On Existence of Solutions to Semilinear Equations : Lack of Classical Compactness

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1 Introduction and Main Results

In this work, we are concerned with the problem of existence of a function u satisfying the semilinear elliptic equation

$$\begin{cases} -\triangle u = u^{2^*-1} + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1}$$

where $n \geq 3$, $2^* = \dfrac{2n}{n-2}$ and $\Omega \subset \mathbb{R}^n$ is a bounded open subspace of $\mathbb{R}^n$ with C^2 boundary.

This equation emerges from a wide range of variational problems, the most famous of which is Yamabe's problem:

$$\begin{cases} -4\dfrac{n-1}{n-2}\triangle u = R'(x)u^{\frac{n+2}{n-2}} - R(x)u & \text{on } M, \\ u > 0 & \text{on } M. \end{cases}$$

where M is a n-dimensional Riemannian manifold and $R(x)$ is the scalar curvature of M.

We investigate our problem by studying the functional

$$Q_\lambda(u) := \frac{\displaystyle\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u^2 dx}{\left(\displaystyle\int_\Omega |u|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}} \tag{2}$$

and its infimum

$$J_\lambda = \inf_{u \in C_c^\infty(\Omega)} Q_\lambda(u) \tag{3}$$

on an appropriate function space. As a matter of fact, every minimizer u of Q_λ satisfies the Euler Lagrange relation

$$\int_\Omega \nabla u \cdot \nabla v dx = \mu \int_\Omega u^{\frac{n+2}{n-2}} v dx + \lambda \int_\Omega uv dx, \forall v \in C_c^\infty(\Omega)$$

for a positive Lagrange multiplier μ

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We start by defining the principal eigenvalue λ_1 of the Laplace operator $-\triangle$ in Ω with Dirichlet boundary conditions

$$0 < \lambda_1 = \lambda_1(\Omega) = \inf_{\phi \in C_c^\infty(\Omega)} \frac{\displaystyle\int_\Omega |\nabla \phi|^2 dx}{\displaystyle\int_\Omega \phi^2 dx}$$

The principal eigenvalue λ_1 along with its corresponding eigenfunction $\phi_1 \in C^\infty(\Omega)$ solve the linear equation

$$\begin{cases} -\triangle \phi_1 = \lambda_1 \phi_1 & \text{in } \Omega \\ \phi_1|_{\partial\Omega} = 0 \end{cases}$$

Also, ϕ_1 can be chosen to be strictly positive in Ω.

Related to problem (1) is the Sobolev Inequality. The Sobolev Inequality in all of $\mathbb{R}^n$ for $n \geq 3$ reads:

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq S_n \left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} , u \in C_c^\infty(\mathbb{R}^n) \tag{4}$$

where the sharp Sobolev constant S_n is given by

$$S_n = \pi n(n-2) \left(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \right)^{\frac{2}{n}}$$

The Sobolev Inequality has as a minimizer in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ the function $U(x) = (1+|x|^2)^{-\frac{n-2}{2}}$ as well as translates and scaled versions of it. One can also show by an approximation argument that the Sobolev Inequality holds in bounded subsets of $\mathbb{R}^n$.The Sobolev Inequality has deep connections with other important analytic and geometric results, such as the Isoperimetric Inequality. For a deeper insight into this matter, see [2].

At this point, our first result, due to E. Lieb, H. Brezis and L. Nirenberg (see [1]) is the following:

Theorem 1 *(Lieb - Brezis - Nirenberg) If $J_\lambda < S_n$, then Q_λ has a minimizer and problem*

$$\begin{cases} -\triangle u = u^{2^*-1} + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

has a solution.

After establishing this result, we will investigate whether $J_\lambda < S_n$ and for this we consider $\lambda \in (0, \lambda_1)$. It turns out that the dimension plays an important role. More specifically, when $n \geq 4$ we have the following theorem:

Theorem 2 *(Existence of Solutions for $n \geq 4$) If $n \geq 4$ and $0 < \lambda < \lambda_1$, then $0 < J_\lambda < S_n$.*

However, when $n = 3$, we only have a full result when Ω is a ball, lets say the unit ball B_1. In this case, $\lambda_1 = \pi^2$ and we have:

Theorem 3 *Let $n = 3$ and $\Omega = B_1$. Then:*

1. *If $\frac{1}{4}\pi^2 < \lambda < \pi^2$, then $0 < J_\lambda < S_3$ and problem (1) has a solution.*

2. *If $0 < \lambda \leq \frac{1}{4}\pi^2$, then*

$$\int_{B_1} |\nabla u|^2 dx \geq \frac{1}{4}\pi^2 \int_{B_1} u^2 dx + S_3 \left(\int_{B_1} u^6 dx \right)^{\frac{1}{3}}, \forall u \in C_c^\infty(B_1)$$

and furthermore problem (1) has no solution.

The next natural question is what happens when $n = 3$ and Ω is not a ball. In this case, there is a theorem of R. Schoen(see [4]) who takes under account the Green function $G_\lambda(x,y)$ of the operator $-\triangle - \lambda$. In particular we write $G_\lambda(x,y) = \dfrac{1}{4\pi|x-y|} + g_\lambda(x,y)$, where the continuous function g_λ is the regular part of the Green function. We now have:

Theorem 4 *(R. Schoen) Let Ω be a bounded domain in $\mathbb{R}^3$ and $\lambda \in (0, \lambda_1)$. If $g_\lambda(x,x) > 0$ for some $x \in \Omega$ then $J_\lambda < S_3$.*

By an analysis of J_λ in $(0, \lambda_1)$ we can see that $J_\lambda = S_3$ for "small" values of λ whereas $J_\lambda < S_3$ for λ sufficiently close to λ_1. It is therefore natural to define

$$\lambda^* = sup\{\lambda \in (0, \lambda_1) | J_\lambda = S_3\}$$

When $\lambda^* < \lambda < \lambda_1$ we have solutions to problem (1) (since $J_\lambda < S_3$). On the other hand, when $0 < \lambda < \lambda^*$ we have nonexistence of a solution. Actually this is the case even when $\lambda = \lambda^*$ as it has been established by O. Druet (see [3]):

Theorem 5 *(O.Druet) If $0 < \lambda \leq \lambda^*$ then Q_λ has no minimizer and problem (1) has no solution.*

In view of Schoen's theorem, this means that $\lambda^* = \inf\{\lambda \in (0, \lambda_1) | g_\lambda(x,x) > 0 \text{ for some } x \in \Omega \}$

While Schoen's result provides a criterion for the solvability of equation (1), it is impossible to express g_λ explicitly for a general bounded domain Ω. However, we will give a criterion for general domains of $\mathbb{R}^3$ via a more direct approach. More specifically, we define

$$\mu^* = \mu^*(\Omega) = \inf_{y \in \Omega} \inf_{u \in C_c^\infty(\Omega)} \frac{\int_\Omega \dfrac{|\nabla u|^2}{|x-y|^2} dx}{\int_\Omega \dfrac{u^2}{|x-y|^2} dx}$$

One can then establish that $0 < \mu^* < \lambda_1$. We next have

Theorem 6 *Let Ω be a bounded domain of $\mathbb{R}^3$. If*

$$\mu^* < \lambda < \lambda_1$$

then $J_\lambda < S_3$ and problem (1) has a solution.

One can actually compare λ^* with μ^* and have $\lambda^* \leq \mu^*$.

<u>OPEN QUESTION:</u> *Is it true that $\lambda^* = \mu^*$?*

In the second section, we will give some basic definitions, and we will establish our main method of looking for solutions of (1). In the third section, we will establish some important non-existence Lemmas. In the fourth section, we will establish our main theorems 1,2 and 3. In the fifth section, we will restrict ourselves to the case $n = 3$ and we will give the proof of theorem 5. Finally, in the sixth section we will present our method in order to prove our theorem 6.

2 The Variational Problem

In this section, we want to establish a way of searching for solutions to the problem

$$\begin{cases} -\triangle u = u^p + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{5}$$

where $1 < p \leq \dfrac{n+2}{n-2}$.

We start by making some definitions

For $q \geq 1$, we define the space of q-integrable functions in Ω :

$$L^q(\Omega) = \{u : \Omega \to \mathbb{R} | \int_\Omega |u|^q dx < +\infty\}$$

and also $L^q_{loc}(\Omega) = \{u : \Omega \to \mathbb{R} | \int_K |u|^q dx < +\infty, \forall \text{ compact } K \subset\subset \Omega\}$

We now turn to the idea of the weak derivative: Let $u \in L^1_{loc}(\Omega)$ and $x = (x_1, ..., x_n) \in \Omega$. We say that the function v_i is the first order weak derivative of u with respect to x_i, $i = 1, ..., n$ if

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} dx = -\int_\Omega v_i \phi dx$$

for every test function ϕ. If such a function exists, we write $\dfrac{\partial u}{\partial x_i} = v_i$ for $i = 1, ..., n$. If all the first order weak derivatives of u exist, we define the weak gradient of u: $\nabla u(x) = (\dfrac{\partial u(x)}{\partial x_1}, ..., \dfrac{\partial u(x)}{\partial x_n})$. From their definition, we see that the weak derivatives of u, if they exist, are a.e. uniquely defined.

Now, from the definition of the principal eigenvalue λ_1, we get the Poincare Inequality:

$$\int_\Omega |\nabla u|^2 dx \geq \lambda_1 \int_\Omega u^2 dx, \forall u \in C_c^\infty(\Omega) \tag{6}$$

Using this inequality, we see that $||u||_{H_0^1} = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$ is a norm of $C_c^\infty(\Omega)$. We now define:

$$H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}^{||.||_{H_0^1}} \tag{7}$$

We can see that $H_0^1(\Omega)$ is a Hilbert space. For a deeper insight in these matters, see Chapter 5 of L.Evans PDEs (2010)

From the Sobolev Inequality, we have the continuous embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for every exponent $1 \leq q \leq 2^*$. Furthermore, by the Rellich Kondrachov Selection Theorem (see e.g. p. 290 of L.Evans PDE), the embedding $H_0^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ is compact $\forall 1 \leq q < 2^*$. This means that $H_0^1(\Omega)$ is a precompact subset of $L^q(\Omega)$ for every $q \in [1, 2^*)$.

However, the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is continuous, but *not* compact. This can be shown by an appropriate example (e.g. we set $n = 3$, $\Omega = B_1$ and $u_\varepsilon(x) = \varepsilon^{\frac{1}{4}} C_\varepsilon (\varepsilon + |x|^2)^{-\frac{1}{2}}$, where $\varepsilon > 0$ and C_ε is an appropriate bounded positive constant. Then as $\varepsilon \to 0+$ one can show that $(u_\varepsilon)_{\varepsilon > 0}$ is uniformly bounded in

$H_0^1(\Omega)$ but has no strongly convergent subsequence in $L^6(\Omega)$). This is why the exponent 2^* is called "critical exponent".

We now introduce the following variational problem: *Let* $1 < p \leq \dfrac{n+2}{n-2}$ *and let*

$$I_\lambda(u) := \frac{\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u^2 dx}{\left(\int_\Omega |u|^{p+1} dx\right)^{\frac{2}{p+1}}} \tag{8}$$

and

$$M_\lambda = \inf_{u \in H_0^1(\Omega)} I_\lambda(u) \tag{9}$$

Note here that $\lambda \mapsto M_\lambda$ is a continuous and non-increasing mapping. Firstly, we will show that $M_\lambda > 0$ for $\lambda < \lambda_1$

To see this, let $\lambda < \lambda_1$ and from the Poincare and Sobolev Inequalities,

$$\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u^2 dx \geq (1 - \frac{\lambda}{\lambda_1}) \int_\Omega |\nabla u|^2 dx \geq C(1 - \frac{\lambda}{\lambda_1}) \left(\int_\Omega |u|^{p+1} dx\right)^{\frac{2}{p+1}}$$

for every $u \in H_0^1(\Omega)$, where $C = C(n, \Omega) > 0$ is the best constant of the inequality

$$\int_\Omega |\nabla u|^2 dx \geq C \left(\int_\Omega |u|^{p+1} dx\right)^{\frac{2}{p+1}}, 1 < p \leq \frac{n+2}{n-2}$$

This means that $I_\lambda(u) \geq C(1 - \frac{\lambda}{\lambda_1}), \forall u \in H_0^1(\Omega) \Rightarrow M_\lambda > 0$. We also get as a byproduct that the operator $-\triangle - \lambda$ is coercive for $\lambda < \lambda_1$.

We now establish that any minimizer of I_λ corresponds to a weak solution of (5)

Lemma 1 *Let* $\lambda < \lambda_1$ *and* u *be an* $H_0^1(\Omega)$ *minimizer of* I_λ. *Then,* u *is a weak solution of problem (5).*

Proof Let $u \in H_0^1(\Omega)$ be a minimizer of I_λ . Note that we can assume that $u \geq 0$ a.e. (orelse we substitute |u| for u). We define the real function

$$f(t) = I_\lambda(u + t\phi), t \in \mathbb{R}, \phi \in H_0^1(\Omega)$$

Then, f is differentiable $\forall t \in \mathbb{R}$. Since u is a minimizer of the functional I_λ (i.e. $I_\lambda(u) = M_\lambda$), this means that $t = 0$ is a minimum point of the function f. Therefore $f'(0) = 0$. Now, $0 = f'(0) = \frac{d}{dt} I_\lambda(u + t\phi)|_{t=0}$ and after computing this leads to

$$\int_\Omega \nabla u \cdot \nabla \phi dx = M_\lambda \int_\Omega u^p \phi dx + \lambda \int_\Omega u \phi dx, \forall \phi \in H_0^1(\Omega)$$

After setting $u \mapsto ku$ and choosing the appropriate $k > 0$ (we keep the symbol u) this means that

$$\int_\Omega \nabla u \cdot \nabla \phi dx = \int_\Omega u^p \phi dx + \lambda \int_\Omega u \phi dx, \forall \phi \in H_0^1(\Omega) \tag{10}$$

But this means that u is a weak solution to the Euler Lagrange equation $-\triangle u = u^p + \lambda u$ a.e. in Ω □

Actually, there is much more that can be said about the regularity of the weak solution u: *If* $\partial\Omega \in C^2$, *then* $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$.

