})} \leq 0,$$

which imply by (4.1.86) that $g^+ = 0$ and the lower bound follows.

For the upper bound we first note by (4.1.75) that

$$\frac{\int_{B_R^c} \frac{1}{|x|^{2a_n}} |\nabla u|^2 + (\frac{1}{4}\frac{1}{|x|^{2a_n+2}} - \frac{1}{d^2|x|^{2a_n}}) u^2 dx}{\int_{B_R^c} \frac{u^2}{|x|^{2a_n}(1+d^{2+\sigma})} dx} \geq C \frac{\left(\int_\Omega \frac{|u|^{\frac{2n}{n-2}}}{|x|^{\frac{2na_n}{n-2}}} dx\right)^{\frac{n-2}{n}} - \int_{B_R^c} \frac{u^2}{|x|^{2a_n}(1+d^{2+\sigma})} dx}{\int_{B_R^c} \frac{u^2}{|x|^{2a_n}(1+d^{2+\sigma})} dx}$$

$$\geq C \frac{\left(\int_\Omega \frac{|u|^{\frac{2n}{n-2}}}{|x|^{\frac{2na_n}{n-2}}} dx\right)^{\frac{n-2}{n}}}{\left(\int_{B_R^c} \frac{1}{(1+d^{2+\sigma})^{\frac{n}{2}}} dx\right)^{\frac{2}{n}} \left(\int_\Omega \frac{|u|^{\frac{2n}{n-2}}}{|x|^{\frac{2na_n}{n-2}}} dx\right)^{\frac{n-2}{n}}} - 1 \to \infty, \quad as \ \ R \to \infty \quad \forall u \in C_0^\infty(B_R^c) \tag{4.1.104}$$

Since $\frac{1}{4}(\frac{1}{d^2|x|} - \frac{1}{|x|^3}) \leq \frac{2R_0}{4|x|^2 d^2}$, for $|x| > R_0 = \sup_{x \in \partial\Omega} |x|$ we have two cases:
Case 1:
If $0 < \sigma < 1$ then as before we see that $(L - \frac{\lambda}{|x|^{2a_n}(1+d^{2+\sigma})})(1 - C_1|x|^{-\sigma}) \geq 0$ for $C_1 > 0$ big enough and $(L - \frac{\lambda_1}{|x|^{2a_n}(1+d^(2+\sigma))})\psi_1 = 0$. We next choose $\varepsilon > 0$ big enough so that $g(x) = \varepsilon\psi_1 - (1 - C_1|x|^{-\sigma}) \leq 0$ on ∂B_R^c. Case 2:
If $\sigma \geq 1$ we note that $(L - \frac{\lambda}{|x|^{2a_n}(1+d^{2+\sigma})})(1 - C_1|x|^{-1}) \geq 0$ for $C_1 > 0$ big enough and $(L - \frac{\lambda}{|x|^{2a_n}(1+d^{2+\sigma})})\psi_1 = 0$. We next choose $\varepsilon > 0$ big enough so that $g(x) = \varepsilon\psi_1 - (1 - C_1|x|^{-1}) \leq 0$ on ∂B_R^c.
Thus in both cases, since g^+ is a test function we have

$$\int_{B_R^c} \frac{1}{|x|^{2a_n}} \nabla g \nabla g^+ dx + (\frac{1}{4}\frac{1}{|x|^{2a_n+2}} \frac{1}{d^2|x|^{2a_n}}) g g^+ dx - \int_{B_R^c} \frac{\lambda}{|x|^{2a_n}(1 + d^{2+\sigma})} g g^+ dx \leq 0,$$

from which it follows

$$\frac{\int_{B_R^c} \frac{1}{|x|^{2a_n}} |\nabla g^+|^2 + (\frac{1}{4}\frac{1}{|x|^{2a_n+2}} \frac{1}{d^2|x|^{2a_n}}) g^{+2} dx}{\int_{B_R^c} \frac{g^{+2}}{|x|^{2a_n}(1+d^{2+\sigma})}} \leq \lambda.$$

This contradicts with (4.1.104) unless $g^+ = 0$ from which follows the upper bound for ϕ.
For $n = 3$ the only difference is that in (4.1.104) we use (4.1.85) instead of (4.1.94). □

Remark: Let us now prove that the functions g^+ of the above theorem are tests functions.
First we consider the function $a(t) = 1$ if $t \leq 1$ and $a(t) = 0$ if $t \geq 2$. Then the function $\zeta_r(x) = a(\frac{|x|}{r})$, for $r > R$, it is a $H_0^1(B_{2r} \setminus B_R)$ function. Then we set $g_m = \min\{g^+, m\}$ and $\phi_{r,\varepsilon,m} = \zeta_r |x|^{-\varepsilon} g_m$. Note now that $\phi_{r,\varepsilon,m} \in H_0^1(B_{2r} \setminus B_R)$. First we claim that $\phi_{r,\varepsilon,m} \to |x|^{-\varepsilon} g_m$, as $r \to \infty$ with respect the norm (4.1.86). To this end, it is enough to show that

$$\int_{B_R^c} \frac{|\nabla((1 - \zeta_r)(|x|^{-\varepsilon} g_m))|^2}{|x|^{2a_n}} dx \to 0, \quad \text{as} \quad r \to \infty.$$

Indeed

$$\int_{B_R^c} \frac{|\nabla((1-\zeta_r)(|x|^{-\varepsilon} g_m))|^2}{|x|^{2a_n}} dx$$
$$\leq C\Big(\int_{B_R^c} \frac{|\nabla(1-\zeta_r)|^2 |(|x|^{-\varepsilon} g_m)^2}{|x|^{2a_n}} dx + \varepsilon^2 \int_{B_R^c} \frac{|(1-\zeta_r) g_m|^2}{|x|^{2a_n+2+\varepsilon}} dx + \int_{B_R^c} \frac{|\nabla g_m|^2 (1-\zeta_r)^2 |x|^{-2\varepsilon}}{|x|^{2a_n}} dx\Big) \tag{4.1.105}$$

It is not difficult to show that the second and third integral in the above inequality go to zero as r goes to infinity. Finally for the first integral in (4.1.105), we have

$$\begin{aligned} \int_{B_R^c} \frac{|\nabla((1-\zeta_r)(|x|^{-\varepsilon} g_m))|^2}{|x|^{2a_n}} dx &\leq \frac{mC(n)}{r^2} \int_r^{2r} r^{1-\varepsilon-2\sqrt{\frac{(n-2)^2}{4}-\frac{1}{4}}} dr \\ &= mC(n) \frac{r^{2-1-\varepsilon-2\sqrt{\frac{(n-2)^2}{4}-\frac{1}{4}}}}{r^2} \to 0, \quad \text{as } r \to \infty \end{aligned} \tag{4.1.106}$$

and the claim follows. By the same way we can prove that $|x|^{-\varepsilon} g_m \to g_m$ as $\varepsilon \to 0$ and $g_m \to g^+$ as $m \to \infty$. Thus we reach to conclusion that $g^+ \in \mathfrak{D}_0^{1,2}(\Omega; |x|, d)$ and $g^+ = 0$ in B_R. In particular, we show that (by definition of $W_0^1(\Omega; |x|, d)$) there exist a sequence $u_m \in C_0^\infty(B_R^c)$ such that $u_m \to u$ with respect the norm (4.1.86), that is g^+ is a test function.

4.2 Hardy Sobolev Inequalities In Domains Above the Graphs of $C^{1,1}$ Functions

In this section we will prove Hardy-Sobolev type inequalities in domains above the graphs of $C^{1,1}$ functions. More precisely, let $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying the conditions $|\nabla\Gamma| < \lambda$ and $\Gamma \in C^{1,1}(\mathbb{R}^{n-1})$. We then call the set

$$\Omega = \{(x', x_n) \in \mathbb{R}^n : x_n > \Gamma(x')\},$$

a domain above the graph of a $C^{1,1}$ function.
The half space $\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$ is an example of a domain above the graph of $C^{1,1}$ function. Especially, we have the Hardy-Maz'ya-Sobolev inequality in half space (for $n \geq 3$)

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx - \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2} dx \geq C(n)\Big(\int_{\mathbb{R}^n_+} |u|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}}, \quad \forall u \in C_0^\infty(\mathbb{R}^n_+). \tag{4.2.107}$$

We note here that the inequality (4.2.107) is valid for $n = 3$ without using some logarithmic function as in exterior domains. Thus, the proof of Hardy-Maz'ya-Sobolev inequality in domain above above the graphs of $C^{1,1}$ functions is different from the proof in exterior domains.
Set $d(x) = \inf_{y \in \partial\Omega} |x - y|$ and $\delta(x) = x_n - \Gamma(x')$. Then we can easily prove that $k\delta(x) \leq d(x) \leq \delta(x)$, where $k = \frac{1}{1+\mu}$. Then we have the following Theorem

Theorem 4.2.1. *Let $n \geq 3$ and Ω be the domain above the graph of $C^{1,1}$ function which satisfies $-\Delta d \geq 0$. Then there exists a positive constant C which depends only on n and μ, such that*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx \geq C(n,\mu)\Big(\int_\Omega |u|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}}. \qquad \forall u \in C_0^\infty(\Omega). \tag{4.2.108}$$

proof: Set $u = d^{\frac{1}{2}} v$ then (4.2.108) becomes equivalent to

$$\int_\Omega d|\nabla v|^2 dx - \frac{1}{2}\int_\Omega -\Delta d u^2 dx \geq C\Big(\int_\Omega d^{\frac{n}{n-2}}|u|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}.$$

Since $-\Delta d \geq 0$ and $k\delta(x) \leq d(x) \leq \delta(x)$, it is enough to prove

$$\int_\Omega \delta|\nabla v|^2 dx \geq C(k)\Big(\int_\Omega \delta^{\frac{n}{n-2}}|v|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}.$$

But by inequality (4.2.107) if we set $u = x_n^{\frac{1}{2}} v$ we have that

(4.2.109) $$\int_{\mathbb{R}^n_+} y_n|\nabla_y v|^2 dy \geq C(n)\Big(\int_{\mathbb{R}^n_+} y_n^{\frac{n}{n-2}}|v|^{\frac{2n}{n-2}}dy\Big)^{\frac{n-2}{n}} \quad \forall v \in C_0^\infty(\mathbb{R}^n_+).$$

Now set in (4.2.109) $x_i = y_i$ for $i = 1, ..., n-1$ and $x_n = y_n + \Gamma(y')$ then $\nabla_{y'} v = \nabla_{x'} v + v_{x_n}\nabla_{x'}\Gamma$ and $v_{y_n} = v_{x_n}$, thus,

$$C(\mu)|\nabla_x v| \leq |\nabla_y v| \leq c(\mu)|\nabla_x v|$$

and by (4.2.109) we have

$$\int_\Omega \delta|\nabla v|^2 dx \geq C(\mu)\Big(\int_\Omega \delta^{\frac{n}{n-2}}|v|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}},$$

which is the desired result. □

Lemma 4.2.2. *Let a, b, p and q be such that $1 \leq p < n$, $p < q \leq \frac{pn}{n-p}$ and $b = a - 1 + \frac{q-p}{qp}n$. Then for any $\eta > 0$, there holds:*

$$\lambda\eta^{-\frac{1-\lambda}{\lambda}}\|x_n^a v\|_{L^{\frac{np}{n-p}}}(\mathbb{R}^n_+) + (1-\lambda)\eta\|x_n^{a-1}v\|_{L^p(\mathbb{R}^n_+)} \geq \|x_n^b v\|_{L^q(\mathbb{R}^n_+)}, \quad \forall u \in C_0^\infty(\mathbb{R}^n_+)$$

where

(4.2.110) $$0 < \lambda = \frac{n(q-p)}{qp} \leq 1.$$

proof: For $p^* = \frac{np}{n-p}$ and λ as (4.2.2) we use Hölder inequality to obtain:

$$\begin{aligned}\int_{\mathbb{R}^n_+} x_n^{qb} v^q dx &= \int_{\mathbb{R}^n_+} x_n^{a\lambda q} v^{\lambda q} d^{qb-a\lambda}|v|^{q(1-\lambda)}dx \\ &\leq \Big(x_n^{ap^*}|v|^{p^*}\Big)^{\frac{\lambda q}{p^*}}\Big(\int_{\mathbb{R}^n_+} x_n^{p(a-1)}|v|^p dx\Big)^{\frac{(1-\lambda)q}{p}} \Leftrightarrow\end{aligned}$$

$$\|x_n^b v\|_{L^q(\mathbb{R}^n_+)} \leq \|x_n^a v\|^\lambda_{L^{\frac{np}{n-p}}(\mathbb{R}^n_+)}\|x_n^{a-1}v\|^{1-\lambda}_{L^p(\mathbb{R}^n_+)}.$$

Now use the fact that $x^\lambda y^{1-\lambda} \leq \lambda\eta^{-\frac{1-\lambda}{\lambda}}x + (1-\lambda)\eta y$, for any $\eta > 0$, to reach to the desired result.
hfill□

Theorem 4.2.3. *Let $n \geq 3$. Then the following inequality is valid*

$$\int_{\mathbb{R}^n_+} x_n|\nabla u|^2dx \geq \Big(\int_{\mathbb{R}^n_+} x_n u^{\frac{2(n+1)}{n-1}}dx\Big)^{\frac{n-1}{n+1}}, \quad \forall u \in C_0^\infty(\mathbb{R}^n_+)$$

where C is a positive constant which depends only on dimension n.

proof: By Lemma 4.2.2, if we choose $p = 1$ and $\eta = 1$ we have the following inequality

$$\|x_n^b v\|_{L^q(\mathbb{R}^n_+)} \leq \lambda \|x_n^a v\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n_+)} + (1-\lambda)\|x_n^{a-1}v\|_{L^1(\mathbb{R}^n_+)}. \tag{4.2.111}$$

Now, for any $a \neq 0$ we have

$$\int_{\mathbb{R}^n_+} x_n^{a-1}|v|dx = \frac{1}{a}\int_{\mathbb{R}^n_+} \nabla x_n^a \nabla x_n |v|dx = -\frac{1}{a}\int_{\mathbb{R}^n_+} x_n^a \nabla d \nabla |v|dx \leq \frac{1}{|a|}\int_{\mathbb{R}^n_+} x_n^a|\nabla v|dx. \tag{4.2.112}$$

By Sobolev inequality and (4.2.112) we have

$$S_n\|dx_n^a v\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n_+)} \leq \int_{\mathbb{R}^n_+} |\nabla x_n^a v|dx \leq a\int_{\mathbb{R}^n_+} x_n^{a-1}|v|dx + \int_{\mathbb{R}^n_+} x_n^a|\nabla v|dx \leq 2\int_{\mathbb{R}^n_+} x_n^a|\nabla v|dx, \tag{4.2.113}$$

where $S_n = n\pi^{\frac{1}{2}}(\Gamma(1+\frac{n}{2}))^{-\frac{1}{n}}$ see [Ma] p189. Thus by (4.2.111), (4.2.112) and (4.2.113) we have

$$\Big(\int_{\mathbb{R}^n_+} x_n^{bq}|v|^q\Big)^{\frac{1}{q}} \leq (\frac{2\lambda}{S_n} + \frac{1-\lambda}{|a|})\int_{\mathbb{R}^n_+} x_n^a|\nabla v|dx. \tag{4.2.114}$$

Now, replace v by u^s in the above inequality to obtain

$$\Big(\int_{\mathbb{R}^n_+} x_n^{bq}|u|^{sq}\Big)^{\frac{1}{q}} \leq (\frac{2}{S_n} + \frac{1}{a})s\int_{\mathbb{R}^n_+} x_n^a|u|^{s-1}|\nabla u|dx \leq (\frac{2}{S_n} + \frac{1}{a})s\Big(\int_{\mathbb{R}^n_+} x_n^a|\nabla u|^2dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^n_+} x_n^a|u|^{2s-2}dx\Big)^{\frac{1}{2}} \tag{4.2.115}$$

and the result follows, if we choose $a = 1,\ q = \frac{n+1}{n}\ \lambda = \frac{n}{n+1}$ and $s = \frac{2n}{n-1}$. □

Finally, we prove a Hardy-Sobolev type inequality which is of independent interest.

Theorem 4.2.4. *Let $n \geq 3$ and Ω be the domain above the graph of $C^{1,1}$ function which satisfies $-\Delta d \geq 0$. Then there exists a positive constant C which depends only on n and μ, such that*

$$\int_\Omega |\nabla u|^2dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx \geq C(\mu,n)\Big(\int_\Omega du^{\frac{2(n+1)}{n-1}}dx\Big)^{\frac{n-1}{n+1}}, \quad \forall u \in C_0^\infty(\Omega). \tag{4.2.116}$$

proof: The proof is same as Theorem 4.2.1. The only difference is that we use Theorem 4.2.3. □

Chapter 5

Harnack Inequality and Heat Kernel Estimates

Throughout this chapter we assume that $n \geq 3$ and Ω is an exterior domain i.e complement of a smooth compact domain. For our purposes here, smooth means C^2 and we consider exterior domains not containing the origin.
The main goal of this chapter is to prove a parabolic Harnack type inequality for the positive solutions of the problem

$$u_t = \Delta u + \frac{u}{4d^2} \quad \text{in} \quad \Omega \times (0, T]. \tag{5.0.1}$$

Also we prove sharp two side estimates for the heat kernel corresponding to problem (5.0.1).
The strategy which we follow is:
First, we consider the minimizing problem in section 4.1.4.

$$\lambda_1 = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}}{\int_\Omega \frac{u^2}{1+d^{2+\sigma}}},$$

where $\sigma > 0$. We have proven in that $\lambda_1 \in \mathbb{R}$. Also, in section 4.1.4 we have proven the existence of a ground state function $\phi \in H^1_{loc}(\Omega)$ which solves

$$-\Delta u - \frac{u}{4d^2} = \lambda_1 \frac{u}{d^{2+\sigma}} \quad \text{in} \quad \Omega, \tag{5.0.2}$$

in the weak sense. In addition, we have shown that the function ϕ has the following properties
1. if $d(x) \leq C_0$ then there exist $C_1,\ C_2$ such that $C_1 d^{1/2}(x) \leq \phi(x) \leq C_2 d^{1/2}(x)$, where C_0 is small enough.

2. if $C_0 \leq d(x) \leq R_0$ then there exist $C_3,\ C_4$ such that $C_3 \leq \phi(x) \leq C_4$ where R_0 is big enough.

3. if $d(x) \geq R_0$ then there exist $C_5,\ C_6$ such that $C_5|x|^{-a_n} \leq \phi(x) \leq C_6|x|^{-a_n}$, where $a_n = \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - \frac{1}{4}}$.

Now if we set $u = v\phi$ in problem (5.0.1) we have

$$v_t = L_\phi v = \frac{div(\phi^2 \nabla v)}{\phi^2} - \lambda_1 \frac{v}{1 + d^{2+\sigma}}, \qquad \text{in } \Omega \times (0, T]. \tag{5.0.3}$$

Thus it is enough to prove a boundary parabolic Harnack inequality for the positive solutions of the problem (5.0.3). We will prove it by Moser's iteration technique (see [SC2] for a simple case.)
Especially we prove

Theorem 5.0.5. *Let v be a non-negative solution of (5.0.3). Then there exist constant A such that the following estimate is valid for all $x,\ y \in \Omega$ and all $0 < s < t < T$.*

$$v(s,y) \leq v(t,x)\exp\left(A\left(1+\frac{t-s}{R^2}+\frac{t-s}{s}+\frac{|x-y|^2}{t-s}\right)\right),$$

where the constant $R > 0$ is small enough and depends only on $\partial\Omega$ and the constant C_0.