In order to establish regularity of an H_0^1 weak solution u of (5), we have to make two make two steps: Firstly, we examine the interior regularity of u (that is regularity on compact subsets of Ω) and then we move to the boundary regularity, where we carefully use the assumption that $\partial\Omega$ is C^2. The exact details of this methods are beyond the scope of this work and will therefore be omitted. One can look e.g. pages 326-344 of L. Evans PDEs (2010) for a detailed account.

We now have:

Proposition 2 *Let $\lambda < \lambda_1$ and $u \in H_0^1(\Omega)$ be a minimizer of I_λ. Then, up to a constant, u is a classical solution of (5).*

Proof Let $u \in H_0^1(\Omega)$ be a minimizer of I_λ, and thus a weak solution to (5) . Since $\partial\Omega$ is assumed to be C^2, $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Since we can assume that $u \geq 0$ a.e., by continuity $u \geq 0$ everywhere in Ω. Also, by the Strong Maximum Principle for Elliptic Equations, since $u = 0$ on $\partial\Omega$, u can't vanish in Ω unless it is constant. This means that $u > 0$ in Ω and this completes the proof $\square$

From now on, any solution of (5) will mean a minimizer of I_λ which belongs in $C^2(\Omega) \cap C^1(\overline{\Omega})$

3 Some Nonexistence Results

In this chapter, we want to investigate how the parameters p and λ contribute to the existence (or the nonexistence), of a positive solution to the problem

$$\begin{cases} -\triangle u = u^p + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

We first prove a nonexistence result:

Lemma 3 *Let $1 < p \leq \dfrac{n+2}{n-2}$. If $\lambda \geq \lambda_1$, then problem*

$$\begin{cases} -\triangle u = u^p + \lambda u & \textit{in } \Omega \\ u > 0 & \textit{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

has no solution.

Proof

Suppose on the contrary that for some $\lambda \geq \lambda_1$ we can find a solution u to this problem. Let $\phi_1 \in C^\infty(\Omega)$ be the eigenfunction of $-\triangle$ corresponding to λ_1 , with $\phi_1 > 0$ in Ω. Then, by integration by parts, we have:

$$-\int_\Omega (\triangle u)\phi_1 dx = -\int_\Omega u(\triangle\phi_1)dx = \lambda_1 \int_\Omega u\phi_1 dx = \int_\Omega u^p\phi_1 dx + \lambda \int_\Omega u\phi_1 dx > \lambda \int_\Omega u\phi_1 dx.$$

Thus, $\lambda < \lambda_1$ which leads to a contradiction. $\square$

We also prove a result of Pohozaev (see [5])

Lemma 4 *(Pohozaev Identity) Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of the problem*

$$\begin{cases} -\triangle u = g(u) & in\ \Omega \\ u > 0 & in\ \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where g is continuous. Then ,we have that:

$$\frac{2-n}{2}\int_\Omega ug(u)dx + n\int_\Omega G(u)dx = \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2(x\cdot\nu)dS. \tag{11}$$

where $G(u) = \displaystyle\int_0^u g(t)dt$ and ν stands for the outer unit normal of $\partial\Omega$.

Proof Let u be a C^2 function up to the boundary, such that $-\triangle u = g(u)$.
We multiply equation $-\triangle u = g(u)$ with the factor $x \cdot \nabla u$, and integrate over Ω. Thus, we have the equation

$$-\int_\Omega \triangle u(\nabla u\cdot x)dx = \int_\Omega g(u)(\nabla u \cdot x)dx. \tag{12}$$

Now, the left hand side of (3) is

$$-\sum_{i,j=1}^n \int_\Omega u_{x_i x_i} x_j u_{x_j} dx.$$

After integrating by parts, this equals:

$$\sum_{i,j=1}^n \int_\Omega u_{x_i}(x_j u_{x_j})_{x_i} dx - \sum_{i,j=1}^n \int_{\partial\Omega} u_{x_i}\nu^i x_j u_{x_j} dS$$

where ν^i is the i-th component of the unit normal ν. Now:

$$\sum_{i,j=1}^n \int_\Omega u_{x_i}(x_j u_{x_j})_{x_i} dx = \sum_{i,j=1}^n \int_\Omega u_{x_i}\delta_{ij}u_{x_j}dx + \sum_{i,j=1}^n \int_\Omega u_{x_i}x_j u_{x_j x_i}dx =$$

$$= \sum_{i=1}^n \int_\Omega u_{x_i}^2 dx + \sum_{j=1}^n \int_\Omega \left(\sum_{i=1}^n u_{x_i}u_{x_i x_j}\right) x_j dx = \int_\Omega |\nabla u|^2 dx + \sum_{j=1}^n \int_\Omega \left(\frac{|\nabla u|^2}{2}\right)_{x_j} x_j dx$$

Integrating by parts once again, we find that

$$\int_\Omega \left(\frac{|\nabla u|^2}{2}\right)_{x_j} x_j dx = \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2 x_j \nu^j dS - \frac{1}{2}\int_\Omega |\nabla u|^2 dx.$$

As a result,

$$\sum_{i,j=1}^n \int_\Omega u_{x_i}(x_j u_{x_j})_{x_i} dx = (1-\frac{n}{2})\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2(\nu\cdot x)dS.$$

Since $u = 0$ on $\partial\Omega$, $\nabla u(x)$ is parallel to the unit normal $\nu(x)$ at any point $x \in \partial\Omega$. Also, since $u > 0$ in Ω, this means that $\nu(x) = -\dfrac{\nabla u(x)}{|\nabla u(x)|}$ for every point $x \in \partial\Omega$ where the gradient is nonzero. Thus, we see that

$$\sum_{i,j=1}^n \int_{\partial\Omega} u_{x_i}\nu^i x_j u_{x_j} dS = -\int_{\partial\Omega}|\nabla u|^2(\nu\cdot x)dS.$$

Thus, the left-hand side of (3) becomes

$$\frac{2-n}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2(\nu\cdot x)dS.$$

Also, the right-hand side of (3) equals

$$\sum_{j=1}^{n}\int_{\Omega} g(u)u_{x_j}x_j dx = \sum_{j=1}^{n}\int_{\Omega}(G(u))_{x_j}x_j dx$$

Integrating by parts, since $G(0)=0$, we have

$$\sum_{j=1}^{n}\int_{\Omega} g(u)u_{x_j}x_j dx = -n\int_{\Omega} G(u)dx.$$

Putting it all together,

$$\frac{2-n}{2}\int_{\Omega}|\nabla u|^2dx - \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2(\nu\cdot x)dS = -n\int_{\Omega}G(u)dx \Leftrightarrow \frac{2-n}{2}\int_{\Omega}|\nabla u|^2dx + n\int_{\Omega}G(u)dx = \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2(\nu\cdot x)dS$$

Furthermore, if we multiply equation $-\triangle u = g(u)$ with u and integrate over Ω, we have that

$$\int_{\Omega}|\nabla u|^2dx = \int_{\Omega}ug(u)dx.$$

Thus,

$$\frac{2-n}{2}\int_{\Omega}ug(u)dx + n\int_{\Omega}G(u)dx = \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2(x\cdot\nu)dS.$$

□

We now introduce the notion of a star shaped set. A set $\Omega \subseteq \mathbb{R}^n$ is called star shaped if there is a point $x_0 \in \Omega$, such that $\forall x \in \Omega$ the line segment $[x, x_0]$ connecting x and x_0 lies totally in Ω. We are now ready to prove a second nonexistence result:

Lemma 5 *If Ω is starshaped, $\lambda \leq 0$ and $p \geq \frac{n+2}{n-2}$, then equation*

$$\begin{cases} -\triangle u = u^p + \lambda u & in\ \Omega \\ u > 0 & in\ \Omega \\ u|_{\partial\Omega} = 0 & \end{cases}$$

has no solution

Proof We apply Pohozaev identity in the case of $g(u) = u|u|^{p-1} + \lambda u$. (hence $G(u) = \frac{|u|^{p+1}}{p+1} + \frac{\lambda}{2}u^2$) and we get:

$$\left(\frac{2-n}{2} + \frac{n}{p+1}\right)\int_{\Omega}|u|^{p+1}dx + \lambda\int_{\Omega}u^2dx = \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)|\nabla u|^2dS.$$

Now, if Ω is starshaped, we know that $x\cdot\nu > 0$ for almost every x in the boundary of Ω. Also, by Hopf's Lemma, since $\partial\Omega$ is C^2, we have that $\frac{\partial u}{\partial\nu}(x) > 0$ for every $x \in \Omega$. This means that $\int_{\partial\Omega}(x\cdot\nu)|\nabla u|^2dS > 0$

This means that

$$\left(\frac{2-n}{2} + \frac{n}{p+1}\right)\int_{\Omega}|u|^{p+1}dx + \lambda\int_{\Omega}u^2dx > 0$$

and the result follows □

In view of Lemmas 3 and 5, we see that if we hope to establish a solution to the problem

$$\begin{cases} -\triangle u = u^p + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

with Ω starshaped and $1 < p \leq \frac{n+2}{n-2}$, our only possibilities are:

- $1 < p < \frac{n+2}{n-2}$ and $\lambda < \lambda_1$ (Subcritical case)
- $p = \frac{n+2}{n-2}$ and $0 < \lambda < \lambda_1$ (Critical case)

Let us see now how the standard argument goes in the case $1 < p < \frac{n+2}{n-2}$. This problem can be solved by standard arguments of Functional Analysis.

Proposition 6 *(Existence of Solutions in the Subcritical case) Let* $1 < p < \frac{n+2}{n-2}$ *and* $\lambda < \lambda_1(\Omega)$*. Then, the problem*

$$\begin{cases} -\triangle u = u^p + \lambda u & \textit{in } \Omega \\ u > 0 & \textit{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

has a solution.

Proof Let $(u_j)_{j\in\mathbb{N}} \subset H_0^1(\Omega)$ be a minimizing sequence for I_λ, with $\int_\Omega |u_j|^{p+1}dx = 1$, $\forall\ j \in \mathbb{N}$. Thus,

$$\int_\Omega |\nabla u_j|^2 dx - \lambda \int_\Omega u_j^2 dx \to M_\lambda.$$

From the Poincare inequality

$$\int_\Omega |\nabla u_j|^2 dx - \lambda \int_\Omega u_j^2 dx \geq (1 - \frac{\lambda}{\lambda_1}) \int_\Omega |\nabla u_j|^2 dx$$

and as a result $(u_j)_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$.

Since $H_0^1(\Omega)$ is a Hilbert space, and thus reflexive, there is a subsequence of $(u_j)_{j\in\mathbb{N}}$ (still denoted by the same symbol) which converges weakly to a function $u \in H_0^1(\Omega)$.

We now use the Rellich-Kondrachov Compactness Theorem and extract a subsequence of u_j that converges strongly in L^{p+1}. Since $\int_\Omega |u_j|^{p+1}dx = 1\ \forall\ j \in \mathbb{N}$, this means that

$$\int_\Omega |u|^{p+1} dx = 1.$$

Extracting yet another subsequence of u_j, we have that

$$\int_\Omega u_j^2 dx \to \int_\Omega u^2 dx$$

We note here that although we can't ensure the strong convergence of the sequence (u_j) in $H_0^1(\Omega)$, we have the property of lower semi-continuity:

$$\int_\Omega |\nabla u|^2 dx \leq \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^2 dx \Rightarrow \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u^2 dx \leq M_\lambda.$$

Since by construction, the inequality $\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u^2 dx \geq M_\lambda$ is trivial,

$$\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u^2 dx = M_\lambda$$

with $\int_\Omega |u|^{p+1} dx = 1$. This means that u is a minimizer of I_λ. □

We now ask ourselves what happens in the critical case $p = 2^* - 1$. This will be answered in the next section.

4 Investigating the Critical case

In this chapter, we concentrate our attention to the equation

$$\begin{cases} -\triangle u = u^{\frac{n+2}{n-2}} + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{13}$$

More specifically, we will prove an interesting Lemma of Brezis and Lieb which be of great importance to our work, and then we will give the proof of Theorems 2,3 and 4.

Before we start, we will make some computations which will be crucial to our work: We recall from the introduction the function $U(x) = (1+|x|^2)^{-\frac{n-2}{2}}$ and we define

$$K_1 =: \int_{\mathbb{R}^n} |\nabla U(x)|^2 dx, K_2 =: \left(\int_{\mathbb{R}^n} |U(x)|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}, K_3 =: \int_{\mathbb{R}^n} U^2(x) dx.$$

We will now use the Gamma function $\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt, x > 0$ and we will have:

Lemma 7 *For every $n \geq 3$, we have:*

$$\begin{cases} K_1 = \frac{\omega_n (n-2)^2}{2} \dfrac{\Gamma(\frac{n+2}{2})\Gamma(\frac{n-2}{2})}{\Gamma(n)} \\ K_2 = \left(\frac{1}{2}\omega_n \dfrac{\Gamma^2(\frac{n}{2})}{\Gamma(n)}\right)^{\frac{n-2}{n}} \end{cases} \tag{14}$$

and since U is a minimizer of the Sobolev Inequality in all of $\mathbb{R}^n$, we have that $\frac{K_1}{K_2} = S_n$

Furthermore, for $n \geq 5$,

$$K_3 = \frac{1}{2}\omega_n \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-4}{2})}{\Gamma(n-2)}$$

Proof We recall that for $x, y > 0$, Γ satisfies the functional equation

$$\int_0^{+\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Let $n \geq 3$. We now use the functional equation for Γ and compute

$$\int_0^{+\infty} \frac{r^{n+1}}{(1+r^2)^n} dr = \frac{1}{2} \int_0^{+\infty} \frac{t^{\frac{n}{2}}}{(1+t)^n} dt = \frac{1}{2} \int_0^{+\infty} \frac{t^{\frac{n+2}{2}-1}}{(1+t)^n} dt = \frac{1}{2} \frac{\Gamma(\frac{n+2}{2})\Gamma(\frac{n-2}{2})}{\Gamma(n)} \tag{15}$$

Also,

$$\int_0^{+\infty} \frac{r^{n-1}}{(1+r^2)^n} dr = \frac{1}{2} \int_0^{+\infty} \frac{t^{\frac{n}{2}-1}}{(1+t)^n} dt = \frac{1}{2} \frac{\Gamma^2(\frac{n}{2})}{\Gamma(n)} \tag{16}$$

We have

$$\int_{\mathbb{R}^n} |\nabla U(x)|^2 dx = (n-2)^2 \int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^n} dx = (n-2)^2 \omega_n \int_0^{+\infty} \frac{r^{n+1}}{(1+r^2)^n} dr = \frac{\omega_n (n-2)^2}{2} \frac{\Gamma(\frac{n+2}{2})\Gamma(\frac{n-2}{2})}{\Gamma(n)}$$

Furthermore

$$\left(\int_{\mathbb{R}^n} |U(x)|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} = \left(\omega_n \int_0^{+\infty} \frac{r^{n-1}}{(1+r^2)^n} dr \right)^{\frac{n-2}{n}} = \left(\frac{1}{2} \omega_n \frac{\Gamma^2(\frac{n}{2})}{\Gamma(n)} \right)^{\frac{n-2}{n}}$$

We now let $n \geq 5$ (since the integral $\displaystyle\int_0^{+\infty} \frac{r^{n-1}}{(1+r^2)^{n-2}}$ is not convergent for $n \leq 4$) and we have:

$$\int_0^{+\infty} \frac{r^{n-1}}{(1+r^2)^{n-2}} = \frac{1}{2} \int_0^{+\infty} \frac{t^{\frac{n-2}{2}-1}}{(1+t)^{n-2}} dt = \frac{1}{2} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-4}{2})}{\Gamma(n-2)}$$

As a result

$$K_3 = \frac{1}{2} \omega_n \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-4}{2})}{\Gamma(n-2)}, n \geq 5$$

An application of those formulas in the case $n = 3$ gives

$$\int_0^{+\infty} \frac{r^4}{(1+r^2)^3} dr = \frac{1}{2} \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)} = \frac{3\pi}{16}$$

and

$$\int_0^{+\infty} \frac{r^2}{(1+r^2)^3} dr = \frac{1}{2} \frac{\Gamma^2(\frac{3}{2})}{\Gamma(3)} = \frac{\pi}{16}$$

As a result

$$\begin{cases} K_1 = \displaystyle\int_{\mathbb{R}^3} |\nabla U(x)|^2 dx = \omega_3 \int_0^{+\infty} \frac{r^4}{(1+r^2)^3} dr = \omega_3 \frac{3\pi}{16} \\ K_2 = \left(\displaystyle\int_{\mathbb{R}^3} |U(x)|^6 dx \right)^{\frac{1}{3}} = \left(\omega_3 \displaystyle\int_0^{+\infty} \frac{r^2}{(1+r^2)^3} dr \right)^{\frac{1}{3}} = \left(\omega_3 \frac{\pi}{16} \right)^{\frac{1}{3}} \end{cases}$$

□

At this point, we have to note that the Banach spaces L^q with $q > 1$ are reflexive, but *not* compact. This means that given a uniformly bounded sequence $(u_j)_{j \in \mathbb{N}}$ of $L^q(\Omega)$, $q > 1$ one can extract a weakly convergent subsequence, but in general one *cannot* extract a strongly convergent one. However, the following Lemma of Brezis and Lieb (see [6]) provides a quantitive measure of how much a weakly convergent sequence fails to be strongly convergent.

Lemma 8 *(Brezis–Lieb) Let $q > 0$ and let $(f_j)_{j \in \mathbb{N}} \subseteq L^q(\Omega)$ be a sequence of functions which is uniformly bounded in $L^q(\Omega)$. Suppose that $f_j \to f$ pointwise a.e. Then, we have that*

$$\lim_{j \to \infty} \left(||f_j||_q^q - ||f_j - f||_q^q \right) = ||f||_q^q.$$

Proof At first, the proof will be carried out in the special case $q = 6$.

$$||f_j - f||_6^6 = \int_\Omega (f_j - f)^6 = \int_\Omega f_j^6 - 6\int_\Omega f_j^5 f + 15\int_\Omega f_j^4 f^2 - 20\int_\Omega f_j^3 f^3 + 15\int_\Omega f_j^2 f^4 - 6\int_\Omega f_j f^5 + \int_\Omega f^6$$

Thus,

$$||f_j||_6^6 - ||f_j - f||_6^6 = 6\left(\int_\Omega f_j^5 f + \int_\Omega f_j f^5\right) - 15\left(\int_\Omega f_j^4 f^2 + \int_\Omega f_j^2 f^4\right) + 20\int_\Omega f_j^3 f^3 - \int_\Omega f^6$$

It remains to show that every factor of the form

$$\int_\Omega f_j^k f^{6-k}$$

converges to $\int_\Omega f^6$ as $j \to \infty$.

But this is a direct consequence of the Banach-Alaoglu Theorem: Let $k = 1, ..., 5$. Since $(f_j)_{j\in\mathbb{N}} \in L^6(\Omega)$, $(f_j^k)_{j\in\mathbb{N}} \in L^{\frac{6}{k}}(\Omega)$. Since $L^{\frac{6}{k}}(\Omega)$ is reflexive (because $\frac{6}{k} > 1$), and $f_j^k \to f^k$ pointwise a.e., there is a subsequence of (f_j) (still denoted by the same symbol) such that

$$\int_\Omega f_j^k g \to \int_\Omega f^k g,$$

$\forall\ g \in L^{\frac{6}{6-k}}(\Omega)$, since

$$\frac{k}{6} + \frac{6-k}{6} = 1.$$

But the function f^{6-k} belongs to $L^{\frac{6}{6-k}}(\Omega)$ (since f belongs to $L^6(\Omega)$), and thus

$$\int_\Omega f_j^k f^{6-k} \to \int_\Omega f^k f^{6-k} = \int_\Omega f^6.$$

Let us now prove the general case $q > 0$. Let $(f_j)_{j\in\mathbb{N}} \subseteq L^q(\Omega)$ be a sequence of functions which is uniformly bounded and converges pointwise a.e. to a function f. At first, we state that we have the following inequality: For every $a, b \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$,

$$||a + b|^q - |a|^q| \le \varepsilon|a|^q + C_\varepsilon|b|^q \tag{17}$$

where $C_\varepsilon > 0$ is a constant dependent only on ε and q. This inequality is obvious if $0 < q \le 1$ (since $||a + b|^q - |a|^q| \le |b|^q$) but also follows easily if $q > 1$ as one establishes it for every rational exponent $q > 1$ and then for real exponents via a density argument. Let

$$W_{\varepsilon,n}(x) = [||f_j(x)|^q - |f_j(x) - f(x)|^q - |f(x)|^q| - \varepsilon|f_j(x) - f(x)|^q]_+$$

where $[a]_+ = max\{0, a\}$. We see that as $j \to +\infty$, $W_{\varepsilon,n} \to 0$ a.e. On the other hand, from (17):

$$||f_j(x)|^q - |f_j(x) - f(x)|^q - |f(x)|^q| \le ||f_j(x)|^q - |f_j(x) - f(x)|^q| + |f(x)|^q \le \varepsilon|f_j(x)|^q + C_\varepsilon|f(x)|^q + |f(x)|^q$$

Therefore, $W_{\varepsilon,n}(x) \le (C_\varepsilon + 1)|f(x)|^q \in L^1(\Omega)$. By Dominated Convergence Theorem, $\int_\Omega W_{\varepsilon,n}(x)dx \to 0$ By construction,

$$||f_j(x)|^q - |f_j(x) - f(x)|^q - |f(x)|^q| \le W_{\varepsilon,n}(x) + \varepsilon|f_j(x) - f(x)|^q$$

As a result,

$$\int_\Omega ||f_j(x)|^q - |f_j(x) - f(x)|^q - |f(x)|^q|dx \le \int_\Omega W_{\varepsilon,n}(x)dx + \varepsilon\int_\Omega |f_j(x) - f(x)|^q dx$$

This means that for a positive constant C (independent of ε),

$$\limsup_{j\to+\infty}\int_\Omega ||f_j(x)|^q - |f_j(x)-f(x)|^q - |f(x)|^q|dx \le C\varepsilon$$

Letting $\varepsilon \to 0^+$, we actually get that $|f_j(x)|^q - |f_j(x)-f(x)|^q - |f(x)|^q \to 0$ strongly in $L^1(\Omega)$. This is an even stronger result than what we aimed to show. □

At this point, we are ready to establish Theorem 1.

Proof of Theorem 1

Throughout the proof , various subsequences will be denoted by the same symbol.

Suppose that for some $\lambda \in (0, \lambda_1)$ we have that $J_\lambda < S_n$.

Let $(u_j)_{j\in\mathbb{N}} \subseteq H_0^1(\Omega)$ be a minimizing sequence for Q_λ. We normalize it such that

$$||u_j||_{2^*} = 1. \tag{18}$$

Then as $j \to +\infty$, we have that

$$||\nabla u_j||_2^2 - \lambda||u_j||_2^2 = J_\lambda + o(1). \tag{19}$$

From (18), since Ω is bounded, we get by Holder's inequality that $||u_j||_2$ is uniformly bounded. From (19), $||\nabla u_j||_2$ is also uniformly bounded. Thus, (u_j) is bounded in $H_0^1(\Omega)$ and since $H_0^1(\Omega)$ is reflexive, we can extract a subsequence of (u_j), such that, for a function $u \in H_0^1(\Omega)$:

- $u_j \rightharpoonup u$ weakly in $H_0^1(\Omega)$.
- $u_j \to u$ strongly in $L^2(\Omega)$ (from the Rellich-Kondrachov Theorem)
- $u_j \to u$ pointwise a.e. in Ω.

Since the property of lower semicontinuity holds,

$$||u||_{2^*} \le \liminf_j ||u_j||_{2^*} = 1.$$

Let $v_j := u_j - u$, so that $v_j \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and $v_j \to 0$ pointwise a.e.

From the definition of S_n and (18), we have $||\nabla u_j||_2 \ge S_n$. From (19), this means that $\lambda||u||_2^2 \ge S_n - J_\lambda > 0$ and therefore $u \neq 0$.

From (19) and using the fact that $u_j \rightharpoonup u$ weakly in $H_0^1(\Omega)$., we have:

$$||\nabla u||_2^2 + ||\nabla v_j||_2^2 - \lambda||u||_2^2 = ||\nabla u||_2^2 + ||\nabla(u-u_j)||_2^2 - \lambda||u||_2^2 = 2||\nabla u||_2^2 - 2(\nabla u, \nabla u_j)_2 + ||\nabla u_j||_2^2 - \lambda||u||_2^2$$

$$= o(1) + ||\nabla u_j||_2^2 - \lambda||u||_2^2$$

and since $||u_j||_2 = ||u||_2 + o(1)$ as $j \to +\infty$ we get:

$$||\nabla u||_2^2 + ||\nabla v_j||_2^2 - \lambda||u||_2^2 = J_\lambda + o(1) \Leftrightarrow ||\nabla v_j||_2^2 = \lambda||u||_2^2 - ||\nabla u||_2^2 + J_\lambda + o(1) \tag{20}$$

Now, if we use Lemma 8 of Brezis and Lieb for $(u_j)_{j\in\mathbb{N}} = (u+v_j)_{j\in\mathbb{N}}$ (which is allowed since u and v_j are bounded in $L^{2^*}(\Omega)$ and $v_j \to 0$ pointwise a.e.) , we get:

$$||u+v_j||_{2^*}^{2^*} = ||u||_{2^*}^{2^*} + ||v_j||_{2^*}^{2^*} + o(1)$$

and therefore , by the normalization made in (18)

$$1 = ||u||_{2^*}^{2^*} + ||v_j||_{2^*}^{2^*} + o(1).$$

Since $||u||_{2^*} \leq 1$ and $||v_j||_{2^*} \leq 1$ for large j, we have that

$$1 \leq ||u||_{2^*}^2 + ||v_j||_{2^*}^2 + o(1).$$

From the Sobolev inequality, we then get that:

$$1 \leq ||u||_{2^*}^2 + \frac{1}{S_n}||\nabla v_j||_2^2 + o(1).$$

We multiply by $J_\lambda > 0$ and get:

$$J_\lambda \leq J_\lambda||u||_{2^*}^2 + \frac{J_\lambda}{S_n}||\nabla v_j||_2^2 + o(1).$$

and therefore, since $J_\lambda < S_n$,

$$J_\lambda < J_\lambda||u||_{2^*}^2 + ||\nabla v_j||_2^2 + o(1).$$

Using (20) , we have that

$$J_\lambda < J_\lambda||u||_{2^*}^2 + \lambda||u||_2^2 - ||\nabla u||_2^2 + J_\lambda + o(1) \Leftrightarrow ||\nabla u||_2^2 - \lambda||u||_2^2 < J_\lambda||u||_{2^*}^2 + o(1)$$

and letting $j \to +\infty$ we finally get

$$||\nabla u||_2^2 - \lambda||u||_2^2 \leq J_\lambda||u||_{2^*}^2$$

That means that

$$Q_\lambda(u) \leq J_\lambda$$

Since the reverse inequality is trivial by the definition of J_λ, it follows that $Q_\lambda(u) = J_\lambda$. Therefore, u is a minimizer of Q_λ. □

Remark The result we just stated is quite remarkable: It shows that below the energy level S_n, some form of compactness holds.