By the above theorem we have the following corollary

Corollary 5.0.6. *Let u be a non-negative solution of (5.0.1). Then there exist constant A such that the following estimate is valid for all $x,\ y \in \Omega$ and all $0 < s < t < T$.*

$$\frac{u(s,y)}{\phi(y)} \leq \frac{u(t,x)}{\phi(x)}\exp\left(A\left(1+\frac{t-s}{R^2}+\frac{t-s}{s}+\frac{|x-y|^2}{t-s}\right)\right),$$

where the constant $R > 0$ is small enough and depends only on $\partial\Omega$ and the constant C_0.

Finally we prove the sharp two side estimates for the heat kernel $h_\phi(t,x,y)$ corresponding to problem (5.0.3).

Theorem 5.0.7. *Let $a_n = \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - \frac{1}{4}}$. Then, there exist positive constants $A_1,\ A_2,\ C_1,\ C_2$ and t_0 such that for all $x,\ y \in \Omega$ and all $0 < t < t_0$ the heat kernel $h_\phi(x,t,y)$ satisfies*

$$C_1\Big[\frac{|x|^{2a_n+1}|y|^{2a_n+1}}{\max\{d(x),r\}\max\{d(y),r\}}\Big]^{\frac{1}{2}} t^{-\frac{n}{2}}\exp\Big(-A_1\frac{|x-y|^2}{t}\Big)$$

$$\leq h_\phi(t,x,y) \leq C_2\Big[\frac{|x|^{2a_n+1}|y|^{2a_n+1}}{\max\{d(x),r\}\max\{d(y),r\}}\Big]^{\frac{1}{2}} t^{-\frac{n}{2}}\exp\Big(-A_2\frac{|x-y|^2}{t}\Big).$$

Now we note that the heat kernel $h(t,x,y)$ corresponding to the problem (5.0.1) satisfies

$$h(t,x,y) = \phi(x)\phi(y)h_\phi(t,x,y). \tag{5.0.4}$$

Also by properties (1-4) of ϕ, we have that there exist c_1 and c_2 such that

$$c_1\frac{d(x)}{|x|^{a_n+1}} \leq \phi(x) \leq c_2\frac{d(x)}{|x|^{a_n+1}}. \tag{5.0.5}$$

Thus by (5.0.4), (5.0.5) and the previous theorem we have

Corollary 5.0.8. *Let $n \geq 3$, then there exist positive constants $C_1,\ C_2,\ A_1,\ A_2$ and $t_0 > 0$ such that for all $x,\ y \in \Omega$ and all $0 < t < t_0$ the heat kernel $h(x,t,y)$ satisfies*

$$C_1\Big[\min\Big(\frac{d(x)}{\sqrt{t}},1\Big)\min\Big(\frac{d(y)}{\sqrt{t}},1\Big)\Big]^{\frac{1}{2}} t^{-\frac{n}{2}}\exp\Big(-A_1\frac{|x-y|^2}{t}\Big)$$

$$\leq h(t,x,y) \leq C_2\Big[\min\Big(\frac{d(x)}{\sqrt{t}},1\Big)\min\Big(\frac{d(y)}{\sqrt{t}},1\Big)\Big]^{\frac{1}{2}} t^{-\frac{n}{2}}\exp\Big(-A_2\frac{|x-y|^2}{t}\Big).$$

In the rest of this chapter when we meet the function ϕ, we always mean the function ϕ which we refer above.

5.1 Doubling Property, Poincare and Moser inequality

5.1.1 Doubling Property

Consider now the space $\mathfrak{D}_0^{1,2}(\Omega;\phi)$ which is the completion of $C_0^\infty(\Omega)$ function under the norm,

$$||u||_{\mathfrak{D}_0^{1,2}(\Omega;\phi)} = \int_\Omega \phi(|\nabla u|^2 + \frac{u^2}{1+d^{2+\sigma}})dx.$$

In the sequel we will use the following local representation of the boundary of Ω. There exist a finite number m of coordinate systems (y_i', y_n), $y_i' = (y_{i1}, ..., y_{in-1})$ and the same number m of functions $a_i(y_i')$ defined on the closure cubs, $\Delta_i := \{y_i' : |y_{ij}| \le \beta\}$ $for\ j = 1, ..., n-1,\ i \in \{1, .., m\}$ so that for each point $x \in \partial\Omega$ there is at least i such that $x = (x_i', a_i(x_i'))$. The function a_i satisfies the Lipschitz condition on $\overline{\Delta_i}$ with constant $A > 0$, that is

$$|a_i(y_i') - a_i(z_i') \le A|y_i' - z_i'|,$$

for $y_i', z_i' \in \overline{\Delta_i}$. Moreover there exists a positive constant $b < 1$ such that the set B_i is defined for any $i \in \{1, .., m\}$ by the relation $B_i = \{(y_i', y_{in}) :\ a_i(y_i') \le y_{in} \le a_i(y_i') + b\}$ and $\Gamma_i = B_i \cap \partial\Omega = \{(y_i', y_{in}) : y_i' \in \Delta_i, y_{in} = a_i(y_i')\}$. Finally let us observe for any $y \in \Delta_i$ where someone can make the following inequality on the distance function

$$(1 + A)^{-1}(y_{in} - a_i(y_i')) \le d(y) \le y_{in} - a_i(y_i').$$

Let us now define the balls which we will use to prove some Poincare, weighted Poincare and Nash inequalities. More precisely we have the following definition

Definition 5.1.1. *Let $\gamma \in (1, 2)$*
For any $x \in \Omega$ and for any $0 < r < \frac{C_0}{2\gamma} < b$, we define the ball centered at x and having radius r as follows.
(i) If $d(x) \le \gamma r$ then

$$\mathfrak{B}(x, r) = \{(y_i', y_{in}) : |y_i' - x_i'| \le r, d(x) - r \le y_{in} - a_i(y_i') \le r + d(x)\},$$

where $i \in \{1, ..., m\}$ is uniquely defined by the point $\overline{x} \in \partial\Omega$ such that $|x - \overline{x}| = d(x)$, that is by the projection of the center x onto $\partial\Omega$.
(ii) If $d(x) \ge \gamma r$ then $\mathfrak{B}(x, r) = B(x, r)$ the Euclidean ball centered at x.
We also define by

$$V(x, r) = \int_{\mathfrak{B}(x,r)\cap\Omega} \phi^2(y)dy,$$

the volume of the "ball" centered at x and having radius r.

We first derive a sharp volume estimate.

Lemma 5.1.2. *Let $n \ge 3$ and Ω be an exterior domain not containing the origin. Then there exist positive constants d_1 and d_2 such that for any $x \in \Omega$ and $0 < r < \frac{C_0}{2\gamma} < b$, we have*

$$d_1 \frac{\max\{d(x), r\}}{|x|^{2a_n+1}} r^n \le V(x, r) \le \frac{\max\{d(x), r\}}{|x|^{2a_n+1}} r^n,$$

where $a_n = \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - \frac{1}{4}}$.

proof: To prove the Lemma we consider four cases.
Case 1. $d(x) \le \frac{C_0}{2\gamma}$ and $d(x) \ge \gamma r$. In this case we have $\mathfrak{B}(x, r) = B(x, r) \subset \Omega$. Due to the fact that for any $y \in B(x, r)$, we

have

$$\frac{\gamma-1}{\gamma}d(x)\leq d(x)-r\leq d(y)\leq d(x)+r\leq\frac{\gamma+1}{\gamma}d(x),$$

we obtain

$$\begin{aligned}\int_{B(x,r)}\phi^2(y)dy\leq C_2\int_{B(x,r)}d(y)dy &\leq C_2w_n\frac{\gamma+1}{\gamma}d(x)r^n=C_2w_n\frac{\gamma+1}{\gamma}\max\{d(x),r\}r^n\\ &\leq C_2w_n\frac{\gamma+1}{\gamma}(P+\frac{C_0}{2\gamma})^{a_n+1}\frac{\max\{d(x),r\}}{|x|^{2a_n+1}}r^n,\end{aligned}$$

where $P=\sup\{|x|:x\in\partial\Omega\}$. On the other hand we also have

$$C_1w_n\frac{\gamma-1}{\gamma}\max\{d(x),r\}r^n\leq\int_{B(x,r)}\phi^2(y)dy\Rightarrow$$

$$C_1w_n\frac{\gamma-1}{\gamma}p^{2a_n+1}\frac{\max\{d(x),r\}}{|x|^{2a_n+1}}r^n\leq V(x,r),$$

where $p=\inf\{|x|:x\in\partial\Omega\}$.

Case 2.
Considering now $d(x)\leq\frac{C_0}{2\gamma}$ and $d(x)\leq\gamma r$. Then we have (for some $i\in\{1,...,m\}$ where we omit the subscript i for convenience)

$$\begin{aligned}V(x,r)=\int_{\mathfrak{B}(x,r)\cap\Omega}\phi^2(y)dy &\leq C_2\int_{max\{d(x)+a(y')-r,a(y')\}}^{d(x)+a(y')+r}(y_n-a(y'))dy_ndy'\\ &\leq C_2(d(x)+r)\int_{max\{d(x)+a(y')-r,a(y')\}}^{d(x)+a(y')+r}dy_ndy'\leq C_2(d(x)+r)w_{n-1}r^n\\ &\leq 2C_2w_{n-1}\max\{d(x),r\}r^n\leq 2C_2w_{n-1}(P+\frac{C_0}{2\gamma})^{2a_n+1}\frac{\max\{d(x),r\}}{|x|^{2a_n+1}}r^n.\end{aligned}$$

On the other hand we have

$$\begin{aligned}V(x,r) &\geq C_1(1+A)^{-1}\int_{max\{d(x)+a(y')-r,a(y')\}}^{d(x)+a(y')+r}(y_n-a(y'))dy_ndy'\\ &\geq C_1(1+A)^{-1}\int_{max\{\gamma r+a(y')-r,a(y')\}}^{d(x)+a(y')+r}(y_n-a(y'))dy_ndy'\\ &\geq C_1(1+A)^{-1}w_{n-1}(\gamma-1)r^n(d(x)+(2-\gamma)r)\\ &\geq C_1(1+A)^{-1}w_{n-1}(\gamma-1)(2-\gamma)r^n\max\{d(x),r\}\\ &\geq C_1(1+A)^{-1}w_{n-1}(\gamma-1)(2-\gamma)p^{2a_n+1}\frac{\max\{d(x),r\}}{|x|^{2a_n+1}}r^n\end{aligned}$$

Case 3.
$\frac{C_0}{2\gamma}\leq d(x)\leq 4R$,

$$V(x,r)=\int_{B(x,r)}\phi^2(y)dy\leq\int_{B(x,r)}C_4dy\leq C_4w_nr^n\leq\frac{2\gamma(4R)^{2a_n+1}}{C_0}C_4w_n\frac{\max\{d(x),r\}}{|x|^{2a_n+1}}r^n.$$

Also, we have

$$V(x,r)\geq C_3w_nr^n\geq\frac{1}{\gamma(4R)^{2a_n+1}}C_3w_n\frac{\max\{d(x),r\}}{|x|^{2a_n+1}}r^n.$$

Case 4.

$d(x) \geq 4R$

$$\begin{aligned} V(x,r) = \int_{B(x,r)} \phi^2(y)dy &\leq C_6 \int_{B(x,r)} |y|^{-2a_n} dy \leq C_6 \frac{1}{(|x|-r)^{2a_n}} \int_{B(x,r)} dy \\ &\leq C_6 w_n 2^{2a_n} \frac{1}{|x|^{2a_n}} r^n \leq C_6 w_n 2^{2a_n} \frac{4R}{4R-P} \frac{\max\{d(x),r\}}{|x|^{2a_n+1}} r^n, \end{aligned}$$

where $P = \sup\{|x| : x \in \partial\Omega\}$. Also, on the other hand we have

$$V(x,r) \geq C_5 \frac{1}{(|x|+r)^{2a_n}} \int_{B(x,r)} dy \geq C_5 w_n 2^{-2a_n} \frac{\max\{d(x),r\}}{|x|^{2a_n+1}} r^n.$$

□

From the previous Lemma someone can easily deduce the doubling property which reads as follows:

Corollary 5.1.3. ***Doubling property.*** *Ω be an exterior domain not containing the origin. Then there exist positive constants C and β such that for any $x \in \mathbb{R}^n \setminus \Omega$ and $0 < r < \beta$ we have*

$$V(x,2r) \leq CV(x,r).$$

5.1.2 Poincare Types Inequalities

We begin this section with the proof some Poincaré type inequalities. We begin with the following weighted Poincare inequality, the proof of which is [Mo]. We give it for convenience to the reader.

Lemma 5.1.4. *Let $n \geq 2$, $U \subset \mathbb{R}^n$ be a smooth bounded convex domain. Also let Φ be non-negative continuous function with support in U with the following property.*
If for any $x,\ y \in U$ we have

$$\Phi(x) \leq \Phi(y)$$

then

(5.1.6) $$\Phi(x) \leq \Phi(tx + (1-t)y) \quad \forall\ t \in [0,1].$$

Then $\forall\ f \in C^\infty(U)$ we have

$$\min_{\xi\in\mathbb{R}} \int_U |f(y)-\xi|^2 \Phi^2(y)dy \leq cB \int_U |\nabla f(y)|^2 \Phi^2(y)dy,$$

where $B = \sup_{x,y\in supp(\Phi)} |x-y|$ and

$$c = \frac{\max_{x\in U} \phi(x)}{2\int_U \Phi(x)dx} \int_{\Phi>0} dx.$$

Where the minimum above is assumed for

$$k = \frac{\int_U f\Phi(x)dx}{\int_U \Phi(x)dx}$$

proof: We note that

(5.1.7) $$\int_U \int_U |f(x)-f(y)|^2 \Phi(x)\Phi(y)dxdy = 2\int_U \Phi(x)dx \int_U |f(y)-k|^2 \Phi(y)dy,$$

where
$$k = \frac{\int_U f(x)\Phi(x)dx}{\int_U \Phi dx}.$$
Let $x, y \in U$ such that $0 < \Phi(x) \leq \Phi(y)$. Then we have
$$|f(x) - f(y)|^2\Phi(x)\Phi(y) = \Big(\int_0^{|x-y|} \nabla f(x+rw)\cdot w dr\Big)^2 \Phi(x)\Phi(y),$$
where $w = \frac{y-x}{|x-y|}$.
$$\begin{aligned}
|f(x) - f(y)|^2\Phi(x)\Phi(y) &\leq \int_0^{|x-y|} \sqrt{\Phi(x+rw)}\frac{\nabla f(x+rw)\cdot w}{\sqrt{\Phi(x+rw)}} dr\Phi(x)\Phi(y)\\
&\leq \int_0^{|x-y|} \Phi(x+rw)|\nabla f(x+rw)|^2 dr \int_0^{|x-y|}\frac{1}{\Phi(x+rw)}dr\Phi(x)\Phi(y)\\
&\leq \int_0^{|x-y|} \Phi(x+rw)|\nabla f(x+rw)|^2 dr|x-y|\max\Phi\\
&\leq B\max\Phi\int_0^{|x-y|} \Phi(x+rw)|\nabla f(x+rw)|^2 dr,
\end{aligned}$$
where in the above inequalities we have used the Hölder inequality and (5.1.6).
Now letting $z = y - x$ and integrating with respect x we have
$$\int_U |f(x) - f(x+z)|^2\Phi(x)\Phi(x+z)dx \leq B\max\Phi\int_U\int_0^{|z|}\Phi(x+r\frac{z}{|z|})|\nabla f(x+r\frac{z}{|z|})|^2 drdx.$$
Now set $V(y) = \Phi(y)|\nabla f(y)|^2$ if $y \in U$, $V(y) = 0$ otherwise. Then we have
$$\begin{aligned}
\int_U |f(x) - f(x+z)|^2\Phi(x)\Phi(x+z)dx &\leq B\max\Phi\int_U\int_0^{|z|} V(x+r\frac{z}{|z|})drdx\\
&= B\max\Phi\int_{\mathbb{R}^n}\int_0^{|z|} V(x+r\frac{z}{|z|})drdx\\
&= B\max\Phi\int_{\mathbb{R}^n}\int_0^{|z|} V(y)drdy\\
&= |z|B\max\Phi\int_{\mathbb{R}^n} V(y)dy \leq B^2\max\Phi\int_{\mathbb{R}^n} V(y)dy
\end{aligned}$$
Integrating now over z we have
$$\int_U\int_U |f(x) - f(y)|^2\Phi(x)\Phi(y)dxdy \leq B^2\max\Phi\int_{\Phi>0}dx\int_U|\nabla f(y)|^2\Phi(y)dy.$$

Finally, combining the above inequality and (5.1.7) we have the desired result. □

Also we have,

Theorem 5.1.5. ***Local Poincaré inequality.*** *Let $n \geq 2$ and Ω be an exterior domain not containing the origin. Then there exist positive constants $C = C(n, \gamma, \Omega)$ and β such that for any $x_0 \in \Omega$, we have*
$$\inf_{\xi\in\mathbb{R}}\int_{\mathfrak{B}(x_0,r)\cap\Omega}|f(y) - \xi|^2\phi^2(y)dy \leq C\int_{\mathfrak{B}(x_0,r)\cap\Omega}|\nabla f(y)|^2\phi^2(y)dy, \quad \forall f \in C^\infty(\mathfrak{B}(x_0, r)\cap\Omega)$$

proof: We consider only the case $d(x) \leq \gamma r$ for some $\gamma \in (1,2)$. Since in other cases we have $c\phi(x) \leq \phi(y) \leq C\phi(x)$ for any $x \in \mathfrak{B}(x_0, r)\cap\Omega$ and we can reach easily to the desired result.