We now ask ourselves if we can actually find $\lambda \in (0, \lambda_1)$ such that $J_\lambda < S_n$. At first, we will give an answer to this question for the case $n \geq 4$.

Proof of Theorem 2 In order to study the behavior of J_λ, we introduce the family of test functions

$$u_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon + |x|^2)^{\frac{n-2}{2}}},$$

$\varepsilon > 0$, where ϕ is a fixed smooth function such that $\phi \equiv 1$ in a neighborhood of the origin.

The crucial step is to evaluate the ratio $Q_\lambda(u_\varepsilon)$ for those particular functions. The reason behind the usefulness of the family $(u_\varepsilon)_{\varepsilon>0}$ is that they are is some way a scaled version of the functions $U_\varepsilon(x) = (\varepsilon + |x|^2)^{-\frac{n-2}{2}}$, which, as discussed in the introduction, are minimizers of the Sobolev Inequality in all of $\mathbb{R}^n$. We have

$$\nabla u_\varepsilon(x) = \frac{\nabla\phi(x)}{(\varepsilon + |x|^2)^{\frac{n-2}{2}}} - (n-2)\frac{\phi(x)x}{(\varepsilon + |x|^2)^{\frac{n}{2}}}$$

and thus

$$|\nabla u_\varepsilon(x)|^2 = \frac{|\nabla\phi(x)|^2}{(\varepsilon + |x|^2)^{n-2}} - 2(n-2)\frac{\phi(x)(x \cdot \nabla\phi(x))}{(\varepsilon + |x|^2)^{n-1}} + (n-2)^2\frac{|x|^2\phi^2(x)}{(\varepsilon + |x|^2)^n}.$$

We integrate over Ω and get

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = \int_\Omega \frac{|\nabla \phi|^2(x)}{(\varepsilon+|x|^2)^{n-2}}dx - 2(n-2)\int_\Omega \frac{\phi(x)(x\cdot\nabla\phi(x))}{(\varepsilon+|x|^2)^{n-1}}dx + (n-2)^2\int_\Omega \frac{|x|^2\phi^2(x)}{(\varepsilon+|x|^2)^n}dx.$$

Since $\phi\equiv 1$ near 0 ,we can assume that for some appropriately small $\delta>0$, $\nabla\phi\equiv 0$ in B_δ. Thus,

$$\int_\Omega \frac{|\nabla\phi(x)|^2}{(\varepsilon+|x|^2)^{n-2}}dx = \int_{\Omega\setminus B_\delta} \frac{|\nabla\phi(x)|^2}{(\varepsilon+|x|^2)^{n-2}}dx$$

In $\Omega\setminus B_\delta$, we have that

$$\frac{|\nabla\phi(x)|^2}{(\varepsilon+|x|^2)^{n-2}} \le \frac{|\nabla\phi(x)|^2}{(\varepsilon+\delta^2)^{n-2}} \le \frac{1}{\delta^{2(n-2)}}|\nabla\phi(x)|^2$$

Thus,

$$\int_\Omega \frac{|\nabla\phi(x)|^2}{(\varepsilon+|x|^2)^{n-2}}dx = O(1).$$

Using the same argument, we see that

$$\int_\Omega \frac{\phi(x)(x\cdot\nabla\phi(x))}{(\varepsilon+|x|^2)^{n-1}}dx = O(1).$$

As a result,

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = (n-2)^2\int_\Omega \frac{|x|^2}{(\varepsilon+|x|^2)^n}dx + O(1)$$

Since the integral

$$\int_{\mathbb{R}^n} \frac{|x|^2}{(\varepsilon+|x|^2)^n}dx$$

is finite (this is standard due to the fact that the function $\frac{|x|^2}{(\varepsilon+|x|^2)^n}$ is asymptotically equal to $\frac{1}{|x|^{2(n-1)}}$ as $|x|\to\infty$),

$$\int_{\mathbb{R}^n} \frac{|x|^2}{(\varepsilon+|x|^2)^n}dx = \int_\Omega \frac{|x|^2}{(\varepsilon+|x|^2)^n}dx + O(1).$$

Thus,

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = (n-2)^2\int_{\mathbb{R}^n} \frac{|x|^2}{(\varepsilon+|x|^2)^n}dx + O(1).$$

Making the substitution $x=\varepsilon^{\frac{1}{2}}y$, we have that:

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = (n-2)^2\int_{\mathbb{R}^n} \frac{\varepsilon|y|^2}{(\varepsilon+\varepsilon|y|^2)^n}\varepsilon^{\frac{n}{2}}dy + O(1) = (n-2)^2\varepsilon^{-\frac{n-2}{2}}\int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^n}dy + O(1)$$

We observe now that

$$(n-2)^2\int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^n}dy = \int_{\mathbb{R}^n} |\nabla U(x)|^2 dx$$

Thus,

$$\int_\Omega |\nabla u_\varepsilon(x)|^2 dx = K_1\varepsilon^{-\frac{n-2}{2}} + O(1). \tag{21}$$

We now compute

$$\int_\Omega |u_\varepsilon(x)|^{\frac{2n}{n-2}}dx = \int_\Omega \frac{\phi^{\frac{2n}{n-2}(x)}(x)}{(\varepsilon+|x|^2)^n}dx = \int_\Omega \frac{\phi^{\frac{2n}{n-2}}(x)-1}{(\varepsilon+|x|^2)^n}dx + \int_\Omega \frac{1}{(\varepsilon+|x|^2)^n}dx$$

Since $\phi\equiv 1$ near 0, we have:

$$\int_\Omega |u_\varepsilon(x)|^{\frac{2n}{n-2}}dx = O(1) + \int_{\mathbb{R}^n} \frac{1}{(\varepsilon+|x|^2)^n}dx$$

Making the substitution $y = \varepsilon^{\frac{1}{2}} x$ once again , we see that

$$\int_{\mathbb{R}^n} \frac{1}{(\varepsilon + |x|^2)^n} dx = \varepsilon^{-\frac{n}{2}} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^n} dy = \varepsilon^{-\frac{n}{2}} \int_{\mathbb{R}^n} |U(x)|^{\frac{2n}{n-2}} dx$$

Thus,

$$\int_\Omega |u_\varepsilon(x)|^{\frac{2n}{n-2}} dx = \varepsilon^{-\frac{n}{2}} \int_{\mathbb{R}^n} |U(x)|^{\frac{2n}{n-2}} dx + O(1) = \varepsilon^{-\frac{n}{2}} \left(\int_{\mathbb{R}^n} |U(x)|^{\frac{2n}{n-2}} dx + O(\varepsilon^{\frac{n}{2}}) \right)$$

and from (19) we finally get

$$\left(\int_\Omega |u_\varepsilon(x)|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} = K_2 \varepsilon^{-\frac{n-2}{2}} + O(\varepsilon) \tag{22}$$

We now compute :

$$\int_\Omega |u_\varepsilon(x)|^2 dx = \int_\Omega \frac{\phi^2(x) - 1}{(\varepsilon + |x|^2)^{n-2}} dx + \int_\Omega \frac{1}{(\varepsilon + |x|^2)^{n-2}} dx = \int_\Omega \frac{1}{(\varepsilon + |x|^2)^{n-2}} dx + O(1) = \int_{\mathbb{R}^n} \frac{1}{(\varepsilon + |x|^2)^{n-2}} dx + O(1)$$

- When $n \geq 5$, we make the substitution $r = \varepsilon^{\frac{1}{2}} y$ and from the previous Lemma we have that

$$\int_\Omega |u_\varepsilon(x)|^2 dx = \varepsilon^{-\frac{n-4}{2}} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}} dy + O(1) = \varepsilon^{-\frac{n-4}{2}} \int_{\mathbb{R}^n} |U(x)|^2 dx + O(1) = K_3 \varepsilon^{-\frac{n-4}{2}} + O(1) \tag{23}$$

- When $n = 4$, the problem is more delicate because the integral $\int_{\mathbb{R}^4} \frac{1}{(1 + |x|^2)^2} dx$ is not convergent, and therefore we can't bound $\int_\Omega |u_\varepsilon(x)|^2 dx$ in the same way. However we can do the following: Let B_{R_1} and B_{R_2} be balls centered at the origin, such that $B_{R_1} \subseteq \Omega \subseteq B_{R_2}$. Thus,

$$\int_{|x| \leq R_1} \frac{1}{(\varepsilon + |x|^2)^2} dx \leq \int_\Omega \frac{1}{(\varepsilon + |x|^2)^2} dx \leq \int_{|x| \leq R_2} \frac{1}{(\varepsilon + |x|^2)^2} dx.$$

  With the use of spherical coordinates, we have:

$$\int_{|x| \leq R} \frac{1}{(\varepsilon + |x|^2)^2} dx = \omega_4 \int_0^R \frac{r^3}{(\varepsilon + r^2)^2} dr = \frac{1}{2} \omega_4 |log\varepsilon| + O(1)$$

  This means that

$$\int_\Omega |u_\varepsilon(x)|^2 dx = \frac{1}{2} \omega_4 |log\varepsilon| + O(1) \tag{24}$$

Let us now suppose that $n \geq 5$. From estimates (20),(21) and (22) we get:

$$Q_\lambda(u_\varepsilon) = \frac{\int_\Omega |\nabla u_\varepsilon(x)|^2 dx - \lambda \int_\Omega u_\varepsilon^2(x) dx}{\left(\int_\Omega |u_\varepsilon(x)|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}} = \frac{K_1 \varepsilon^{-\frac{n-2}{2}} - \lambda K_3 \varepsilon^{-\frac{n-4}{2}} + O(1)}{K_2 \varepsilon^{-\frac{n-2}{2}} + O(\varepsilon)}$$

Thus by (20)

$$Q_\lambda(u_\varepsilon) = \frac{K_1 - \lambda K_3 \varepsilon + O(\varepsilon^{\frac{n-2}{2}})}{K_2 + O(\varepsilon^{\frac{n}{2}})} = \frac{K_1}{K_2} - \lambda \frac{K_3}{K_2} \varepsilon + O(\varepsilon^{\frac{n-2}{2}}) = S_n - \lambda \frac{K_3}{K_2} \varepsilon + O(\varepsilon^{\frac{n-2}{2}})$$

When $n = 4$ we have from (20),(21) and (23)

$$Q_\lambda(u_\varepsilon) = \frac{\int_\Omega |\nabla u_\varepsilon(x)|^2 dx - \lambda \int_\Omega u_\varepsilon^2(x) dx}{\left(\int_\Omega |u_\varepsilon(x)|^4 dx \right)^{\frac{1}{2}}} = \frac{K_1 \varepsilon^{-1} - \frac{1}{2} \omega_4 |log\varepsilon| + O(1)}{K_2 \varepsilon^{-1} + O(\varepsilon)} = \frac{K_1 - \frac{1}{2} \omega_4 \varepsilon |log\varepsilon| + O(\varepsilon)}{K_2 + O(\varepsilon^2)}$$

and again by (20) we have that

$$Q_\lambda(u_\varepsilon) = \frac{K_1}{K_2} - \frac{1}{2}\frac{\omega_4}{K_2}\varepsilon|log\varepsilon| + O(\varepsilon) = S_4 - \frac{1}{2}\frac{\omega_4}{K_2}\varepsilon|log\varepsilon| + O(\varepsilon)$$

We conclude that

$$Q_\lambda(u_\varepsilon) = \begin{cases} S_n - \lambda\frac{K_3}{K_2}\varepsilon + O(\varepsilon^{\frac{n-2}{2}}) & n \geq 5 \\ S_4 - \frac{1}{2}\frac{\omega_4}{K_2}\varepsilon|log\varepsilon| + O(\varepsilon) & n = 4 \end{cases} \tag{25}$$

and in any case, we see that $Q_\lambda(u_\varepsilon) < S_n$ for appropriately small $\varepsilon > 0$. □

We now turn to the case $n = 3$. As we will see, this problem is quite delicate and depends greatly on the geometry of the domain Ω. As a result, for the rest of this section we will restrict ourselves to the assumption that Ω is a ball B.

Proof of Theorem 3, Part 1 Without loss of generality, we assume that B is the unit ball B_1 of $\mathbb{R}^3$, and thus the first eigenvalue of the Laplacian is $\lambda_1(B_1) = \pi^2$ (with corresponding eigenfunction $|x|^{-1}sin(\pi|x|)$). Once again, the key lies in estimating the ratio $Q_\lambda(u)$ for

$$u = u_\varepsilon(r) = \frac{\phi(r)}{(\varepsilon + r^2)^{\frac{1}{2}}}, \tag{26}$$

with $r = |x|$ and $\varepsilon > 0$, where ϕ is a fixed smooth function such that $\phi(0) = 1$, $\phi^{'}(0) = 0$ and $\phi(1) = 0$.

We have $u_\varepsilon'(r) = \dfrac{\phi^{'}(r)}{(\varepsilon + r^2)^{\frac{1}{2}}} - \dfrac{r\phi(r)}{(\varepsilon + r^2)^{\frac{3}{2}}}$. Hence, since u_ε is radial,

$$\int_{B_1} |\nabla u_\varepsilon|^2 dx = \omega_3 \int_0^1 \left(\frac{|\phi^{'}(r)|^2}{\varepsilon + r^2} - \frac{2r\phi(r)\phi^{'}(r)}{(\varepsilon + r^2)^2} + \frac{r^2\phi^2(r)}{(\varepsilon + r^2)^3} \right) r^2 dr.$$

Since $\phi(1) = 0$, after integrating by parts, we find that

$$-2\int_0^1 \frac{r^3\phi(r)\phi^{'}(r)}{(\varepsilon + r^2)^2} dr = -\int_0^1 (\phi^2(r))^{'} \frac{r^3}{(\varepsilon + r^2)^2} dr = \int_0^1 \phi^2(r) \left(\frac{3r^2}{(\varepsilon + r^2)^2} - \frac{4r^4}{(\varepsilon + r^2)^3} \right) dr$$

Thus,

$$\int_\Omega |\nabla u_\varepsilon(x)|^2 dx = \omega_3 \int_0^1 \frac{r^2|\phi^{'}(r)|^2}{\varepsilon + r^2} dr + 3\omega_3\varepsilon \int_0^1 \frac{r^2\phi^2(r)}{(\varepsilon + r^2)^3} dr.$$

Moreover,since $\phi^{'}(0) = 0$, we get that

$$\int_0^1 \frac{r^2|\phi^{'}(r)|^2}{(\varepsilon + r^2)} dr = \int_0^1 |\phi^{'}(r)|^2 dr + O(\varepsilon). \tag{27}$$

Since $\phi^{'}(0) = 0$ and $\phi(0) = 1$, we can assume that $\phi(r) - 1 = O(r^2)$ near 0. Thus,

$$\int_0^1 \frac{r^2[\phi^2(r) - 1]}{(\varepsilon + r^2)^2} dr = O(\int_0^1 \frac{r^4}{(\varepsilon + r^2)^3} dr).$$

After the change of variables $r = \varepsilon^{\frac{1}{2}}t$, we see that $\displaystyle\int_0^1 \frac{r^4}{(\varepsilon + r^2)^3} dr = O(\varepsilon^{-\frac{1}{2}})$. Thus,

$$\int_0^1 \frac{r^2\phi^2(r)}{(\varepsilon + r^2)^3} dr = \int_0^1 \frac{r^2}{(\varepsilon + r^2)^3} dr + O(\varepsilon^{-\frac{1}{2}}).$$

After the same change of variables, we have that

$$\int_0^1 \frac{r^2}{(\varepsilon+r^2)^3}dr = \varepsilon^{-\frac{3}{2}}\int_0^{\varepsilon^{-\frac{1}{2}}} \frac{t^2}{(1+t^2)^3}dt = \varepsilon^{-\frac{3}{2}}\int_0^{+\infty} \frac{t^2}{(1+t^2)^3}dt + O(1). \tag{28}$$

Combining (27) and (28) , we get that

$$\int_{B_1} |\nabla u_\varepsilon(x)|^2dx = 3\omega_3\varepsilon^{-\frac{1}{2}}\int_0^{+\infty} \frac{t^2}{(1+t^2)^3}dt + \omega_3\int_0^1 |\phi'(t)|^2dt + O(\varepsilon^{\frac{1}{2}}) = 3\frac{\pi}{16}\omega_3\varepsilon^{-\frac{1}{2}} + \omega_3\int_0^1 |\phi'(t)|^2dt + O(\varepsilon^{\frac{1}{2}})$$

and thus

$$\int_{B_1} |\nabla u_\varepsilon(x)|^2dx = K_1\varepsilon^{-\frac{1}{2}} + \omega_3\int_0^1 |\phi'(t)|^2dt + O(\varepsilon^{\frac{1}{2}}) \tag{29}$$

We now calculate

$$\int_{B_1} u_\varepsilon^6(x)dx = \omega_3\int_0^1 \frac{r^2\phi^6(r)}{(\varepsilon+r^2)^3}dr = \omega_3\int_0^1 \frac{r^2[\phi^6(r)-1]}{(\varepsilon+r^2)^3}dr + \omega_3\int_0^1 \frac{r^2}{(\varepsilon+r^2)^3}dr.$$

As before, we can assume that $\phi(r)-1 = O(r^2)$ near 0. Thus,

$$\omega_3\int_0^1 \frac{r^2[\phi^6(r)-1]}{(\varepsilon+r^2)^3}dr = O(\int_0^1 \frac{r^4}{(\varepsilon+r^2)^3}dr) = O(\varepsilon^{\frac{1}{2}}).$$

Also, using the substitution $r = \varepsilon^{\frac{1}{2}}y$, we have:

$$\omega_3\int_0^1 \frac{r^2}{(\varepsilon+r^2)^3}dr. = \omega_3\varepsilon^{-\frac{3}{2}}\int_0^{\varepsilon^{-\frac{1}{2}}} \frac{t^2}{(1+t^2)^3}dt = \omega_3\varepsilon^{-\frac{3}{2}}\int_0^{+\infty} \frac{t^2}{(1+t^2)^3}dt + O(1).$$

Thus,

$$\int_{B_1} u_\varepsilon^6(x)dx = \varepsilon^{-\frac{3}{2}}\left(\omega_3\int_0^{+\infty} \frac{t^2}{(1+t^2)^3}dt + O(\varepsilon)\right) = \varepsilon^{-\frac{3}{2}}\left(\omega_3\frac{\pi}{16} + O(\varepsilon)\right)$$

As a result,

$$\left(\int_{B_1} u_\varepsilon^6(x)dx\right)^{\frac{1}{3}} = K_2\varepsilon^{-\frac{1}{2}} + O(\varepsilon^{\frac{1}{2}}) \tag{30}$$

Finally,

$$\int_{B_1} u_\varepsilon^2(x)dx = \omega_3\int_0^1 \frac{r^2\phi^2(r)}{(\varepsilon+r^2)}dr = -\omega_3\varepsilon\int_0^1 \frac{\phi^2(r)}{(\varepsilon+r^2)}dr + \omega_3\int_0^1 \phi^2(r)dr$$

We then have $\int_0^1 \frac{\phi^2(r)}{(\varepsilon+r^2)}dr = O(\int_0^1 \frac{1}{(\varepsilon+r^2)}dr) = O(\varepsilon^{-\frac{1}{2}})$.

As a result,

$$\int_{B_1} u_\varepsilon^2(x)dx = \omega_3\int_0^1 \phi^2(r)dr + O(\varepsilon^{\frac{1}{2}}) \tag{31}$$

Combining estimates (29)-(31) we have that for small $\varepsilon > 0$:

$$Q_\lambda(u_\varepsilon) = \frac{K_1\varepsilon^{-\frac{1}{2}} + \omega_3\left(\int_0^1 |\phi'(t)|^2dt - \lambda\int_0^1 \phi^2(t)dt\right) + O(\varepsilon^{\frac{1}{2}})}{K_2\varepsilon^{-\frac{1}{2}} + O(\varepsilon^{\frac{1}{2}})} = \frac{K_1 + \omega_3\varepsilon^{\frac{1}{2}}\left(\int_0^1 |\phi'(t)|^2dt - \lambda\int_0^1 \phi^2(t)dt\right) + O(\varepsilon)}{K_2 + O(\varepsilon)}$$

We choose $\phi(t) = cos(\frac{\pi}{2}t)$. Then, $\phi(1) = \phi^{'}(0) = 0$ and $\phi(0) = 1$. Also,

$$\int_0^1 |\phi'(t)|^2 dt = \frac{\pi^2}{4} \int_0^1 \phi^2(t) dt.$$

This means that for different constants $C > 0$

$$Q_\lambda(u_\varepsilon) = \frac{K_1 + C\varepsilon^{\frac{1}{2}} \left(\frac{\pi^2}{4} - \lambda\right) + O(\varepsilon)}{K_2 + O(\varepsilon)} = S_3 + C\varepsilon^{\frac{1}{2}} \left(\frac{\pi^2}{4} - \lambda\right) + O(\varepsilon)$$

Thus, if $\lambda > \dfrac{\pi^2}{4}$, we get that $Q_\lambda(u_\varepsilon) < S_3$ for small $\varepsilon > 0$. Hence, $J_\lambda < S_3$.

Since $\dfrac{\pi^2}{4}$ is exactly one fourth of the first eigenvalue of the Laplacian in the unit ball, we have the result. □

Proof of Theorem 3, Part 2 Without loss of generality, we assume again that $B = B_1$, and thus $\lambda_1(B_1) = \pi^2$.

Let u be a classic solution of

$$\begin{cases} -\triangle u = u^5 + \lambda u, & \text{in } B_1 \\ u > 0, & \text{in } B_1 \\ u|_{\partial B_1} = 0 \end{cases}$$

Since B_1 is a ball, by a result of B.Gidas, Wei-Ming Ni and L.Nirenberg(see [6]), u is radially symmetric. We write $u(x) = u(r)$, where $r = |x|$. Turning to spherical coordinates, u satisfies:

$$\begin{cases} -u^{''} - \frac{2}{r} u^{'} = u^5 + \lambda u, & \text{on } (0,1) \\ u^{'}(0) = u(1) = 0 \end{cases} \tag{32}$$

Let ψ be a smooth function such that $\psi(0) = 0$.

We multiply equation (32) by $r^2\psi u'$ and obtain:

$$-\int_0^1 u'' r^2 \psi u' dr - 2\int_0^1 (u')^2 r\psi dr = \int_0^1 u^5 r^2 \psi u' dr + \lambda \int_0^1 u r^2 \psi u' dr$$

After integrating by parts, we find that

$$\int_0^1 |u'|^2 (\frac{1}{2} r^2 \psi' - r\psi) dr - \frac{1}{2} |u'(1)|^2 \psi(1) = -\frac{1}{6} \int_0^1 u^6 (2r\psi + r^2\psi') dr - \frac{\lambda}{2} \int_0^1 u^2 (2r\psi + r^2 \psi') dr \tag{33}$$

We then multiply equation (32) by $(\frac{1}{2} r^2 \psi' - r\psi)u$ and, after integrating by parts, we obtain:

$$\int_0^1 |u'|^2 (\frac{1}{2} r^2 \psi' - r\psi) dr - \frac{1}{4} \int_0^1 u^2 r^2 \psi''' dr = \int_0^1 u^6 (\frac{1}{2} r^2 \psi' - r\psi) dr + \lambda \int_0^1 u^2 (\frac{1}{2} r^2 \psi' - r\psi) dr. \tag{34}$$

Combining (33) and (34) we get:

$$\int_0^1 u^2 (\lambda \psi' + \frac{1}{4} \psi''') r^2 dr = \frac{2}{3} \int_0^1 u^6 (r\psi - r^2 \psi') dr + \frac{1}{2} |u'(1)|^2 \psi(1). \tag{35}$$

From Pohozaev's result, we know that if Ω is starshaped, equation (1) has no solution if $\lambda \leq 0$. Thus, we assume that $0 < \lambda \leq \dfrac{\pi^2}{4}$.

We choose $\psi(r) = sin((4\lambda)^{\frac{1}{2}}r)$ (the choice is valid since $\psi'(0) = \psi(1) = 0$) and we have the following: $\psi(1) \geq 0$,

$$\lambda\psi' + \frac{1}{4}\psi''' = 0$$

and

$$r\psi - r^2\psi' = rsin((4\lambda)^{\frac{1}{2}}r) - r^2(4\lambda)^{\frac{1}{2}}cos((4\lambda)^{\frac{1}{2}}r) > 0$$

on $(0,1]$ (since $sint - tcost > 0$, $\forall\ t \in (0,\pi]$)

Therefore, if we insert this particular ψ in equation (36), we get that the left hand side is equal to zero, but the right hand side is strictly positive, which leads to a contradiction.

As a result, equation (1) has no solution if $0 < \lambda \leq \frac{1}{4}\lambda_1$. This particularly means that $J_\lambda \geq S_3$ for $\lambda = \dfrac{1}{4}\lambda_1$ and as a result we get the inequality

$$\int_{B_1} |\nabla u|^2 dx \geq \frac{1}{4}\lambda_1 \int_{B_1} u^2 dx + S_3 \left(\int_{B_1} u^6 dx\right)^{\frac{1}{3}}, \forall u \in C_c^\infty(\Omega)$$

□

The next natural question is what changes when Ω is not a ball. This will be the subject of the next section.

5 Non Radial Domains of $\mathbb{R}^3$

From now on, Ω is an open domain of $\mathbb{R}^3$ with C^2 boundary, which is not necessarily a ball. In order to gain information from the previous case, we will now need the notion of the "Symmetric Decreasing Rearrangement" of a nonnegative measurable function.

For a bounded domain $\Omega \in \mathbb{R}^n$ we define Ω^* as the open ball centered at the origin, such that $Vol(\Omega) = Vol(\Omega^*)$ For a nonnegative measurable function $u : \Omega \to [0, +\infty)$ we define its Symmetric Decreasing Rearrangement $u^* : \Omega^* \to [0, +\infty)$ as:

$$u^*(x) = \int_0^\infty \mathbf{1}_{\{f(x)>t\}^*} dt$$

provided of course that $Vol(\{f(x) > t\}) < +\infty$ for every $t \geq 0$.