In our case we have that $\phi^2(x) \le C_2(x_n - a(x')) = C_2\Phi(x)$, thus it is enough to prove that

$$\inf_{\xi\in\mathbb{R}} \int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)-\xi|^2\phi(y)dy \le C_n \int_{\mathfrak{B}(x_0,r)\cap\Omega} |\nabla f(y)|^2\phi(y)dy.$$

We note that

$$\begin{aligned}&\int_{\mathfrak{B}(x_0,r)\cap\Omega}\int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(x)-f(y)|^2\Phi(x)\Phi(y)dxdy\\ &= 2\int_{\mathfrak{B}(x_0,r)\cap\Omega}\Phi(x)dx\int_{\mathfrak{B}(x_0,r)\cap\Omega}|f(y)-k|^2\Phi(y)dy,\end{aligned}$$

where

$$k = \frac{\int_{\mathfrak{B}(x_0,r)\cap\Omega} f(x)\Phi(x)dx}{\int_{\mathfrak{B}(x_0,r)\cap\Omega}\Phi(x)dx}.$$

First we assume $x_n \le y_n$ then we have for any $t \in [0,1]$

$$x_n \le tx_n + (1-t)y_n. \tag{5.1.8}$$

Then we have

$$|f(x)-f(y)|^2\Phi(x)\Phi(y) = \Big(\int_0^{|x-y|}\nabla f(x+rw)\cdot w dr\Big)^2\Phi(x)\Phi(y),$$

where $w = \frac{y-x}{|x-y|}$.

$$\begin{aligned}|f(x)-f(y)|^2\Phi(x)\Phi(y) &\le \int_0^{|x-y|}\sqrt{\Phi(x+rw)}\frac{\nabla f(x+rw)\cdot w}{\sqrt{\Phi(x+rw)}}dr\Phi(x)\Phi(y)\\ &\le \int_0^{|x-y|}\Phi(x+rw)|\nabla f(x+rw)|^2dr\int_0^{|x-y|}\frac{1}{\Phi(x+rw)}dr\Phi(x)\Phi(y)\\ &\le \int_0^{|x-y|}\Phi(x+rw)|\nabla f(x+rw)|^2drx_n|x-y|\\ &\le (\gamma+1)r^2\int_0^{|x-y|}\Phi(x+rw)|\nabla f(x+rw)|^2dr,\end{aligned}$$

where in the above inequalities we have used the Hölder inequality and the notations (5.1.8).
Now letting $z = y-x$ and integrating with respect x we have

$$\int_{\mathfrak{B}(x_0,r)\cap\Omega}|f(x)-f(x+z)|^2\Phi(x)\Phi(x+z)dx \le (\gamma+1)r^2\int_{\mathfrak{B}(x_0,r)\cap\Omega}\int_0^{|z|}\Phi(x+r\frac{z}{|z|})|\nabla f(x+r\frac{z}{|z|})|^2drdx.$$

Now set $V(y) = \Phi(y)|\nabla f(y)|^2$ if $y \in \mathfrak{B}(x_0,r)\cap\Omega$, $V(y) = 0$ otherwise. Then we have

$$\int_{\mathfrak{B}(x_0,r)\cap\Omega}|f(x)-f(x+z)|^2\Phi(x)\Phi(x+z)dx \le (\gamma+1)r^2\int_{\mathfrak{B}(x_0,r)\cap\Omega}\int_0^{|z|}V(x+r\frac{z}{|z|})drdx$$

$$= (\gamma+1)r^2\int_{\mathbb{R}^n}\int_0^{|z|}V(x+r\frac{z}{|z|})drdx = (\gamma+1)r^2\int_{\mathbb{R}^n}\int_0^{|z|}V(y)drdy = |z|(\gamma+1)r^2\int_{\mathbb{R}^n}V(y)dy.$$

Integrating now over z we get

$$\int_{\mathfrak{B}(x_0,r)\cap\Omega}\int_{\mathfrak{B}(x_0,r)\cap\Omega}|f(x)-f(y)|^2\Phi(x)\Phi(y)dxdy \le (\gamma+1)r^{n+3}w_n\int_{\mathfrak{B}(x_0,r)\cap\Omega}|\nabla f(y)|^2\Phi(y)dy.$$

Thus,
$$\int_{\mathfrak{B}(x_0,r)\cap\Omega}|f(y)-k|^2\Phi(y)dy \le \frac{w_n(\gamma+1)r^{n+1}}{\int_{\mathfrak{B}(x_0,r)\cap\Omega}\Phi(x)dx}\int_{\mathfrak{B}(x_0,r)\cap\Omega}|\nabla f(y)|^2\Phi(y)dy.$$

Now let us estimate the following integral

$$\begin{aligned}\int_{\mathfrak{B}(x_0,r)\cap\Omega}\Phi(x)dx &= \int_{B(x_0',r)}\int_{\max\{d(x)-r,0\}}^{d(x)+r} x_n dx_n dx' \\ &\ge w_{n-1}\int_{d(x)}^{d(x)+r} x_n dx_n \ge w_{n-1}\frac{r^2}{2}.\end{aligned}$$

We deduce the desired result.

$$\int_{\mathfrak{B}(x_0,r)\cap\Omega}|f(y)-k|^2\Phi(y)dy \le \frac{w_n(\gamma+1)r^{n+3}}{\frac{w_{n-1}}{2}r^{n+1}}\int_{\mathfrak{B}(x_0,r)\cap\Omega}|\nabla f(y)|^2\Phi(y)dy.$$

□

Finally we have

Theorem 5.1.6. ***Local weighted Poincare inequality.*** *Let $n \ge 2$ and Ω be an exterior domain not containing the origin. Then there exist positive constants $C = C(n,\gamma,\Omega)$ and β such that for any $x_0 \in \Omega$ with $d(x_0) < \gamma r < \beta$ for some $\gamma \in (1,2)$, we have for any $f \in C_0^\infty(\Omega)$*

$$\inf_{\xi\in\mathbb{R}}\int_{\mathfrak{B}(x_0,r)\cap\Omega}|f(y)-\xi|^2\Phi(y)dy \le C\int_{\mathfrak{B}(x_0,r)\cap\Omega}|\nabla f(y)|^2\Phi(y)dy,$$

where
$$\Phi(x) = \left(1-\frac{|x'-x_0'|}{r}\right)^{+2}(x_n - a(x'))\left(1-\frac{|x_n-a(x')-d(x_0)|}{r}\right)^{+2}.$$

proof: We note that
$$\begin{aligned}&\int_{\mathfrak{B}(x_0,r)\cap\Omega}\int_{\mathfrak{B}(x_0,r)\cap\Omega}|f(x)-f(y)|^2\Phi(x)\Phi(y)dxdy \\ &= 2\int_{\mathfrak{B}(x_0,r)\cap\Omega}\Phi(x)dx\int_{\mathfrak{B}(x_0,r)\cap\Omega}|f(y)-k|^2\Phi(y)dy,\end{aligned}$$

where
$$k = \frac{\int_{\mathfrak{B}(x_0,r)\cap\Omega}f(x)\Phi(x)dx}{\int_{\mathfrak{B}(x_0,r)\cap\Omega}\Phi(x)dx}.$$

First we assume $x_n \le y_n$ then we have for any $t \in [0,1]$

$$x_n \le tx_n + (1-t)y_n$$

and
$$\left(1-\frac{|tx'(1-t)y'-x_0'|}{r}\right)^+ \ge \left(1-\frac{|x'-x_0'|}{r}\right)^+$$

or
$$\left(1-\frac{|tx'(1-t)y'-x_0'|}{r}\right)^+ \ge \left(1-\frac{|y'-x_0'|}{r}\right)^+$$

and
$$\left(1-\frac{|tx_n+(1-t)y_n-d(x_0)|}{r}\right)^+ \ge \left(1-\frac{|x_n-d(x_0)|}{r}\right)^+$$

or

$$\left(1-\frac{|tx_n+(1-t)y_n-d(x_0)|}{r}\right)^+\geq\left(1-\frac{|y_n-d(x_0)|}{r}\right)^+. \tag{5.1.9}$$

Then we get

$$|f(x)-f(y)|^2\Phi(x)\Phi(y)=\left(\int_0^{|x-y|}\nabla f(x+rw)\cdot wdr\right)^2\Phi(x)\Phi(y),$$

where $w=\frac{y-x}{|x-y|}$.

$$|f(x)-f(y)|^2\Phi(x)\Phi(y)\leq\int_0^{|x-y|}\sqrt{\Phi(x+rw)}\frac{\nabla f(x+rw)\cdot w}{\sqrt{\Phi(x+rw)}}dr\Phi(x)\Phi(y)$$

$$\leq\int_0^{|x-y|}\Phi(x+rw)|\nabla f(x+rw)|^2dr\int_0^{|x-y|}\frac{1}{\Phi(x+rw)}dr\Phi(x)\Phi(y)$$

$$\leq\int_0^{|x-y|}\Phi(x+rw)|\nabla f(x+rw)|^2drx_n|x-y|\leq(\gamma+1)r^2\int_0^{|x-y|}\Phi(x+rw)|\nabla f(x+rw)|^2dr,$$

where in the above inequalities we have used the Hölder inequality and the notations (5.1.9).
Now letting $z=y-x$ and integrating with respect x we have

$$\int_{\mathfrak{B}(x_0,r)\cap\Omega}|f(x)-f(x+z)|^2\Phi(x)\Phi(x+z)dx\leq(\gamma+1)r^2\int_{\mathfrak{B}(x_0,r)\cap\Omega}\int_0^{|z|}\Phi(x+r\frac{z}{|z|})|\nabla f(x+r\frac{z}{|z|})|^2drdx.$$

Now set $V(y)=\Phi(y)|\nabla f(y)|^2$ if $y\in\mathfrak{B}(x_0,r)\cap\Omega$, $V(y)=0$ otherwise. We then get

$$\int_{\mathfrak{B}(x_0,r)\cap\Omega}|f(x)-f(x+z)|^2\Phi(x)\Phi(x+z)dx\leq(\gamma+1)r^2\int_{\mathfrak{B}(x_0,r)\cap\Omega}\int_0^{|z|}V(x+r\frac{z}{|z|})drdx$$

$$=(\gamma+1)r^2\int_{\mathbb{R}^n}\int_0^{|z|}V(x+r\frac{z}{|z|})drdx=(\gamma+1)r^2\int_{\mathbb{R}^n}\int_0^{|z|}V(y)drdy=|z|(\gamma+1)r^2\int_{\mathbb{R}^n}V(y)dy.$$

Integrating now over z we have

$$\int_{\mathfrak{B}(x_0,r)\cap\Omega}\int_{\mathfrak{B}(x_0,r)\cap\Omega}|f(x)-f(y)|^2\Phi(x)\Phi(y)dxdy\leq(\gamma+1)r^{n+3}w_n\int_{\mathfrak{B}(x_0,r)\cap\Omega}|\nabla f(y)|^2\Phi(y)dy.$$

Thus we have

$$\int_{\mathfrak{B}(x_0,r)\cap\Omega}|f(y)-k|^2\Phi(y)dy\leq\frac{w_n(\gamma+1)r^{n+1}}{\int_{\mathfrak{B}(x_0,r)\cap\Omega}\Phi(x)dx}\int_{\mathfrak{B}(x_0,r)\cap\Omega}|\nabla f(y)|^2\Phi(y)dy.$$

Now let us estimate the following integral

$$\int_{\mathfrak{B}(x_0,r)\cap\Omega}\Phi(x)dx=\int_{B(x_0',r)}\left(1-\frac{|x'-x_0'|}{r}\right)^{+^2}\int_{\max\{d(x)-r,0\}}^{d(x)+r}y_n\left(1-\frac{|x_n-d(x_0)|}{r}\right)^{+^2}dx_ndx'$$

First we estimate from below the following integral.

$$\int_{B(x_0',r)}\left(1-\frac{|x'-x_0'|}{r}\right)^{+^2}dx'=\int_0^r\int_{\partial B(x_0,s)}s^{n-2}(1-\frac{s}{r})^2dS_{y'}ds=w_{n-1}C(n)r^{n-1}.$$

Let us estimate from below the following integral

$$\int_{\max\{d(x)-r,0\}}^{d(x)+r}x_n\left(1-\frac{|x_n-d(x_0)|}{r}\right)^{+^2}dx_n\geq\int_{d(x_0)}^{d(x_0)+\frac{r}{2}}x_n\left(1-\frac{|x_n-d(x_0)|}{r}\right)^{+^2}dx_n\geq\frac{1}{32}r^2.$$

Thus we have the desired result.

$$\int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)-k|^2\Phi(y)dy \le \frac{w_n(\gamma+1)r^{n+3}}{c(n)r^{n+1}} \int_{\mathfrak{B}(x_0,r)\cap\Omega} |\nabla f(y)|^2\Phi(y)dy.$$

□

5.1.3 Moser Inequality

In this section we will give the Moser inequality, which proof is similar as in [FMoT3]. We will give it for convenience to the reader.

Theorem 5.1.7. *Let $n \ge 3$ and Ω be an exterior domain not containing the origin. Then there exist positive constants C and β such that for any $\nu \ge n+1$, $x_0 \in \Omega$ and $f \in C_0^\infty(\mathfrak{B}(x_0,r)\cap\Omega)$ we have*

$$\int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)|^{2(1+\frac{2}{\nu})}\phi^2(y)dy$$

$$\le C_M r^2 V(x,r)^{-\frac{2}{\nu}} \int_{\mathfrak{B}(x_0,r)\cap\Omega} |\nabla f(y)|^2\phi^2(y)dy\Big(\int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)|^2\phi^2(y)dy\Big)^{\frac{2}{\nu}}.$$

proof: We consider only the case where $d(x_0) < \gamma r$, $d(x_0) < \beta$ and $f \in C_0^\infty(\mathfrak{B}(x_0,r)\cap\Omega)$, since $C_1\phi(x) \le \phi(y) \le C_2\phi(x)$ otherwise. First we claim that it is enough to prove

$$\Big(\int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)|^{\frac{2(n+1)}{n-1}}d(y)dy\Big)^{\frac{n-1}{n+1}} \le C\int_{\mathfrak{B}(x_0,r)\cap\Omega} |\nabla f(y)|^2\phi^2(y)dy. \tag{5.1.10}$$

Indeed, if (5.1.10) is valid then

$$\begin{aligned}
\int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)|^{2(1+\frac{2}{n+1})}\phi^2(y)dy &\le C_2\int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)|^{2(1+\frac{2}{\nu})}d(y)dy\\
&= \int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)|^2|f(y)|^{\frac{4}{\nu}}d^{\frac{2}{n+1}}(y)d^{\frac{n-1}{n+1}}(y)dy\\
&\le \Big(\int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)|^{\frac{2(n+1)}{n-1}}d(y)dy\Big)^{\frac{n-1}{n+1}}\Big(\int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)|^2d(y)dy\Big)^{\frac{2}{n+1}}, \qquad (5.1.11)
\end{aligned}$$

as well as for any $\nu > n+1$

$$\begin{aligned}
&\int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)|^{2(1+\frac{2}{\nu})}d(y)dy\Big(\int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)|^2d(y)dy\Big)^{\frac{2(\nu-n-1)}{\nu(n+1)}}\\
&\le \Big(\int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)|^{2(1+\frac{2}{n+1})}d(y)dy\Big)^{\frac{1+\frac{2}{\nu}}{1+\frac{2}{n+1}}}\Big(\int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)|^{2(1+\frac{2}{n+1})}d(y)dy\Big)^{(\frac{2}{n+1}-\frac{2}{\nu})\frac{1}{1+\frac{2}{n+1}}}\\
&\times\Big(\int_{\mathfrak{B}(x_0,r)\cap\Omega} d(y)dy\Big)^{(\frac{2}{n+1}-\frac{2}{\nu})\frac{1}{1+\frac{2}{n+1}}}\Big(\int_{\mathfrak{B}(x_0,r)\cap\Omega} d(y)dy\Big)^{(\frac{2}{n+1}-\frac{2}{\nu})\frac{\frac{2}{n+1}}{1+\frac{2}{n+1}}}\\
&= \int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)|^{2(1+\frac{2}{n+1})}d(y)dy\Big(\int_{\mathfrak{B}(x_0,r)\cap\Omega} d(y)dy\Big)^{\frac{2(\nu-n-1)}{\nu(n+1)}}\\
&\le C_1^{-\frac{2(\nu-n-1)}{\nu(n+1)}}V(x,r)^{2-\frac{2}{\nu}}\int_{\mathfrak{B}(x_0,r)\cap\Omega} |f(y)|^{2(1+\frac{2}{n+1})}d(y)dy,
\end{aligned}$$

thus by (5.1.11) and the above inequality we have

$$\int_{\mathfrak{B}(x_0,r)\cap\mathbb{R}^n\setminus\Omega} |f(y)|^{2(1+\frac{2}{\nu})} d(y)dy$$
$$\leq C(n,\beta) r^2 V(x,r)^{-\frac{2}{\nu}} \Big(\int_{\mathfrak{B}(x_0,r)\cap\mathbb{R}^n\setminus\Omega} |f(y)|^{\frac{2(n+1)}{n-1}} d(y)dy \Big)^{\frac{n-1}{n+1}} \Big(\int_{\mathfrak{B}(x_0,r)\cap\mathbb{R}^n\setminus\Omega} |f(y)|^2 \phi^2(y)dy \Big)^{\frac{2}{\nu}},$$

where by (5.1.10) we have the desired result.
In the sequel we will give the proof of (5.1.10). We will follow closely the argument of [FMaT1]. If $V \subset \mathbb{R}^n$ is any bounded domain $u \in C_0^\infty(V)$, then it is well known that

$$S_n \|u\|_{L^{\frac{n}{n-1}}(V)} \leq \|\nabla u\|_{L^1(V)},$$

where $S_n = n\pi^{\frac{1}{2}}[\Gamma(1+\frac{n}{2})]^{-\frac{1}{2}}$ (see p 189 in [Ma]). Let us fix from now on that $V = \mathfrak{B}(x_0, r) \cap \Omega$ and let us apply the above inequality to $u = d^a f$ for any $f \in C_0^\infty(V)$ and any $a > 0$. Thus we get

$$S_n \|u\|_{L^{\frac{n}{n-1}}(V)} \leq \int_V |\nabla f| d^a + a d^{a-1} |\nabla d||f| dy.$$

Let us remark at this point that boundary terms on $\partial\Omega$ are zero due to the presence of the weight d^a, $a > 0$. To estimate the last term of the right hand side we will make use of an integration by parts, noting that $\nabla d \cdot \nabla d = 1$ a.e.. That is we have

$$\int_V a d^{a-1}|f| dy = a \int_V d^{a-1} \nabla d \cdot \nabla d |f| dy = \int_V \nabla d^a \cdot \nabla d |f| dy = -\int_V d^a \cdot \delta d |f| dy + \int_V d^a \nabla d \cdot \nabla |f| dy.$$

Under our smoothness assumption on Ω we have that $|d\Delta d| \leq c_0\delta$ in Ω_δ, for δ small, say $0 < \delta < \delta_0$, and for same positive constant independent of δ (δ_0, c_0 depending only Ω). Now if $d(x) + r < \delta$ that is if $r < \frac{\delta}{\gamma+1}$, we have that $V \subset \Omega_\delta$ and it follows that

$$a \int_V d^{a-1}|f| dy \leq c_0\delta \int_V d^{a-1}|f| dy + \int_V d^a |\nabla f| dy,$$

consequently for any $r \in (0,\beta)$ with $\beta = \frac{1}{\gamma+1 \min\{\delta_0, \frac{a}{c_0}\}}$ and any $\delta < \frac{a}{c_0}$ the following inequality is true

$$S_n \|d^a u\|_{L^{\frac{n}{n-1}}(V)} \leq (1 + \frac{a}{a - c_0\delta}) \|d^a \nabla u\|_{L^1(V)}. \tag{5.1.12}$$

To proceed we will use the following interpolation inequality (cf. Lemma 4.1 in [FMaT1]):

$$\|d^b u\|_{L^q(V)} \leq \frac{n(q-1)}{q} \|d^a u\|_{L^{\frac{n}{n-1}}(V)} + \frac{q - n(q-1)}{q} \|d^{a-1} u\|_{L^1(V)},$$

for each $1 < q \leq \frac{n}{n-1}$ and $b = a - 1 + \frac{q-1}{q} n$, $a > 0$.
By (5.1.12) and the above inequality, we get for any a, b, q as above the following inequality

$$\|d^b u\|_{L^q(V)} \leq C_1 \|d^a \nabla u\|_{L^1(V)}, \tag{5.1.13}$$

where $C_1 = \frac{n(q-1)}{S_n q}(\frac{a}{a - c_0\delta} + 1) + \frac{q - n(q-1)}{q} \frac{1}{a - c_0\delta_0}$. Let us apply inequality (5.1.13) to $|u|^s$ instead of u then

$$\Big(\int_V d^{bq} |u|^{sq} dx \Big)^{\frac{1}{q}} \leq C_1 s \int_V d^a |u|^{s-1} |\nabla u| dx$$

$$\leq C_1 s\Big(\int_V d^{2a_1}|u|^{2s-2}dx\Big)^{\frac{1}{2}}\Big(\int_V d^{2a_2}|\nabla u|^2 dx\Big)^{\frac{1}{2}},$$

where $a_1 + a + 2 = a$. Now we choose $a_2 = \frac{1}{2}$ and $2a_1 = bq \Leftrightarrow a_1 = \frac{bq}{2}$. Thus $a = \frac{bq}{2} + \frac{1}{2}$ and $b = \frac{bq-1}{2} + \frac{q-1}{q}n \Leftrightarrow b = \frac{-\frac{1}{2}+\frac{q-1}{q}n}{1-\frac{q}{2}}$. Also we choose $s = \frac{2}{2-q}$. Thus the last inequality becomes

$$\Big(\int_V d^{bq}|u|^{sq}dx\Big)^{\frac{1}{q}-\frac{1}{2}} \leq C_1\frac{2-q}{2}\Big(\int_V d|\nabla u|^2 dx\Big)^{\frac{1}{2}}.$$

Now we choose $q = \frac{n+1}{n}$ (then $sq = \frac{2(n+1)}{n-1}$ and $bq = 1$) to give us the desired result. □

5.2 Moser's Iteration

We keep the notation of the previous sections. Set

$$Q = (s - r^2, s) \times \mathfrak{B}(x, r) \cap \Omega$$

$$Q_\delta = (s - \delta r^2, s) \times \mathfrak{B}(x, \delta r) \cap \Omega.$$

Definition 5.2.1. *We will say that $v \in C^1((s - r^2, r) : H^1_\phi(\mathfrak{B}(x, r) \cap \Omega))$ is a weak solution of (5.0.3) if for each $\Phi \in C^1_0((s - r^2, r) : C^\infty_0(\mathfrak{B}(x, r) \cap \Omega))$, for each $s - r^2 < t_1 < t_2 < s$ we have*

$$\int_{t_1}^{t_2}\int_{\mathfrak{B}(x,r)\cap\Omega} v_t\Phi + \nabla v\nabla\Phi + \lambda_1\frac{v\Phi}{1 + d^{2+\sigma}}dmdt = 0,$$

where $dm = \phi^2 dx$ and $\sigma > 0$.

We denote here by $H^1_\phi(\mathfrak{B}(x, r) \cap \Omega)$ the space which consists of all functions $u : \mathfrak{B}(x, r) \cap \Omega \to \mathbb{R}$ such that, ∇u exists in the weak sense and

$$||u||^2_{H^1_\phi(\mathfrak{B}(x,r)\cap\Omega)} = \int_{\mathfrak{B}(x,r)\cap\Omega}|\nabla u|^2\phi^2 dx + \int_{\mathfrak{B}(x,r)\cap\Omega}\frac{u^2}{1 + d^{2+\sigma}}\phi^2 dx < \infty.$$

By Definition 5.2.1 of weak solution, we note that the choice of the test function plays an important role in our analysis. Thus for this reason we have the following theorem which proof is in [FMoT3].

Theorem 5.2.2. *Let $n \geq 2$ and $U \subset \mathbb{R}^n$ be a smooth bounded domain. Then we have*

$$H^1_0(U, d(y)dy) = H^1(U, d(y)dy).$$

Here $H^1(U, d(y)dy)$ denotes the set

$$\{v = v(y) : ||v||^2_{H^1_1} = \int_U d(|\nabla v|^2 + v^2)dy < \infty\}.$$

proof: By Theorem 7.2 in [K] it is well known that the set $C^\infty(\overline{U})$ is dense in $H^1(U, d(y)dy)$. Thus for any $v \in H^1(U, d(y)dy)$ there exists $v_m \in C^\infty(\overline{U})$ such that for $\frac{\varepsilon}{2} > 0$ we have $||v - v_m||_{H^1_1} < \frac{\varepsilon}{2}$ for all $m \geq m(\varepsilon)$. Let us choose

$w := v_{m(\varepsilon)}$ and let us define for $k \geq 2$

$$\phi_k = \begin{cases} 0 & d(x) \leq \frac{1}{k^2} \\ 1 + \frac{\ln(kd(x))}{\ln k} & \frac{1}{k^2} \leq d(x) \leq \frac{1}{k} \\ 1 & d(x) \geq \frac{1}{k} \end{cases}$$

Setting $w_k = w(1 - \phi_k)$, we then have

$$\begin{aligned} \|w - w_k\|^2_{H^1_1} &= \int_U d(|\nabla(w - w_k)|^2 + (w - w_k)^2)dy \\ &\leq 2\int_U d(|\nabla w|^2(1 - \phi_k))dy + 2\int_U d(|\nabla(1 - \phi_k)|^2 w^2)dy \\ &\leq 2\int_{d(x)\leq y} d(|\nabla w|^2)dy + 2\int_{\frac{1}{k^2}\leq d(x)\leq \frac{1}{k}} d(\frac{|\nabla d|^2}{d^2k^2\ln^2 k}w^2)dy \\ &\leq \frac{2}{k}\|w\|_{H^1_1} + \frac{2}{\ln^2 k}\|w\|_{H^1_1} \leq \frac{\varepsilon^2}{4}, \quad \forall k \geq k_0, \end{aligned}$$

where we have choose k_0 large enough. Thus

$$\|v - w_k\|_{H^1_1} \leq \|v - w\|_{H^1_1} + \|w - w_k\|_{H^1_1} \leq \varepsilon,$$

and the desired result follows. □

5.2.1 Properties of Subsolutions

Similarly with Definition 5.2.1, we call a function $v \in C^1((s-r^2, r) : H^1_\phi(\mathfrak{B}(x, r) \cap \Omega))$ subsolution of (5.0.3) if for each $0 \leq \Phi \in C^1_0((s - r^2, r) : C^\infty_0(\mathfrak{B}(x, r) \cap \Omega))$ and for each $s - r^2 < t_1 < t_2 < s$ we have

$$\int_{t_1}^{t_2}\int_{\mathfrak{B}(x,r)\cap\Omega} v_t\Phi + \nabla v\nabla\Phi + \lambda_1\frac{v\Phi}{1 + d^{2+\sigma}}dmdt \leq 0, \tag{5.2.14}$$

where $dm = \phi^2 dx$.

Theorem 5.2.3. *Let $\Omega \subset \mathbb{R}^n$ be an exterior domain not containing the origin, $\nu \geq n + 1$, $\gamma \in (1, 2)$ and $p \geq 0$. Then there exist constant $\beta(\Omega)$ and $C(\nu, \lambda_1, c_0)$ such that for all $x \in \Omega$ with $\gamma r < c_0$ and for any positive subsolution v of (5.0.3) in Q we have the estimate*

$$\sup_{Q_\delta} |v|^p \leq \frac{C}{(\delta' - \delta)^{\nu+2}r^2V(x, r)}\int_{Q_{\delta'}} |v|^p dxdt,$$

for each $0 < \delta < \delta' \leq 1$.

proof: First we consider the case where $d(x) < \gamma r$.
Set $u = v + \varepsilon$, then u is bounded away from zero (at the end of the argument we send ε to origin). Thus by (5.2.14) we have for any $\Phi \in C^\infty_0\mathfrak{B}(x, R) \cap \Omega$

$$\begin{aligned} \int_{\mathfrak{B}(x,R)\cap\Omega} u_t\Phi + \nabla u\nabla\Phi + \lambda_1\frac{u\Phi}{1 + d^{2+\sigma}}dm &\leq \varepsilon|\lambda_1|\int_{\mathfrak{B}(x,R)\cap\Omega}\frac{\Phi u}{1 + d^{2+\sigma}} \\ &\leq |\lambda_1|\int_{\mathfrak{B}(x,R)\cap\Omega} u\Phi dm. \end{aligned} \tag{5.2.15}$$

Let $G : [0 : \infty) \to [0, \infty)$ be a piecewise C^1 function such that $G(s) = as$ for large s and $G(0) = 0$. Assume that G has a non-negative, non-decreasing derivative $G'(s)$. Hence, G is non-decreasing and $G(s) \le sG'(s)$. Finally define $H(s) \ge 0$ by $H(s) = \sqrt{G'(s)}$, $H(0) = 0$. Observe that $H(s) \le sH'(s)$ as well. Due to Theorem 5.2.2 there exists a sequence of functions u_m in $C^\infty(\overline{\mathfrak{B}(x,r) \cap \Omega})$ having compact support in Ω such that $u_k \to u$ in $H^1(\mathfrak{B}(x,r) \cap \Omega, d(y)dy)$. Since $\phi \sim d^{\frac{1}{2}}$, we have that $u_k \to u$ in $H^1_\phi(\mathfrak{B}(x,r) \cap \Omega)$. Hence for any $\forall\ \psi \in C_0^\infty \mathfrak{B}(x,R) \cap \Omega$ and $k \ge 1$ the function $\Phi = \psi^2 G(u_k)$ is an admissible test function, that is, the following holds true:

$$\int_{\mathfrak{B}(x,R)\cap\Omega} v_t \psi^2 G(u_k) + \nabla v \nabla(\psi^2 G(u_k)) + \lambda_1 \frac{v\psi^2 G(u_k)}{1 + d^{2+\sigma}} dm \le |\lambda_1| \int_{\mathfrak{B}(x,R)\cap\Omega} u\psi^2 G(u_k) dm.$$

Passing to the limit $k \to \infty$ we have

$$\int_{\mathfrak{B}(x,R)\cap\Omega} v_t \psi^2 G(u) + \nabla v \nabla(\psi^2 G(u)) + \lambda_1 \frac{v\psi^2 G(u)}{1 + d^{2+\sigma}} dm \le |\lambda_1| \int_{\mathfrak{B}(x,R)\cap\Omega} u\psi^2 G(u) dm \Rightarrow$$

$$\begin{aligned} \int_{\mathfrak{B}(x,R)\cap\Omega} u_t \psi^2 G(u) dm \quad &+ \quad \int_{\mathfrak{B}(x,R)\cap\Omega} \psi^2 G'(u) |\nabla u|^2 + 2\psi G(u) \nabla u \nabla \psi dm \\ &\le \quad 2|\lambda_1| \int_{\mathfrak{B}(x,R)\cap\Omega} u\psi^2 G(u) dm. \end{aligned} \tag{5.2.16}$$

Now

$$2\int_{\mathfrak{B}(x,R)\cap\Omega} \psi G(u) \nabla u \nabla \psi dm \ge -2 \int_{\mathfrak{B}(x,R)\cap\Omega} \psi u G'(u) |\nabla u| |\nabla \psi| dm,$$

where we have used the fact that $G(u) \le uG'(u)$. Finally by Hölder inequality we have

$$2\int_{\mathfrak{B}(x,R)\cap\Omega} \psi G(u) \nabla u \nabla \psi dm \ge -\frac{1}{2} \int_{\mathfrak{B}(x,R)\cap\Omega} \psi^2 G'(u) |\nabla u|^2 dm - C \int_{\mathfrak{B}(x,R)\cap\Omega} \psi^2 u^2 G'(u) |\nabla \psi|^2 dm.$$

Combining now the last inequality and (5.2.16), we have

$$\begin{aligned} \int_{\mathfrak{B}(x,R)\cap\Omega} u_t \psi^2 G(u) dm \quad &+ \quad \frac{1}{2} \int_{\mathfrak{B}(x,R)\cap\Omega} \psi^2 G'(u) |\nabla u|^2 dm \\ &\le \quad C \int_{\mathfrak{B}(x,R)\cap\Omega} \psi^2 u^2 G'(u) |\nabla \psi|^2 dm + 2|\lambda_1| \int_{\mathfrak{B}(x,R)\cap\Omega} u^2 \psi^2 G'(u) dm. \end{aligned} \tag{5.2.17}$$

Then, we note that

$$\begin{aligned} |\nabla(\psi H(u))|^2 &= (\nabla \psi H(u) + \psi H'(u) \nabla u)^2 = |\nabla \psi|^2 H^2(u) + \psi^2 |H'(u)|^2 |\nabla u|^2 + 2\psi H(u) H'(u) \nabla \psi \nabla u \\ &\le 2|\nabla \psi|^2 H^2(u) + 2\psi^2 |H'(u)|^2 |\nabla u|^2 \le 2(|\nabla \psi|^2 u^2 H'^2(u) + \psi^2 |G'(u)| |\nabla u|^2) \\ &\le 2(|\nabla \psi|^2 u^2 |G'(u)| + \psi^2 |G'(u)| |\nabla u|^2). \end{aligned}$$

Hence, we have

$$\int_{\mathfrak{B}(x,R)\cap\Omega} |\nabla(\psi H(u))|^2 dm \le 2 \int_{\mathfrak{B}(x,R)\cap\Omega} |\nabla \psi|^2 u^2 |G'(u)| dm + 2 \int_{\mathfrak{B}(x,R)\cap\Omega} \psi^2 |G'(u)| |\nabla u|^2 dm.$$

Using the above inequality in (5.2.17), we have

$$\begin{aligned}\int_{\mathfrak{B}(x,R)\cap\Omega} u_t\psi^2 G(u)dm &+ \frac{1}{4}\int_{\mathfrak{B}(x,R)\cap\Omega}|\nabla(\psi H(u))|^2 dm\\ &\leq C\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2u^2G'(u)|\nabla\psi|^2dm + 2|\lambda_1|\int_{\mathfrak{B}(x,R)\cap\Omega}u^2\psi^2G'(u)dm.\end{aligned}$$

We note here that, the above integrals are all finite since $G'(s) = s$ and $H(s) = \alpha s$ for s large enough. Now multiplying the last inequality by a function $\chi(t)$, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2F^2(u)\chi dm &+ \frac{1}{4}\int_{\mathfrak{B}(x,R)\cap\Omega}|\nabla(\psi H(u))|^2dm\\ &\leq C\int_{\mathfrak{B}(x,R)\cap\Omega}\chi\psi^2u^2G'(u)|\nabla\psi|^2dm + 2|\lambda_1|\int_{\mathfrak{B}(x,R)\cap\Omega}\chi u^2\psi^2G'(u)dm.\\ &+ \int_{\mathfrak{B}(x,R)\cap\Omega}u\psi^2G(u)\chi_t dm\\ &\leq C\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2u^2G'(u)|\nabla\psi|^2dm + 2|\lambda_1|\int_{\mathfrak{B}(x,R)\cap\Omega}u^2\psi^2G'(u)dm\\ &+ \int_{\mathfrak{B}(x,R)\cap\Omega}|u|^2\psi^2|G'(u)|^2|\chi_t|dm,\end{aligned}\tag{5.2.18}$$

where F is a function such that $2F'(s)F(s) = G(s)$. Given $R > r > 0$, we choose $\chi(t) \in H_0^1(R)$ such that $0 \leq \chi \leq 1, \chi(t) = 1$ in $(s - r^2, \infty), \chi(t) = 0$ in $(-\infty, s - R^2)$ and $|\chi'| \leq \frac{1}{(R-r)^2}$. Also we choose a function $\psi = \xi(|y' - x'|)\xi(|y_n - a(y') - d(x)|)$, where $\xi \in C^\infty(\mathbb{R})$ and satisfies $0 \leq \xi \leq 1$, $\xi(s) = 1$ if $s \leq r$ and $\xi(s) = 0$ if $s > R$. Then clearly we have $|\nabla\psi| \leq \frac{1}{R-r}$. Now, we integrate (5.2.18) from zero to t for some $t \in (s - R^2, s)$ and letting t go to s, we have

$$\begin{aligned}\sup_{t\in(s-r^2,s)}\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2F^2(u)\chi dm &+ \frac{1}{4}\int_{s-r^2}^{s}\int_{\mathfrak{B}(x,R)\cap\Omega}|\nabla(\psi H(u))|^2dmdt\\ &\leq \frac{C + 2|\lambda|c_0}{(R-r)^2}\int_{s-R^2}^{s}\int_{\mathfrak{B}(x,R)\cap\Omega}u^2|G'(u)|dmdt.\end{aligned}\tag{5.2.19}$$

Now fix some large N. Set

$$\begin{aligned}H_N(s) &= \begin{cases} s^{\frac{p}{2}} & s \leq N\\ N^{\frac{p}{2}-1}s & s > N\end{cases}\\ G_N(s) &= \int_0^s|H'(t)|^2dt = \frac{p^2}{4(p-1)}\begin{cases} s^{p-1} & s \leq N\\ N^{\frac{4(p-1)}{p^2}}N^{p-2}(s-N) + N^{p-1} & s > N\end{cases}\\ F_N^2(s) &= \int_0^s|H'(t)|^2dt = \frac{p^2}{4(p-1)}\begin{cases} \frac{s^p}{p} & s \leq N\\ N^{\frac{4(p-1)}{p^2}}N^{p-2}\frac{(s-N)^2}{2} + N^{p-1}(s-N) + \frac{N^p}{p} & s > N\end{cases}\end{aligned}$$

For any $p \geq 2$. These $G_N's,\ H_N's,\ F_N's$ have the required properties and we note that $F_N^2 \geq \frac{p}{4(p-1)}H_N^2$. Thus (5.2.19) becomes

$$\begin{aligned}\frac{p}{4(p-1)}\sup_{t\in(s-r^2,s)}\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2H_N^2(u)\chi dm &+ \frac{1}{4}\int_{s-r^2}^{s}\int_{\mathfrak{B}(x,R)\cap\Omega}|\nabla(\psi H_N(u))|^2dmdt\\ &\leq \frac{C(\lambda_1, c_0)}{(R-r)^2}\int_{s-R^2}^{s}\int_{\mathfrak{B}(x,R)\cap\Omega}u^2|G_N'(u)|dmdt.\end{aligned}\tag{5.2.20}$$

Also we have

$$\int_{\mathfrak{B}(x,r)\cap\Omega}|H_N(u)|^{2+\frac{4}{\nu}}dm = \int_{\mathfrak{B}(x,R)\cap\Omega}|\psi H_N(u)|^{2+\frac{4}{\nu}}dm$$
$$\leq E\Big(\int_{\mathfrak{B}(x,R)\cap\Omega}|\nabla(\psi H_N(u))|^2dm\Big)\Big(\sup_{t\in(s-r^2,s)}\int_{B(x,R)}|\psi H_N(u)|^2dm\Big)^{\frac{2}{\nu}} \Rightarrow$$

$$\int_{s-r^2}^{s}\int_{\mathfrak{B}(x,R)\cap\Omega}|H_N(u)|^{2+\frac{4}{\nu}}dmdt \leq E\Big(\frac{C(\lambda_1,c_0)}{(R-r)^2}\int_{s-R^2}^{s}\int_{\mathfrak{B}(x,R)\cap\Omega}u^2|G_N'(u)|dmdt\Big)^{1+\frac{2}{\nu}},$$

where in the last inequality we have used Theorem 5.1.7 with the constant $E = C_M r^2 V^{\frac{-2}{\nu}}$ and (5.2.20). We note here that we can use Theorem 5.1.7 for the function $\psi H_N(u)$. Since by Theorem 5.2.2 there exists a sequence of functions u_k in $C^\infty(\overline{\mathfrak{B}(x,r)\cap\Omega})$ having compact support in Ω such that $u_k \to \psi^2 H_N(u)$ in $H^1(\mathfrak{B}(x,r)\cap\Omega, d(y)dy)$ and since $\phi \sim d^{\frac{1}{2}}$ we have that $u_k \to u$ in $H^1_\phi(\mathfrak{B}(x,r)\cap\Omega)$. Thus we have,

$$\int_{\mathfrak{B}(x,r)\cap\Omega}|u_k(y)|^{2(1+\frac{2}{\nu})}dm$$
$$\leq C_M r^2 V(x,r)^{-\frac{2}{\nu}}\int_{\mathfrak{B}(x_0,r)\cap\Omega}|\nabla u_k(y)|^2dm\Big(\int_{\mathfrak{B}(x_0,r)\cap\Omega}|u_k(y)|^2dm\Big)^{\frac{2}{\nu}},$$

and passing to the limit $k \to \infty$

$$\int_{\mathfrak{B}(x,r)\cap\Omega}|\psi^2 H_N(u)|^{2(1+\frac{2}{\nu})}dm$$
$$\leq C_M r^2 V(x,r)^{-\frac{2}{\nu}}\int_{\mathfrak{B}(x,r)\cap\Omega}|\nabla(\psi^2 H_N(u))|^2dm\Big(\int_{\mathfrak{B}(x,r)\cap\Omega}|\psi^2 H_N(u)|^2dm\Big)^{\frac{2}{\nu}}.$$