By the construction of u^* it is easy to see that u and $u*$ are equimesurable, that is their corresponding level sets have the same volume: For every $t \geq 0$,

$$Vol(\{u(x) > t\}) = Vol(\{u^*(x) > t\})$$

The Symmetric Decreasing Rearrangement u^* preserves the L^p-norm $\forall p \geq 1$.

$$\int_\Omega |u|^p = \int_{\Omega^*} |u^*|^p$$

$\forall p \geq 1$. The Symmetric Decreasing Rearrangement u^* of an H_0^1 function u belongs in $H_0^1(\Omega^*)$. Also, the L^2 gradient norm decreases:

$$\int_{\Omega^*} |\nabla u^*|^2 dx \leq \int_\Omega |\nabla u|^2 dx$$

(this is a special case of the Polya-Szego Inequality). For a deeper insight into the Symmetric Rearrangement of functions and sets, see Lieb E. , Moss M., Analysis (Graduate Studies in Mathematics) 2nd Edition(2001), Chapter 3, pp. 79-82

We now use the Symmetric Decreasing Rearrangement and prove:

Proposition 9 *Let $\Omega \subset R^3$ be a bounded open domain. Then,*

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\lambda_1(\Omega^*) \int_\Omega u^2 dx + S_3 \left(\int_\Omega u^6 dx \right)^{\frac{1}{3}}$$

for every $u \in H_0^1(\Omega)$

Proof

Let $u \in H_0^1(\Omega)$ be an a.e. positive function. Since the Symmetric Decreasing Rearrangement u^* of u preserves the L^p norms and decreases the gradient norm we have that:

$$Q_\lambda(u) = \frac{\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u^2 dx}{\left(\int_\Omega |u|^6 dx \right)^{\frac{1}{3}}} \geq \frac{\int_{\Omega^*} |\nabla u^*|^2 dx - \lambda \int_{\Omega^*} |u^*|^2 dx}{\left(\int_{\Omega^*} |u^*|^6 dx \right)^{\frac{1}{3}}} = Q_\lambda(u^*)$$

But from Part 2 of Theorem 3 we have established that when Ω is a ball, $J_\lambda \geq S_3$ when $\lambda = \frac{1}{4}\lambda_1$. The result then follows □

By proposition 8, we see that when $n = 3$, for "small" positive values of λ we cannot have the strict inequality $J_\lambda < S_3$ but instead we have $J_\lambda = S_3$. However, for λ sufficiently near to λ_1, we can easily see that $J_\lambda < S_3$ (this can be seen by using the continuity of the mapping $\lambda \mapsto J_\lambda$ and the fact that $J_\lambda = 0$ when $\lambda = \lambda_1$. We can also make this clear with the results we will produce in the next section).

It is therefore natural to define

$$\lambda^* = \lambda^*(\Omega) = max\{\lambda \in (0, \lambda_1) | J_\lambda = S_3\}$$

for any bounded subset Ω of $\mathbb{R}^3$. The constant λ^* has the following property:

1. $J_\lambda = S_3$ when $0 < \lambda \leq \lambda^*$
2. $J_\lambda < S_3$ when $\lambda^* < \lambda < \lambda_1$ and problem (1) has a solution.

Furthermore, when Ω is a ball, then $\lambda^* = \frac{1}{4}\lambda_1$.

The next question is what happens to problem (1) when $0 < \lambda \leq \lambda^*$. The answer to this is Theorem 4 which is due to O. Druet. But first, we will state a very helpful lemma

Lemma 10 *Let u be a smooth function, such that $\int_\Omega u dx = 1$. There exists $(y, t) \in \mathbb{R}^3 \times \mathbb{R}^+$ such that*

$$\begin{cases} F_i(y,t) = \int_\Omega u \frac{2t(x_i - y_i)}{1 + t^2|x - y|^2} dx = 0 \quad \text{for } i = 1, 2, 3 \\ G(y,t) = \int_\Omega u \frac{1 - t^2|x - y|^2}{1 + t^2|x - y|^2} dx = 0 \end{cases}$$

Proof We consider the function $H : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^4$ defined by

$$H(y, s) = (F_1(y, e^s) + y_1, F_2(y, e^s) + y_2, F_3(y, e^s) + y_3, G(y, e^s) + s)$$

We claim that for large values of $|y|^2 + s^2$, H satisfies the inequality $|H(y,s)| \leq |y|^2 + s^2$.

First, we will prove that for large values of $|y_1|$ and $|s|$,

$$(F_1(y, e^s) + y_1)^2 \leq y_1^2$$

To do this, note first that since $\int_\Omega u dx = 1$, we have

$$F_1(y, e^s) + y_1 = \int_\Omega u[\frac{2e^s(x_1 - y_1)}{1 + e^{2s}|x - y|^2} + y_1]dx$$

Again by our normalization, we then have

$$|F_1(y, e^s) + y_1| \leq \sup_{x\in\Omega} |\frac{2e^s(x_1 - y_1)}{1 + e^{2s}|x - y|^2} + y_1|$$

Therefore, it suffices to show that for appropriate values of y_1 and s

$$|\frac{2e^s(x_1 - y_1)}{1 + e^{2s}|x - y|^2} + y_1| \leq |y_1|, \forall x \in \Omega$$

Equivalently, we wish to show that for appropriate y_1 and s

$$-|y_1| - y_1 \leq \frac{2e^s(x_1 - y_1)}{1 + e^{2s}|x - y|^2} \leq |y_1| - y_1, \forall x \in \Omega \tag{36}$$

Note here that

$$-1 \leq \frac{2e^s(x_1 - y_1)}{1 + e^{2s}|x - y|^2} \leq 1 \tag{37}$$

for every $x_1 \in \Omega$, $y_1, s \in \mathbb{R}$

Suppose that $y_1 > 0$. Then, (36) becomes:

$$-2y_1 \leq \frac{2e^s(x_1 - y_1)}{1 + e^{2s}|x - y|^2} \leq 0$$

For the left inequality, from (37), we have to make sure that $-2y_1 \leq -1 \Rightarrow y_1 \geq \frac{1}{2}$. For the right inequality, we have to make sure that $x_1 - y_1 \leq 0$ for every $x \in \Omega$. Thus, we set $y \geq \sup_{x\in\Omega} |x_1|$. Therefore, for $y_1 \geq max\{\frac{1}{2}, \sup_{x\in\Omega} |x_1|\}$ and $s \in \mathbb{R}$, inequality (36) holds for every $x \in \Omega$.

Suppose now that $y_1 < 0$. Then, (36) becomes:

$$0 \leq \frac{2e^s(x_1 - y_1)}{1 + e^{2s}|x - y|^2} \leq 2|y_1|$$

By the exact same arguments and using (37) as before, we get that inequality (36) holds for every $x \in \Omega$ if we set $y_1 \leq -max\{\frac{1}{2}, \sup_{x\in\Omega} |x_1|\}$ and $s \in \mathbb{R}$.

Summing up the two cases, if $|y_1| \geq max\{\frac{1}{2}, \sup_{x\in\Omega} |x_1|\}$ and $s \in \mathbb{R}$, then equation (36) holds. Therefore, for big values of $|y_1|$ and $s \in \mathbb{R}$, we have that $(F_1(y, e^s) + y_1)^2 \leq y_1^2$.

By the exact same arguments, we can show that $(F_2(y,e^s)+y_2)^2 \leq y_2^2$ and $(F_3(y,e^s)+y_3)^2 \leq y_3^2$ for big values of $|y_2|, |y_3|$ and $s \in \mathbb{R}$.

It remains to show that for appropriate y and s, $(G(y,s)+s)^2 \leq s^2$. Again by the fact that $\int_\Omega u dx = 1$, it suffices to show that for appropriate y and s :

$$|\frac{1-e^{2s}|x-y|^2}{1+e^{2s}|x-y|^2}+s| \leq |s|, \forall x \in \Omega$$

But this also follows easily from the fact that $\frac{1-e^{2s}|x-y|^2}{1+e^{2s}|x-y|^2} \to -1$ as $s \to +\infty$ and $\frac{1-e^{2s}|x-y|^2}{1+e^{2s}|x-y|^2} \to 1$ as $s \to -\infty$.

Summing up, we can find appropriately large $|y|$ and $|s|$ such that $|H(y,s)| \leq |y|^2+s^2$. This particularly means that for some appropriate $R>0$, $H(B_R) \subseteq B_R$. From Brouwer's Fixed Point Theorem, since H is continuous, it has a fixed point in B_R. But this fixed point corresponds exactly to the statement of the Lemma. □

Proof of Theorem 4

When $0<\lambda<\lambda^*$ the argument is simplier. Suppose that for a $0<\lambda<\lambda^*$ we have an H_0^1 minimizer of Q_λ. We normalize u such that $\int_\Omega u^6 dx = 1$ and as a result we have

$$\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u^2 dx = J_\lambda = S_3$$

Now pick any μ such that $\lambda<\mu<\lambda^*$. We now have

$$\int_\Omega |\nabla u|^2 dx - \mu \int_\Omega u^2 dx = \int_\Omega |\nabla u|^2 dx - (\mu-\lambda)\int_\Omega u^2 dx - \lambda \int_\Omega u^2 dx < \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u^2 dx = S_3$$

and therefore $J_\mu < S_3$ which is impossible since $\mu<\lambda^*$.

We now turn to the case $\lambda=\lambda^*$. Suppose on the contrary that u is a smooth function which satisfies

$$-\triangle u - \lambda^* u = S_3 u^5 \tag{38}$$

We normalize u such that $\int_\Omega u^6 dx = 1$. Now, for any $\phi \in C^\infty(\mathbb{R}^3)$ since $J_{\lambda^*} = S_3$ we have that for any $\varepsilon>0$,

$$S_3\left(\int_\Omega u^6(1+\varepsilon\phi)^6 dx\right)^{\frac{1}{3}} \leq \int_\Omega |\nabla(u(1+\varepsilon\phi))|^2 dx - \lambda^* \int_\Omega u^2(1+\varepsilon\phi)^2 dx \tag{39}$$

By the normalization we made,

$$\left(\int_\Omega u^6(1+\varepsilon\phi)^6 dx\right)^{\frac{1}{3}} = 1+2\varepsilon\int_\Omega u^6\phi dx + 5\varepsilon^2\int_\Omega u^6\phi^2 dx - 4\varepsilon^2\left(\int_\Omega u^6\phi dx\right)^2 + o(\varepsilon^2) \tag{40}$$

After multiplying (39) by $u(1+\varepsilon\phi)^2$ and integrating we have that:

$$-\int_\Omega u\triangle u(1+\varepsilon\phi)^2 dx - \lambda^*\int_\Omega u^2(1+\varepsilon\phi)^2 dx = S_3\int_\Omega u^6(1+\varepsilon\phi)^2 dx$$

We now integrate by parts and get:

$$-\int_\Omega u\triangle u(1+\varepsilon\phi)^2 dx = \int_\Omega |\nabla u|^2(1+\varepsilon\phi)^2 dx + 2\varepsilon\int_\Omega u\nabla u\cdot\nabla\phi(1+\varepsilon\phi)dx = \int_\Omega |\nabla(u(1+\varepsilon\phi))|^2 dx - \varepsilon^2\int_\Omega u^2|\nabla\phi|^2 dx$$

As a result, we have that:

$$\int_\Omega |\nabla(u(1+\varepsilon\phi))|^2 dx - \lambda^* \int_\Omega u^2(1+\varepsilon\phi)^2 dx = S_3 \int_\Omega u^6(1+\varepsilon\phi)^2 dx + \varepsilon^2 \int_\Omega u^2 |\nabla\phi|^2 dx$$

and again by normalization

$$\int_\Omega |\nabla(u(1+\varepsilon\phi))|^2 dx - \lambda^* \int_\Omega u^2(1+\varepsilon\phi)^2 dx = S_3[1+2\varepsilon\int_\Omega u^6\phi dx + \varepsilon^2\int_\Omega \phi^2 u^6 dx] + \varepsilon^2 \int_\Omega u^2|\nabla\phi|^2 dx \quad (41)$$

Using (41) and (42) in relation (40) and letting $\varepsilon \to 0$, we get:

$$4\int_\Omega u^6\phi^2 dx \le S_3^{-1}\int_\Omega u^2|\nabla\phi|^2 dx + 4\left(\int_\Omega u^6\phi dx\right)^2 \quad (42)$$

In view of Lemma (9), we will now apply relation (43)to each ϕ_i , $i = 1, 2, 3, 4$ where

$$\begin{cases} \phi_i = \dfrac{2t(x_i - y_i)}{1+t^2|x-y|^2} & i = 1,2,3 \\ \phi_4 = \dfrac{1-t^2|x-y|^2}{1+t^2|x-y|^2} \end{cases}$$

Here, $(y, t) \in \mathbb{R}^3 \times \mathbb{R}^+$ is selected such that $\int_\Omega u^6\phi_i dx = 0$ for $i = 1, 2, 3, 4$. Summing what is left, we get:

$$\sum_{i=1}^4 \int_\Omega u^6\phi_i^2 dx \le S_3^{-1}\sum_{i=1}^4\int_\Omega u^2|\nabla\phi_i|^2 dx$$

We have that

$$\sum_{i=1}^4 \phi_i^2 = \frac{(1-t^2|x-y|^2)^2 + 4t^2\sum_{i=1}^4(x_i-y_i)^2}{(1+t^2|x-y|^2)^2} = \frac{1+2t^2|x-y|^2+t^4|x-y|^4}{(1+t^2|x-y|^2)^2} = 1$$

and we also find that

$$\sum_{i=1}^4 |\nabla\phi_i|^2 = \frac{12t^2}{(1+t^2|x-y|^2)^2}$$

As a result, we get that

$$4\int_\Omega u^6 dx \le 12 S_3^{-1}\int_\Omega u^2\frac{t^2}{(1+t^2|x-y|^2)^2}dx$$

and by our normalization:

$$S_3 \le 3\int_\Omega u^2 \frac{t^2}{(1+t^2|x-y|^2)^2}dx$$

By Holder's Inequality:

$$\int_\Omega u^2\frac{t^2}{(1+t^2|x-y|^2)^2}dx \le \left(\int_\Omega u^6 dx\right)^{\frac{1}{3}}\left(\int_\Omega \frac{t^3}{(1+t^2|x-y|^2)^3}dx\right)^{\frac{2}{3}}$$

and again by our normalization

$$S_3 \le 3\left(\int_\Omega \frac{t^3}{(1+t^2|x-y|^2)^3}dx\right)^{\frac{2}{3}} \quad (43)$$

Now, after a change of variables

$$\left(\int_{\mathbb{R}^3}\frac{t^3}{(1+t^2|x-y|^2)^3}dx\right)^{\frac{2}{3}} = \left(\int_{\mathbb{R}^3}\frac{1}{(1+|s-ty|^2)^3}ds\right)^{\frac{2}{3}} = \left(\omega_3\int_0^{+\infty}\frac{r^2}{(1+r^2)^3}dr\right)^{\frac{2}{3}} = \frac{1}{3}S_3$$

from the computations made in Lemma 7 and the value of S_3 discussed in the introduction. Therefore, since Ω is bounded, inequality (43) leads to a contradiction. □

As a result, we see that λ^* "splits" $(0, \lambda_1)$ into two open subintervals: a "nonexistence" interval $(0, \lambda^*]$ where problem (1) has no solution and an "existence" interval (λ^*, λ_1) where $J_\lambda < S_3$ and thus we have solutions to (1).