Hence combining all the above we have,

(5.2.21)
$$\int_{s-r^2}^{s}\int_{\mathfrak{B}(x,r)\cap\Omega}|H_N(u)|^{2+\frac{4}{\nu}}dmdt \leq E\Big(\frac{C(\lambda_1,c_0)}{(R-r)^2}\int_{s-R^2}^{s}\int_{\mathfrak{B}(x,r)\cap\Omega}u^2|G_N'(u)|dmdt\Big)^{1+\frac{2}{\nu}}.$$

Moreover as $N \to \infty$ $H_N(s) \to s$ and $G_N'(s) \to \frac{p^2}{2}s^{p-2}$. Thus the inequality (5.2.21) becomes

(5.2.22)
$$\int_{s-r^2}^{s}\int_{\mathfrak{B}(x,r)\cap\Omega}u^{p(1+\frac{2}{\nu})}dmdt \leq E\Big(\frac{C(\lambda_1,c_0)}{(R-r)^2}\frac{p^2}{2}\int_{s-R^2}^{s}\int_{\mathfrak{B}(x,r)\cap\Omega}u^p dmdt\Big)^{1+\frac{2}{\nu}},$$

provided the integral on the left hand to be bounded. We note that by iteration for $p_0 = p,\ p_1 = p(1+\frac{1}{\nu}), ..., p_i = p\Big(1+\frac{1}{\nu}\Big)^i$ that

$$\int_{s-r''^2}^{s}\int_{B(x,r'')}u^{p_i}dmdt < \infty,\quad \forall\ i \geq 0 \text{ and } r'' < r'.$$

Thus by same argument as before we have for $r < R$

(5.2.23)
$$\int_{s-r^2}^{s}\int_{B(x,r)}u^{p_{i+1}}dmdt \leq E\Big(\frac{C(\lambda_1,c_0)}{(R-r)^2}\frac{p_i^2}{2}\int_{s-R^2}^{s}\int_{B(x,R)}u^{p_i}dmdt\Big)^{1+\frac{2}{\nu}},$$

Now set $\delta_0 = \delta' r$ and $r_i = \delta' - (\delta'-\delta)\sum_{j=1}^{i}2^{-j}$. Then $r_i - r_{i+1} = (\delta'-\delta)2^{-i-1}$ and $p_{i+1} = p_i(1+\frac{2}{\nu})$. Thus inequality (5.2.23) becomes

$$\int\int_{Q_{r_{i+1}}}u^{p_{i+1}}dmdt \leq E\Big(\frac{C(\lambda_1,c_0)2^{2(i+1)}}{r^2(\delta'-\delta)^2}p_i^2\int\int_{Q_{r_i}}u^{p_i}dmdt\Big)^{1+\frac{2}{n}} \Leftrightarrow$$

$$\Big(\int\int_{Q_{r_{i+1}}} u^{p_{i+1}} dmdt\Big)^{\frac{1}{p_{i+1}}} \leq E^{\frac{1}{p_{i+1}}}\Big(\frac{C(\lambda_1,c_0)2^{2(i+1)}}{r^2(\delta'-\delta)^2}\Big)^{\frac{1}{p_i}}\Big(p_i^2\int\int_{Q_{r_i}} u^{p_i} dmdt\Big)^{\frac{1}{p_i}}$$

$$\leq E^{\frac{1}{p_{i+1}}+\frac{1}{p_i}}\Big(\frac{C(\lambda_1,c_0)}{r^2(\delta'-\delta)^2}\Big)^{\frac{1}{p_i}+\frac{1}{p_{i-1}}} 2^{\frac{2(i+1)}{p_i}+\frac{2i}{p_{i-1}}} p_i^{\frac{2}{p_i}} p_{i-1}^{\frac{2}{p_{i-1}}}\Big(\int\int_{Q_{r_{i-1}}} u^{p_{i-1}} dmdt\Big)^{\frac{1}{p_{i-1}}}$$

$$\leq E^{\sum_{j=1}^{i+1}}\Big(\frac{C(\lambda_1,c_0)}{r^2(\delta'-\delta)^2}\Big)^{\frac{1}{p}\sum_{j=0}^{i}\frac{1}{\Theta^j}} 4^{\frac{1}{p}\sum_{j=0}^{i}\frac{j+1}{\Theta^j}} e^{\sum_{j=0}^{i}\Theta^{-j}log(p\Theta^j)}\Big(\int\int_{Q_{r_0}} u^{p_0} dmdt\Big)^{\frac{1}{p_0}},$$

where $\Theta = 1+\frac{2}{\nu}$.Observe now that $r_i \to \delta$ as $i\to\infty$, all sum above are finite and $\sum_{j=0}^{\infty}\Theta^{-j} = \frac{\nu}{2}+1$. Hence we have,

$$\sup_{Q_\delta}|u|^p \leq E^{\frac{\nu}{2}}\frac{C(\lambda_1,p,\nu,c_0)}{(\delta'-\delta)^{\nu+2}r^{\nu+2}}\int_{Q_{\delta'}}|u|^p dmdt, \quad \forall\ p\geq 2.$$

where $E = C_M r^2 V^{-\frac{2}{\nu}}(x,r)$. We note here that the inequality which we used to reach the desired result is (5.2.23) for $p=2$. Thus the function $u\in L^p_{loc}(Q)$ for $p\geq 2$, and we have the inequality (5.2.23) for any $p\geq 2$. Also we note that since $u\in L^\infty_{loc}(Q)$ we can set $G(t)=t^{p-1}$ and by the same arguments to reach to the desired result where the constant is independent on p.

Now we prove the statement for $p\in(0,2)$. We have shown that for any $\Theta\in(0,1)$

$$\sup_{Q_\Theta}|u| \leq \Big(\frac{C(\lambda_1,\nu,c_0)}{((1-\Theta)r)^{n+2}}\int_Q |u|^2 dmdt\Big)^{\frac{1}{2}}.$$

For $p\in(0,2)$ we have

$$\int_Q |u|^2 dxdt \leq \|u\|^{2-p}_{L^\infty(Q)}\int_Q |u|^p dmdt.$$

Hence,

$$\sup_{Q_\Theta}|u| \leq \Big(\frac{C}{((1-\Theta)r)^{n+2}}\|u\|^{2-p}_{L^\infty(Q)}\int_Q |u|^p dxdt\Big)^{\frac{1}{2}} \leq \frac{1}{2}\|u\|_{L^\infty(Q)} + \frac{C}{((1-\Theta)r)^{\frac{n+2}{p}}}\|u\|_{L^p(Q)},$$

where in the last inequality we have used the Hölder inequality. Now set $f(\theta)=\sup_{Q_\theta}|u|$, then for any $\Theta\in(0,1)$ we have

$$f(\Theta) \leq \frac{1}{2}f(1) + \frac{C}{((1-\Theta)r)^{\frac{n+2}{p}}}\|u\|_{L^p(Q)}.$$

We apply the Lemma 5.2.4 to get

$$f(\Theta) \leq \frac{C}{((1-\Theta)r)^{\frac{n}{p}}}\|u\|_{L^p(Q)},$$

which is the desired result for $0<p<2$.
If $d(x)\geq\gamma r$, we do the same approach as before, but for the admissible test function we set $\psi(y)=\xi(\frac{|x-y|}{R})$ instead of $\xi(|y'-x'|)\xi(|y_n-a(y')-d(x)|)$ and we use the fact that $\mathfrak{B}(x,r)=B(x,r)$ (thus we don't need to use of Theorem 5.2.2).
□

Lemma 5.2.4. *Let $f(t)\geq 0$ be bounded in $[t_0,t_1]$ with $t_0\geq 0$. Suppose for $t_0\leq t<s\leq t_1$, we have*

$$f(t) \leq \Theta f(s) + \frac{A}{(s-t)^a} + B,$$

for some $\Theta \in (0,1)$ *and* $a \geq 0$*. Then for any* $t_0 \leq t < s \leq t_1$ *there holds*

$$f(t) \leq C(\Theta, a)(\frac{A}{(s-t)^a} + B).$$

proof: Fix $t_0 \leq t < s \leq t_1$. For some $0 \leq \tau < 1$, we consider the sequence $\{t_i\}$ defined by $t_0 = t$ and $t_{i+1} = t_i + (1-\tau)\tau^i(s-t)$. Note $t_\infty = s$, since $t_{i+1} = t_0 + \sum_{j=0}^{i}(1-\tau)\tau^j(s-t) \to s$.
By iteration

$$f(t) \leq \Theta f(t_1) + \frac{A}{(t_1-t_0)^a} + B \leq ... \leq \Theta^k f(t_k) + [\frac{A}{(s-t)^a} + B]\sum_{i=0}^{k-1}\Theta^i\tau^{-ia},$$

choose $\tau < 1$ such that $\Theta\tau < 1$, that is, $\Theta < t^a < 1$.
As k tend to infinity, we have

$$f(t) \leq C(\Theta)((1-\tau)^{-a}\frac{A}{(s-t)^a} + B).$$

□

5.2.2 Properties of Super Solutions

Similarly with Definition 5.2.1, we call a function $v \in C^1((s-r^2, r) : H^1_\phi(\mathfrak{B}(x,r)\cap\Omega))$ supersolution of (5.0.3) if for each $0 \leq \Phi \in C^1_0((s-r^2, r) : C^\infty_0(\mathfrak{B}(x,r)\cap\Omega))$ and for each $s - r^2 < t_1 < t_2 < s$ we have

$$\int_{t_1}^{t_2}\int_{\mathfrak{B}(x,r)\cap\Omega} v_t\Phi + \nabla v\nabla\Phi + \lambda_1\frac{v\Phi}{1+d^{2+\sigma}}dmdt \geq 0, \tag{5.2.24}$$

where $dm = \phi^2 dx$.

Theorem 5.2.5. *Let* $\Omega \subset \mathbb{R}^n$ *be a exterior domain not containing the origin,* $\nu \geq n+1$, $\gamma \in (1,2)$ *and* $p \geq 0$*. Then there exist constant* $c_0(\Omega)$ *and* $C(\nu, \lambda_1, c_0)$ *such that for all* $x \in \Omega$ *with* $\gamma r < c_0$ *and for any positive supersolution* v *of (5.0.3) in* Q*, we have the estimate*

$$\sup_{Q_\delta}|v|^{-p} \leq \frac{C}{(\delta'-\delta)^{n+2}r^2V(x,r)}\int_{Q_{\delta'}}|v|^{-p}dmdt,$$

for each $0 < \delta < \delta' \leq 1$.

proof: Set $u = v + \varepsilon$, then u is bounded away from zero (at the end of the argument we send ε to origin). Thus by (5.2.24) we have for any $\Phi \in C^\infty_0(\mathfrak{B}(x,R)\cap\Omega^c)$

$$\int_{\mathfrak{B}(x,R)\cap\Omega} u_t\Phi + \nabla u\nabla\Phi + \lambda_1\frac{u\Phi}{1+d^{2+\sigma}}dm \geq \varepsilon|\lambda_1|\int_{\mathfrak{B}(x,R)\cap\Omega}\frac{\Phi u}{1+d^{2+\sigma}}dm$$

$$\geq -|\lambda_1|\int_{\mathfrak{B}(x,R)\cap\Omega} u\Phi dm.$$

We set $\Phi = -\beta u^{\beta-1}\psi^2$, where $0 \leq \psi \in C^\infty_0(\mathfrak{B}(x,R))$ and $\beta < 0$. As $\phi \in H^1_0(\mathfrak{B}(x,R))$, we have $\Phi_{x_i} = -2\beta u^{\beta-1}\psi\psi_{x_i} - \beta(\beta-1)u^{\beta-2}\psi^2 u_{x_i}$. By the same arguments as in Theorem 5.2.3, Φ is a admissible test function (if $d(x) \leq \gamma r$), thus we can use it in (5.2.2) to yield

$$\begin{aligned} -\beta\int_{\mathfrak{B}(x,R)\cap\Omega} u^{\beta-1}\psi^2 u_t dm - 2\beta\int_{\mathfrak{B}(x,R)\cap\Omega}\psi u^{\beta-1}\nabla u\nabla\psi dm &- \beta(\beta-1)\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2 u^{\beta-2}|\nabla u|^2 dm \\ &\geq -2|\lambda_1||\beta|\int_{\mathfrak{B}(x,R)\cap\Omega} u^\beta dm \end{aligned}$$

Now we set $w = u^{\frac{\beta}{2}}$. Then $w_{x_i} = \frac{\beta}{2}u^{\frac{\beta}{2}-1}u_{x_i}$ and the above inequality becomes

$$\begin{aligned} -\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2(w^2)_t dm - 4\int_{\mathfrak{B}(x,R)\cap\Omega}\psi w\nabla w\nabla\psi dm &- 4\frac{\beta-1}{\beta}\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2|\nabla w|^2 dm \\ &\geq -2|\lambda_1||\beta|\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2 w^2 dm \end{aligned} \tag{5.2.25}$$

Now,

$$\begin{aligned} 4\left|\int_{\mathfrak{B}(x,R)\cap\Omega}\psi w\nabla w\nabla\psi dm\right| &\leq 4\int_{\mathfrak{B}(x,R)\cap\Omega}|\psi||w||\nabla w||\nabla\psi| dm \\ &\leq \lambda\int_{\mathfrak{B}(x,R)\cap\Omega}|\psi|^2|\nabla w|^2 dm + C(\lambda)\int_{\mathfrak{B}(x,R)\cap\Omega}|w|^2|\nabla\psi|^2 dm. \end{aligned}$$

Thus by the above inequality and (5.2.25) we have

$$\begin{aligned} \int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2(w^2)_t dm &+ 4\frac{|\beta|+1}{|\beta|}\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2|\nabla w|^2 dm - \lambda\int_{\mathfrak{B}(x,R)\cap\Omega}|\psi|^2|\nabla w|^2 dm \\ &\leq C(\lambda)\int_{\mathfrak{B}(x,R)\cap\Omega}|w|^2|\nabla\psi|^2 dm + 2|\lambda_1||\beta|\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2 w^2 dm. \end{aligned}$$

Finally if we choose $\lambda = 1$ then $4\frac{|\beta|+1}{|\beta|} - \lambda > 1$ and the above inequality becomes

$$\begin{aligned} \int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2(w^2)_t dm &+ \int_{\mathfrak{B}(x,R)\cap\Omega}|\psi|^2|\nabla w|^2 dm \\ &\leq C(\lambda)\int_{\mathfrak{B}(x,R)\cap\Omega}|w|^2|\nabla\psi|^2 dm + 2|\lambda_1||\beta|\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2 w^2 dm. \end{aligned} \tag{5.2.26}$$

Also,

$$\begin{aligned} \int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2(w^2)_t dm &+ \frac{1}{2}\int_{\mathfrak{B}(x,R)\cap\Omega}|\nabla(\psi w)|^2 dm = \int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2(w^2)_t dm + \frac{1}{2}\int_{\mathfrak{B}(x,R)\cap\Omega}|\psi|^2|\nabla w|^2 dm \\ &+ \frac{1}{2}\int_{\mathfrak{B}(x,R)\cap\Omega}|w|^2|\nabla\psi|^2 dm + \int_{\mathfrak{B}(x,R)\cap\Omega}\psi w\nabla\psi\nabla w)|^2 dm \\ &\leq \int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2(w^2)_t dm + \frac{1}{2}\int_{\mathfrak{B}(x,R)\cap\Omega}|\psi|^2|\nabla w|^2 dm + \frac{1}{2\theta}\int_{\mathfrak{B}(x,R)\cap\Omega}|w|^2|\nabla\psi|^2 dm \\ &+ \frac{1}{2}\int_{\mathfrak{B}(x,R)\cap\Omega}|\psi|^2|\nabla w|^2 dm + \frac{1}{2}\int_{\mathfrak{B}(x,R)\cap\Omega}|w|^2|\nabla\psi|^2 dm \\ &\leq C(\lambda)\int_{\mathfrak{B}(x,R)\cap\Omega}|w|^2|\nabla\psi|^2 dm + 2|\lambda_1||\beta|\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2 w^2 dm. \end{aligned}$$

Now working as Theorem 5.2.3 we have

$$\int_{s-r^2}^{s}\int_{\mathfrak{B}(x,R)\cap\Omega}|w|^{2+\frac{4}{\nu}} dm dt \leq E\Big(\frac{C + 2c_0|\beta||\lambda_1|}{(R-r)^2}\int_{s-R^2}^{s}\int_{\mathfrak{B}(x,R)\cap\Omega}w^2 dm dt\Big)^{1+\frac{2}{\nu}},$$

where $E = C_M r^2 V^{\frac{-2}{\nu}}$ is the constant of Theorem 5.1.7. Replace now $w = u^{\frac{\beta}{2}}$, we get

$$\int_{s-r^2}^{s}\int_{\mathfrak{B}(x,R)\cap\Omega}|u|^{\beta(1+\frac{2}{\nu})} dm dt \leq E\Big(\frac{C + 2c_0|\beta||\lambda_1|}{(R-r)^2}\int_{s-R^2}^{s}\int_{\mathfrak{B}(x,R)\cap\Omega}u^\beta dm dt\Big)^{1+\frac{2}{\nu}}.$$

This is the analogue of Theorem 5.2.3 and the iterative steps give the desired result. □

In order to state the next result we need to introduce the following notation.

$$Q'_\delta = (s - r^2, s - (1-\delta)r^2) \times \mathfrak{B}(x, \delta r) \cap \Omega$$

Theorem 5.2.6. *Fix* $0 < p_0 < \Theta = 1 + \frac{2}{\nu}$, $\nu \geq n+1$, $\gamma \in (1,2)$. *Then there exist constant* $\beta(\Omega)$ *and* $C(\nu, \lambda_1, c_0)$ *such that for all* $x \in \Omega$ *with* $\gamma r < c_0$, *for any* $0 < p < \frac{p_0}{\Theta}$ *and for any positive supersolution* v *of (5.0.3) in* Q, *we have the estimate*

$$\Big(\int_{Q'_\delta} |u|^{p_0}\phi^2 dydt\Big)^{\frac{p_0}{p}} \leq \Big(\frac{A}{(\delta'-\delta)^{(2+\nu)(1+\Theta)} r^2 V^\nu(x,r)}\Big)^{1-\frac{p}{p_0}} \int_{Q'_{\delta'}} |u|^p \phi dydt,$$

for each $0 < \delta < \delta'$.