6 Our Approach

In this section, we will present our approach. More specifically, we will use a more direct argument that the one presented in Section 4 in order to prove a slightly refined existence result (Theorem 6) concerning solutions of (1) in three dimensions. Theorem 6 will be proved after the use of various helpful Lemmas.

We define the following quantities

$$\mu(y) = \inf_{u \in C_c^\infty(\Omega)} \frac{\int_\Omega \frac{|\nabla u|^2}{|x-y|^2} dx}{\int_\Omega \frac{u^2}{|x-y|^2} dx}, y \in \overline{\Omega} \tag{44}$$

and

$$\mu^* = \mu^*(\Omega) = \inf_{y \in \Omega} \mu(y) \tag{45}$$

Lemma 11 *We have that* $0 < \mu^* < \lambda_1$.

Proof We recall the classical three-dimensional Hardy Inequality:

$$\frac{1}{4} \int_\Omega \frac{u^2}{|x-y|^2} dx \leq \int_\Omega |\nabla u|^2 dx$$

for every $u \in C_c^\infty(\Omega), x \in \overline{\Omega}$ (the constant $\frac{1}{4}$ is sharp when $x \in \Omega$.) Using the fact that Ω is bounded , we have that $|x-y| \leq diam(\Omega), \forall x, y \in \overline{\Omega}$. Therefore, we get that

$$\frac{1}{(2diam(\Omega))^2} \int_\Omega \frac{u^2}{|x-y|^2} dy \leq \int_\Omega \frac{|\nabla u|^2}{|x-y|^2} dy, \forall u \in C_c^\infty(\Omega)$$

After taking infima, we get that $\mu^* \geq \frac{1}{(2diam(\Omega))^2} > 0$. For the second inequality, let $u(x) = |x-y|\phi_1(x)$ where $\phi_1 \in C^\infty(\Omega)$ is the eigenfunction corresponding to λ_1 as in the introduction. After calculating we get that

$$\frac{\int_\Omega \frac{|\nabla u|^2}{|x-y|^2} dx}{\int_\Omega \frac{u^2}{|x-y|^2} dx} = \frac{\int_\Omega |\nabla \phi_1|^2 dx}{\int_\Omega \phi_1^2 dx} = \lambda_1$$

Although u is not smooth, this calculation actually shows that $\mu(y) \leq \lambda_1$ for every $y \in \overline{\Omega}$. $\square$

As a byproduct of Lemma 11 we get that $||u|| = \left(\int_\Omega \frac{|\nabla u|^2}{|x-y|^2} dx \right)^{\frac{1}{2}}$ is a norm of $C_c^\infty(\Omega)$.

We now define the following space

$$H_0^1(\Omega, |x-y|^{-2}) = \overline{C_c^\infty(\Omega)}^{||.||}$$

One can see that $H_0^1(\Omega, |x-y|^{-2}) \subset H_0^1(\Omega)$ is a Hilbert space. Intuitively, $H_0^1(\Omega, |x-y|^{-2})$ consists of all H_0^1 functions which do not grow too rapidly near y.

We will also define the following quantities, which will be essential in the following lemma:

$$C_\rho(x) := \inf_{u \in C_c^\infty(B_\rho(x))} \frac{\displaystyle\int_{B_\rho(x)} \frac{|\nabla u|^2}{|y-x|^2} dy}{\displaystyle\int_{B_\rho(x)} \frac{u^2}{|y-x|^2} dy}, x \in \Omega$$

and

$$\Lambda(x) := \lim_{\rho \to 0^+} C_\rho(x)$$

We will now give the proof of an important result concerning the behavior of the function μ. The proof uses the ideas of Filippas and Tertikas (see [8]).

Lemma 12 *If $\mu(y) < \Lambda(y)$ for some $y \in \Omega$, then $\mu(y)$ has a minimizer in $H_0^1(\Omega, |x-y|^{-2})$. In particular, $\Lambda(y) = +\infty$ for every $y \in \Omega$.*

Proof Without loss of generality, we assume that $y = 0 \in \Omega$. Let $\rho > 0$ sufficiently small so that $B_\rho \subset \Omega$. Let $\mu(0) < \Lambda(0)$ and $(u_j)_{j\in\mathbb{N}}$ be a minimizing sequence for $\mu(0)$. We normalize it such that

$$\int_\Omega \frac{u_j^2}{|x|^2} dx = 1$$

This means that as $j \to +\infty$,

$$\int_\Omega \frac{|\nabla u_j|^2}{|x|^2} dx \to \mu(0)$$

and as a result $(u_j)_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega, |x|^{-2})$. Since $H_0^1(\Omega, |x|^{-2})$ is a Hilbert space, this means that there is a subsequence of $(u_j)_{j\in\mathbb{N}}$ (still denoted by the same symbol) and a function $u \in H_0^1(\Omega, |x|^{-2})$, such that $u_j \rightharpoonup u$ in the weak sense. Furthermore, $u_j \to u$ strongly in $L^2(\Omega \setminus B_\rho)$. Let $w_j = u_j - u$. We now have that as $j \to +\infty$,

$$1 = \int_\Omega \frac{w_j^2}{|x|^2} dx + \int_\Omega \frac{u^2}{|x|^2} dx + o(1) \tag{46}$$

and similarly

$$\mu(0) = \int_\Omega \frac{|\nabla w_j|^2}{|x|^2} dx + \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx + o(1) \tag{47}$$

By (45) and the definition of $\mu(0)$ we now get

$$\mu(0) \geq \int_\Omega \frac{|\nabla w_j|^2}{|x|^2} dx + \mu(0) \int_\Omega \frac{u^2}{|x|^2} dx + o(1) \tag{48}$$

and also

$$\mu(0) \geq \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx \tag{49}$$

Since $\mu(0) < \Lambda(0)$, by definition of $\Lambda(0)$, we have for sufficiently small $\rho > 0$

$$\mu(0) < C_\rho(0) = \inf_{v \in C_c^\infty(B_\rho)} \frac{\displaystyle\int_{B_\rho} \frac{|\nabla v|^2}{|x|^2} dx}{\displaystyle\int_{B_\rho} \frac{v^2}{|x|^2} dx} \tag{50}$$

Let $\phi \in C_c^\infty(B_\rho)$ be a smooth cutoff function such that $0 \leq \phi \leq 1$ and $\phi = 1$ in $B_{\rho/2}$. We write $w_j = \phi w_j + (1-\phi) w_j$ and we have as $j \to +\infty$

$$\int_\Omega \frac{|\nabla w_j|^2}{|x|^2} dx = \int_\Omega \frac{|\nabla(\phi w_j)|^2}{|x|^2} dx + \int_\Omega \frac{|\nabla((1-\phi) w_j)|^2}{|x|^2} dx + 2\int_\Omega \frac{\phi(1-\phi)|\nabla w_j|^2}{|x|^2} dx + o(1)$$

and as a result

$$\int_\Omega \frac{|\nabla w_j|^2}{|x|^2} dx \geq \int_\Omega \frac{|\nabla(\phi w_j)|^2}{|x|^2} dx + o(1) \tag{51}$$

From (48) and the fact that $\phi w_j \in H_0^1(B_\rho, |x|^{-2})$ we get that

$$\int_\Omega \frac{|\nabla(\phi w_j)|^2}{|x|^2} dx \geq C_\rho(0) \int_\Omega \frac{(\phi w_j)^2}{|x|^2} dx$$

But

$$\int_\Omega \frac{w_j^2}{|x|^2} dx - \int_\Omega \frac{(\phi w_j)^2}{|x|^2} dx = \int_{\Omega \setminus B_{\rho/2}} \frac{(\phi w_j)^2}{|x|^2} dx = o(1)$$

and in view of (49) we obtain

$$\int_\Omega \frac{|\nabla w_j|^2}{|x|^2} dx \geq C_\rho(0) \int_\Omega \frac{w_j^2}{|x|^2} dx + o(1)$$

Taking (44) into account, this means that

$$\int_\Omega \frac{|\nabla w_j|^2}{|x|^2} dx \geq C_\rho(0) \left(1 - \int_\Omega \frac{u^2}{|x|^2} dx\right) + o(1)$$

and by (46) we now get

$$(\mu(0) - C_\rho(0)) \left(1 - \int_\Omega \frac{u^2}{|x|^2} dx\right) \geq 0$$

and because of our assumption

$$\int_\Omega \frac{u^2}{|x|^2} dx \geq 1$$

From this and (47) we finally arrive at

$$0 < \frac{\displaystyle\int_\Omega \frac{|\nabla u|^2}{|x|^2} dx}{\displaystyle\int_\Omega \frac{u^2}{|x|^2} dx} \leq \mu(0)$$

and since the reverse inequality is obvious, u is a minimizer of $\mu(0)$.

We will now prove that $\Lambda(x) = +\infty$ for every $x \in \Omega$. Let $x \in \Omega$. Without loss of generality, we assume that $x = 0$.

Let $u \in C_c^\infty(B_1)$.

We make the change of variables $y = \frac{x}{\rho}$, with $\rho > 1$.

Let $v(y) = u(x)$. Then , $v \in C_c^\infty(B_{\frac{1}{\rho}})$

Also, $|\nabla u|^2(x) = \dfrac{1}{\rho^2} |\nabla v|^2(y)$

As a result,

$$\frac{\displaystyle\int_{B_1} \frac{|\nabla u|^2}{|x|^2} dx}{\displaystyle\int_{B_1} \frac{u^2 dx}{|x|^2}} = \frac{\displaystyle\int_{B_{\frac{1}{\rho}}} \frac{\frac{1}{\rho^2}|\nabla v|^2}{\rho^2 |y|^2} dy}{\displaystyle\int_{B_{\frac{1}{\rho}}} \frac{v^2 dy}{\rho^2 |y|^2}} = \frac{1}{\rho^2} \frac{\displaystyle\int_{B_{\frac{1}{\rho}}} \frac{|\nabla v|^2}{|y|^2} dy}{\displaystyle\int_{B_{\frac{1}{\rho}}} \frac{v^2 dy}{|y|^2}} \Rightarrow \frac{\displaystyle\int_{B_{\frac{1}{\rho}}} \frac{|\nabla v|^2}{|y|^2} dy}{\displaystyle\int_{B_{\frac{1}{\rho}}} \frac{v^2 dy}{|y|^2}} = \rho^2 \frac{\displaystyle\int_{B_1} \frac{|\nabla u|^2}{|x|^2} dx}{\displaystyle\int_{B_1} \frac{u^2 dx}{|x|^2}}, \forall \rho > 1$$

After taking infima, we have $C_{\frac{1}{\rho}}(0) = \rho^2 C_1(0)$. Letting $\rho \to +\infty$, we see that $\Lambda(0) = +\infty$ $\square$

We actually believe that an even stronger result holds: *μ^* is minimized for some $y_0 \in \overline{\Omega}$ and for some $u \in H_0^1(\Omega, |x-y_0|^{-2})$. Furthermore, y_0 is actually an interior point of Ω.* However, we will not try to prove this claim in the present work.

We now take under account the family of functions $u_{\varepsilon,y}(x) = \dfrac{\phi(x)}{(\varepsilon + |x-y|^2)^{\frac{1}{2}}}$, where $y \in \Omega$,and $\phi \in C_c^\infty(\Omega)$ is an arbitrary smooth function with compact support within Ω, which is constant near y. We prove the following essential lemma:

Lemma 13 *Let $\varepsilon > 0$, $y \in \Omega$, $\phi \in C_c^\infty(\Omega)$ be a smooth function with compact support within Ω which is constant near y, and*

$$u_{\varepsilon,y}(x) = \frac{\phi(x)}{(\varepsilon + |x-y|^2)^{\frac{1}{2}}}.$$

Then, there is a positive constant C (independent of ε), such that

$$Q_\lambda(u_{\varepsilon,y}) = S_3 + C\varepsilon^{\frac{1}{2}}\Big(\int_\Omega \frac{|\nabla\phi|^2}{|x-y|^2}dx - \lambda\int_\Omega \frac{\phi^2}{|x-y|^2}dx\Big) + O(\varepsilon) \tag{52}$$

as $\varepsilon \to 0^+$.

Proof We assume that $y \in \Omega$ and $\phi \equiv 1$ near y. We have that

$$\nabla u_{\varepsilon,y}(x) = \frac{\nabla\phi(x)}{(\varepsilon + |x-y|^2)^{\frac{1}{2}}} - \frac{\phi(x)(x-y)}{(\varepsilon + |x-y|^2)^{\frac{3}{2}}}$$

Thus,

$$\int_\Omega |\nabla u_{\varepsilon,y}(x)|^2 dx = \int_\Omega \left(\frac{|\nabla\phi(x)|^2}{\varepsilon + |x-y|^2} - 2\frac{\phi(x)\nabla\phi(x)\cdot(x-y)}{(\varepsilon + |x-y|^2)^2} + \frac{\phi^2(x)|x-y|^2}{(\varepsilon + |x-y|^2)^3} \right) dx$$

We now observe that

$$-2\int_\Omega \frac{\phi(x)\nabla\phi(x)\cdot(x-y)}{(\varepsilon + |x-y|^2)^2}dx = -\int_\Omega \frac{\nabla(\phi^2(x))\cdot(x-y)}{(\varepsilon + |x-y|^2)^2}dx = \int_\Omega \phi^2(x)\Big[\frac{3}{(\varepsilon + |x-y|^2)^2} - \frac{4|x-y|^2}{(\varepsilon + |x-y|^2)^3}\Big]dx$$

This means that

$$\int_\Omega |\nabla u_{\varepsilon,y}(x)|^2 dx = \int_\Omega \frac{|\nabla\phi(x)|^2}{\varepsilon + |x-y|^2}dx + 3\varepsilon\int_\Omega \frac{\phi^2(x)}{(\varepsilon + |x-y|^2)^3}dx \tag{53}$$

Now,

$$\int_\Omega \frac{|\nabla\phi(x)|^2}{\varepsilon + |x-y|^2}dx = \int_\Omega \frac{|\nabla\phi(x)|^2}{|x-y|^2}dx - \varepsilon\int_\Omega \frac{|\nabla\phi(x)|^2}{(\varepsilon + |x-y|^2)|x-y|^2}dx$$

But since $\nabla\phi \equiv 0$ near y, we have $\displaystyle\int_\Omega \frac{|\nabla\phi(x)|^2}{(\varepsilon + |x-y|^2)|x-y|^2}dx = O(1)$ and as a result

$$\int_\Omega \frac{|\nabla\phi(x)|^2}{\varepsilon + |x-y|^2}dx = \int_\Omega \frac{|\nabla\phi(x)|^2}{|x-y|^2}dx + O(\varepsilon) \tag{54}$$

Moreover,

$$\int_\Omega \frac{\phi^2(x)}{(\varepsilon + |x-y|^2)^3}dx = \int_\Omega \frac{\phi^2(x) - 1}{(\varepsilon + |x-y|^2)^3}dx + \int_\Omega \frac{1}{(\varepsilon + |x-y|^2)^3}dx = \int_\Omega \frac{1}{(\varepsilon + |x-y|^2)^3}dx + O(1).$$

We have that $\int_\Omega \frac{1}{(\varepsilon+|x-y|^2)^3}dx = \int_{\mathbb{R}^3} \frac{1}{(\varepsilon+|x-y|^2)^3}dx + O(1)$ and with the change of variables $r=\varepsilon^{\frac{1}{2}}t$, $(r=|x-y|)$ we get:

$$\int_{\mathbb{R}^3} \frac{1}{(\varepsilon+|x-y|^2)^3}dx = \omega_3 \int_0^{+\infty} \frac{r^2}{(\varepsilon+r^2)^3}dr = \omega_3 \varepsilon^{-\frac{3}{2}} \int_0^{+\infty} \frac{t^2}{(1+t^2)^3}dt$$

As a result,

$$\int_\Omega \frac{\phi^2(x)}{(\varepsilon+|x-y|^2)^3}dx = 3\omega_3 \varepsilon^{-\frac{1}{2}} \int_0^{+\infty} \frac{t^2}{(1+t^2)^3}dt + O(\varepsilon) \tag{55}$$

Using (54) and (55) in (53) we get that as $\varepsilon \to 0^+$,

$$\int_\Omega |\nabla u_{\varepsilon,y}(x)|^2 dx = K_1 \varepsilon^{-\frac{1}{2}} + \int_\Omega \frac{|\nabla \phi(x)|^2}{|x-y|^2}dx + O(\varepsilon) \tag{56}$$

where $K_1 = 3\omega_3 \int_0^{+\infty} \frac{t^2}{(1+t^2)^3}dt = \frac{3\pi}{16}$ as computed in Lemma 7.

Also,

$$\int_\Omega u_{\varepsilon,y}^6(x)dx = \int_\Omega \frac{\phi^6(x)}{(\varepsilon+|x-y|^2)^3}dx = \int_\Omega \frac{\phi^6(x)-1}{(\varepsilon+|x-y|^2)^3}dx + \int_\Omega \frac{1}{(\varepsilon+|x-y|^2)^3}dx$$

Since $\phi \equiv 1$ near y,

$$\int_\Omega u_{\varepsilon,y}^6(x)dx = \int_\Omega \frac{1}{(\varepsilon+|x-y|^2)^3}dx + O(1) = \int_{\mathbb{R}^3} \frac{1}{(\varepsilon+|x-y|^2)^3}dx + O(1) = \varepsilon^{-\frac{3}{2}}\omega_3 \int_0^{+\infty} \frac{t^2}{(1+t^2)^3}dt + O(1) \Rightarrow$$

$$\Rightarrow \int_\Omega u_{\varepsilon,y}^6(x)dx = \varepsilon^{-\frac{3}{2}} \left(\omega_3 \int_0^{+\infty} \frac{t^2}{(1+t^2)^3}dt + O(\varepsilon^{\frac{3}{2}}) \right)$$

Thus,

$$\left(\int_\Omega u_{\varepsilon,y}^6(x)dx \right)^{\frac{1}{3}} = \varepsilon^{-\frac{1}{2}} \left(\omega_3 \int_0^{+\infty} \frac{t^2}{(1+t^2)^3}dt + O(\varepsilon^{\frac{3}{2}}) \right)^{\frac{1}{3}} = K_2 \varepsilon^{-\frac{1}{2}} + O(\varepsilon^{\frac{1}{2}}) \tag{57}$$

where

$$K_2 = \left(\omega_3 \int_0^{+\infty} \frac{t^2}{(1+t^2)^3}dt \right)^{\frac{1}{3}} = (\frac{\pi}{16})^{\frac{1}{3}}$$

Furthermore,

$$\int_\Omega u_{\varepsilon,y}^2 dx = \int_\Omega \frac{\phi^2(x)}{\varepsilon+|x-y|^2}dx = \int_\Omega \frac{\phi^2(x)}{|x-y|^2}dx - \varepsilon \int_\Omega \frac{\phi^2(x)}{(\varepsilon+|x-y|^2)|x-y|^2}dx \tag{58}$$

Let $R>0$ such that $\Omega \subseteq B_R(y)$. Since $\phi \equiv 1$ near y, we have that

$$\int_\Omega \frac{\phi^2(x)}{(\varepsilon+|x-y|^2)|x-y|^2}dx = O(1) + \int_{B_R(y)} \frac{1}{(\varepsilon+|x-y|^2)|x-y|^2}dx$$

By the usual computations,

$$\int_{B_R(y)} \frac{1}{(\varepsilon+|x-y|^2)|x-y|^2}dx = \omega_3 \int_0^R \frac{1}{\varepsilon+r^2}dr = \omega_3 \varepsilon^{-\frac{1}{2}} \int_0^{R\varepsilon^{-\frac{1}{2}}} \frac{1}{1+t^2}dt = O(\varepsilon^{-\frac{1}{2}})$$

Coming back to (58), this means that

$$\int_\Omega u_{\varepsilon,y}^2 dx = \int_\Omega \frac{\phi^2(x)}{|x-y|^2}dx + O(\varepsilon^{\frac{1}{2}}) \tag{59}$$

Using (56),(57) and (59) we get that as $\varepsilon \to 0^+$

$$Q_\lambda(u_{\varepsilon,y}) = \frac{K_1\varepsilon^{-\frac{1}{2}} + \int_\Omega \frac{|\nabla\phi(x)|^2}{|x-y|^2}dx - \lambda\int_\Omega \frac{\phi^2(x)}{|x-y|^2}dx + O(\varepsilon^{\frac{1}{2}})}{K_2\varepsilon^{-\frac{1}{2}} + O(\varepsilon^{\frac{1}{2}})} = \frac{K_1 + \varepsilon^{\frac{1}{2}}[\int_\Omega \frac{|\nabla\phi(x)|^2}{|x-y|^2}dx - \lambda\int_\Omega \frac{\phi^2(x)}{|x-y|^2}dx] + O(\varepsilon)}{K_2 + O(\varepsilon)}$$

Finally,

$$Q_\lambda(u_{\varepsilon,y}) = S_3 + K_2^{-1}\varepsilon^{\frac{1}{2}}[\int_\Omega \frac{|\nabla\phi(x)|^2}{|x-y|^2}dx - \lambda\int_\Omega \frac{\phi^2(x)}{|x-y|^2}dx] + O(\varepsilon) \tag{60}$$

Lemma 14 *The space of locally flat smooth functions is dense in* $H_0^1(\Omega, |x-y|^{-2})$.

Proof

Let $x \in \Omega$. Without loss of generality, let $x = 0$.

Since the space $C_c^\infty(\Omega)$ is by construction dense in $H_0^1(\Omega, |x|^{-2})$, it suffices to show that every smooth function can be approximated by smooth functions which are flat in a neighborhood of the origin. Let $\phi \in C_c^\infty(\Omega)$ and $\varepsilon > 0$ such that B_ε is entirely contained in Ω.

It is now standard that we can find a function $\phi_\varepsilon \in C_c^\infty(\Omega)$, such that $\phi|_{\Omega\setminus B_\varepsilon} = \phi_\varepsilon|_{\Omega\setminus B_\varepsilon}$ and $\phi_\varepsilon = \phi(0)$ in $B_{\frac{\varepsilon}{2}}$. This means that ϕ_ε is a locally flat smooth function.

Now,

$$\int_\Omega \frac{|\nabla(\phi_\varepsilon - \phi)|^2}{|x|^2}dx = \int_{B_\varepsilon} \frac{|\nabla(\phi_\varepsilon - \phi)|^2}{|x|^2}dx \le \sup_{B_\varepsilon}\{||\nabla\phi_\varepsilon|^2 - |\nabla\phi|^2|\}\int_{B_\varepsilon} \frac{1}{|x|^2}dx$$

We choose ϕ_ε such that $||\nabla\phi_\varepsilon|^2 - |\nabla\phi|^2|$ remains uniformly bounded as $\varepsilon \to 0^+$ and since $\int_{B_\varepsilon} \frac{1}{|x|^2}dx = 4\pi\varepsilon$ we have that $\phi_\varepsilon \to \phi$ in the norm of $H_0^1(\Omega, |x|^{-2})$ as $\varepsilon \to 0^+$. $\square$

We will now use Lemmas 13 and 14 to give the proof of our main theorem 6.

Proof of Theorem 6

Let $\mu^* < \lambda < \lambda_1$. Then, by definition of μ^*, there is a point $y \in \Omega$ and a function $u \in H_0^1(\Omega, |x-y|^{-2})$, such that

$$\frac{\int_\Omega \frac{|\nabla u|^2}{|x-y|^2}dx}{\int_\Omega \frac{u^2}{|x-y|^2}dy} < \lambda$$

In view of lemma 14, there is a smooth function ϕ which is constant near y, such that

$$\frac{\int_\Omega \frac{|\nabla\phi|^2}{|x-y|^2}dx}{\int_\Omega \frac{\phi^2}{|x-y|^2}dx} < \lambda \Leftrightarrow \int_\Omega \frac{|\nabla\phi|^2}{|x-y|^2}dx - \lambda\int_\Omega \frac{\phi^2}{|x-y|^2}dx < 0.$$

In view of Lemma 13, we use this particular function ϕ in $u_{\varepsilon,y}(x) = \frac{\phi(x)}{(\varepsilon + |x-y|^2)^{\frac{1}{2}}}$. As a result, $Q_\lambda(u_{\varepsilon,y}) < S_3 + O(\varepsilon)$ and therefore, for small enough $\varepsilon > 0$, $Q_\lambda(u_\varepsilon) < S_3 \Rightarrow J_\lambda < S_3$ $\square$

Moreover, as somebody would expect, μ^* has the following interesting properties:

Lemma 15 *If B is a ball, then $\mu^*(B) = \frac{1}{4}\lambda_1(B) = \lambda^*(B)$.*

Proof Without loss of generality, we assume again that $B = B_1$. Thus, $\lambda_1(B) = \pi^2$. In Theorem 6, we have seen that $J_\lambda < S_3$, $\forall\ \lambda \in (\mu^*(B), \lambda_1(B))$. By continuity of the mapping $\lambda \mapsto J_\lambda$, this means that $J_{\mu^*(B)} \leq S_3$. On the other hand, we know from Part B of Theorem 3 that we have the inequality $J_{\frac{1}{4}\lambda_1(B)} \geq S_3$. By monotonicity of J_λ, this means that $\mu^*(B) \geq \dfrac{1}{4}\lambda_1(B)$. For the reverse inequality , it suffices to choose $y = 0$ and insert $cos(\frac{\pi}{2}|x|) \in H_0^1(\Omega, ||x||^{-2})$ as a test function. We then have that $\mu^*(B) \leq \dfrac{\displaystyle\int_B \frac{|\nabla(cos(\frac{\pi}{2}|x|))|^2}{|x|^2}dx}{\displaystyle\int_B \frac{(cos(\frac{\pi}{2}|x|))^2}{|x|^2}dx}$. Turning to spherical coordinates , this means that

$$\mu^*(B) \leq \frac{\displaystyle\int_0^1 (-\frac{\pi}{2}sin(\frac{\pi}{2}r))^2 dr}{\displaystyle\int_0^1 cos^2(\frac{\pi}{2}r)dr} = \frac{\pi^2}{4}\frac{\displaystyle\int_0^1 sin^2(\frac{\pi}{2}r)dr}{\displaystyle\int_0^1 cos^2(\frac{\pi}{2}r)dr} = \frac{\pi^2}{4} = \frac{\lambda_1(B)}{4}.$$

As a result, we have that $\mu^*(B) = \dfrac{\lambda_1(B)}{4}$. □

Lemma 16 *We have that $\mu^*(\Omega) \geq \mu^*(\Omega^*)$.*

Proof

This statement is a direct consequence of the fact that $J_{\mu^*(\Omega)} \leq S_3$, $J_{\mu^*(\Omega^*)} \geq S_3$, and that the mapping $\lambda \mapsto J_\lambda$ is non-increasing. □

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