proof: First the case $d(x) < \gamma r$

Set $u = v + \varepsilon$, then u is bounded away from zero (at the end of the argument we send ε to origin). Thus by the definition of supersolutions we have for any $\Phi \in C_0^\infty \mathfrak{B}(x,R) \cap \Omega$

$$\begin{aligned} \int_{\mathfrak{B}(x,R)\cap\Omega} u_t\Phi + \nabla v \nabla \Phi + \lambda_1 \frac{u\Phi}{1+d^{2+\sigma}} dm &\geq \varepsilon|\lambda_1| \int_{\mathfrak{B}(x,R)\cap\Omega} \frac{\Phi u}{1+d^{2+\sigma}} dm \\ &\geq -|\lambda_1| \int_{\mathfrak{B}(x,R)\cap\Omega} u\Phi dm. \end{aligned} \tag{5.2.27}$$

We set $\Phi = \beta u^{\beta-1}\psi^2$, where $0 \leq \psi \in C_0^\infty(\mathfrak{B}(x,R))$ and $0 < \beta < \frac{p_0}{\Theta}$. As $\phi \in H_0^1(\mathfrak{B}(x,R))$, we have $\Phi_{x_i} = 2\beta u^{\beta-1}\psi\psi_{x_i} + \beta(\beta-1)u^{\beta-2}\psi^2 u_{x_i}$. By the same arguments as in Theorem 5.2.3, Φ is a admissible test function , thus we can use it in (5.2.27) to yield

$$\begin{aligned} \beta \int_{\mathfrak{B}(x,R)\cap\Omega} u^{\beta-1}\psi^2 u_t dm + 2\beta \int_{\mathfrak{B}(x,R)\cap\Omega} \psi u^{\beta-1}\nabla u \nabla\psi dm &+ \beta(\beta-1)\int_{\mathfrak{B}(x,R)\cap\Omega} \psi^2 u^{\beta-2}|\nabla u|^2 dm \\ &\geq -2|\lambda_1| \int_{\mathfrak{B}(x,R)\cap\Omega} u^\beta dm \end{aligned}$$

Now we set $w = u^{\frac{\beta}{2}}$. Then $w_{x_i} = \frac{\beta}{2} u^{\frac{\beta}{2}-1} u_{x_i}$ and the above inequality becomes

$$\begin{aligned} \int_{\mathfrak{B}(x,R)\cap\Omega} \psi^2 (w^2)_t dm + 4\int_{\mathfrak{B}(x,R)\cap\Omega} \psi w \nabla w \nabla \psi dm &+ 4\frac{\beta-1}{\beta}\int_{\mathfrak{B}(x,R)\cap\Omega} \psi^2|\nabla w|^2 dm \\ &\geq -2|\lambda_1| \int_{\mathfrak{B}(x,R)\cap\Omega} \psi^2 w^2 dm \end{aligned} \tag{5.2.28}$$

Now,

$$\begin{aligned} 4|\int_{\mathfrak{B}(x,R)\cap\Omega} \psi w \nabla w \nabla\psi dm| &\leq 4\int_{\mathfrak{B}(x,R)\cap\Omega} |\psi||w||\nabla w||\nabla\psi| dm \\ &\leq \lambda \int_{\mathfrak{B}(x,R)\cap\Omega} |\psi|^2|\nabla w|^2 dm + C(\lambda)\int_{\mathfrak{B}(x,R)\cap\Omega} |w|^2|\nabla\psi|^2 dm. \end{aligned}$$

Thus by the above inequality and (5.2.28) we have

$$\begin{aligned} -\int_{\mathfrak{B}(x,R)\cap\Omega} \psi^2(w^2)_t dm &+ (1-\frac{p_0}{\Theta})\int_{\mathfrak{B}(x,R)\cap\Omega} \psi^2|\nabla w|^2 dm - \lambda\int_{\mathfrak{B}(x,R)\cap\Omega} |\psi|^2|\nabla w|^2 dm \\ &\leq C(\lambda)\int_{\mathfrak{B}(x,R)\cap\Omega} |w|^2|\nabla\psi|^2 dm + 2|\lambda_1|\int_{\mathfrak{B}(x,R)\cap\Omega} \psi^2 w^2 dm. \end{aligned} \tag{5.2.29}$$

Finally if we choose $\lambda = \frac{1-\frac{p_0}{\Theta}}{2}$ then the above inequality becomes

$$(5.2.30)\quad \begin{aligned} -\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2(w^2)_t dm \quad &+ \quad \frac{1-\frac{p_0}{\Theta}}{2}\int_{\mathfrak{B}(x,R)\cap\Omega}|\psi|^2|\nabla w|^2 dm \\ &\leq \quad C(p_0,\nu)\int_{\mathfrak{B}(x,R)\cap\Omega}|w|^2|\nabla\psi|^2 dm + 2|\lambda_1|\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2 w^2 dm. \end{aligned}$$

Also,

$$\begin{aligned} -\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2(w^2)_t dm \quad &+ \quad \frac{1-\frac{p_0}{\Theta}}{4}\int_{\mathfrak{B}(x,R)\cap\Omega}|\nabla(\psi w)|^2 dm \\ &= \quad -\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2(w^2)_t dm + \frac{1-\frac{p_0}{\Theta}}{4}\int_{\mathfrak{B}(x,R)\cap\Omega}|\psi|^2|\nabla w|^2 dm \\ &+ \quad \frac{1-\frac{p_0}{\Theta}}{4}\int_{\mathfrak{B}(x,R)\cap\Omega}|w|^2|\nabla\psi|^2 dm + \frac{1-\frac{p_0}{\Theta}}{4}\int_{\mathfrak{B}(x,R)\cap\Omega}\psi w\nabla\psi\nabla w dm \end{aligned}$$

$$\begin{aligned} \leq -\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2(w^2)_t dm \quad &+ \quad \frac{1-\frac{p_0}{\Theta}}{2}\int_{\mathfrak{B}(x,R)\cap\Omega}|\psi|^2|\nabla w|^2 dm + \frac{1-\frac{p_0}{\Theta}}{2}\int_{\mathfrak{B}(x,R)\cap\Omega}|w|^2|\nabla\psi|^2 dm \\ &\leq \quad C(p_0,\nu)\int_{\mathfrak{B}(x,R)\cap\Omega}|w|^2|\nabla\psi|^2 dm + 2|\lambda_1|\int_{\mathfrak{B}(x,R)\cap\Omega}\psi^2 w^2 dm. \end{aligned}$$

Now multiply the above inequality by a bounded function $\chi^2(t) \in C^\infty(\mathbb{R})$ to reach

$$(5.2.31)\quad \begin{aligned} -\frac{d}{dt}\int_{\mathfrak{B}(x,R)\cap\Omega}\chi^2\psi^2 w^2 dy \quad &+ \quad \frac{1-\frac{p_0}{\Theta}}{2}\int_{\mathfrak{B}(x,R)\cap\Omega}\chi^2|\nabla(\psi w)|^2 dy \\ &\leq \quad C(\nu,p_0)\|\chi\|_{L^\infty}(\|\chi\|_{L^\infty}\|\nabla\psi\|_{L^\infty}^2 + \|\chi'\|_{L^\infty} + |\lambda_1|)\int_{\mathfrak{B}(x,R)\cap\Omega}|w|^2 dy. \end{aligned}$$

we choose $\chi(t) \in C^1(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi(t) = 1$ in $(-\infty, s-(1-\delta)r^2)$, $\chi(t) = 0$ in $(s-(1-\delta')r^2, \infty)$ and $|\chi'| \leq \frac{1}{r^2(\delta'-\delta)^2}$. Also we choose a function $\psi = \xi(|y'-x'|)\xi(|y_n - a(y') - d(x)|)$, where $\xi \in C^\infty(\mathbb{R})$ and satisfies $0 \leq \xi \leq 1$, $\xi(s) = 1$ if $s \leq \delta r$ and $\xi(s) = 0$ if $s > \delta' r$. Now, we integrate (5.2.31) from t to $s-(1-\delta')r^2$ for some $t \in (s-r^2, s-(1-\delta')r^2)$ and letting t go to $s-r^2$ we have

$$\begin{aligned} \sup_{t\in(s-r^2,s-(1-\delta)r^2)}\int_{\mathfrak{B}(x,R)\cap\Omega}\chi^2\psi^2 w^2 dm \quad &+ \quad \frac{1-\frac{p_0}{\Theta}}{2\theta}\int_{s-r^2}^{s-(1-\delta')r^2}\int_{\mathfrak{B}(x,R)\cap\Omega}\chi^2|\nabla(\psi w)|^2 dm \\ &\leq \quad \frac{C(\nu,p_0,c_0)}{(\delta'-\delta)^2 r^2}\int_{s-r^2}^{s-(1-\delta')r^2}\int_{\mathfrak{B}(x,R)\cap\Omega}|w|^2 dm, \end{aligned}$$

thus we have as in Theorem (5.2.3),

$$\int\int_{Q'_\delta}|w|^{(1+\frac{2}{n})}dmdt \leq E\Big(\frac{C(n,\theta,p_0)}{r^2(\delta'-\delta)^2}\int_{Q'_{\delta'}}w^2 dmdt\Big)^{1+\frac{2}{n}} \Leftrightarrow$$

$$(5.2.32)\quad \int\int_{Q'_\delta}|u|^{\beta\Theta}dmdt \leq E\Big(\frac{C(\nu,p_0,c_0)}{r^2(\delta'-\delta)^2}\int_{Q'_{\delta'}}u^\beta dmdt\Big)^{\Theta},$$

for all $0 < \beta < \frac{p_0}{\Theta}$ and for $E = C_M r^2 V^{-\frac{2}{\nu}}(x,r)$ Define now $p_i = p_0\Theta^{-i}$, $r_i = \delta' - (\delta'-\delta)\sum_{j=1}^{i}2^{-j}$ and $r_0 = \delta'$. Now since

$p^i\Theta^{j-1} < \frac{p_0}{\Theta}$, we have by (5.2.32) for any $j = 1, .., i$

$$\int\int_{Q'_{r_i}} |u|^{p_i\Theta^j} dydt \leq E\Big(\frac{2^{2j}C(\nu, c_0, p_0)}{r^2(\delta' - \delta)^2}\int_{Q'_{r_{i-1}}} u^{p_i\Theta^{j-1}} dydt\Big)^{\Theta}$$
$$\leq E^{\sum_{k=0}^{j-1}\Theta^k}\Big(\frac{C(\nu, c_0, p_0)}{r^2(\delta' - \delta)^2}\Big)^{\sum_{k=1}^{j}\Theta^k} 4^{\sum_{k=1}^{j}(j-k+1)\Theta^k}\Big(\int_{Q'_{r_{i-j+1}}} u^{p_i} dydt\Big)^{\Theta^j}.$$

The above inequality holds for all $j = 1, ..., i$. Thus for $j = i$ we have

$$\int\int_{Q'_{r_i}} |u|^{p_0} dydt \leq E^{\sum_{k=0}^{j-1}\Theta^k}\Big(\frac{C(\nu, c_0, p_0)}{r^2(\delta' - \delta)^2}\Big)^{\sum_{k=1}^{i}\Theta^k} 4^{\sum_{k=1}^{i}(i-k+1)\Theta^k}\Big(\int_{Q'_{r_0}} u^{p_i} dydt\Big)^{\Theta^i}.$$

Finally we note that
$\sum_{k=0}^{i}(i - k + 1)\Theta^k \leq (\frac{\nu}{2})^3\Theta(\frac{p_0}{p_i} - 1)$
$r_i > \delta$
$\sum_{k=0}^{j-1}\Theta^k = \frac{\nu}{2}(\frac{p_0}{p_i} - 1)$
$\sum_{k=1}^{i}\Theta^k = (1 + \frac{\nu}{2})((\frac{p_0}{p_i} - 1))$.
Thus we have

$$\int\int_{Q'_{\delta}} |u|^{p_0} dydt \leq E^{\frac{n}{2}(\frac{p_0}{p_i}-1)}\Big(\frac{C(\nu, c_0, p_0)}{r^2(\delta' - \delta)^2}\Big)^{(1+\frac{n}{2})(\frac{p_0}{p_i}-1)}\Big(\int_{Q'_{r_0}} u^{p_i} dydt\Big)^{\frac{p_0}{p_i}}.$$

To obtain the desired inequality for any $p \in (0, \frac{p_0}{\Theta})$, let $i \geq 2$ be the integer such that $p_i \leq p \leq p_{i-1}$. Then $\frac{1}{p_i} - \frac{1}{p_0} \leq (1 + \Theta)(\frac{1}{p} - \frac{1}{p_0})$. Thus, by Jensen's inequality we have

$$\int\int_{Q'_{\delta}} |u|^{p_0} dmdt \leq E^{\frac{n}{2}}\Big(\frac{C(n, \theta, p_0)}{(\delta' - \delta)^2}\Big)^{(1+\Theta)(1+\frac{\nu}{2})(\frac{p_0}{p}-1)}\Big(\frac{1}{r^2}\Big)^{(1+\frac{\nu}{2})(\frac{p_0}{p_i}-1)}\Big(\int_{Q'_{r_0}} u^{p_i} dmdt\Big)^{\frac{p_0}{p_i}}$$

$$\leq E^{\frac{\nu}{2}}\Big(\frac{C(\nu, c_0, p_0)}{(\delta' - \delta)^2}\Big)^{(1+\Theta)(1+\frac{\nu}{2})(\frac{p_0}{p}-1)}\Big(\frac{1}{r^2}\Big)^{(1+\frac{\nu}{2})(\frac{p_0}{p_i}-1)} V^{\frac{p_0}{p_i}-\frac{p_0}{p}}(x, r)\Big(\int_{Q'_{r_0}} u^{p} dydt\Big)^{\frac{p_0}{p}},$$

which is the desired result since $E = C_M r^2 V^{\frac{-2}{\nu}}$. If $d(x) \geq \gamma r$, we do the same approach as before. But for the admissible test function we set $\psi(y) = \xi(\frac{|x-y|}{R})$ instead of $\xi(|y' - x'|)\xi(|y_n - a(y') - d(x)|)$ and we use the fact that $\mathfrak{B}(x, r) = B(x, r)$ (thus we not need the use of Theorem 5.2.2). □

5.3 Harnack Inequality

In the following lemma we see the importance of Theorem 5.1.6.

Lemma 5.3.1. *Let v be any positive supersolution of problem (5.0.3) in $(s - r^2, s) \times \mathfrak{B}(x, r) \cap \Omega$, where $\gamma r < c_0$ and $\gamma \in (1, 2)$. Then there exists a constant $c = c(u, \eta, c_0)$ such that, for all $\lambda > 0$,*

$$\mu(\{(t, z) \in K_+ : \log v < -\lambda - c\}) \leq Cr^{2+n}\lambda^{-1}$$

and

$$\mu(\{(t, z) \in K_- : \log v > \lambda - c\}) \leq Cr^{2+n}\lambda^{-1},$$

where $\mu = \phi^2 dxdt$ $K_+ = (s - \eta r^2, s) \times \mathfrak{B}(x, \delta r) \cap \Omega$ and $K_- = (s - r^2, s - \eta r^2) \times \mathfrak{B}(x, \delta r) \cap \Omega$. Here the constant C is independent of $\lambda > 0$, v, s and the radius r.

proof: Note that δ and η play somewhat different roles here. The parameter δ is used to stay away from the boundary of the ball $\mathfrak{B}$. The parameter η is used to define a fixed point $s' = s - \eta r^2$ in the interval $(s - r^2, s)$ away from $s - r^2$ and s.

Let us first observe that (by changing δ) we can assume that v is a super-solution in $(s - r^2, s) \times \mathfrak{B}(x, \delta r) \cap \Omega^c$ where $\mathfrak{B}'$ is a concentric ball larger than $\mathfrak{B}(x, r)$. We set $w = -\log u$, where $u = v + \varepsilon$. Then, for any non-negative function $\psi \in C_0^\infty(B')$, we have

$$\begin{aligned}
\partial_t(\int_{\mathfrak{B}(x,r)\cap\Omega} \psi^2 w dm) &= -\int_{\mathfrak{B}(x,r)\cap\Omega} \frac{\psi^2}{u} u_t dm \\
&\leq \int_{\mathfrak{B}(x,r)\cap\Omega} \nabla u \nabla(\frac{\psi^2}{u}) dm + 2|\lambda_1| \int_{\mathfrak{B}(x,r)\cap\Omega} \psi^2 dm \\
&\leq 2\theta \int_{\mathfrak{B}(x,r)\cap\Omega} \psi |\nabla w||\nabla\psi| dm - \int_{\mathfrak{B}(x,r)\cap\Omega} \psi^2 |\nabla w|^2 dm + 2|\lambda_1| \int_{\mathfrak{B}(x,r)\cap\Omega} \psi^2 dm \\
&\leq -\frac{1}{2} \int_{\mathfrak{B}(x,r)\cap\Omega} \psi^2 |\nabla w|^2 dm + C \int_{\mathfrak{B}(x,r)\cap\Omega} |\nabla\psi|^2 dm + 2|\lambda_1| \int_{\mathfrak{B}(x,r)\cap\Omega} \psi^2 dm \Rightarrow
\end{aligned}$$

(5.3.33)
$$\partial_t(\int_{\mathfrak{B}(x,r)\cap\Omega} \psi^2 w dm) + \frac{1}{2} \int_{\mathfrak{B}(x,r)\cap\Omega} \psi^2 |\nabla w|^2 dm \leq C(\|\nabla\psi\|_\infty^2 + 2|\lambda_1|)\mu(supp\psi).$$

Here we have two cases.

First case $d(x) < \gamma r$.

We choose $\psi(y) = (1 - |x - y|/r)_+ (1 - \frac{|y_n - a(y') - d(x)|}{r})_+$. By Theorem 5.1.6 we have

$$\int_{\mathfrak{B}(x,r)\cap\Omega} |w - W|^2 \psi^2 dm \leq A_0 r^2 \int_{\mathfrak{B}(x,r)\cap\Omega} |\nabla w|^2 \psi^2 dm,$$

with

$$W = \frac{\int_{\mathfrak{B}(x,r)\cap\Omega} w\psi^2 dm}{\int_{\mathfrak{B}(x,r)\cap\Omega} \psi^2 dm}.$$

Second case $d(x) > \gamma r$.

Here, we choose $\psi(y) = (1 - |x - y|/r)_+$. Due to the fact that $c\phi(x) \leq \phi(y) \leq C\phi(x)$ for any $y \in \mathfrak{B}(x, r) \cap \Omega = B(x, r)$, by Lemma (5.1.4), we have for $\psi^2(y) = \Phi(y)$

$$\int_{\mathfrak{B}(x,r)\cap\Omega} |w - W|^2 \psi^2 dm \leq A_0 r^2 \int_{\mathfrak{B}(x,r)\cap\Omega} |\nabla w|^2 \psi^2 dm,$$

with

$$W(t) = \frac{\int_{\mathfrak{B}(x,r)\cap\Omega} w\psi^2 dm}{\int_{\mathfrak{B}(x,r)\cap\Omega} \psi^2 dm}.$$

Now using the fact that

$$\int_{\delta B(x,r)} \psi^2 dx \geq C(\delta) V(x, r).$$

By (5.3.33), $1 \leq \frac{c_0}{r}$ and the weighted Poincaré inequalities, we have

$$W_t + \frac{C(\delta)}{Vr^2} \int_{\mathfrak{B}(x,\delta r)\cap\Omega} |w - W|^2 \psi^2 dm$$

$$\leq \frac{\partial_t \int_{\mathfrak{B}(x,\delta r)\cap\Omega} w\psi^2 dm}{C(\delta) V} + \frac{C(\delta)}{V(x, r) r^2} \int_{\mathfrak{B}(x,\delta r)\cap\Omega} |w - W|^2 \psi^2 dm \leq A_2(c_0, \lambda_1) r^{-2},$$

for some constants $A_1,\ A_2 > 0$. Rewrite this inequality as

$$\partial_t W' + (A_1 r^2 V(x,r))^{-1} \int_{\mathfrak{B}(x,\delta r)\cap\Omega^c} |w' - W'|^2 \psi^2 dm \leq 0, \tag{5.3.34}$$

where

$$w'(t,z) = w(t,z) - A_2 r^{-2}(t-s'),$$

$$W'(t) = W(t) - A_2 r^{-2}(t-s'),$$

with $s' = s - \eta r^2$.

Now, set $c(u,\eta) = W'(s')$ and

$$\Omega_t^+(\lambda) = \{z \in B(x,\delta r) : w'(t,z) > c + \lambda\}$$

$$\Omega_t^-(\lambda) = \{z \in B(x,\delta r) : w'(t,z) < c - \lambda\}.$$

Then, if $t > s'$,

$$w'(t,z) - W'(t) \geq \lambda + c - W'(s') > \lambda,$$

in $\Omega_t^+(\lambda)$, because $c = W'(s')$ and $\partial_t W' \leq 0$. Using this in (5.3.34) we obtain

$$\partial_t W' + (Cr^2 V(x,r))^{-1} |\lambda + c - W'(t)|^2 |\Omega_t^+(\lambda)| \leq 0,$$

or equivalently,

$$-Cr^2 V \partial_t ((|\lambda + c - W'(t)|)^{-1}) \geq m(\Omega_t^+(\lambda)).$$

Integrating from s' to s, we obtain

$$\mu(\{(t,z) \in K_+ : w'(t,z) > c + \lambda\}) \leq Cr^2 V(|\lambda + c - W'(s')|)^{-1} \leq Cr^2 V(x,r)\lambda^{-1}$$

and returning to $-\log u = w = w' + A_2 r^{-2}(t-s')$

$$\begin{aligned}
\mu(\{(t,z) \in K_+ : \log u(t,z) < -c - \lambda\}) &\leq \mu(\{(t,z) \in K_+ : \log u(t,z) + < -c - \frac{\lambda}{2}\}) \\
&+ \mu(\{(t,z) \in K_+ : A_2 r^{-2}(t-s') > \frac{\lambda}{2}\}) \\
&\leq Cr^2 V \lambda^{-1} + \mu(\{(t,z) \in K_+ : A_2 r^{-2}(t-s') > \frac{\lambda}{2}\}).
\end{aligned}$$

Now consider two cases.

1. $0 < \lambda \leq 2\eta A_2$, then

$$\begin{aligned}
\mu(\{(t,z) \in K_+ : A_2 r^{-2}(t-s') > \frac{\lambda}{2}\}) &= \mu(\{(t,z) \in K_+ : t > \frac{r^2\lambda}{2A_2} + s - \eta r^2\}) \\
&\leq (-\frac{r^2\lambda}{2A_2} + \eta r^2)_+ V(x,r) \leq 2\frac{\eta^2 A_2 r^2}{\lambda} V(x,r).
\end{aligned}$$

2. $\lambda \geq 2A_2$, then

$$(\frac{-r^2\lambda}{2A_2} + \eta r^2)_+ = 0$$

Thus, in all cases we have

$$\mu(\{(t,z) \in K_+ : \log u < -\lambda - c\}) \leq Cr^2 V(x,r)\lambda^{-1}.$$

This proves the first inequality in this Lemma. Working with Ω_t^- instead of Ω_t^+, we obtain the second inequality by the same argument. The result follows by sending ε to origin. □

Let us prove an abstract lemma which we use in the following theorem.

Lemma 5.3.2. *Fix* $0 < \delta < 1$*. Let* $\gamma,\ C$ *be positive constants and* $0 < \alpha_0 \leq \infty$*. Let* f *be a positive measurable function on* $U_1 = U$ *which satisfies,*

$$\|f\|_{\alpha_0,U_{\sigma'}} \leq [C(\sigma-\sigma')^{-\gamma}\mu(U)^{-1}]^{1/\alpha-1/\alpha_0}\|f\|_{\alpha,U_\sigma},$$

for all $\sigma,\ \sigma',\ \alpha$ *such that* $0 < \delta \leq \sigma' < \sigma \leq 1$ *and* $0 < \alpha \leq \min\{1, \alpha_0/2\}$*. Assume further that* f *satisfies*

$$\mu(\log(f) > \lambda) \leq C\mu(U)\lambda^{-1}$$

for all $\lambda > 0$*. Then*

$$\|f\|_{\alpha_0,U_\delta} \leq A\mu(U)^{1/\alpha_0},$$

where A *depends only on* $\delta,\ ,\gamma,\ C$ *and a lower bound on* α_0*.*

proof: For the proof, assume without loss of generality that $\mu(U) = 1$ and $\|f\|_{\alpha_0,U_\sigma} > 1$, also we assume that :

$$\psi = \psi(\sigma) = \log(\|f\|_{\alpha_0,U_\delta}) \geq A_1 > 0, \quad for\ \ 0 < \delta \leq \sigma < 1.$$

Where A_1 depends only on a lower bound on α_0, which we will determine later.
Decomposing now U_σ into the sets where $\log(f) > \psi/2$ and where $\log(f) \leq \psi/2$, we get

$$\begin{aligned}\|f\|_{\alpha,U_\delta} = \Big(\int_{U_\sigma}|f|^\alpha d\mu\Big)^{1/\alpha} &\leq \Big(\int_{U_\sigma\cap\{\log(f)>\psi/2\}}|f|^\alpha d\mu\Big)^{1/\alpha} + \Big(\int_{U_\sigma\cap\{\log(f)\leq\psi/2\}}|f|^\alpha d\mu\Big)^{1/\alpha}\\ &\leq \|f\|_{\alpha_0,U_\delta}\mu(\log(f) > \psi/2)^{1/\alpha-1/\alpha_0} + e^{\psi/2} \leq e^{\psi}\Big(\frac{2C}{\psi}\Big)^{1/\alpha-1/\alpha_0} + e^{\psi/2}. \qquad (5.3.35)\end{aligned}$$

Here, we have used successively the Hölder inequality and the second hypothesis of the Lemma. Now, we want to choose α so that the two terms in the right-hand side of (5.3.35) are equal and $0 < \alpha \leq \min\{1, \alpha_0/2\}$. This is possible if

$$\Big(\frac{2C}{\psi}\Big)^{1/\alpha-1/\alpha_0} = e^{-\psi/2} \Leftrightarrow 1/\alpha - 1/\alpha_0 = (-\psi/2)\Big(\log\frac{2C}{\psi}\Big)^{-1} \Rightarrow$$

$$1/\alpha \geq 1/\alpha_0 + C = \frac{1+\alpha_0 C}{\alpha_0} \Rightarrow \alpha \leq \frac{\alpha_0}{1+\alpha_0 C} \leq \min\{1, \alpha_0/2\}$$

and the last inequality is certainly satisfied when

$$\psi \geq A_1 \geq \max\{2C, 1/\alpha'\}, \qquad (5.3.36)$$

where α' is a lower bound on α_0.
Assuming that (5.3.36) holds and that α has been chosen as above, then we obtain

$$\|f\|_{\alpha_0,U_\sigma} \leq 2e^{\psi/2}. \qquad (5.3.37)$$

The first hypothesis of the Lemma and (5.3.37) yield

$$\psi(\sigma') \leq (1/\alpha - 1/\alpha_0)\log(C(\sigma-\sigma')^{-\gamma}) + \psi/2 + log2, \quad for < \delta < \sigma' < \sigma \leq 1.$$

By our choice of α, specified above, we have

$$\psi(\sigma') \leq \psi/2(\frac{\log(C(\sigma-\sigma')^{-\gamma})}{\log\psi/2C}+1)+log2.$$

On the one hand, if

$$\psi \geq 2C^3(\sigma-\sigma')^{-2\gamma}, \tag{5.3.38}$$

we have

$$\psi(\sigma') \leq (3/4)\psi + \log 2.$$

On the other hand, if one of the hypotheses (5.3.36), (5.3.38) made on ψ is not satisfied, we have

$$\psi(\sigma') \leq \psi \leq A_1 + 2C^3(\sigma-\sigma')^{-2\gamma}.$$

Thus, in all cases, we obtain

$$\psi(\sigma') \leq \psi \leq A_2 + 2C^3(\sigma-\sigma')^{-2\gamma}, \tag{5.3.39}$$

where A_2 depend only on C and on a lower bound on α_0. For any sequence

$$0 < \delta = \sigma_0 < \sigma_1 < ... < \sigma_i \leq 1,$$

an iteration of (5.3.39) yield

$$\psi(\sigma_0) \leq (3/4)^i\psi(\sigma_i) + A_2\sum_{j=0}^{i}(3/4)^j(\sigma_{j+1}-\sigma_j)^{-2\gamma}$$

and while i tends to infinity, the last inequality becomes

$$\psi(\sigma_0) \leq A_2\sum_{j=0}^{\infty}(3/4)^j(\sigma_{j+1}-\sigma_j)^{-2\gamma}$$

Now, if we set $\sigma = 1-(1+j)^{-1}(1-\delta)$, we have

$$\sum_{j=0}^{\infty}(3/4)^j(\sigma_{j+1}-\sigma_j)^{-2\gamma} \leq C(1-\delta)^{-2\gamma},$$

and the desired bound follows. □

The following Theorem states that positive super-solutions satisfy a weak form of Harnack inequality. For any fixed $\tau > 0$, $\delta \in (0,1)$ and $x \in M$, $s, r > 0$ define

$$\begin{aligned} Q_- &= (s-(3+\delta)r^2/4, s-(3-\delta)r^2/4) \times \mathfrak{B}(x,\delta r) \cap \Omega, \\ Q'_- &= (s-r^2/4, s-(3-\delta)r^2/4) \times \mathfrak{B}(x,\delta r) \cap \Omega, \\ Q_+ &= (s-(1+\delta)r^2/4, s) \times \mathfrak{B}(x,\delta r) \cap \Omega. \end{aligned}$$

Recall also that $Q = Q(x,s,r) = (s-r^2) \times \mathfrak{B}(x,r) \cap \Omega$. Let us now prove a lower bound for positive supersolutions.

Theorem 5.3.3. *Fix $p_0 \in (0, 1+\frac{2}{\nu})$. Then there exists a constant A such that, for $x \in \Omega$, $s \in \mathbf{R}$, $0 < r < c_0$ and any* *positive function supersolution u of problem (5.0.3) in Q, we have*

$$\left(\frac{1}{\mu(Q'_-)}\int_{Q'_-} u^{p_0} d\mu\right)^{1/p_0} \le A \inf_{Q_+}\{u\},$$

where $\mu = \phi^2 dxdt$.

proof: Fix a non-negative super-solution u. Let $c(u,\eta)$ be the constant given by Lemma 5.3.1 applied to u with $\eta = 1/2$. Set $\upsilon = e^c u$. Set also

$$Q_1 = (s - r^2, s - (1/2)r^2) \times \mathfrak{B}(x,r) \cap \Omega,\ \ Q_\sigma = (s - r^2, s - (3-\sigma)r^2/4) \times \mathfrak{B}(x,\sigma r) \cap \Omega.$$

By Theorem 5.2.6 we have

$$\left(\int_{Q_{\sigma'}} u^{p_0} d\mu\right)^{p/p_0} \le \left[\frac{A(p_0,n)}{(\delta'-\delta)^{(2+n)(1+\frac{2}{n})} r^2 V(x,r)}\right]^{1-p/p_0} \int_{Q_\sigma} u^p d\mu,$$

for all $0 < \delta < \sigma' < \sigma < 1$ and $0 < p < p_0(1+\frac{2}{n})^{-1}$. By lemma 5.3.1 we have

$$|\log \upsilon > \lambda| = |\log u > \lambda - c| \le C\mu(Q_1)\lambda^{-1}.$$

Thus we can apply Lemma 5.3.2 to conclude that $\int_{Q_\delta} \upsilon^{p_0} dxdt \le A_1 \mu(Q)$, that is

(5.3.40) $$\left(\frac{1}{\mu(Q'_-)}\int_{Q'_-} (e^c u)^{p_0} d\mu\right)^{1/p_0} \le A_1.$$

Set now $\upsilon = e^{-c}u^{-1}$, where $c = c(u)$ is the same constant as above, given by Lemma 5.3.1, applied to u with $\eta = 1/2$. This time set

$$Q'_1 = (s - (1/2)r^2, s) \times \mathfrak{B}(x,r),\ \ Q'_\sigma = (s - (1+\sigma)r^2/4, s) \times \mathfrak{B}(x,\sigma r).$$

By Theorem 5.2.3 we have

$$\sup_{Q'_{\sigma'}}\{\upsilon^p\} \le \frac{A(p,\nu)}{(\sigma'-\sigma)^{2+n} r^2 V(x,r)} \int_{Q'_\sigma} \upsilon^p d\mu,$$

for all $0 < \delta < \sigma < \sigma' < 1$ and $0 < p < \infty$. By Lemma 5.3.1, we also have

$$\mu(\log \upsilon > \lambda) \le Cm'(Q'_1)\lambda^{-1}.$$

Thus we can apply Lemma 5.3.2 to conclude that $sup_{Q_\delta}\{\upsilon\} \le A_2\mu(Q)$, that is

(5.3.41) $$\sup_{Q_+}\{(e^c u)^{-1}\} \le A_2.$$

Multiplying (5.3.40)and (5.3.41), we obtain

$$\left(\frac{1}{\mu(Q'_-)}\int_{Q'_-} u^{p_0} dxdt\right)^{1/p_0} \le A \inf_{Q_+}\{u\},$$

which is the desired inequality. □

Theorem 5.3.4. *Fix* $0 < \delta < 1$, *then there exists a constant A such that, for* $x \in \Omega,\ s \in \mathbb{R},\ 0 < r < c_0$ *and any positive* *solution v of problem (5.0.3) in $Q = (s - r^2) \times \mathfrak{B}(x, r) \cap \Omega$, we have*

$$\sup_{Q_-}\{v\} \le A \inf_{Q_+}\{v\},$$

where

$$Q_- = (s - (3 + \delta)r^2/4, s - (3 - \delta)r^2/4) \times \mathfrak{B}(x, \delta r) \cap \Omega$$

$$Q_+ = (s - (1 + \delta)r^2/4, s) \times \mathfrak{B}(x, \delta r) \cap \Omega.$$

proof: This follows immediately from Theorems 5.3.3 and 5.2.3. □

Corollary 5.3.5. *Let $R = \frac{C_0}{4\gamma}$ be the constant of Lemma 5.1.2. Let u be a non-negative solution of $(\partial_t + L_\phi)u = 0$ in $(0, T) \times \Omega$, $T > 0$. Then there exist constant A such that the following estimate is valid for all x, $y \in \Omega$ and all $0 < s < t < T$.*

$$\log \frac{u(s, x)}{u(t, y)} \le A\Big(1 + \frac{t - s}{R^2} + \frac{t - s}{s} + \frac{|x - y|^2}{t - s}\Big).$$

proof: Now, by our assumption on Ω we can assume that there exist a length curve $\gamma : [a, b] \to \Omega$, such that, $\gamma(a) = y$, $\gamma(b) = x$ and $||\gamma|| \le C_0|x - y|$ where C_0 depends on diameter of Ω. Then

$$\phi_1(x) - \phi_1(y) = \int_a^b \frac{d}{dt}(\phi_1(\gamma(t)))dt = \int_a^b \nabla\phi_1(\gamma(t))\dot{\gamma}(t)dt$$

$$\le \int_a^b |\dot{\gamma}(t)|dt \le C_0|x - y|.$$

Connect the points x y by balls $\mathfrak{B}_0, ..., \mathfrak{B}_{k-1}$ of radius $\frac{r}{2}$ and centers $x_0, .., x_{k-1}$ with $x_0, ..., x_{k-1} \in \gamma$ and $x_{i+1} \in \overline{\mathfrak{B}_i}$, $0 \le i \le k - 1$ with $x_k = y$. This is possible as soon as,

$$kr \ge 2|\gamma|. \tag{5.3.42}$$

The values of r and k are to be chosen later. Let $t_0 = s$. $t_i = s + r^2 i$, $0 \le i \le k$. Now choose r to satisfy the following conditions:
(i) $r^2 = (t - s)/k$ so that $t_k = t$. Note that this implies $t - s \ge r^2$.

(ii) $r \le R$ and $r^2 \le s$ so that u is a solution $(\partial_t + L_\phi)u = 0$ in each of the cylinder $(t_i - r^2, t_{i+1}) \times 2\mathfrak{B}_i$ $0 \le i \le k - 1$.

Then, applying Theorem 5.3.4 successively in $(t_i - r^2, t_i) \times 2\mathfrak{B}_i$, $0 \le i \le k - 1$, we obtain

$$u(t_0, x_0) \le A_0 u(t_1, x_1) \le A_0^2 u(t_2, x_2) \le ... \le A_0^k u(t_k, x_k),$$

that is,

$$u(s, x) \le A_0^k u(t, y).$$

Now, (5.3.42) is satisfied if $k \ge d^2/(t - s)$ because $kr = \sqrt{k(t - s)}$ by (i). Similarly, (ii) is satisfied as soon as

$k \geq (t-s)\max\{1/R^2, 1/s\}$. Thus, we can choose k of order

$$1 + \frac{t-s}{R^2} + \frac{t-s}{s} + \frac{d^2}{t-s}.$$

This gives the desired inequality. □

Definition 5.3.6. *We will say that* $v \in C^1((s-r^2, r) : H^1(\mathfrak{B}(x,r) \cap \Omega))$ *is a weak solution of (5.0.1) if for each* $\Phi \in C_0^1((s-r^2, r) : C_0^\infty(\mathfrak{B}(x,r) \cap \Omega))$, *for each* $s - r^2 < t_1 < t_2 < s$ *we have*

$$\int_{t_1}^{t_2} \int_{\mathfrak{B}(x,r)\cap\Omega} v_t\Phi + \nabla v \nabla \Phi + \lambda_1 \frac{v\Phi}{4d^2} dxdt = 0.$$

Corollary 5.3.7. *Fix* $0 < \delta < 1$, *then there exists a constant A such that, for* $x \in \Omega^c$, $s \in \mathbb{R}$, $0 < r < c_0$ *and any positive solution u of problem (5.0.1) in* $Q = (s - r^2) \times \mathfrak{B}(x,r) \cap \Omega$, *we have*

$$\sup_{Q_-}\{\frac{u}{\phi}\} \leq A \inf_{Q_+}\{\frac{u}{\phi}\},$$

where

$$Q_- = (s - (3+\delta)r^2/4, s - (3-\delta)r^2/4) \times \mathfrak{B}(x, \delta r) \cap \Omega$$

$$Q_+ = (s - (1+\delta)r^2/4, s) \times \mathfrak{B}(x, \delta r) \cap \Omega.$$

proof: If we set $u = \phi v$, then we note that v is a non-negative weak solution of problem (5.0.3). Thus by Theorem 5.3.4 we have

$$\sup_{Q_-}\{v\} \leq A \inf_{Q_+}\{v\}$$

and the result follows. □

5.4 Localized Heat Kernel Bounds

We recall the first eigenvalue of the problem

$$-\infty < \lambda_1 = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}}{\int_\Omega \frac{u^2}{1+d^{2+\sigma}}},$$

where $\sigma > 0$. Also, we recall the ground state function ϕ in the introduction of this chapter. Let $h_\phi(t,x,y)$ be the respective heat kernel of the following problem

$$(5.4.43) \qquad v_t = -(-L_\phi v) = \frac{div(\phi^2 \nabla v)}{\phi^2} - \lambda_1 \frac{v}{1 + d^{2+\sigma}}, \quad in \ (0,T] \times \Omega.$$

We note here that if $\lambda_1 < 0$, then we set $\widetilde{h_\phi}(t,x,y) = e^{\lambda_1 t} h_\phi(t,x,y)$. Then $\widetilde{h_\phi}(t,x,y)$ is the heat kernel of the problem

$$(5.4.44) \qquad v_t = \frac{div(\phi^2 \nabla v)}{\phi^2} - \lambda_1 \frac{v}{1 + d^{2+\sigma}} + \lambda_1 v, \quad in \ (0,T] \times \Omega,$$

i.e. $\widetilde{h_\phi}(t,x,y)$ is positivity-preserving for all $0 \leq t < \infty$. Thus without loss of generality we assume that $\lambda_1 > 0$.
In this section we prove the two side estimates for the heat kernel h_ϕ.

5.4.1 Localized Heat Kernel Upper Bounds

For any function $\phi_1 \in C_0^\infty(\Omega)$ with $\|\nabla\phi_1\|_\infty \le 1$ and any complex number a, consider the semigroup defined by

$$H_t^{a,\phi_1} f(x) = e^{-a\phi_1(x)} \int h_\phi(t,x,y) e^{a\phi_1(y)} f(x) dm = e^{-a\phi_1(x)} H_t(e^{a\phi_1} f)(x).$$

It is clear that this is a well-defined semigroup of operators on the spaces $L^p(\Omega, m)$. Its infinitesimal generator is given by

$$-A_{a,\phi_1} f = -e^{-a\phi_1} \delta(e^{a\phi_1} f).$$

When a is real this semigroup preserves positivity but there is no reason that it contracts $L^p(\Omega, m)$, for any $1 \le p \le \infty$. It is not self-adjoint but its adjoint simply H_t^{-a,ϕ_1}. The next lemma estimates the norm of this semigroups on $L^2(\Omega, m)$.

Lemma 5.4.1. *For any function $\phi_1 \in C_0^\infty(\mathbb{R}^n)$ with $\|\nabla\phi_1\|_\infty \le 1$ and any real number a, the semigroup $(H_t^{a,\phi_1})_{t>0}$ satisfies*

$$\forall t > 0, \quad \|H_t^{a,\phi_1}\|_{2\to 2} \le e^{a^2 t}.$$

proof: Set $u(t) = \|H_t^{a,\phi_1} f\|_2^2,\ f \in L^2(\Omega, m)$. Then u has derivative

$$u'(t) = -2 < A_{a,\phi} H_t^{a,\phi_1} f, H_t^{a,\phi_1} f > .$$

Thus, it suffices to show that

$$< A_{a,\phi_1} f, f > \ge -a^2 \|f\|_2^2, \tag{5.4.45}$$

for all $f \in C_0^\infty(\Omega)$. If this holds, we have $u' \le a^2 u$, that is $u(t) \le e^{a^2 t} u(0) = e^{a^2 t}\|f\|_2^2$, the desired inequality. To prove (5.4.45), write

$$\begin{aligned} < A_{a,\phi_1} f, f > = < -e^{-a\phi_1} L_\phi(e^{a\phi_1} f), f > &= \int_\Omega \nabla(e^{a\phi} f)\nabla(e^{-a\phi} f) dm + \lambda_1 \int_\Omega \frac{f^2}{1+d^{2+\sigma}} dm \\ &\ge \int_\Omega |\nabla f|^2 dm - a^2 \int_\Omega |\nabla\phi|^2 |f|^2 dm \ge -a^2 \int_\Omega |f|^2 dm. \end{aligned}$$

Where we have used that $\lambda_1 \ge 0$ and the fact that $|\nabla\phi| \le 1$. This proves the Lemma. □

Theorem 5.4.2. *There exists a constant A such that, for any $\varepsilon \in (0,1)$, for $\gamma \in (1,2)$ and any two balls $\mathfrak{B}_1 = \mathfrak{B}(x, r_1),\ \mathfrak{B}_2 = \mathfrak{B}(y, r_2)$ (see Definition 5.1.1), we have*

$$h_\phi(t,x,y) \le \frac{C}{[V(x,r_1)V(x,r_2)]^{1/2}} \exp\Big(-\frac{|x-y|^2}{4t} + \varepsilon(\gamma+2)\frac{|x-y|}{\sqrt{t}}\Big),$$

for all $t \ge \varepsilon^{-2}\max\{r_1^2, r_2^2\}$.

proof: Let $(H_t^{a,\phi_1})_{t>0}$ be defined as above. By Lemma (5.4.1) we have

$$\|H_t^{a,\phi_1}\|_{2\to 2} \le e^{a^2 t}.$$

Fix $x, y \in \Omega$ and $r_1,\ r_2 > 0$ and χ_1 (resp χ_2) be the function equal to 1 on $\mathfrak{B}_1 = \mathfrak{B}(x, r_1)$ (resp $\mathfrak{B}_2 = \mathfrak{B}(y, r_2)$) and equal

0 otherwise. Then

$$\int\int_{(\xi,\zeta)\in\mathfrak{B}_1\times\mathfrak{B}_2} h(t,\xi,\zeta)e^{-a(\phi_1(\xi)-\phi_1(\zeta))}dm(\xi)dm(\zeta) = <\chi_1, H_t^{a,\phi_1}\chi_2>$$
$$\leq \|H_t^{a,\phi_1}\|_{2\to 2}\|\chi_1\|_2\|\chi_2\|_2 \leq e^{a^2t}V^{\frac{1}{2}}(x,r_1)V^{\frac{1}{2}}(y,r_2).$$

Now, by our assumption on Ω we can assume that there exist a length curve $\gamma : [a,b] \to \Omega$, such that, $\gamma(a) = y,\ \gamma(b) = x$ and $\|\gamma\| \leq C_0|x-y|$ where C_0 depends on diameter of Ω. Then

$$\begin{aligned}\phi_1(x) - \phi_1(y) = \int_a^b \frac{d}{dt}(\phi_1(\gamma(t)))dt &= \int_a^b \nabla\phi_1(\gamma(t))\dot{\gamma}(t)dt \\ &\leq \int_a^b |\dot{\gamma}(t)|dt \leq C_0|x-y|.\end{aligned}$$

Thus,

$$\begin{aligned}&\int\int_{(\xi,\zeta)\in\mathfrak{B}_1\times\mathfrak{B}_2} h_\phi(t,\xi,\zeta)dm(\xi)dm(\zeta)\\ &= \int\int_{\mathfrak{B}_1\times\mathfrak{B}_2} h_\phi(t,\xi,\zeta)e^{-a(\phi(\xi)-\phi(\zeta))}e^{a(\phi_1(\xi)-\phi_1(\zeta))}e^{-a(\phi_1(x)-\phi_1(y))}e^{a(\phi_1(x)-\phi_1(y))}dm(\xi)dm(\zeta)\\ &\leq V^{\frac{1}{2}}(x,r_1)V^{\frac{1}{2}}(y,r_2)\exp(a^2t + a(\phi_1(x)-\phi_1(y)) + (\gamma+2)|a|(r_1+r_2)),\end{aligned} \tag{5.4.46}$$

since $d(x,\xi) \leq (\gamma+2)r_1$ and $d(y,\zeta) \leq (\gamma+2)r_2$ (see Definition 5.1.1). Without loss of generality we assume that $r_2 \geq r_1$. As $u(s,\zeta) \to h_\phi(s,\xi,\zeta)$ is a positive solution of $(\partial_t + L_\phi)u = 0$ in $(0,\infty)\times\Omega$, assuming that $t \geq r_2^2$ and applying Theorem (5.2.3) with $p = 1$, we obtain

$$\sup_{s\in(t-\frac{r_1^2}{4},t)} (h_\phi(s,\xi,y)) \leq \frac{C}{r_2^2V(y,r_2)}\int_{t-r_2}^t\int_{\mathfrak{B}_2} h_\phi(s,\xi,\zeta)dm(\zeta)ds.$$

Thus, by the above inequality and (5.4.46) we have

$$\int_{\mathfrak{B}_1}\sup_{s\in(t-\frac{r_1^2}{4},t)}(h_\phi(s,\xi,y))dm(\xi) \leq \frac{CV^{\frac{1}{2}}(x,r_1)}{V^{\frac{1}{2}}(y,r_2)}\exp(a^2t + a(\phi_1(x)-\phi_1(y)) + (\gamma+2)(\gamma+2)|a|(r_1+r_2)). \tag{5.4.47}$$

By the same token, working with the variable ξ and assuming $t \geq r_1^2$, we get

$$\begin{aligned}h_\phi(t,x,y) &\leq \frac{C}{r_1^2V(x,r_1)}\int_{t-r_1^2/4}^t\int_{\mathfrak{B}_1} h_\phi(s,\xi,y)dm(\xi)ds\\ &\leq \frac{C}{V^{\frac{1}{2}}(x,r_1)V^{\frac{1}{2}}(y,r_2)}\exp(a^2t + a(\phi_1(x)-\phi_1(y)) + (\gamma+2)|a|(r_1+r_2)),\end{aligned}$$

where we have used (5.4.47). Taking $a = -\frac{(\phi(x)-\phi(y))}{2t}$ and assuming $t \geq \varepsilon^{-2}\max\{r_1^2, r_2^2\}$ we obtain

$$h_\phi(t,x,y) \leq \frac{C}{[V(x,r_1)V(y,r_2)]^{\frac{1}{2}}}\exp\Big(-\frac{|\phi_1(x)-\phi_1(y)|^2}{4t} + \varepsilon(\gamma+2)\frac{|\phi_1(x)-\phi_1(y)|}{\sqrt{t}}\Big).$$

Taking (as we may) a sequence of $\phi_i \in C_0^\infty(\mathbb{R}^n)$ with $|\nabla\phi_i| \leq 1$ and

$$\phi_i(x) - \phi_i(y) \to |x-y|,$$

finally gives

$$h_\phi(t,x,y) \le \frac{C}{[V(x,r_1)V(y,r_2)]^{\frac{1}{2}}} \exp\Big(-\frac{|x-y|^2}{4t} + \varepsilon(\gamma+2)\frac{|x-y|}{\sqrt{t}}\Big),$$

which is the desired result. □

Corollary 5.4.3. *Let $R = \frac{C_0}{4\gamma}$ be the constant of Lemma 5.1.2. Then there exist constant A such that the following upper bound for is valid for all $x,\ y \in \Omega$ and all $0 < t < R^2$.*

$$h_\phi(t,x,y) \le \frac{A}{[V(x, \frac{t}{\sqrt{t}+|x-y|})V(y, \frac{t}{\sqrt{t}+|x-y|})]^{1/2}} \exp\Big(-\frac{|x-y|^2}{4t}\Big)$$

proof: This follows from applying Theorem 5.4.2 with $\mathfrak{B}_1 = \mathfrak{B}(x,r_1),\ \mathfrak{B}_2 = \mathfrak{B}(y,r_2),\ r_1 = r_2 = \varepsilon\sqrt{t},\ \varepsilon = (\frac{\sqrt{t}}{1+|x-y|})$. □

By Lemma (5.1.2) we have

$$\frac{V(x,\ \sqrt{t})}{V(x, \frac{t}{\sqrt{t}+|x-y|})} \le \Big(\frac{\sqrt{t}+|x-y|}{\sqrt{t}}\Big)^{n+1},$$

thus, we can deduce from the bound above a slightly less precise but nicer looking estimate, namely, for all $x,\ y \in \Omega,\ 0 < t < R^2$.

$$h_\phi(t,x,y) \le A\frac{(\frac{\sqrt{t}+|x-y|}{\sqrt{t}})^{n+1}}{[V(x,\ \sqrt{t})V(y,\ \sqrt{t})]^{\frac{1}{2}}} \exp\Big(-\frac{|x-y|^2}{4t}\Big).$$

Furthermore, note

$$\Big(\frac{\sqrt{t}+|x-y|}{\sqrt{t}}\Big)^{n+1} \le C_n \exp\Big(c_n\frac{|x-y|}{\sqrt{t}}\Big) \le C(n,\varepsilon)\exp\Big(\varepsilon\frac{|x-y|^2}{t}\Big),$$

for all $\varepsilon > 0$. Thus we have

(5.4.48) $$h_\phi(t,x,y) \le \frac{A}{[V(x,\ \sqrt{t})V(y,\ \sqrt{t})]^{\frac{1}{2}}} \exp\Big(-C\frac{|x-y|^2}{4t}\Big).$$

5.4.2 Heat Kernel Lower Bounds

The Harnack inequalities of section (4.3) easily yield heat kernel lower bounds. First, we have the following on-diagonal bound.

Theorem 5.4.4. *Let $R = \frac{C_0}{4\gamma}$ be the constant of Lemma 5.1.2. Then there exist constant A such that the following lower bound is valid for all $x \in \Omega$ and all $0 < t < R^2$.*

$$h_\phi(t,x,x) \ge \frac{c}{V(x,\ \sqrt{t})}.$$

proof: Fix $0 < t < R^2$. Let $\mathfrak{B} = \mathfrak{B}(x,\ \sqrt{t}) \cap \Omega$. Let ζ be a smooth function such that $0 \le \zeta \le 1,\ \zeta = 1$ in $\mathfrak{B}(x, \frac{\sqrt{t}}{4})$ and $\zeta = 0$ in $\mathbb{R}^n \setminus \mathfrak{B}(x, \frac{\sqrt{t}}{2})$. Also let η be a non-negative solution of

$$-\mathrm{div}(\phi^2\nabla\eta) + \lambda_1\frac{\eta\phi^2}{1+d^{2+\sigma}} = 0 \quad \text{in } \mathfrak{B}(x, \tfrac{\sqrt{t}}{2}) \cap \Omega$$

$$\eta = 1 \qquad \text{on } \partial(\mathfrak{B}(x, \tfrac{\sqrt{t}}{2}) \cap \Omega)$$

By Harnack inequality we have for any $y \in \mathfrak{B}(x, \frac{\sqrt{t}}{2}) \cap \Omega$

$$\frac{1}{C}\eta(y) \leq \eta(x) \leq C\eta(y),$$

letting y to go to a point of the boundary we have

$$\frac{1}{C} \leq \eta(x) \leq C.$$

Define

$$u(t,y) = \begin{cases} H_t\Phi(y), & \text{if } t > 0 \\ \phi & \text{, if } t \leq 0 \end{cases},$$

where $\Phi = \zeta\eta$. Obviously, this function satisfies

$$(\partial_t + L_\phi)u = 0,$$

on $(-\infty, \infty) \times \mathfrak{B}(x, \frac{\sqrt{t}}{4})$. Applying Corollary (5.3.5), first to u and then to the heat kernel $(s, y) \mapsto h(s, x, y)$, we get

$$\begin{aligned} \frac{1}{C} &\leq u(0,x) \leq Au(t/4,x) = A\int_{\mathfrak{B}(x,\frac{\sqrt{t}}{2})\cap\Omega} h_\phi(t/4,x,y)\Phi(y)dm(y) \\ &\leq CA^2\int_{\mathfrak{B}(x,\sqrt{t})} h_\phi(t,x,x)dm(y) \leq CA^2V(x,\sqrt{t})h_\phi(t,x,x). \end{aligned}$$

This gives

$$h_\phi(t,x,x) \geq C^{-2}A^{-2}V(x,\sqrt{t})^{-1},$$

as desired. □

Theorem 5.4.5. *Let $R = \frac{C_0}{4\gamma}$ be the constant of Lemma 5.1.2. Then there exist constant A such that for all $x,\ y \in \Omega$ and all $0 < t < R^2$ the heat kernel $h_\phi(x,t,y)$ satisfies*

$$h_\phi(t,x,y) \geq \frac{a}{V(x,\sqrt{t})}\exp\Big(-A\frac{|x-y|^2}{t}\Big).$$

proof: Apply Corollary (5.3.5) to $u(s,y) = h(s,x,y)$ with x fixed and $s = \frac{t}{4}$. This gives

$$h_\phi(t,x,y) \geq Ah_\phi(\frac{t}{4},x,y)\exp\Big(-A\frac{|x-y|^2}{t}\Big).$$

The result follows by Theorem (5.4.4) and Lemma 5.1.2. □

Consider now the heat kernel $h(t,x,y)$ of $u_t = \Delta u + \frac{u}{4d^2}$. Then we have the following theorem

Theorem 5.4.6. *Let $R = \frac{C_0}{4\gamma}$ be the constant of Lemma 5.1.2. Then there exist constant A such that for all $x,\ y \in \Omega$ and all $0 < t < R^2$ the heat kernel $h(x,t,y)$ satisfies*

$$\begin{aligned} &C_1\Big[\min\Big(\frac{d(x)}{\sqrt{t}},1\Big)\min\Big(\frac{d(y)}{\sqrt{t}},1\Big)\Big]^{\frac{1}{2}} t^{-\frac{n}{2}}\exp\Big(-A_1\frac{|x-y|^2}{t}\Big) \\ &\leq h(t,x,y) \leq C_2\Big[\min\Big(\frac{d(x)}{\sqrt{t}},1\Big)\min\Big(\frac{d(y)}{\sqrt{t}},1\Big)\Big]^{\frac{1}{2}} t^{-\frac{n}{2}}\exp\Big(-A_2\frac{|x-y|^2}{t}\Big). \end{aligned}$$

proof: We note here that $h(t,x,y) = \phi(x)\phi(y)h_\phi(t,x,y)$. For $h_\phi(t,x,y)$ we have the following estimate

$$\frac{a_1}{V^{\frac{1}{2}}(x,\ \sqrt{t})V^{\frac{1}{2}}(y,\ \sqrt{t})}\exp\Big(-A_1\frac{|x-y|^2}{t}\Big) \le h_\phi(t,x,y) \le \frac{a_2}{V^{\frac{1}{2}}(x,\ \sqrt{t})V^{\frac{1}{2}}(y,\ \sqrt{t})}\exp\Big(-A_2\frac{|x-y|^2}{t}\Big).$$

We also have

$$C_1\frac{d(x)}{|x|^{2a_n+1}} \le \phi^2(x) \le C_2\frac{d(x)}{|x|^{2a_n+1}},$$

where $a_n = \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - \frac{1}{4}}$. Using now the Lemma 5.1.2 we have

$$C_1\min\Big(\frac{d(x)}{\sqrt{t}}, 1\Big)t^{\frac{n}{2}} \le \frac{\phi^2(x)}{V(x,\ \sqrt{t})} \le C_2\min\Big(\frac{d(x)}{\sqrt{t}}, 1\Big)t^{\frac{n}{2}}.$$

Thus combining all above we have the desired result. □

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